ASYMPTOTIC ANALYSIS OF A ROTARY WAVE IN A CYLINDRICAL TANK

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Summary The free surface flow in a vertical, cylindrical, rotating container will be considered when axis-symmetric angular and radial shear stress distributions are applied on the free surface. It is well known that in the inviscid case a rotary wave which can be described by a flow potential exists. We want to determine the stability limit of the axis-symmetric base state with respect to the rotary wave for small Ekman and Froude numbers. Under the above assumption, the critical flow conditions can be determined analytically. The results can be verified experimentally using a vertical, cylindrical container partially filled with water where the top lid rotates with a given angular velocity. Thus, the induced air flow will exert shear stresses in the angular and radial direction onto the water surface. Above a certain threshold, rotary waves can be observed.

PROBLEM FORMULATION

Using a Francis turbine in phase condenser mode, the rotor sets the air above the water level in the suction pipe into a rotation. If the angular speed of the rotor is above a critical value a rotary wave in the water forms. The angular wave speed is almost independent of the rotor speed, and the amplitude of the wave can be considerably large. Thus, we want to explain the excitation mechanism using asymptotic analysis of this wave. In particular, we want to determine critical conditions when the axial-symmetric surface of the water becomes unstable, and the rotary wave develops. Since we are interested in an analytical solution, we define a simplified model problem.

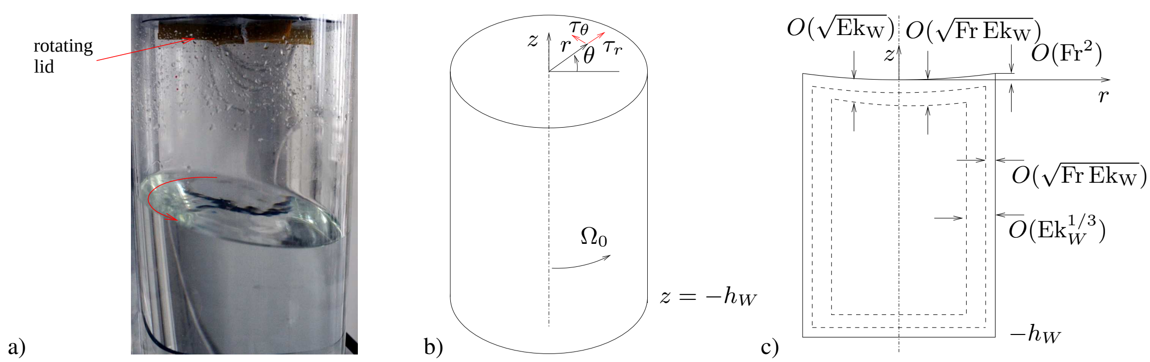

Figure 1: a) Rotatry wave in a vertical, cylindrical container induced by a rotating lid, b) coordinate system and applied shear stresses c) boundary-layers relevant for the stability analysis

Consider a vertical cylindrical tank with radius R partially filled with water. Assume that on the free surface shear stresses in azimuthal $\tau_\theta = \tau_{\theta,0} r/R$ and inward radial $\tau_r = \tau_{r,0} r/R$ direction are applied. These stresses model the action of the rotating air flow onto the water. The shear stress distributions are assumed to be axis-symmetric and depend linearly on the distance from the axis of the container.

For an inviscid fluid, the rotary gravity wave given by the flow potential $\phi = J_1(\mu_1 r)\sin(\theta - \omega_0 t)e^{\mu_1 z}$ is well known, see figure 1a, where J_1 is the Bessel function with index 1, μ_1 is the smallest zero of its derivative, ω_0 is the dimensionless angular wave speed, see [1]. Here and in the following, r and z denote the dimensionless radial and vertical coordinate. They are referred to the radius R of the container. The angular velocity is made dimensionless with $\sqrt{g/R}$, where, and g is the gravity acceleration. All other quantities like time, velocity are scaled accordingly. For the definition of the coordinate system and the boundary conditions, see figure 1b.

To facilitate a complete analytical solution, we assume that the container rotates with given angular velocity Ω_0.Thus, we define the following dimensionless numbers,

$$\mathrm{Ek}_W = \frac{\nu}{R^2\Omega_0}, \quad \mathrm{Fr} = \frac{\Omega_0}{\sqrt{g/R}} \quad \delta = \sqrt{\mathrm{Ek_W}}\frac{\tau_{\theta,0}}{\rho\nu\Omega_0}, \quad \tau = \frac{\tau_{r,0}}{\tau_{\theta,0}}.$$

where ρ, and ν are the density and the viscosity of the fluid, respectively. Assuming $\delta \ll 1$ and $\mathrm{Ek_W} \ll 1$ the base flow can be obtained by linearizing the Navier-Stokes equation around the solid body rotation with angular speed Ω_0. Thus, it is given

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by a solid body rotation in the core (with a perturbed angular speed) and boundary-layers of width $\sqrt{\mathrm{Ek_W}}$ at the free surface and the bottom of the container, see [2]

$$u_B = \mathrm{Fr}\,\delta r \hat{u}_B(\eta), \quad v_B = \mathrm{Fr}\, r(1+\delta(\bar{v}_B + \hat{v}_B(\eta))), \quad w_B = \mathrm{Fr}\,\delta\sqrt{\mathrm{Ek}_W}(\bar{w}_B + \hat{w}_B(\eta)), \qquad \eta = z/\sqrt{\mathrm{Ek}_W}.$$

Here, u_B, v_B, W_B denote the velocity components of the base flow in the radial, azimuthal, and vertical direction. Since the boundary-layer of the base flow along the container wall does not play a role in the following asymptotic stability analysis, the corresponding terms are omitted here.

ASYMPTOTIC STABILITY ANALYSIS

We linearize the governing equations (Navier-Stokes equations, the kinematic and dynamic boundary conditions, respectively) around the base flow. We are looking for a rotary eigensolution in the double limit

$\mathrm{Ek_W} \to 0$, $\mathrm{Fr} \to 0$, $\delta << 1$, $\tau = O(1)$. Thus, the asymptotic ansatz for the radial velocity component is of the form

$$u \sim \left[\left(u_0^{(s)} + \sqrt{\mathrm{Ek_W Fr}}u_G^{(s)} + \mathrm{Fr}\,u_1^{(s)} + \mathrm{Fr}^2\,u_2^{(s)}\right)\sin(\theta-\omega t) + \left(\sqrt{\mathrm{Ek_W Fr}}u_G^{(c)} + \mathrm{Fr}\,u_1^{(c)} + \mathrm{Fr}^2\,u_2^{(c)}\right)\cos(\theta-\omega t)\right]e^{g\,t}$$

with the dimensionless angular wave speed $\omega = \omega_0 + \sqrt{\mathrm{Ek_W Fr}}\omega_G + \mathrm{Fr}\omega_1 + ...$, and the dimensionless growth rate $g = -d\sqrt{\mathrm{Ek_W Fr}} + \mathrm{Fr}g_1 + \mathrm{Fr}^2 g_2 + ...$

Note that we have written in the ansatz only the order of magnitude of the terms needed for the discussion of the stability. The terms $u_G^{(s)}$, $u_1^{(s)}$, u_2 have to be expanded with respect to $\mathrm{Ek_W}$ and δ and they may have boundary-layers of thickness $\sqrt{\mathrm{Ek_W Fr}}$ and $\sqrt{\mathrm{Ek_W}}$, respectively. The leading order term is given by the inviscid rotary wave supplemented by boundary layers at the wall. In the following we will shortly discuss the role of the terms in the expansion.

The inviscid rotary wave does not satisfy the no-slip boundary conditions at the container wall and bottom. Thus, boundary-layers of thickness $\sqrt{\mathrm{Ek_W Fr}}$ will develop along the wall, see figure 1c, and the bottom causing a velocity component normal to the wall of the order $\sqrt{\mathrm{Ek_W Fr}}$ at the outer edge of the boundary layer. Thus, a secondary flow of the same order is induced in the core region. Applying the kinematic, and dynamic boundary condition in the normal direction at the free surface yields a condition for the decay rate d and the shift of the angular velocity ω_G.

At the free surface, the shear stresses are prescribed. The rotary wave displaces the free surface. Thus, a boundary-layer correction of the radial and azimuthal component of the rotary wave of the order $\delta\mathrm{Fr}^{3/2}/\sqrt{\mathrm{Ek_W}}$ is necessary. This, however, induces a vertical flow velocity of order $\mathrm{Fr}^2\delta$ and depending on the sign a growth or decay rate of the wave of the same order. A detailed analysis shows that this correction is positive (growth rate) if the radial component of the surface velocity is negative.

In the core region of the flow, the rotary wave interacts with the base flow causing an additional correction to the wave speed and the growth/decay rate. However, the corrections to the growth rate are only created in the boundary layer of the base flow (width $\sqrt{\mathrm{Ek_W}}$). It turns out that only the second order term g_2 is different from zero. The contribution of this terms of the same order $\delta\mathrm{Fr}$ as the correction of the growth rate induced by the dynamic boundary condition in the radial direction.

Under the above assumptions all this contributions to growth/decay rate can be evaluated analytically yielding our main result

$$g \sim -\sqrt{\mathrm{Fr\,Ek_W}}d(h_W) + \mathrm{Fr}\sqrt{\mathrm{Ek_W}}\tilde{g}_1 - \mathrm{Fr}^2\delta\hat{u}_B(0)\tilde{g}_2(h_W),$$

where d, and $\tilde{g}_2$ are positive constants depending only on the dimensionless water depth h_W. Setting the growth rate equal to zero, conditions for neutral stability are obtained. Thus, for neutral stability the radial surface velocity $\hat{u}_B(0)$ of the base flow has to be negative.

NON-ROTATING CONTAINER AND EXPERIMENTAL VERIFICATION

In the case of a non-rotating container, see figure 1a, the main ideas of the present analysis hold. However, the equations have to be solved numerically. Formally, we have to set $\delta = 1$ and set $\Omega_0 = (\tau_{\theta,0}^2/\rho^2\nu R^2)^{1/3}$, a scale for the angular velocity induced by the angular shear stress at the surface. The angular speed of the wave can be measured easily from a video. The angular velocity of the core region can be estimated by the time an immersed particle needs for a complete revolution. The ratio of the angular velocity of the flow in the core region to the angular wave speed is of the order Fr. In case of neutral stability, we have $\mathrm{Fr} \sim (\mathrm{Ek_W\,Fr})^{1/4} = (\frac{\nu}{\mathrm{R}^2\sqrt{\mathrm{g/R}}})^{1/4}$ which is in agreement with experimental observations.

References

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