Purely Catalytic P Systems over Integers and Their Generative Power

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Abstract. We further investigate the computing power of the recently introduced P systems with $\mathbb{Z}$-multisets (also known as hybrid sets) as generative devices. These systems apply catalytic rules in the maximally parallel way, even consuming absent non-catalysts, thus effectively generating vectors of arbitrary (not just non-negative) integers. The rules may only be made inapplicable by dissolution rules. However, this releases the catalysts into the immediately outer region, where new rules might become applicable to them. We discuss the generative power of this model. Finally, we consider the variant with mobile catalysts.

1 Introduction

Membrane systems (cell-like, with symbol objects) have traditionally been viewed as collections of hierarchically arranged multiset processors, e.g., see [12]. In the list of open problems disseminated in 2015, see [11], Gheorghe Păun suggested to go beyond the traditional setting where symbol multiplicities in multisets are restricted to non-negative integers. In [6] generalized multisets are defined as taking multiplicities from arbitrary finitely generated, totally ordered commutative groups.

In [3], a different approach is taken: only catalytic rules are allowed, and the applicability of a rule only depends on the presence of the corresponding catalyst in the given region. Consuming an absent non-catalyst makes its multiplicity negative. While in [3] it was already established that such a model is not universal, we found it interesting to investigate its generative power more precisely.

Since the number of catalysts remains finite and does not change throughout the computation, this induces a finite set of "rule teams " which can be applied

© Springer International Publishing AG 2017
A. Leporati et al. (Eds.): CMC 2016, LNCS 10105, pp. 67–82, 2017.
DOI: 10.1007/978-3-319-54072-6_5

in parallel in one step. The virtual absence of applicability conditions and the finiteness of the "teams" hints at the possibility of seeing them as integer vectors; in this case the P system itself can be seen as evolving by sequentially adding such vectors (possibly having negative components) to the contents of its membranes. In [2], this general model is compared with vector addition systems (see [5,9] for standard definitions; adapted to allow negative vector components in [8]) and blind register machines, e.g., see [7].

Here we return to the particular model from [3], discussing the lower bound of its generative power and giving some results on the variants with target indications and with multiple membrane dissolutions.

2 Preliminaries

The reader is assumed to be familiar with the basic notions of formal languages and membrane computing; see [13] for a comprehensive introduction and the webpage [14] of P systems.

We only recall a few basic notations as they are used in this paper: $\mathbb{N}$ denotes the set of non-negative integers; $\mathbb{Z}$ denotes the set of integers. Multisets over the set of objects O can be seen as functions from O to $\mathbb{N}$; thus, the set of all multisets over O is $\mathbb{N}^O$ (the set of all functions from O to $\mathbb{N}$). In membrane computing, the set of all multisets over O is commonly denoted by $O^\circ = \mathbb{N}^O$, while the multisets themselves are represented by strings in O^*, keeping in mind that the order of symbols is not relevant.

2.1 Extending Multisets

To represent also negative multiplicities, multisets over a set of objects O have to be extended to $\mathbb{Z}$. A $\mathbb{Z}$-multiset, allowing integer multiplicities (called a *hybrid set* in [4]) then is from $\mathbb{Z}^O$; it can be represented by a string in $(O \cup O^-)^*$, where $O^- = \{a^- \mid a \in O\}$ is a set of symbols that represents objects in multiplicity "negative one". Note that, as opposed to P systems with matter/antimatter [1], symbol a^- here is not an actual object, but simply a convenient way to represent a deficit of a, and the actual multiplicity of a represented by a string w is $|w|_a - |w|_{a^-}$. We also do not distinguish between notations a^{-k} and $(a^-)^k$. The superscript $^-$ can be used as a morphism, producing a multiset with opposite multiplicities, e.g., $(a^k)^-$ represents the same $\mathbb{Z}$-multiset as a^{-k}. As the strings here are only used to represent $\mathbb{Z}$-multisets, we may write an equality sign between the strings representing the same $\mathbb{Z}$-multiset. For conciseness, let us use the notation $O^\bullet = (O \cup O^-)^*$. Finally, since it will be always clear from the context, we may call an element of $O^\bullet$ "multiset", omitting the word "representing". Assuming an order is fixed on O, for $u \in O^\bullet$, vector $(|u|_a - |u|_{a^-})_{a \in O}$ is denoted by $\psi_O(u)$; the subscript O may be omitted when it is clear from the context. This vector is called the *Parikh image* of u.

2.2 Linear Sets

The *linear* set generated by a set of vectors $A = \{\mathbf{a}_i \mid 1 \leq i \leq d\} \subset_{fin} \mathbb{Z}^n$ (here $A \subset_{fin} B$ indicates that A is a finite subset of B) and an offset $\mathbf{a}_0 \in \mathbb{Z}^n$ is defined as follows:

$$\langle A, \mathbf{a}_0 \rangle_{\mathbb{N}} = \left\{ \mathbf{a}_0 + \sum\nolimits_{i=1}^{d} k_i \mathbf{a}_i \mid k_i \in \mathbb{N},\ 1 \leq i \leq d \right\}.$$

If the offset $\mathbf{a}_0$ is the zero vector, we will call the corresponding linear set *homogeneous*; we also will use the short notation $\langle A \rangle_{\mathbb{N}} = \langle A, \mathbf{0} \rangle_{\mathbb{N}}$.

We use the notation $\mathbb{Z}^n LIN_{\mathbb{N}} = \{\langle A, \mathbf{a}_0 \rangle_{\mathbb{N}} \mid A \subset_{fin} \mathbb{Z}^n,\ \mathbf{a}_0 \in \mathbb{Z}\}$, to refer to the class of all linear sets. Semilinear sets are defined as finite unions of linear sets. We use the notations $\mathbb{Z}^n SLIN_{\mathbb{N}}$ to refer to the classes of semilinear sets of n-dimensional vectors. In case no restriction is imposed on the dimension, n is replaced by $*$. We may omit n if $n = 1$. A finite union of linear sets which only differ in the starting vectors is called *uniform* semilinear:

$$\mathbb{Z}^n SLIN_{\mathbb{N}}^{U} = \left\{ \bigcup\nolimits_{\mathbf{b} \in B} \langle A, \mathbf{b} \rangle_{\mathbb{N}} \mid A \subset_{fin} \mathbb{Z}^n,\ B \subset_{fin} \mathbb{Z}^n \right\}$$

Let us denote such a set by $\langle A, B \rangle_{\mathbb{N}}$.

Note that a uniform semilinear set $\langle A, B \rangle_{\mathbb{N}}$ can be seen as a pairwise sum of the finite set B and the homogeneous linear set $\langle A \rangle_{\mathbb{N}}$:

$$\langle A, B \rangle_{\mathbb{N}} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \langle A \rangle_{\mathbb{N}}, \mathbf{b} \in B\}.$$

This observation immediately yields the conclusion that the sum of two uniform semilinear sets $\langle A_1, B_1 \rangle_{\mathbb{N}}$ and $\langle A_2, B_2 \rangle_{\mathbb{N}}$ is uniform semilinear as well and can be computed in the following way:

$$\langle A_1, B_1 \rangle_{\mathbb{N}} + \langle A_2, B_2 \rangle_{\mathbb{N}} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \langle A_1 \cup A_2 \rangle_{\mathbb{N}}, \mathbf{b} \in B_1 + B_2\}.$$

3 Purely Catalytic P Systems over Integers

In purely catalytic P systems over integers the set of objects is a disjoint union of catalysts C and the regular objects O. The regular objects are allowed to have any integer multiplicity, while the catalysts are only allowed to appear in a non-negative number of copies.

Formally, a *(generating) purely catalytic P system over integers* Π is defined as a construct

$$\Pi = (O, C, \mu, w_1, \cdots, w_n, R_1, \cdots, R_n, i_0)$$

where

- O is a finite set of objects,
- $C \subset O$ is the set of catalysts,
- μ is a membrane structure of n hierarchically arranged membranes, labeled by 1 to n,

– R_i, $1 \leq i \leq n$, is the set of rules associated with membrane region i, and
– i_0 is the output membrane.

The rules can be of the two following types:

– *catalytic rules*: $cu \to cv$, where $c \in C$ and $u, v \in O^*$;
– *catalytic rules with dissolution*: $cu \to cv\delta$, where $c \in C$, $u, v \in O^*$, and $\delta \notin C \cup O$ is the symbol indicating membrane dissolution.

The rules applied in parallel cannot involve more catalysts than available in the system; the multiplicities of regular objects, on the other hand, do not influence the applicability of rules. An application of a rule $cu \to cv$ in a region containing cw, $c \in C$, $u, v \in O^*$, $w \in O^\bullet$, produces $cw(cu)^- cv = cwv(u^-)$, or, in terms of vectors, ignoring the catalyst, the vector $\psi(w) + \psi(v) - \psi(u)$ represents the contents of that region after the rule has been applied. An application of a rule $cu \to cv\delta$ produces the same effect, and then dissolves the enclosing membrane, moving the contents of the dissolved membrane into the parent membrane.

Purely catalytic P systems over integers evolve under the maximally parallel semantics, so each catalyst induces the application of exactly one rule (nondeterministically chosen), unless the given region has no rules associated with this catalyst. A computation is called successful if no rule is applicable any more, i.e., if the system *halts*.

By $\mathbb{Z}^d O_{\mathbb{Z}} P_m(pcat_k, \delta)$ we denote the family of sets of d-dimensional vectors of integers generated by purely catalytic P systems over integers with dissolution, at most m membranes and at most k catalysts. If any of parameters d, m, k is unbounded, it is replaced by $*$ in this notation.

We also use notations for extended features (listed in parentheses in the notation of the sets of $\mathbb{Z}$-vectors generated by the corresponding families of P systems). Target indications, denoted by *tar*, allow the non-catalysts to be sent to a different membrane. In the right side of the rules, sending object a is written by (a, tar), where $tar \in \{out\} \cup \{in_j \mid 1 \leq j \leq m\}$; j here is the label of an immediately inner membrane. In this paper, we write $tar_{\mathbb{Z}}$ in the notation of a set of $\mathbb{Z}$-vectors generated by a family of P systems; this generalization reflects the possibility to assign targets even to negative multiplicities of objects.

Another feature is *mobile catalysts*, e.g., see [10], i.e., targets may also be associated to the catalysts, and thus the catalysts move across the membrane structure; we denote this feature by $mpcat_k$ since the systems we consider are purely catalytic. We use the plus sign between the features of catalytic mobility and dissolution when it is allowed for the *same* rule to move a catalyst and to dissolve the membrane currently containing it.

4 Results

Before elaborating our results and giving characterizations of various families of linear sets by purely catalytic P systems, we discuss some general observations which allow us to simplify these systems.

4.1 Observations and Simplifications

First, we would like to explicitly allow rules of the form $c \to cx$, $(c \in C, x \in O^\bullet)$, i.e., the multiset of regular objects in the left side being empty. This does not change the model, since any $\mathbb{Z}$-multiset x can be written as $u(v^-)$, $u, v \in O^*$, and, fixing some $a \in O$, $c \to cx$ is equivalent to $cau \to av$. Moreover, any rule $cu \to cv$ is equivalent to $c \to cu(v^-)$, so it suffices to only consider rules of types $c \to cx$ and $c \to cx\delta$ $(c \in C, x \in O^\bullet)$.

Second, we claim that it is enough to start with a single catalyst in every region. To show that, we will consider that membrane i of the P system contains the catalysts $c_{i,k}$, $1 \leq k \leq n_i$, in the *initial* configuration. Now we will define the sets $X_{i,k,j}$ of right-hand sides of catalytic rules of membrane j involving the catalysts initially located in membrane i:

$$X_{i,k,j} = \{x \in O^\bullet \cup O^\bullet\delta \mid (c_{i,k} \to c_{i,k}x) \in R_j\},$$

where R_j is the set of rules associated with membrane j. To simplify subsequent explanations, we will adopt the following convention: If, for a given i and j, there exists such a k that $X_{i,k,j} \neq \emptyset$, then we will replace all empty $X_{i,k',j}$ by $\{\lambda\}$.

We now remark that the catalysts $c_{i,k}$ initially present in membrane i will always stay together, because dissolution cannot separate them. We will replace each such group ("band") by a new catalyst having the combined effect of the group. More formally, we will replace all the catalysts $c_{i,k}$ initially present in membrane i by one new catalyst c_i, and the rules associated with membrane j by the following set:

$$R'_j = \{c_i \to c_i x_1 \cdots x_{n_i} \mid x_k \in X_{i,k,j}, 1 \leq k \leq n_i, 1 \leq i \leq m\},\ 1 \leq j \leq m,$$

where m is the number of membranes of the membrane system. Note that R'_j contains rules for every catalyst c_i representing the original "band" from membrane i which may have an effect in region j.

The argument in the previous paragraph shows that we can replace multiple catalysts in a region by a single one. On the other hand, having no catalyst in some region is equivalent to having one catalyst with no associated rules. Therefore, without restricting the generality, in the following we assume that in the initial configuration of an arbitrary purely catalytic P system over integers, each membrane region i, $1 \leq i \leq m$, contains precisely one catalyst, and we can call it c_i.

Third, notice that the symbols may only travel from the inner membranes to the outer ones, so if the output region i_0 is not the skin, only the contents of the membrane substructure inside i_0 (including i_0) is relevant for the result. The only way in which a membrane i not contained within i_0 could influence the evolution of the system is by preventing it from halting. If the computations inside the substructure of membrane i halt after i_0, but in a finite number of steps, then this only influences the moment when we are allowed to retrieve the result, but not the result itself. If the computations inside the substructure of membrane i never halt, then the result of the system is always empty. If both

halting in a finite number of steps or never halting are an option, then it is sufficient to only consider the halting computations, and, as we have just shown, in this case what happened inside the substructure of membrane i does not have any influence on the result of the whole system.

According to this reasoning, the membranes not contained within the output membrane i_0 may influence the power of the system only trivially (by reducing its result to the empty set). We will therefore assume that the output region is always the skin.

Fourth, every elementary membrane having no rules associated to the catalysts available there may be removed from the system without affecting the result (unless it is the output membrane, in which case a singleton is generated, which is a degenerate case), so in the following we assume that every elementary membrane has some applicable rules. Clearly, the P system will not reach the halting until all these membranes are dissolved.

Consider this reasoning starting from the elementary membranes, by induction. Take any non-elementary membrane i which becomes elementary during a computation. Assume i is not dissolved (i.e., it has no rules associated to any of the catalysts that were placed within the membrane substructure inside i, including i), but it is not the output membrane. Then all the computations in the membrane substructure inside i, including i, do not contribute to the result, and can be removed from the system without affecting the result.

As a summary of the fourth observation, without restricting the generality (except, possibly the degenerate cases generating the empty set or some singleton), we may assume that any purely catalytic P system over integers has applicable rules associated to all elementary membranes, and all membranes except the skin must be dissolved at some moment during the computation.

Finally, for every region except the skin, a catalyst c_i without associated rules is equivalent to a catalyst with a rule $c_i \to c_i$. Hence, without restricting the generality, we may assume that each catalyst is *never* idle before it reaches the skin membrane. Clearly, (excluding the degenerate case generating the empty set), the skin should have no rules associated to any catalyst of the system.

We would like to note that even without pruning the membrane structure by removing membrane substructures not contributing to the result, the membrane structure obtained at halting (if at all reachable) is unique.

We recall that in [2], the following generalized approach is taken – there is a finite number of reachable membrane structures. These could be used as states of a sequential P system, which may be obtained, separately for each membrane structure, by combining the behavior of all catalysts in all regions of the P system. Indeed, having fixed a reachable membrane structure, we know which membranes have been dissolved, and thus the resulting location of each catalyst. Then, for each catalyst, associated rules in its current location are considered and combined, similarly to the second observation above, but globally. Having obtained a sequential system, the catalyst is no longer needed. Then, in [2] it was shown that such a generalization is nothing else but a sequential blind vector addition system with states, and it was claimed that it characterizes precisely the family of all semilinear vectors of integers.

Indeed, in this way any purely catalytic P system over integers can be substituted by a sequential blind vector addition system with states, so the upper bound of the family of all semilinear sets of vectors of integers, or, equivalently, the family of all integer vector sets, generated by blind register machines, holds. However, the reverse is not necessarily true, i.e., it does not follow that for any sequential blind vector addition system with states there would exist an equivalent purely catalytic P system over integers.

In the present paper we investigate the particularities of how dissolution affects the computation, and the lower bounds.

4.2 Generative Power

Since, by definition, the output region cannot be dissolved and any other applicable rule can never be stopped, single-membrane purely catalytic P systems over integers are degenerate (we will not mention these degenerate cases any more when considering multiple membranes):

Theorem 1. $\mathbb{Z}^d O_{\mathbb{Z}} P_1(pcat_*, \delta) = \{\emptyset\} \cup \{\{\mathbf{a}\} \mid \mathbf{a} \in \mathbb{Z}^d\}$.

With two membranes, a characterization is still straightforward:

Theorem 2. $\mathbb{Z}^d O_{\mathbb{Z}} P_2(pcat_*, \delta) = \mathbb{Z}^d SLIN_{\mathbb{N}}^U$.

Proof. Let A be the finite set of vectors corresponding to the non-dissolving rules in the elementary membranes, and let B be the finite set of sums of two vectors: the one corresponding to the initial configuration and vectors corresponding to the dissolving rules in the elementary membrane; the skin should have no rules. If the catalyst in the elementary membrane is c_2, then the correspondence mentioned above is $c_2 \to c_2 x \leftrightarrow \psi(x)$, and similarly with dissolution. An arbitrary computation of a P system consists of an arbitrary number of applications of non-dissolving rules and one application of a dissolving rule. Hence, the resulting vector sums up from the "initial" vector, one arbitrary "dissolving" vector, and an arbitrary linear combination of "non-dissolving" vectors. □

It is worth noting that, by a similar reasoning, for a P system with multiple membranes, if the chronological order of dissolving the membranes is fixed, the result is still $\mathbb{Z}^d SLIN_{\mathbb{N}}^U$. Indeed, each combination of rules (one for each catalyst) yields one vector, so all such possible combinations of non-dissolving rules yield a finite set of vectors, and multiple non-dissolving steps yield a linear set generated by these vectors. Thus, over the whole computation the result sums up from the initial configuration, a finite number of dissolution vectors, and a finite number of linear sets corresponding to the membrane structures reached during that computation. Since the total number of chronological orders of dissolving membranes is bounded, we infer the following known result:

Theorem 3. $\mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_*, \delta) \subseteq \mathbb{Z}^d SLIN_{\mathbb{N}}$.

P systems with more than two membranes

Even with three membranes, in case two of them are elementary, the power of such purely catalytic P systems over integers is still $\mathbb{Z}^d SLIN_{\mathbb{N}}^U$, but for a different reason: each elementary membrane contributes with its uniform semilinear set, and a sum of two uniform semilinear sets is still uniform semilinear.

Let us now examine a P system with three nested membranes – as we will see, the minimal number to obtain a set which is not in $\mathbb{Z}^d SLIN_{\mathbb{N}}^U$. Let the vector obtained by joining the initial contents of all membranes be $\mathbf{a_0}$. In the elementary membrane 3, there are two sets of integers associated with the catalyst c_3: the set of non-dissolving vectors is denoted by $A_{3,3}$, the set of dissolving vectors by $B_{3,3}$. In the middle membrane 2, associated with the catalyst c_2, we have the sets of non-dissolving and dissolving vectors $A_{2,2}$ and $B_{2,2}$. When c_3 arrives in membrane 2, other rules may become applicable there with this catalyst, and we denote the corresponding sets of integers associated with catalyst c_3 in membrane 2 by $A_{3,2}$ and $B_{3,2}$. As we may assume the P system to be reduced according to the observations above, we collect the result of halting computations in the skin membrane and therefore, no rules are assigned to any catalyst there. Let us now see what the resulting vector set may be built from, besides $\mathbf{a_0}$.

In each non-dissolving computation step, in the inner two membranes, an element of $A_{3,3}$ is added to the elementary membrane 3 and an element of $A_{2,2}$ is added to the middle membrane 2; throughout every successful computation, nothing is added any more in the skin membrane, although finally all objects will arrive to the skin to constitute the result of a halting computation. Hence, during that non-dissolving phase of a computation, we get an arbitrary element from $\langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}}$. Then there are three possibilities:

1. Both membranes 2 and 3 are dissolved at the same moment, hence, by such halting computations in total we get
$$\mathbf{a_0} + \langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} + (B_{2,2} + B_{3,3}) .$$
2. If membrane 2 is dissolved first, then the system continues by only applying the rules in membrane 3 (no rules are applicable in the skin membrane), and eventually dissolving membrane 3, we have obtained an element from $(B_{2,2} + A_{3,3}) + \langle A_{3,3} \rangle_{\mathbb{N}} + B_{3,3}$ during this phase of the computation, hence, by such halting computations in total we get
$$\mathbf{a_0} + \langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} + \big((B_{2,2} + A_{3,3}) + \langle A_{3,3} \rangle_{\mathbb{N}} + B_{3,3}\big) .$$
As the zero-vector is an element of $\langle A_{3,3} \rangle_{\mathbb{N}}$, we observe that this expression can be combined with the previous one, thus in sum yielding
$$\mathbf{a_0} + \langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} + B_{3,3} + B_{2,2} + \langle A_{3,3} \rangle_{\mathbb{N}} .$$
3. If membrane 3 is dissolved first, then both catalysts are active in membrane 2; eventually dissolving it, this phase of the computation yields an element from
$$B_{3,3} + \langle A_{2,2} + A_{3,2} \rangle_{\mathbb{N}} + ((A_{3,2} + B_{2,2}) \cup (A_{2,2} + B_{3,2}) \cup (B_{3,2} + B_{2,2})) .$$

The three expressions $(A_{3,2} + B_{2,2}), (A_{2,2} + B_{3,2}), (B_{3,2} + B_{2,2})$ describe the three possibilities for dissolving membrane 2. Hence, by such halting computations in total we get

$$\mathbf{a_0} + \langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} + B_{3,3} \; + \\ \langle A_{2,2} + A_{3,2} \rangle_{\mathbb{N}} + ((A_{3,2} + B_{2,2}) \cup (A_{2,2} + B_{3,2}) \cup (B_{3,2} + B_{2,2})) .$$

Therefore, the total set of integer vectors generated by such a purely catalytic P system over integers with three nested membranes is

$$\mathbf{a_0} + B_{3,3} + \langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} + \Big(\big(B_{2,2} + \langle A_{3,3} \rangle_{\mathbb{N}}\big) \cup \\ \big(\langle A_{2,2} + A_{3,2} \rangle_{\mathbb{N}} + ((A_{3,2} + B_{2,2}) \cup (A_{2,2} + B_{3,2}) \cup (B_{3,2} + B_{2,2})) \big) \Big),$$

An immediate reduction of this formula is to omit $\mathbf{a_0}$ as it can be included in $B_{3,3}$, hence, from now on we omit $\mathbf{a_0}$. Let us denote the resulting expression by

$$M(B_{3,3}, B_{2,2}, B_{3,2}, A_{3,3}, A_{2,2}, A_{3,2}) .$$

We also note that the power of the nested case subsumes the power of the case with two elementary membranes, as in that case we only get some uniform semilinear set, which formally can also be described by

$$M(B, \emptyset, \emptyset, A, \emptyset, \emptyset) ,$$

although, using our notation, the correct formula would be

$$\big(\langle A_{2,2} \rangle_{\mathbb{N}} + B_{2,2}\big) + \big(\langle A_{3,3} \rangle_{\mathbb{N}} + B_{3,3}\big) .$$

Therefore the power of all purely catalytic P systems over integers with three membranes can be described as follows:

$$\mathbb{Z}^d O_{\mathbb{Z}} P_3 (pcat_*, \delta) = \{M(B_{3,3}, B_{2,2}, B_{3,2}, A_{3,3}, A_{2,2}, A_{3,2}) \mid \\ B_{3,3}, B_{2,2}, B_{3,2}, A_{3,3}, A_{2,2}, A_{3,2} \subset_{fin} \mathbb{Z}^d\},$$

where $M(B_{3,3}, B_{2,2}, B_{3,2}, A_{3,3}, A_{2,2}, A_{3,2})$ is the expression defined above.

Unfortunately, it is not obvious what more could be simplified in this expression, hence, to achieve our goal of showing how to get a non-uniform semilinear set, we have to consider some particular cases.

The expression $M(B_{3,3}, B_{2,2}, B_{3,2}, A_{3,3}, A_{2,2}, A_{3,2})$ contains the three bounded B-terms $B_{3,3}$, $B_{2,2}$, $B_{3,2}$, but the three independent finite sets of vectors $A_{3,3}$, $A_{2,2}$, $A_{3,2}$ appear as the three unbounded terms $\langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}}$, $\langle A_{2,2} + A_{3,2} \rangle_{\mathbb{N}}$, $\langle A_{3,3} \rangle_{\mathbb{N}}$ which are not independent any more. It is, however, possible to separate them in some particular cases.

Let us choose $A_{3,3}$ to be a singleton, i.e., $A_{3,3} := \{\mathbf{e}\}$ for some vector $\mathbf{e}$, and $A_{2,2} = -A_{3,3} := \{-\mathbf{e}\}$, as well as, for some non-empty set of vectors C, $A_{3,2} := C - A_{2,2}$. Hence, the first unbounded term $\langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}}$ does not

contribute any more, as $\langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} = \langle \mathbf{0} \rangle_{\mathbb{N}} = \mathbf{0}$, where $\mathbf{0}$ denotes the zero-vector. Moreover, we get $\langle A_{2,2} + A_{3,2} \rangle_{\mathbb{N}} = \langle C \rangle_{\mathbb{N}}$ and $\langle A_{3,3} \rangle_{\mathbb{N}} = \langle \{\mathbf{e}\} \rangle_{\mathbb{N}}$.

If we choose $B_{3,3} = B_{2,2} = B_{3,2} = \{\mathbf{0}\}$, the expression for M simplifies to

$$\langle \{\mathbf{e}\} \rangle_{\mathbb{N}} \cup \Big(\langle C \rangle_{\mathbb{N}} + \big((C - \{-\mathbf{e}\}) \cup \{-\mathbf{e}\} \big) \Big)$$

or to $\langle \{\mathbf{e}\} \rangle_{\mathbb{N}} \cup \Big(\langle C \rangle_{\mathbb{N}} + \big((C + \{\mathbf{e}\}) \cup \{-\mathbf{e}\} \big) \Big)$, respectively, which can be seen as the union of a homogenous linear set and a uniform semilinear set. As e and C can be chosen independently from each other, we conclude

$$\mathbb{Z}^d O_{\mathbb{Z}} P_n(pcat, \delta) \supsetneq \mathbb{Z}^d SLIN_{\mathbb{N}}^U, \quad \text{for all} \quad n \geq 3.$$

Alternatively, to avoid dealing with the union of three cases when membrane 2 is divided last, we may choose $B_{2,2} = A_{2,2}$ and $B_{3,2} = A_{3,2}$; then the last parenthesis in the general expression of set M simply reduces to $A_{2,2} + A_{3,2}$. Let us define $C := A_{2,2} + A_{3,2}$ and, moreover, choose $B_{3,3} := \{\mathbf{0}\}$, $A_{3,3} := \{\mathbf{e}\}$, and $A_{2,2} := \{-\mathbf{e}\}$, then we get the reduced expression

$$(\{-\mathbf{e}\} + \langle \{\mathbf{e}\} \rangle_{\mathbb{N}}) \cup \big(\langle C \rangle_{\mathbb{N}} + (C) \big),$$

Since $\mathbf{0} \in \langle \{\mathbf{e}\} \rangle_{\mathbb{N}} - \{\mathbf{e}\}$ and $(\langle C \rangle_{\mathbb{N}} + C) \cup \{\mathbf{0}\} = \langle C \rangle_{\mathbb{N}}$ as well as $\{-\mathbf{e}\} + \langle \{\mathbf{e}\} \rangle_{\mathbb{N}} = -\{\mathbf{e}\} \cup \langle \{\mathbf{e}\} \rangle_{\mathbb{N}}$, in this case we can rewrite M to

$$-\{\mathbf{e}\} \cup \langle \{\mathbf{e}\} \rangle_{\mathbb{N}} \cup \langle C \rangle_{\mathbb{N}},$$

which is the "clean union" of two arbitrary homogeneous linear sets, such that the first one has only one generator, only "contaminated" by the union with the opposite vector of that singleton generator.

Another interesting variant is to choose $B_{3,2} = \emptyset$, i.e., catalyst c_3 has no associated dissolution rules in membrane region 2. Then the general expression of set M is immediately simplified to

$$B_{3,3} + B_{2,2} + \langle A_{3,3} + A_{2,2} \rangle_{\mathbb{N}} + \Big(\langle A_{3,3} \rangle_{\mathbb{N}} \cup \big(\langle A_{2,2} + A_{3,2} \rangle_{\mathbb{N}} + A_{3,2} \big) \Big).$$

Choosing $A_{3,3} := \{\mathbf{e}\}$, $A_{2,2} := \{-\mathbf{e}\}$, $A_{3,2} := C + \{\mathbf{e}\}$, and $B_{3,3} + B_{2,2} := \{-\mathbf{e}\}$, the expression for the set M becomes

$$\{-\mathbf{e}\} + \Big(\langle \{\mathbf{e}\} \rangle_{\mathbb{N}} \cup \big(\langle C \rangle_{\mathbb{N}} + C + \{\mathbf{e}\} \big) \Big),$$

which can be reduced to $\{-\mathbf{e}\} \cup \langle \{\mathbf{e}\} \rangle_{\mathbb{N}} \cup \langle C \rangle_{\mathbb{N}}$, i.e., we again have obtained an "almost clean union" as already deduced above. Finally, we notice that we can equivalently write our last set of integers as

$$\langle \{\mathbf{e}\}, -\mathbf{e} \rangle_{\mathbb{N}} \cup \langle C \rangle_{\mathbb{N}}.$$

We can extend the approach we used for deriving M to construct precise characterizations of the vector languages generated by purely catalytic P systems over integers with an arbitrary membrane structure of more than three membranes.

For every membrane i, except the skin membrane, of such a P system Π, according to the previous observations and simplifications, we may assume exactly one catalyst to be found there at the beginning; moreover, as all the inner membranes in the skin should be able to contribute to a result, we also may assume that no catalyst arriving in such a membrane will ever be able to be idle there, i.e., at least the rule $c_j \to c_j$ can assumed to be assigned to c_j in membrane i.

For every catalyst c_i and every membrane j, $2 \leq i, j \leq n$, we define the following sets of vectors:

- $A_{i,j}$ contains the vectors corresponding to the right-hand sides of *non-dissolving* rules associated with the catalyst c_i in membrane j;
- $B_{i,j}$ contains the vectors corresponding to the right-hand sides of *dissolving* rules associated with the catalysts c_i in membrane j.

In fact, we only need these sets $A_{i,j}$ and $B_{i,j}$ if membrane i is located somewhere within membrane j in the membrane structure μ of the P system.

For a fixed order of dissolution of membranes, the set of vectors generated by Π then can be written as a linear combination of various specific sets $A_{i,j}$ and $B_{i,j}$ plus the initial vector $\mathbf{a_0}$. The final vector language generated by Π can be characterized as a union of such linear combinations. Actually writing down such a characterization is a matter of technicality, but we refrain from doing it to avoid boring the reader with very tedious details.

4.3 Communication

We would like to remark that adding target indications to the regular objects should not increase the power of purely catalytic P systems over integers. Indeed, looking at a purely catalytic P system over integers, it is easily decidable which membranes will eventually be dissolved. Hence, the only question is whether the contents of a region specified by target, after possible dissolutions, will be in the output. There is no need to examine the future of a moved regular object, since the resources in purely catalytic P systems over integers are unbounded, and we can view this copy of a moved object as staying in that region until the end of the computation.

However, if also the catalysts are allowed to have target indications associated, it does make a difference. We claim the following characterizations.

Theorem 4. *For all $k \geq 1$,*

$$\begin{aligned} &\mathbb{Z}^d O_{\mathbb{Z}} P_*(mpcat_k, tar_{\mathbb{Z}}) = \mathbb{Z}^d SLIN_{\mathbb{N}}, \\ &\mathbb{Z}^d O_{\mathbb{Z}} P_*(mpcat_k + \delta) = \mathbb{Z}^d SLIN_{\mathbb{N}}, \quad \textit{and} \\ &\mathbb{Z}^d O_{\mathbb{Z}} P_*(mpcat_*, \delta) = \mathbb{Z}^d SLIN_{\mathbb{N}}. \end{aligned}$$

Proof. The upper bound in either case is easy to see because the number of possible arrangements of catalysts across the given membrane structure (and any possible structures obtained from it by membrane dissolutions) is bounded. Hence, purely catalytic P systems over integers with mobile catalysts are still not more powerful than blind vector-addition systems with states, which characterize $\mathbb{Z}^* SLIN_{\mathbb{N}}$, see [2].

We now proceed to the $\supseteq$ inclusions and consider an arbitrary semilinear set $\bigcup_{1 \le i \le m} \langle A_i, \mathbf{b}_i \rangle_{\mathbb{N}}$, where for each i, $1 \le i \le m$, $A_i \cup \{\mathbf{b}_i\} \subset_{fin} \mathbb{Z}^d$.

We first construct the following purely catalytic P system over integers Π_1 showing the first inclusion $\mathbb{Z}^d O_{\mathbb{Z}} P_*(mpcat_1, tar_{\mathbb{Z}}) \supseteq \mathbb{Z}^d SLIN_{\mathbb{N}}$.

$$
\begin{aligned}
\Pi_1 &= (O, C = \{c\}, \mu, w_1, \cdots, w_{2m+1}, R_1, \cdots, R_{2m+1}, i_0 = 1) \text{ where} \\
O &= \{a_i \mid 1 \le i \le d\} \cup \{c\}, \\
\mu &= [\,[\,[\,\,]_{m+2}\,]_2 \cdots [\,[\,\,]_{2m+1}\,]_{m+1}\,]_1, \\
w_1 &= c, \\
w_{i+1} &= \lambda,\ 1 \le i \le 2m, \\
R_1 &= \{c \to (c, in_{i+1}) v_i \mid 1 \le i \le m,\ \psi(v_i) = \mathbf{b}_i\}, \\
R_{i+1} &= \{c \to c(v, out) \mid \psi(v) \in A_i\} \cup \{c \to (c, in_{m+i+1})\},\ 1 \le i \le m, \\
R_{m+i+1} &= \emptyset,\ 1 \le i \le m.
\end{aligned}
$$

The work of Π_1 consists of a non-deterministic choice of the i-th linear set to be generated, by moving the single catalyst c from the skin membrane into membrane $i+1$, at the same time producing $\mathbf{b}_i$ in the skin membrane. After having sent an arbitrary combination of vectors from A_i to the skin membrane, the catalyst enters membrane $m+i+1$, getting disabled there, and the system halts.

For showing the second inclusion $\mathbb{Z}^d O_{\mathbb{Z}} P_*(mpcat_1 + \delta) \supseteq \mathbb{Z}^d SLIN_{\mathbb{N}}$, where the catalyst may move, but at the same time may dissolve its surrounding membrane, we construct the purely catalytic P system over integers Π_2:

$$
\begin{aligned}
\Pi_2 &= (O, C = \{c\}, \mu, w_1, \cdots, w_{m+2}, R_1, \cdots, R_{m+2}, i_0 = 1) \text{ where} \\
O &= \{a_i \mid 1 \le i \le d\} \cup \{c\}, \\
\mu &= [\,[\,[\,\,]_3 \cdots [\,\,]_{m+2}\,]_2\,]_1, \\
w_1 &= \lambda, \\
w_2 &= c, \\
w_{i+2} &= \lambda,\ 1 \le i \le m, \\
R_1 &= \emptyset, \\
R_2 &= \{c \to (c, in_{i+2}) v_i \delta \mid 1 \le i \le m,\ \psi(v_i) = \mathbf{b}_i\}, \\
R_{i+2} &= \{c \to cv \mid \psi(v) \in A_i\} \cup \{c \to c\delta\},\ 1 \le i \le m.
\end{aligned}
$$

In Π_2, the single catalyst c from the second membrane is sent into membrane $i+2$, at the same time dissolving membrane 2, thus also producing $\mathbf{b}_i$ in the skin membrane. Now a linear combination of vectors from A_i is generated directly in

membrane $i+1$, and is released into the skin upon the dissolution of membrane $i+1$. As the skin membrane contains no rules, the system must halt.

If moving the catalyst and dissolving the surrounding membrane are not allowed to be carried out within one single rule, a more complicated construction is needed, involving two catalysts, with the second catalyst being needed in multiple copies depending on m. For proving the inclusion

$$\mathbb{Z}^d O_{\mathbb{Z}} P_*(mpcat_*, \delta) \supseteq \mathbb{Z}^d SLIN_{\mathbb{N}},$$

we now construct the following purely catalytic P system over integers Π_3:

$$\begin{aligned}
\Pi_3 &= (O, C = \{c, c'\}, \mu, w_1, \cdots, w_{3m+1}, R_1, \cdots, R_{3m+1}, i_0 = 1) \text{ where} \\
O &= \{a_i \mid 1 \le i \le d\} \cup \{c, c'\}, \\
\mu &= [\,[\,[\,[\,\,]_{2m+2}\,]_{m+2}\,]_2 \cdots [\,[\,[\,\,]_{3m+1}\,]_{2m+1}\,]_{m+1}\,]_1, \\
w_1 &= c, \\
w_{i+1} &= \lambda,\ 1 \le i \le 2m, \\
w_{2m+1+i} &= c',\ 1 \le i \le m, \\
R_1 &= \{c \to (c, in_{i+1})v_i \mid 1 \le i \le m,\ \psi(v_i) = \mathbf{b}_i\}, \\
R_{i+1} &= \{c \to cv \mid \psi(v) \in A_i\} \\
&\quad \cup \{c \to (c, in_{m+i+1}),\ c' \to c'\delta\},\ 1 \le i \le m, \\
R_{m+i+1} &= \{c \to (c, in_{2m+i+1}),\ c' \to (c', out)\},\ 1 \le i \le m, \\
R_{2m+i+1} &= \{c \to c\delta\},\ 1 \le i \le m.
\end{aligned}$$

With each linear set i, $1 \le i \le n$, three nested membranes are associated ($i+1$, $m+i+1$, and $2m+i+1$). At the beginning, catalyst c is sent from the skin membrane into membrane $i+1$, at the same time producing $\mathbf{b}_i$ in the skin membrane. After having produced an arbitrary combination of vectors from A_i in membrane $i+1$, the catalyst enters membrane $m+i+1$; observe that membrane $i+1$ is not dissolved yet. Then, c enters the elementary membrane $2m+i+1$ and dissolves it, releasing catalyst c' into the surrounding membrane $m+i+1$. Clearly, c cannot reenter membrane $2m+i+1$, which no longer exists, so it has no applicable associated rules. Catalyst c', however, is sent out to membrane $i+1$, and now can dissolve it, which releases all generated regular objects to the skin and halts the computation.

In sum, we have shown that using mobile catalysts we get characterizations for the semilinear sets over integers, i.e., for $\mathbb{Z}^d SLIN_{\mathbb{N}}$, with also using dissolution (together with moving catalysts or as a separate operation) or with using targets for integer values (not only for a positive number of objects). □

5 Multiple Dissolution

In this section we focus on extensions of usual dissolution which allow dissolving multiple nested membranes in one step by adding multiple instances of the dissolution symbol δ. Thus, if the multiset δ^n is introduced into a membrane h

at depth d, membrane h and $n-1$ of its parent membranes are dissolved and the contents of h is transferred into its n-th parent membrane h_p. One may now consider two different semantics for the contents of the $n-1$ dissolved parent membranes of h:

- *forgetting semantics* (δ^*) — the contents of the intermediate dissolved membranes is discarded, and
- *conservative semantics* (δ'^*) — the contents of the intermediate dissolved membranes is merged with that of h and moved into the n-th parent membrane h_p.

Multiple dissolutions allow characterizing semilinear sets relatively easily, even if only one catalyst is available:

Lemma 1. *For all $k \geq 1$,*

$$\mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_k, \delta^*) \cap \mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_k, \delta'^*) \supseteq \mathbb{Z}^d SLIN_{\mathbb{N}}.$$

Proof. To prove the inclusion $\mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_k, \delta^*) \supseteq \mathbb{Z}^d SLIN_{\mathbb{N}}$, consider an arbitrary semilinear set $L = \bigcup_{1 \leq i \leq m} \langle A_i, b_i \rangle_{\mathbb{N}}$, with $A_i \cup \{b_i\} \subset_{fin} \mathbb{Z}^d$. We now construct the following purely catalytic P system over integers generating this semilinear set:

$$\begin{aligned}
\Pi_4 &= (O, C = \{c\}, \mu, w_1, \cdots, w_{m+2}, R_1, \cdots, R_{m+2}, i_0 = 1), \text{ where} \\
O &= \{a_i \mid 1 \leq i \leq d\} \cup \{c\}, \\
\mu &= [\,[\cdots[\;\;]_{m+2} \cdots\,]_2\,]_1, \\
w_k &= \lambda, 1 \leq k < m+2, \\
w_{m+2} &= c, \\
R_1 &= \emptyset, \\
R_{k+1} &= \{c \to cv \mid \psi(v) \in A_k\} \cup \{c \to cw\delta^k \mid \psi(w) = b_k\}, 1 \leq k \leq m, \\
R_{m+2} &= \{c \to \delta^{m-k+1} \mid 1 \leq k \leq m\}.
\end{aligned}$$

Π_4 contains a linear membrane structure: the skin membrane has no rules, the innermost membrane contains some dissolution rules, while the other membranes contain rules representing the generators of one of the linear sets forming L.

Initially, the only catalyst c is located in the innermost membrane. In the first step, c non-deterministically dissolves 1 to m membranes, thereby choosing one of the linear sets of L. Then, by the rules of R_{k+1}, $1 \leq k \leq m$, c adds some generators from A_k to the initially empty membrane. At a certain moment, c decides to stop the generation and uses the rule $c \to cw\delta^k$ to add the offset b_k and to dissolve the remaining membranes, thus putting the generated result into the skin.

Remark now that in Π_4 almost all membranes stay empty during the whole evolution. Specifically, whenever multiple membranes are dissolved, all the parent membranes are empty. This means that choosing a different semantics for

multiple dissolution, i.e., having in mind the primed delta, also allows for covering all semilinear sets; hence, we immediately also infer

$$\mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_1, \delta'^*) \supseteq \mathbb{Z}^d SLIN_{\mathbb{N}}.$$

This observation completes the proof. □

Finally, note that the discussion from Subsect. 4.2 establishing the upper bound at semilinear sets of vectors can be adapted for both semantics of multiple dissolutions. Alternatively, to establish this upper bound, one could show that any evolution of a purely catalytic P system over integers can be simulated by a blind register machine, see [2]. In both cases, we may infer the following characterization result:

Theorem 5. *For all* $k \geq 1$,

$$\mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_k, \delta^*) = \mathbb{Z}^d O_{\mathbb{Z}} P_*(pcat_k, \delta'^*) = \mathbb{Z}^d SLIN_{\mathbb{N}}.$$

Multiple membrane dissolutions in a row are also considered in [2]. In that work, however, one rule may influence multiple membranes (or even all of them) at a time, and is inapplicable if one of the membranes it affects has already been dissolved. This is an additional means of global synchronisation which allows for a simpler the proof in [2].

6 Conclusions

We have reproved that the power of purely catalytic P systems over integers is contained in the family of all semilinear sets of vectors of integers. We then have shown that with one membrane purely catalytic P systems over integers give degenerate results, and with two membranes they exactly characterize the family of all uniform semilinear sets of vectors of integers. With more membranes, this equality becomes a strict inclusion, and specific unions of linear sets with different base vectors have been obtained. More specifically, for any vector $\mathbf{e} \in \mathbb{Z}^d$ and any finite set $C \subseteq \mathbb{Z}^d$, purely catalytic P systems over integers can generate

$$\langle \{\mathbf{e}\}, -\mathbf{e} \rangle_{\mathbb{N}} \cup \langle C \rangle_{\mathbb{N}}.$$

The most interesting open question remaining is whether $\mathbb{Z}^* O_{\mathbb{Z}} P_*(pcat_*, \delta)$ is closed under union. While in almost all cases in membrane computing closure under union is trivial, e.g., by making a non-deterministic choice in the first step of the computation, the current situation is rather surprising.

Finally, we have considered the variants with mobile catalysts, and showed a few combinations of features leading to characterizations of semilinear sets of $\mathbb{Z}$-vectors.

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