Modeling finite deformations of an axially moving elastic plate with a mixed Eulerian-Lagrangian kinematic description

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Abstract

We consider the motion of a flexible plate across a domain, bounded by two parallel lines. Kinematically prescribed velocities of the plate, entering the domain and leaving it, may vary in space and time. The corresponding deformation of the plate is quasistatically analyzed using the geometrically nonlinear model of a Kirchhoff shell with a mixed Eulerian-Lagrangian kinematic description. In contrast to the formulations, available in the literature, both the in-plane and the out-of-plane deformations are unknown a priori and may be arbitrarily large. The particles of the plate travel across a finite element mesh, which remains fixed in the axial direction. The evident advantage of the approach is that the boundary conditions need to be applied at fixed edges of the finite elements. In the paper, we present the mathematical formulation and demonstrate its consistency by comparing the solution of a benchmark problem against results, obtained with conventional Lagrangian finite elements.

Keywords: Axially moving plates, nonlinear theory of shells, multiplicative decomposition, Eulerian-Lagrangian description, finite element method

1. Introduction

The problem of mathematical modeling of nonlinear deformations of axially moving structures is both challenging and practically important. Numerous papers deal with the transverse vibrations of axially moving beams and strings, see a review paper Ref. [1]. While an extension towards nonlinearly coupled in-plane and out-of-plane vibrations of a moving plate is presented in Ref. [4], this model is incapable of representing arbitrarily deformed configurations of the plate. Moreover, the use of Lagrange equations of motion to an open system with influx and outflux of the mass is not justified in the latter reference.

Large axial deformation and bending of a beam, which can move across a fixed domain, is treated by the authors of Ref. [5] using a suitable change of variables. We apply a similar technique for the quasistatic modeling of finite deformations of a plate moving across a given domain in the direction x. The velocities of the plate are prescribed at two boundaries of the domain $x = 0$ and $x = L$, see Fig. 1.

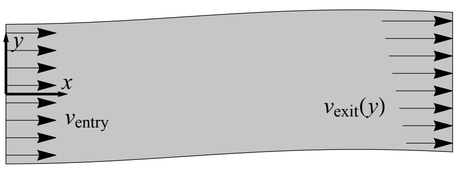

Figure 1: Quasistatic deformation of a plate with prescribed velocities at the boundaries

2. Mathematical model

In the present study we assume the velocity v_{entry}, with which the plate is entering the domain, to be constant. In the future, arbitrary velocity profiles may be incorporated into the model using the notion of intrinsic strains and the technique of multiplicative decomposition of the deformation gradient, Ref. [6]. The varying velocity profile $v_{\text{exit}}(y)$, with which the material particles of the plate are leaving the domain at $x = L$, leads to the time varying deformation of the plate. Searching for a sequence of quasistatic equilibrium states of the elastic plate, we need to minimize the total energy of the active region of the plate, which is currently residing in the considered domain. Not going into details concerning the time integration, which is intended to be discussed in future publications, we focus on the kinematic modeling of the deformation of the plate.

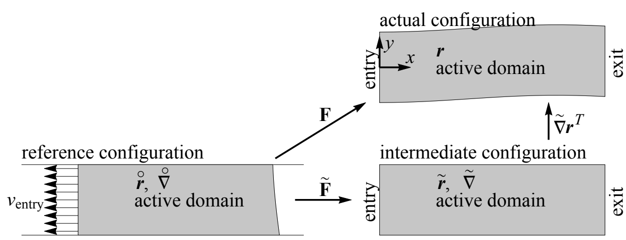

Figure 2: Two-stage mapping from the reference configuration to the actual one: the intermediate configuration is fixed in space

*This work has been partially carried out at LCM GmbH as part of a K2 project. K2 projects are financed using funding from the Austrian COMET-K2 programme. The COMET K2 projects at LCM are supported by the Austrian federal government, the federal state of Upper Austria, the Johannes Kepler University and all of the scientific partners which form part of the K2-COMET Consortium.

The plane reference configuration $0 \le \mathring{y} \le w$ is straight (w is the undeformed width and $\mathring{\boldsymbol{r}} = \mathring{x}\boldsymbol{i} + \mathring{y}\boldsymbol{j}$ is the position vector in the reference configuration), see Fig. 2. The present mixed Eulerian-Lagrangian kinematic description makes use of a fixed intermediate configuration with the position vector $\tilde{\boldsymbol{r}}$ such, that the mapping of the positions of particles from the reference configuration to the actual one $\boldsymbol{r} = \boldsymbol{r}(\mathring{\boldsymbol{r}})$ comprises two stages:

$$\boldsymbol{r} = \tilde{\boldsymbol{r}} + u_y(\tilde{\boldsymbol{r}})\boldsymbol{j} + u_z(\tilde{\boldsymbol{r}})\boldsymbol{k}, \quad \tilde{\boldsymbol{r}} = \mathring{\boldsymbol{r}} + u_x(\tilde{\boldsymbol{r}})\boldsymbol{i}. \tag{1}$$

The simplicity of this description essentially distinguishes it from the known Arbitrary Lagrangian-Eulerian formulation, Ref. [2]. All fields are functions of the place in the fixed intermediate configuration, in which a finite element discretization is performed.

We apply the classical Kirchhoff shell model, see Refs. [3, 7]. The total gradient of deformation of the plate with the differential operator of the intermediate configuration $\tilde{\nabla}$ results in the form

$$\mathbf{F} = \mathring{\nabla}\boldsymbol{r}^T = \tilde{\nabla}\boldsymbol{r}^T \cdot \tilde{\mathbf{F}}, \quad \tilde{\mathbf{F}} = \left(\mathbf{I}_2 - \boldsymbol{i}\tilde{\nabla}u_x\right)^{-1}. \tag{2}$$

Here $\mathbf{I}_2 = \boldsymbol{ii} + \boldsymbol{jj}$ is the in-plane identity tensor, and the expression for the gradient of deformation from the reference to the intermediate configuration $\tilde{\mathbf{F}}$ follows from $\mathbf{I}_2 = \mathring{\nabla}\mathring{\boldsymbol{r}} = \tilde{\mathbf{F}}^T \cdot \tilde{\nabla}(\tilde{\boldsymbol{r}} - u_x\boldsymbol{i})$. The strain measures of a classical shell

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}_2\right), \quad \mathbf{K} = \mathbf{F}^T \cdot \mathbf{b} \cdot \mathbf{F} \tag{3}$$

feature the actual second metric tensor $\mathbf{b} = -\nabla\boldsymbol{n}$, and after mathematical transformations we express the tensor of bending strains with the operator of the intermediate configuration:

$$\mathbf{K} = \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{K}} \cdot \tilde{\mathbf{F}}, \quad \tilde{\mathbf{K}} = \tilde{\nabla}\tilde{\nabla}\boldsymbol{r} \cdot \boldsymbol{n}. \tag{4}$$

Now, the strain energy per unit area in the reference configuration is computed as a quadratic form

$$U = \frac{1}{2}\left(A_1(\mathrm{tr}\,\mathbf{E})^2 + A_2\mathbf{E}\cdot\cdot\mathbf{E} + D_1(\mathrm{tr}\,\mathbf{K})^2 + D_2\mathbf{K}\cdot\cdot\mathbf{K}\right) \tag{5}$$

with known coefficients, Ref. [3, 7]. The total strain energy

$$U_\Sigma = \int\limits_0^L \int\limits_{-w/2}^{w/2} U(\det\tilde{\mathbf{F}})^{-1}\mathrm{d}\tilde{y}\,\mathrm{d}x \tag{6}$$

is integrated in the intermediate configuration using the finite element discretization of displacements u_x, u_y, u_z and minimized.

3. Numerical benchmark problem

Prior to modeling the axial motion, we test the formulation by seeking the equilibrium of a trapezoidal plate of the width w and side lengths L and $L + u_{x0}$, see Fig. 3. The inclined edge is rotated parallel to the right one by kinematically prescribed displacements u_x and u_y such, that the actual configuration is bounded by the lines $x = 0$ and $x = L$. The mapping Eq. (1) is thus possible with the intermediate configuration $0 \le x \le L$, $0 \le \tilde{y} \le w$, which is discretized using C^1 continuous finite element approximation of displacements, presented in Refs. [7, 8].

The compressed shell buckles out of plane, and the region with $u_z < 0$ is "shadowed" by the gray initial configuration in Fig. 3. The transverse edges of the finite element mesh remain parallel in the deformed configuration. This corresponds to Eq. (1), as the mapping $\boldsymbol{r}(\tilde{\boldsymbol{r}})$ includes only u_y and u_z.

The considered parameters of the model in SI system are $L = 1$, $w = 0.4$, thickness of the plate $5 \cdot 10^{-3}$, Young's modulus $2.1 \cdot 10^{11}$ and Poisson's ratio 0.3. In Table 1 we summarized the maximum and minimum values of the out-of-plane displacements, computed for various discretizations using the present method and the conventional shell finite elements with Lagrangian description, discussed in the above references. The current implementation of the mixed Eulerian-Lagrangian finite element formulation using *Wolfram Mathematica* is yet restricted concerning the size of the mesh, but one can conclude that the results converge to the same solution.

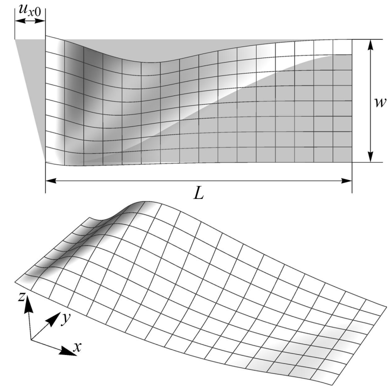

Figure 3: Deformation of a trapezoidal plate, seen from above (together with the undeformed configuration) and from the side

Table 1: Mesh convergence and comparison of the mixed Eulerian-Lagrangian and traditional Lagrangian frameworks

Discretization,	Mixed E.-L.		Lagrangian	
$n_x \times n_y$	$\min u_z$	$\max u_z$	$\min u_z$	$\max u_z$
4×2	-0.07270	0.18577	-0.07322	0.18138
8×4	-0.05846	0.18427	-0.05831	0.18256
16×8	-0.05542	0.18319	-0.05527	0.18259
32×16	—	—	-0.05490	0.18262

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