Numerical and experimental comparison of the energy transfer in a parametrically excited system

Andreas Fichtinger\textsuperscript{1} and Horst Ecker\textsuperscript{2}
Institute of Mechanics and Mechatronics, Vienna University of Technology, Vienna, AT
E-mail: \textsuperscript{1}andreas.fichtinger@tuwien.ac.at
E-mail: \textsuperscript{2}horst.ecker@tuwien.ac.at

Abstract. This study is based on experimental results obtained from a test rig for a 2-DOF vibrational system. The main feature of this test rig is an electromechanical actuator which enables the variation of a mechanical stiffness parameter. The vibrational system is excited by this device, resulting in a parametrically excited system. In this study, the test rig is operated such that an initial deflection is applied to the masses, but no external excitation acts on the system. The actuator coils are driven by a current following a harmonic function defined by amplitude and frequency. The latter defines the parametric excitation frequency (PEF). When the mechanical system performs free vibrations, the parametric excitation may initiate an energy transfer between the two vibrational modes of the system. This transfer depends primarily on the PEF. For certain values of the PEF only one of the vibrational modes is affected. At other frequencies both modes are influenced and a continuous intermodal energy transfer is observed. A numerical model for the test rig was established and allows a comparison of experimental and numerical results. The total system energy as well as the modal energies are calculated from the measured and from the simulated signals. Interesting experimental observations are reported concerning the transfer of energy, and are in good agreement with simulated results.

1. Introduction
It is well known that linear time-variant systems (LTV-systems) exhibit a rich dynamical behavior \cite{1, 2}. This holds already for the most simple systems one can think of, with just two system states and as described mathematically by Mathieu’s famous equation. If a mechanical system is considered, Mathieu’s equation represents the mathematical model for a free vibrating single mass, supported by a spring with time-periodic stiffness parameter. Hence, such LTV-systems are also known as parametrically excited systems. Countless contributions have been published on the dynamic stability of single degree-of-freedom-systems (SDOF-systems) with parametric stiffness excitation. Such systems may become dynamically unstable within small frequency-intervals related to the frequency of the parametric excitation. This phenomenon is known as ”parametric resonance”, although it is, strictly speaking, a loss of stability within a narrow frequency interval and not a true resonance phenomenon.

Because of the mathematical and experimental challenges one is already confronted when dealing with the most simple SDOF-systems, multi-degree-of-freedom-systems (MDOF-systems) have been rarely investigated analytically or numerically in the past, and even more rarely by experiment. However, in a paper, published by A. Tondl \cite{3} in 1998, a completely new feature
of MDOF-systems with parametric excitation has been presented. This reference sparked an increasing and still on-going interest in the dynamical behavior of such systems.

While at first, the focus of studies has been on the suppression of self-excited vibrations due to the newly discovered "anti-resonance"-phomenon [4] and the comprehensive study of PE-systems operating at parametric combination frequencies [5], [6], the interest now concerns the understanding of the underlying mechanisms. In this context, the energy flow into and out-of the system, as well as within the system, plays a major role. Understanding this energy flow is essential for the utilization of beneficial features of PE-excited systems, but also to avoid triggering troubles due to PE-excitation.

In a numerical study [7] Ecker and Pumhössel investigated the energy flow in a MDOF-torsion vibration system with parametric excitation. It turned out, that the vibrational modes of the system are of utmost importance, since energy is either pumped into a mode, or extracted from a mode, or moved between modes. Although for a LTV-MDOF-system the term "vibrational mode" needs to be redefined (due to the time-periodicity of relevant system parameters), it is shown that the modes of the abbreviated LTI-system are still meaningful and can be used, provided that the parametric excitation amplitude is (rather) small.

To supplement the numerical and analytical studies and efforts, an already existing test rig was modified and enhanced to make it suitable for investigating the energy transfer due to parametric excitation. The following sections explain the key features of the test stand, introduce the numerical simulation model of the test stand, and show and compare measured and calculated results from experimental work.

2. 2-DOF - test rig with parametrically excited stiffness actuator

A 2-DOF test rig (2DTR) has been developed in prior works at the Institute of Mechanics and Mechatronics [8]. In the current configuration, the 2DTR allows to harmonically change one of the stiffness parameters of the rig. Figure 1 shows the 2DTR consisting of two masses, linear springs, eddy current brakes and the electro-magnetic stiffness actuator. Mass 1 defining the first DOF can be adjusted manually to the needs by adding or removing cylinders made of steel. Mass 2 representing the second DOF moves inside the magnetic actuator and cannot be modified. Both masses are attached to wires hanging down from the ceiling. The length of the wires is approximately two meters and allows the masses to move like pendula with a minimum of drag. The vibration of the system is measured by laser sensors at each mass which continuously determine the actual deflection.

Figure 1. 2-DOF Test rig with electromagnetic actuator for parametric excitation of one stiffness parameter.
The electro-magnetic actuator is build from a ferromagnetic tube with permanent magnets attached inside at both ends. Inside the tube a copper coil is mounted on a carbon fibre pipe which can move freely through holes on both ends of the case. Cables are lead inside the tube to power the coil with a current from an external power supply, see [9].

![Figure 2. Schematic of the electromagnetic actuator to provide a time-periodic stiffness](image)

A simplified model for the non-linear stiffness \( k_{PE} \) of the actuator was determined in [9] of the following form:

\[
k_{PE}(i(t), x) = i(t) \left( k_{I0} + k_{I2} x^2 \right)
\]

with \( i(t) \), the time dependent current, \( x \), the DOF, and \( k_{I0}, k_{I2} \) as constants which describe the stiffness dependency of current and position.

3. Numerical model of the 2-DOF vibration system

For the numerical simulation of the 2-DOF system a mathematical model including the inherently nonlinear parametric excitation was implemented in MATLAB. The model consists of two discrete masses, including all additional masses of the connecting and supporting elements. Therefore the mass matrix is diagonal. The stiffness matrix is build up of three different sources of stiffness. First, there are the linear springs between the two masses and each mass and the frame of the 2DTR, which can be represented in a symmetrical stiffness matrix. The stiffness of the electromagnetic actuator is represented in the matrix at the position for absolute stiffness of the 2nd DOF. Finally the restoring forces of the pendulum motion have to be considered in the stiffness matrix. They can be linearized because of the small maximum displacements of the masses, and are added to the diagonal elements of the stiffness matrix. To compensate any position offsets an additional vector \( x_{v0} \) is used in the model equation.

\[
M \ddot{x} + C \dot{x} + K(t, x) x = 0
\]

\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix} \ddot{x} + \begin{bmatrix}
c_{01} & 0 \\
0 & c_{02}
\end{bmatrix} \dot{x}
+ \begin{bmatrix}
k_{11} + k_{12} + \frac{m_1 g}{L} & -k_{12} \\
-k_{12} & k_{01} + k_{12} + \frac{m_2 g}{L} + \frac{m_1 g}{L} + i(t)(k_{I0} + k_{I2} x^2)
\end{bmatrix} (x - x_{v0}) = 0
\]

A thorough system identification had to be carried out. Although the mass parameters could be easily determined, identification of the stiffness parameters \( k_{i,j} \) was already non-trivial. As expected, the identification of the damping parameters \( c_{0,i} \) was the most difficult part of the procedure, and only satisfactory results were achieved.
4. Energy calculation

The energy present in the system was analyzed by calculating the total energy level and split up into so-called “quasi-modal energy signals”. Because of the non-linear pendulum motion and also the non-linear parametric excitation of the stiffness a straight-forward modal transformation of the signals is not possible. The modal analysis was carried out by using the average values for the stiffness parameters and therefore the expression “quasi-modal” is used here for this transformation.

For the energy calculation position and velocity data for each DOF is needed. Since, there are no particular speed sensors installed on the 2DTR, the velocity was derived by central differences of the filtered position signals.

The linear potential energy (springs and linearized pendulum stiffness) and the kinetic energy are calculated as:

\[
E_{pot,\text{lin}} = \frac{1}{2} x^T K_{\text{lin}} x \quad \text{and} \quad E_{\text{kin}} = \frac{1}{2} \dot{x}^T M \dot{x}
\]  

(3)

The force of the magnetic actuator is \( F(i(t), x_2) = k_{PE}(i(t), x_2) x_2 \). Thus, after integration of the force, the potential energy of the magnetic actuator can be written as

\[
E_{pot,\text{mag}} = i(t) \left( k_{I0} \frac{x_2^2}{2} + k_{I2} \frac{x_2^4}{4} \right)
\]  

(4)

See [9] for details about the model in use for the non-linear relationship between coil-current, displacement and force created by the actuator.
5. Comparison of measurement and simulation in the frequency-domain

The model parameters were obtained first by measurements and then numerically with a least-square fitting procedure. The parameters for the system can be found in the Appendix in Tab. A1. To verify the parameters and the 2DOF-model, simulations and experiments were carried out in a wide range of PEFs. The start of the PEF range is at 0.05 rad/s (0.008 Hz) and goes up to 15.5 rad/s (2.4669 Hz) in 0.05 rad/s steps. Within this range all important frequencies (eigenfrequencies, parametric resonance and anti-resonance frequencies, etc.) of the parametric excited 2DOF-system are included. For each PEF a Fourier Analysis of both position signals was computed and then arranged into a frequency map.

![Figure 4. Frequency map of the MEASURED position signal of the 2nd DOF, red lines indicate aborted tests due to large displacements](image1)

![Figure 5. Frequency map of the SIMULATED position signal of the 2nd DOF](image2)

The diagrams in Figs. 4 and 5 show a very good agreement of experimental and simulation data. The two eigenfrequencies can be seen clearly as vertical lines in the frequency maps. Many additional frequencies can be seen that do not exist in linear, time-invariant systems and vary with the PEF. In Fig. 6 the change and dependencies of these frequencies are shown schematically in the investigated range of PEFs.

Also in the following figures Fig. 7 and Fig. 8, that show the positions over time and the spectrum of the vibration at a single PEF, a good agreement of the model behaviour and the experiments can be seen. Due to the low frequency vibrations of mass 1, a minor phase error becomes evident only after 40 seconds. Vibrations of mass 2 are dominated by a higher frequency, so the inevitable phase error is visible after about 10 seconds. In addition to that, the actual simulation of the actuator forces may also be different from the measurements and will consequently have a minor impact on the results.

Another way to compare measurement and simulation is to compare the time that is needed to reach a certain energy level. The following Fig. 9 shows the decay time down to a level of 20% of the initial energy level. The diagram contains markers at special frequencies that are
Figure 6. Main frequencies in the parametrically excited system

Figure 7. Measured and simulated position signals over time with a PEF of 0.5968 Hz

Figure 8. Measured and simulated spectra of the position signal with a PEF of 0.5968 Hz

described in Tab. 1. At the parametric anti-resonance frequencies ($\Omega_2 - \Omega_1$ and $\frac{\Omega_2 - \Omega_1}{2}$) the increased damping behaviour results in strongly decreased decay times. The absolute decay time level is much lower for the simulation. The cause for this difference is non-linear damping behaviour at the second mass, that was not considered in the damping model.
6. Energy and intermodal energy transfer

Table 1. Investigated PEFs of the 2-DOF-system

<table>
<thead>
<tr>
<th>Name</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st eigenfrequency</td>
<td>$\Omega_1$</td>
</tr>
<tr>
<td>2nd eigenfrequency</td>
<td>$\Omega_2$</td>
</tr>
<tr>
<td>half combination resonance freq.</td>
<td>$\frac{\Omega_2+\Omega_1}{2}$</td>
</tr>
<tr>
<td>PEF with best damping behaviour</td>
<td>$\sim (\Omega_2 - \Omega_1)$</td>
</tr>
</tbody>
</table>

The energy of the vibration is investigated here at four particular PEFs. These are the first and the second eigenfrequency and two of the combination-PEFs. The PEF values can be found in the Tab. 1. All simulations and measurements were done with the same initial conditions for the free vibration: $x_{0,1} = 20\text{mm}$ and $x_{0,2} = 8\text{mm}$, which mainly excite the first quasi-mode of the system. The current amplitude of the harmonic stiffness variation was chosen to be approximately at 60% of the mean component.
6.1. PEF at 1st eigenfrequency
In this case the PE-frequency is chosen to be the 1st eigenfrequency of the (linearized) system. The signals were recorded for a period of 300 seconds, starting at the beginning of the vibrations, and then analyzed. In Fig. 10 one can see that the modulation of the decreasing total energy is slightly different for the simulation and the measurement values, but the global behaviour is still similar. The next Fig. 11 shows the filtered, quasi-modal energies. It can be seen clearly that the dominant modulation of the first eigenmode energy is at a very low frequency, like for the total energy, whereas the second eigenmode energy shows changes with a much higher frequency ($\sim f_{PE}/2$). The last figures Fig. 12 and Fig. 13 show the quasi-modal energies in a modal energy plane. There it can be seen that only a small amount of energy is transferred between the two modes. The line colours are used to indicate the time period, divided into sections of 100 seconds.

Figure 10. Total energy of the simulated and measured 2-DOF-system with a PEF of 0.8674 Hz at the 1st EF

Figure 11. Modal energies of the simulated and measured 2-DOF-system with a PEF of 0.8674 Hz at the 1st EF

Figure 12. Measured modal energies in the modal plane with a PEF of 0.8674 Hz at the 1st EF

Figure 13. Simulated modal energies in the modal plane with a PEF of 0.8674 Hz at the 1st EF
6.2. PEF at 2nd eigenfrequency

Setting the PEF to the 2nd EF (2.1327 Hz) the system exhibits a completely different behavior compared to the previous case. First, the total energy approaches a non-zero value within the investigated time span and the variation has a much higher frequency, see Fig. 14. The diagrams of Fig. 15 showing the modal energies clearly show that this strong variation appears only for the energies of the second eigenmode. Therefore, the modal energy plane diagrams 16 and 17 are dominated by vertically oriented paths, thus there is almost no exchange of energy between the modes. While the energy of the first mode is decreasing the energy of the second mode has strong variations at the beginning and is converging to a non-zero energy level. The seemingly large difference of this energy level between measurement and simulation can be explained by the simple linear damping model which was used in the simulation. The actual damping for large displacements is strongly nonlinear. However, the simulation model was not further developed to close the gap between measurement and simulation and this is most likely the cause for the remaining deviations. Further research will be needed to address this issue.

Figure 14. Simulated and measured total energy with a PEF of 2.1327 Hz (2nd EF)

Figure 15. Simulated and measured modal energies with a PEF of 2.1327 Hz (2nd EF)

Figure 16. Measured modal energies in the modal plane with PEF of 2.1327 Hz (2nd EF)

Figure 17. Simulated modal energies in the modal plane with PEF of 2.1327 Hz (2nd EF)
6.3. PEF at half combination resonance frequency

The combination resonance frequency investigated now is the average value of the two eigenfrequencies $\Omega_{PE} = \frac{\Omega_1 + \Omega_2}{2}$. The energy diagrams show again a significant modulation of the energies over time composed from different frequencies. The variation with the lowest frequency is for both modes in phase, so that there is permanent energy transfer in and out of the entire system. The modulation at a high frequency is out of phase with a phase shift of $180^\circ$. This phase shift can be seen easily in the modal energy plane, where the energy paths are not mainly vertically oriented any more, but in a diagonal direction. Thus, there is a permanent exchange of energy between the modes additionally to the variation of the total energy. In the simulation the average energy level seems to reach a stationary level, whereas the measured energy levels are still decreasing within the observed period.

Figure 18. Total energy of the simulated and measured 2-DOF-system with a PEF of 1.5040 Hz at the half combination resonance frequency

Figure 19. Modal energies of the simulated and measured 2-DOF-system with a PEF of 1.5040 Hz at the half combination resonance frequency

Figure 20. Measured modal energies in the modal plane with a PEF of 1.5040 HZ at the half combination resonance frequency

Figure 21. Simulated modal energies in the modal plane with a PEF of 1.5040 HZ at the half combination resonance frequency
6.4. PEF near a combination anti-resonance frequency

In this case the PE-frequency is set to 1.3289 Hz, which is the frequency with the highest increase of damping. This value was found experimentally and is very close to the calculated anti-resonance PEF, which is, according to the established theory $\Omega_2 - \Omega_1 (= 1.2653 \text{ Hz})$.

In Fig. 22 the total energy is plotted over time. The overall fast decrease of total energy is obvious. However, the maximum energy peak that occurs shortly after the beginning is twice as high as the initial energy of the system! The diagrams of the modal energies, Fig. 23, show once more that the modal energies are out of phase with a phase shift of 180 degrees. This can also be seen very clearly in the modal plane diagram, where the energy paths are all running diagonal. The local energy minima are almost zero during this permanent exchange of energy between the modes. The modal energy plane diagram shows also the fast decay of the vibration.

Figure 22. Total energy of the simulated and measured 2-DOF-system with a PEF of 1.3289 Hz (best damping behavior)

Figure 23. Modal energies of the simulated and measured 2-DOF-system with a PEF of 1.3289 Hz (best damping behavior)

Figure 24. Measured modal energies in the modal plane with a PEF of 1.3289 HZ (best damping behavior)

Figure 25. Simulated modal energies in the modal plane with a PEF of 1.3289 HZ (best damping behavior)
7. Conclusions

The numerical 2-DOF model employed in this study predicts the behavior of the 2-DOF test rig very well, even though only a linear damping was considered. The frequency spectra of measurement and simulation both include signals at various frequencies beside the eigenfrequencies. Since those signals change with the PE-frequency they are harmonics of the PEF. These harmonics cross the (constant) eigenfrequencies exactly at the anti-resonance PEFs. Another phenomenon, termed “frequency splitting” is observed in these cases.

Energy transfer between the vibrational modes was predicted by simulation and could be observed in the experiment. The (modal) energy characteristics depend heavily on the PEF. At PEFs that induce permanent energy transfer between the modes, also a change of the total energy can be observed always.

Parameter of the 2DOF test rig

The following Tab. A1 lists the numerically identified parameters of the 2DOF system which were obtained with a least square fitting algorithm.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$ (kg)</td>
<td>1,110</td>
</tr>
<tr>
<td>$k_{01}$ (N/m)</td>
<td>17,513</td>
</tr>
<tr>
<td>$c_{01}$ (mN/ms)</td>
<td>6,089</td>
</tr>
<tr>
<td>$x_{v0,1}$ (mm)</td>
<td>0,086</td>
</tr>
<tr>
<td>$m_2$ (kg)</td>
<td>14,010</td>
</tr>
<tr>
<td>$k_{02}$ (N/m)</td>
<td>18,500</td>
</tr>
<tr>
<td>$c_{02}$ (mN/ms)</td>
<td>-0,199</td>
</tr>
<tr>
<td>$x_{v0,2}$ (mm)</td>
<td>11,004</td>
</tr>
<tr>
<td>$k_{12}$ (N/m)</td>
<td>75,560</td>
</tr>
<tr>
<td>$k_{I0}$ (N/Am)</td>
<td>63906,974</td>
</tr>
</tbody>
</table>

References