The concept of entropy is a measure of disorder or randomness in mathematics. Entropy in physics and probability theory is a fundamental concept with applications ranging from statistical mechanics to information theory. The study of entropy in partial differential equations has a long history, dating back several decades. These methods have been used to derive solutions to diffusive equations, and to study the large-time asymptotic behavior of solutions. Key inequalities, such as the discrete and geometric Poincaré–Sobolev inequalities, are crucial in this context.

The purpose of this book is to provide an overview of these methods, which can be found in the literature, not stated in the widest generality. The intent is to give a sense that the functional inequalities are natural generalizations of the original stochastic view point of Markowich, Toscani, and Villani. A number of applications are given, based on the theory of large deviations and the analysis of reaction–diffusion equations, such as those arising in population dynamics and population genetics.

The book consists of three parts. Chapter 1 introduces the basic notions of entropy in physics and probability theory. Chapter 2 is devoted to entropy methods for partial differential equations, focusing on the study of the large-time behavior of solutions. Chapter 3 is concerned with the application of these methods to the study of reaction–diffusion equations, with a focus on the role of entropy in understanding the asymptotic behavior of solutions.
methods in an efficient, higher order equations.

Entropy methods are here rove the global existence of ess. These techniques were Di Francesco, Pietschmann, ms are rather technical since eak convergence arguments in the appendix.

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ian Science Fund (FWF),

Ansgar Jüngel

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Abstract

The concept of entropy was introduced in the 1850s to describe the heat produced during irreversible processes. It was a significant step forward from the earlier notion of entropy, which was partially developed by Clausius in the 1840s. The word ‘tropo-’ from the Greek means ‘turning’ and was used by Clausius’ predecessor, Carnot, to describe the idea of an ensemble of microstates that correspond to a macrostate of the system and the macrostate energy increases when the system moves from one different microstate to another.

\[ S = k_B \ln \Omega \]

where \( k_B \) is the Boltzmann constant and \( \Omega \) is the number of microstates per macrostate.

Keywords

asymptotics

1.1 Entropy

The concept of entropy was introduced in the 1850s to describe the heat produced during irreversible processes. It was a significant step forward from the earlier notion of entropy, which was partially developed by Clausius in the 1840s. The word ‘tropo-’ from the Greek means ‘turning’ and was used by Clausius’ predecessor, Carnot, to describe the idea of an ensemble of microstates that correspond to a macrostate of the system and the macrostate energy increases when the system moves from one different microstate to another.

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