Roman Kuznets

# Craig Interpolation via Hypersequents

**Abstract:** In this paper, we describe a novel constructive method of proving the Craig interpolation property (CIP) based on cut-free hypersequent calculi and apply the method to prove the CIP for the modal logic S5.

**Keywords:** Craig interpolation, Hypersequent, Structural proof theory, Modal logic

**Mathematics Subject Classification 2010:** 03B45, 03C40, 03F07, 03B70

## 1 Introduction

Interpolation is one of the standard properties of a logical system, studied alongside decidability, complexity, and semantic completeness. Interpolation is closely related to algebraic properties such as amalgamation (see [Gabbay and Maksimova, 2005] for an overview and the history of the subject). The logic L is said to have the *Craig interpolation property* (CIP) if, whenever $\mathsf{L} \vdash A \to B$, there exists a formula $C$ "in the common language" of $A$ and $B$ such that $\mathsf{L} \vdash A \to C$ and $\mathsf{L} \vdash C \to B$. The formula $C$ is then called the *interpolant* of $A$ and $B$. In this paper, we consider modal logics, hence, the "common language" simply means having the same propositional variables.

One of the methods for proving the CIP constructively and efficiently[1], is by employing a cut-free (or, more generally, an analytic) proof system and by constructing an interpolant by induction on the derivation of $A \to B$ (properly represented in this proof system). Such a method based on sequent calculi is well known and had been used for many a system, e.g., for classical and intuitionistic propositional logics and for many modal logics. However, sequent calculi do not seem to be expressive enough to capture many interesting modal logics. In fact, one of the first modal logics to be considered in modern times, S5, has so far resisted all attempts at being captured by a cut-free sequent calculus. Moreover, it was shown by Lellmann and Pattinson [2013] that such a calculus does not exist under reasonable restrictions on the type of rules used.

1 Rather than by the exhaustive search of all potential interpolants.

**Roman Kuznets:** Institut für Computersprachen, Technische Universität Wien,
e-mail: roman@logic.at

The simplest extension of the sequent formalism that is known to be capable of capturing S5 is the formalism of hypersequents, which was first introduced by Minc [1971]. (Mints[2] used the term "cortege" rather than "hypersequent" and employed a slightly more cumbersome notation than we are used to now.[3]) Hypersequents under their proper name were later independently rediscovered by Pottinger [1983] and Avron [1987].

It is, thus, natural to generalize the constructive method of showing the CIP from sequents to hypersequents. In his seminal survey, Avron [1996] writes:

> The only rule [...] which brings moments of synchronization into proofs is external contraction. [I]ts presence is the explanation why in hypersequential calculi cut-elimination usually does not imply the Craig interpolation theorem.

According to Avron, the rule of external contraction

$$\mathsf{EC}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

is the main obstacle to such a generalization. However, the modal logic S5, captured by a hypersequent calculus with the external contraction rule EC, is known to enjoy the CIP. Thus, EC does not, in general, prevent the CIP from holding.

In a joint work [Fitting and Kuznets, 2015], we developed a method of generalizing the syntactic proof of the CIP from sequents to nested sequents. While hypersequents are not exactly subsystems of nested sequents because several of hypersequent rules are not present in nested calculi, it is reasonable to view hypersequents as substructures of nested sequents. Indeed, a hypersequent is a set (multiset, sequence) of ordinary sequents, whereas a nested sequent is a tree of sequents. Both hypersequents and nested sequents are *internal calculi*, meaning that each hypersequent and each nested sequent can be translated into a formula. Based on the commonly used translations, components of a hypersequent correspond to children of the root of a nested sequent. Thus, it seems reasonable to expect that the method for proving the CIP via nested sequents from [Fitting and Kuznets, 2015] can be adapted to hypersequents. In this paper, we show that this is indeed the case and that the external contraction does not present any difficulties for our method. We develop the method using a hypersequent calculus for S5 from [Avron, 1996] as an example.

---

**2** Spelled "Minc" in the translation.

**3** The full text of the original Russian version of 1968 is available at http://mi.mathnet.ru/eng/tm/v98/p88.

## 2 The Modal Logic S5

Being the fifth of Lewis's systems, the modal logic S5 is one of the oldest modern modal logics in existence and is often used, for instance, as a logic of knowledge [Fagin et al., 1995]. Its language, which we will call the *modal language*, is defined by the grammar

$$A ::= P \mid \bot \mid (A \to A) \mid \Box A,$$

where $P$ is taken from a countably infinite set Prop of propositional variables. The Boolean constant $\top$, the Boolean connectives $\neg$, $\wedge$, and $\vee$, and the modality $\Diamond$ are defined in the standard way. We use the standard conventions regarding the omission of redundant parentheses.

The modal logic S5 is axiomatized by all instances of propositional tautologies in the modal language above and the following modal axioms:

- t: $\Box A \to A$;
- 4: $\Box A \to \Box\Box A$;
- 5: $\neg\Box A \to \Box\neg\Box A$.

The epistemic reading of these axioms, when $\Box A$ is understood as "$A$ is known," states the factivity of knowledge and the positive and negative introspection of the knowledge agent. The inference rules are *modus ponens* and *modal necessitation*:

$$\text{MP}\ \frac{A \to B \quad A}{B} \qquad \text{and} \qquad \text{Nec}\ \frac{A}{\Box A}.$$

The logic S5 has a particularly simple Kripke semantics:

**Definition 2.1** (Kripke frames and models). *A* Kripke frame *(for* S5*) is* $(W, \sim)$*, a pair of a non-empty set* $W$ *and of an equivalence relation* $\sim$ *on* $W$*. Elements of* $W$ *are called* worlds. *If* $u \sim v$*, we say that the world* $u$ *is* indistinguishable *from the world* $v$*. A* Kripke model *(for* S5*) is a triple* $(W, \sim, V)$ *where* $(W, \sim)$ *is a Kripke frame (for* S5*) and* $V\colon W \to 2^{\text{Prop}}$ *is a valuation that assigns to each world from* $W$ *the set of propositional variables true at this world.*

**Definition 2.2** (Truth and validity in Kripke models). *Given a Kripke model* $\mathcal{M} = (W, \sim, V)$*, the relation of truth between modal formulas and worlds* $w$ *in this model is defined as follows:*

*for each propositional variable* $P \in \text{Prop}$*:* $\quad \mathcal{M}, w \Vdash P \quad$ *iff* $\quad P \in V(w)$;

$\mathcal{M}, w \nVdash \bot$;

$\mathcal{M}, w \Vdash A \to B \quad$ *iff* $\quad \mathcal{M}, w \nVdash A$ *or* $\mathcal{M}, w \Vdash B$;

$\mathcal{M}, w \Vdash \Box A \quad$ *iff* $\quad \mathcal{M}, v \Vdash A$ *for all* $v \in W$ *such that* $v \sim w$.

*We say that $A$ is* valid in $\mathcal{M}$, *written* $\mathcal{M} \Vdash A$, *if* $\mathcal{M}, w \Vdash A$ *for all* $w \in W$.

**Theorem 2.3** (Completeness of Kripke semantics for S5, [Fagin et al., 1995]). *For any modal formula $A$,*

$$\mathsf{S5} \vdash A \qquad \textit{iff} \qquad A \textit{ is valid in all Kripke models for } \mathsf{S5}.$$

# 3 Hypersequents for S5

**Definition 3.1** (Sequents and hypersequents). *A* sequent *is a figure $\Gamma \Rightarrow \Delta$ where $\Gamma$ and $\Delta$ are finite (possibly empty) multisets of modal formulas. A* hypersequent $\mathcal{G}$ *is a finite (possibly empty) sequence*

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

*of sequents $\Gamma_i \Rightarrow \Delta_i$, called (sequent)* components *of $\mathcal{G}$. We define the length $\|\mathcal{G}\|$ of a hypersequent $\mathcal{G}$ to be the number of sequent components it contains.*

We denote finite multisets of formulas by $\Gamma, \Delta, \Pi, \Sigma$, etc. and denote hypersequents by $\mathcal{G}, \mathcal{H}$, etc. For a multiset of formulas $\Gamma$, we define $\Box\Gamma := \{\Box C \mid C \in \Gamma\}$.

**Definition 3.2** (Hypersequent system for S5). *The hypersequent system* HS5 *for the logic* S5 *is presented in Figure 1. It is essentially the one presented in [Avron, 1996]. The modifications are slight and clearly do not affect the cut-free completeness of the system. We list the differences between our presentation compared to that by Avron [1996]:*

- *sequent components consist of pairs of multisets of formulas rather than pairs of sequences of formulas (a hypersequent, however, remains a sequence of components);*
- *Boolean connectives are restricted to $\bot$ and $\to$;*
- *in [Avron, 1996] rules can be applied to any component of a hypersequent, whereas we restrict the applications to the last component of the hypersequent, except for the external exchange rule;*
- *it is not specified what type of propositional rules, additive or multiplicative, is used in [Avron, 1996].*

To state the completeness of the hypersequent calculus HS5 with respect to the logic S5, we use a translation from hypersequents to formulas:

$$\mathsf{id}\ \frac{}{A \Rightarrow A} \qquad \mathsf{id}_\bot\ \frac{}{\bot \Rightarrow}$$

$$\to\Rightarrow\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta} \qquad \Rightarrow\to\ \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B}$$

$$\mathsf{W}\Rightarrow\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \qquad \Rightarrow\mathsf{W}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \qquad \mathsf{EW}\ \frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

$$\mathsf{C}\Rightarrow\ \frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \qquad \Rightarrow\mathsf{C}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \qquad \mathsf{EC}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

$$\mathsf{Ex}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Lambda \Rightarrow \Theta \mid \mathcal{H}}{\mathcal{G} \mid \Lambda \Rightarrow \Theta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}$$

$$\Box\Rightarrow\ \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta} \qquad \Rightarrow\Box\ \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow A}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box A} \qquad \mathsf{MS}\ \frac{\mathcal{G} \mid \Box\Lambda, \Gamma \Rightarrow \Box\Phi, \Delta}{\mathcal{G} \mid \Box\Lambda \Rightarrow \Box\Phi \mid \Gamma \Rightarrow \Delta}$$

**Fig. 1.** Cut-free hypersequent system HS5 for the modal logic S5 (following [Avron, 1996]).

**Definition 3.3** (Formula translation). *The formula translation of a hypersequent is defined as*

$$\underline{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n} := \bigvee_{i=1}^{n} \Box\big(\bigwedge \Gamma_i \to \bigvee \Delta_i\big).$$

*As usual, the empty disjunction is understood to be $\bot$ and the empty conjunction is $\top$. (Strictly speaking, the order of formulas in $\bigwedge \Gamma_i$ and $\bigvee \Delta_i$ has to be specified even though all possible end results are pairwise provably equivalent in* S5. *To be formally correct, we assume an arbitrary but fixed order on modal formulas and use it in the conjunctions/disjunctions of multisets $\Gamma_i$ and $\Delta_i$.)*

**Theorem 3.4** (Completeness of HS5, [Avron, 1996]). *For any hypersequent* $\mathcal{G}$,

$$\mathsf{HS5} \vdash \mathcal{G} \qquad \Longleftrightarrow \qquad \mathsf{S5} \vdash \underline{\mathcal{G}}.$$

**Corollary 3.5.** *For arbitrary modal formulas A and B,*

$$\mathsf{HS5} \vdash A \Rightarrow B \qquad \Longleftrightarrow \qquad \mathsf{S5} \vdash A \to B.$$

# 4 Preparing for Interpolation

Interpolation is always performed between two entities, e.g., between formulas $A$ and $B$. The last corollary of the preceding section shows that we can equivalently interpolate between the antecedent and consequent of a single-component hypersequent. However, as in the case of two-sided sequents, such a division does not remain stable along a hypersequent derivation because the $\to$-introducing rules move formulas from one side of $\Rightarrow$ to the other one, which affects the set of variables common between the antecedent(s) and the consequent(s) in unpredictable ways. Instead, we must supply a hypersequent with an extra layer of structure, splitting all formulas, antecedent and consequent alike, into *left* formulas, i.e., eventually contributing to $A$, and *right* formulas, eventually contributing to $B$ in the endsequent $A \Rightarrow B$:

**Definition 4.1** (Split hypersequents). *A* split hypersequent $\widetilde{\mathcal{G}}$ *is a hypersequent where each antecedent and each consequent is partitioned into two multisets by a semicolon:*

$$\Gamma_1; \Pi_1 \Rightarrow \Delta_1; \Sigma_1 \mid \dots \mid \Gamma_n; \Pi_n \Rightarrow \Delta_n; \Sigma_n$$

*(the semicolon can be omitted if an antecedent or a consequent is empty, i.e., we write $\Gamma; \Pi \Rightarrow$ instead of $\Gamma; \Pi \Rightarrow\ ;\ $). For the split hypersequent $\widetilde{\mathcal{G}}$ above, its left (right) sides are obtained by dropping all right (left) formulas:*

$$L\widetilde{\mathcal{G}} := \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n,$$
$$R\widetilde{\mathcal{G}} := \Pi_1 \Rightarrow \Sigma_1 \mid \dots \mid \Pi_n \Rightarrow \Sigma_n.$$

*As before, the length of a split hypersequent is the number of its components, i.e., for the split hypersequent above $\|\widetilde{\mathcal{G}}\| := n$. It is obvious that $\|\widetilde{\mathcal{G}}\| = \|L\widetilde{\mathcal{G}}\| = \|R\widetilde{\mathcal{G}}\|$.*

Given that we plan to find interpolants between the left and right formulas of a given split hypersequent, we need to split all the rules of the calculus HS5.

**Definition 4.2** (Split hypersequent calculus SHS5). *The calculus* SHS5 *is presented in Figures 2 and 3.*

$$\mathsf{EW}\ \frac{\widetilde{\mathcal{G}}}{\widetilde{\mathcal{G}} \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma} \qquad \mathsf{EC}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma}{\widetilde{\mathcal{G}} \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma}$$

$$\mathsf{Ex}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma \mid \Lambda;\Phi \Rightarrow \Theta;\Psi \mid \widetilde{\mathcal{H}}}{\widetilde{\mathcal{G}} \mid \Lambda;\Phi \Rightarrow \Theta;\Psi \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma \mid \widetilde{\mathcal{H}}}$$

$$\Box^{\mathsf{l}} \Rightarrow\ \frac{\widetilde{\mathcal{G}} \mid \Gamma, A;\Pi \Rightarrow \Delta;\Sigma}{\widetilde{\mathcal{G}} \mid \Gamma, \Box A;\Pi \Rightarrow \Delta;\Sigma} \qquad \Rightarrow \Box^{\mathsf{l}}\ \frac{\widetilde{\mathcal{G}} \mid \Box\Gamma;\Box\Pi \Rightarrow A;}{\widetilde{\mathcal{G}} \mid \Box\Gamma;\Box\Pi \Rightarrow \Box A;}$$

$$\Box^{\mathsf{r}} \Rightarrow\ \frac{\widetilde{\mathcal{G}} \mid \Gamma;\Pi, A \Rightarrow \Delta;\Sigma}{\widetilde{\mathcal{G}} \mid \Gamma;\Pi, \Box A \Rightarrow \Delta;\Sigma} \qquad \Rightarrow \Box^{\mathsf{r}}\ \frac{\widetilde{\mathcal{G}} \mid \Box\Gamma;\Box\Pi \Rightarrow ;A}{\widetilde{\mathcal{G}} \mid \Box\Gamma;\Box\Pi \Rightarrow ;\Box A}$$

$$\mathsf{MS}\ \frac{\widetilde{\mathcal{G}} \mid \Box\Lambda, \Gamma;\Box\Theta, \Pi \Rightarrow \Box\Phi, \Delta;\Box\Psi, \Sigma}{\widetilde{\mathcal{G}} \mid \Box\Lambda;\Box\Theta \Rightarrow \Box\Phi;\Box\Psi \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma}$$

**Fig. 2.** Cut-free split hypersequent system SHS5 for the modal logic S5: modal and external structural rules.

It is easy to see that the split system is nothing but the split of the original system:

**Theorem 4.3** (Equivalence of HS5 and SHS5). *For arbitrary finite multisets $\Gamma$, $\Pi$, $\Delta$, and $\Sigma$ of modal formulas,*

$$\mathsf{SHS5} \vdash \Gamma;\Pi \Rightarrow \Delta;\Sigma \qquad \Longleftrightarrow \qquad \mathsf{HS5} \vdash \Gamma \cup \Pi \Rightarrow \Delta \cup \Sigma.$$

**Proof:** Both directions are proved by induction on the depth of the derivation. The crucial observation is that each split rule of SHS5 becomes an ordinary rule of HS5 if one takes the union of left and right formulas separately in each antecedent and each consequent. Vice versa, for each split of the conclusion of any rule of HS5, there is a split of the premise(s) that turns this rule into a rule of SHS5. □

**Corollary 4.4.** *For arbitrary modal formulas $A$ and $B$,*

$$\mathsf{SHS5} \vdash A;\ \Rightarrow ;B \qquad \Longleftrightarrow \qquad \mathsf{S5} \vdash A \to B.$$

$$\mathsf{id}^{\mathsf{ll}}\,\frac{}{A;\ \Rightarrow A;}\qquad \mathsf{id}^{\mathsf{rl}}\,\frac{}{;A \Rightarrow A;}\qquad \mathsf{id}^{\mathsf{lr}}\,\frac{}{A;\ \Rightarrow\ ;A}\qquad \mathsf{id}^{\mathsf{rr}}\,\frac{}{;A \Rightarrow\ ;A}$$

$$\mathsf{id}^{\mathsf{l}}_{\bot}\,\frac{}{\bot;\ \Rightarrow}\qquad \mathsf{id}^{\mathsf{r}}_{\bot}\,\frac{}{;\bot \Rightarrow}$$

$$\Rightarrow\to^{\mathsf{l}}\,\frac{\widetilde{\mathcal{G}} \mid \Gamma, A; \Pi \Rightarrow \Delta, B; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta, A \to B; \Sigma}\qquad \Rightarrow\to^{\mathsf{r}}\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A \Rightarrow \Delta; \Sigma, B}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma, A \to B}$$

$$\to^{\mathsf{l}}\Rightarrow\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta, A; \Sigma \quad \widetilde{\mathcal{G}} \mid \Gamma, B; \Pi \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma, A \to B; \Pi \Rightarrow \Delta; \Sigma}$$

$$\to^{\mathsf{r}}\Rightarrow\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma, A \quad \widetilde{\mathcal{G}} \mid \Gamma; \Pi, B \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A \to B \Rightarrow \Delta; \Sigma}$$

$$\mathsf{W}^{\mathsf{l}}\Rightarrow\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma, A; \Pi \Rightarrow \Delta; \Sigma}\qquad \Rightarrow\mathsf{W}^{\mathsf{l}}\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta, A; \Sigma}$$

$$\mathsf{W}^{\mathsf{r}}\Rightarrow\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A \Rightarrow \Delta; \Sigma}\qquad \Rightarrow\mathsf{W}^{\mathsf{r}}\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma, A}$$

$$\mathsf{C}^{\mathsf{l}}\Rightarrow\,\frac{\widetilde{\mathcal{G}} \mid \Gamma, A, A; \Pi \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma, A; \Pi \Rightarrow \Delta; \Sigma}\qquad \Rightarrow\mathsf{C}^{\mathsf{l}}\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta, A, A; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta, A; \Sigma}$$

$$\mathsf{C}^{\mathsf{r}}\Rightarrow\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A, A \Rightarrow \Delta; \Sigma}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A \Rightarrow \Delta; \Sigma}\qquad \Rightarrow\mathsf{C}^{\mathsf{r}}\,\frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma, A, A}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma, A}$$

**Fig. 3.** Cut-free split hypersequent system SHS5 for the modal logic S5: propositional and internal structural rules

**Proof:** The statement follows from Corollary 3.5 and Theorem 4.3. □

**Remark 4.5.** *The same reasons that compelled us to split each component of a hypersequent explain why the cut rule cannot be processed by a constructive proof of the CIP based on the induction on a sequent-like derivation. The cut rule violates the subformula property by removing a formula completely from each of the two premises. This removal can shrink the set of common variables in unpredictable ways, thus, rendering the interpolants from the induction hypothesis suddenly unusable.*

Splitting a hypersequent is no different from splitting a sequent and is standard for constructive proofs of the CIP. Thus, by itself, it does not yet help to extend the method to hypersequents. The crucial idea behind our method of proving the CIP is that interpolation should be done on the component level, i.e., instead of having a formula interpolant for the whole hypersequent, responsible, in particular, for encoding the hypersequent structure and its transformations, we allow each sequent component to have its own formula interpolant and combine these componentwise interpolants by explicit operations creating structures that parallel the structure of the hypersequent being interpolated. This view of interpolation signifies a departure from the very definition of interpolation. Thus, we must both present the intuition behind our view and demonstrate that the two definitions coincide for the final result of our interpolation procedure (but not during the intermediate stages, where our interpolation statements cannot be translated to the standard ones). To this end, we present an alternative semantics for hypersequents that provides the intuition for the structure of our interpolants.

**Definition 4.6** (Connected worlds). *Let $\mathcal{M} = (W, \sim, V)$ be a Kripke model. A sequence of worlds $\vec{w} = w_1, \dots, w_n$ from $W$ is called* $\mathcal{M}$-connected *if $w_1 \sim w_i$ for each $2 \leq i \leq n$. It immediately follows that $w_i \sim w_j$ for all $1 \leq i \leq j \leq n$.*

**Definition 4.7** (Componentwise semantics). *Let $\mathcal{M} = (W, \sim, V)$ be a Kripke model. A sequent $\Gamma \Rightarrow \Delta$ holding at a world $w \in W$ of $\mathcal{M}$ is defined as follows:*

$$\mathcal{M}, w \vDash \Gamma \Rightarrow \Delta \quad\Longleftrightarrow\quad \begin{array}{l}\mathcal{M}, w \nVdash A \text{ for some } A \in \Gamma \quad \text{or}\\ \mathcal{M}, w \Vdash B \text{ for some } B \in \Delta.\end{array}$$

*Let $\mathcal{G}$ be a hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ and $\vec{w} = w_1, \dots, w_n$ be a sequence of length $\|\mathcal{G}\|$ of worlds from $W$. The hypersequent $\mathcal{G}$ holding on the sequence $\vec{w}$ in $\mathcal{M}$ is defined as follows:*

$$\mathcal{M}, \vec{w} \vDash \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n \quad\Longleftrightarrow\quad \mathcal{M}, w_i \vDash \Gamma_i \Rightarrow \Delta_i \text{ for some } 1 \leq i \leq n.$$

*(In particular, we define $\mathcal{M}, \varepsilon \nvDash \varepsilon$, i.e., the empty hypersequent does not hold on the empty sequence of worlds.)*

**Definition 4.8** (Componentwise validity). *A hypersequent $\mathcal{G}$ is* componentwise valid *if $\mathcal{M}, \vec{w} \vDash \mathcal{G}$ for any Kripke model $\mathcal{M}$ and any $\mathcal{M}$-connected sequence $\vec{w}$ of length $\|\mathcal{G}\|$.*

**Lemma 4.9** (Equivalence of two semantics). *A hypersequent $\mathcal{G}$ is componentwise valid iff its formula interpretation is valid in all Kripke models.*

**Proof:** Let $\mathcal{G}$ be a hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. We prove that it is componentwise invalid iff its formula interpretation is invalid. Our first observation is that for any component $\Gamma \Rightarrow \Delta$, any Kripke model $\mathcal{M} = (W, \sim, V)$, and any world $w \in W$, we have

$$\mathcal{M}, w \nvDash \Gamma \Rightarrow \Delta \qquad \Longleftrightarrow \qquad \mathcal{M}, w \nVdash \bigwedge \Gamma \to \bigvee \Delta. \tag{1}$$

In the following sequence of statements, each statement is equivalent to the previous one.

1. $\underline{\mathcal{G}}$ is invalid;
2. there is a Kripke model $\mathcal{M} = (W, \sim, V)$ and there is a world $v \in W$ such that $\mathcal{M}, v \nVdash \bigvee_{i=1}^{n} \Box(\bigwedge \Gamma_i \to \bigvee \Delta_i)$; (by definitions of $\underline{\mathcal{G}}$ and of validity for formulas)
3. there exists a Kripke model $\mathcal{M} = (W, \sim, V)$, a world $v \in W$, and worlds $w_1, \ldots, w_n \in W$ such that $v \sim w_i$ and $\mathcal{M}, w_i \nVdash \bigwedge \Gamma_i \to \bigvee \Delta_i$ for each $i = 1, \ldots, n$; (by definition of $\Vdash$)
4. there exists a Kripke model $\mathcal{M} = (W, \sim, V)$ and an $\mathcal{M}$-connected sequence $\vec{w} = w_1, \ldots, w_n$ such that $\mathcal{M}, w_i \nVdash \bigwedge \Gamma_i \to \bigvee \Delta_i$ for each $i = 1, \ldots, n$; ($v$ can be set to $w_1$ because $\sim$ is an equivalence relation)
5. there exists a Kripke model $\mathcal{M} = (W, \sim, V)$ and an $\mathcal{M}$-connected sequence $\vec{w} = w_1, \ldots, w_n$ such that $\mathcal{M}, w_i \nvDash \Gamma_i \Rightarrow \Delta_i$ for each $i = 1, \ldots, n$; (by (1))
6. there exists a Kripke model $\mathcal{M} = (W, \sim, V)$ and an $\mathcal{M}$-connected sequence $\vec{w}$ of length $n$ such that $\mathcal{M}, \vec{w} \nvDash \mathcal{G}$; (by definition of $\vDash$)
7. $\mathcal{G}$ is componentwise invalid. (by definition of componentwise validity)

□

Having transferred validity to the level of sequent components, we now define the objects that will serve as interpolants.

**Definition 4.10** (Hyperformulas and g-hyperformulas). *An* hyperformula *is a sequence of modal formulas, written in the hypersequent notation: $C_1 \mid \ldots \mid C_n$. By analogy with sequence of worlds, we use vector notation $\vec{C}$ for hyperformulas and define the length of a hyperformula to be the number of formulas in it, i.e., $\|C_1 \mid \ldots \mid C_n\| := n$.*

*Each hyperformula is a* generalized hyperformula *of the same length. If* $\mho_1$ *and* $\mho_2$ *are generalized hyperformulas of length* $n$ *each, then* $(\mho_1 \circledvee \mho_2)$ *and* $(\mho_1 \circledwedge \mho_2)$ *are also generalized hyperformulas of length* $n$*. For brevity's sake, we sometimes call generalized hyperformulas simply* g-hyperformulas.

**Definition 4.11** (Componentwise semantics for (g-)hyperformulas). *Let a triple* $\mathcal{M} = (W, \sim, V)$ *be a Kripke model. Let* $\vec{C} = C_1 \mid \ldots \mid C_n$ *be a hyperformula and* $\vec{w} = w_1, \ldots, w_n$ *be a sequence of length* $\|\vec{C}\|$ *of worlds from* $W$*. We define*

$$\mathcal{M}, \vec{w} \vDash C_1 \mid \ldots \mid C_n \qquad \textit{iff} \qquad \mathcal{M}, w_i \vDash C_i \textit{ for some } 1 \leq i \leq n.$$

*For arbitrary g-hyperformulas* $\mho_1$ *and* $\mho_2$ *and arbitrary sequence* $\vec{w}$ *of worlds from* $W$ *such that* $\|\mho_1\| = \|\mho_2\| = \|\vec{w}\|$*, we define*

$$\mathcal{M}, \vec{w} \vDash \mho_1 \circledwedge \mho_2 \quad \textit{iff} \quad \mathcal{M}, \vec{w} \vDash \mho_1 \textit{ and } \mathcal{M}, \vec{w} \vDash \mho_2;$$
$$\mathcal{M}, \vec{w} \vDash \mho_1 \circledvee \mho_2 \quad \textit{iff} \quad \mathcal{M}, \vec{w} \vDash \mho_1 \textit{ or } \mathcal{M}, \vec{w} \vDash \mho_2.$$

*Thus, with respect to the componentwise semantics,* $\circledwedge$ *and* $\circledvee$ *on hyperformulas correspond to* $\wedge$ *and* $\vee$ *on formulas respectively.*

*Two g-hyperformulas* $\mho_1$ *and* $\mho_2$ *are called* componentwise equivalent, *written* $\mho_1 \dashv\vDash \mho_2$*, iff* $\|\mho_1\| = \|\mho_2\|$ *and, for any Kripke model* $\mathcal{M}$ *and any* $\mathcal{M}$*-connected sequence* $\vec{w}$ *of length* $\|\mho_1\|$*, we have*

$$\mathcal{M}, \vec{w} \vDash \mho_1 \qquad \Longleftrightarrow \qquad \mathcal{M}, \vec{w} \vDash \mho_2.$$

**Definition 4.12** (Componentwise interpolant). *A* componentwise interpolant *of a split hypersequent* $\widetilde{\mathcal{G}}$ *is a g-hyperformula* $\mho$ *such that* $\|\mho\| = \|\widetilde{\mathcal{G}}\|$ *and for each Kripke model* $\mathcal{M}$ *and each* $\mathcal{M}$*-connected sequence* $\vec{w}$ *with* $\|\vec{w}\| = \|\mho\|$*,*

- *if* $\mathcal{M}, \vec{w} \nvDash \mho$*, then* $\mathcal{M}, \vec{w} \vDash L\widetilde{\mathcal{G}}$*;*
- *if* $\mathcal{M}, \vec{w} \vDash \mho$*, then* $\mathcal{M}, \vec{w} \vDash R\widetilde{\mathcal{G}}$*;*
- $\mho$ *contains only propositional variables common to* $L\widetilde{\mathcal{G}}$ *and* $R\widetilde{\mathcal{G}}$*.*

*If* $\mho$ *is a componentwise interpolant of* $\widetilde{\mathcal{G}}$*, we write* $\widetilde{\mathcal{G}} \longleftarrow \mho$*.*

**Lemma 4.13** (Translation from componentwise to formula interpolants). *If a g-hyperformula* $C^1 \circledwedge \cdots \circledwedge C^m$ *of length 1 is a componentwise interpolant of a split hypersequent* $A; \ \Rightarrow \ ; B$*, then the formula* $C^1 \wedge \cdots \wedge C^m$ *is an interpolant of the formulas* $A$ *and* $B$*.*

**Proof:** Indeed, $L(A; \ \Rightarrow \ ; B)$ is $A \Rightarrow$ and $R(A; \ \Rightarrow \ ; B)$ is $\Rightarrow B$. Thus, the formulas $C^j$ contain only propositional variables common to $A$ and $B$.

Let $\mathcal{M} = (W, \sim, V)$ be a Kripke model and $w \in W$ be a world in it. We have

$$\mathcal{M}, w \nVdash \bigwedge_{i=1}^{m} C^i \quad\Longrightarrow\quad \mathcal{M}, w \nvDash \bigotimes_{i=1}^{m} C^i \quad\Longrightarrow\quad \mathcal{M}, w \vDash A \Rightarrow \quad\Longrightarrow\quad \mathcal{M}, w \nVdash A,$$

$$\mathcal{M}, w \Vdash \bigwedge_{i=1}^{m} C^i \quad\Longrightarrow\quad \mathcal{M}, w \vDash \bigotimes_{i=1}^{m} C^i \quad\Longrightarrow\quad \mathcal{M}, w \vDash\ \Rightarrow B \quad\Longrightarrow\quad \mathcal{M}, w \Vdash B.$$

Hence, by the completeness of S5, we have

$$\mathsf{S5} \vdash A \to C^1 \wedge \cdots \wedge C^m \qquad \text{and} \qquad \mathsf{S5} \vdash C^1 \wedge \cdots \wedge C^m \to B. \qquad \square$$

**Lemma 4.14** (External disjunction elimination).

1. *For arbitrary modal formulas* $A_1, \ldots, A_n, B_1, \ldots, B_n$,

$$(A_1 \mid \ldots \mid A_n) \oslash (B_1 \mid \ldots \mid B_n) \dashv\vDash A_1 \vee B_1 \mid \ldots \mid A_n \vee B_n.$$

2. *Any g-hyperformula can be transformed to a componentwise equivalent external conjunction of hyperformulas without changing the set of propositional variables occurring in it.*

In principle, any g-hyperformula of length 1 can be translated into a formula in this way, but translating $\otimes$ is sufficient because with respect to the componentwise semantics the external conjunction and disjunction $\otimes$ and $\oslash$ behave like $\wedge$ and $\vee$ with respect to Boolean semantics. In particular, any g-hyperformula can be transformed to a componentwise equivalent DNF or CNF using the standard algorithm. The simple proof of the following lemma is left to the reader.

**Lemma 4.15** (External disjunction elimination).

1. *For arbitrary modal formulas* $A_1, \ldots, A_n, B_1, \ldots, B_n$,

$$(A_1 \mid \ldots \mid A_n) \oslash (B_1 \mid \ldots \mid B_n) \dashv\vDash A_1 \vee B_1 \mid \ldots \mid A_n \vee B_n.$$

2. *Any g-hyperformula can be transformed to a componentwise equivalent external conjunction of hyperformulas without changing the set of propositional variables occurring in it.*

# 5 The Interpolation Algorithm

We divide our description of the algorithm for constructing interpolants into three parts. First, we present the propositional and structural rules in Figures 4 and 5 respectively and prove correctness for the most interesting cases. The modal rules (presented later in Figure 6) require additional auxiliary lemmas.

$$\mathsf{id}^{\mathsf{ll}}\ \frac{}{A;\ \Rightarrow A;\ \longleftarrow\ \bot} \qquad \mathsf{id}^{\mathsf{rl}}\ \frac{}{;A \Rightarrow A;\ \longleftarrow\ \neg A}$$

$$\mathsf{id}^{\mathsf{lr}}\ \frac{}{A;\ \Rightarrow ;A\ \longleftarrow\ A} \qquad \mathsf{id}^{\mathsf{rr}}\ \frac{}{;A \Rightarrow ;A\ \longleftarrow\ \top}$$

$$\mathsf{id}^{\mathsf{l}}_{\bot}\ \frac{}{\bot;\ \Rightarrow\ \ \longleftarrow\ \bot} \qquad \mathsf{id}^{\mathsf{r}}_{\bot}\ \frac{}{;\bot \Rightarrow\ \ \longleftarrow\ \top}$$

$$\to^{\mathsf{l}}\Rightarrow\ \frac{\widetilde{\mathcal{G}} \mid \Gamma;\Pi \Rightarrow \Delta, A;\Sigma\ \longleftarrow\ \mho_1 \quad \widetilde{\mathcal{G}} \mid \Gamma, B;\Pi \Rightarrow \Delta;\Sigma\ \longleftarrow\ \mho_2}{\widetilde{\mathcal{G}} \mid \Gamma, A \to B;\Pi \Rightarrow \Delta;\Sigma\ \longleftarrow\ \mho_1 \circledvee \mho_2}$$

$$\to^{\mathsf{r}}\Rightarrow\ \frac{\widetilde{\mathcal{G}} \mid \Gamma;\Pi \Rightarrow \Delta;\Sigma, A\ \longleftarrow\ \mho_1 \quad \widetilde{\mathcal{G}} \mid \Gamma;\Pi, B \Rightarrow \Delta;\Sigma\ \longleftarrow\ \mho_2}{\widetilde{\mathcal{G}} \mid \Gamma;\Pi, A \to B \Rightarrow \Delta;\Sigma\ \longleftarrow\ \mho_1 \circledwedge \mho_2}$$

For rules $\to^{\mathsf{l}}\Rightarrow$ and $\to^{\mathsf{r}}\Rightarrow$, it is required that $\|\mho_1\| = \|\mho_2\| = \|\widetilde{\mathcal{G}}\| + 1$.

**Fig. 4.** Interpolation algorithm: Propositional rules. For the propositional rules not depicted above, the given interpolant for the premise is to be used as an interpolant for the conclusion.

**Remark 5.1.** *Note that the empty (split) hypersequent can never occur in a derivation in* HS5 *(or in* SHS5*), thus, we need not care how to properly define transformations to CNF and DNF for g-hyperformulas of length* 0*.*

**Lemma 5.2** (Algorithm correctness I: propositional and structural rules). *All the rules depicted in Figure 4 produce a componentwise interpolant for the conclusion of the rule whenever componentwise interpolants (transformed to a proper form) are given for all the premises. Further, any componentwise interpolant of the premise of a propositional or structural rule not depicted in Figure 4 is also a componentwise interpolant for the conclusion of the same rule.*

**Proof:** We consider several representative cases, leaving the rest to the reader. Throughout the proof we assume $\mathcal{M} = (W, \sim, V)$ to be an arbitrary Kripke model, $w \in W$ to be an arbitrary world from it, and $\vec{w}$ to be an arbitrary $\mathcal{M}$-connected sequence of worlds of appropriate length. We also omit the model from the $\Vdash$ statements about formulas and from $\vDash$ statements about hyperformulas and hypersequents.

**Rule** $\mathsf{id}^{\mathsf{rl}}$ $\dfrac{}{\;;A \Rightarrow A;\; \longleftarrow \neg A}$. It is clear that all propositional variables in $\neg A$ are common between $\;\Rightarrow\; A$ and $A \;\Rightarrow\;$. We need to consider arbitrary $\mathcal{M}$-connected sequences of length 1, i.e., arbitrary worlds. We have

$$\begin{array}{lllllll} w \nvDash \neg A & \Longrightarrow & w \nVdash \neg A & \Longrightarrow & w \vDash \;\Rightarrow A & \Longrightarrow & w \vDash L(\;;A \Rightarrow A;\;), \\ w \vDash \neg A & \Longrightarrow & w \Vdash \neg A & \Longrightarrow & w \vDash A \Rightarrow\; & \Longrightarrow & w \vDash R(\;;A \Rightarrow A;\;). \end{array}$$

**Rules not depicted in Figure 4.** They all work in the same way. It is sufficient to note that, for all of them, if one side, left or right, of the premise hypersequent holds on $\vec{w}$ then the same side of the conclusion hypersequent also holds on $\vec{w}$ (and, of course, to verify the common-variable condition).

**Rule** $\rightarrow^{\mathsf{l}} \Rightarrow$ $\dfrac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta, A; \Sigma \longleftarrow \mho_1 \quad \widetilde{\mathcal{G}} \mid \Gamma, B; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \mho_2}{\widetilde{\mathcal{G}} \mid \Gamma, A \rightarrow B; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \mho_1 \oslash \mho_2}$.

Assume that $\mho_1$ and $\mho_2$ are interpolants of the premises, in particular, $\|\mho_1\| = \|\mho_2\| = \|\widetilde{\mathcal{G}}\| + 1$, making $\mho_1 \oslash \mho_2$ well defined. It is easy to see that the common-variable condition for the conclusion is fulfilled. Let $\vec{w} = \vec{v}, u$. For the left side,

$$\begin{array}{c} \vec{w} \nvDash \mho_1 \oslash \mho_2 \quad \Longrightarrow \quad \vec{w} \nvDash \mho_1 \;\text{ and }\; \vec{w} \nvDash \mho_2 \quad \Longrightarrow \\ \vec{w} \vDash L\widetilde{\mathcal{G}} \mid \Gamma \Rightarrow \Delta, A \;\text{ and }\; \vec{w} \vDash L\widetilde{\mathcal{G}} \mid \Gamma, B \Rightarrow \Delta \quad \Longrightarrow \\ \left(\vec{v} \vDash L\widetilde{\mathcal{G}} \;\text{ or }\; u \vDash \Gamma \Rightarrow \Delta \;\text{ or }\; u \Vdash A\right) \text{ and} \\ \left(\vec{v} \vDash L\widetilde{\mathcal{G}} \;\text{ or }\; u \vDash \Gamma \Rightarrow \Delta \;\text{ or }\; u \nVdash B\right) \quad \Longrightarrow \\ \left(\vec{v} \vDash L\widetilde{\mathcal{G}} \;\text{ or }\; u \vDash \Gamma \Rightarrow \Delta \;\text{ or }\; u \nVdash A \rightarrow B\right) \quad \Longrightarrow \\ \vec{w} \vDash L\widetilde{\mathcal{G}} \mid \Gamma, A \rightarrow B \Rightarrow \Delta. \end{array}$$

The argument for the right side is even simpler. If $\vec{w} \vDash \mho_1 \oslash \mho_2$, then at least one of $\mho_1$ and $\mho_2$ holds on $\vec{w}$ making the right side of the corresponding premise hypersequent true on $\vec{w}$. But the right side of the conclusion hypersequent is the same as that of both premise hypersequents.

**Rule** EC $\dfrac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid A^j \mid B^j)}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid A^j \vee B^j)}$.

Recall that it was this rule that was identified as the main obstacle to using hypersequents for proving the CIP in [Avron, 1996]. Once again, the common-variable condition presents no difficulties. Let $\vec{w} = \vec{v}, u$ be a sequence of length $\|\widetilde{G}\| + 1$. The crucial observations for this case are that

$$\vec{v}, u \vDash \vec{C}^j \mid A^j \vee B^j \qquad \Longleftrightarrow \qquad \vec{v}, u, u \vDash \vec{C}^j \mid A^j \mid B^j$$

for each $j = 1, \ldots, m$ and that

$$\vec{v}, u, u \vDash S(\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma) \quad \Longrightarrow \quad \vec{v}, u \vDash S(\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma)$$

for $S$ being either $L$ or $R$. □

---

$$\mathsf{EW}\ \frac{\widetilde{\mathcal{G}} \longleftarrow \bigotimes_{j=1}^{m} \vec{C}^j}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid \bot)}$$

$$\mathsf{EC}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid A^j \mid B^j)}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid A^j \vee B^j)}$$

$$\mathsf{Ex}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \Lambda; \Phi \Rightarrow \Theta; \Psi \mid \widetilde{\mathcal{H}} \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid A^j \mid B^j \mid \vec{D}^j)}{\widetilde{\mathcal{G}} \mid \Lambda; \Phi \Rightarrow \Theta; \Psi \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \widetilde{\mathcal{H}} \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid B^j \mid A^j \mid \vec{D}^j)}$$

In order to apply the rules EW, EC, and Ex, the interpolant of the premise must be first transformed to an external conjunction of hyperformulas by Lemma 4.15. In addition, it is required that

$$\|\widetilde{G}\| = \|\vec{C}^j\| \text{ and } \|\widetilde{\mathcal{H}}\| = \|\vec{D}^j\| \text{ for each } j = 1, \ldots, m.$$

---

**Fig. 5.** Interpolation algorithm: Structural rules. For the structural rules not depicted above, the given interpolant for the only premise is to be used as an interpolant for the conclusion.

Restricting some interpolants to be external conjunctions of hyperformulas in Figure 4 is for convenience more than out of necessity. The same structural transformations applied to arbitrary g-hyperformulas would have worked equally well, but would have been more cumbersome to describe. The modal rules, in contrast, require logical transformations that affect the structure of the interpolant, making it necessary to impose an even more restrictive interpolant format to process the $\Rightarrow \Box^{\mathrm{l}}$ rule. Our immediate goal is to show that this special format can always be achieved. We formulate this as a lemma but leave its simple proof to the reader.

**Lemma 5.3** (Separation of a g-hyperformula component). *Let* $\bot^n$ *be an abbreviation for* $\underbrace{\bot \mid \ldots \mid \bot}_{n}$.

1. *For any hyperformula* $\vec{C}$ *of length n and any formula A,*

$$\vec{C} \mid A \quad \dashv\vDash \quad (\bot^n \mid A) \otimes (\vec{C} \mid \bot).$$

2. *For arbitrary formulas* $A^1, \ldots, A^m$ *with* $m \geq 2$ *and any* $n \geq 0$,

$$\bigotimes_{j=1}^{m}(\bot^n \mid A^j) \quad \dashv\vDash \quad \bot^n \mid \bigwedge_{j=1}^{m} A^j.$$

3. *For any hyperformula* $\vec{C}$ *of length n,*

$$\vec{C} \mid \bot \quad \dashv\vDash \quad (\bot^n \mid \top) \otimes (\vec{C} \mid \bot).$$

4. *For any g-hyperformula* $\mho$ *of length* $n+1$, *there is a componentwise equivalent g-hyperformula of the form*

$$\bigovee_{j=1}^{m}\left((\bot^n \mid B^j) \otimes \bigotimes_{k_j=1}^{l_j}(\vec{C}^{j,k_j} \mid \bot)\right)$$

*with the same set of propositional variables.*

Finally, before getting our hands dirty with the modal rules, we formulate auxiliary statements that will make the main arguments more transparent by separating tedious technical details to stand-alone lemmas.

**Lemma 5.4** (Boxed formulas invariant within a connected component). *Let* $\mathcal{M} = (W, \sim, V)$ *be an arbitrary Kripke model and u and z be worlds from W.*

$$u \sim z \qquad \Longrightarrow \qquad (u \Vdash \Box G \quad \Longleftrightarrow \quad z \Vdash \Box G). \tag{2}$$

**Proof:** By a standard semantic argument. □

**Lemma 5.5** (Hypersequent necessitation in one component). *Let* $\mathcal{M} = (W, \sim, V)$ *be an arbitrary Kripke model. For any hypersequent* $\mathcal{H}$, *any finite multiset* $\Xi$ *of modal formulas, any modal formula F, any sequence* $\vec{v}$ *of worlds from W, and any world* $u \in W$ *such that* $\|\mathcal{H}\| = \|\vec{v}\|$, *we have*

1. $\mathcal{M}, \vec{v}, u' \vDash \mathcal{H} \mid \Box\Xi \Rightarrow$ *for some* $u' \sim u$ $\Longrightarrow$ $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow;$
2. $\mathcal{M}, \vec{v}, u' \vDash \mathcal{H} \mid \Box\Xi \Rightarrow F$ *for all* $u' \sim u$ $\Longrightarrow$ $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow \Box F.$

**Proof:** We distinguish three possibilities:

1. $\mathcal{M}, \vec{v} \vDash \mathcal{H}$. It follows that $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow$ and $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow \Box F$.
2. $\mathcal{M}, u' \not\Vdash \Box G$ for some $G \in \Xi$ and some $u' \sim u$. By Lemma 5.4, $\mathcal{M}, u \not\Vdash \Box G$. It follows immediately that $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow$ and $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow \Box F$.
3. If $\mathcal{M}, \vec{v} \nvDash \mathcal{H}$ and $\mathcal{M}, u' \Vdash \Box G$ for all $G \in \Xi$ and all $u' \sim u$, then assumption of part 1 of the lemma is not satisfied. However, whenever the assumption of part 2 is satisfied, we have $\mathcal{M}, u' \Vdash F$ for all $u' \sim u$. Thus, $\mathcal{M}, u \Vdash \Box F$ and $\mathcal{M}, \vec{v}, u \vDash \mathcal{H} \mid \Box\Xi \Rightarrow \Box F$. □

**Lemma 5.6** (Algorithm correctness II: modal rules). *All the rules depicted in Figure 6 produce a componentwise interpolant for the conclusion of the rule whenever componentwise interpolants (in a proper form) are given for the premise. Further, any componentwise interpolant of the premise of a modal rule not depicted in Figure 6 is also a componentwise interpolant for the conclusion of the same rule.*

**Proof:** We omit the rule $\Box^{\mathrm{r}} \Rightarrow$ because it is very similar to the case of $\Box^{\mathrm{l}} \Rightarrow$ we demonstrate. Throughout the proof we assume $\mathcal{M} = (W, \sim, V)$ to be an arbitrary Kripke model, $w$ to be an arbitrary world from $W$, and $\vec{w}$ to be an arbitrary connected sequence of worlds from $W$ of appropriate length. We also omit the model from the $\Vdash$ statements about formulas and from $\vDash$ statements about hyperformulas and hypersequents. Finally, we omit the trivial proofs that the common-variables condition is satisfied for all the rules.

**Rule** $\Box^{\mathrm{l}} \Rightarrow$.

We need to show that any interpolant of the premise hypersequent is also an interpolant of the conclusion hypersequent. The only change from the premise to the conclusion is that $A$ is replaced with $\Box A$ in the antecedent. Thus, it is sufficient to observe that, due to the reflexivity of $\sim$,

$$w \not\Vdash A \qquad \Longrightarrow \qquad w \not\Vdash \Box A.$$

**Rule** MS
$$\frac{\widetilde{\mathcal{G}} \mid \Box\Lambda, \Gamma; \Box\Theta, \Pi \Rightarrow \Box\Phi, \Delta; \Box\Psi, \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid A^j)}{\widetilde{\mathcal{G}} \mid \Box\Lambda; \Box\Theta \Rightarrow \Box\Phi; \Box\Psi \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \longleftarrow \bigotimes_{j=1}^{m} (\vec{C}^j \mid \bot \mid A^j)}.$$

The key observations are (2) and that, for any sequence $\vec{v}$ of worlds of length $\|\widetilde{\mathcal{G}}\|$, for arbitrary worlds $u$ and $z$, and for each $j = 1, \ldots, m$,

$$\vec{v}, u \vDash \vec{C}^j \mid A^j \qquad \Longleftrightarrow \qquad \vec{v}, z, u \vDash \vec{C}^j \mid \bot \mid A^j$$

The requirement for the sequence of worlds to be connected is used here for the first time to allow for the application of (2).

$$\textbf{Rule} \Rightarrow \Box^{\mathrm{r}} \quad \frac{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow \,; A \;\longleftarrow\; \bigotimes_{j=1}^{m} (\vec{C}^j \mid B^j)}{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow \,; \Box A \;\longleftarrow\; \bigotimes_{j=1}^{m} (\vec{C}^j \mid \Box B^j)}.$$

Let $\vec{w} = \vec{v}, u$. For the left side, let us assume that $\vec{w} \nvDash \bigotimes_{j=1}^{m} (\vec{C}^j \mid \Box B^j)$. Then there is $1 \leq J \leq m$ such that $\vec{v} \nvDash \vec{C}^J$ and $u \nVdash \Box B^J$. It follows that there exists a world $u' \in W$ such that $u \sim u'$ and $u' \nVdash B^J$. Thus, on the $\mathcal{M}$-connected sequence $\vec{v}, u'$, the given interpolant $\bigotimes_{j=1}^{m} (\vec{C}^j \mid B^j)$ of the premise hypersequent does not hold, making the left side of the premise hypersequent true on the same sequence: $\vec{v}, u' \vDash L\widetilde{\mathcal{G}} \mid \Box\Gamma \Rightarrow$ . By Lemma 5.5.1, we have $\vec{w} \vDash L\widetilde{\mathcal{G}} \mid \Box\Gamma \Rightarrow$ for the left side of the conclusion hypersequent.

For the right side, assume that $\vec{w} \vDash \bigotimes_{j=1}^{m} (\vec{C}^j \mid \Box B^j)$. Then for each $1 \leq j \leq m$ either $\vec{v} \vDash \vec{C}^j$ or $u \Vdash \Box B^j$. Since $u \Vdash \Box B^j$ implies $u' \Vdash B^j$ for all $u' \sim u$, it follows that $\vec{v}, u' \vDash \bigotimes_{j=1}^{m} (\vec{C}^j \mid B^j)$ for all $u' \sim u$, making the right side of the premise hypersequent true on all these sequences, which are $\mathcal{M}$-connected: $\vec{v}, u' \vDash R\widetilde{\mathcal{G}} \mid \Box\Pi \Rightarrow A$ for all $u' \sim u$. By Lemma 5.5.2, $\vec{w} \vDash R\widetilde{\mathcal{G}} \mid \Box\Pi \Rightarrow \Box A$ for the right side of the conclusion hypersequent.

$$\textbf{Rule} \Rightarrow \Box^{\mathrm{l}} \quad \frac{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow A; \;\longleftarrow\; \bigovee_{j=1}^{m} \left( (\bot^n \mid B^j) \oslash \bigotimes_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right)}{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow \Box A; \;\longleftarrow\; \bigovee_{j=1}^{m} \left( (\bot^n \mid \diamond B^j) \oslash \bigotimes_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right)}.$$

Let us consider $\vec{w} = \vec{v}, u$. For the left side, assume that

$$\vec{w} \nvDash \bigovee_{j=1}^{m} \left( (\bot^n \mid \diamond B^j) \oslash \bigotimes_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right).$$

In other words, for each $j = 1, \ldots, m$ there exists $1 \leq K_j \leq l_j$ such that

$$u \nVdash \diamond B^j \qquad \text{or} \qquad \vec{v} \nVdash \vec{C}^{j,K_j}.$$

It follows that for each $u' \sim u$ and for each $j = 1, \dots, m$,

$$u' \nVdash B^j \qquad \text{or} \qquad \vec{v} \nVdash \vec{C}^{j,K_j}.$$

In other words, for each $u' \sim u$,

$$\vec{v}, u' \nvDash \bigovee_{j=1}^{m} \left( (\bot^n \mid B^j) \oslash \bigowedge_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right),$$

making the left side of the premise hypersequent true on all such sequences, which are $\mathcal{M}$-connected: for all $u' \sim u$, we have $\vec{v}, u' \vDash L\widetilde{\mathcal{G}} \mid \Box\Gamma \Rightarrow A$. By Lemma 5.5.2, we have $\vec{w} \vDash L\widetilde{\mathcal{G}} \mid \Box\Gamma \Rightarrow \Box A$, i.e., the left side of the conclusion hypersequent holds on $\vec{w}$.

For the right side, assume that

$$\vec{w} \vDash \bigovee_{j=1}^{m} \left( (\bot^n \mid \diamond B^j) \oslash \bigowedge_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right).$$

In other words, for some $1 \leq J \leq m$,

$$u \Vdash \diamond B^J \qquad \text{and} \qquad \text{for all } 1 \leq k_J \leq l_J \quad \vec{v} \Vdash \vec{C}^{J,k_J}.$$

It follows that, for some $u' \sim u$,

$$u' \Vdash B^J \qquad \text{and} \qquad \text{for all } 1 \leq k_J \leq l_J \quad \vec{v} \Vdash \vec{C}^{J,k_J}.$$

In other words, for some $u' \sim u$,

$$\vec{v}, u' \vDash \bigovee_{j=1}^{m} \left( (\bot^n \mid B^j) \oslash \bigowedge_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right),$$

making the right side of the premise hypersequent true on the sequence $\vec{v}, u'$, which is $\mathcal{M}$-connected: we have

$$\vec{v}, u' \vDash R\widetilde{\mathcal{G}} \mid \Box\Pi \Rightarrow$$

for this $u'$. By Lemma 5.5.1, we have

$$\vec{w} \vDash L\widetilde{\mathcal{G}} \mid \Box\Pi \Rightarrow$$

i.e., the right side of the conclusion hypersequent holds on $\vec{w}$. □

Putting all together, we conclude that

**Theorem 5.7.** S5 *enjoys the CIP.*

**Proof:** Let S5 $\vdash A \to B$. By Corollary 4.4, we have SHS5 $\vdash A; \Rightarrow ; B$. By Lemmas 5.2 and 5.6, we can construct a componentwise interpolant $\mho$ of $A; \Rightarrow ; B$. By Lemma 4.15, this $\mho$ can be efficiently transformed to another componentwise interpolant $\mho'$, which is an external conjunction of hyperformulas of length 1. By Lemma 4.13, this $\mho'$ can be efficiently transformed to a formula interpolant $C$ of $A$ and $B$. □

$$\Rightarrow \Box^{\mathsf{l}} \frac{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow A; \quad \longleftarrow \quad \bigovee_{j=1}^{m} \left( (\bot^n \mid B^j) \owedge \bigowedge_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right)}{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow \Box A; \quad \longleftarrow \quad \bigovee_{j=1}^{m} \left( (\bot^n \mid \diamond B^j) \owedge \bigowedge_{k_j=1}^{l_j} (\vec{C}^{j,k_j} \mid \bot) \right)}$$

$$\Rightarrow \Box^{\mathsf{r}} \frac{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow ; A \quad \longleftarrow \quad \bigowedge_{j=1}^{m} (\vec{C}^j \mid B^j)}{\widetilde{\mathcal{G}} \mid \Box\Gamma; \Box\Pi \Rightarrow ; \Box A \quad \longleftarrow \quad \bigowedge_{j=1}^{m} (\vec{C}^j \mid \Box B^j)}$$

$$\mathsf{MS} \frac{\widetilde{\mathcal{G}} \mid \Box\Lambda, \Gamma; \Box\Theta, \Pi \Rightarrow \Box\Phi, \Delta; \Box\Psi, \Sigma \quad \longleftarrow \quad \bigowedge_{j=1}^{m} (\vec{C}^j \mid A^j)}{\widetilde{\mathcal{G}} \mid \Box\Lambda; \Box\Theta \Rightarrow \Box\Phi; \Box\Psi \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \quad \longleftarrow \quad \bigowedge_{j=1}^{m} (\vec{C}^j \mid \bot \mid A^j)}$$

For $\Rightarrow \Box^{\mathsf{r}}$ and MS, the interpolant of the premise must be transformed to an external conjunction of hyperformulas. The rule $\Rightarrow \Box^{\mathsf{l}}$ requires a more elaborate form guaranteed by Lemma 5.3. It is required that

$$\|\widetilde{\mathcal{G}}\| = \|\vec{C}^j\| \text{ for each } j = 1, \dots, m, \tag{MS}$$

and that

$$\|\widetilde{\mathcal{G}}\| = \|\vec{C}^{j,k_j}\| = n \text{ for each } j = 1, \dots, m \text{ and } k_j = 1, \dots, l_j, \tag{$\Rightarrow \Box^{\mathsf{l}}$}$$

where $\bot^n = \underbrace{\bot \mid \dots \mid \bot}_{n}$.

**Fig. 6.** Interpolation algorithm: Modal rules. For the modal rules $\Box^{\mathsf{l}} \Rightarrow$ and $\Box^{\mathsf{r}} \Rightarrow$ not depicted above, the given interpolant for the premise is to be used as an interpolant for the conclusion.

**Example 5.8.** *To illustrate our method, we apply it to a properly adapted sample derivation from [Avron, 1996] in Figure 7. The rules for handling* $\neg$ *are easily derivable from those we presented for* $\rightarrow$. *Note that the interpolant* $P \vee \Box\bot$ *of* $P$ *and* $\Box\neg\Box\neg P$ *constructed by the algorithm contains redundancies: the formula* $P$ *alone could serve as an interpolant.*

$$\mathsf{id}^{lr}\ \dfrac{}{P;\ \Rightarrow\ ;P \longleftarrow P}$$

$$\neg^{r}\Rightarrow\ \dfrac{P;\ \Rightarrow\ ;P \longleftarrow P}{P;\neg P \Rightarrow\ \longleftarrow P}$$

$$\Box^{r}\Rightarrow\ \dfrac{P;\neg P \Rightarrow\ \longleftarrow P}{P;\Box\neg P \Rightarrow\ \longleftarrow P}$$

$$\mathsf{MS}\ \dfrac{P;\Box\neg P \Rightarrow\ \longleftarrow P}{;\Box\neg P \Rightarrow\ \mid P;\ \Rightarrow\ \longleftarrow \bot \mid P}$$

$$\mathsf{Ex}\ \dfrac{;\Box\neg P \Rightarrow\ \mid P;\ \Rightarrow\ \longleftarrow \bot \mid P}{P;\ \Rightarrow\ \mid ;\Box\neg P \Rightarrow\ \longleftarrow P \mid \bot}$$

$$\Rightarrow\neg^{r}\ \dfrac{P;\ \Rightarrow\ \mid ;\Box\neg P \Rightarrow\ \longleftarrow P \mid \bot}{P;\ \Rightarrow\ \mid\ \Rightarrow ;\neg\Box\neg P \longleftarrow P \mid \bot}$$

$$\Rightarrow\Box^{r}\ \dfrac{P;\ \Rightarrow\ \mid\ \Rightarrow ;\neg\Box\neg P \longleftarrow P \mid \bot}{P;\ \Rightarrow\ \mid\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \mid \Box\bot}$$

$$\mathsf{Ex}\ \dfrac{P;\ \Rightarrow\ \mid\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \mid \Box\bot}{\Rightarrow ;\Box\neg\Box\neg P \mid P;\ \Rightarrow\ \longleftarrow \Box\bot \mid P}$$

$$\Rightarrow\mathsf{W}^{r}\ \dfrac{\Rightarrow ;\Box\neg\Box\neg P \mid P;\ \Rightarrow\ \longleftarrow \Box\bot \mid P}{\Rightarrow ;\Box\neg\Box\neg P \mid P;\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow \Box\bot \mid P}$$

$$\mathsf{Ex}\ \dfrac{\Rightarrow ;\Box\neg\Box\neg P \mid P;\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow \Box\bot \mid P}{P;\ \Rightarrow ;\Box\neg\Box\neg P \mid\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \mid \Box\bot}$$

$$\mathsf{W}^{l}\Rightarrow\ \dfrac{P;\ \Rightarrow ;\Box\neg\Box\neg P \mid\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \mid \Box\bot}{P;\ \Rightarrow ;\Box\neg\Box\neg P \mid P;\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \mid \Box\bot}$$

$$\mathsf{EC}\ \dfrac{P;\ \Rightarrow ;\Box\neg\Box\neg P \mid P;\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \mid \Box\bot}{P;\ \Rightarrow ;\Box\neg\Box\neg P \longleftarrow P \vee \Box\bot}$$

**Fig. 7.** Application of our algorithm to the derivation of $P \Rightarrow \Box\neg\Box\neg P$ from [Avron, 1996].

# 6 Conclusion and Future Work

To the best of our knowledge, we have presented the first method of proving interpolation constructively by induction on a hypersequent derivation. The method was developed for the classical modal logic S5. We plan to extend this method to

- various other classical hypersequent systems,
- grafted sequent calculi, recently developed in [Kuznets and Lellmann, 2016], which combine hypersequent and nested sequent calculi,
- non-classical hypersequent systems, especially ones for intermediate logics.

**Acknowledgment:** This work is funded by the Austrian Science Fund (FWF) grant M 1770-N25.

# References


Arnon Avron. A constructive analysis of **RM**. *Journal of Symbolic Logic*, 52(4):939–951, December 1987. 10.2307/2273828.

Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In Wilfrid Hodges, Martin Hyland, Charles Steinhorn, and John Truss, editors, *Logic: From Foundations to Applications: European Logic Colloquium*, pages 1–32. Clarendon Press, 1996.

Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.

Melvin Fitting and Roman Kuznets. Modal interpolation via nested sequents. *Annals of Pure and Applied Logic*, 166(3):274–305, March 2015. 10.1016/j.apal.2014.11.002.

Dov M. Gabbay and Larisa Maksimova. *Interpolation and Definability: Modal and Intuitionistic Logic*, volume 46 of *Oxford Logic Guides*. Clarendon Press, 2005.

Roman Kuznets and Björn Lellmann. Grafting hypersequents onto nested sequents. *Logic Journal of the IGPL*, In Press, 2016. 10.1093/jigpal/jzw005.

Björn Lellmann and Dirk Pattinson. Correspondence between modal Hilbert axioms and sequent rules with an application to S5. In Didier Galmiche and Dominique Larchey-Wendling, editors, *Automated Reasoning with Analytic Tableaux and Related Methods, 22nd International Conference, TABLEAUX 2013, Nancy, France, September 16–19, 2013, Proceedings*, volume 8123 of *Lecture Notes in Computer Science*, pages 219–233. Springer, 2013. 10.1007/978-3-642-40537-2_19.

G. E. Minc. On some calculi of modal logic. In V. P. Orevkov, editor, *The Calculi of Symbolic Logic. I*, volume 98 of *Proceedings of the Steklov Institute of Mathematics*, pages 97–124. AMS, 1971. Originally published in Russian in 1968.

Garrel Pottinger. Uniform, cut-free formulations of $T$, $S_4$, and $S_5$. *Journal of Symbolic Logic*, 48 (3):900, September 1983. 10.2307/2273495. Abstract.