# Proving Craig and Lyndon Interpolation Using Labelled Sequent Calculi*

Roman Kuznets

Institut für Computersprachen
TU Wien
roman@logic.at

**Abstract.** Interpolation is a fundamental logical property with applications in mathematics, computer science, and artificial intelligence. In this paper, we develop a general method of translating a semantic description of modal logics via Kripke models into a constructive proof of the Lyndon interpolation property (LIP) via labelled sequents. Using this method we demonstrate that all frame conditions representable as Horn formulas imply the LIP and that all 15 logics of the modal cube, as well as the infinite family of transitive Geach logics, enjoy the LIP.

**Keywords:** Craig interpolation, Lyndon interpolation, labelled sequents, modal logic, Geach formulas

## 1 Introduction

Interpolation is a fundamental logical property with applications in mathematics, computer science, and artificial intelligence. For instance, *uniform interpolation* is related to *variable forgetting*. The Craig Interpolation Property (CIP) states that, for any valid fact $A \to B$ of the logic, there must exist an *interpolant* $C$ in the common language of $A$ and $B$ such that both $A \to C$ and $C \to B$ are valid. The CIP is used, e.g., to prove correctness of algorithms for reasoning about knowledge bases with overlap in content [1]. The Lyndon Interpolation Property (LIP) strengthens the CIP by requiring that not just *propositional atoms* in $C$ but their *literals*, i.e., polarized propositional atoms, be common to $A$ and $B$. The LIP and CIP are known to imply the Beth definability property, which can be applied to rewritings in description logics [4], commonly used in knowledge representation [18].

In this paper, we develop a general method of translating a semantic description of a modal logic (with classical propositional background) via Kripke models into a constructive proof of the LIP. Hence, the *common language* is to be understood as common literals. While we formulate our results for the LIP, they are directly applicable to the weaker CIP too. The proof-theoretic method

* This material is based upon work supported by the Austrian Science Fund (FWF) Lise Meitner Grant M 1770-N25.

of proving the LIP is to construct an interpolant by induction on a derivation of (a representation of) $A \to B$ in a suitable analytic sequent calculus. The method is modular: if the sequent system is strengthened by an extra rule, only this additional rule needs to be checked to extend the LIP to the resulting stronger logic.

Until recently, a major weakness of the method was the limited expressivity of analytic sequent calculi. Recent advances extended the reach of the method to nested sequents ([6]) and hypersequents ([12]). These results were unified and generalized to a wide range of *internal* sequent-like formalisms in [13]. In this paper, we develop a similar method for the *external* formalism of labelled sequents[1], which is strictly more expressive [8] and was just recently shown in [5] to capture all modal logics complete w.r.t. first-order definable frame conditions. Moreover, labelled sequent rules can be effectively generated from these frame conditions. In this paper, we harness this strength by outlining sufficient criteria on the frame conditions to guarantee the LIP. We also provide an algorithm for constructing an interpolant.

The paper is structured as follows. In Sect. 2, we describe the formalism of labelled sequents (closely following [16]) and outline the method of proving the LIP using labelled sequents. In Sect. 3, we show how to construct an interpolant for all the labelled rules of the basic normal modal logic $\mathsf{K}$. In Sect. 4, we prove that all logics complete w.r.t. quantifier-free Horn formulas enjoy the LIP and argue that the restriction to Horn clauses is essential. We also extend these results to labelled sequents with equality atoms. In Sect. 5, we extend the method to several common types of Horn-like geometric rules and apply our findings to the infinite family of Geach logics. Section 6 contains related work, a summary of our results, and a discussion of future research.

## 2 Interpolation for Labelled Sequent Calculi

**Definition 1 (Labelled sequent).** *A* labelled sequent*, from now on a* sequent*, is an object $\Gamma \Rightarrow \Delta$ with $\Gamma$ and $\Delta$ being multisets*[2] *of* labelled formulas $\mathsf{w}:A$ *and* relational atoms $\mathsf{wRo}$*, where* $\mathsf{w}$ *and* $\mathsf{o}$ *are labels from a fixed countable set* $\mathsf{Lab}$ *and $A$ is a modal formula in negation normal form (NNF)*[3].

**Definition 2 (Kripke model).** *A* Kripke frame *is $(W, R)$ where $W \neq \varnothing$ and $R \subseteq W \times W$. A* Kripke model *$\mathcal{M}$ is $(W, R, V)$ where $V : \mathsf{Prop} \to 2^W$ is a function on the set* $\mathsf{Prop}$ *of propositional atoms. The satisfaction relation between $w \in W$ and modal formulas is defined recursively: $\mathcal{M}, w \Vdash P$ iff $w \in V(P)$; $\mathcal{M}, w \Vdash \overline{P}$ iff $w \notin V(P)$; $\wedge$ and $\vee$ behave classically; $\mathcal{M}, w \Vdash \Box A$ iff $\mathcal{M}, u \Vdash A$ whenever $wRu$; $\mathcal{M}, w \Vdash \Diamond A$ iff $\mathcal{M}, u \Vdash A$ for some $u$ such that $wRu$.*

[1] Unlike internal formalisms, external ones cannot generally be translated into formulas, typically because of the essential use of semantic elements, e.g., Kripke worlds.

[2] The method also works for sequence- and set-based sequents.

[3] NNF is used here to simplify the notation rather than out of necessity and means that negation is restricted to propositional atoms, creating two literals $P$ and $\overline{P}$ for each atom. Primary connectives are $\wedge$, $\vee$, $\Box$, and $\Diamond$. Negation $\overline{A}$ is a function of a formula $A$ defined via De Morgan laws. $A \to B := \overline{A} \vee B$.

| | | |
|---|---|---|
| $\mathsf{w}:P, \Gamma \Rightarrow \Delta, \mathsf{w}:P$ | $\mathsf{w}:\overline{P}, \Gamma \Rightarrow \Delta, \mathsf{w}:\overline{P}$ | $\mathsf{w}:\bot, \Gamma \Rightarrow \Delta$ |
| $\mathsf{w}:P, \mathsf{w}:\overline{P}, \Gamma \Rightarrow \Delta$ | $\Gamma \Rightarrow \Delta, \mathsf{w}:P, \mathsf{w}:\overline{P}$ | $\Gamma \Rightarrow \Delta, \mathsf{w}:\top$ |

**Table 1.** Initial sequents

$$\frac{\mathsf{w}:A, \mathsf{w}:B, \Gamma \Rightarrow \Delta}{\mathsf{w}:A \wedge B, \Gamma \Rightarrow \Delta} L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \mathsf{w}:A \quad \Gamma \Rightarrow \Delta, \mathsf{w}:B}{\Gamma \Rightarrow \Delta, \mathsf{w}:A \wedge B} R\wedge$$

$$\frac{\mathsf{w}:A, \Gamma \Rightarrow \Delta \quad \mathsf{w}:B, \Gamma \Rightarrow \Delta}{\mathsf{w}:A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \Delta, \mathsf{w}:A, \mathsf{w}:B}{\Gamma \Rightarrow \Delta, \mathsf{w}:A \vee B} R\vee$$

**Table 2.** Propositional rules for NNF

**Definition 3 (Labelled semantics).** *An* interpretation into a Kripke model *$\mathcal{M} = (W, R, V)$ is a map $[\![\cdot]\!]\colon \mathsf{Lab} \to W$ from labels to worlds. $\vDash [\![\Gamma \Rightarrow \Delta]\!]$, if the following holds: if $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash A$ for each $\mathsf{w} : A \in \Gamma$ and $[\![\mathsf{w}]\!] R [\![\mathsf{o}]\!]$ for each* $\mathsf{wRo} \in \Gamma$*, then $\mathcal{M}, [\![\mathsf{u}]\!] \Vdash B$ for some $\mathsf{u}:B \in \Delta$. A sequent $\Gamma \Rightarrow \Delta$ is* valid in a class $\mathcal{C}_\mathsf{L}$ of Kripke models*, written $\mathcal{C}_\mathsf{L} \vDash \Gamma \Rightarrow \Delta$, if $\vDash [\![\Gamma \Rightarrow \Delta]\!]$ for each $\mathcal{M} \in \mathcal{C}_\mathsf{L}$ and each interpretation $[\![\cdot]\!]$ into $\mathcal{M}$.*

The rules of the calculus **SK** for the basic normal modal logic $\mathsf{K}$ can be found in Tables 1–3 (this calculus is a trivial modification of the calculus **G3K** from [16, Table 11.5] for the NNF language). As is standard, we omit initial sequents $\mathsf{wRo}, \Gamma \Rightarrow \Delta, \mathsf{wRo}$, which do not affect completeness because the satisfaction relation ignores relational atoms in the consequent. Unless stated otherwise, from now on $\mathcal{C}_\mathsf{L}$ stands for an arbitrary class of Kripke models.

We replace a formula-level interpolation statement with a sequent-level Componentwise Interpolation Property (CWIP). While the concept of the CWIP for (labelled) sequents is the same as for nested sequents and hypersequents, the labelled notation facilitates a much simpler presentation. Interpolants are objects of the following type:

**Definition 4 (Multiformula).** *The grammar*

$$\mho ::= \mathsf{w}:C \mid (\mho \owedge \mho) \mid (\mho \ovee \mho)$$

*defines* multiformulas*, where $\mathsf{w}:C$ is a labelled formula. For an interpretation $[\![\cdot]\!]$ into a model $\mathcal{M}$, we say*

1. *$\vDash [\![\mathsf{w}:C]\!]$ iff $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash C$;*
2. *$\vDash [\![\mho_1 \ovee \mho_2]\!]$ iff $\vDash [\![\mho_i]\!]$ for some $i = 1, 2$;*
3. *$\vDash [\![\mho_1 \owedge \mho_2]\!]$ iff $\vDash [\![\mho_i]\!]$ for each $i = 1, 2$.*

*Thus, the external $\owedge$ and $\ovee$ on multiformulas correspond to $\wedge$ and $\vee$ on formulas.*

**Definition 5 ($\mathsf{Ant}_\mathrm{f}$, $\mathsf{Con}_\mathrm{f}$, $\mathsf{Ant}_\mathrm{r}$).** *For an interpretation $[\![\cdot]\!]$ into a model $\mathcal{M}$ and a multiset $\Gamma$ of labelled formulas and relational atoms, we write*

| | | |
|---|---|---|
| $\vDash [\![\mathsf{Ant}_\mathrm{f}(\Gamma)]\!]$ | *iff* | $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash A$ *for each* $\mathsf{w}:A \in \Gamma$, |
| $\vDash [\![\mathsf{Ant}_\mathrm{r}(\Gamma)]\!]$ | *iff* | $[\![\mathsf{w}]\!] R [\![\mathsf{o}]\!]$ *for each* $\mathsf{wRo} \in \Gamma$*, and* |
| $\vDash [\![\mathsf{Con}_\mathrm{f}(\Gamma)]\!]$ | *iff* | $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash A$ *for some* $\mathsf{w}:A \in \Gamma$. |

$$\frac{\mathsf{o}:A, \mathsf{w}:\Box A, \mathsf{wRo}, \Gamma \Rightarrow \Delta}{\mathsf{w}:\Box A, \mathsf{wRo}, \Gamma \Rightarrow \Delta} L\Box \qquad \frac{\mathsf{wRo}, \Gamma \Rightarrow \Delta, \mathsf{o}:A}{\Gamma \Rightarrow \Delta, \mathsf{w}:\Box A} R\Box$$

$$\frac{\mathsf{wRo}, \mathsf{o}:A, \Gamma \Rightarrow \Delta}{\mathsf{w}:\Diamond A, \Gamma \Rightarrow \Delta} L\Diamond \qquad \frac{\mathsf{wRo}, \Gamma \Rightarrow \Delta, \mathsf{w}:\Diamond A, \mathsf{o}:A}{\mathsf{wRo}, \Gamma \Rightarrow \Delta, \mathsf{w}:\Diamond A} R\Diamond$$

**Table 3.** Modal rules. For $L\Diamond$ and $R\Box$, the eigenvariable $\mathsf{o}$ does not occur in the conclusion

| | | |
|---|---|---|
| $\mathsf{w}:P, \Gamma \xRightarrow{\mathsf{w}:P} \Delta, \mathsf{w}:P$ | $\mathsf{w}:\overline{P}, \Gamma \xRightarrow{\mathsf{w}:\overline{P}} \Delta, \mathsf{w}:\overline{P}$ | $\mathsf{w}:\bot, \Gamma \xRightarrow{\mathsf{w}:\bot} \Delta$ |
| $\mathsf{w}:P, \mathsf{w}:\overline{P}, \Gamma \xRightarrow{\mathsf{w}:\bot} \Delta$ | $\Gamma \xRightarrow{\mathsf{w}:\top} \Delta, \mathsf{w}:P, \mathsf{w}:\overline{P}$ | $\Gamma \xRightarrow{\mathsf{w}:\top} \Delta, \mathsf{w}:\top$ |

**Table 4.** Interpolating initial sequents

**Definition 6 (CWIP).** *A multiformula $\mho$ is a $\mathcal{C}_{\mathsf{L}}$-*interpolant *of $\Gamma \Rightarrow \Delta$, written $\Gamma \overset{\mho}{\Rightarrow} \Delta$, if all of the following conditions hold:*
1. *each label $\mathsf{w}$ occurring in $\mho$ occurs in $\Gamma$ or in a labelled formula from $\Delta$;*
2. *each literal $P$ or $\overline{P}$ occurring in $\mho$ occurs in both $\Gamma$ and $\Delta$;*
3. *for any interpretation $[\![\cdot]\!]$ into a model $\mathcal{M} \in \mathcal{C}_{\mathsf{L}}$ with $\models [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma)]\!]$:*

$$\models [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma)]\!] \qquad \textit{implies} \qquad \models [\![\mho]\!] \ , \tag{1}$$

$$\models [\![\mho]\!] \qquad \textit{implies} \qquad \models [\![\mathsf{Con}_{\mathrm{f}}(\Delta)]\!] \ . \tag{2}$$

*A calculus* **SL** *has the CWIP w.r.t. $\mathcal{C}_{\mathsf{L}}$ iff every* **SL***-derivable sequent has a $\mathcal{C}_{\mathsf{L}}$-interpolant.*

The modularity of the proof-theoretic method follows from the trivial

**Fact 7** *If $\mho$ is a $\mathcal{C}_{\mathsf{L}}$-interpolant of $\Gamma \Rightarrow \Delta$, it is also a $\mathcal{C}'_{\mathsf{L}}$-interpolant of the same sequent w.r.t. any class $\mathcal{C}'_{\mathsf{L}} \subseteq \mathcal{C}_{\mathsf{L}}$.*

**Definition 8 (Duality).** *We say that a labelled calculus* **SL** *has the* duality property *whenever*

$$\mathbf{SL} \vdash \mathsf{w}:A, \Gamma \Rightarrow \Delta \qquad \textit{iff} \qquad \mathbf{SL} \vdash \Gamma \Rightarrow \Delta, \mathsf{w}:\overline{A} \ .$$

**Theorem 9 (Reducing LIP to CWIP).** *Let* **SL** *be a labelled calculus for a logic $\mathsf{L}$ such that* **SL** *has the duality property and invertible rule $R\vee$ and such that both $\mathsf{L}$ and* **SL** *are sound and complete (*adequate*) w.r.t. $\mathcal{C}_{\mathsf{L}}$. If* **SL** *has the CWIP w.r.t. $\mathcal{C}_{\mathsf{L}}$, then $\mathsf{L}$ has the LIP.*

*Proof.* Assume that **SL** satisfies the CWIP and $\mathsf{L} \vdash A \to B$. Then $\mathcal{C}_{\mathsf{L}} \models A \to B$ by soundness of $\mathsf{L}$ and $\mathbf{SL} \vdash \Rightarrow \mathsf{w}:\overline{A} \vee B$ by completeness of **SL**. $\mathbf{SL} \vdash \Rightarrow \mathsf{w}:\overline{A}, \mathsf{w}:B$ by invertibility of $R\vee$ and $\mathbf{SL} \vdash \mathsf{w}:A \Rightarrow \mathsf{w}:B$ by duality. By CWIP, $\mathsf{w}:A \overset{\mho}{\Rightarrow} \mathsf{w}:B$ for some $\mho$ that has only $\mathsf{w}$ as a label. It is easy to see that $\mathsf{w}:A \xRightarrow{\mathsf{w}:C} \mathsf{w}:B$ for $C$ obtained from $\mho$ by omitting all labels and replacing $⊘$ and $⦸$ with $\wedge$ and $\vee$ respectively. It immediately follows that $C$ is a Lyndon interpolant of $A \to B$. $\Box$

*Remark 10.* Only a derivation of $\mathsf{w}\!:\!A \Rightarrow \mathsf{w}\!:\!B$ is needed for the reduction. Since relational atoms cannot occur in the consequents in such derivations, from now on we allow only labelled formulas in consequents.

## 3 Interpolation Basis: Basic Normal Modal Logic $\mathsf{K}$

The modularity of the proof-theoretic method means that each sequent rule can be treated separately as long as the logic and its labelled calculus satisfy the conditions of Theorem 9. As all labelled calculi we consider extend **SK** for the basic normal modal logic $\mathsf{K}$, we start by describing interpolant transformations for all rules of **SK**. Table 4 presents interpolants for all initial sequents from Table 1. Since many single-premise rules require no change in the interpolant, we describe a sufficient condition for this to happen:

**Definition 11 (Local rules).** *A rule*

$$\frac{\Gamma_p \Rightarrow \Delta_p}{\Gamma_c \Rightarrow \Delta_c}\,\mathsf{r}$$

*is called* $\mathcal{C}_{\mathsf{L}}$-local *if*

1. *each label from the premise occurs in the conclusion;*
2. *each literal from $\Gamma_p$ (from $\Delta_p$) occurs in $\Gamma_c$ (in $\Delta_c$);*
3. *for any interpretation $[\![\cdot]\!]$ into any $\mathcal{M} \in \mathcal{C}_{\mathsf{L}}$,*
   (a) $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_c)]\!]$ *implies* $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_p)]\!]$*;*
   (b) $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_c)]\!]$ *and* $\vDash [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma_c)]\!]$ *imply* $\vDash [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma_p)]\!]$*;*
   (c) $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_c)]\!]$ *and* $\vDash [\![\mathsf{Con}_{\mathrm{f}}(\Delta_p)]\!]$ *imply* $\vDash [\![\mathsf{Con}_{\mathrm{f}}(\Delta_c)]\!]$.

*Example 12.* The rules $L\wedge$ and $R\vee$ from Table 2 and $L\Box$ and $R\Diamond$ from Table 3 are $\mathcal{C}_{\mathsf{L}}$-local for any $\mathcal{C}_{\mathsf{L}}$.

**Lemma 13 (Local).** *Given a $\mathcal{C}_{\mathsf{L}}$-local rule, each $\mathcal{C}_{\mathsf{L}}$-interpolant of the rule's premise $\Gamma_p \Rightarrow \Delta_p$ is also a $\mathcal{C}_{\mathsf{L}}$-interpolant of its conclusion $\Gamma_c \Rightarrow \Delta_c$.*

*Proof.* Let $\mho$ be a $\mathcal{C}_{\mathsf{L}}$-interpolant of $\Gamma_p \Rightarrow \Delta_p$. The conditions on labels and on common literals for $\Gamma_c \Rightarrow \Delta_c$ are inherited from the premise by the definition of local rules. Consider an interpretation $[\![\cdot]\!]$ into an $\mathcal{M} \in \mathcal{C}_{\mathsf{L}}$ with $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_c)]\!]$. Then $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_p)]\!]$. If $\vDash [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma_c)]\!]$, then $\vDash [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma_p)]\!]$, and, hence, $\vDash [\![\mho]\!]$. If $\vDash [\![\mho]\!]$, then $\vDash [\![\mathsf{Con}_{\mathrm{f}}(\Delta_p)]\!]$, and, hence, $\vDash [\![\mathsf{Con}_{\mathrm{f}}(\Delta_c)]\!]$. □

**Lemma 14 ($R\wedge$, $L\vee$).**

1. *If $\mho_1$ and $\mho_2$ are $\mathcal{C}_{\mathsf{L}}$-interpolants of the premises of the rule $R\wedge$, then $\mho_1 \owedge \mho_2$ is a $\mathcal{C}_{\mathsf{L}}$-interpolant of its conclusion.*
2. *If $\mho_1$ and $\mho_2$ are $\mathcal{C}_{\mathsf{L}}$-interpolants of the premises of the rule $L\vee$, then $\mho_1 \ovee \mho_2$ is a $\mathcal{C}_{\mathsf{L}}$-interpolant of its conclusion.*

*Proof.* Similar to that of Lemma 13. □

**Lemma 15** **($L\Diamond$, $R\Box$).** *Let* $\mathsf{o} \neq \mathsf{w}$ *and* $\mathsf{o}$ *occur in neither* $\Gamma$ *nor* $\Delta$. *If*

$$\mho_p = \bigovee_{i=1}^{n} \left( \bigowedge_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\owedge\; \bigowedge_{k=1}^{l_i} \mathsf{o} : C_{ik} \right)$$

*is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of the* $L\Diamond$*'s premise* $\mathsf{wRo}, \mathsf{o}\!:\!A, \Gamma \Rightarrow \Delta$, *then*

$$\mho_c = \bigovee_{i=1}^{n} \left( \bigowedge_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\owedge\; \mathsf{w} : \left( \Diamond \bigwedge_{k=1}^{l_i} C_{ik} \right)\right)$$

*is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of its conclusion* $\mathsf{w}\!:\!\Diamond A, \Gamma \Rightarrow \Delta$. *If*

$$\mho_p = \bigowedge_{i=1}^{n} \left( \bigovee_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\ovee\; \bigovee_{k=1}^{l_i} \mathsf{o} : C_{ik} \right)$$

*is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of the* $R\Box$*'s premise* $\mathsf{wRo}, \Gamma \Rightarrow \Delta, \mathsf{o}\!:\!A$, *then*

$$\mho_c = \bigowedge_{i=1}^{n} \left( \bigovee_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\ovee\; \mathsf{w} : \left( \Box \bigvee_{k=1}^{l_i} C_{ik} \right)\right)$$

*is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of its conclusion* $\Gamma \Rightarrow \Delta, \mathsf{w}\!:\!\Box A$.
*W.l.o.g.* $\mho_p$ *is assumed to be in DNF or CNF respectively, which is achieved by the standard conversion method applied to* $\ovee$ *and* $\owedge$.

**Definition 16.** *Given an interpretation* $[\![\cdot]\!]$ *into a model* $\mathcal{M} = (W, R, V)$, *a sequence of distinct labels* $\mathbf{o} = \mathsf{o}_1, \dots, \mathsf{o}_n$ *from* $\mathsf{Lab}$, *and a sequence of worlds* $\boldsymbol{u} = u_1, \dots, u_n$ *from* $W$, *a new interpretation* $[\![\cdot]\!]^{\boldsymbol{u}}_{\mathbf{o}}$ *into* $\mathcal{M}$ *is defined as follows:*

$$[\![\mathsf{o}_i]\!]^{\boldsymbol{u}}_{\mathbf{o}} := u_i \ , \qquad [\![\mathsf{w}]\!]^{\boldsymbol{u}}_{\mathbf{o}} := [\![\mathsf{w}]\!] \text{ if } \mathsf{w} \notin \{\mathsf{o}_1, \dots, \mathsf{o}_n\} \ .$$

*Proof (of Lemma 15).* The proofs for the two rules are similar. We give the one for $L\Diamond$. The label and common literal conditions are clearly satisfied. Consider any interpretation $[\![\cdot]\!]$ into an $\mathcal{M} = (W, R, V) \in \mathcal{C}_{\mathsf{L}}$ such that $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\mathsf{w}\!:\!\Diamond A, \Gamma)]\!]$.

Assume $\vDash [\![\mathsf{Ant}_{\mathsf{f}}(\mathsf{w}\!:\!\Diamond A, \Gamma)]\!]$. Since $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash \Diamond A$, there is $u \in W$ such that $[\![\mathsf{w}]\!] R u$ and $\mathcal{M}, u \Vdash A$. Clearly, $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\mathsf{wRo}, \mathsf{o}\!:\!A, \Gamma)]\!]^u_{\mathsf{o}}$ because $\mathsf{o}$ does not occur in $\Gamma$. Since $\vDash [\![\mathsf{Ant}_{\mathsf{f}}(\mathsf{wRo}, \mathsf{o}\!:\!A, \Gamma)]\!]^u_{\mathsf{o}}$, for some disjunct $1 \leq i \leq n$ of $\mho_p$

$$\vDash \left[\!\!\left[ \bigowedge_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\owedge\; \bigowedge_{k=1}^{l_i} \mathsf{o} : C_{ik} \right]\!\!\right]^u_{\mathsf{o}} , \tag{3}$$

in particular, $\mathcal{M}, u \Vdash C_{ik}$ for all $k = 1, \dots, l_i$. Given that $[\![\mathsf{w}]\!] R u$, we see that $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash \Diamond \wedge_{k=1}^{l_i} C_{ik}$.[4] Thus, $\vDash \left[\!\!\left[ \owedge_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\owedge\; \mathsf{w} : \left( \Diamond \wedge_{k=1}^{l_i} C_{ik} \right) \right]\!\!\right]^u_{\mathsf{o}}$. Further,

[4] It also holds for $l_i = 0$: the empty conjunction is $\top$ and $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash \Diamond \top$.

given that neither $\mathsf{w}$ nor any of $\mathsf{w}_{ij}$ is $\mathsf{o}$, we have $\vDash [\![\mho_c]\!]$, which completes the proof of (1) for the conclusion of $L\Diamond$.

Assume now that $\vDash [\![\mho_c]\!]$. Then $\vDash \left[\!\!\left[ \oslash_{j=1}^{m_i} \mathsf{w}_{ij} : D_{ij} \;\; \oslash \;\; \mathsf{w} : \left(\Diamond \wedge_{k=1}^{l_i} C_{ik}\right) \right]\!\!\right]$ for some disjunct $1 \leq i \leq n$ of $\mho_c$. In particular, $\mathcal{M}, [\![\mathsf{w}]\!] \Vdash \Diamond \wedge_{k=1}^{l_i} C_{ik}$. Thus, there is $u \in W$ such that $[\![\mathsf{w}]\!] Ru$ and $\mathcal{M}, u \Vdash C_{ik}$ for all $k = 1, \ldots, l_i$. Again we have $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\mathsf{wRo}, \mathsf{o} : A, \Gamma)]\!]^u_{\mathsf{o}}$ and (3) holds for one of disjuncts of $\mho_p$. It follows that $\vDash [\![\mathsf{Con}_{\mathrm{f}}(\Delta)]\!]^u_{\mathsf{o}}$ and, since $\mathsf{o}$ does not occur in $\Delta$, also $\vDash [\![\mathsf{Con}_{\mathrm{f}}(\Delta)]\!]$. □

**Corollary 17.** $\mathsf{K}$ *enjoys the LIP.*

*Proof.* The CWIP for **SK** w.r.t. to the class $\mathcal{K}$ of all Kripke models follows from Table 4 and Lemmas 13–15. Adequacy of $\mathsf{K}$ w.r.t. $\mathcal{K}$ is due to Kripke [11]. Invertibility of all rules of **SK** including $R\vee$ is proved in [16]. The height-preserving duality property is proved by induction on the derivation depth. □

## 4 Mathematical Rules with or without Equality Atoms

Now that the minimal modal logic having Kripke semantics is dealt with, we start considering frame conditions that preserve the LIP. In this section, we explore the exact scope of our method for quantifier-free frame conditions, which generate *mathematical* rules [16]. As noted in [16, Prop. 6.8], any quantifier-free property of Kripke frames can be represented as $P_1 \wedge \ldots \wedge P_m \to Q_1 \vee \ldots \vee Q_n$ where $P_i$ and $Q_j$ are relational atoms. It is, however, clear that the case of $n \geq 2$ cannot be generally treated by our or, indeed, any other method. The logic $\mathsf{S4.3}$ of transitive, reflexive, and connected frames does not enjoy even Craig interpolation [14]. Later, we successfully deal with reflexivity and transitivity, hence, it is *connectedness* $(wRo \wedge wRu \to oRu \vee uRo)$ that is to blame for the breakdown of interpolation. Thus, we concentrate on the cases of $n \leq 1$, or Horn clauses. For $n = 0$, restricting a class $\mathcal{C}_{\mathsf{L}}$ by a frame condition $w_1Ru_1 \wedge \ldots \wedge w_mRu_m \to \bot$ corresponds to adding initial sequents $\mathsf{w}_1\mathsf{R}\mathsf{u}_1, \ldots, \mathsf{w}_m\mathsf{R}\mathsf{u}_m, \Gamma \Rightarrow \Delta$ to the labelled calculus (for some 1-1 onto map of metavariables $w_1, u_1, \ldots, w_m, u_m$ onto labels $\mathsf{w}_1, \mathsf{u}_1, \ldots, \mathsf{w}_m, \mathsf{u}_m$. In particular, the adequacy of the labelled calculus is preserved ([16]).

**Lemma 18.** *If all frames in* $\mathcal{C}_{\mathsf{L}}$ *satisfy*

$$w_1Ru_1 \wedge \ldots \wedge w_mRu_m \to \bot \;\; ,$$

*then* $\mathsf{w}_1 : \bot$ *is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of* $\mathsf{w}_1\mathsf{R}\mathsf{u}_1, \ldots, \mathsf{w}_m\mathsf{R}\mathsf{u}_m, \Gamma \Rightarrow \Delta$.

*Proof.* Similar to initial sequents $\mathsf{w} : \bot, \Gamma \Rightarrow \Delta$ for **SK**. □

**Definition 19.** *A labelled rule has the* subterm property *if each label from each premise, except for* eigenvariables*, occurs in the conclusion. Restricting a rule* $\mathsf{r}$ *to those instances that have the subterm property yields the rule* $\mathsf{r}^{\dagger}$.

| Reflexivity | Transitivity | |
|---|---|---|
| $wRw$ | $wRo \wedge oRr \to wRr$ | |
| $\dfrac{\mathsf{wRw}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref^{\dagger}$ | $\dfrac{\mathsf{wRr}, \mathsf{wRo}, \mathsf{oRr}, \Gamma \Rightarrow \Delta}{\mathsf{wRo}, \mathsf{oRr}, \Gamma \Rightarrow \Delta}\ Trans$ | $\dfrac{\mathsf{wRw}, \mathsf{wRw}, \Gamma \Rightarrow \Delta}{\mathsf{wRw}, \Gamma \Rightarrow \Delta}\ Trans^{*}$ |
| Symmetry | Euclideanness | |
| $wRo \to oRw$ | $wRo \wedge wRr \to oRr$ | |
| $\dfrac{\mathsf{oRw}, \mathsf{wRo}, \Gamma \Rightarrow \Delta}{\mathsf{wRo}, \Gamma \Rightarrow \Delta}\ Sym$ | $\dfrac{\mathsf{oRr}, \mathsf{wRo}, \mathsf{wRr}, \Gamma \Rightarrow \Delta}{\mathsf{wRo}, \mathsf{wRr}, \Gamma \Rightarrow \Delta}\ Eucl$ | $\dfrac{\mathsf{oRo}, \mathsf{wRo}, \Gamma \Rightarrow \Delta}{\mathsf{wRo}, \Gamma \Rightarrow \Delta}\ Eucl^{*}$ |

**Table 5.** Common Horn frame conditions

It was shown in [16, Th. 11.27, Cor. 11.29] that restricting a class of frames by a Horn clause $w_1Ru_1 \wedge \ldots \wedge w_mRu_m \to vRz$ for $n = 1$ corresponds to adding to the labelled calculus both

– the rule

$$\frac{\mathsf{vRz}, \mathsf{w}_1\mathsf{Ru}_1, \ldots, \mathsf{w}_m\mathsf{Ru}_m, \Gamma \Rightarrow \Delta}{\mathsf{w}_1\mathsf{Ru}_1, \ldots, \mathsf{w}_m\mathsf{Ru}_m, \Gamma \Rightarrow \Delta}\ Math^{\dagger} \tag{4}$$

with the subterm property, i.e., restricted to instances with both $\mathsf{v}$ and $\mathsf{z}$ occurring in the conclusion, and

– rules obtained from it by the *closure condition*, i.e., by contracting identical relational atoms $\mathsf{w}_i\mathsf{Ru}_i$ and $\mathsf{w}_j\mathsf{Ru}_j$ from the conclusion in both premise and conclusion for those instances of the rule that contain such identical atoms.

*Example 20.* Common examples of such Horn restrictions and their corresponding rules can be found in Table 5. Rules $Trans^*$ and $Eucl^*$ are added due to the closure condition and correspond to $\mathsf{wRo} = \mathsf{oRr}$ in *Trans* and $\mathsf{wRo} = \mathsf{wRr}$ in *Eucl* respectively. Note that all rules but *Ref* already have the subterm property.

**Lemma 21.** *An interpolant transformation for a rule* $\mathsf{r}$ *is also applicable for the rules* $\mathsf{r}^*$ *obtained from* $\mathsf{r}$ *by the closure condition. More precisely, applying the interpolant transformation for* $\mathsf{r}$ *to an interpolant for the premise of a rule* $\mathsf{r}^*$ *yields an interpolant for the conclusion of* $\mathsf{r}^*$*.*

*Proof.* This observation follows from the fact that the definition of componentwise interpolant is not sensitive to the multiplicities of relational atoms. □

Thus, from now on we consider $\mathsf{r}^*$-variants obtained from $\mathsf{r}$ by the closure condition to be "instances" of $\mathsf{r}$ and do not mention them explicitly.

**Lemma 22 (Horn).** *If all frames in* $\mathcal{C}_{\mathsf{L}}$ *satisfy*

$$w_1Ru_1 \wedge \ldots \wedge w_mRu_m \to vRz \ \ ,$$

*then a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of the premise of* (4) *is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of its conclusion.*

*Proof.* This follows from Lemma 13 as (4) (and all its contracted versions) is $\mathcal{C}_{\mathsf{L}}$-local, e.g., the locality condition 1. follows from the subterm property. □

$$\frac{\mathsf{w} = \mathsf{w}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\dagger \qquad \frac{\mathsf{oRr}, \mathsf{w} = \mathsf{o}, \mathsf{wRr}, \Gamma \Rightarrow \Delta}{\mathsf{w} = \mathsf{o}, \mathsf{wRr}, \Gamma \Rightarrow \Delta} \qquad \frac{\mathsf{wRr}, \mathsf{o} = \mathsf{r}, \mathsf{wRo}, \Gamma \Rightarrow \Delta}{\mathsf{o} = \mathsf{r}, \mathsf{wRo}, \Gamma \Rightarrow \Delta}$$

$$\frac{\mathsf{o} = \mathsf{r}, \mathsf{w} = \mathsf{o}, \mathsf{w} = \mathsf{r}, \Gamma \Rightarrow \Delta}{\mathsf{w} = \mathsf{o}, \mathsf{w} = \mathsf{r}, \Gamma \Rightarrow \Delta} \qquad \frac{\mathsf{o} : A, \mathsf{w} = \mathsf{o}, \mathsf{w} : A, \Gamma \Rightarrow \Delta}{\mathsf{w} = \mathsf{o}, \mathsf{w} : A, \Gamma \Rightarrow \Delta}$$

**Table 6.** Rules for equality atoms († means the subterm property restriction)

The formalism of labelled sequents can be enriched with equality atoms $\mathsf{w} = \mathsf{o}$ and the rules in Table 6 without affecting the adequacy results ([16, Sect. 11.6]). Equality atoms can be treated the same way as relational atoms, e.g., $\vDash [\![\mathsf{Ant}_{\mathrm{r}}\Gamma]\!]$ now means that $[\![\mathsf{w}]\!]R[\![\mathsf{o}]\!]$ for each $\mathsf{wRo} \in \Gamma$ and $[\![\mathsf{w}]\!] = [\![\mathsf{o}]\!]$ for each $\mathsf{w} = \mathsf{o} \in \Gamma$. It follows from the definition of local rules that

**Lemma 23.** *All rules from Table 6 are $\mathcal{C}_\mathsf{L}$-local for any $\mathcal{C}_\mathsf{L}$.*

Further, it is easy to see that the proof of Lemma 13 directly applies also to labelled calculi with equality. Using the same construction of labelled rules from Horn clauses as $Math^\dagger$ in the previous section and assuming w.l.o.g. that no equality atoms occur among $P_i$, we can prove that such rules with the subterm property are still $\mathcal{C}_\mathsf{L}$-local in the presence of equality atoms:

**Lemma 24.** *If all frames in $\mathcal{C}_\mathsf{L}$ satisfy*

$$w_1 R u_1 \wedge \ldots \wedge w_m R u_m \to v = z \enspace ,$$

*then a $\mathcal{C}_\mathsf{L}$-interpolant of the premise of* (4) *with* $\mathsf{v} = \mathsf{z}$ *instead of* $\mathsf{vRz}$ *is also a $\mathcal{C}_\mathsf{L}$-interpolant of its conclusion.*

## 5 Geometric Rules

Dyckhoff and Negri [5] showed how to geometrize any first-order frame condition. Once again, we restrict our attention to single-conclusion canonical geometric implications

$$w_1 R o_1 \wedge \ldots \wedge w_m R o_m \to \exists y_1 \ldots \exists y_k \big(Q_1(\boldsymbol{y}) \wedge \ldots \wedge Q_l(\boldsymbol{y})\big) \enspace ,$$

where $y_j \notin \{w_1, o_1, \ldots, w_m, o_m\}$ for any pairwise distinct $y_1, \ldots, y_k = \boldsymbol{y}$ and, w.l.o.g., $Q_i(\boldsymbol{y})$ are relational atoms. They correspond to the rules

$$\frac{Q_1(\mathbf{y}), \ldots, Q_l(\mathbf{y}), \mathsf{w}_1\mathsf{Ro}_1, \ldots, \mathsf{w}_m\mathsf{Ro}_m, \Gamma \Rightarrow \Delta}{\mathsf{w}_1\mathsf{Ro}_1, \ldots, \mathsf{w}_m\mathsf{Ro}_m, \Gamma \Rightarrow \Delta}\,Geom^\dagger \tag{5}$$

where the eigenvariables $\mathsf{y}_1, \ldots, \mathsf{y}_k = \mathbf{y}$ do not occur in the conclusion. We consider first a subset of such rules that we call *telescopic*.

### 5.1 Telescopic Rules

**Definition 25 (Telescopic).** Telescopic *frame conditions have the form*

$$\bigwedge_{i=1}^{m} w_i R o_i \quad \rightarrow \quad \exists y_1 \ldots \exists y_k \big( x R y_1 \wedge y_1 R y_2 \wedge \ldots \wedge y_{k-1} R y_k \big) \tag{6}$$

*where* $\{y_1, \ldots, y_k\} \cap \{x, w_1, o_1, \ldots, w_m, o_m\} = \varnothing$.

Corresponding rules $\dfrac{\mathsf{x}\mathsf{R}\mathsf{y}_1, \mathsf{y}_1\mathsf{R}\mathsf{y}_2, \ldots, \mathsf{y}_{k-1}\mathsf{R}\mathsf{y}_k, \mathsf{w}_1\mathsf{R}\mathsf{o}_1, \ldots, \mathsf{w}_m\mathsf{R}\mathsf{o}_m, \Gamma \Rightarrow \Delta}{\mathsf{w}_1\mathsf{R}\mathsf{o}_1, \ldots, \mathsf{w}_m\mathsf{R}\mathsf{o}_m, \Gamma \Rightarrow \Delta}\ \mathit{Tele}^{\dagger}$ have $\mathsf{x}$ occurring in the conclusion and pairwise distinct eigenvariables $\mathsf{y}_1, \ldots, \mathsf{y}_k$ (and may generate contracted versions by the closure condition).

**Lemma 26.** *Let all frames in a class* $\mathcal{C}_{\mathsf{L}}$ *satisfy* (6)*. For any* $\mathcal{C}_{\mathsf{L}}$*-interpolant*[5]

$$\mho_p = \bigovee_{i=1}^{n} \left( \bigowedge_{b=1}^{m_i} \mathsf{u}_{ib} : D_{ib} \ \owedge \ \bigowedge_{j=1}^{k} \mathsf{y}_j : C_{ij} \right)$$

*of the the premise of* $\mathit{Tele}^{\dagger}$*, we have that*

$$\mho_c = \bigovee_{i=1}^{n} \left( \bigowedge_{b=1}^{m_i} \mathsf{u}_{ib} : D_{ib} \ \owedge \ \mathsf{x} : \mathbb{T}_i \right)$$

*is a* $\mathcal{C}_{\mathsf{L}}$*-interpolant of the rule's conclusion, where*

$$\mathbb{T}_i := \Diamond(C_{i,1} \wedge \Diamond(C_{i,2} \wedge \Diamond(\ldots \wedge \Diamond(C_{i,k-1} \wedge \Diamond C_{ik}) \ldots))) \ .$$

*Proof.* Let us abbreviate the premise and conclusion sequents as $\Gamma_p \Rightarrow \Delta$ and $\Gamma_c \Rightarrow \Delta$ respectively. The common literal condition is clearly preserved. Eigenvariables $\mathsf{y}_j$ occur neither in $\Gamma_c \Rightarrow \Delta$ nor in $\mho_c$. Consider an interpretation $[\![\cdot]\!]$ into a model $\mathcal{M} = (W, R, V) \in \mathcal{C}_{\mathsf{L}}$ such that $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_c)]\!]$.

Assume $\vDash [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma_c)]\!]$. Since $[\![\mathsf{w}_l]\!] R [\![\mathsf{o}_l]\!]$ for all $l$, by (6) there are $y_j \in W$ such that $[\![\mathsf{x}]\!] R y_1 R \ldots R y_{k-1} R y_k$. Since $\mathsf{y}_j$ does not occur in $\Gamma_c \Rightarrow \Delta$, it follows that $\vDash [\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_p)]\!]^{\boldsymbol{y}}_{\mathsf{y}}$ and $\vDash [\![\mathsf{Ant}_{\mathrm{f}}(\Gamma_p)]\!]^{\boldsymbol{y}}_{\mathsf{y}}$. Thus, for some disjunct $1 \le i \le n$ of $\mho_p$,

$$\vDash \left[\!\!\left[ \bigowedge_{b=1}^{m_i} \mathsf{u}_{ib} : D_{ib} \ \owedge \ \bigowedge_{j=1}^{k} \mathsf{y}_j : C_{ij} \right]\!\!\right]^{\boldsymbol{y}}_{\mathsf{y}} , \tag{7}$$

in particular, $\mathcal{M}, y_j \Vdash C_{ij}$ for all $j = 1, \ldots, k$ for this $i$. It is easy to show by induction that $\mathcal{M}, y_j \Vdash C_{ij} \wedge \Diamond(C_{i,j+1} \wedge \Diamond(\ldots \wedge \Diamond(C_{i,k-1} \wedge \Diamond C_{ik}) \ldots))$ culminating in $\mathcal{M}, [\![\mathsf{x}]\!] \Vdash \mathbb{T}_i$. Since neither of $\mathsf{u}_{ib}$ coincides with any of $\mathsf{y}_j$, it follows that $\vDash [\![\mho_c]\!]$.

[5] For each eigenvariable $\mathsf{y}_j$ we have collected all formulas labelled with $\mathsf{y}_j$ within each disjunct into one labelled formula by transforming $\mathsf{v}\!:\!A \owedge \mathsf{v}\!:\!B$ into $\mathsf{v}\!:\!(A \wedge B)$ if more than one formula has this label or by adding $\mathsf{y}_j\!:\!\top$ if no formula has.

Assume now that $\vDash [\![\mho_c]\!]$. Then $\vDash [\![\bigotimes_{b=1}^{m_i} \mathsf{u}_{ib} : D_{ib} \ \oslash \ \mathsf{x} : \mathbb{T}_i]\!]$ holds for some $1 \leq i \leq n$. In particular, $\mathcal{M}, [\![\mathsf{x}]\!] \Vdash \mathbb{T}_i$. Thus, there exist worlds $y_j \in W$ such that $[\![\mathsf{x}]\!] R y_1 R y_2 R \dots R y_{k-1} R y_k$ and $\mathcal{M}, y_j \Vdash C_{ij}$ for all $j = 1, \dots, k$. Again, $\vDash [\![\mathsf{Ant}_\mathrm{r}(\Gamma_p)]\!]^{\boldsymbol{y}}_{\mathsf{y}}$ and, hence, (7) holds for some disjunct of $\mho_p$. It follows that $\vDash [\![\mathsf{Con}_\mathrm{f}(\Delta)]\!]^{\boldsymbol{y}}_{\mathsf{y}}$. Since none of $\mathsf{y}_j$ occurs in $\Delta$, we have $\vDash [\![\mathsf{Con}_\mathrm{f}(\Delta)]\!]$. $\square$

*Example 27.* The simplest and most familiar example of a telescopic frame condition is *seriality*: $\exists y(xRy)$. The corresponding rule is $\dfrac{\mathsf{xRy}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} Ser^\dagger$ where $\mathsf{x}$ occurs in the conclusion and the eigenvariable $\mathsf{y}$ doesn't. Thus, for any class $\mathcal{C}_\mathsf{L}$ whose models have serial frames, if $\bigcirc\!\!\!\!\vee_{i=1}^{n} (\bigotimes_{b=1}^{m_i} \mathsf{u}_{ib} : D_{ib} \ \oslash \ \mathsf{y} : C_i)$ is a $\mathcal{C}_\mathsf{L}$-interpolant of $\mathsf{xRy}, \Gamma \Rightarrow \Delta$, then $\bigcirc\!\!\!\!\vee_{i=1}^{n} (\bigotimes_{b=1}^{m_i} \mathsf{u}_{ib} : D_{ib} \ \oslash \ \mathsf{x} : \Diamond C_i)$ is a $\mathcal{C}_\mathsf{L}$-interpolant of $\Gamma \Rightarrow \Delta$, where $\mathsf{x}$ occurs in $\Gamma$ or $\Delta$, $\mathsf{y}$ does not, and $\mathsf{y}$ does not coincide with any of $\mathsf{u}_{ib}$. This is essentially the same transformation as used for $L\Diamond$.

### 5.2 Non-telescopic Geometric Rules

While $\Diamond$ helps describe one accessible world, more complex configurations of eigenvariables are hard to describe by modal formulas. Consider *convergence* $wRo_1 \wedge wRo_2 \rightarrow \exists y(o_1Ry \wedge o_2Ry)$, a single-conclusion canonical geometric implication that cannot be handled using Lemma 26. It is not clear which formulas are to be true at $w$, $o$, $o_1$, and $o_2$ in order to ensure that the interpolant information from the conclusion can be lifted to the premise. For instance, for the case of convergence, $\mathsf{o}_i : \Diamond C$ only describes a world satisfying $C$ and accessible from $[\![\mathsf{o}_i]\!]$. It is not clear how to pinpoint a world satisfying $C$ and simultaneously accessible from $[\![\mathsf{o}_1]\!]$ and $[\![\mathsf{o}_2]\!]$. Indeed, $\mathsf{o}_1 : \Diamond C \oslash \mathsf{o}_2 : \Diamond C$ only implies that each of the two worlds has an accessible world, $o_1'$ and $o_2'$ respectively, satisfying $C$ but cannot guarantee that $o_1' = o_2'$. To overcome this difficulty, we use a convergence-like property to find a third $C$-world $y$ accessible from both $o_1'$ and $o_2'$ and a transitivity-like property to ensure that $y$ is directly accessible from both original worlds $[\![\mathsf{o}_1]\!]$ and $[\![\mathsf{o}_2]\!]$.

In this section, we outline general conditions and an interpolant transformation that enable us to carry the interpolation proof beyond geometric rules whose eigenvariables form disjoint telescopes. While the conditions themselves are a bit technical, they can be viewed as weakened forms of transitivity and convergence adapted to the particulars of a given sequent rule. As a result, both density and convergence become amenable to our method in presence of some additional frame properties.

W.l.o.g. we assume that each $Q_j(\boldsymbol{y})$ is a relational atom containing an occurrence of one of $y_j$'s because eigenvariable-free conjuncts can be pulled out and handled using Lemma 22. We demonstrate interpolation for frame conditions

$$\bigwedge_{i=1}^{m} w_i R o_i \quad \rightarrow \quad \exists y_1 \dots \exists y_k \bigwedge_{j=1}^{l} x_j R e_j \ , \tag{8}$$

where each $x_j Re_j$ contains a $y_i$ and each $y_i$ occurs among $x_j$'s and $e_j$'s. The corresponding rule is $\dfrac{\mathsf{x}_1\mathsf{Re}_1,\ldots,\mathsf{x}_l\mathsf{Re}_l,\mathsf{w}_1\mathsf{Ro}_1,\ldots,\mathsf{w}_m\mathsf{Ro}_m,\Gamma\Rightarrow\Delta}{\mathsf{w}_1\mathsf{Ro}_1,\ldots,\mathsf{w}_m\mathsf{Ro}_m,\Gamma\Rightarrow\Delta}\,\mathsf{GI}^\dagger$ with eigenvariables $\mathsf{y}_1,\ldots,\mathsf{y}_k$ where each $\mathsf{x}_j$ and $\mathsf{e}_j$ that is not an eigenvariable must occur in the conclusion sequent.

**Definition 28 (Conmap and premap).** *An interpretation* $[\![\cdot]\!]$ *into* $\mathcal{M}$ *is called an* r-conmap *(an* r-premap*) for a rule* $\dfrac{\Gamma_p\Rightarrow\Delta_p}{\Gamma_c\Rightarrow\Delta_c}\,\mathsf{r}$ *if* $\vDash[\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_c)]\!]$ $(\vDash[\![\mathsf{Ant}_{\mathrm{r}}(\Gamma_p)]\!])$.

**Lemma 29.** *If* $\mathcal{M}$ *is a model satisfying* (8)*, any* $\mathsf{GI}^\dagger$*-conmap* $[\![\cdot]\!]$ *into* $\mathcal{M}$ *can be modified into a* $\mathsf{GI}^\dagger$*-premap* $[\![\cdot]\!]^{\boldsymbol{y}}_{\mathbf{y}}$ *into* $\mathcal{M}$.

*Proof.* It immediately follows from (8). □

**Definition 30 (Interpolable rule).** *Let all frames of a class* $\mathcal{C}_\mathsf{L}$ *satisfy* (8)*. A rule* $\mathsf{GI}^\dagger$ *is* $\mathcal{C}_\mathsf{L}$-interpolable *for an order* $\langle\mathsf{y}_1,\ldots,\mathsf{y}_k\rangle$ *on its eigenvariables if a parent function* $\mathsf{par}\colon\mathsf{Lab}\to\mathsf{Lab}$ *exists satisfying the following three properties:*

- *for each* $\mathsf{y}_j$*, there is* $i$ *such that* $\mathsf{par}(\mathsf{y}_j)\mathsf{R}\mathsf{y}_j=\mathsf{x}_i\mathsf{Re}_i$ *where* $\mathsf{x}_i$ *must either occur in the conclusion of* $\mathsf{GI}^\dagger$ *or be* $\mathsf{y}_{j'}$ *for some* $j'<j$*;* (connectedness)

*given any model* $\mathcal{M}=(W,R,V)\in\mathcal{C}_\mathsf{L}$*, any* $\mathsf{GI}^\dagger$*-conmap* $[\![\cdot]\!]$ *into* $\mathcal{M}$ *and any* $\mathsf{GI}^\dagger$*-premap* $[\![\cdot]\!]^{y_1,\ldots,y_k}_{\mathsf{y}_1,\ldots,\mathsf{y}_k}$ *into* $\mathcal{M}$*, for each* $j=1,\ldots,k$

- *if* $y_jRy'_j$*, there is a* $\mathsf{GI}^\dagger$*-premap* $[\![\cdot]\!]^{y_1,\ldots,y_{j-1},y'_j,\ldots,y'_k}_{\mathsf{y}_1,\ldots,\mathsf{y}_{j-1},\mathsf{y}_j,\ldots\mathsf{y}_k}$ *into* $\mathcal{M}$*;* (pushability)
- *if* $[\![\mathsf{par}(\mathsf{y}_j)]\!]^{y_1,\ldots,y_k}_{\mathsf{y}_1,\ldots,\mathsf{y}_k}Rz_l$ *for all* $1\le l\le s$*, there exists* $y'_j$ *such that* $z_lRy'_j$ *for all* $1\le l\le s$ *and a* $\mathsf{GI}^\dagger$*-premap* $[\![\cdot]\!]^{y_1,\ldots,y_{j-1},y'_j,\ldots,y'_k}_{\mathsf{y}_1,\ldots,\mathsf{y}_{j-1},\mathsf{y}_j,\ldots\mathsf{y}_k}$ *into* $\mathcal{M}$*.* (conjoinability)

**Definition 31 (Geach properties).** *The Scott–Lemmon generalizations of the Geach covergence axiom are known to correspond to the* $hijk$-convergence properties $wR^hv\wedge wR^ju\to\exists y(vR^iy\wedge uR^ky)$ *[7, Sect. 9]. We only consider the cases of* $h,i,j,k\ge1$*. Each* $hijk$*-convergence property can be written as a canonical geometric implication:*

$$
\begin{aligned}
&wRv_1\wedge\ldots\wedge v_{h-1}Rv \quad\wedge\quad wRu_1\wedge\ldots\wedge u_{j-1}Ru \qquad\to\\
&\exists z_1\ldots\exists z_{i-1}\exists y_1\ldots\exists y_{k-1}\exists y(vRz_1\wedge\ldots\wedge z_{i-1}Ry\wedge uRy_1\wedge\ldots\wedge y_{k-1}Ry)\ .
\end{aligned}\tag{9}
$$

It is tedious but not hard to prove the following two lemmas:

**Lemma 32.** *If all frames in* $\mathcal{C}_\mathsf{L}$ *are* $hijk$*-convergent and transitive (shift-transitive if* $h,j\ge2$*), then* $\mathsf{GI}^\dagger$ *for the case of* (8) *being the* $hijk$*-convergence property* (9) *is* $\mathcal{C}_\mathsf{L}$*-interpolable for the order* $\langle\mathsf{z}_1,\ldots,\mathsf{z}_{i-1},\mathsf{y}_1,\ldots,\mathsf{y}_{k-1},\mathsf{y}\rangle$.

**Lemma 33.** *Let* $m<n$ *and all frames in* $\mathcal{C}_\mathsf{L}$ *be transitive, Euclidean, and* $(n,m)$*-transitive, i.e., satisfy* $wR^mx\to wR^nx$*. Then taking the frame condition* (8) *to be* $wRv_1\wedge\ldots\wedge v_{m-1}Rx\to\exists y_1\ldots\exists y_{n-1}(wRy_1\wedge\ldots\wedge y_{n-1}Rx)$*, the rule* $\mathsf{GI}^\dagger$ *is* $\mathcal{C}_\mathsf{L}$*-interpolable for the order* $\langle\mathsf{y}_1,\ldots,\mathsf{y}_{n-1}\rangle$.

**Definition 34 (Transformation for interpolable rules).** *For*

$$\mho = \bigotimes_{r=1}^{s} \bigovee_{b=1}^{t_r} \mathsf{v}_{rb} : D_{rb}$$

*in CNF and arbitrary labels* $\mathsf{y}$ *and* $\mathsf{x}$ *such that* $\mathsf{y} \neq \mathsf{x}$*,*

$$\mathsf{rem}\,(\mathsf{y},\, \mathsf{x},\, \mho) \;:=\; \bigotimes_{r=1}^{s} \left( \mathsf{x} : \Diamond\Box \bigvee_{\mathsf{v}_{rb}=\mathsf{y}} D_{rb} \quad \owedge \quad \bigovee_{\mathsf{v}_{rb}\neq\mathsf{y}} \mathsf{v}_{rb} : D_{rb} \right) \;.$$

*It is clear that* $\mathsf{y}$ *does not occur in* $\mathsf{rem}(\mathsf{y}, \mathsf{x}, \mho)$*. Let a rule* $\mathsf{GI}^\dagger$ *be* $\mathcal{C}_\mathsf{L}$*-interpolable for the order* $\langle \mathsf{y}_1, \ldots, \mathsf{y}_k \rangle = \mathbf{y}$ *and parent function* $\mathsf{par}$*. For each* $j = 0, \ldots, k$*,*

$$\mathsf{rem}_j(\mathbf{y}, \mathsf{GI}^\dagger, \mho) := \begin{cases} \mho & \text{if } j = k, \\ \mathsf{rem}(\mathsf{y}_{j+1}, \mathsf{par}(\mathsf{y}_{j+1}), \mathsf{rem}_{j+1}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)) & \text{if } j \leq k-1. \end{cases} \tag{10}$$

*Note that* $\mathsf{rem}_j(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$ *is in CNF and* $\mathsf{y}_{j+1}, \ldots, \mathsf{y}_k$ *don't occur in it. Finally,* $\mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho) := \mathsf{rem}_0(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$ *and contains no eigenvariables of* $\mathsf{GI}^\dagger$*.*

**Lemma 35.** *Let* $\mathcal{C}_\mathsf{L}$ *satisfy* (8) *and* $\mathsf{GI}^\dagger$ *be* $\mathcal{C}_\mathsf{L}$*-interpolable for the order* $\mathbf{y} = \langle \mathsf{y}_1, \ldots, \mathsf{y}_k \rangle$ *and a parent function* $\mathsf{par}$*. Then for any* $\mathcal{C}_\mathsf{L}$*-interpolant* $\mho$ *of the premise of* $\mathsf{GI}^\dagger$ *in CNF,* $\mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$ *is a* $\mathcal{C}_\mathsf{L}$*-interpolant of the conclusion of* $\mathsf{GI}^\dagger$*.*

*Proof.* Let $\Gamma_p \Rightarrow \Delta$ and $\Gamma_c \Rightarrow \Delta$ be the premise and conclusion of $\mathsf{GI}^\dagger$. Let $\Gamma_p \overset{\mho}{\Rightarrow} \Delta$ for some $\mho$ in CNF, let $\mathcal{M} \in \mathcal{C}_\mathsf{L}$, and let $[\![\cdot]\!]$ be a $\mathsf{GI}^\dagger$-conmap into $\mathcal{M}$. The label and common language conditions are satisfied because of the subterm property and the absence of eigenvariables in $\mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$ and because no labelled formula is changed by $\mathsf{GI}^\dagger$ and $\mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$ has the same literals as $\mho$ respectively.

Given $\vDash [\![\mathsf{Ant}_\mathsf{f}(\Gamma_c)]\!]$, let us show $\vDash \left[\!\!\left[\mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)\right]\!\!\right]$. We abbreviate $\mho_j := \mathsf{rem}_j(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$. It can be proved by induction on $j = k, \ldots, 0$ that $\vDash [\![\mho_j]\!]^{y_1,\ldots,y_j}_{\mathsf{y}_1,\ldots,\mathsf{y}_j}$ for any $\mathsf{GI}^\dagger$-premap $[\![\cdot]\!]^{\boldsymbol{y}}_{\mathbf{y}}$ into $\mathcal{M}$. In particular, $\vDash [\![\mho_0]\!]$ for any $\mathsf{GI}^\dagger$-premap $[\![\cdot]\!]^{\boldsymbol{y}}_{\mathbf{y}}$ into $\mathcal{M}$. It remains to note that such premaps exist by Lemma 29 and that $\mho_0 = \mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)$. This completes the proof of (1).

Given $\vDash \left[\!\!\left[\mathsf{rem}(\mathbf{y}, \mathsf{GI}^\dagger, \mho)\right]\!\!\right]$, let us show $\vDash [\![\mathsf{Con}_\mathsf{f}(\Delta)]\!]$. We can prove by induction on $j = 0, \ldots, k$ that there is a $\mathsf{GI}^\dagger$-premap $[\![\cdot]\!]^{y_1^1,\ldots,y_j^j,y_{j+1}^j,\ldots,y_k^j}_{\mathsf{y}_1,\ldots,\mathsf{y}_j,\mathsf{y}_{j+1},\ldots,\mathsf{y}_k}$ into $\mathcal{M}$ such that $\vDash [\![\mho_j]\!]^{y_1^1,\ldots,y_j^j}_{\mathsf{y}_1,\ldots,\mathsf{y}_j}$. In particular, since $\mho = \mho_k$, we have $\vDash [\![\mho]\!]^{\boldsymbol{y}}_{\mathbf{y}}$ for $\boldsymbol{y} = y_1^1, \ldots, y_k^k$. Since $\Gamma_p \overset{\mho}{\Rightarrow} \Delta$, it follows that $\vDash [\![\mathsf{Con}_\mathsf{f}(\Delta)]\!]^{\boldsymbol{y}}_{\mathbf{y}}$. But $\Delta$ contains no eigenvariables. Hence, $\vDash [\![\mathsf{Con}_\mathsf{f}(\Delta)]\!]$. This completes the proof of (2). $\square$

**Corollary 36.** *Modal logics complete w.r.t. Kripke models defined via*

- *Horn properties, including reflexivity, transitivity, symmetry, Euclideanness,* $(1, n)$*-transitivity, and functionality, as well as the shift versions thereof,*

– *telescopic properties, including seriality, and*
– *properties generating interpolable rules, including hijk-convergence with* $h, i, j, k \geq 1$ *(in presence of transitivity or, for* $h, j \geq 2$*, shift transitivity); density (in presence of transitivity and Euclideanness);* $(n, m)$*-transitivity for* $m < n$ *(in presence of transitivity and Euclideanness)*

*enjoy LIP. In particular, logics with LIP proved using labelled sequents include all 15 logics of the so-called modal cube from [7, Sect. 8],* $\mathsf{K4.2}$*,* $\mathsf{S4.2}$*, and* $\mathsf{K4}_{1,n}$*, as well as the infinite family of non-degenerate Geach logics over* $\mathsf{K4}$ *and almost the full family of Geach logics over* $\mathsf{K5}$ *(due to the shift transitivity of the latter).*

## 6 Related Work, Conclusion, and Future Work

The body of work on interpolation is so great and so varied that it is hopeless to try giving even a restricted overview of the field. While this is the first result on proving interpolation using labelled sequent calculi, there were several recent advances in other proof formalisms. Brotherston and Goré [3] developed a method of using display calculi for proving interpolation for displayable substructural logics. Bílková [2] and Herzig and Mengin [9] used nested sequent calculi and resolution respectively to show the stronger and, consequently, rarer uniform interpolation. Pattinson [17] provided a blanket proof of uniform interpolation for the somewhat restricted class of rank-1 modal logics. Iemhoff [10] connected the existence of ordinary sequent calculi to the property of uniform interpolation, which can be used to show the absence of such sequent calculi, but can only prove uniform interpolation for logics with sequent systems.

By using a non-constructive method based on duality theory, Marx proved a similar but slightly weaker result than ours [15, Cor. B.4.1]: a non-constructive proof of Craig interpolation for logics defined by frame conditions given by universal Horn sentences, compared to our constructive proof of Lyndon interpolation for the same logics. It would be interesting to compare the semantic restrictions of his method with those of our method. Perhaps, a more exact upper bound and a better description of both methods' applicability area(s) can be obtained by such comparative analysis.

We developed a constructive and modular method of proving the Lyndon (and Craig) Interpolation Property for modal logics by using labelled sequent calculi. The method is sufficient to establish the LIP for all frame conditions described by quantifier-free Horn formulas. For geometric formulas, the method generally requires additional conditions similar to transitivity and convergence in nature, but is still sufficient to tackle an infinite family of standard modal logics.

Many questions remain open. The extension to multimodal logics and to first-order languages is long overdue. Intuitionistic systems have so far evaded this method. Logics like $\mathsf{GL}$ can be captured by labelled sequents even though they are not first-order definable. Thus, our method should extend to them too.

**Acknowledgments** I am grateful to M. Fitting, whose idea started this interpolation project. I thank S. Negri for encouragement, V. Sikimić for procuring a source not available online, Y. Venema and M. Marx for valuable information on the non-constructive method. I am deeply indebted to B. Lellmann, who is always ready to listen and has provided many inspiring suggestions for improving this paper. I thank the anonymous reviewers for the suggestions on terminology.

## References

1. Amir, E., McIlraith, S.: Partition-based logical reasoning for first-order and propositional theories. Artificial Intelligence **162**(1–2) (2005) 49–88
2. Bílková, M.: A note on uniform interpolation proofs in modal deep inference calculi. In Bezhanishvili, N., Löbner, S., Schwabe, K., Spada, L., eds.: TbiLLC 2009. LNAI, vol. 6618. Springer (2011) 30–45
3. Brotherston, J., Goré, R.: Craig interpolation in displayable logics. In Brünnler, K., Metcalfe, G., eds.: TABLEAUX 2011. LNCS, vol. 6793. Springer (2011) 88–103
4. ten Cate, B., Franconi, E., Seylan, İ.: Beth definability in expressive description logics. Journal of Artificial Intelligence Research **48**(1) (2013) 347–414
5. Dyckhoff, R., Negri, S.: Geometrisation of first-order logic. Bulletin of Symbolic Logic **21**(2) (2015) 123–163
6. Fitting, M., Kuznets, R.: Modal interpolation via nested sequents. Annals of Pure and Applied Logic **166**(3) (2015) 274–305
7. Garson, J.: Modal logic. In Zalta, E.N., ed.: The Stanford Encyclopedia of Philosophy (2016) `http://plato.stanford.edu/entries/logic-modal/`
8. Goré, R., Ramanayake, R.: Labelled tree sequents, tree hypersequents and nested (deep) sequents. In Bolander, T., Braüner, T., Ghilardi, S., Moss, L., eds.: Advances in Modal Logic, Volume 9. College Publications (2012) 279–299
9. Herzig, A., Mengin, J.: Uniform interpolation by resolution in modal logic. In Hölldobler, S., Lutz, C., Wansing, H., eds.: JELIA 2008. LNAI, vol. 5293. Springer (2008) 219–231
10. Iemhoff, R.: Uniform interpolation and sequent calculi in modal logic. Preprint 325, Logic Group Preprint Series (2015)
11. Kripke, S.A.: Semantical analysis of modal logic I: Normal modal propositional calculi. Zeitschrift für mathematische Logik und Grundlagen der Mathematik **9**(5–6) (1963) 67–96
12. Kuznets, R.: Craig interpolation via hypersequents. In Probst, D., Schuster, P., eds.: Concepts of Proof in Mathematics, Philosophy, and Computer Science. Ontos Mathematical Logic, vol. 6. De Gruyter (2016) 193–214
13. Kuznets, R.: Interpolation method for multicomponent sequent calculi. In Artemov, S., Nerode, A., eds.: LFCS 2016. LNCS, vol. 9537. Springer (2016) 202–218
14. Maksimova, L.L.: Absence of the interpolation property in the consistent normal modal extensions of the Dummett logic. Algebra and Logic **21**(6) (1982) 460–463
15. Marx, M., Venema, Y.: Multi-Dimensional Modal Logic. Applied Logic Series, vol. 4. Springer (1997)
16. Negri, S., von Plato, J.: Proof Analysis: A Contribution to Hilbert's Last Problem. Cambridge University Press (2011)
17. Pattinson, D.: The logic of exact covers: Completeness and uniform interpolation. In: 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science. IEEE (2013) 418–427
18. Thomason, R.: Logic and Artificial Intelligence. In Zalta, E.N., ed.: The Stanford Encyclopedia of Philosophy (2014) `http://plato.stanford.edu/entries/logic-ai/`