SOBRA - Shielding Optimization for BRAchytherapy

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Abstract. In this paper, we study a combinatorial problem arising in the development of innovative treatment strategies and equipment using tunable shields in internal radiotherapy. From an algorithmic point of view, this problem is related to circular integer word decomposition into circular binary words under constraints. We consider several variants of the problem, depending on constraints and parameters and present exact algorithms, polynomial time approximation algorithms and NP-hardness results.

1 Introduction

In France, every year, almost 200,000 patients are treated by radiotherapy as part of their cancer treatment. This kind of therapy uses ionizing radiation aiming at controlling or killing malignant cells as a curative procedure or as part of adjuvant therapy and is widely used (in 2/3 of the cancer treatments). While internal radiotherapy treatments are currently widespread and considered as routine, there is still room for related innovative developments. The aim is to concentrate the radiation beams as precisely as possible towards the tumor site while sparing as much as possible the nearby healthy tissues, such as skin or vital organs (the so-called organs at risk from radiation).

Brachytherapy – also sometimes named Curietherapy – refers to a short distance (*brachys* in Greek) treatment of cancer with radiation from small, encapsulated radionuclide sources (also called *seeds*). These radioactive seeds are used to deliver a high dose to the tissues close to the source. It is characterized by strong dose gradients, *i.e.*, the dose becomes negligible in a very short distance from the source (about 10 % decay per mm) [7]. Such a treatment is given by placing sources directly into or near the volume to be treated. The dose is then

This research was partially supported by PoPRA project funded by Conseil Régional d'Aquitaine and European FEDER and IdEx Bordeaux. We thank Maxence Ronzié for initial assistance in the very first step of the project.

V. Mäkinen et al. (Eds.): IWOCA 2016, LNCS 9843, pp. 309–320, 2016.
DOI: 10.1007/978-3-319-44543-4_24

delivered continuously, either over a short period of time (temporary implants) or over the lifetime of the source to a complete decay (permanent implants). There are many different techniques and sources available.

In this contribution, we focus on *High Dose Rate* (HDR) implants. HDR brachytherapy is a form of internal radiation which temporarily exposes abnormal tissue to a high amount of radiation. Under Computed Tomography and Fluoroscopy guidance, a bronchoscope or a needle is used to deliver a catheter into a position at the tumor site. The other end of this catheter is connected to a computerized machine. This machine passes a small radioactive metal seed through the catheter. The catheter guides the seed to the tumor site. The seed moves step by step through the catheter in order to cover the whole tumor site. The time spent at each position – also known as *dwell time* – is used to control the radiation dose distribution accross the tumor. The overall effect of HDR brachytherapy is to deliver short and precise amounts of high-dose radiation to a tumor while minimizing healthy tissue exposure. After a series of treatment sessions, the catheter is removed leaving no radioactive seeds in the body.

One of the main drawbacks of this technique comes from the lack of precise modulation of the irradiation field and thus of conformation to the shape of the tumor site. In this paper, we aim at studying the benefit of an innovative modulation technique in brachytherapy using tunable shields (as done in external radiotherapy). This approach will allow accumulating both the temporal modulation currently used and the shielding modulation. The aim is to provide treatment of better accuracy by adapting more precisely to the tumor shape. Indeed, currently, the modulation of the radiation source is done by controlling the time spent at each position by the source along the catheter. The main problem is that, at any position, the irradiation is uniform and can be represented as a cylinder surrounding the catheter. This shape does not always conform to the relative placement of the tumor and organs at risk (i.e., in the radiation field). In this contribution, we consider modulating a unique radioactive source using a gear inspired by external radiotherapy.

The use of the shield will allow to preserve, for a given position along the catheter, some part of the surrounding area. The so-called rotating shield brachytherapy (RSBT) was conceptually proposed by Ebert in 2002 [3]. In RSBT, the dose is delivered through a partially shielded radiation source in an optimized step-shot fashion (as done in classical brachytherapy treatment) to improve tumor dose conformity. The intensity of radiation is modulated by the amount of time the shield is pointed in a given direction. RSBT [5,6,13] and other intensity-modulated brachytherapy techniques such as dynamic modulated brachytherapy (DMBT) [10–12] were further studied with the aim of improving intracavitary brachytherapy dose distributions for rectal and cervical cancer. We will first focus on a peculiar type of shield which have been briefly described in the patent [9] and studied in [4]. It corresponds to a set of shield segments forming a cylinder that can be individually retracted to produce circumferentially limited radiation output, directed radially. According to the way the sources are introduced in the patient body, and the physical constraints of the material, it is not possible to build sector of size as small, and thus as high resolution, as wanted. Therefore, using the possible rotation of the equipment, the aim is to find a sequence of sectors configurations that allows delivering a dose distribution as near as possible to the prescribed one. The corresponding algorithmic aspects are unexplored and the goal of this paper is to conduct an algorithmic study which will guide the final development of the equipment. From an algorithmic point of view, the problem is related to circular integer word decomposition into circular binary words under constraints. In Sect. 2 we formally introduce the considered problem and we present an overview of the results.

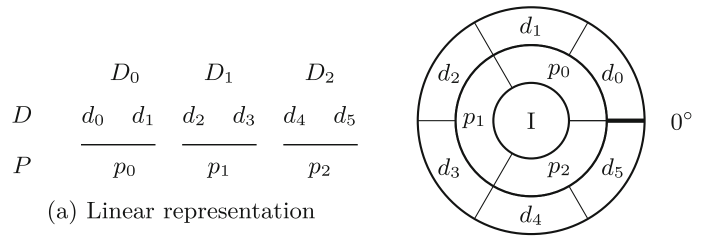
2 A Formal Model for HDR Brachytherapy with Shields

Considering each dwell position of the irradiation source (denoted I), our main objective is to deliver to each part of the surrounding volume its proper irradiation dose. For this purpose, we will use a paddle-based shielding equipment P of K paddles (also referred as sectors for ease) that can stop the radiation going through when they are not retracted. We will consider the surrounding volume to be treated as a circular volume of interest divided in N subvolumes. In the following, a treatment plan for a given dwell position will be defined as a sequence of T shield configurations $\big((P^1, \tau^1), (P^2, \tau^2), \dots, (P^T, \tau^T)\big)$ where P^t, $1 \le t \le T$, is a paddle configuration and τ^t is its dwell time. Each paddle configuration is represented as a binary string $P^t = p_0^t p_1^t \dots p_{K-1}^t$ where p_k^t represents the state (open or closed) of the sector k of P^t. An open sector of the shield (paddle retracted allowing radiation going through) is represented by a 1, while a closed one (paddle is out and radiation is stopped) is represented by a 0.

For each given step (P^t, τ^t) in the treatment plan, a corresponding *received dose* D^t by the surrounding volume is defined as a string of integers $D^t = d_0^t d_1^t \dots d_{N-1}^t$ where d_n^t corresponds to the total irradiation time the subvolume n was exposed to during this step. Roughly, it corresponds to the contribution of the corresponding treatment step to the whole treatment plan. For ease, when parameters P^t and τ^t are not needed for comprehension we may omit them and only write D. Regarding the entire treatment plan, we will denote *the prescribed doses* as a string of nonnegative integers $\hat{D} = \hat{d}_0 \hat{d}_1 \dots \hat{d}_{N-1}$ where $\hat{d}_n$ corresponds to the total irradiation time needed to achieve the right dose for the subvolume n. We will moreover denote *the total received doses* as a string of integers $D = d_0 d_1 \dots d_{N-1}$, such that for all $d_n \in D$, $d_n = \sum_{1 \le t \le T} d_n^t$.

For ease and without loss of generality, we assume that each shield sector is associated to $w = N/K$ consecutive subvolumes, and, for simplicity, that K divides N (so w is an integer). By default, each shield sector p_k will be associated to $D_k = D[k \cdot w,\ (k+1) \cdot w - 1] = d_{k \cdot w} d_{k \cdot w+1} \dots d_{k \cdot w + w - 1}$ of length w (see example Fig. 1a). We can remark that $D = D_0 D_1 \dots D_K$. Informally, one may see P and D as circular strings, P placed inside D and representing a mask that can stop the radiation from going through (see Fig. 1b, with a counterclockwise indexation).

Let us consider the practical case where one is applying a given shielded configuration (represented by P) on a patient (represented by D) for a given



(a) Linear representation

(b) Circular representation

Fig. 1. Relation between P and D ($K = 3$, $N = 6$)

amount of time τ (expressed in a given unit of time). Let us denote $D(P, \tau) = d_0 d_1 \ldots d_{N-1}$ the string of integers obtained by applying radiation for a time τ to D through the mask P. We consider that $p_k = 1$ (resp. $p_k = 0$) denotes applying radiation (resp. no radiation applied) to the area D_k. Moreover, $d_n = \tau$ (resp. $d_n = 0$) if radiation is applied to the volume n for a time τ (resp. if no radiation is applied there). In other words, each subvolume associated to an open sector (represented by a 1) is irradiated τ units of time, while volume associated to a closed sector (represented by a 0) is left in its previous state.

One may consider several variants of the problem, depending on constraints and parameters. First of all, the shield configuration can be considered as fixed or dynamic (one fixed mask or a minimal number of chosen masks) and provided with or without rotation capabilities (this last property is not considered here). These properties are related to manufacturing purposes and constraints. We moreover consider allowing or not irradiation overdoses ($d_n > \hat{d}_n$). Indeed, in practice, it is convenient to overdose a tumor region while one should try to not overdose regions of organs at risk. From a combinatorial point of view, there are two parameters that alter the overall treatment time; namely, the sum of the irradiation times and the number of configurations (as a transition between two configurations will require some time). In the following, we will consider variants of the problem based on the previous observations. In the first two variants, the input consists of only one shield configuration that is given and fixed. The goal is to decide what is the optimum amount of radiation that can be applied when allowing or disallowing overdoses. As proven in Sect. 3, these variants of the problem are polynomial time solvable.

*Problem 1 (*FixMask*).* Given a prescribed dose represented as a string of non-negative integers $\hat{D} = \hat{d}_0 \hat{d}_1 \ldots \hat{d}_{N-1}$ and a fixed shield configuration represented as a binary string $P = p_0 p_1 \ldots p_{K-1}$, find the dwell time τ minimizing $\sum_{n=0}^{N-1} |\hat{d}_n - d_n|$ with $D = D(P, \tau)$.

While in the FixMask variant of the problem, $\hat{d}_n - d_n$ can be negative – that is overdoses are allowed – in the FixMask$^+$ variant, we moreover impose that $\forall n < N$, $d_n \leq \hat{d}_n$ – thus, forbidding overdoses. We now consider variants of the problem where multiple shield configurations are allowed. As mentioned previously, two different criteria can be optimized in such a treatment plan. One would like to either achieve the optimal difference between the prescribed dose and the actual total delivered dose using a minimal number of shield configurations or given an upper bound on the number of shield configurations, achieving the minimum reachable difference. Formally, the problems are defined as follows.

Problem 2 ($\text{MinFixMasks}_{\text{OPT}}$*).* Given two nonnegative integers K and *diff* and a string of integers $\hat{D} = \hat{d}_0\hat{d}_1 \dots \hat{d}_{N-1}$ (with N being a multiple of K), find a treatment plan $\big((P^1, \tau^1), (P^2, \tau^2), \dots, (P^T, \tau^T)\big)$ minimizing T such that $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right| \leq \mathit{diff}$, where $\forall d_n \in D$, $d_n = \sum_{1 \leq t \leq T} d_n^t$.

Problem 3 ($\text{MinFixMasks}_{\text{BOUND}}$*).* Given two nonnegative integers K and $T_{\max}$ and a string of integers $\hat{D} = \hat{d}_0\hat{d}_1 \dots \hat{d}_{N-1}$ (with N being a multiple of K), find a treatment plan $\big((P^1, \tau^1), (P^2, \tau^2), \dots, (P^T, \tau^T)\big)$ where $T < T_{\max}$ minimizing $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right|$, where $\forall d_n \in D$, $d_n = \sum_{1 \leq t \leq T} d_n^t$.

Similarly to FixMask^+, in $\text{MinFixMasks}^+_{\text{OPT}}$ and $\text{MinFixMasks}^+_{\text{BOUND}}$ variants of the problem, we moreover impose that $\forall n < N$, $d_n \leq \hat{d}_n$ – thus, forbidding overdoses.

Our results can be summarized as follows. We will show in Sect. 3 that the problems FixMask and FixMask^+ can be solved in polynomial time. We will then show in Sect. 4 that all of $\{\text{MinFixMasks}_{\text{OPT}}, \text{MinFixMasks}^+_{\text{OPT}}, \text{MinFixMasks}_{\text{BOUND}}, \text{MinFixMasks}^+_{\text{BOUND}}\}$ can be solved in quasi-polynomial time if $\hat{d}_{max}$ is bounded by a polynomial in the number of prescribed doses, where $\hat{d}_{max}$ is the maximum prescribed dose to a subvolume of the patient. In the same section we will also show that the problems $\text{MinFixMasks}_{\text{OPT}}$ and $\text{MinFixMasks}^+_{\text{OPT}}$ can be approximated in polynomial time within a factor of $\log \hat{d}_{max}$ of the optimum. Finally, we will show in Sect. 5 that the problems $\text{MinFixMasks}_{\text{OPT}}$ and $\text{MinFixMasks}_{\text{BOUND}}$ are NP-complete.

3 Polynomial Results

In this section, we show that the variants of the problem where the shield configuration is given and fixed are solvable in polynomial time. Clearly, for a fixed masked, the doses associated to closed paddles cannot be brought closer to the corresponding prescribed doses and will thus not be considered.

Theorem 1. FixMask^+ *can be solved in* $\mathcal{O}(N)$ *time.*

Proof. Because we are not allowed to apply overdoses, we obtain that the maximum and also the optimum irradiation time is equal to the minimum of all prescribed doses $\hat{d}_j$ of $\hat{D}$ for which the corresponding paddle is open. Since the minimum of these doses can be obtained in linear time, the result follows. □

The main observation required to show that FixMask can also be solved in polynomial time is given in the following lemma, which can be considered folklore and is stated here only for the convenience of the reader.

Lemma 1. *For a sequence S of natural numbers and a natural number x, consider the function $f(x)$ such that $f(x) = \sum_{s \in S} |s - x|$. Then $f(x)$ has a unique minimum, which is only reached by any number x in between the at most two medians of S. Moreover, for any x not between the at most two medians of S, the function $f(x)$ decreases with the distance of x to a median of S.*

The above lemma implies that an optimum dwell time for an instance of FixMask is a median of the subsequence of $\hat{D}$ containing all prescribed doses for which the paddles are open.

Theorem 2. FixMask *can be solved in $\mathcal{O}(N)$.*

Proof. Because of Lemma 1 the best possible value that we can achieve for $\sum_{n=0}^{N-1} |\hat{d}_n - d_n|$ is obtained by setting the dwell time τ_0 to any median of the subsequence of $\hat{D}$ containing only the prescribed doses for which the paddles are open (in the given mask). It is known [1] that a median of n numbers can be found in linear time. □

4 Quasi-polynomial Algorithms for MinFixMasks

In this section, we present exact algorithms for all variants of the MinFixMasks problem. The presented algorithms run in quasi-polynomial time if the values of the prescribed patient doses are bounded by a polynomial in the number of prescribed doses. As a by-product we show that the problems MinFixMasks$_{\text{OPT}}$ and MinFixMasks$^{+}_{\text{OPT}}$ can be approximated in polynomial-time within a factor of $\log \hat{d}_{\max}$ of the optimum where $\hat{d}_{\max}$ is the maximum prescribed dose to a subvolume of the patient, *i.e.*, $\hat{d}_{\max} := \max_{\hat{d}_n \in \hat{D}} \hat{d}_n$. We first show that it is sufficient to consider treatment plans where the applied dwell times are pairwise distinct.

Lemma 2. *For any instance of* MinFixMasks$_{\text{OPT}}$*,* MinFixMasks$^{+}_{\text{OPT}}$*,* MinFixMasks$_{\text{BOUND}}$*, and* MinFixMasks$^{+}_{\text{BOUND}}$ *there is an optimal solution $\big((P^1, \tau^1), (P^2, \tau^2), \ldots, (P^T, \tau^T)\big)$ satisfying $\tau^i \neq \tau^j$ for every i and j with $1 \leq i \neq j \leq T$.*

Proof. Let $\mathcal{P} = \big((P^1, \tau^1), \ldots, (P^T, \tau^T)\big)$ be an optimal solution of an instance $\mathcal{I}$ of any of the mentioned variants of the MinFixMasks problem. We will show that we can transform $\mathcal{P}$ into an equivalent treatment plan that does not use any dwell time more than once. Let i and j with $1 \leq i \neq j \leq T$ be such that $\tau^i = \tau^j$. Let (i) $\tau^j_* = 2\tau^j$, (ii) the binary string P^i_* be obtained by the *XOR* of binary strings P^i and P^j, and (iii) the binary string P^j_* be obtained by the *AND* of binary strings P^i and P^j. Then the treatment plan obtained from $\mathcal{P}$ by replacing (P^i, τ^i) with (P^i_*, τ^i) and (P^j, τ^j) with (P^j_*, τ^j_*) is also an optimal solution of $\mathcal{I}$. Moreover, by applying this procedure iteratively we eventually obtain an optimal solution of $\mathcal{I}$ such that all dwell times are pairwise distinct. □

Let S be a set of dwell times. We say that S is *complete* if it contains a subset S' for every number $1 \leq i \leq \hat{d}_{max}$ such that $i = \sum_{s \in S'} s$. We say that a treatment plan is S*-restricted* if it uses only dwell times from S and each of them at most once.

Lemma 3. *Let S be a set of dwell times. Then an S-restricted treatment plan minimizing $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right|$ can be found in time $O((\hat{d}_{max})^2|S| + Kw + K\hat{d}_{max})$. Moreover, the same applies to an S-restricted treatment satisfying the additional constraint that $\hat{d}_n - d_n \geq 0$ for every $0 \leq n \leq N-1$. Finally, if S is complete then the S-restricted treatment plans returned by the above algorithms are optimal among all (not necessarily S-restricted) treatment plans.*

Lemma 4. *There is a treatment plan minimizing $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right|$ using at most $\lfloor \log \hat{d}_{max} \rfloor + 1$ steps. Moreover, such a treatment plan can be found in polynomial time. The same holds for a treatment plan minimizing $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right|$ under the additional constraint that $\hat{d}_n - d_n \geq 0$ for every n with $0 \leq n \leq N-1$.*

Proof. Because the set $S = \{\, 2^i : 0 \leq i \leq \lfloor \log \hat{d}_{max} \rfloor \,\}$ is complete and has size $\lfloor \log \hat{d}_{max} \rfloor + 1$, this follows immediately from Lemma 3. $\square$

Because any non-trivial instance of $\textsc{MinFixMasks}_{\textsc{opt}}$ and $\textsc{MinFixMasks}^{+}_{\textsc{opt}}$ require at least one step, we obtain the following corollary from the above lemma.

Corollary 1. $\textsc{MinFixMasks}_{\textsc{opt}}$ *and* $\textsc{MinFixMasks}^{+}_{\textsc{opt}}$ *can be approximated in polynomial time within a factor of* $\log \hat{d}_{max}$ *of the optimum.*

We are now ready to show our main theorem of this section.

Theorem 3. $\textsc{MinFixMasks}_{\textsc{opt}}$, $\textsc{MinFixMasks}^{+}_{\textsc{opt}}$, $\textsc{MinFixMasks}_{\textsc{bound}}$, *and* $\textsc{MinFixMasks}^{+}_{\textsc{bound}}$ *can be solved in time $O(\hat{d}_{\max}^{\lfloor \log \hat{d}_{max} \rfloor + 1}((\hat{d}_{max})^2(\lfloor \log \hat{d}_{max} \rfloor + 1) + Kw + K\hat{d}_{max}))$.*

Proof. The algorithm goes over all sets S containing at most $\lfloor \log \hat{d}_{max} \rfloor + 1$ (respectively at most $\min\{\lfloor \log \hat{d}_{max} \rfloor + 1, T_{\max}\}$ in the case of $\textsc{MinFixMasks}_{\textsc{bound}}$ and $\textsc{MinFixMasks}^{+}_{\textsc{bound}}$) dwell times between 1 and $\hat{d}_{\max}$. For every such set S, the algorithm then uses Lemma 3 to compute the optimal (the meaning of optimal here depends on the considered problem) S-restricted treatment plan. Finally, in the case of $\textsc{MinFixMasks}_{\textsc{opt}}$ and $\textsc{MinFixMasks}^{+}_{\textsc{opt}}$ the algorithm returns a shortest treatment plan satisfying $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right| \leq$ *diff* and in the case of $\textsc{MinFixMasks}_{\textsc{bound}}$ and $\textsc{MinFixMasks}^{+}_{\textsc{bound}}$ returns a treatment plan minimizing $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right|$ found for any of the considered sets S. The stated running time of the algorithm follows because there are at most $\hat{d}_{\max}^{\lfloor \log \hat{d}_{max} \rfloor + 1}$ such sets S and because of Lemma 3 for each set S, we require time at most $O(\hat{d}_{max})^2|S| + Kw + K\hat{d}_{max})$. The correctness of the algorithm follows from Lemmas 2, 3 and 4. $\square$

Corollary 2. MinFixMasks$_{\text{OPT}}$, MinFixMasks$^{+}_{\text{OPT}}$, MinFixMasks$_{\text{BOUND}}$, *and* MinFixMasks$^{+}_{\text{BOUND}}$ *can be solved in quasi-polynomial time if* $\hat{d}_{max}$ *is bounded by a polynomial in the number of prescribed doses.*

5 Hardness of MinFixMasks$_{\text{opt}}$ and MinFixMasks$_{\text{bound}}$

In this section, we show that MinFixMasks$_{\text{OPT}}$ and MinFixMasks$_{\text{BOUND}}$ are NP-complete already for $w = 2$ (recall that $w = N/K$). Observe that the decision version of the problems MinFixMasks$_{\text{OPT}}$ and MinFixMasks$_{\text{BOUND}}$ are the same, i.e., given a target sequence $\hat{D} = \hat{d}_0 \ldots \hat{d}_{N-1}$ and integers K, *diff*, and $T_{\max}$, determine whether there is a treatment plan $\big((P^1, \tau^1), (P^2, \tau^2), \ldots, (P^T, \tau^T)\big)$ such that $T \leq T_{\max}$ and $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right| \leq \mathit{diff}$.

Our proof uses a reduction from the Monotone 1-3 SAT problem (proven to be NP-complete in [8]): given a boolean formula $\phi = \{c_1, c_2, \ldots\}$ in 3-CNF of $|\phi|$ clauses built on a set $V = \{v_1, v_2, \ldots\}$ of $|V|$ variables, such that its clauses contain only unnegated literals, does there exist a truth assignment on V satisfying ϕ such that each clause is satisfied by exactly one of its three literals?

Given any instance $(\phi,\ V)$ of Monotone 1-3 SAT problem, we build an instance of the decision version of MinFixMasks$_{\text{OPT}}$ and MinFixMasks$_{\text{BOUND}}$ as follows. For all $i \in [1, |V|]$, let q_i be an integer value computed using the following recurrence formula: $q_i = 1 + 2 \times \sum_{j=1}^{i-1}(1 + q_j)$ with $q_1 = |V|$. For each variable $v_i \in V$, we build the sequence $V_i = (q_i, 1+q_i)$. For each clause $c_m = (v_a, v_b, v_c) \in \phi$, we build the sequence C_m composed of two copies of $(q_a + q_b + q_c + 2)$. For each pair (v_i, v_j), $i < j \leq |V|$, we build the sequence $V_{i,j} = (q_i + q_j, q_i + q_j + 2)$. Let $V_{*,j}$ be the concatenation of $V_{1,j}, V_{2,j}, \ldots V_{j-1,j}$. The sequence $\hat{D}$ is obtained by concatenating in order $V_1\ V_2 \ldots V_{|V|}\ C_1\ C_2 \ldots C_{|\phi|}\ V_{*,2}\ V_{*,3} \ldots V_{*,|V|}$. We finally set $K = |V| + |\phi| + \frac{|V|\cdot(|V|-1)}{2}$, $N = 2 \cdot K$ (*i.e.*, $w = 2$), $\mathit{diff} = |V|^2$, and $T_{\max} = |V|$. An illustration is given in Fig. 2.

$$\underbrace{5\ 6}_{V_1}\ \underbrace{13\ 14}_{V_2}\ \underbrace{41\ 42}_{V_3}\ \underbrace{125\ 126}_{V_4}\ \underbrace{377\ 378}_{V_5}\quad \underbrace{61^2}_{C_1}\ \underbrace{545^2}_{C_2}\ \underbrace{517^2}_{C_3}\ \underbrace{145^2}_{C_4}\quad \underbrace{18\ 20}_{V_{1,2}}\ \underbrace{46\ 48}_{V_{1,3}}\ \ldots\ \underbrace{502\ 504}_{V_{4,5}}$$

Fig. 2. Example of an instance of MinFixMasks considering the boolean formula $\phi = (v_1, v_2, v_3) \wedge (v_3, v_4, v_5) \wedge (v_2, v_4, v_5) \wedge (v_1, v_2, v_4)$ which only admits one optimal solution ($v_1 = v_5 =$ **true** and $v_2 = v_3 = v_4 =$ **false**). For ease of notation, v^x will denote x occurrences of the element v (thus 61^2 corresponds to 61 61) and most elements $V_{i,j}$ have been omitted.

Let us start by showing some important properties for any treatment plan of the constructed instance. Let $\big((P^1, \tau^1), (P^2, \tau^2), \ldots, (P^T, \tau^T)\big)$ be a treatment plan that is a solution for an instance constructed from the given formula. We say that a step t *contributes* to a sequence V_i if the block V_i is irradiated at step

t (the i^{th} bit of its mask P^t is set to 1) and t *minimizes* a sequence V_i if the step t is the last one of the treatment plan contributing to V_i: at step $t-1$ and before, V_i did not reach its minimum yet, at step $t+1$ and after, V_i cannot be lowered. Note that a sequence is minimized at exactly one step.

Lemma 5. *For every treatment plan it holds that* $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right| \geq \mathit{diff} = |V|^2$*. Moreover, any treatment plan for which* $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right| = \mathit{diff}$*, uses at least* $|V|$ *steps. Finally, for every treatment plan that uses at most* $|V|$ *steps, it holds that every step minimizes at most one sequence* V_i*.*

Using the reduction defined above, we are now ready to show the main theorem of this section.

Theorem 4. *The* MinFixMasks$_{\text{OPT}}$ *and* MinFixMasks$_{\text{BOUND}}$ *problems are NP-complete when* $w = 2$*.*

Proof. Clearly, both problems are contained in NP, since there is always an optimal solution of length at most $\lfloor \log \hat{d}_{max} \rfloor + 1$ (see also Lemma 4).

We will show the correctness of the reduction from Monotone 1-3 SAT to the decision versions of MinFixMasks$_{\text{OPT}}$ and MinFixMasks$_{\text{BOUND}}$ given above the theorem.

($\Rightarrow$) Let τ be an assignment satisfying ϕ such that each clause is satisfied by exactly one of its literals. We will construct a treatment plan $\mathcal{P} = \big((P^1, \tau^1), (P^2, \tau^2), \ldots, (P^{|V|}, \tau^{|V|})\big)$ satisfying $\sum_{n=0}^{N-1} \left|\hat{d}_n - d_n\right| = \mathit{diff} = |V|^2$ as follows.

For all $1 \leq n \leq |V|$, τ^n is defined by setting $\tau^n = q_n$ if $\tau(v_n) = \mathbf{true}$ and setting $\tau^n = 1 + q_n$ otherwise. Each P^n is obtained by concatenating three substrings corresponding to the V_i's, C_m's and $V_{i,j}$'s as follows: $P^n = P^n_{\text{V}} P^n_{\text{C}} P^n_{\text{V}_*}$ where $P^n_{\text{V}} = 0^{n-1}10^{|V|-n}$, $P^n_{\text{C}} = \texttt{In}(n,1)\texttt{In}(n,2)\ldots\texttt{In}(n,|\phi|)$ ($\texttt{In}(n,m)$ is 1 if $v_n \in c_m$, 0 otherwise) and $P^n_{\text{V}_*} = P^n_{\text{V}_{*,2}} \ldots P^n_{\text{V}_{*,|V|}}$. Each $P^n_{V_{*,i}}$ is defined accordingly to i and n as $P^n_{V_{*,i}} = 0^{i-1}$ if $i < n$; $P^n_{V_{*,i}} = 1^{n-1}$ if $i = n$ and $P^n_{V_{*,i}} = 0^{n-1}10^{i-1-n}$ otherwise.

By construction, $\mathcal{P}$ applies total dwell time of either q_n or $1 + q_n$ each V_n. Moreover, any C_m corresponding to a clause (v_a, v_b, v_c) receives a total dwell time of $q_a + q_b + q_c + 2$, since by hypothesis exactly one of $\{v_a, v_b, v_c\}$ is true in our assignment: that is either $q_a + (1 + q_b) + (1 + q_c)$ or $(1 + q_a) + q_b + (1 + q_c)$ or $(1 + q_a) + (1 + q_b) + q_c$. Finally, a total dwell time $\tau^i + \tau^j$ such that $q_i + q_j \leq \tau^i + \tau^j \leq q_i + q_j + 2$ has been applied to each $V_{i,j}$, lowering its cost to 2. Thus, any Monotone 1-3 SAT solution over ϕ gives us an optimal solution for our instance of MinFixMasks using $|V|$ shield configurations.

($\Leftarrow$) Let $\mathcal{P} = \big((P^1, \tau^1), (P^2, \tau^2), \ldots, (P^T, \tau^T)\big)$ be a solution, i.e., it holds that $\sum_{n=0}^{N-1} |\hat{d}_n - d_n| \leq \mathit{diff} = |V|^2$ and $T \leq T_{\max}$. It follows from Lemma 5 that $\sum_{n=0}^{N-1} |\hat{d}_n - d_n| = \mathit{diff} = |V|^2$ and moreover $T = T_{\max} = |V|$.

Because we have $|V|$ sequences V_i that need to be minimized at some step of $\mathcal{P}$ and $\mathcal{P}$ has $|V|$ steps, we obtain from Lemma 5 that exactly one sequence

V_i is minimized at any step of $\mathcal{P}$. W.l.o.g., we can assume that $\mathcal{P}$ is ordered in such a way that each sequence V_i is minimized at step i. In the next proposition, we prove that the dwell time at step i can take one of two possible values, thus corresponding to a true/false assignment of variable i.

Proposition 1. *For any i, $\tau^i \in \{q_i, 1+q_i\}$.*

Proof. We prove the result by induction. Because of our assumption on the ordering of $\mathcal{P}$, we obtain that $q_i - \sum_{j=1}^{i-1} \tau^j \leq \tau^i \leq 1 + q_i$ for each step i. Thus, for the first induction step, it holds that $q_1 \leq \tau^1 \leq 1 + q_1$, so $\tau^1 \in \{q_1, 1+q_1\}$.

Considering the step j, and the sequence $V_{j-1,j} = (q_{j-1} + q_j, q_{j-1} + q_j + 1)$, no step after j can contribute to $V_{j-1,j}$ since

$$\begin{aligned}\tau^{j+1} \quad &\geq \quad q_{j+1} - \sum_{i=1}^{j}(1+q_i) \quad \geq \quad 1 + 2 \times \sum_{i=1}^{j}(1+q_i) - \sum_{i=1}^{j}(1+q_i) \\ &\qquad\qquad\qquad\qquad\qquad\quad > \quad q_{j-1} + q_j + 2\end{aligned}$$

Moreover, $\sum_{i=1}^{j-1} \tau^i \leq \sum_{i=1}^{j-1}(1+q_i) < q_j$, so the contribution of step j is mandatory to minimize $V_{j-1,j}$ which induces that

$$\begin{aligned}\tau^j \quad &\geq \quad q_{j-1} + q_j - \sum_{i=1}^{j-1} \tau^i \quad \geq \quad q_{j-1} + q_j - \sum_{i=1}^{j-1}(1+q_i) \\ &\qquad\qquad\qquad\qquad\qquad\quad \geq \quad q_j - 1 - \sum_{i=1}^{j-2}(1+q_i)\end{aligned}$$

Suppose now that there exists $k \geq 2$ such that $\tau^j \geq q_{j-1} - 1 - \sum_{i=1}^{j-k}(1+q_i)$. Consider then $V_{j-k,j} = (q_{j-k} + q_j, q_{j-k} + q_j + 2)$. Applying a similar reasoning as before, we conclude that the contribution of step j is mandatory, and, with our last lower bound over τ^j that

$$\begin{aligned}q_{j-k} + q_j - \tau^j \quad &\leq \quad q_{j-k} + q_j - \left(q_j - 1 - \sum_{i=1}^{j-k}(1+q_i)\right) \\ &\leq \quad q_{j-k} + 1 + \sum_{i=1}^{j-k}(1+q_i) \quad < \quad q_{j-k+1}\end{aligned}$$

Thus, steps strictly between $j-k$ and j cannot contribute. Therefore, the only steps able to contribute are j and 1 to $j-k$:

$$\begin{aligned}\tau^j \quad &\geq \quad q_{j-k} + q_j - \sum_{i=1}^{j-k} \tau^i \quad \geq \quad q_{j-k} + q_j - \sum_{i=1}^{j-k}(1+q_i) \\ &\qquad\qquad\qquad\qquad\qquad\quad \geq \quad q_j - 1 - \sum_{i=1}^{j-(k+1)}(1+q_i)\end{aligned}$$

We obtain a greater lower bound for τ^j. This reasoning can be applied as long as $V_{j-k,j}$ exists, that is as long as $j - k \geq 1$. The last application ($k = j - 1$, $k + 1 = j$) leads to

$$\tau^j \quad \geq \quad q_j - 1 - \sum_{i=1}^{j-j}(1 + q_i) \quad \geq \quad q_j - 1$$

On the whole, we obtain $\tau^j \in \{q_j - 1, q_j, 1 + q_j\}$. Moreover, if $\tau^j = q_j - 1$, then step j is not enough to minimize $V_j = (q_j, 1 + q_j)$ (an amount of 1 or 2 is missing). But we can only use the dwell times of the treatment plan, and the lowest one is $\tau^1 \in \{|V|, |V| + 1\}$, where $|V|$ is the number of variables in ϕ (so $|V| \geq 3$). Thus, $\tau^j = q_j - 1$ is impossible. This leads to $\tau^j \in \{q_j, 1 + q_j\}$ for any step j. ■

To complete our proof, it remains to show that a sequence $C_m = (q_a + q_b + q_c + 2, q_a + q_b + q_c + 2)$ cannot be minimized by other steps, except those corresponding to an assignment to the variables a, b and c.

Proposition 2. *Minimizing a sequence $C_m = (q_a + q_b + q_c + 2, q_a + q_b + q_c + 2)$ implies the contribution of exactly the steps a, b, and c.*

Proof. W.l.o.g. let $a < b < c$. To minimize C_m, we need to apply a total amount of exactly $\tau = q_a + q_b + q_c + 2$. Since $\tau^{c+1} \geq q_{c+1} \geq 1 + 2 \times \sum_{i=1}^{c}(1 + q_i) > \tau$, step $c + 1$ or higher cannot contribute to C_m. Thus the contribution of step c is mandatory since $\sum_{i=1}^{c-1} \tau^i \leq \sum_{i=1}^{c-1}(1 + q_i) < 1 + 2 \times \sum_{i=1}^{c-1}(1 + q_i) < q_c$. Similarly, $\tau^{b+1} \geq q_{b+1} > \tau - \tau^c$, so steps strictly between b and c cannot contribute, and $\sum_{i=1}^{b-1} \tau^i < q_b \leq \tau - \tau^c$ inducing that the contribution of step b is mandatory. Finally, $\tau^{a+1} \geq q_{a+1} > \tau - \tau^c - \tau^b$, so steps strictly between a and b cannot contribute, implying that the contribution of step a is mandatory since $\sum_{i=1}^{a-1} \tau^i < q_a \leq \tau - \tau^c - \tau^b$. ■

Gathering the previous results, we have an optimal solution to our MIN-FIXMASKS instance if and only if each sequence C_m corresponding to a clause (v_a, v_b, v_c) receives exactly the dwell times received by the sequences V_a, V_b and V_c. Moreover, each of theses V_i receives either q_i or $1 + q_i$ as a (total) dwell time. Finally, minimizing C_m implies that exactly one of the three V_i receives the lowest of its two possible values. This corresponds to a truth assignment over ϕ such that each of its clauses contains exactly one true variable. □

6 Conclusions and Future Work

We gave the first rigorous algorithmic study of the recently introduced rotating shield brachytherapy. Our analysis led to efficient algorithms as well matching hardness results. For future work we plan to explore further variants of the problem, e.g., variants resulting from a rotation of the shield.

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