Numerical Relativity & AdS/CFT Correspondence

Based on work with Daniel Grumiller, Stefan Stricker 1506.02658 (JHEP); & Wilke van der Schee, Philipp Stanzer (16XX.XXXXX)

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Quark-gluon plasma (QGP) is a deconfined phase of quarks and gluons produced in heavy ion collision (HIC) experiments at RHIC and LHC.
Why AdS/CFT?

The QGP produced in HIC's behaves like a **strongly coupled liquid** rather than a **weakly coupled gas**.
AdS/CFT correspondence

**AdS/CFT correspondence:** [Maldacena 97]

**Type IIB string theory** on $\text{AdS}_5 \times S^5$ is equivalent to $\mathcal{N}=4$ super symmetric $\text{SU}(N_c)$ **Yang-Mills theory** in 4D.

**Supergravity limit:**

Strongly coupled large $N_c$ $\mathcal{N}=4$ $\text{SU}(N_c)$ SYM theory is equivalent to **classical supergravity** on $\text{AdS}_5$.

**Strategy:**

- Use $\mathcal{N}=4$ SYM as **toymodel** for QCD in the strongly coupled regime.
- Build a **gravity model** dual to HICs, like colliding **gravitational shock waves**.
- Switch on the computer and solve the 5-dim. gravity problem **numerically**.
- Use the **holographic dictionary** to compute **observables in the 4 dim. field theory** form those gravity result.
Solving the Einstein equations on asymptotically AdS

We want to solve the 5 dim. vacuum Einstein equations (EE) with negative cosmological constant $\Lambda$.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

Eddington-Finkelstein gauge (light-like slicing) decouples the EE into a nested set of ODEs. (method of characteristics)

These ODEs can be efficiently solved with a spectral method.

The time evolution is done with a 4th order Runge-Kutta method (RK4).

AdS is not globally hyperbolic – need IC's & BC's to formulate a well defined initial value problem (IVP).

- **BC's**: boundary metric is 4-dim Minkovski
  = background metric of the boundary QFT
- **IC's**: two gravitational shock waves in AdS
  = Lorentz contracted nuclei in the QFT
Holographic thermalization

Thermalization = Black hole formation

$T_{\mu\nu}$

$g_{\mu\nu}$
Entanglement entropy

**Divide** the system into **two parts** A,B. The total Hilbert space factorizes:

\[ \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \]

The **reduced density matrix** of A is obtained by the trace over \( \mathcal{H}_B \):

\[ \rho_A = \text{Tr}_B \rho \]

**Entanglement entropy** is defined as the **von Neumann entropy** of \( \rho_A \):

\[ S_A = -\text{Tr}_A \rho_A \log \rho_A \]
Entanglement entropy in a two quantum bit system

Consider a quantum system of two spin 1/2 dof's. Observer Alice has only access to one spin and Bob to the other spin.

A product state (not entangled) in a two spin 1/2 system:

$$|\psi\rangle = \frac{1}{2}(|\uparrow_A\rangle + |\downarrow_A\rangle) \otimes (|\uparrow_B\rangle + |\downarrow_B\rangle)$$

A (maximally) entangled state in a two spin 1/2 system:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle)$$

Entanglement entropy is a measure for entanglement in a quantum system.
Holographic entanglement entropy

Within AdS/CFT entanglement entropy can be computed from the area of minimal (extremal) surfaces in the gravity theory.

\[ S_A = \frac{\text{Area}(\Sigma)}{4G_N} \]

[Ryu-Takayanagi 06, Hubeny-Rangamani-Takayanagi 07]
Holographic entanglement entropy

- In practice computing extremal co-dim. 2 hyper-surfaces is numerically involved. [work in progress: CE-Grumiller-Khavari]

- Can we somehow simplify our lives?

Yes we can!

Minimal surface for a star boundary region (red) in AdS4 computed with Surface Evolver
Entanglement entropy from geodesics

Consider a **stripe region** of infinite extend in **homogeneous directions** of the geometry. The **entanglement entropy** is prop. to the **geodesics length** in an **auxiliary spacetime**.

\[ S_A = \text{const.} \frac{\text{Length}(\Gamma)}{4G_N} \]

\[ \tilde{g}_{\mu\nu} = \Omega(z, t, x)^2 g_{\mu\nu} \]

**Diagram:**
- **D dim. CFT**
- **D+1 dim. GR**
- **homogeneous directions**
- **geodesic \( \Gamma \)**
- **UV (IR) cut off in CFT (GR)**
- **t=const.**
- **L**
Numerics: relax, don't shoot!

Geodesic equation as two point boundary value problem.

\[ \ddot{X}^\mu(\tau) + \Gamma^\mu_{\alpha\beta} \dot{X}^\alpha(\tau) \dot{X}^\beta(\tau) = 0 \]

BCs: \((V(\pm1), Z(\pm1), X(\pm1)) = (t_0, 0, L/2)\)

- There are two standard numerical methods for solving two point boundary value problems. [see Numerical Recipes]
  - Shooting: Very sensitive to initialization on asymptotic AdS spacetimes.
  - Relaxation: Converges very fast if good initial guess is provided.
Holographic shock wave collisions

HIC is modeled by two colliding sheets of energy with infinite extend in transverse direction and Gaussian profile in beam direction. [Chesler-Yaffe 10]
Wide vs. narrow shocks

Two qualitatively different dynamical regimes

- **Wide** shocks (~RHIC): full stopping

- **Narrow** shocks (~LHC): transparency

[Solana-Heller-Mateos-van der Schee 12]
Geodesics and apparent horizon
Entanglement entropy


t = const.

\[ S_A = \text{const.} \frac{\text{Length}(\Gamma)}{4G_N} \]

[CE-Grumiller-Van der Schee-Stanzer-Stricker 16XX.XXXXX]
Is the numerics right?

Narrow shocks

\[
\frac{(S_{EE}-S_{EE,vac})}{\mu^3}
\]

\[
\mu_L = 0.5
\]

\[
\mu_t = 0.5
\]

Gridsize | \( S_{\text{reg}} \)
---|---
50 | 0.0794411
80 | 0.112574
100 | 0.114726
200 | 0.115453
300 | 0.115452
400 | 0.115451

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Summary

- **AdS/CFT** allows to study the **real time dynamics** of **strongly coupled QFT's** by solving the IVP of (classical) **supergravity** theories.

- **Entanglement entropy** may serve as an **order parameter** for the **full stopping–transparency transition** in holographic shock wave collisions. [CE-Grumiller-Van der Schee-Stanzer-Stricker 16XX.XXXXX]

Work in progress

- **Going beyond supergravity**: string corrections, semi-holography, … [CE-Mukhopadhyay-Preiss-Rebhan-Stricker]

- On the field theory side the **null energy condition** (NEC) is **violated** in narrow shock wave collisions. The **quantum null energy condition** (QNEC) is conjectured to give an upper bound for this violation.

\[ \langle T_{kk} \rangle \geq S'' \]  

[CE-Grumiller-Van der Schee-Stanzer]