J Glob Optim
DOI 10.1007/s10898-017-0575-0

The Glowinski–Le Tallec splitting method revisited in the framework of equilibrium problems in Hilbert spaces

Phan Tu Vuong[1,2] · **Jean Jacques Strodiot**[3,4]

Received: 26 January 2017 / Accepted: 1 October 2017

Abstract In this paper, we introduce a new approach for solving equilibrium problems in Hilbert spaces. First, we transform the equilibrium problem into the problem of finding a zero of a sum of two maximal monotone operators. Then, we solve the resulting problem using the Glowinski–Le Tallec splitting method and we obtain a linear rate of convergence depending on two parameters. In particular, we enlarge significantly the range of these parameters given rise to the convergence. We prove that the sequence generated by the new method converges to a global solution of the considered equilibrium problem. Finally, numerical tests are displayed to show the efficiency of the new approach.

Keywords Maximal monotone operator · Glowinski–Le Tallec splitting method · Equilibrium problem · Nash equilibrium · Global convergence

1 Introduction

The equilibrium problem, also called the Ky Fan inequality problem [13], has been recently reconsidered by Blum, Muu and Oettli in [5,27,28]. This is a very general problem because it includes, among others, the optimization problem, the variational inequality, the saddle point

Dedicated to the memory of Professor Van Hien Nguyen.

✉ Jean Jacques Strodiot
jjstrodiot@fundp.ac.be

Phan Tu Vuong
vuong.phan@tuwien.ac.at

1 Institute of Statistics and Mathematical Methods in Economics, Vienna University of Technology, Vienna, Austria

2 HCMC University of Technology and Education, Ho Chi Minh City, Vietnam

3 Institute for Computational Science and Technology - HCMC (ICST), Ho Chi Minh City, Vietnam

4 University of Namur, Namur, Belgium

Published online: 09 October 2017

problem, the Nash equilibrium problem in noncooperative games, the fixed point problem. The interest of this problem is that it unifies all these particular problems in a convenient way. Many methods have been proposed for solving equilibrium problems as the projection methods [22,24,26,34], the proximal point methods [17,25], the extragradient methods with or without linesearch [30,35,36,38], the bundle methods [29]. We refer the readers to [4] and the references quoted therein, where an excellent survey of the existing results is presented.

The strategy used in this paper consists in transforming the equilibrium problem into the problem of finding a zero of a sum of two maximal monotone operators. The first one is multivalued and corresponds to the normal cone associated with the feasible set of the equilibrium problem. The second one is single-valued and coincides with the derivative with respect to the second variable of the equilibrium function. For solving this problem, we propose to use the Glowinski–Le Tallec splitting method introduced in [14] and whose convergence has been studied in [15] for finite dimensional spaces. In this paper, our aim is first to prove the linear convergence of the Glowinski–Le Tallec splitting method not only in the framework of Hilbert spaces but also for a larger range of parameters than the one used in [15]. Then, in the second part, these new convergence results are applied to the equilibrium problem.

The paper is organized as follows: In Sect. 2, some preliminary results are recalled. Linear convergence of the Glowinski–Le Tallec splitting method is established in Sect. 3 following the value of the parameters. In Sect. 4, applications of the Glowinski–Le Tallec splitting method to equilibrium problems are discussed and some numerical results are reported. Finally, some conclusions are discussed in the last section.

2 Preliminaries

Let H be a real Hilbert space endowed with an inner product and its induced norm denoted $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and let f be a function from $C \times C$ to $\mathbb{R}$ such that $f(x, x) = 0$ for all $x \in C$. The equilibrium problem associated with f, in the sense of [5], is denoted $EP(f, C)$, and consists in finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0 \quad \text{for every } y \in C.$$

The set of solutions of $EP(f, C)$ is supposed to be nonempty and denoted $Sol(f, C)$. Here we also assume that the function $f(x, \cdot) : C \to \mathbb{R}$ is convex and differentiable at x for every $x \in C$. Many methods have been proposed in the literature for solving such a problem, see for example [4,34,38] and the references quoted therein. Setting, for every $x \in C$,

$$Ax = N_C(x) \quad \text{and} \quad Bx = \nabla_2 f(x, x),$$

where $N_C(x)$ denotes the normal cone to C at x, it is easy to see that the operators A and B are maximal monotone with A multivalued and B single-valued [3]. So the equilibrium problem $EP(f, C)$ is equivalent to the following problem:

$$(P) \qquad \text{Find } x^* \in H \quad \text{such that} \quad 0 \in Ax^* + Bx^*.$$

Indeed, we obtain immediately the following equivalences:

$$x^* \in Sol(f, C) \Longleftrightarrow x^* \in \arg\min_{y \in C} f(x^*, y) \Longleftrightarrow 0 \in \nabla_2 f(x^*, x^*) + N_C(x^*).$$

A large variety of methods can be found in the literature for solving problem (P) when A and B are maximal monotone with A multivalued and B single-valued. Most of them are based on the forward-backward scheme [7–9,31,39] where at each iteration a forward step for B is alternated with a backward step for A as follows:

$$x^{k+1} = J_{\lambda A}(I - \lambda B)x^k.$$

Here λ is some positive steplength, I is the identity operator and $J_{\lambda A} = (I+\lambda A)^{-1}$ is the resolvent operator of A. This operator is single-valued [3,7,9,12]. Among the well known other schemes, let us mention the Peaceman–Rachford scheme [20,32], the Douglas–Rachford scheme [6,10,11,16,20,33] and the Glowinski–Le Tallec scheme [14]. This last method has been introduced by Glowinski and Le Tallec and applied by them for solving, among others, elastoviscoplasticity, liquid crystal, eigenvalue computation problems. The corresponding Glowinski–Le Tallec iteration can be described as a triple forward-backward iteration as follows:

$$x^{k+1} = J_{\lambda_1 A}(I - \lambda_1 B)J_{\lambda_2 B}(I - \lambda_2 A)J_{\lambda_1 A}(I - \lambda_1 B)x^k$$

where $\lambda_1, \lambda_2 > 0$. Since for every maximal monotone operator T defined on H and for every $\mu, \nu > 0$, we have the following identity:

$$I - \nu T = \frac{\nu}{\mu}\left(\gamma J_{\mu T} - I\right)(I + \mu T)$$

where $\gamma = 1 + \dfrac{\mu}{\nu}$, we can rewrite the Glowinski–Le Tallec iteration as

$$x^{k+1} = J_{\lambda_1 A}(I - \lambda_1 B)J_{\lambda_2 B}\left(\frac{\lambda_2}{\lambda_1}\right)\left(\alpha J_{\lambda_1 A} - I\right)(I - \lambda_1 B)x^k$$

where $\alpha = 1 + \dfrac{\lambda_1}{\lambda_2}$. Written under this form, we immediately see that this formula is well defined for a multivalued operator A and a single-valued operator B. The proof of convergence of this scheme to a solution $x^* \in Sol(f, C)$ has been given by Haubruge et al. [15] for finite dimentional spaces. Our aim in this paper is to prove the linear convergence of the Glowinski–Le Tallec method in the framework of Hilbert spaces and with a larger range of parameters. Once obtained, these results will be applied for solving equilibrium problems. Moreover, numerical tests will be reported to show the efficiency of the new method. However, before starting all these results, we need to recall some preliminaries.

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for every $y \in C$. We know that P_C is a nonexpansive mapping from H onto C and for every $x \in H$, $u \in C$, the following property holds

$$u = P_C x \Longleftrightarrow \langle x - u, v - u\rangle \le 0 \quad \forall v \in C.$$

Moreover, if we set $Au = N_C(u)$ for every $u \in C$, then for any $x \in H$ and $\lambda > 0$, we have the following equivalences:

$$\begin{aligned} u = J_{\lambda A}x &\Longleftrightarrow u = (I + \lambda A)^{-1}x \\ &\Longleftrightarrow x \in u + \lambda N_C(u) \\ &\Longleftrightarrow x - u \in \lambda N_C(u) \end{aligned}$$

$$\begin{aligned}
&\iff \frac{1}{\lambda}\langle x-u, v-u\rangle \le 0 \quad \forall v \in C \\
&\iff \langle x-u, v-u\rangle \le 0 \quad \forall v \in C \\
&\iff u = P_C x. \qquad (2.1)
\end{aligned}$$

On the other hand, let $T : H \to 2^H$ be a multivalued operator. The graph of T, the effective domain of T and the inverse of T are defined, respectively, by

$$\begin{aligned}
Gr(T) &= \{(x, y) \in H \times H \mid y \in Tx\}; \\
D(T) &= \{x \in H \mid Tx \neq \emptyset\}; \\
T^{-1}y &= \{x \in H \mid y \in Tx\} \quad \forall y \in H.
\end{aligned}$$

Let us also recall some well-known definitions useful in the sequel [3].

Definition 2.1 The operator $T : H \to 2^H$ is said to be

(a) strongly monotone if there exists $\gamma > 0$ such that

$$\langle u-v, x-y\rangle \ge \gamma\|x-y\|^2 \quad \forall (x,u), (y,v) \in Gr(T);$$

(b) monotone if

$$\langle u-v, x-y\rangle \ge 0 \quad \forall (x,u), (y,v) \in Gr(T);$$

(c) maximal monotone if it is monotone and its graph is not strictly contained in the graph of any other monotone operator.

Definition 2.2 The single-valued operator $F : D \subseteq H \to H$ is said to be

(a) nonexpansive if

$$\|Fx - Fy\| \le \|x-y\| \quad \forall x, y \in D;$$

(b) firmly nonexpansive if

$$\|Fx - Fy\|^2 \le \|x-y\|^2 - \|(I-F)x - (I-F)y\|^2 \quad \forall x, y \in D;$$

(c) co-coercive if there exists $\sigma > 0$ such that

$$\langle Fx - Fy, x-y\rangle \ge \sigma\|Fx - Fy\|^2 \quad \forall x, y \in D;$$

(d) Lipschitz continuous if there exists $L > 0$ such that

$$\|Fx - Fy\| \le L\|x-y\| \quad \forall x, y \in D.$$

The following lemmas are useful to establish our main results.

Lemma 2.1 ([11]) *Let T be a maximal monotone operator defined on H. Then, for any $\lambda > 0$, the resolvent $J_{\lambda T}$ is single-valued, firmly nonexpansive, and everywhere defined.*

Lemma 2.2 ([3], Proposition 23.11) *Let T be a strongly maximal monotone operator defined on H with modulus $\eta > 0$. Then, for any $\lambda > 0$, the resolvent $J_{\lambda T}$ is Lipschitz continuous with constant $\frac{1}{1+\lambda\eta}$.*

Lemma 2.3 *Let T be a co-coercive operator defined on H with modulus $\sigma > 0$. Then, for $0 < \lambda \le 2\sigma$, the operator $I - \lambda T$ is nonexpansive. If, in addition, T is strongly monotone with modulus $\delta > 0$, then the operator $I - \lambda T$ is Lipschitz continuous with modulus $L = \sqrt{1 - \lambda(2\sigma - \lambda)\delta^2}$.*

Proof Let $x, y \in H$, we have

$$\begin{aligned}\|(I-\lambda T)x-(I-\lambda T)y\|^2 &= \|x-y\|^2 - 2\lambda\langle x-y, Tx-Ty\rangle + \lambda^2\|Tx-Ty\|^2 \\ &\leq \|x-y\|^2 - \lambda(2\sigma-\lambda)\|Tx-Ty\|^2 \quad (2.2)\\ &\leq \|x-y\|^2. \quad (2.3)\end{aligned}$$

This means that $I-\lambda T$ is nonexpansive.
If in addition, T is strongly monotone with modulus $\delta > 0$, using Cauchy-Schwarz inequality, we have, for every $x, y \in H$, that

$$\delta\|x-y\|^2 \leq \langle x-y, Tx-Ty\rangle \leq \|x-y\|\|Tx-Ty\|.$$

Hence

$$\delta\|x-y\| \leq \|Tx-Ty\|.$$

Substituting the last inequality into (2.2), we obtain

$$\|(I-\lambda T)x-(I-\lambda T)y\|^2 \leq (1-\lambda\,(2\sigma-\lambda)\,\delta^2)\|x-y\|^2$$

i.e., the operator $I-\lambda T$ is Lipschitz continuous with modulus

$$L = \sqrt{1-\lambda(2\sigma-\lambda)\delta^2}.$$

□

Remark 2.1 When the operator T is Lipschitz continuous and strongly monotone with modulus $l > 0$ and $\delta > 0$, respectively, we have that T is co-coercive with modulus $\sigma = \frac{\delta}{l^2}$. So, in this situation, from Lemma 2.3, we have that for $0 < \lambda \leq \frac{2\delta}{l^2}$, the operator $I-\lambda T$ is Lipschitz continuous with modulus $L = \sqrt{1-\lambda(\frac{2\delta}{l^2}-\lambda)\delta^2}$. □

Lemma 2.4 *Let T be an operator which is co-coercive with modulus $\sigma > 0$ and strongly monotone with modulus $\delta > 0$. Then $\delta\sigma \leq 1$.*

Proof For every $x, y \in H$, since T is σ co-coercive, we have, using Cauchy-Schwarz inequality, that

$$\|Tx-Ty\|\|x-y\| \geq \langle Tx-Ty, x-y\rangle \geq \sigma\|Tx-Ty\|^2$$

or equivalently,

$$\|x-y\| \geq \sigma\|Tx-Ty\|. \quad (2.4)$$

On the other hand, since T is δ strongly monotone, using again Cauchy-Schwarz inequality, we obtain

$$\|Tx-Ty\|\|x-y\| \geq \langle Tx-Ty, x-y\rangle \geq \delta\|x-y\|^2$$

which leads to

$$\|Tx-Ty\| \geq \delta\|x-y\|. \quad (2.5)$$

Combining the two inequalities (2.4) and (2.5), we obtain the conclusion. □

3 Convergence of the Glowinski–Le Tallec splitting method

Let H be a real Hilbert space and let $\{x^k\}$ be the sequence generated by the Glowinski–Le Tallec splitting iteration

$$x^{k+1} = J_{\lambda_1 A}(I - \lambda_1 B) J_{\lambda_2 B} \left(\frac{\lambda_2}{\lambda_1}\right) \left(\alpha J_{\lambda_1 A} - I\right) (I - \lambda_1 B) x^k \quad k \in \mathbb{N} \tag{3.1}$$

where $x^0 \in H$, $\alpha = 1 + \frac{\lambda_1}{\lambda_2}$ and $\lambda_1, \lambda_2 > 0$. Here A and B are supposed to be maximal monotone operators on H with A multivalued and B single-valued. The following proposition plays a key-role in the convergence analysis of the sequence generated by the Glowinski–Le Tallec splitting method. Since its proof is similar to the one given in [15] for finite dimentional spaces, it will not be given here.

Theorem 3.1 *Let T be a maximal monotone operator defined on H, and let μ and ν be two positive real numbers. Set $\gamma = 1 + \frac{\mu}{\nu}$. Then, the operator $\gamma J_{\mu T} - I$ is Lipschitz continuous with constant $L = \max\{1, \frac{\mu}{\nu}\}$. In particular, when $\mu = \nu$, the operator $2J_{\mu T} - I$ is nonexpansive. If, in addition, T is co-coercive with modulus $\tau > 0$ and if $\mu \leq \nu < 2\tau$, then $\gamma J_{\mu T} - I$ is Lipschitz continuous with constant $L = \frac{\mu}{\nu} \leq 1$.*

Now we are in a position to discuss the linear convergence of the Glowinski–Le Tallec splitting method. First let us recall that a sequence $\{x^k\} \subset H$ converges linearly to $x \in H$ if there exists a number $r \in (0, 1)$ and an index k_0 such that $\|x^{k+1} - x\| \leq r\|x^k - x\|$ for all $k \geq k_0$. The number r is called the linear rate of convergence.

In the next theorem we consider the case when $0 < \lambda_2 \leq \lambda_1 \leq 2\sigma$, where σ is the modulus of co-coercivity of operator B. In that case, our result improves significantly the convergence ratio found by Haubruge et al. in [15], Theorem 2.2.

Theorem 3.2 *Suppose that the operator B is co-coercive with modulus $\sigma > 0$ and strongly monotone with modulus $\delta > 0$. If $0 < \lambda_2 \leq \lambda_1 \leq 2\sigma$, then the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method converges linearly to a solution of (P) at the linear rate*

$$r = \left[1 - \lambda_1(2\sigma - \lambda_1)\delta^2\right] \frac{1}{1 + \lambda_2 \delta} < 1. \tag{3.2}$$

Proof We prove this theorem by considering separately the following operators in the Glowinski–Le Tallec scheme (3.1):

$$J_{\lambda_1 A}, \quad I - \lambda_1 B, \quad J_{\lambda_2 B}, \quad \alpha J_{\lambda_1 A} - I.$$

First we have, using Lemma 2.1, that the resolvent $J_{\lambda_1 A}$ is nonexpansive. Since the operator B is σ co-coercive and δ strongly monotone, and since $0 < \lambda_1 \leq 2\sigma$, it follows from Lemma 2.3 that the operator $I - \lambda_1 B$ is Lipschitz continuous with modulus

$$r_1 = \sqrt{1 - \lambda_1(2\sigma - \lambda_1)\delta^2}.$$

On the other hand, since the operator B is δ strongly monotone, from Lemma 2.2 we obtain that the resolvent $J_{\lambda_2 B}$ is Lipschitz continuous with modulus

$$r_2 = \frac{1}{1 + \lambda_2 \delta}.$$

Finally, since $\lambda_2 \leq \lambda_1$, we can conclude by using Theorem 3.1 that the operator $\alpha J_{\lambda_1 A} - I$ with $\alpha = 1 + \frac{\lambda_1}{\lambda_2}$ is Lipschitz continuous with modulus

$$r_3 = \max\{1, \frac{\lambda_1}{\lambda_2}\} = \frac{\lambda_1}{\lambda_2}.$$

Gathering the above operators, we deduce that the operator

$$G = J_{\lambda_1 A}(I - \lambda_1 B) J_{\lambda_2 B} \left(\frac{\lambda_2}{\lambda_1}\right) \left(\alpha J_{\lambda_1 A} - I\right) (I - \lambda_1 B)$$

is contractive with modulus

$$r = \left[1 - \lambda_1(2\sigma - \lambda_1)\delta^2\right] \frac{1}{1 + \lambda_2 \delta} < 1.$$

So the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method converges linearly to a solution of (P) at the linear rate r. □

Remark 3.1 The smallest value for r is obtained when $\lambda_1 = \lambda_2 = \sigma$ and is equal to

$$r^* = (1 - \sigma^2\delta^2)\frac{1}{1 + \sigma\delta} = 1 - \sigma\delta.$$

This means that, if $\sigma\delta = 1$, then we obtain immediately the solution of problem (P) after one iteration by choosing $\lambda_1 = \lambda_2 = \sigma$. □

Remark 3.2 When the operator B is Lipschitz continuous and strongly monotone with modulus $l > 0$ and $\delta > 0$, respectively, the operator B is co-coercive with modulus $\sigma = \frac{\delta}{l^2}$. So, using Remark 2.1, we have that the operator $I - \lambda_1 B$ is Lipschitz continuous for $0 < \lambda_1 \leq \frac{2\delta}{l^2}$ with modulus

$$r_1 = \sqrt{1 - \lambda_1 \left(\frac{2\delta}{l^2} - \lambda_1\right) \delta^2}.$$

In this situation the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method converges linearly to a solution of (P) at the linear rate (3.2) with $\sigma = \frac{\delta}{l^2}$. □

Remark 3.3 In [15, Theorem 2.2], it was proved that if B is co-coercive with modulus $\sigma > 0$ and B^{-1} is Lipschitz continuous with modulus $\delta_1 > 0$ and $0 < \lambda_2 \leq \lambda_1 \leq 2\sigma$, then the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method converges linearly to a solution of (P) at the linear rate

$$c_1 = \sqrt{1 - \lambda_1(2\sigma - \lambda_1)/\delta_1^2}.$$

It is easy to see that, the optimal value of c_1 is

$$c_1^* = \sqrt{1 - \sigma^2/\delta_1^2} \tag{3.3}$$

when $\lambda_1 = \sigma$. □

On the other hand, when B is co-coercive with modulus $\sigma > 0$ and B^{-1} is Lipschitz continuous with modulus $\delta_1 > 0$, we have that

$$\langle Bx - By, x - y\rangle \geq \sigma \|Bx - By\|^2 \geq \frac{\sigma}{\delta_1^2}\|x - y\|^2 \quad \forall x, y \in H$$

implying the strong monotonicity of B with modulus $\frac{\sigma}{\delta_1^2}$. Therefore, as a consequence of Theorem 3.2, we have

Corollary 3.1 *Suppose that the operator B is co-coercive with modulus $\sigma > 0$ and B^{-1} is Lipschitz continuous with modulus $\delta_1 > 0$. If $0 < \lambda_2 \leq \lambda_1 \leq 2\sigma$, then the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method converges linearly to a solution of (P) at the linear rate*

$$r = \left[1 - \lambda_1(2\sigma - \lambda_1)\frac{\sigma^2}{\delta_1^4}\right]\frac{1}{1 + \frac{\lambda_2\sigma}{\delta_1^2}} < 1.$$

Moreover, the optimal rate is

$$r^* = 1 - \frac{\sigma^2}{\delta_1^2} \tag{3.4}$$

when $\lambda_1 = \lambda_2 = \sigma$. □

Remark 3.4 We observe that

$$r^* = 1 - \sigma^2/\delta_1^2 = \left(c_1^*\right)^2 < c_1^*.$$

This means that the optimal rate given in (3.4) is much better than the one given in (3.3) in the sense that it allows us to divide at least by two the number of iterations to obtain the same accuracy. □

Now we consider the case $\lambda_1 \leq \lambda_2$. The following result has not been examined in [15].

Theorem 3.3 *Suppose that the operator A is strongly monotone with modulus $\eta > 0$ and the operator B is co-coercive with modulus $\sigma > 0$. If $0 < \lambda_1 \leq \min\{\lambda_2, 2\sigma\}$, then the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method satisfies the following inequality:*

$$\|x^{k+1} - x^*\| \leq \frac{\lambda_2}{\lambda_1(1 + \lambda_1\eta)}\|x^k - x^*\| \quad \forall k \in \mathbb{N}, \tag{3.5}$$

where x^* is the unique solution of (P).
Moreover, if $\lambda_2 \in [\lambda_1, \lambda_1(1 + \lambda_1\eta))$, then the sequence $\{x^k\}$ converges linearly to x^* at the linear rate

$$s = \frac{\lambda_2}{\lambda_1(1 + \lambda_1\eta)} < 1.$$

Proof We prove this theorem by considering separately the following operators in the Glowinski–Le Tallec scheme (3.1):

$$J_{\lambda_1 A}, \quad I - \lambda_1 B, \quad J_{\lambda_2 B}, \quad \alpha J_{\lambda_1 A} - I.$$

The operator A being strongly maximal monotone with modulus $\eta > 0$, it follows from Lemma 2.2 that the resolvent $J_{\lambda_1 A}$ is Lipschitz continuous with modulus

$$s_1 = \frac{1}{1 + \lambda_1\eta}.$$

On the other hand, the operator B being co-coercive maximal monotone with modulus $\sigma > 0$, one can apply Lemma 2.3 and Lemma 2.1 to obtain that the operators $I - \lambda_1 B$ and $J_{\lambda_2 B}$ are nonexpansive. Since $\lambda_1 \le \lambda_2$, it is easy to see from Theorem 3.1 that the operator $\alpha J_{\lambda_1 A} - I$ is Lipschitz continuous with modulus

$$s_2 = \max\{1, \frac{\lambda_1}{\lambda_2}\} = 1.$$

Gathering the above operators, we deduce that the operator

$$G = J_{\lambda_1 A}(I - \lambda_1 B) J_{\lambda_2 B} \left(\frac{\lambda_2}{\lambda_1}\right) \left(\alpha J_{\lambda_1 A} - I\right)(I - \lambda_1 B)$$

is Lipschitz continuous with modulus

$$s = \frac{\lambda_2}{\lambda_1(1 + \lambda_1 \eta)}.$$

Then the inequality (3.5) holds. Now, if $\lambda_2 \in [\lambda_1, \lambda_1(1 + \lambda_1 \eta))$, then

$$s = \frac{\lambda_2}{\lambda_1(1 + \lambda_1 \eta)} < 1$$

and the proof is complete. □

Remark 3.5 This result shows that the convergence rate of the sequence generated by the Glowinski–Le Tallec splitting method depends on both λ_1 and λ_2 in the case when $0 < \lambda_1 \le \min\{\lambda_2, 2\sigma\}$. Furthermore the smallest ratio is obtained when $\lambda_1 = \lambda_2 = 2\sigma$ and is equal to

$$s = \frac{1}{1 + 2\sigma\eta}.$$

□

Remark 3.6 In the proof of Theorem 3.3, we observe that the strong monotonicity of A is only used to obtain the contraction of the operator $J_{\lambda_1 A}$. In some cases, the contraction of $J_{\lambda_1 A}$ can be obtained without assuming the strong monotonicity of A, and the conclusions of Theorem 3.3 still hold. For example, when C is a strongly convex set[1] and $A = N_C$ is the normal cone operator to C, the resolvent J_{λ_1} coincides for all $\lambda_1 > 0$, with the projection P_C onto C which is a contraction onto $H \setminus C$ [2, Theorem 2.2]. This is obtained without assuming that the operator $A = N_C$ is strongly monotone. Note also that, when C is a strongly convex set, the normal cone N_C is not strongly monotone in general because $N_C(x) = \{0\}\ \forall x \in \mathrm{int}(C)$. It is only strongly monotone on the boundary of C [37, Proposition 2.9]. □

In the situation when the operator B is co-coercive and strongly monotone, we obtain the following result:

Theorem 3.4 *Suppose that the operator B is co-coercive with modulus $\sigma > 0$ and strongly monotone with modulus $\delta > 0$. If $0 < \lambda_1 \le \min\{\lambda_2, 2\sigma\}$, the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method satisfies the following inequality:*

$$\|x^{k+1} - x^*\| \le \frac{\lambda_2}{1 + \lambda_2 \delta} \frac{\left[1 - \lambda_1(2\sigma - \lambda_1)\delta^2\right]}{\lambda_1} \|x^k - x^*\| \quad \forall k \in \mathbb{N},$$

where x^* is the unique solution of (P).

[1] We recall that a nonempty subset $C \subset H$ is called strongly convex of radius $R > 0$ if it can be represented as the intersection of closed balls of radius $R > 0$, i.e. there exists a subset $X \subset H$ such that $C = \cap_{x \in X} \mathbf{B}(x, R)$, see e.g. [2,37].

Moreover, if

$$\rho = \frac{\lambda_2}{1+\lambda_2\delta}\frac{\left[1-\lambda_1(2\sigma-\lambda_1)\,\delta^2\right]}{\lambda_1} < 1 \tag{3.6}$$

then the sequence $\{x^k\}$ *converges linearly to* x^*.

Proof We prove this theorem by considering separately the following operators in the Glowinski–Le Tallec scheme (3.1):

$$J_{\lambda_1 A},\quad I-\lambda_1 B,\quad J_{\lambda_2 B},\quad \alpha J_{\lambda_1 A}-I.$$

Since A is maximal monotone, the resolvent $J_{\lambda_1 A}$ is nonexpansive thanks to Lemma 2.1. The operator B being co-coercive with modulus $\sigma > 0$ and strongly monotone with modulus $\delta > 0$, it follows from Lemma 2.3 that the operator $I-\lambda_1 B$ is Lipschitz continuous with modulus

$$\rho_1 = \sqrt{1-\lambda_1(2\sigma-\lambda_1)\delta^2}.$$

Furthermore, from Lemma 2.2, the resolvent $J_{\lambda_2 B}$ is Lipschitz continuous with modulus

$$\rho_2 = \frac{1}{1+\lambda_2\delta}.$$

Finally, since $\lambda_1 \le \lambda_2$, it is easy to see from Theorem 3.1, that the operator $\alpha J_{\lambda_1 A}-I$ is Lipschitz continuous with modulus

$$\rho_3 = \max\{1, \frac{\lambda_1}{\lambda_2}\} = 1.$$

Gathering the operators, we deduce that the operator

$$G = J_{\lambda_1 A}(I-\lambda_1 B)J_{\lambda_2 B}\left(\frac{\lambda_2}{\lambda_1}\right)\left(\alpha J_{\lambda_1 A}-I\right)(I-\lambda_1 B)$$

is Lipschitz continuous with modulus

$$\rho = \frac{\lambda_2}{1+\lambda_2\delta}\frac{\left[1-\lambda_1(2\sigma-\lambda_1)\delta^2\right]}{\lambda_1}.$$

If

$$\rho = \frac{\lambda_2}{1+\lambda_2\delta}\frac{\left[1-\lambda_1(2\sigma-\lambda_1)\delta^2\right]}{\lambda_1} < 1$$

then the sequence $\{x^k\}$ converges linearly to a solution of (P). □

Now we examine when inequality (3.6) holds. In that purpose we observe that

$$\rho = \frac{\lambda_2}{1+\lambda_2\delta}\frac{\left[1-\lambda_1(2\sigma-\lambda_1)\delta^2\right]}{\lambda_1} < \frac{\lambda_2}{1+\lambda_2\delta}\frac{1}{\lambda_1}. \tag{3.7}$$

So, to obtain that $\rho < 1$, we can choose λ_2 such that the right-hand side of (3.7) is less than 1, i.e., that

$$(1-\lambda_1\delta)\lambda_2 \le \lambda_1. \tag{3.8}$$

If $\lambda_1 \ge \dfrac{1}{\delta}$, then inequality (3.8) holds for all $\lambda_2 > 0$. On the other side, if $\lambda_1 < \frac{1}{\delta}$, then (3.8) holds for $0 < \lambda_2 \le \frac{\lambda_1}{1-\delta\lambda_1}$.

Remark 3.7 When the operator B is Lipschitz continuous and strongly monotone with modulus l and δ, respectively, the operator B is co-coercive with modulus $\sigma = \frac{\delta}{l^2}$ and the conclusion of Theorem 3.4 is still valid. □

From Theorems 3.2 and 3.4, we have the following corollary.

Corollary 3.2 *Suppose that the operator B is co-coercive with modulus $\sigma > 0$ and strongly monotone with modulus $\delta > 0$. Assume that $\frac{1}{2} \le \sigma\delta \le 1$. If $\lambda_1 \in \left[\frac{1}{\delta}, 2\sigma\right]$, then for every $\lambda_2 > 0$, the sequence $\{x^k\}$ generated by the corresponding Glowinski–Le Tallec splitting method converges linearly to a solution of (P). If $0 < \lambda_1 < \frac{1}{\delta}$, then the same conclusion holds for every $\lambda_2 \in \left(0, \frac{\lambda_1}{1-\delta\lambda_1}\right]$.*

Proof Let $\lambda_1 \in \left[\frac{1}{\delta}, 2\sigma\right]$. If $\lambda_2 \ge \lambda_1$ then $0 < \lambda_1 \le \min\{\lambda_2, 2\sigma\}$ and $\rho < 1$. So the conclusion follows from Theorem 3.4. If $0 < \lambda_2 < \lambda_1$, then apply Theorem 3.2 to get the conclusion. The proof is similar when $\lambda_1 \in \left(0, \frac{1}{\delta}\right)$. □

The following example illustrates the case $\frac{1}{2} \le \sigma\delta \le 1$.

Example 1 We consider the affine variational inequality [21,23]:

$$\text{Find } x^* \text{ such that } \langle Mx^* + q, y - x^*\rangle \ge 0, \text{ for all } y \in C$$

where C is a nonempty convex set of $\mathbb{R}^n$ and M is a symmetric positive definite matrix of order n. We choose, for every $x \in C$, $Ax = N_C x$, where N_C denotes the normal cone of C, and $Bx = Mx + q$. Let λ_{min} and λ_{max} be the smallest and largest eigenvalue of M, respectively. Then the operator B is co-coercive with modulus $\sigma = \frac{1}{\lambda_{max}}$ and strongly monotone with modulus $\delta = \lambda_{min}$ on C. Indeed, for every $x, y \in C$, we have

$$\begin{aligned}\langle Bx - By, x - y\rangle &= \langle M(x-y), x-y\rangle \\ &\ge \frac{1}{\|M\|}\langle M(x-y), M(x-y)\rangle \\ &= \frac{1}{\lambda_{max}}\langle M(x-y), M(x-y)\rangle \\ &= \frac{1}{\lambda_{max}}\|Bx - By\|^2, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\langle Bx - By, x - y\rangle &= \langle M(x-y), x-y\rangle \\ &\ge \lambda_{min}\|x-y\|^2.\end{aligned}$$

In this case, if $\lambda_{max} \le 2\lambda_{min}$, then $\dfrac{1}{2} \le \sigma\delta \le 1$. Furthermore, if we choose λ_1 such that $\lambda_1 \in \left[\frac{1}{\delta}, 2\sigma\right] = \left[\frac{1}{\lambda_{min}}, \frac{2}{\lambda_{max}}\right]$, then the sequence $\{x^k\}$ generated by the corresponding Glowinski–Le Tallec splitting method converges linearly for all $\lambda_2 > 0$.
In particular, if $\lambda_{min} = \lambda_{max} = \lambda$, then it follows from Remark 3.1, that the solution of the affine variational inequality can be obtained after one iteration by choosing $\lambda_1 = \lambda_2 = \frac{1}{\lambda}$.
In the next section, we give some numerical tests to show that the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting method converges more quickly when the parameter λ_2 is increasing and approaches $+\infty$.

From Theorems 3.3 and 3.4, we easily deduce the following corollary:

Corollary 3.3 *Suppose that all the assumptions of Theorem 3.4 are satisfied. Suppose in addition that the operator A is strongly monotone with modulus* $\eta > 0$ *and that* $\frac{1}{\delta} \leq \lambda_1 \leq 2\sigma$. *Then the sequence* $\{x^k\}$ *generated by the Glowinski–Le Tallec splitting method converges linearly to a solution of (P) at the linear rate*

$$\frac{\lambda_2}{1+\lambda_2\delta}\,\frac{\left[1-\lambda_1(2\sigma-\lambda_1)\delta^2\right]}{\lambda_1(1+\lambda_1\eta)} < 1. \tag{3.10}$$

Moreover, the smallest ratio is obtained when $\lambda_1 = \lambda_2 = \sigma$ *and is equal to*

$$\frac{1-\sigma\delta}{1+\sigma\eta}.$$

4 Application to equilibrium problems

In this section, we apply the Glowinski–Le Tallec splitting method for solving the equilibrium problem $EP(f, C)$:

$$\text{Find } x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0 \quad \text{for every } y \in C.$$

Here we assume that C is a nonempty closed convex subset of H and that f is a function from $C \times C$ into $\mathbb{R}$ such that $f(x, x) = 0$ for all $x \in C$. We also assume that the function $f(x, \cdot) : C \to \mathbb{R}$ is convex and differentiable at x for all $x \in C$. Moreover, we suppose that the derivative $x \to \nabla_2 f(x, x)$ is co-coercive with modulus $\sigma > 0$ for all $x \in C$ i.e.,

$$\langle \nabla_2 f(x, x) - \nabla_2 f(y, y), x - y \rangle \geq \sigma \|\nabla_2 f(x, x) - \nabla_2 f(y, y)\|^2 \quad \forall x, y \in C.$$

Several methods have been proposed for solving a finite-dimensional equilibrium problem satisfying this assumption, see for example [34] and the references quoted therein.

In view of using the Glowinski–Le Tallec splitting method for solving problem $EP(f, C)$, we define the operators A and B as follows:

$$Ax = N_C(x) \quad \text{and} \quad Bx = \nabla_2 f(x, x) \quad \text{for every } x \in C$$

where $N_C(x)$ denotes the normal cone to C at x.

With these notations, problem $EP(f, C)$ is equivalent to the problem

$$(P) \qquad \text{Find } x^* \in H \quad \text{such that} \quad 0 \in A(x^*) + B(x^*).$$

Since $J_{\lambda_1 A} = P_C$, the projection onto C, the Glowinski–Le Tallec splitting iteration for solving the equilibrium problem can be expressed as

$$x^{k+1} = P_C(I - \lambda_1 B)\, J_{\lambda_2 B}\left(\frac{\lambda_2}{\lambda_1}\right)(\alpha P_C - I)\,(I - \lambda_1 B)\, x^k \tag{4.1}$$

where $\alpha = 1 + \frac{\lambda_1}{\lambda_2}$ and λ_1, λ_2 are positive real numbers.

For convenience, we can replace (4.1) by the system

$$\begin{cases} \bar{y}^k = P_C\,(x^k - \lambda_1 B x^k), \\ y^k = \alpha \bar{y}^k - (x^k - \lambda_1 B x^k), \\ z^k = [\,I + \lambda_2 B\,]^{-1} \left(\dfrac{\lambda_2}{\lambda_1} y^k \right), \\ x^{k+1} = P_C\,(z^k - \lambda_1 B z^k). \end{cases}$$

Under this form, it is easy to prove that if $x^k = \bar{y}^k$, then x^k is a solution of $EP(f, C)$. Indeed, if $x^k = \bar{y}^k = P_C(x^k - \lambda_1 B x^k)$, we have, using the basic property of the projection operator, that

$$\langle x^k - (x^k - \lambda_1 B x^k), y - x^k \rangle \geq 0 \quad \forall y \in C.$$

Since $\lambda_1 > 0$ and $Bx^k = \nabla_2 f(x^k, x^k)$, the last inequality can be rewritten as

$$\langle \nabla_2 f(x^k, x^k), y - x^k \rangle \geq 0 \quad \forall y \in C.$$

On the other hand, since the function $f(x^k, \cdot)$ is convex, we have

$$f(x^k, y) - f(x^k, x^k) \geq \langle \nabla_2 f(x^k, x^k), y - x^k \rangle \quad \forall y \in C.$$

Combining the last two inequalities, and noting that $f(x^k, x^k) = 0$, we obtain

$$f(x^k, y) \geq 0 \quad \forall y \in C.$$

But this means that $x^k \in Sol(f, C)$.
Next, when we assume that the sequence $\{x^k\}$ generated by (4.1) is infinite, we can obtain from Theorems 3.2 and 3.4 that if the operator B defined by $Bx = \nabla_2 f(x, x)$ for every $x \in C$ is co-coercive with modulus $\sigma > 0$ and strongly monotone with modulus $\delta > 0$, then the sequence $\{x^k\}$ generated by the Glowinski–Le Tallec splitting iteration (4.1) converges linearly to x^*, the unique solution of problem $EP(f, C)$ when $0 < \lambda_2 \leq \lambda_1 \leq \sigma$. The same result holds when $0 < \lambda_1 \leq \max\{\lambda_2, 2\sigma\}$ provided that (3.6) is satisfied.

As pointed out by one of the referees, when the bifunction f is differentiable with respect to the second variable, the equilibrium problem $EP(f, C)$ can be considered as the following variational inequality:

$$\text{Find } x^* \in C \quad \text{such that} \quad \left\langle Bx^*, y - x^* \right\rangle \geq 0 \quad \forall y \in C$$

where the operator B is defined for each $x \in C$ by $Bx = \nabla f(x, x)$.
This problem can be solved by using the gradient projection method [1,18] or the extragradient method [36]. In [1, Theorem 3.1], the authors proved that if B is δ-strongly monotone and L-Lipschitz continuous, then the sequence $\{x^k\}$ generated by the gradient projection method, namely

$$x^0 \in C \quad \text{and} \quad x^{k+1} = P_C\,(x^k - \lambda B x^k) \quad \forall k$$

where $\lambda \in (0, 2\delta/L^2)$, converges linearly to the unique solution x^* of the equilibrium problem at the linear rate

$$r_1 = \sqrt{1 - \lambda(2\delta - \lambda L^2)}.$$

Furthermore, the optimal rate is $r_1^* = \sqrt{1 - \frac{\delta^2}{L^2}}$ and is obtained when $\lambda = \delta/L^2$.

On the other hand, observe that if B is δ-strongly monotone and L-Lipschitz continuous, then it is co-coercive with modulus $\sigma := \delta/L^2$.

Therefore, the optimal rate obtained by the Glowinski–Le Tallec method (see Theorem 3.2 and Remark 3.2) is given by

$$r^* = 1 - \sigma\delta = 1 - \frac{\delta^2}{L^2}$$

which is much smaller than the optimal rate given by the gradient projection method. This will be confirmed by numerical results given below.

Now for ending this section, we will apply the Glowinski–Le Tallec splitting method for solving numerically the equilibrium problem $EP(f, C)$. In this purpose, the corresponding algorithms are coded in MATLAB and the stopping criterion $\|x^k - \bar{y}^k\| \leq \epsilon$ is chosen for all test problems with $\epsilon = 10^{-6}$. Furthermore, each time, two different starting points are considered with five different values for λ_1 and λ_2. We perform all computations on a Windows Desktop with an Intel(R) Core(TM) i7-2600 CPU at 3.4GHz and 8.00 GB of memory. The number of iterations and the CPU time needed to get a solution are reported in a table for each problem. We also compare the efficiency of the Glowinski–Le Tallec method (GLM) with the gradient projection method (GPM) [1,18], the extragradient method (EGM) [36] as well as the relaxed projection method (RPM) presented in [34].

Problem 1 The bifunction f of the equilibrium problem comes from the Cournot–Nash equilibrium model considered in [36]. It is defined for each $x, y \in \mathbb{R}^5$, by

$$f(x, y) = \langle Px + Qy + r, y - x\rangle$$

where $r \in \mathbb{R}^5$, and P and Q are two square matrices of order 5 such that $P + Q$ is symmetric positive definite. It is easy to see that for each $x \in C$ the function $f(x, \cdot)$ is convex and differentiable over C, and that

$$Bx = \nabla_2 f(x, x) = (P + Q)x + r.$$

Then for every $\lambda > 0$ we have

$$J_{\lambda B}(x) = (I + \lambda B)^{-1}(x) = [I + \lambda(P + Q)]^{-1}(x - \lambda r).$$

Furthermore, the constraint set is defined as

$$C = \{x \in \mathbb{R}^5 \mid \sum_{i=1}^{5} x_i \geq 0, \quad -5 \leq x_i \leq 5, \quad i = 1, 2, 3, 4, 5\}$$

and the vector r and the matrices P and Q are chosen as follows:

$$r = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix}; \quad P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}; \quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this problem, the operator B is co-coercive with modulus $\sigma = 0.1256$ and strongly monotone with modulus $\delta = 1.8983$. Consequently, the sequence of iterates generated by the Glowinski–Le Tallec algorithm converges linearly to the solution of the problem. Moreover if we choose $\lambda_1 \in [\sigma, 2\sigma]$, then the sequence $\{x^k\}$ converges linearly to the solution x^* for all $\lambda_2 > 0$. From Table 1, we can see that the sequence generated by the Glowinski–Le Tallec

Table 1 The results of Problem 1

Starting point	Parameters	Number of iterations	CPU time (s)
$(1, 3, 1, 1, 2)^T$	$\lambda_1 = 0.2,\ \lambda_2 = 0.1$	12	0.5616
	$\lambda_1 = 0.1,\ \lambda_2 = 0.2$	17	0.1092
	$\lambda_1 = 0.1,\ \lambda_2 = 2.0$	7	0.0312
	$\lambda_1 = 0.2,\ \lambda_2 = 5.0$	5	0.0312
	$\lambda_1 = 0.2,\ \lambda_2 = 50.0$	3	0.0468
$(-1, 0, 2, 3, 1)^T$	$\lambda_1 = 0.2,\ \lambda_2 = 0.2$	12	0.0624
	$\lambda_1 = 0.1,\ \lambda_2 = 0.2$	18	0.0936
	$\lambda_1 = 0.1,\ \lambda_2 = 3.0$	7	0.0312
	$\lambda_1 = 0.2,\ \lambda_2 = 5.0$	5	0.0468
	$\lambda_1 = 0.2,\ \lambda_2 = 50.0$	3	0.0156

splitting method converges more quickly when the parameter λ_2 is increasing. Finally, the obtained solution for this problem is

$$x^* = (-0.725388, 0.803109, 0.72000, -0.866667, 0.200000)^T.$$

Problem 2 The River basin pollution game given in [19] consists of three players with payoff functions:

$$\phi_j(x) = u_j x_j^2 + 0.01 x_j (x_1 + x_2 + x_3) - v_j x_j, \quad j = 1, 2, 3,$$

where $u = (0.01, 0.05, 0.01)$ and $v = (2.90, 2.88, 2.85)$ and the constraints are given by

$$\begin{cases} x_1, x_2, x_3 \geq 0, \\ 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100, \\ 2.291x_1 + 1.5625x_2 + 2.8125x_3 \leq 100. \end{cases}$$

In this problem, we define

$$f(x, y) = \sum_{j=1}^{3} [\phi_j(y_j|x) - \phi_j(x)],$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and

$$(y_1|x) = (y_1, x_2, x_3), \quad (y_2|x) = (x_1, y_2, x_3), \quad (y_3|x) = (x_1, x_2, y_3).$$

Then, we have

$$B(x) = \nabla_2 f(x, x) = \begin{bmatrix} 0.04 & 0.01 & 0.01 \\ 0.01 & 0.12 & 0.01 \\ 0.01 & 0.01 & 0.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 2.90 \\ 2.88 \\ 2.85 \end{bmatrix}$$

The operator B is co-coercive with modulus $\sigma = 8.1467$ and strongly monotone with modulus $\delta = 0.03$. The Nash equilibrium point obtained for this game is

$$x^* = (21.144795, 16.027853, 2.725963)^T.$$

The results obtained by the Glowinski–Le Tallec splitting method are reported in Table 2. Again, we see that the sequence $\{x^k\}$ converges more quickly when the parameter λ_2 is increasing.

Table 2 The results of the River basin pollution problem

Starting point	Parameters	Number of iterations	CPU time (s)
$(0, 0, 0)^T$	$\lambda_1 = 15,\ \lambda_2 = 7$	12	0.2808
	$\lambda_1 = 8,\ \lambda_2 = 8$	20	0.0624
	$\lambda_1 = 15,\ \lambda_2 = 10$	11	0.0468
	$\lambda_1 = 5,\ \lambda_2 = 15$	21	0.1248
	$\lambda_1 = 15,\ \lambda_2 = 150$	8	0.0624
$(1, 3, 2)^T$	$\lambda_1 = 16,\ \lambda_2 = 8$	11	0.0468
	$\lambda_1 = 8,\ \lambda_2 = 8$	20	0.1248
	$\lambda_1 = 7,\ \lambda_2 = 18$	17	0.1092
	$\lambda_1 = 15,\ \lambda_2 = 15$	10	0.0624
	$\lambda_1 = 16,\ \lambda_2 = 80$	7	0.0156

Problem 3 We consider the well-known Rosen-Suzuki optimization problem and its reformulation as an equilibrium problem [34]. The equilibrium function f is given for each $x, y \in \mathbb{R}^4$ by $f(x, y) = \phi(y) - \phi(x)$ with the function ϕ defined for $x = (x_1, x_2, x_3, x_4)$ by

$$\phi(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4.$$

The constraint set is given by $C = \{x \in \mathbb{R}^4 \mid g_i(x) \leq 0, i = 1, 2, 3\}$, where

$$\begin{aligned} g_1(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8, \\ g_2(x) &= x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10, \\ g_3(x) &= 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5. \end{aligned}$$

The optimal solution of this problem is $x^* = (0, 1, 2, -1)^T$. Let us note that here the operator

$$B(x) = \nabla_2 f(x, x) = (2x_1 - 5, 2x_2 - 5, 4x_3 - 21, 2x_4 + 7)^T$$

is co-coercive with modulus $\sigma = 0.25$ and strongly monotone with modulus $\delta = 2.0$. The results obtained by using the Glowinski–Le Tallec splitting method on this problem are reported in Table 3. Here we can observe that the best choice for the parameter λ_1 is to take it equal to 0.5.

In the next table, we compare the Glowinski–Le Tallec method (GLM) with the Gradient Projection method (GPM) [1], the Extragradient method (EGM) [36, Algorithm 1] and the Relaxed Projection method (RPM) [34]. The number of iterations and the CPU time (s) needed to get a solution are reported. As in [34,36], we choose the starting point $x_0 = (1, 3, 1, 1, 2)^T$ for Problem 1, $x_0 = (0, 0, 0)^T$ for Problem 2 and $x_0 = (5, -5, 5, -5)^T$ for Problem 3. For (GLM), we define $\lambda_1 = \lambda_2 = \sigma$ and for (GPM) and (EGM) we choose the optimal stepsize, that is, $\lambda = \sigma$ where σ is the co-coercivity modulus of each problem. The stepsizes and parameters for (RPM) are taken as in [34]. For Problem 3, we choose $\alpha_k = 0.25(k^2 - 1)/k^2$, which is the best value given in [34, Table 4]. Observe that, as showed in the above tables, the performance of (GLM) can be improved by choosing different stepsizes λ_1, λ_2. For example, results for $\lambda_1 = 0.2, \lambda_2 = 0.5$ for Problem 1, and $\lambda_1 = 15, \lambda_2 = 10$ for Problem 2 are displayed in the Fig. 1 below.

Table 3 The results of the Rosen–Suzuki optimization problem

Starting point	Parameters	Number of iterations	CPU time (s)
$(1, -1, 2, -3)^T$	$\lambda_1 = 0.4,\ \lambda_2 = 0.3$	7	0.2028
	$\lambda_1 = 0.5,\ \lambda_2 = 0.3$	1	0.0468
	$\lambda_1 = 0.4,\ \lambda_2 = 0.25$	6	0.1716
	$\lambda_1 = 0.3,\ \lambda_2 = 0.5$	11	0.3276
	$\lambda_1 = 0.5,\ \lambda_2 = 7.0$	1	0.0468
$(2, 2, -2, -5)^T$	$\lambda_1 = 0.3,\ \lambda_2 = 0.2$	8	0.2028
	$\lambda_1 = 0.4,\ \lambda_2 = 0.25$	10	0.2808
	$\lambda_1 = 0.3,\ \lambda_2 = 0.6$	13	0.3588
	$\lambda_1 = 0.25,\ \lambda_2 = 0.25$	11	0.2964
	$\lambda_1 = 0.5,\ \lambda_2 = 0.5$	15	0.3432

Table 4 Comparison of Glowinski–Le Tallec method (GLM) with Gradient Projection method (GPM), Extragradient method (EGM) and Relaxed Projection method (RPM)

	GLM		GPM		EGM		RPM	
	Iter	CPU time	Iter	CPU time	Iter	CPU time	Iter	CPU time
Problem 1	17	0.1404	46	0.0936	23	0.5148	41	0.2808
Problem 2	20	0.1248	52	0.156	26	0.6864	25	0.1092
Problem 3	4	0.1092	5	0.078	8	1.07641	13	0.4368

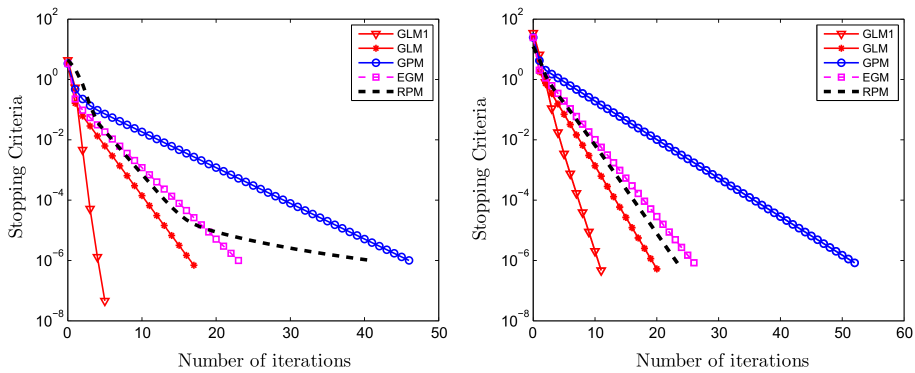

Fig. 1 Comparison of different methods for Problem 1 (left) with the parameters chosen above, $\lambda_1 = 0.2$, $\lambda_2 = 0.5$ for (GLM1), and for Problem 2 (right) with $\lambda_1 = 15$, $\lambda_2 = 10$ for (GLM1)

5 Conclusion

In this paper, the linear rate of convergence of the Glowinski–Le Tallec splitting method was studied for finding a zero of the sum of two maximal monotone operators in Hilbert spaces. The aim was to apply this method for solving equilibrium problems and to obtain numerical results showing the efficiency of the new approach. It is proved that the sequence generated by the new method converges to a global solution of the considered equilibrium problem.

Theoretical results are confirmed by numerical experiments. The comparison of the new method with some others is also presented. It seems that the Glowinski–Le Tallec splitting has a very good numerical performance both in terms of the number of iterations and the CPU times, especially when the parameter λ_2 increases and approaches $+\infty$. Many other test problems should be considered and other choices of the parameters λ_1 and λ_2 should be studied to improve the performance of this method.

Acknowledgements The authors would like to thank the Associate Editor and the two anonymous referees for their useful remks, comments and suggestions that allowed to improve substantially the original version of this paper. This work was mostly carried out when the first author was a PhD student working at the Institute for Computational Science and Technology—Ho Chi Minh City, Vietnam. This research was supported by this Institute and partly, for the first author, by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) Grant 101.01-2017.315 and the Austrian Science Foundation (FWF), Grant P26640-N25.

References

1. Anh, P.N., Muu, L.D., Nguyen, V.H., Strodiot, J.J.: On the contraction and nonexpansiveness properties of the marginal mappings in generalized variational inequalities involving co-coercive operartors. In: Eberhard, A., Hadjisavvas, N., Luc, D.T. (eds.) Generalized Convexity, Generalized Monotonicity and Applications. Nonconvex Optimization and Its Applications, vol. 77, pp. 89–111. Springer, Boston (2005)
2. Balashov, M.V., Golubev, M.O.: About the Lipschitz property of the metric projection in the Hilbert space. J. Math. Anal. Appl. **394**, 545–551 (2012)
3. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, Berlin (2010)
4. Bigi, G., Castellani, M., Pappalardo, M., Passacantando, M.: Existence and solution methods for equilibria. Eur. J. Oper. Res. **227**, 1–11 (2013)
5. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
6. Briceno-Arias, L.: A Douglas–Rachford splitting method for solving equilibrium problems. Nonlinear Anal. **75**, 6053–6059 (2012)
7. Chen, G., Rockafellar, R.T.: Convergence and Structure of Forward–backward Splitting Methods. Technical Report, Department of Applied Mathematics, University of Washington (1990)
8. Chen, G., Rockafellar, R.T.: Extended Forward–backward Splitting Methods and Convergence. Technical Report, Department of Applied Mathematics, University of Washington (1990)
9. Chen, G., Rockafellar, R.T.: Forward–Backward Splitting Methods in Lagrangian Optimization. Technical Report, Department of Applied Mathematics, University of Washington (1992)
10. Douglas, J., Rachford, H.H.: On the numerical solution of the heat conduction problem in 2 and 3 space variables. Trans. Am. Math. Soc. **82**, 421–439 (1956)
11. Eckstein, J., Bertsekas, D.P.: On the Douglas–Rachford splitting method and the proximal point algorithm for maximal monotone operators. Math. Program. **55**, 293–318 (1992)
12. Eckstein, J., Svaiter, B.F.: A family of projective splitting methods for the sum of two maximal monotone operators. Math. Program. **111**, 173–199 (2008)
13. Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequality III, pp. 103–113. Academic Press, New York (1972)
14. Glowinski, R., Le Tallec, P.: Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics. SIAM, Philadelphia (1989)
15. Haubruge, S., Strodiot, J.J., Nguyen, V.H.: Convergence analysis and applications of the Glowinski–Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators. J. Optim. Theory Appl. **97**, 645–673 (1998)
16. He, B., Yuan, X.: On the convergence rate of Douglas–Rachford operator splitting method. Math. Program. **153**, 715–722 (2015)
17. Iusem, A.N., Sosa, W.: On the proximal point method for equilibrium problems in Hilbert spaces. Optimization **59**, 1259–1274 (2010)
18. Khanh, P.D., Vuong, P.T.: Modified projection method for strongly pseudomonotone variational inequalities. J. Glob. Optim. **58**, 341–350 (2014)

19. Krawczyk, J.B., Uryasev, S.: Relaxation algorithms to find Nash equilibria with economic applications. Environ. Model. Assess. **5**, 63–73 (2000)
20. Lions, P.L., Mercier, B.: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. **16**, 964–979 (1979)
21. Luo, Z.Q., Tseng, P.: Error bound and convergence analysis of matrix splitting algorithms for the affine variational inequality problem. SIAM J. Optim. **2**, 43–54 (1992)
22. Lyashko, S.L., Semenov, V.V., Voitova, T.A.: Low cost modification of Korpelevich's methods for monotone equilibrium problems. Cybern. Syst. Anal. **47**, 631–639 (2011)
23. Marcotte, P., Wu, J.H.: On the convergence of projection methods: application to the decomposition of affine variational inequalities. J. Optim. Theory Appl. **85**, 347–362 (1995)
24. Mastroeni, G.: On auxiliary principle for equilibrium problems. In: Daniele, P., Gianessi, F., Maugeri, A. (eds.) Equilibrium Problems and Variational Models, pp. 289–298. Kluwer, Dordrecht (2003)
25. Moudafi, A.: On the convergence of splitting proximal methods for equilibrium problems in Hilbert spaces. J. Math. Anal. Appl. **359**, 508–513 (2009)
26. Muu, L.D., Quoc, T.D.: Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash–Cournot equilibrium model. J. Optim. Theory Appl. **142**, 185–204 (2009)
27. Muu, L.D.: Stability property of a class of variational inequalities. Optimization **15**, 347–353 (1984)
28. Muu, L.D., Oettli, W.: Convergence of an adaptive penalty scheme for finding constraint equilibria. Nonlinear Anal. **18**, 1159–1166 (1992)
29. Nguyen, T.T.V., Strodiot, J.J., Nguyen, V.H.: A bundle method for solving equilibrium problems. Math. Program. **116**, 529–552 (2008)
30. Nguyen, T.T.V., Strodiot, J.J., Nguyen, V.H.: The interior proximal extragradient method for solving equilibrium problems. J. Glob. Optim. **44**, 175–192 (2009)
31. Passty, G.B.: Ergodic convergence to a zero of the sum of monotone operators in Hilbert Space. J. Math. Anal. Appl. **72**, 383–390 (1979)
32. Peaceman, D.H., Rachford, H.H.: The numerical solution of parabolic elliptic differential equations. SIAM J. Appl. Math. **3**, 28–41 (1955)
33. Phan, H.M.: Linear convergence of the Douglas–Rachford method for two closed sets. Optimization **65**, 369–385 (2016)
34. Scheimberg, S., Santos, P.S.M.: A relaxed projection method for finite dimensional equilibrium problems. Optimization **60**, 1193–1208 (2011)
35. Strodiot, J.J., Vuong, P.T., Nguyen, T.T.V.: A class of shrinking projection extragradient methods for solving non-monotone equilibrium problems in Hilbert spaces. J. Glob. Optim. **64**, 159–178 (2016)
36. Tran, D.Q., Le Dung, M., Nguyen, V.H.: Extragradient algorithms extended to equilibrium problems. Optimization **57**, 749–776 (2008)
37. Vial, J.-P.: Strong and weak convexity of sets and functions. Math. Oper. Res. **8**, 231–259 (1983)
38. Zeng, L.C., Yao, J.Y.: Modified combined relaxation method for general monotone equilibrium problems in Hilbert spaces. J. Optim. Theory Appl. **131**, 469–483 (2006)
39. Zhu, C.: Asymptotic convergence nnalysis of the forward-backward splitting algorithm. Math. Oper. Res. **20**, 449–464 (1995)