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Analytical evaluation of heterogeneous cellular networks under flexible user association and frequency reuse☆

Mehdi Fereydooni[a], Masoud Sabaei*,[a], Mehdi Dehghan[a], Gita Babazadeh Eslamlou[b], Markus Rupp[b]

[a] Amirkabir University of Technology, Iran
[b] Technische Universität Wien, Austria

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ABSTRACT

Offloading mobile users from highly loaded macro base stations (BSs) to lightly-loaded small cell BSs is critical for utilizing the full potential of heterogeneous cellular networks (HCNs). However, to alleviate the signal-to-interference-plus-noise ratio (SINR) degradation of so called biased users, offloading needs to be activated in conjunction with an efficient interference management mechanism. Fractional frequency reuse (FFR) is an attractive interference management technique due to its bandwidth efficiency and its suitability to orthogonal frequency division multiple access based cellular networks. This paper introduces a general mathematical model to study the potential benefit of load balancing in conjunction with two main types of FFR interference coordination: Strict-FFR and soft frequency reuse (SFR)- in the downlink transmissions of HCNs. For some special but realistic cases we were able to reduce the rather complex general mathematical expressions to much simpler closed-forms that reveal the basic properties of BS density on the overall coverage probability. We show that although Strict-FFR outperforms the SFR mechanism in terms of SINR and rate coverage probability, it fails to provide the same spectral efficiency. Finally, we present a novel resource allocation mechanism based on the BSs bias values and FFR thresholds that achieves an even higher minimum user throughput and rate coverage probability.

1. Introduction

Cellular network operators are forced to increase their network capacity in order to cope with the rapidly rising demand on data rate. In the recent years the mobile data usage has grown up to 200% per annum [1]. Introducing new tiers that comprise of base stations (BSs) with smaller transmission ranges (called small cell BSs), is a potential and cost effective approach to increase the capacity of cellular networks [2].

The design of cellular networks with optimal parameter settings requires to have efficient methods and models to analyze the performance of heterogeneous cellular networks (HCNs). Because of the uncontrolled nature of small cell BS distributions in HCNs, conventional models such as Wyner [3] and hexagonal models are considered to be obsolete for modeling. A recent approach to describe the random nature of BS locations is to apply point process theories, leveraging techniques from stochastic geometry [4]. Its accuracy in abstracting realistic BS deployments has been validated in numerous contributions [5–7]. In [5], the authors showed that even in a single tier network Poisson point processes (PPPs) provide an accuracy at least as high as grid models. However, the grid model does not exhibit the same level of analytical tractability as PPPs. Owing to their advantages in tractability and accuracy, PPPs have been extensively employed to model and analyze HCNs [8] in recent years.

The authors in [5] derived important performance metrics such as coverage probability and average rate for a given system model in a single-tier cellular network. Their analysis is generalized in [9] for a K-tier cellular network. They proved that in open access and interference-limited networks with identical target signal-to-interference-plus-noise ratio (SINR) for all tiers the overall coverage probability is independent of the number of tiers and the density of BSs. The model is further developed in [10] to incorporate a user-based load definition into the analysis. In [11], the authors presented a framework to evaluate the coverage probability of indoor users in urban two-tier cellular networks. In this model, environments are partitioned by walls with a certain penetration loss to distinguish between the outdoor BSs in line

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* Corresponding author.
E-mail addresses: fereydooni@aut.ac.ir (M. Fereydooni), sabaei@aut.ac.ir (M. Sabaei), dehghan@aut.ac.ir (M. Dehghan), gita.babazadeh@nt.tuwien.ac.at (G. Babazadeh Eslamlou), markus.rupp@nt.tuwien.ac.at (M. Rupp).

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of sight (LOS) and non-line of sight (NLOS). They showed that an increasing number of small cell BSs can reduce the impact of the building safeguard against the aggregate interference.

Most of the users in HCNs are assigned to macro BSs due to the lower transmission power of small cell BSs [12]. Hence, the network encounters an imbalanced load distribution among its tiers. A common approach to tackle this issue is biasing the users, as proposed by 3rd Generation Partnership Project (3GPP) in Release 10 [13]. Biasing follows the idea of pushing the user assignment towards low-power BSs by artificially increasing their coverage region.

In [14,15] the authors considered the problem of user association and power allocation in the downlink of HCNs with the goal of maximizing the network energy efficiency. The authors in [16] evaluated the performance of cellular networks under an interference coordination mechanism when the users are associated to the closest BS. In [17], the authors developed a model to compute the coverage probability and the average throughput in a K-tier cellular network using a flexible user association. They showed that biasing the users without an efficient interference coordination mechanism always reduces the overall coverage probability of HCNs. It is a direct result of forcing the users with a lower SINR to a BS. In [18] the authors proposed a model to evaluate the performance of a two-tier cellular network under a simple resource partitioning mechanism. They assumed that a ξ fraction of resources is allocated to the macro cell users and the unbiased small cell users. The remaining $(1-\xi)$ fraction of the resources, in which the macro cell shuts down the transmission, is assigned to the biased small cell users. Since the interference level in the fraction of resources for the biased users is lower than the shared part, those users experience a better SINR distribution.

The interference coordination mechanism under consideration in [18] is not bandwidth efficient due to reserving a fixed fraction of resources solely for the biased users. Achieving a high spectral efficiency, the use of the total bandwidth in all cells, is one of the key objectives of long-term evolution (LTE) systems [19]. Fractional frequency reuse (FFR) is a popular interference coordination strategy due to its good spectral efficiency and its suitability to orthogonal frequency division multiple access (OFDMA) based cellular networks [20–22]. It is included in the 3GPP-LTE standard since Release 8 [23]. The FFR mechanism partitions the cell into two regions (interior and edge regions) and applies different frequency reuse factors to each region [24,25]. In this paper we consider two main types of FFR mechanism: Strict-FFR and soft frequency reuse (SFR).

Strict-FFR mechanism: In a Strict-FFR system the entire frequency band W is partitioned into a common part $W_{\mathrm{In}}^{\mathrm{FFR}}=\beta W$ and a reuse part $W_{\mathrm{e}}^{\mathrm{FFR}}=(1-\beta)W$ where $0\le\beta\le1$. The common part of resources is shared by the cell interior users of each tier. The reuse part is divided among the network tiers and $W_{\mathrm{e},k}^{\mathrm{FFR}}$ is utilized by the kth tier, where $W_{\mathrm{e}}^{\mathrm{FFR}}=\sum_{k=1}^{K}W_{\mathrm{e},k}^{\mathrm{FFR}}$. Hence, the cell edge users of a different tier use a disjoint set of resources. Furthermore, the reuse part of the kth tier is further partitioned into Δ_k sub-bands and each BS randomly chooses one sub-band to transmit to the cell edge users.

SFR mechanism: In an SFR system the entire frequency band in the kth tier is divided into Δ_k sub-bands. One of the Δ_k sub-bands is randomly allocated to the cell edge users of each tier, and the cell interior users share the rest of the sub-bands with the edge users of other cells. Since the BS employs the sub-bands used for the edge users of other cells to connect to its own cell interior users, the downlink data of cell interior users is typically transmitted with a lower transmit power to decrease the interference level of the other cells. Each tier has two possible power levels, i.e., a high power level P_k and a low power level m_kP_k where $0<m_k<1$. The BS assigns a high power P_k to transmit to its cell edge users and a low power m_kP_k for the cell interior users. Since the BS uses two different power levels to transmit to cell edge and cell interior users in an SFR mechanism, we divide the users into two groups based on the BS power level. a) Category I user: the user which is located close to a BS and is associated to that BS even though the BS uses a low transmit power level m_kP_k. b) Category II user: the user that is associated to a BS applying a high power level P_k. In such case if the BS apply a low power level m_kP_k, the user of this tier receives the maximum long-term biased received power from BSs of other tiers. A rigorous mathematical description is presented in Section 2.

The author in [26] proposed a frequency resource allocation mechanism to decrease the interference of a two-tier cellular network. In this model the macro cell BSs use the FFR mechanism for the frequency resource allocation. The small cell BSs located in the cell edge region utilize the whole frequency band while the small cell BSs located in the interior region use different sub-bands from the macro cell interior to mitigate the interference level. In [27] the overall coverage probability and the average rate of HCNs with an FFR interference management technique is derived when the mobile users are associated to the nearest BS of each tier.

Contributions: The main contribution of this work is the development of a general analytical model to evaluate the performance of HCNs downlink transmissions under a flexible user association and two main types of FFR interference coordination: Strict-FFR and SFR. We extend the work in [27] by presenting the per-tier coverage probability of cell edge and cell interior users. In the design of cellular networks it is important to find the minimum achievable user rate and the number of cell edge and cell interior users for the optimal setting of network parameters such as the frequency reuse threshold. But [20,27] just provide the average ergodic rate of networks. In this paper, we derive the per-tier average rate of the cell edge and the cell interior users as well as average number of cell edge and cell interior users of each tier, rate coverage probability, and the minimum achievable user rate. Although the mathematical results for very general scenarios are not in closed form, we provide closed form expressions for the coverage probability of users under several special cases. They in turn allow a deeper insight in the performance dependency on various parameters. Last but not least a contribution of this paper is proposing a novel resource allocation mechanism based on the cell edge and cell interior load of a BS and the user SINR distribution that achieves higher minimum user throughput and rate coverage probability.

In this paper, we present many novel observations and provide design insights. Particularly, our analysis demonstrates that even in an interference limited network when all tiers experience the same path loss exponent and we employ an unbiased user association, the coverage probability of the network is not independent of the BSs density. This result contradicts recent observations which show the coverage probability is independent of the BS density and the number of tiers [5,9,17]. Also, we observe that in an SFR mechanism with a low power control and a high frequency reuse factor, the users are always in coverage with almost sure probability. We show by our simulation examples that in an interference limited network, when all tiers experience the same path loss exponent, the average ergodic rate is only loosely correlated to the BS density. Furthermore, we show that the overall coverage probability of Strict-FFR outperforms the SFR mechanism, while the SFR mechanism exhibits better performance in terms of average ergodic rate and minimum average user rate.

The rest of the paper is organized as follows: Section 2 describes the system model employed along with deriving some important network metrics such as the user association probability, and distance distribution. The coverage probability of the cellular network is computed in Section 3. We then continue with other important performance metrics, such as the average ergodic rate of HCNs, average user throughput and the rate distribution in Section 4. Section 5 presents closed form expressions and proposes a novel resource allocation mechanism. Numerical results are presented in Section 6 before the paper is concluded in Section 7.

2. System model

In this paper, we consider downlink transmissions in OFDMA based

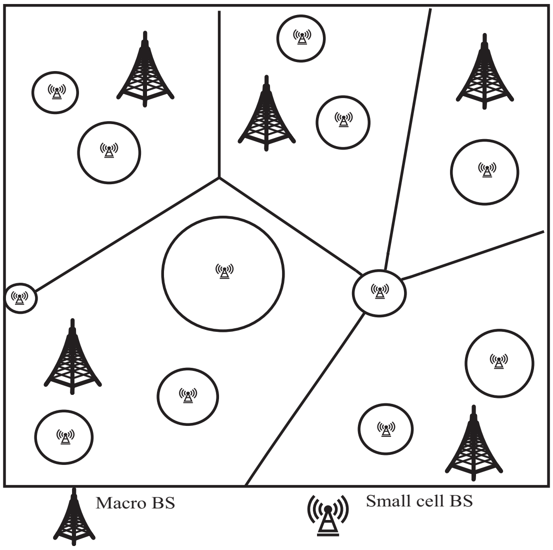


Fig. 1. Topology of a two-tier network.

K-tier cellular networks. Fig. 1 shows the network architecture in a sample two-tier cellular network. Let us denote k (where $k \in \mathscr{K}$, and $\mathscr{K} = \{1, 2, 3, ..., K\}$) the index of tier k, and assume that the BSs in tier k are spatially distributed as a PPP, $\phi_k \in \mathbb{R}^2$, of density λ_k. The locations of the users in the network are modeled by another independent homogeneous PPP, $\phi_u \in \mathbb{R}^2$, with a non-zero density λ_u. Without loss of generality, we perform all of the analysis on a typical mobile user located at the origin, which is possible via a striking property of PPPs explained by the Slivnyak theorem [28]. Each BS in tier k is offered a maximum transmit power P_k, and τ_k denotes the tier's SINR threshold. More precisely, a mobile user can reliably communicate with a BS in the kth tier, only if the downlink SINR of the BS at the mobile user is greater than τ_k. The noise power is represented by σ^2 and the distance based path loss function is $l(y) = \|y\|^{-\alpha_k}$, where α_k is the path loss exponent for the BSs of tier k. The downlink SINR of a user when it connects to a BS $x \in \phi_k$ is

$$\mathrm{SINR}(k, P_k) = \frac{P_k h_{kx} y_{kx}^{-\alpha_k}}{I + \sigma^2}, \tag{1}$$

while the interference at such user is computed by

$$I = \sum_{j=1}^{K} \sum_{z \in \phi_j \setminus x} P_j h_{jz} y_{jz}^{-\alpha_j}, \tag{2}$$

where h_{jz} denotes the random fading, following an exponential distribution with unit mean ($h_{jz} \sim \exp(1)$). Furthermore, y_{jz} is the distance between the user under consideration and BS z in tier j. Table 1 summarizes the paper notations.

We consider a Strict-FFR and an SFR interference management mechanism, and similar to [20] we divide the users into cell edge and cell interior users by means of a certain frequency reuse threshold $\tau_{k,\,\mathrm{FR}}$. It should be noted that, a user in tier k is considered as cell interior user if it's received SINR exceeds a threshold $\tau_{k,\,\mathrm{FR}}$, otherwise it is considered as cell edge user.

2.1. Tier association probability and distance distribution

Each user chooses a BS as its serving BS, if it provides the maximum long-term biased received power [17]. The user association policy is given by:

$$u \in \mathscr{U}_i \quad \text{if} \quad i = \arg\max_{i \in \mathscr{K}} P_i B_i R_i^{-\alpha_i} \quad , \tag{3}$$

where $\mathscr{U}_i$ denotes the user's set of BSs of the ith tier and B_i is a positive bias value which is identical for all the BSs of tier i. Employing a bias value $B_i > 1$ by the BSs of the ith tier, extends its coverage area.

Strict-FFR mechanism: The tier association probability A_i, the number of user associated to tier i $N(i)$, and the probability distribution function (PDF) of distance between the user and it's associated BS $f_i(x)$ under the Strict-FFR system are obtained via [17, Lemma 1], [17, Lemma 2] and [17, Lemma 3], respectively.

$$A_i = \int_0^\infty \exp\left(\sum_{k=1}^{K} -\lambda_k \pi \left(\frac{B_k P_k}{B_i P_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right) 2\pi r \lambda_i dr, \tag{4}$$

$$N(i) = \int_0^\infty \exp\left(\sum_{k=1}^{K} -\lambda_k \pi \left(\frac{B_k P_k}{B_i P_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right) 2\pi r \lambda_u dr, \tag{5}$$

$$f_i(x) = \frac{2\pi\lambda_i x}{A_i} \exp\left(-\pi \sum_{k=1}^{K} \lambda_k \left(\frac{B_k P_k}{B_i P_i}\right)^{2/\alpha_k} x^{\frac{2\alpha_k}{\alpha_i}}\right). \tag{6}$$

SFR mechanism: In an SFR system the BS uses two different power levels to transmit to cell edge and cell interior users. Since the user association mechanism is based on the maximum long-term biased received power, the two power levels of the SFR mechanism have a direct influence on the user association. Consider the case in which based on user association policy (3), the typical user chooses the BS at tier i as serving BS, i.e., $P_i B_i R_i^{-\alpha_i} > \max_{k \in \mathscr{K},\, k \neq i} P_k B_k R_k^{-\alpha_k}$. Since each tier has two transmit power levels, the user could be in two different states. In one case, the user is located somewhere that even if the BSs of tier i use a low transmit power level $m_i P_i$, while all the BSs of other tiers use a high transmit power level P_k, $k \in \mathscr{K}$, $k \neq i$, again tier i provides the maximum long-term biased received power for the user, i.e.,

$$m_i P_i B_i R_i^{-\alpha_i} > \max_{k \in \mathscr{K},\, k \neq i} P_k B_k R_k^{-\alpha_k}.$$

Let us call this type of users Category I users.

In the second case, when the BSs of tier i apply a low power level $m_i P_i$ the typical user of this tier receives its maximum long-term biased

Table 1
Notation of frequently used parameters.

Symbol	Meaning	Symbol	Meaning
P_k	Maximum transmit power of the kth tier	m_k	Power control factor of the kth tier
B_k	Bias value of the kth tier	τ_k	SINR threshold of the kth tier
$\tau_{k,\,\mathrm{FR}}$	Frequency reuse threshold of the kth tier	γ_k	Rate threshold of the kth tier
λ_k	Density of the kth tier BSs distribution	λ_u	Density of the user distribution
ϕ_k	Point process of the kth tier BSs	ϕ_u	Point process of users
Δ_k	Reuse factor of the kth tier	R_k	Distance between the user and its nearest BS
$y_{k,\,z}$	Distance between the user and the BS z	σ^2	Noise power
$\mathscr{K}$	Set of the network tiers	$\mathscr{U}$	Set of the users
$h_{k,\,z}$	Fading between the user and the BS z	α_k	Path loss exponent of the kth tier
β	Strict-FFR cell interior user resource fraction	ρ_{total}	Total number of sub-bands
W	Channel bandwidth	K	Number of tiers
$W_{\mathrm{in}}^{\mathrm{FFR}}$	Bandwidth of Strict-FFR cell interior user	$W_{\mathrm{e},k}^{\mathrm{FFR}}$	Bandwidth of Strict-FFR kth tier cell edge user

received power from BSs of tier k where $k \neq i$,

$P_iB_iR_i^{-\alpha_i} > \max_{k\in\mathscr{K},\, k\neq i} P_kB_kR_k^{-\alpha_k}$, and $m_iP_iB_iR_i^{-\alpha_i} < \max_{k\in\mathscr{K},\, k\neq i} P_kB_kR_k^{-\alpha_k}$.

Let us call this type of users Category II users. Both user categories are thus forming disjoint sets.

Since a high transmit power level is reserved just for the cell edge users and the users of Category II are associated with the BS at the high power level, all the users of Category II should be considered as the cell edge user. Considering Category II user as interior user means the BS applies a low transmit power level for this user. However, if the BS applies a low power level for this user, based on the user association policy, the user will change its serving BS (i.e., it will be connected to new tier) and it will not remain in the Category II user set of this tier anymore. However, the users of Category I could be either cell interior or cell edge users based on the received SINR. We should compute the association probability of each category before computing the coverage probability.

Lemma 2.1. *The probability of a user association with tier i as a Category I user is given by*

$$A_{1,i} = \int_0^\infty \exp\left(\sum_{k=1,k\neq i}^{K} -\lambda_k\pi\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right)2\pi r\lambda_i \exp(-\lambda_i\pi r^2)dr. \tag{7}$$

Proof. The proof is carried out along the lines of [17, Lemma 1]. □

Lemma 2.2. *The association probability of users with tier i as Category II user is given by:*

$$\begin{aligned}A_{2,i} &= \int_0^\infty \exp\left(\sum_{k=1}^{K} -\lambda_k\pi\left(\frac{B_kP_k}{B_iP_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right)2\pi r\lambda_i dr\\ &\quad -\int_0^\infty \exp\left(\sum_{k=1,k\neq i}^{K} -\lambda_k\pi\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right)2\pi r\lambda_i\\ &\quad \exp(-\lambda_i\pi r^2)dr.\end{aligned} \tag{8}$$

Proof. The proof is obtained by applying the definition of Category II users and considering the fact that the users in Categories I and II are disjoint, i.e., $A_{2,i} = A_i - A_{1,i}$. □

From (8) we can observe that when m_i goes to one, the association probability $A_{2,\,i}$ goes to zero and all the users are considered as Category I.

Corollary 2.1. *If $\alpha_k = \alpha$ the association probabilities simplify to*

$$A_{1,i} = \frac{1}{\sum_{k=1,k\neq i}^{K} \frac{\lambda_k}{\lambda_i}\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha} + 1}, \tag{9}$$

$$A_{2,i} = \frac{1}{\sum_{k=1}^{K} \frac{\lambda_k}{\lambda_i}\left(\frac{B_kP_k}{B_iP_i}\right)^{2/\alpha}} - \frac{1}{\sum_{k=1,k\neq i}^{K} \frac{\lambda_k}{\lambda_i}\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha} + 1}. \tag{10}$$

Now, we compute the PDF of the distance between a user and its associated BS when the user is associated to a BS in tier i. Similar to the derivation of the association probability, the PDF of the distance between a user and its serving BS is separately computed for each category.

Lemma 2.3. *The distance statistics between a typical user in Category I and its serving BS is*

$$f_{1,i}(x) = \frac{2\pi\lambda_i x\exp(-\pi\lambda_i x^2)}{A_{1,i}} \exp\left(-\pi\sum_{k=1,k\neq i}^{K} \lambda_k x^{\frac{2\alpha_k}{\alpha_i}}\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha_k}\right). \tag{11}$$

Proof. The proof is carried out along the lines of [17, Lemma 3]. □

Lemma 2.4. *The distance statistics between a Category II user and its serving BS is*

$$\begin{aligned}f_{2,i}(x) &= \frac{2\pi x\lambda_i}{A_{2,i}}\left[\exp\left(-\pi\sum_{k=1}^{K}\lambda_k\left(\frac{B_kP_k}{B_iP_i}\right)^{2/\alpha_k} x^{\frac{2\alpha_k}{\alpha_i}}\right)\right.\\ &\quad \left. -\exp\left(-\pi\sum_{k=1,k\neq i}^{K}\lambda_k\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha_k} x^{\frac{2\alpha_k}{\alpha_i}}\right)\exp(-\pi\lambda_i x^2)\right].\end{aligned} \tag{12}$$

Proof. Given that the user is a member of Category II of tier i, we initially compute the cumulative distribution function (CDF) of the distance between a user and its serving BS

$$F_{i,2}(x) = 1 - \mathbb{P}[R_i > x|u \in \mathscr{U}_{i,2}] = 1 - \frac{\mathbb{P}[R_i > x,\, u \in \mathscr{U}_{i,2}]}{\mathbb{P}[u \in \mathscr{U}_{i,2}]}. \tag{13}$$

We calculate the numerator of (13) as

$$\begin{aligned}\mathbb{P}[R_i > x,\, u \in \mathscr{U}_{i,2}] &= \int_x^\infty 2\pi r\lambda_i\left\{\exp\left(-\pi\sum_{k=1}^{K}\lambda_k\left(\frac{B_kP_k}{B_iP_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right)\right.\\ &\quad -\exp\left(-\pi\sum_{k=1,k\neq i}^{K}\lambda_k\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{2/\alpha_k} r^{\frac{2\alpha_k}{\alpha_i}}\right)\exp\\ &\quad \left.(-\pi\lambda_i r^2)\right\}dr,\end{aligned} \tag{14}$$

plugging (14) and (8) into (13) and derivation of the results with respect to x, results in (12). □

3. Coverage probability

The coverage probability is the probability that the SINR of a user located at the origin is greater than a predefined target SINR value. In the following we derive the coverage probability for two different interference management mechanisms.

3.1. SFR interference coordination

This section provides the coverage probability of the cell edge users of a K-tier cellular network under biasing and an SFR interference management as well as the coverage probability of the cell interior users. Regarding our system model, cell edge users are divided into two categories and we shall separately compute the coverage probability of each category. Since the two categories are disjoint and each user is associated with at most one of them, the coverage probability of a cell edge user is computable by the law of total probability.

Theorem 3.1. *The coverage probability of tier i cell edge user in a K-tier cellular network with the SFR interference coordination mechanism and biased user association is*

$$\mathscr{S}_{\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i) = A_{1,i}S_{1,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i) + A_{2,i}S_{2,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i), \tag{15}$$

where

$$\begin{aligned}S_{1,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i) &= \int_0^\infty\left[\delta_i(\tau_i)\prod_{k=1}^{K}Q(\Psi_{k,i}(\tau_i),\omega_{k,i}) - Q'\left(\Psi_{i,i}(\tau_i),\Psi_{i,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\right)\right.\\ &\quad \prod_{k=1,k\neq i}^{K}Q\left(\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right),\omega_{k,i}\right)\prod_{k=1,k\neq i}^{K}Q(\Psi_{k,i}(\tau_i),\omega_{k,i})\delta_i\left(\tau_i\right.\\ &\quad \left.\left. + \frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\right]f_{1,i}(r)dr,\end{aligned}$$

$$S_{2,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i) = \int_0^\infty\left(\delta_i(\tau_i)\prod_{k=1}^{K}Q(\Psi_{k,i}(\tau_i),\omega'_{k,i})\right)f_{2,i}(r)dr,$$

and

$$\Psi_{k,i}(x)=\frac{r^{\alpha_i}}{P_i}\eta_k P_k x,\quad \eta_k=\frac{m_k(\Delta_k-1)+1}{\Delta_k},$$
$$\delta_i(x)=\exp\left(-\frac{r^{\alpha_i}\sigma^2}{P_i}x\right),\qquad Q(a_k,\omega)$$
$$=\exp\left(-2\pi\lambda_k\int_\omega^\infty\frac{y}{1+(a_k)^{-1}y^{\alpha_k}}dy\right),$$
$$Q^{'}(a_k,b_k)=\exp\left(-2\pi\lambda_i\int_r^\infty\left(1-\frac{1}{1+a_k y^{-\alpha_i}}\frac{1}{1+b_k y^{-\alpha_i}}\right)ydy\right),$$
$$\omega_{k,i}=\left(\frac{r^{\alpha_i}B_kP_k}{m_iB_iP_i}\right)^{1/\alpha_k},\ \omega^{'}_{k,i}=\left(\frac{r^{\alpha_i}B_kP_k}{B_iP_i}\right)^{1/\alpha_k}.$$

Proof. See Appendix A. □

The coverage probability of tier i cell interior users $S_{\mathrm{In}}^{\mathrm{SFR}}(i,\tau_i)$ is computed by following the same procedure. A cell interior user is a user whose received SINR in the shared sub-band is higher than the frequency reuse threshold $\tau_{i,\ \mathrm{FR}}$. As mentioned in Section 2, only those users that lie in Category I can be classified as cell interior users. Using this fact we compute the coverage probability of cell interior users.

Theorem 3.2. *The coverage probability of tier i cell interior user in a K-tier cellular network with an SFR interference coordination mechanism and biased user association is*

$$S_{\mathrm{In}}^{\mathrm{SFR}}(i,\tau_i)=\int_0^\infty\prod_{k=1}^{K}Q\left(\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right),\omega_{k,i}\right)\prod_{k=1}^{K}Q\left(\Psi_{k,i}\left(\frac{\tau_i}{m_i}\right),\omega_{k,i}\right)\delta_i\left(\frac{\tau_i}{m_i}+\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)f_{1,i}(r)dr. \tag{16}$$

Proof. The proof is carried out along the lines of Theorem 3.1. □

Using Theorems 3.1 and 3.2, the overall coverage probability under SFR is

$$S^{\mathrm{SFR}}=\sum_{i=1}^{K}S_{\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i)+A_{1,i}S_{\mathrm{In}}^{\mathrm{SFR}}(i,\tau_i). \tag{17}$$

3.2. Strict-FFR interference coordination

In this section we provide the coverage probability of HCNs under a flexible user association and a Strict-FFR interference coordination mechanism.

Theorem 3.3. *The coverage probability of ith tier cell edge user in a K-tier cellular network with biased user association and Strict-FFR is*

$$S_{\mathrm{e}}^{\mathrm{FFR}}(i,\tau_i)=\int_0^\infty\Bigg[\delta_i(\tau_i)\exp\left(-2\pi\lambda_i\int_r^\infty\left(1-Q^{''}(y)\right)ydy\right)-\delta_i(\tau_i+\tau_{i,\mathrm{FR}})\prod_{k=1,k\neq i}^{K}Q(\widehat{\Psi}_{k,i}(\tau_{i,\mathrm{FR}}),\omega^{'}_{k,i})\exp\left(-2\pi\lambda_i\int_r^\infty\left[1-\frac{Q^{''}(y)}{1+\widehat{\Psi}_{i,i}(\tau_{i,\mathrm{FR}})y^{-\alpha_i}}\right]ydy\right)\Bigg]f_i(r)dr,$$

where

$$\widehat{\Psi}_{k,i}(x)=\frac{P_kr^{\alpha_i}}{P_i}x,\qquad Q^{''}(y)=\left(1-\frac{1}{\Delta_i}\left(1-\frac{1}{1+\widehat{\Psi}_{i,i}(\tau_i)y^{-\alpha_i}}\right)\right).$$

Proof. See Appendix B. □

Theorem 3.4. *The coverage probability of ith tier cell interior user in a K-tier cellular network with biased user association and Strict-FFR is*

$$S_{\mathrm{In}}^{\mathrm{FFR}}(i,\tau_i)=\int_0^\infty\left[\delta_i(\tau_i+\tau_{i,\mathrm{FR}})\prod_{k=1}^{K}Q(\widehat{\Psi}_{k,i}(\tau_i),\omega^{'}_{k,i})\prod_{k=1}^{K}Q(\widehat{\Psi}_{k,i}(\tau_{i,\mathrm{FR}}),\omega^{'}_{k,i})\right]f_i(r)dr. \tag{18}$$

Proof. The proof is carried out along the lines of Theorem 3.3. □

Using Theorems 3.3 and 3.4, the overall coverage probability under the Strict-FFR mechanism is

$$S^{\mathrm{FFR}}=\sum_{i=1}^{K}A_i(S_{\mathrm{e}}^{\mathrm{FFR}}(i,\tau_i)+S_{\mathrm{In}}^{\mathrm{FFR}}(i,\tau_i)). \tag{19}$$

4. Spectral efficiency

4.1. Average ergodic rate

Finding the average ergodic rate (average cell throughput) is important for the planning and design of the cellular network. The average ergodic rate of network under an SFR interference management $\mathscr{R}^{\mathrm{SFR}}$ is given as

$$\mathscr{R}^{\mathrm{SFR}}=\sum_{i=1}^{K}(\mathscr{R}_{\mathrm{e}}^{\mathrm{SFR}}(i)+\mathscr{R}_{\mathrm{In}}^{\mathrm{SFR}}(i)). \tag{20}$$

The average ergodic rate of tier i cell edge users $\mathscr{R}_{\mathrm{e}}^{\mathrm{SFR}}(i)$ and tier i cell interior users $\mathscr{R}_{\mathrm{In}}^{\mathrm{SFR}}(i)$ are computed by

$$\mathscr{R}_{\mathrm{e}}^{\mathrm{SFR}}(i)=A_{1,i}\mathscr{R}_{1,\mathrm{e}}^{\mathrm{SFR}}(i)+A_{2,i}\mathscr{R}_{2,\mathrm{e}}^{\mathrm{SFR}}(i), \tag{21}$$

$$\mathscr{R}_{\mathrm{In}}^{\mathrm{SFR}}(i)=A_{1,i}\mathscr{R}_{\mathrm{In}}^{\mathrm{SFR}}(i), \tag{22}$$

where $\mathscr{R}_{1,\mathrm{e}}^{\mathrm{SFR}}(i)$, $\mathscr{R}_{2,\mathrm{e}}^{\mathrm{SFR}}$, and $\mathscr{R}_{\mathrm{In}}^{\mathrm{SFR}}(i)$ are the rate of cell edge users in Category I, cell edge users in Category II, and the rate of cell interior users of tier i under an SFR interference coordination mechanism, respectively.

Theorem 4.1. *The average ergodic rate of network under an SFR interference coordination mechanism is equal to*

$$\mathscr{R}^{\mathrm{SFR}}=\sum_{i=1}^{K}A_{1,i}\int_0^\infty S_{1,\mathrm{e}}^{\mathrm{SFR}}(i,\exp(\nu)\ -1)d\nu+\sum_{i=1}^{K}A_{2,i}\int_0^\infty S_{2,\mathrm{e}}^{\mathrm{SFR}}(i,\exp(\nu)\ -1)d\nu+\sum_{i=1}^{K}A_{1,i}\int_0^\infty S_{\mathrm{SFR,In}}(i,\exp(\nu)\ -1)d\nu. \tag{23}$$

Proof. The average ergodic rate of tier i cell edge user is given as

$$\mathscr{R}_{\mathrm{e}}^{\mathrm{SFR}}(i)=A_{1,i}\underbrace{\mathbb{E}[\log(1+\mathrm{SINR}'(i,P_i)),\mathrm{SINR}(i,m_iP_i)<\tau_{i,\mathrm{FR}}|u\in\mathscr{U}_{1,i}]}_{A}+A_{2,i}\mathbb{E}[\log(1+\mathrm{SINR}'(i,P_i))|u\in\mathscr{U}_{2,i}], \tag{24}$$

since the SINR value is positive, the first part of (24) is computed as

$$A=\int_0^\infty\mathbb{P}[\mathrm{SINR}'(i,P_i)>\exp(\nu)\ -1,\mathrm{SINR}(i,m_iP_i)<\tau_{i,\mathrm{FR}}|u\in\mathscr{U}_{1,i}]d\nu,$$

$$\overset{a}{=}\int_0^\infty S_{1,\mathrm{e}}^{\mathrm{SFR}}(i,\exp(\nu)-1)d\nu.$$

The final term (a) is obtained by using the results of Theorem 3.1. By following the same approach for the other parts, we reach (23). □

Theorem 4.2. *The average ergodic rate of network under a Strict-FFR interference coordination mechanism is equal to*

$$\mathscr{R}^{\mathrm{FFR}} = \sum_{i=1}^{K} A_i \int_0^\infty S_{\mathrm{e}}^{\mathrm{FFR}}(i, \exp(\nu) - 1) d\nu + \sum_{i=1}^{K} A_i \int_0^\infty S_{\mathrm{In}}^{\mathrm{FFR}}(i, \exp(\nu) - 1) d\nu. \quad (25)$$

Proof. The proof is carried out along the lines of Theorem 4.1. □

4.2. Average user throughput

Another important metric considered in the design of the cellular networks is the *average user throughput*. Assuming the resources are fairly shared between the users associated to a BS, the average cell edge user throughput of tier i under SFR is given as

$$\mathscr{R}_{\mathrm{u,e}}^{\mathrm{SFR}}(i) = \frac{\mathscr{R}_{\mathrm{e}}^{\mathrm{SFR}}(i)}{\Delta_i N_{\mathrm{e}}^{\mathrm{SFR}}(i)}. \quad (26)$$

Also, the average user throughput of cell interior users of tier i is computed as

$$\mathscr{R}_{\mathrm{u,In}}^{\mathrm{SFR}}(i) = \frac{(\Delta_i - 1)\mathscr{R}_{\mathrm{In}}^{\mathrm{SFR}}(i)}{\Delta_i N_{\mathrm{In}}^{\mathrm{SFR}}(i)}, \quad (27)$$

where $N_{\mathrm{e}}^{\mathrm{SFR}}(i)$ and $N_{\mathrm{In}}^{\mathrm{SFR}}(i)$ are the average number of cell edge and cell interior users associated to a BS of tier i under an SFR mechanism, respectively.

Lemma 4.1. *The average number of cell edge users of a BS in tier i under an SFR mechanism is given by*

$$N_{\mathrm{e}}^{\mathrm{SFR}}(i) = \frac{\lambda_u}{\lambda_i}\left[A_{2,i} + A_{1,i} - A_{1,i}\int_0^\infty \left(\delta_i\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\prod_{k=1}^{K} Q\left(\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right), \omega_{k,i}\right)\right) f_{1,i}(r) dr\right]. \quad (28)$$

Proof. Under an SFR mechanism we have two types of cell edge users. The SINR of Category I cell edge users falls below the frequency reuse threshold. Consequently the number of Category I cell edge users is

$$N_{1,\mathrm{e}}^{\mathrm{SFR}}(i) = \frac{\lambda_u A_{1,i}}{\lambda_i}\left[1 - \int_0^\infty \left(\delta_i\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\prod_{k=1}^{K} Q\left(\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right), \omega_{k,i}\right)\right) f_{1,i}(r) dr\right]. \quad (29)$$

All users that lie in Category II are considered as cell edge users, i.e., $N_{2,\mathrm{e}}^{\mathrm{SFR}}(i) = \frac{\lambda_u A_{2,i}}{\lambda_i}$. The total number of tier i cell edge users is $N_{\mathrm{e}}^{\mathrm{SFR}}(i) = N_{1,\mathrm{e}}^{\mathrm{SFR}}(i) + N_{2,\mathrm{e}}^{\mathrm{SFR}}(i)$. □

Similarly, we can compute the mean number of interior users associated to a BS in tier i.

Lemma 4.2. *The average number of users associated as cell interior users of a BS in tier i under an SFR mechanism is*

$$N_{\mathrm{In}}^{\mathrm{SFR}}(i) = \frac{\lambda_u}{\lambda_i}A_{1,i}\int_0^\infty \left(\delta_i\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\prod_{k=1}^{K} Q\left(\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right), \omega_{k,i}\right)\right) f_{1,i}(r) dr. \quad (30)$$

Plugging (28), and (21) into (26) provides the average cell edge user throughput, and the average cell interior user throughput is derived by putting (30), and (22) into (27). Finally, using (31), we compute the minimum rate achievable by each user

$$\mathscr{R}_{\min}^{\mathrm{SFR}} = \min_{i\in\mathscr{K}}(\mathscr{R}_{\mathrm{u,e}}^{\mathrm{SFR}}(i), \mathscr{R}_{\mathrm{u,In}}^{\mathrm{SFR}}(i)). \quad (31)$$

The average cell edge user throughput under a Strict-FFR interference coordination is

$$\mathscr{R}_{\mathrm{u,e}}^{\mathrm{FFR}}(i) = \frac{W_{\mathrm{e},i}^{\mathrm{FFR}}\mathscr{R}_{\mathrm{e}}^{\mathrm{FFR}}(i)}{\Delta_i W N_{\mathrm{e}}^{\mathrm{FFR}}(i)}, \quad (32)$$

where

$$N_{\mathrm{e}}^{\mathrm{FFR}}(i) = \frac{\lambda_u A_i}{\lambda_i}\left[1 - \int_0^\infty \delta_i(\tau_{i,\mathrm{FR}})\prod_{k=1}^{K} Q(\Psi_{k,i}(\tau_{i,\mathrm{FR}}), \omega_{k,i}')f_i(r) dr\right]. \quad (33)$$

The average cell interior user throughput under a Strict-FFR interference coordination is given as

$$\mathscr{R}_{\mathrm{u,In}}^{\mathrm{FFR}}(i) = \frac{W_{\mathrm{In}}^{\mathrm{FFR}}\mathscr{R}_{\mathrm{In}}^{\mathrm{FFR}}(i)}{W N_{\mathrm{In}}^{\mathrm{FFR}}(i)}, \quad (34)$$

where

$$N_{\mathrm{In}}^{\mathrm{FFR}}(i) = \frac{\lambda_u A_i}{\lambda_i}\int_0^\infty \delta_i(\tau_{i,\mathrm{FR}})\prod_{k=1}^{K} Q(\Psi_{k,i}(\tau_{i,\mathrm{FR}}), \omega_{k,i}')f_i(r) dr. \quad (35)$$

4.3. Rate coverage probability

We now compute the rate coverage probability of a user located at the origin with the same approach as before for computing the coverage probability. The following theorem provides the per-tier rate coverage probability under an SFR interference coordination.

Theorem 4.3. *The rate coverage probability of tier i under the SFR interference coordination for a rate threshold* γ_i *is given as*

$$\mathscr{P}^{\mathrm{SFR}}(i, \gamma_i) = A_{1,i}\mathscr{P}_{1,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i) + A_{2,i}\mathscr{P}_{2,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i) + A_{1,i}\mathscr{P}_{\mathrm{In}}^{\mathrm{SFR}}(i, \gamma_i), \quad (36)$$

where

$$\mathscr{P}_{1,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i) = S_{1,\mathrm{e}}^{\mathrm{SFR}}\left(i, \exp\left(\frac{\gamma_i \Delta_i N_{1,\mathrm{e}}^{\mathrm{SFR}}(i)}{W}\right) - 1\right),$$

$$\mathscr{P}_{2,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i) = S_{2,\mathrm{e}}^{\mathrm{SFR}}\left(i, \exp\left(\frac{\gamma_i \Delta_i N_{2,\mathrm{e}}^{\mathrm{SFR}}(i)}{W}\right) - 1\right),$$

$$\mathscr{P}_{\mathrm{In}}^{\mathrm{SFR}}(i, \gamma_i) = S_{\mathrm{In}}^{\mathrm{SFR}}\left(i, \exp\left(\frac{\gamma_i \Delta_i N_{\mathrm{In}}^{\mathrm{SFR}}(i)}{W(\Delta_i - 1)}\right) - 1\right).$$

Proof. To compute the rate coverage probability of tier i, let us first concentrate on the rate coverage probability of Category I cell edge users of tier i, i.e., $\mathscr{P}_{1,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i)$

$$\begin{aligned}\mathscr{P}_{1,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i) &= \mathbb{P}\left[\frac{W}{\Delta_i N_{1,\mathrm{e}}^{\mathrm{SFR}}(i)}\log(1 + \mathrm{SINR}'(i, P_i)) \geq \gamma_i, \mathrm{SINR}(i, m_i P_i) < \tau_{i,\mathrm{FR}} \middle| u \in \mathscr{U}_{1,i}\right], \\ &= \mathbb{P}\left[\mathrm{SINR}'(i, P_i) \geq \exp\left(\frac{\gamma_i \Delta_i N_{1,\mathrm{e}}^{\mathrm{SFR}}(i)}{W}\right) - 1, \mathrm{SINR}(i, m_i P_i) < \tau_{i,\mathrm{FR}} \middle| u \in \mathscr{U}_{1,i}\right], \\ &= S_{1,\mathrm{e}}^{\mathrm{SFR}}\left(i, \exp\left(\frac{\gamma_i \Delta_i N_{1,\mathrm{e}}^{\mathrm{SFR}}(i)}{W}\right) - 1\right),\end{aligned}$$

the rate coverage probability of Category II cell edge users $\mathscr{P}_{2,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i)$ and the cell interior users, $\mathscr{P}_{\mathrm{In}}^{\mathrm{SFR}}(i, \gamma_i)$, are obtained by following the same approach as computing $\mathscr{P}_{1,\mathrm{e}}^{\mathrm{SFR}}(i, \gamma_i)$. □

Theorem 4.4. *The rate coverage probability of tier i under a Strict-FFR interference coordination for a rate threshold* γ_i *is*

$$\mathscr{P}^{\mathrm{FFR}}(i,\gamma_i)=A_i\mathscr{P}_{\mathrm{e}}^{\mathrm{FFR}}(i,\gamma_i)+A_i\mathscr{P}_{\mathrm{In}}^{\mathrm{FFR}}(i,\gamma_i), \tag{37}$$

where

$$\mathscr{P}_{\mathrm{e}}^{\mathrm{FFR}}(i,\gamma_i)=S_{\mathrm{e}}^{\mathrm{FFR}}\left(i,\exp\left(\frac{\gamma_i\Delta_i N_{\mathrm{e}}^{\mathrm{FFR}}(i)}{W_{\mathrm{e},i}^{\mathrm{FFR}}}\right)-1\right),$$

$$\mathscr{P}_{\mathrm{In}}^{\mathrm{FFR}}(i,\gamma_i)=S_{\mathrm{In}}^{\mathrm{FFR}}\left(i,\exp\left(\frac{\gamma_i N_{\mathrm{In}}^{\mathrm{FFR}}(i)}{W_{\mathrm{In}}^{\mathrm{FFR}}}\right)-1\right).$$

5. Discussion of the results

5.1. Special cases of interest

As the general coverage probability has not been derived in a closed form in the previous section, this section provides more insight by some special cases where $\alpha=4$ for all the tiers and $\sigma^2=0$. Because of the high BS density in HCNs, in general the noise power is negligible compared to the interference power (wireless networks are interference limited systems). Besides, the choice of the path loss exponent $\alpha=4$ is commonly accepted in practice as long as users are not too close to the BS. Due to the space limitation we just provide explicit results for special cases of cell edge users under an SFR mechanism; corresponding results for the cell interior users of an SFR mechanism and a Strict-FFR mechanism are easily computable by employing the same approach.

Corollary 5.1. *Consider $\sigma^2=0$ and $\alpha=4$. The coverage probability of ith tier cell edge users in Category I under an SFR mechanism in Theorem 3.1 is*

$$S_{1,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i)=\frac{\lambda_i}{A_{1,i}(C_2(i,\tau_i)+C_4(i))}-\frac{\lambda_i}{A_{1,i}\left(C_1(i)+C_3(i,\tau_i)+C_3\left(i,\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)+C_4(i)\right)}, \tag{38}$$

where

$$C_1(i)=\frac{\tau_i^{-3/2}\arctan\left(\sqrt{\frac{\eta_i\tau_{i,\mathrm{FR}}}{m_i}}\right)-\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)^{-3/2}\arctan(\sqrt{\tau_i\eta_i})}{(\lambda_i)^{-1}\sqrt{\frac{m_i}{\eta_i\tau_{i,\mathrm{FR}}\tau_i}}\left(\frac{1}{\tau_i}-\frac{m_i}{\tau_{i,\mathrm{FR}}}\right)},$$

$$\begin{aligned}
C_2(i,x)&=\sum_{k=1}^{K}\lambda_k\left(\frac{\eta_k P_k x}{P_i}\right)^{1/2}\arctan\left\{\left(\frac{m_iB_i\eta_k x}{B_k}\right)^{1/2}\right\},\\
C_3(i,x)&=\sum_{k=1,k\neq i}^{K}\lambda_k\left(\frac{\eta_k P_k x}{P_i}\right)^{1/2}\arctan\left\{\left(\frac{m_iB_i\eta_k x}{B_k}\right)^{1/2}\right\},\\
C_4(i)&=\left(\lambda_i+\sum_{k=1,k\neq i}^{K}\lambda_k\left(\frac{B_kP_k}{m_iB_iP_i}\right)^{1/2}\right),\qquad C_5(i)=\sum_{k=1}^{K}\lambda_k\left(\frac{B_kP_k}{B_iP_i}\right)^{1/2}.
\end{aligned}$$

Corollary 5.2. *Consider $\sigma^2=0$ and $\alpha=4$. The coverage probability of ith tier cell edge users in Category II under a SFR interference coordination mechanism in Theorem 3.1 is*

$$S_{2,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i)=\frac{\frac{\lambda_i}{A_{2,i}}}{\sqrt{m_i}C_2\left(i,\frac{\tau_i}{m_i}\right)+C_5(i)}-\frac{\frac{\lambda_i}{A_{2,i}}}{\sqrt{m_i}C_2\left(i,\frac{\tau_i}{m_i}\right)+C_4(i)}. \tag{39}$$

This result is opposing [5,9,17], where it is argued that in interference limited networks when all tiers have the same path loss exponent, the coverage probability is independent of the BS density and the number of tiers. The reason for this difference is that the other work does not employ any interference coordination mechanism among the users. As evident from the coverage probability expressions (38) and (39), by employing an FFR interference coordination, even in an unbiased user association mechanism the overall coverage probability depends indeed on the BS distribution density λ_i.

Corollary 5.3. *Consider $\sigma^2=0$, $\alpha=4$, and $m_i=\epsilon$ (ϵ is an infinitely small value). The coverage probability of users under an SFR interference coordination mechanism is simplified to*

$$S^{\mathrm{SFR}}(i,\tau_i)=\frac{\sum_{k=1}^{K}\lambda_k\left(\frac{B_kP_k}{B_iP_i}\right)^{1/2}}{\sum_{k=1}^{K}\lambda_k\left(\frac{B_kP_k}{B_iP_i}\right)^{1/2}\left(\left(\frac{B_i\tau_i}{\Delta_kB_k}\right)^{1/2}\arctan\left\{\left(\frac{B_i\tau_i}{\Delta_kB_k}\right)^{1/2}\right\}+1\right)}. \tag{40}$$

We observe that by using larger values for reuse factor Δ_k of all tiers, the coverage probability increases as well. When the frequency reuse factor Δ_k of all tiers goes up, the coverage probability of tier i becomes $S^{\mathrm{SFR}}(i,\tau_i)=1$. By a sufficient increase of Δ_k, almost all users are covered by the BSs. Besides, in case of general Δ_k values, the overall coverage probability only loosely depends on the BS density.

5.2. Resource allocation

The optimal resource allocation among the cell edge and cell interior users is one of the main concerns in FFR literature. Most of the former works determined optimal values of FFR system parameters by utilizing advanced techniques such as convex optimization [29–31], game-theoretic approaches [32], or exhaustive search [33].

In this paper, we determine the number of sub-bands of cell edge and cell interior users depending on the chosen bias values and FFR thresholds of different tiers. Considering ρ_{total} as the total number of available sub-bands, ρ_{e} is the number of cell edge user sub-bands and ρ_{In} the number of cell interior user sub-bands, the optimal resource allocation of Strict-FFR is

$$\beta=\frac{\sum_{k=1}^{K}N_{\mathrm{In}}^{\mathrm{FFR}}(k)}{\sum_{k=1}^{K}N(k)}, \tag{41}$$

$$\rho_{\mathrm{In}}^{\mathrm{FFR}}=\lceil\beta\rho_{\mathrm{total}}\rceil, \tag{42}$$

$$\rho_{\mathrm{e}}^{\mathrm{FFR}}(i)=\left\lfloor\frac{(\rho_{\mathrm{total}}-\rho_{\mathrm{In}}^{\mathrm{FFR}})N_{\mathrm{e}}^{\mathrm{FFR}}(i)}{\Delta_i\sum_{k=1}^{K}N_{\mathrm{e}}^{\mathrm{FFR}}(k)}\right\rfloor, \tag{43}$$

while the optimal resource allocation of SFR is

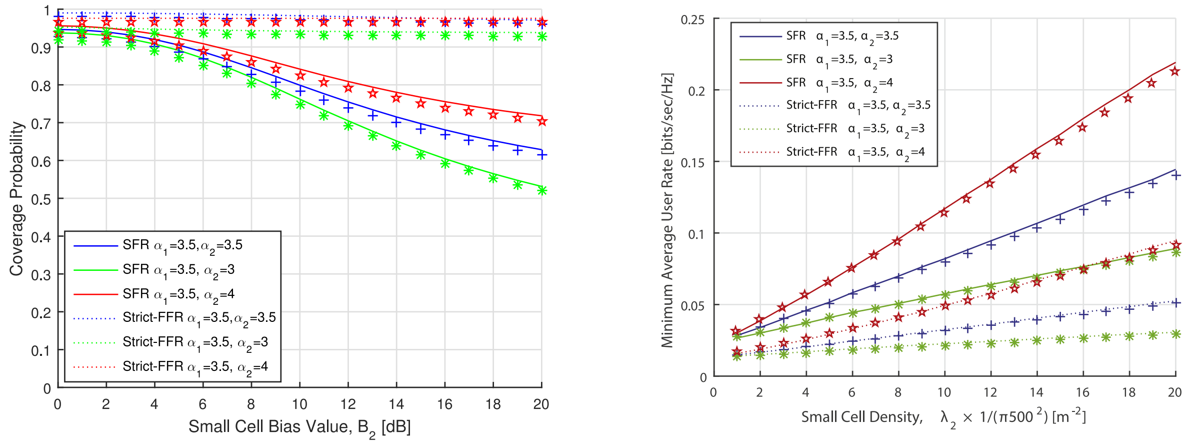
$$\rho_{\mathrm{In}}^{\mathrm{SFR}}=\lceil\rho_{\mathrm{total}}-\rho_{\mathrm{e}}^{\mathrm{SFR}}(i)\rceil, \tag{44}$$

$$\rho_{\mathrm{e}}^{\mathrm{SFR}}(i)=\left\lfloor\frac{\rho_{\mathrm{total}}}{\Delta_i}\right\rfloor. \tag{45}$$

Intuitively, applying this resource allocation mechanism will enforce a more efficient resource allocation among the users based on the number of cell edge and cell interior users and the SINR distribution. Hence, the user will experience a higher throughput and rate coverage probability.

6. Numerical results

For the verification of our results we carried out Monte Carlo simulations under 3GPP compliant scenarios. In particular, we consider a two-tier cellular network (macro cell BSs and small cell BSs). In our simulation of two-tier cellular networks, we assume $W=10$ MHz and for both tiers we set the SINR threshold to -3 dB, the frequency reuse threshold to 3 dB, and the rate threshold to 1 Mbit/s. The macro cell BSs are distributed with density $\lambda_1=\frac{1}{\pi 500^2}$ m^{-2} over a field of $10\,000\times10\,000$ m^2 with a user density of $\lambda_u=100\lambda_1$. We also set the bias value of macro cell BSs to $B_1=0$ dB. The macro BSs and small cell BSs transmit powers are set to 46 dBm and 26 dBm, respectively. We compare the results of SFR and Strict-FFR interference coordination



(a) The coverage probability as a function of bias values ($\lambda_2 = 10\lambda_1$).

(b) The average minimum user rate as a function of density ($B_2 = 8$ dB).

Fig. 2. Lines denote results from simulations, and markers refer to results from analysis ($B_1 = 0$ dB, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$, $m_1 = m_2 = 0.1$, $\sigma^2 = -10$ dBm).

mechanisms against the joint resource partitioning and user association (RPUA) mechanism of [18]. In the RPUA simulation we set the resource partitioning fraction to $\xi = 0.47$ which is shown that is optimal in [18]. This means 47% of resources are allocated to the macro cell users along with unbiased small cell users and the rest of resources, i.e., 53%, goes to the biased small cell users.

6.1. Verification of analysis

In this part, we provide numerical results in order to validate the represented analytical expressions. As shown in Fig. 2, the proposed analytical evaluation matches very well the simulation results. Fig. 2(a) compares the coverage probability of the analytical expressions from Section 3 to the simulation results. The overall coverage probability of users under an SFR mechanism decreases with larger bias values. However, the overall coverage probability of a network under a Strict-FFR mechanism remains almost constant at one for different bias values. Since in a Strict-FFR mechanism we have better resource partitioning among the users, we observe higher SINR values compared to an SFR mechanism. Also, we observe that the overall coverage probability of the network is higher when the user experiences a larger path loss from the small cell BSs, specially for larger bias values. When we employ a lower path loss exponent for the small cell BSs, most of the users will receive the highest long-term biased received power from the small cell BSs.

Fig. 2(b) compares the average minimum user rate under varying small cell BS densities. Intuitively, a denser deployment of small cell BSs increases the average minimum user rate under SFR and Strict-FFR mechanisms. A denser deployment of low-power BSs decreases the number of users associated with each BS and consequently increases the available resources for each user. Also clearly under the SFR mechanism, the users experience a higher rate compared to Strict-FFR due to the better spectral efficiency of SFR. The final observation is that when small cell BSs have a larger path loss exponent, a denser deployment of small cell BSs is more effective on the minimum user received rate. A growing density of small cell BS increases the impact of small cell BS interference. Hence, scenarios with larger small cell path loss exponent experience a higher average minimum rate.

6.2. Impact of small cell density and bias value on the rate coverage probability

This part presents numerical results for comparing the rate coverage probability of users under various system settings.

Fig. 3(a) depicts the impact of different bias values on the overall rate coverage probability of a user, computed in (36) and (37). In this simulation the small cell density is set to $\lambda_2 = 10\lambda_1$.

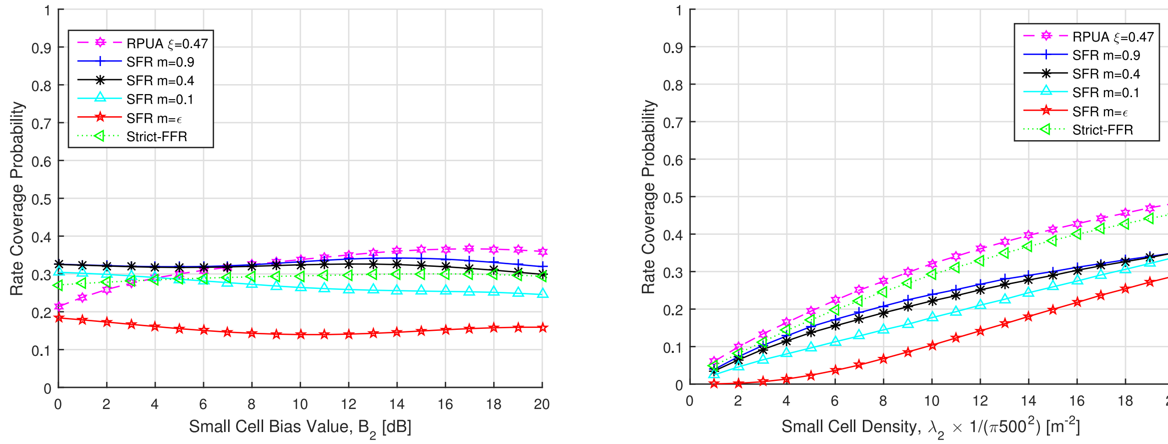
Since in RPUA mechanism a fraction of the resources is dedicated to users with low SINR, this mechanism exhibits the highest increase by using larger bias values.

Fig. 3(b) represents the rate coverage probability of a flexible user association under various small cell BS densities. In this simulation we use a biased user association by employing $B_2 = 8$ dB for all the BSs of the second tier. Clearly employing a higher small cell BS density, increases the rate coverage probability of cellular networks by bringing the users closer to the BSs. Besides, we observe that in dense deployment of small cell BSs the Strict-FFR mechanism outperforms the SFR mechanism in terms of rate coverage probability. Increasing the small cell BS density leads to an increase in the interference. Due to better resource partitioning among the users in Strict-FFR and RPUA mechanisms, they experience a higher increase in the rate coverage probability by a denser deployment of small cell BSs.

6.3. Impact of small cell density and bias value on the average ergodic rate

Fig. 4 compares the average ergodic rate of users under different interference management techniques. Fig. 4(a) depicts the impact of the bias value on the average ergodic rate of the users. An SFR mechanism with power factors $m \in \{0.1, 0.4, 0.9\}$ outperforms the Strict-FFR and RPUA mechanisms in terms of average ergodic rate. Although the RPUA and Strict-FFR mechanisms outperform the SFR mechanism with respect to rate coverage probability, the SFR technique shows a better spectral efficiency due to its more efficient frequency reuse. However, SFR with a power factor ϵ (infinitely small value) does not exhibit the same spectral efficiency by placing almost all users into the cell edge user set.

In Fig. 4(b) by fixing the small cell bias value to $B_2 = 8$ dB, we change the small cell density and compare the average ergodic rate of the users in different interference management techniques. Simulation results show that the average ergodic rate with the SFR and Strict-FFR mechanisms remain almost constant with respect to different small cell densities. Also the RPUA mechanism observes a decrease in the average ergodic rate when employing a higher density. Since by increasing the small cell density most of the users are assigned to the small cell BS without biasing, RPUA wastes resources by allocating a fraction of them to the biased users.



(a) $B_1 = 0$ dB, $\lambda_2 = 10\lambda_1$, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$. (b) $B_1 = 0$ dB, $B_2 = 8$ dB, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$.

Fig. 3. (a) Impact of bias value on the rate coverage probability. (b) Impact of small cell density on the rate coverage probability with $\alpha_1 = \alpha_2 = 4$.

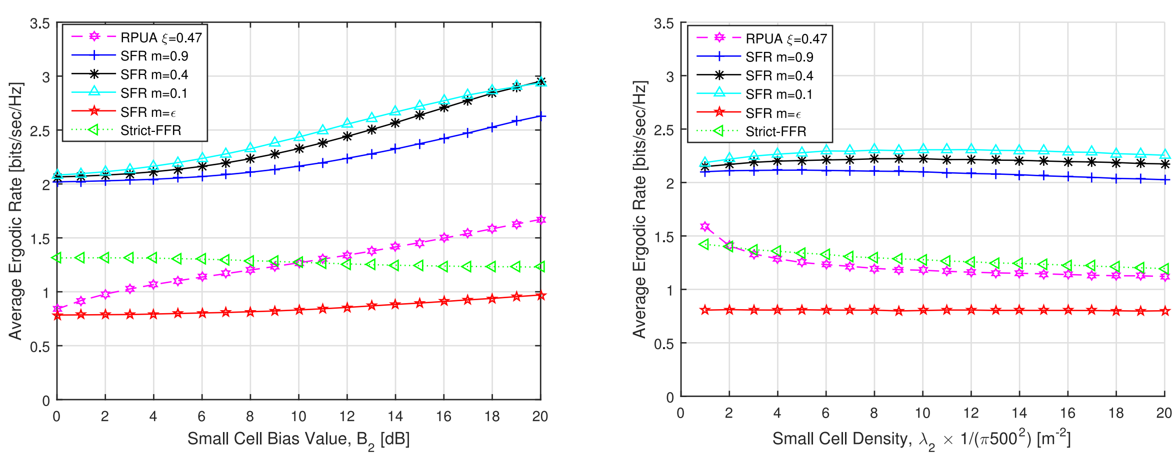
6.4. Impact of small cell density and bias value on the minimum average achievable rate

In Fig. 5(a) we evaluate the impact of biasing on the minimum average user rate as derived in Section 4.B. By employing a higher bias value, more users are pushed from highly loaded macro BSs to lightly loaded small cell BSs. Therefore, the minimum rate of users increases with increasing bias value until it reaches a certain optimal point where the situation changes and a further increase in the bias value decreases the minimum achievable rate of the users by overloading the small cell BSs. Because of a more efficient bandwidth reuse in SFR, its users experience a higher minimum average rate compared to other mechanisms.

In Fig. 5(b) we change the small cell density for the network and display the average minimum user rate when applying a small cell bias value of $B_2 = 8$ dB. Intuitively, by increasing the small cell density, the average minimum user rate increases as well. This is mainly due to decreasing the number of users associated with each BS. Regarding to Section 4.B, by decreasing the user number associated with each BS, the BS has more resources available for each user and we observe a higher average minimum rate. Again the SFR mechanism with $m \in \{0.1, 0.4, 0.9\}$ exhibits the best average minimum user rate because of the better frequency reuse between the BSs. However, by setting $m = \epsilon$, almost all the users lie in the cell edge user set and the network does not benefit from a frequency reuse between different BSs.

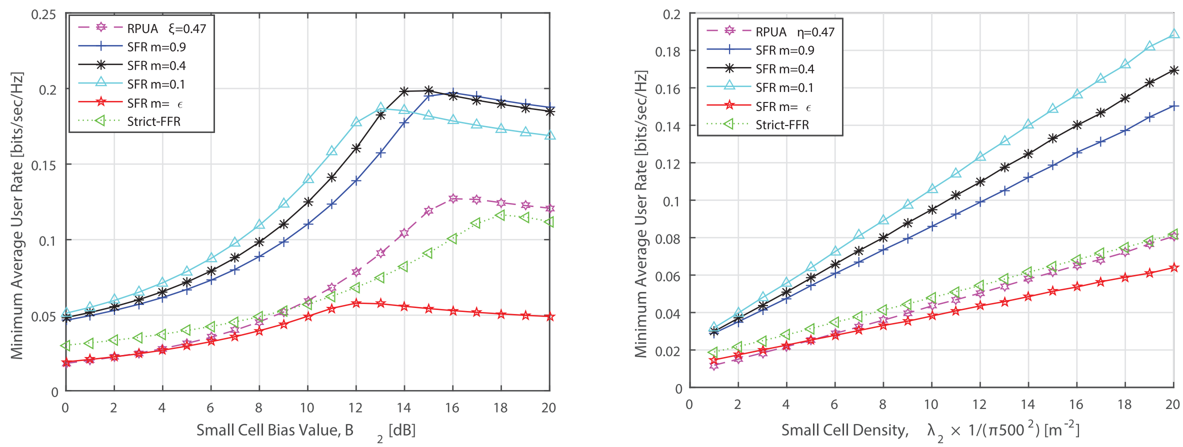
7. Conclusion

This paper proposes a general framework for the performance analysis of HCNs with flexible user association and FFR mechanisms. We evaluate the per-tier and overall coverage probability, as well as the rate coverage probability of such networks, and the average user and cell throughput for the cell edge and the cell interior users. Supported by extensive numerical results, it is shown that SFR outperforms other mechanisms in terms of spectral efficiency and average minimum rate of users. However, because of dedicating a fraction of resources to users with low SINR, RPUA and Strict-FFR provide better rate coverage probability at large bias values. The presented model can be utilized by finding proper optimal values for the design parameters. A natural extension of this work is evaluating the performance of cellular networks by relaxing some assumptions of this paper such as cooperating a load definition into analysis or considering multi antenna BSs.



(a) $B_1 = 0$ dB, $\lambda_2 = 10\lambda_1$, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$. (b) $B_1 = 0$ dB, $B_2 = 8$ dB, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$.

Fig. 4. (a) Impact of small cell bias value on the average ergodic rate. (b) Impact of small cell density on the average ergodic rate with $\alpha_1 = \alpha_2 = 4$.



(a) $B_1 = 0$ dB, $\lambda_2 = 10\lambda_1$, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$. (b) $B_1 = 0$ dB, $B_2 = 8$ dB, $\lambda_1 = \frac{1}{\pi 500^2}\mathrm{m}^{-2}$.

Fig. 5. (a) The average minimum achievable user rate as a function of bias value. (b) The average minimum achievable user rate as a function of small cell BS density with $\alpha_1 = \alpha_2 = 4$.

Appendix A. Proof of Theorem 3.1

Using the law of total probability we have

$$\begin{aligned} S_{\mathrm{e}}^{\mathrm{SFR}} &= \sum_{i=1}^{K} \mathbb{P}(u \in \mathscr{U}_{\mathrm{e},i})\mathbb{P}[\mathrm{SINR}'(i, P_i) > \tau_i | u \in \mathscr{U}_{\mathrm{e},i}], \\ &\overset{a}{=} \sum_{i=1}^{K} \mathbb{P}(u \in \mathscr{U}_{1,i})\mathbb{P}[\mathrm{SINR}'(i, P_i) > \tau_i, \mathrm{SINR}(i, m_i P_i) < \tau_{i,\mathrm{FR}} | u \in \mathscr{U}_{1,i}] \\ &\quad + \mathbb{P}(u \in \mathscr{U}_{2,i})\mathbb{P}[\mathrm{SINR}'(i, P_i) > \tau_i | u \in \mathscr{U}_{2,i}], \end{aligned}$$

where (a) is obtained using the Bayes' theorem and considering the independency of probability of locating user in Category I and $\mathbb{P}[\mathrm{SINR}(i, m_i P_i) < \tau_{i,\mathrm{FR}}]$.

The interference from the BSs of tier k at typical user when the user is associated to the BS x of tier i in common sub-band is equal to I_k. The BSs of the kth tier use the sub-band the same as the BS x with probability $\frac{1}{\Delta_k}$ when they want to transmit to the cell edge users and incorporates in interference of the typical user. Since the BS uses a high power level to transmit to cell edge users, the cell edge users aggregated interference is $I_{\mathrm{e},k} = \sum_{z \in \Phi_k} \frac{1}{\Delta_k} h_{kz} y_{kz}^{-\alpha_k}$. Also the BS takes the remaining sub-bands (except the sub-band that used for its cell edge user) to transmit to the cell interior user with low power level $m_k P_k$. Thus, the interference caused by sending data to the cell interior user is $I_{\mathrm{In},k} = \sum_{z \in \Phi_k} \frac{\Delta_k - 1}{\Delta_k} m_k h_{kz} y_{kz}^{-\alpha_k}$. By considering $\eta_k = \frac{m_k(\Delta_k - 1) + 1}{\Delta_k}$, the aggregate interference is equal to $I_k = I_{\mathrm{e},k} + I_{\mathrm{In},k} = \sum_{z \in \Phi_k} \eta_k h_{kz} y_{kz}^{-\alpha_k}$. To compute the coverage probability of a typical edge user in Category I, by conditioning on distance between the user and associated BS r, we have,

$$\begin{aligned} S_{1,\mathrm{e}}^{\mathrm{SFR}}(i, \tau_i) &= \mathbb{P}\left[\frac{P_i \hat{h}_{i,x} r^{-\alpha_i}}{\sum_{k=1}^{K} \eta_k P_k \hat{I}_k + \sigma^2} > \tau_i, \quad \frac{m_i P_i h_{i,x} r^{-\alpha_i}}{\sum_{k=1}^{K} \eta_k P_k I_k + \sigma^2} < \tau_{i,\mathrm{FR}} \middle| u \in \mathscr{U}_{1,i} \right], \\ &= \mathbb{P}\left[\hat{h}_{i,x} > \frac{\tau_i(\sum_{k=1}^{K} \eta_k P_k \hat{I}_k + \sigma^2)}{P_i} r^{\alpha_i}, h_{i,x} < \frac{\tau_{i,\mathrm{FR}}(\sum_{k=1}^{K} \eta_k P_k I_k + \sigma^2)}{m_i P_i} r^{\alpha_i} \middle| d\ u \in \mathscr{U}_{1,i} \right], \\ &\overset{a}{=} \mathbb{E}_{I,h}\left[\exp\left(-r^{\alpha_i} \frac{\tau_i(\sum_{k=1}^{K} \eta_k P_k \hat{I}_k + \sigma^2)}{P_i} \right) \left(1 - \exp\left(-r^{\alpha_i} \frac{\tau_{i,\mathrm{FR}}(\sum_{k=1}^{K} \eta_k P_k I_k + \sigma^2)}{m_i P_i} \right) \right) \right], \\ &= \mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}\tau_i}{P_i}\sigma^2 \right) \right] \mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}\tau_i}{P_i} \sum_{k=1}^{K} \eta_k P_k \hat{I}_k \right) \right] \\ &\quad - \mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}}{P_i}\left(\tau_i \sum_{k=1}^{K} \eta_k P_k \hat{I}_k + \frac{\tau_{i,\mathrm{FR}} \sum_{k=1}^{K} \eta_k P_k I_k}{m_i} \right) \right) \right] \mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}\sigma^2}{P_i}\left(\tau_i + \frac{\tau_{i,\mathrm{FR}}}{m_i} \right) \right) \right], \end{aligned} \tag{46}$$

$$\overset{b}{=} \delta_i(\tau_i) \underbrace{\mathscr{L}_{\hat{I}_1,\cdots,\hat{I}_K}(\Psi_{1,i}(\tau_i), \cdots, \Psi_{K,i}(\tau_i))}_{A} - \underbrace{\mathscr{L}_{\hat{I}_1,\cdots,\hat{I}_K, I_1,\cdots,I_K}\left(\Psi_{1,i}(\tau_i), \cdots, \Psi_{K,i}(\tau_i), \Psi_{1,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i} \right), \cdots, \Psi_{K,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i} \right) \right)}_{B} \delta_i\left(\tau_i + \frac{\tau_{i,\mathrm{FR}}}{m_i} \right), \tag{47}$$

where (a) follows from the exponential distribution of fading with unit mean, and (b) is obtained by considering $\Psi_{k,i}(x) = \frac{r^{\alpha_i}\eta_k P_k}{P_i} x$ and $\delta_i(x) = \exp\left(-\frac{r^{\alpha_i}\sigma^2}{P_i} x \right)$. The Laplace transform of the second part of (47) is computed as

$$\begin{aligned}B &= \mathbb{E}_{\Phi,h}\left[\exp\left(-\Psi_{i,i}(\tau_i)\hat{I}_i - \Psi_{i,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)I_i - \sum_{k=1,k\neq i}^{K}\Psi_{k,i}(\tau_i)\hat{I}_k - \sum_{k=1,k\neq i}^{K}\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)I_k\right)\right],\\ &= \mathbb{E}_{\Phi,h}\left[\exp\left(-\Psi_{i,i}(\tau_i)\sum_{z\in\Phi_i\setminus x}\hat{h}_{iz}y_{iz}^{-\alpha_i} - \Psi_{i,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\sum_{z\in\phi_i\setminus x}h_{iz}y_{iz}^{-\alpha_i}\right)\right]\\ &\mathbb{E}_{\Phi,h}\left[\prod_{k=1,k\neq i}^{K}\exp\left(-\Psi_{k,i}(\tau_i)\sum_{z\in\Phi_k}\hat{h}_{kz}y_{kz}^{-\alpha_k}\right)\right]\mathbb{E}_{\Phi,h}\left[\prod_{k=1,k\neq i}^{K}\exp\left(-\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)\sum_{z\in\phi_k}h_{kz}y_{kz}^{-\alpha_k}\right)\right],\end{aligned}$$

using the Laplace transform of exponential random variables with unit mean, we have,

$$\begin{aligned}&= \mathbb{E}_{\Phi}\left[\prod_{z\in\Phi_i\setminus x}\frac{1}{1+\Psi_{i,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)y_{iz}^{-\alpha_i}}\frac{1}{1+\Psi_{i,i}(\tau_i)y_{iz}^{-\alpha_i}}\right]\\ &\prod_{k=1,k\neq i}^{K}\mathbb{E}_{\Phi}\left[\prod_{z\in\Phi_k}\frac{1}{1+\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)y_{kz}^{-\alpha_k}}\right]\prod_{k=1,k\neq i}^{K}\mathbb{E}_{\Phi}\left[\prod_{z\in\Phi_k}\frac{1}{1+\Psi_{k,i}(\tau_i)y_{kz}^{-\alpha_k}}\right].\end{aligned}$$

Assuming $Q(a_k,\omega) = \exp\left(-2\pi\lambda_k\int_{\omega}^{\infty}\frac{y}{1+(a_k)^{-1}y^{\alpha_k}}dy\right)$, and applying the probability generating functional (PGFL) of a PPP [28], we reach

$$= \exp\left(-2\pi\lambda_i\int_r^{\infty}\left[1-\frac{1}{1+\Psi_{i,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right)y^{-\alpha_i}}\frac{1}{1+\Psi_{i,i}(\tau_i)y^{-\alpha_i}}\right]ydy\right)\prod_{k=1,k\neq i}^{K}Q\left(\Psi_{k,i}\left(\frac{\tau_{i,\mathrm{FR}}}{m_i}\right),\omega_{k,i}\right)\prod_{k=1,k\neq i}^{K}Q(\Psi_{k,i}(\tau_i),\omega_{k,i}), \tag{48}$$

if $u\in\mathscr{U}_{1,i}$ then we have $\omega_{k,i} = \left(\frac{r^{\alpha_i}B_kP_k}{m_iB_iP_i}\right)^{1/\alpha_k}$. Following from the second Laplace transform of (47), the first Laplace transform is given as

$$A = \prod_{k=1}^{K}\exp\left(-2\pi\lambda_k\int_{\omega_{k,i}}^{\infty}\frac{y}{1+(\Psi_{k,i}(\tau_i))^{-1}y^{\alpha_k}}dy\right). \tag{49}$$

The coverage probability of edge users in Category I is obtained by combining (48) and (49) into (47). By following the same procedure, for the cell edge users in Category II we have,

$$\begin{aligned}S_{2,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i) &= \mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}\tau_i(\sum_{k=1}^{K}\eta_kP_k\hat{I}_k+\sigma^2)}{P_i}\right)\right],\\ &= \delta_i(\tau_i)\prod_{k=1}^{K}2\pi\lambda_k\int_{\omega'_{k,i}}^{\infty}\frac{y}{1+(\Psi_{k,i}(\tau_i))^{-1}y^{\alpha_k}}dy,\\ S_{2,\mathrm{e}}^{\mathrm{SFR}}(i,\tau_i) &= \int_0^{\infty}\left(\delta_i(\tau_i)\prod_{k=1}^{K}Q(\Psi_{k,i}(\tau_i),\omega'_{k,i})\right)f_{2,i}(r)dr.\end{aligned} \tag{50}$$

If $u\in\mathscr{U}_{2,i}$ we have $\omega'_{k,i} = \left(\frac{r^{\alpha_i}B_kP_k}{B_iP_i}\right)^{1/\alpha_k}$.

Appendix B. Proof of Theorem 3.3

Similar to the proof of Theorem 3.1 and applying the law of total probability, the coverage probability of cell edge user under the Strict-FFR system is defined as,

$$\begin{aligned}S_{\mathrm{e}}^{\mathrm{FFR}}(i,\tau_i) &= \mathbb{P}[\mathrm{SINR}'(i,P_i)>\tau_i,\ \mathrm{SINR}(i,P_i)<\tau_{i,\mathrm{FR}}|u\in\mathscr{U}_i], = \mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}\tau_i}{P_i}\sigma^2\right)\right]\mathbb{E}_{\Phi,h}[\exp(-r^{\alpha_i}\tau_i\hat{I}_i)]\\ &-\mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}}{P_i}\left(\tau_iP_i\hat{I}_i+\tau_{i,\mathrm{FR}}\sum_{k=1}^{K}P_kI_k\right)\right)\right]\mathbb{E}_{\Phi,h}\left[\exp\left(-\frac{r^{\alpha_i}\sigma^2}{P_i}(\tau_i+\tau_{i,\mathrm{FR}})\right)\right],\\ &= \delta_i(\tau_i)\mathscr{L}_{\hat{I}_i}(\hat{\Psi}_{i,i}(\tau_i))\underbrace{-\mathscr{L}_{\hat{I}_i,I_1,\cdots,I_K}(\hat{\Psi}_{i,i}(\tau_i),\hat{\Psi}_{1,i}(\tau_{i,\mathrm{FR}}),\cdots,\hat{\Psi}_{K,i}(\tau_{i,\mathrm{FR}}))\delta_i(\tau_i+\tau_{i,\mathrm{FR}})}_{A},\end{aligned} \tag{51}$$

where $\hat{\Psi}_{k,i}(x) = \frac{P_kr^{\alpha_i}}{P_i}x$. The second Laplace transform part of (51) is computed as

$$\begin{aligned}A= &\ \mathbb{E}_{\Phi,h}\left[\exp\left(-\hat{\Psi}_{i,i}(\tau_i)\sum_{z\in\Phi_i\setminus x}\mathbf{1}(\rho_x=\rho_z)\hat{h}_{iz}y_{iz}^{-\alpha_i}\right)\exp\left(-\hat{\Psi}_{i,i}(\tau_{i,\mathrm{FR}})\sum_{z\in\phi_i}h_{iz}y_{iz}^{-\alpha_i}\right)\right]\prod_{k=1,k\neq i}^{K}\mathbb{E}_{\Phi,h}\left[\exp\left(-\hat{\Psi}_{k,i}(\tau_{i,\mathrm{FR}})\sum_{z\in\phi_k}h_{kz}y_{kz}^{-\alpha_k}\right)\right],\\ =&\ \mathbb{E}_{\Phi}\left[\prod_{z\in\Phi_i\setminus x}\frac{1}{1+\hat{\Psi}_{i,i}(\tau_{i,\mathrm{FR}})y_{iz}^{-\alpha_i}}\left(1-\frac{1}{\Delta_i}\left(1-\frac{1}{1+\hat{\Psi}_{i,i}(\tau)y_{iz}^{-\alpha_i}}\right)\right)\right]\prod_{k=1,k\neq i}^{K}\mathbb{E}_{\Phi}\left[\prod_{z\in\Phi_k}\frac{1}{1+\hat{\Psi}_{k,i}(\tau_{i,\mathrm{FR}})y_{kz}^{-\alpha_k}}\right].\end{aligned}$$

where $\mathbf{1}(\rho_x = \rho_z)$ takes on the value 1 when BS x and BS z use the same sub-band. Using the PGFL of PPPs and following the procedure of Theorem 3.1, we find the coverage probability of cell edge users under the Strict-FFR mechanism.

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