Finite-Blocklength Linear Superposition Coding for the Gaussian MAC with Quantized Feedback

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***Abstract*—We propose a linear transceiver scheme for the asymmetric two-user Gaussian multiple access channel with quantized feedback in the practically important regime of finite blocklength. Our scheme is an extension of results obtained for the asymptotic (infinite blocklength) regime. The quantized feedback link is modeled via the information bottleneck method. Our block-feedback superposition coding scheme splits the transmit power between an Ozarow-like linear feedback code and a conventional code that ignores the feedback. We study the achievable sum rate for fixed error probabilities as a function of the blocklength. We use asymptotically optimal power allocation and further optimize the superposition via finding the optimal trade-off in the blocklengths that maximizes the achievable sum rate of the constituent schemes of the superposition.**

I. Introduction

Advances in coding continue to close the performance gap between real-world communication systems and theoretical capacity limits. These fundamental limits of communication systems were usually studied in the asymptotic regime of blocklengths that tend to infinity. These asymptotic results constitute ultimate performance benchmarks that can be achieved only theoretically. By contrast, due to latency and complexity constraints real-world communication systems operate at blocklengths that typically are far away from the asymptotic regime. Practical fundamental performance limits for finite blocklength have been established by Polyanskiy [1], [2]. We use these results as a basis for investigating the finite blocklength performance limits of a linear superposition coding scheme for the asymmetric Gaussian multiple access channel (MAC) with quantized feedback.

Perfect feedback is known to enhance the capacity of the MAC [3]. For the single-user additive white Gaussian noise channel, Schalkwijk and Kailath proposed a remarkably simple linear feedback scheme that achieves capacity and yields an error probability that decreases doubly exponentially in the blocklength [4], [5]. Ozarow extended the Schalkwijk-Kailath scheme to the two-user Gaussian MAC with perfect feedback [6] and proved it to achieve the feedback capacity. Since the assumption of perfect feedback is unrealistic, later work focused on noisy feedback. Gastpar extended Ozarow's scheme to noisy feedback [7], [8] and Lapidoth and Wigger showed that non-perfect feedback is always beneficial [9]. Unfortunately, many of the schemes proposed do not have the simplicity of the original Ozarow scheme and the achievable rate regions are very hard to analyze. We previously proposed a simple, Ozarow-like superposition coding scheme for the infinite blocklength Gaussian MAC with quantized feedback [10], [11], were we optimized the power allocation to maximize the achievable sum rate. In this paper, we extend this work to the finite-blocklength regime and we find the optimal trade-off for the blocklengths of the constituent schemes.

The remainder of this paper is organized as follows. Section II summarizes our contributions and introduces the MAC model and proposed coding scheme including the block feedback. Section III provides further relevant background. In Section IV we formulate and solve the optimal resource allocation problem. Numerical results are presented in Section V and Section VI concludes the paper.

Notation: We use boldface letters for column vectors and upright sans-serif letters for random variables. The identity matrix is denoted by $\boldsymbol{I}$. Expectation is written as $\mathrm{E}\{\cdot\}$ and a Gaussian distribution with mean μ and variance σ^2 is denoted by $\mathcal{N}(\mu, \sigma^2)$. We use the notation $I(\cdot;\cdot)$ for the mutual information [12]. All logarithms are to base 2.

II. Contributions

The main contributions in this paper pertain to adaptations of the linear superposition coding scheme developed in our previous work, which address two important real-world constraints:

- Practical restrictions regarding latency and complexity necessitate a finite blocklength for real-world coding schemes. We thus propose a block feedback scheme that superimposes feedback-based encoding and conventional (non-feedback) coding. In contrast to other block feedback schemes (e.g., [13], [14]), our approach involves a systematic combination of the two constituent codes that allows their separation via successive cancellation in finite time.
- The feedback link in real communication systems can hardly be assumed to be perfect; at least it is quantized (rate-limited). As in our previous work on the asymptotic regime [10], [11] we model the feedback quantization in terms of channel output compression via the information bottleneck principle [15]–[18], which allows the receiver to use the quantization noise as side-information.

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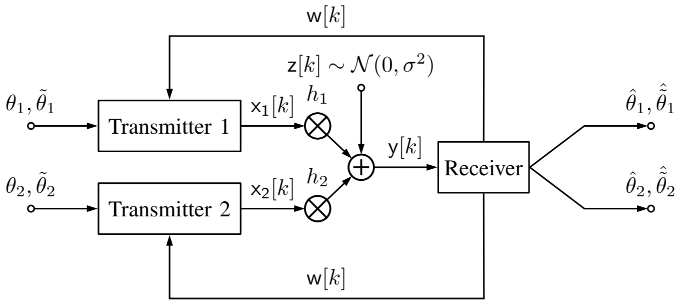

Figure 1: Two-user Gaussian MAC with common feedback link.

In the asymptotic regime we found that allocating the powers to maximize the achievable rates amounts to a difference of convex functions (DC) problem [19]. Here we show that in the finite blocklength regime the maximization of the achievable rates (for a prescribed blocklength and error probability) involves an additional combinatorial optimization problem to find the optimal splitting of the overall block between the two schemes. We solve this optimization problem numerically and show that the superposition scheme still yields larger achievable sum rates than each of the two constituent schemes.

A. MAC Model

We study the two-user asymmetric Gaussian MAC with quantized feedback (see Fig. 1) in the finite blocklength regime. Here, the two transmitters send independent length-n Gaussian signals $x_i[k]$, $i = 1, 2$, and the receiver observes the signal

$$y[k] = h_1 x_1[k] + h_2 x_2[k] + z[k], \tag{3}$$

where $z[k] \sim \mathcal{N}(0, \sigma^2)$ is i.i.d. additive Gaussian noise. The channel gains h_1 and h_2 are assumed to be known by both transmitters and by the receiver. We impose the average transmit power constraint (here, expectation is with respect to the messages and the channel noise)

$$\frac{1}{n}\sum_{k=1}^{n} \mathrm{E}\big\{x_1^2[k] + x_2^2[k]\big\} \leq P. \tag{4}$$

In order to enable successive cancellation decoding, the overall transmission block of length n is split into $\tilde{N}$ subblocks of length N each, i.e.,

$$n = \tilde{N}N. \tag{5}$$

We rearrange each transmit signal $x_i[k]$ into a $\tilde{N} \times N$ matrix $\boldsymbol{X}^{(i)}$ (see (1)) to fit into this block structure,

$$[\boldsymbol{X}^{(i)}]_{j,l} = x_i[(j-1)N + l], \tag{6}$$

Each row of $\boldsymbol{X}^{(i)}$ represents one subblock that encodes one conventional codeword. By contrast, the feedback-based codewords are encoded in the columns of $\boldsymbol{X}^{(i)}$, i.e., over corresponding time slots in all subblocks.

The noise is rearranged in the same way as

$$[\boldsymbol{Z}]_{j,l} = z[(j-1)N + l]. \tag{7}$$

The received signal in matrix form reads

$$\boldsymbol{Y} = h_1\boldsymbol{X}^{(1)} + h_2\boldsymbol{X}^{(2)} + \boldsymbol{Z}. \tag{8}$$

B. Proposed Coding Scheme

The proposed coding scheme superimposes a feedback-based coding scheme with a conventional coding scheme that completely ignores the feedback. Throughout this paper we assume that we can perfectly separate these superimposed codes. The receiver first decodes the conventional code and then cancels it from the receive signal. This type of decoding is usually referred to as successive cancellation and requires that the conventional codeword is completely received before it can be cancelled. At first glance, this seems to contradict the assumption that the received signal is fed back to the transmitters in each time instance. This seeming contradiction can be resolved using the subblock structure introduced above along with per subblock feedback.

More specifically, let $\boldsymbol{w}_l^{(j-1)} = (w[1]\dots w[(j-1)N+l])^\top$ denote the past quantized feedback up to time $k = (j-1)N+l$ (i.e., the lth time slot in subblock j). The linear superposition codewords are obtained as (see (2))

$$[\boldsymbol{X}^{(i)}]_{j,l} = \varphi_{i,l}(\theta_{i,j}) + \tilde{\varphi}_{i,j}\big(\tilde{\theta}_{i,l}, \boldsymbol{w}_l^{(j-1)}\big). \tag{9}$$

Here, $\theta_{i,j}$, $j = 1, \dots, \tilde{N}$ and $\tilde{\theta}_{i,l}$, $l = 1, \dots, N$ are independent messages of user i. Throughout the paper, superscript

$$\boldsymbol{X}^{(i)} = \begin{pmatrix} x_i[1] & x[2] & \dots & x_i[N] \\ x_i[N+1] & x_i[N+2] & \dots & x_i[2N] \\ x_i[2N+1] & x_i[2N+2] & \dots & x_i[3N] \\ \vdots & \vdots & \ddots & \vdots \\ x_i[(\tilde{N}-1)N+1] & x_i[(\tilde{N}-1)N+2] & \dots & x_i[\tilde{N}N] \end{pmatrix} \tag{1}$$

$$= \underbrace{\begin{pmatrix} \boxed{\varphi_{i,1}(\theta_{i,1}) \quad \varphi_{i,2}(\theta_{i,1}) \quad \dots \quad \varphi_{i,N}(\theta_{i,1})} \\ \boxed{\varphi_{i,1}(\theta_{i,2}) \quad \varphi_{i,2}(\theta_{i,2}) \quad \dots \quad \varphi_{i,N}(\theta_{i,2})} \\ \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad\quad \vdots \\ \boxed{\varphi_{i,1}(\theta_{i,\tilde{N}}) \quad \varphi_{i,2}(\theta_{i,\tilde{N}}) \quad \dots \quad \varphi_{i,N}(\theta_{i,\tilde{N}})} \end{pmatrix}}_{\text{message } \theta_{i,j} \text{ is conventionally encoded in row (subblock) } j} + \underbrace{\begin{pmatrix} \boxed{\begin{matrix} \tilde{\varphi}_{i,1}\big(\tilde{\theta}_{i,1}, 0\big) \\ \tilde{\varphi}_{i,2}\big(\tilde{\theta}_{i,1}, \boldsymbol{w}_1^{(1)}\big) \\ \vdots \\ \tilde{\varphi}_{i,\tilde{N}}\big(\tilde{\theta}_{i,1}, \boldsymbol{w}_1^{(\tilde{N}-1)}\big) \end{matrix}} & \boxed{\begin{matrix} \tilde{\varphi}_{i,1}\big(\tilde{\theta}_{i,2}, 0\big) \\ \tilde{\varphi}_{i,2}\big(\tilde{\theta}_{i,2}, \boldsymbol{w}_2^{(1)}\big) \\ \vdots \\ \tilde{\varphi}_{i,\tilde{N}}\big(\tilde{\theta}_{i,2}, \boldsymbol{w}_2^{(\tilde{N}-1)}\big) \end{matrix}} & \begin{matrix} \dots \\ \dots \\ \ddots \\ \dots \end{matrix} & \boxed{\begin{matrix} \tilde{\varphi}_{i,1}\big(\tilde{\theta}_{i,N}, 0\big) \\ \tilde{\varphi}_{i,2}\big(\tilde{\theta}_{i,N}, \boldsymbol{w}_N^{(1)}\big) \\ \vdots \\ \tilde{\varphi}_{i,\tilde{N}}\big(\tilde{\theta}_{i,N}, \boldsymbol{w}_N^{(\tilde{N}-1)}\big) \end{matrix}} \end{pmatrix}}_{\text{message } \tilde{\theta}_{i,l} \text{ is feedback-based encoded in column } l} \tag{2}$$

tilde indicates quantities based on the exploitation of the channel output feedback. The messages are uniformly drawn from finite sets with cardinalities $\mathcal{M}_i = 2^{NR_i}$, $\widetilde{\mathcal{M}}_i = 2^{\tilde{N}\tilde{R}_i}$, $i = 1, 2$. Furthermore, $\varphi_{i,l} : \mathcal{M}_i \to \mathbb{R}$ denotes a conventional encoder with power constraint

$$\frac{1}{N}\sum_{l=1}^{N} \mathrm{E}\{\varphi_{i,l}^2(\theta_{i,j})\} \leq P_i, \tag{10}$$

where the expectation is over all possible transmit messages. The conventional encoder completely ignores the feedback signal. Similarly, $\tilde{\varphi}_{i,j} : \widetilde{\mathcal{M}}_i \times \mathbb{R}^{k-1} \to \mathbb{R}$ is the feedback-based encoder with power constraint

$$\frac{1}{\tilde{N}}\sum_{j=1}^{\tilde{N}} \mathrm{E}\Big\{\tilde{\varphi}_{i,j}^2\big(\tilde{\theta}_{i,l}, \mathbf{w}_l^{(j-1)}\big)\Big\} \leq \tilde{P}_i, \tag{11}$$

with the expectation being over all possible transmit messages and channel realizations. This encoder has causal access to the quantized feedback and works as in the original Ozarow scheme (see Section III-B).

The above (sub)block encoding and feedback scheme enables us to decode a conventional codeword after each subblock and to cancel this component in order to decode the feedback-based components, like in standard successive cancellation. Each conventional codeword is decoded after N time slots whereas the feedback-based codewords are spread over $\tilde{N}$ blocks and decoded after $N(\tilde{N}-1)$ time slots. Thus, the coding scheme with blocklength $n = \tilde{N}N$ has an effective average blocklength of

$$\nu = \frac{(R_1 + R_2)N + (\tilde{R}_1 + \tilde{R}_2)N(\tilde{N}-1)}{R_1 + R_2 + \tilde{R}_1 + \tilde{R}_2} < n. \tag{12}$$

C. Feedback Quantization

In our model, the channel output at the receiver is quantized before being fed back to the transmitters. More specifically, the receiver performs successive cancellation, i.e., it subtracts the estimates $\varphi_{i,l}(\hat{\theta}_{i,j})$ of the conventional codewords in subblock j from the received signal (i.e., from the jth row of $\boldsymbol{Y}$ in (8)) and then quantizes the residual,

$$y'_{j,l} = y[(j-1)N + l] - h_1\,\varphi_{1,l}(\hat{\theta}_{1,j}) - h_2\,\varphi_{2,l}(\hat{\theta}_{2,j}), \tag{13a}$$

$$w[(j-1)N + l] = \mathcal{Q}\big(y'_{j,l}\big). \tag{13b}$$

The quantization $\mathcal{Q}(\cdot)$ is modeled as an information bottleneck, i.e., the mutual information between compressed received signal and transmit signal is maximized under a rate constraint. It was shown in [17] that rate-information optimal channel output compression amounts to additive Gaussian quantization noise, whose variance in our setup is given by

$$\sigma_q^2 = \sigma^2 \frac{1 + (h_1^2\tilde{P}_1 + h_2^2\tilde{P}_2)/\sigma^2}{2^{2R} - 1}. \tag{14}$$

Here, R denotes the quantization rate. For the MAC with noisy feedback the extra noise on the feedback channel is detrimental since it is not known by any node. In our model the transmitters also receive degraded versions of the channel output, but the receiver knows the quantization error of the feedback signal (having itself performed the quantization). With this observation, our scheme can be reduced to an equivalent MAC with perfect feedback in which the quantization error represents additional channel noise. The knowledge of this extra noise is exploited at the receiver.

III. Background

A. Gaussian Information Bottleneck

The feedback in our scheme is obtained by quantizing the output of the Gaussian MAC (after subtracting the estimates of the conventional codewords, cf. (13)). We thus briefly review Gaussian channel output compression [17] based on the Gaussian information bottleneck (GIB) [15], [16].

Consider the Markov chain $\mathbf{x} - \mathbf{y} - \mathbf{w}$ where $\mathbf{x}$ is the channel input, $\mathbf{y}$ is the channel output, and $\mathbf{w}$ is the feedback signal obtained by compressing $\mathbf{y}$. We assume that $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian random vectors with zero mean and full-rank covariance matrix. The trade-off between compression rate and relevant information is captured by the rate-information function $I: \mathbb{R}_+ \to [0, I(\mathbf{x}; \mathbf{y})]$, which is defined as

$$I(R) \triangleq \max_{p(\mathbf{w}|\mathbf{y})} I(\mathbf{x}; \mathbf{w}) \quad \text{subject to } I(\mathbf{y}; \mathbf{w}) \leq R. \tag{15}$$

Here $p(\mathbf{w}|\mathbf{y})$ denotes the conditional distribution of $\mathbf{w}$ given $\mathbf{y}$. The channel input $\mathbf{x}$ is the relevance variable, $I(\mathbf{x}; \mathbf{w})$ is the relevant information in $\mathbf{w}$ about $\mathbf{x}$, and $I(\mathbf{y}; \mathbf{w})$ is the compression rate. The rate-information function thus quantifies the maximum amount of relevant information that can be preserved when the compression rate is at most R. The definition (15) is similar to rate-distortion theory, only that the minimization of distortion is replaced with a maximization of the relevant information.

In [17] it was shown that optimal compression of $\mathbf{y}$ in the sense of the rate-information function yields an equivalent channel $p(\mathbf{w}|\mathbf{x})$ which is also Gaussian. Therefore, a Gaussian input distribution $p(\mathbf{x})$ satisfying the power constraint of the channel $p(\mathbf{y}|\mathbf{x})$ with equality is capacity-achieving also for the channel $p(\mathbf{w}|\mathbf{x})$.

The resulting rate-information function at SNR $\gamma = P/\sigma^2$ can be shown to equal [17, Theorem 5]

$$I(R) = C(\gamma) - \frac{1}{2}\log(1 + 2^{-2R}\gamma), \tag{16}$$

with the AWGN capacity

$$C(\gamma) \triangleq \frac{1}{2}\log(1 + \gamma). \tag{17}$$

Thus, the rate-information function approaches channel capacity as the compression rate R goes to infinity. Equivalently, we can write $I(R) = C(\eta)$, where

$$\eta = \gamma\frac{1 - 2^{-2R}}{1 + 2^{-2R}\gamma} \leq \gamma \tag{18}$$

is the equivalent SNR of the channel $p(\mathbf{w}|\mathbf{x})$. Rate-information optimal channel output compression can thus be modeled via additive Gaussian quantization noise with variance

$$\sigma_q^2 = \sigma^2\frac{1 + \gamma}{2^{2R} - 1}. \tag{19}$$

Our two-user MAC model in (3) can be rewritten as a multiple-input, single-output model in [18, Section V.-C.] so that (19) becomes (14).

B. Asymptotically Optimal Power Allocation

Ozarow derived the perfect-feedback capacity of the two-user Gaussian MAC by extending the Schalkwijk-Kailath scheme. The key concept of this scheme is the iterative refinement of the message estimate. In the first two time slots, both transmitters alternately send their raw messages. The remaining time slots are used to transmit updates of the message estimates at the receiver. For infinite blocklength, this simple linear scheme is capacity-achieving. On top of the Ozarow feedback scheme we superimpose the conventional coding scheme (cf. Section II-B), resulting in a sum capacity of [11]

$$C_S = \max_{\mathbf{1}^\top \boldsymbol{p} \leq P} \bar{C}(\boldsymbol{p}), \tag{20}$$

where $\boldsymbol{p} = (P_1, P_2, \tilde{P}_1, \tilde{P}_2)^\top$ is the power allocation vector and

$$\begin{aligned}\bar{C}(\boldsymbol{p}) = C&\left(\frac{h_1^2 P_1}{\sigma^2 + h_1^2 \tilde{P}_1 + h_2^2 \tilde{P}_2}\right)\\ &+ C\left(\frac{h_2^2 P_2}{\sigma^2 + h_1^2 P_1 + h_1^2 \tilde{P}_1 + h_2^2 \tilde{P}_2}\right)\\ &+ C^{\mathrm{FB}}(h_1^2 \tilde{P}_1, h_2^2 \tilde{P}_2, \sigma^2 + \sigma_q^2).\end{aligned} \tag{21}$$

Here, C^{FB} is the Ozarow capacity [6]. Without loss of generality, we assume that the data of user 2 is decoded before that of user 1. The first two terms in (21) reflect the classical Gaussian MAC capacity with successive cancellation. We showed in [11] that $\bar{C}(\boldsymbol{p})$ can be split into a concave part $\mathsf{C}^{\cap}(\boldsymbol{p})$ and a convex part $\mathsf{C}^{\cup}(\boldsymbol{p})$ and thus

$$C_S = \max_{\mathbf{1}^\top \boldsymbol{p} \leq P} \mathsf{C}^{\cap}(\boldsymbol{p}) + \mathsf{C}^{\cup}(\boldsymbol{p}). \tag{22}$$

This sum rate maximization problem can be solved by difference of convex functions (DC) programming [20].

IV. Optimization

Optimization in the context of finite blocklength means finding the optimum subblock splitting $(N, \tilde{N})$ that maximizes the achievable rates under the constraint $n = \tilde{N}N$ for prescribed blocklength n and error probability.

A. Error Probabilities

The error probabilities $\tilde{P}_{e_i}, i = 1, 2$ of the feedback based scheme are known to decrease doubly exponentially in the blocklength $\tilde{N}$ [6], whereas the error probabilities $P_{e_i}, i = 1, 2$ of the conventional coding scheme decrease only exponentially in the blocklength N [1], [2]. Due to the monotonicity of $\tilde{P}_{e_i}(\tilde{N})$ and $P_{e_i}(N)$ and the relation $n = \tilde{N}N$, decreasing the error probability of one component (via larger blocklength) increases the error probability of the other component (due to shorter blocklength).

The block structure of the superposition coding couples the error probabilities; if the conventional codeword is decoded incorrectly, it cannot be cancelled successively from the superposition and therefore negatively affects the decoding of the feedback based codeword.

1) Error probability of feedback based coding component: The error probability expression of the feedback based coding was bounded by Ozarow [6] for perfect feedback. Adapted to our superposition scheme with quantized feedback we have

$$\begin{aligned}\tilde{P}_{e_i}(\tilde{R}_i, \tilde{N}) \leq 2Q&\left\{\frac{\sigma_s^2}{2\sqrt{\alpha_{i,2}}(\sigma_s^2 + h_i^2 \tilde{P}_i(1 - \rho^{*2}))} \times\right.\\ &\left.\exp\left[\tilde{N}\left(\frac{1}{2}\log\left(\frac{\sigma_s^2 + h_i^2 \tilde{P}_i(1 - \rho^{*^2})}{\sigma_s^2}\right) - \tilde{R}_i\right)\right]\right\},\end{aligned} \tag{23}$$

$i = 1, 2$, with appropriately chosen $\alpha_{i,2}$. ρ^* is the optimal correlation between the feedback components of the transmit signals $\tilde{\varphi}_{1,j}$ and $\tilde{\varphi}_{2,j}$ that maximizes the achievable sum rate. σ_s^2 is the resulting (interference) noise variance, which is given by

$$\sigma_s^2 = \sigma^2 + \sigma_q^2 + P_1 P_{e_1} + P_2 P_{e_2} \tag{24}$$

in this superposition configuration. P_{e_i} are the error probabilities of the conventional coding scheme and the $P_i P_{e_i}$ terms are due to the decoding errors of the conventional coding scheme that cause additional interference. Due to the block coding structure this additional interference is i.i.d. and can be modeled to be Gaussian as well if $\tilde{N}$ is not too small. We choose $\alpha_{i,2}$ as in [6]

$$\alpha_{i,2} = \frac{\sigma_s^2 + \sigma_v^2}{12 h_i^2 \tilde{P}_i}, \tag{25}$$

where σ_v^2 is the variance of an initial common randomness v known to both transmitters. σ_v^2 is chosen such that the correlation remains constant over time,

$$\rho_2 = \frac{\mathrm{E}\{\epsilon_{1,2,l}\epsilon_{2,2,l}\}}{\sqrt{\alpha_{1,2}\alpha_{2,2}}} = \frac{\sigma_v^2}{\sigma_s^2 + \sigma_v^2} \triangleq \rho^*. \tag{26}$$

Here, $\alpha_{i,j}$ are the variances of the current message estimation errors $\epsilon_{i,j,l}$,

$$\epsilon_{i,j,l} = \tilde{\theta}_{i,l} - \hat{\tilde{\theta}}_{i,l}(\boldsymbol{y}'^{((j-1)N+l)}). \tag{27}$$

Note that $\alpha_{i,j}$ is identical for all l because $\tilde{\theta}_{i,l}$, $l = 1, \ldots, N$ are i.i.d.. The initial estimation errors simply yield

$$\epsilon_{1,2,l} = \tilde{\theta}_{1,l} - \hat{\tilde{\theta}}_{1,l}(\boldsymbol{y}'^{(N+l)}) = \tilde{\theta}_{1,l} - \frac{y'[l]}{\sqrt{12 h_1^2 \tilde{P}_1}}, \tag{28}$$

$$\epsilon_{2,2,l} = \tilde{\theta}_{2,l} - \hat{\tilde{\theta}}_{2,l}(\boldsymbol{y}'^{(N+l)}) = \tilde{\theta}_{2,l} - \frac{y'[N+l]}{\sqrt{12 h_2^2 \tilde{P}_2}}, \tag{29}$$

since the first two timeslots (of the feedback scheme) are exclusively used to transmit the messages from both transmitters alternately,

$$y'[l] = \sqrt{12 h_1^2 \tilde{P}_1}\tilde{\theta}_{1,l} + z[l] + v, \tag{30}$$

$$y'[N+l] = \sqrt{12 h_2^2 \tilde{P}_2}\tilde{\theta}_{2,l} + z[N+l] + v \tag{31}$$

Using these relations we directly get the second equality in (26) and then solving (26) for σ_v^2 yields

$$\sigma_v^2 = \sigma_s^2 \frac{\rho^*}{1 - \rho^*}. \tag{32}$$

Finally,

$$\alpha_{i,2} = \frac{\sigma_s^2}{12 h_i^2 \tilde{P}_i (1 - \rho^*)}. \tag{33}$$

2) Error probability of conventional coding component: The error probability expression of the conventional coding scheme can be obtained by adapting the results for the AWGN channel in the finite blocklength regime obtained by Polyanskiy in [1, Theorem 54]. In our superposition scenario we have

$$P_{e_i}(R_i, N) \leq 2Q\left\{\frac{1}{\sqrt{V_i}}\left[\sqrt{N}(C_i - R_i) + \frac{1}{\sqrt{N}}\frac{1}{2}\log N\right]\right\}, \tag{34}$$

$i = 1, 2$, with modified channel dispersion $V_i = h_i^2 P_i/2\,(h_i^2 P_i + 2(\sigma^2 + h_1^2 \tilde{P}_2 + h_1^2 \tilde{P}_2))/(h_i^2 P_i + \sigma^2 + h_1^2 \tilde{P}_2 + h_1^2 \tilde{P}_2)^2 \log^2 e$. Note that here the capacity of the conventional coding scheme is affected by the interference of the feedback based coding and results in a reduced capacity $C_i = \frac{1}{2}\log\left(1 + h_i^2 P_i/(\sigma^2 + h_1^2 \tilde{P}_2 + h_1^2 \tilde{P}_2)\right)$. In our MAC scenario the optimal power allocation ensures that the additional sum rate constraint holds.

B. Achievable Rates over Block Length and Constant Error Probability

Instead of investigating the error probability as a function of the blocklength we are now interested in the achievable rates over blocklength and constant error probability. This is a more practical approach, since usually applications require a specific rate and maximal error probability.

The achievable rate of the feedback based code is found by solving (23) for $\tilde{R}_i$,

$$\tilde{R}_i(\tilde{P}_{e_i}, \tilde{N}) \leq \frac{1}{2}\log\left(\frac{\sigma_s^2 + h_i^2 \tilde{P}_i(1-\rho^{*^2})}{\sigma_s^2}\right) - \frac{1}{\tilde{N}}\log\left[Q^{-1}\left\{\frac{\tilde{P}_{e_i}}{2}\right\}\frac{2\sqrt{\alpha_{i,2}}(\sigma_s^2 + h_i^2 \tilde{P}_i(1-\rho^{*^2}))}{\sigma_s^2}\right], \tag{35}$$

$i = 1, 2$, and the achievable rate of the conventional code is found by solving (34) for R_i,

$$R_i(P_{e_i}, N) \leq C_i - \frac{1}{\sqrt{N}}\sqrt{V_i}Q^{-1}\left\{\frac{P_{e_i}}{2}\right\} + \frac{1}{2}\log \sqrt[N]{N}, \tag{36}$$

$i = 1, 2$.

C. Optimization Problem

We want to find the blocklength allocation that maximizes the achievable sum rate

$$R_s = \sum_{i=1}^{2}\left(R_i(P_{e_i}, N) + \tilde{R}_i(\tilde{P}_{e_i}, \tilde{N})\right), \tag{37}$$

while keeping the effective average error probability

$$\bar{P}_e = \frac{1}{R_s}\sum_{i=1}^{2}\left(R_i(P_{e_i}, N)P_{e_i} + \tilde{R}_i(\tilde{P}_{e_i}, \tilde{N})\tilde{P}_{e_i}\right) \tag{38}$$

below a certain value ϵ. $\tilde{R}_i(\tilde{P}_{e_i}, \tilde{N})$ and $R_i(P_{e_i}, N)$ are given by (35) and (36), respectively. Generally, this leads to the non-convex and partly combinatorial optimization problem

$$\begin{aligned} \text{maximize} \quad & R_s \\ \text{subject to} \quad & \bar{P}_e \leq \epsilon \\ & \tilde{N}N = n. \end{aligned} \tag{39}$$

A simpler approach is to restrict each of the individual error probabilities to be smaller than ϵ. This is indeed more reasonable, since the data from the two transmitters usually carries independent information and thus requires certain error rates. Thus, the optimization simplifies to

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{2}\left(R_i(P_{e_i}, N) + \tilde{R}_i(\tilde{P}_{e_i}, \tilde{N})\right) \\ \text{subject to} \quad & P_{e_i} \leq \epsilon, \qquad i = 1, 2 \\ & \tilde{P}_{e_i} \leq \epsilon, \qquad i = 1, 2 \\ & \tilde{N}N = n, \end{aligned} \tag{40}$$

which is still a non-convex and partly combinatorial optimization problem.

We tackle both optimizations by initially finding the optimal transmit power allocation and then optimize the blocklength allocation. The optimal power allocation is found by maximizing the achievable sum rate, i.e. finding the capacity of our linear superposition coding scheme. This was previously studied in [10], [11]. We then use this (asymptotically) optimal power allocation to find the optimal blocklength allocation. Generally, this approach might not be equivalent to the global optimization since the asymptotically optimal power allocation is not necessarily optimal for finite blocklengths. This is due to the fact that both constituent schemes do not show the same error decay behaviour. The feedback coding scheme has a doubly exponential decreasing error probability, whereas the conventional coding scheme only decreases exponentially in blocklength. Thus, one has to take this into account when optimizing the power allocation. Unfortunately, joint optimization of power allocation and blocklength allocation seems intractable. Nevertheless, numerical results show that this consecutive optimization outperforms pure conventional or feedback coding in most regions.

V. NUMERICAL RESULTS

For what follows, we work with the approximation $\sigma_s^2 \approx \sigma^2 + \sigma_q^2$, i.e., $P_{e_i} \ll (\sigma^2 + \sigma_q^2)/P_i$. In the regime of interest this approximation is sufficiently accurate. This simplifies the numerical evaluation since it allows us to independently analyze the practically decoupled error probability expressions.

A. Impact of Blocklength Allocation

In this section we start with (asymptotically) optimal power allocations (Table I) and assume a symmetric channel, i.e., identical channel gains. A similar analysis for asymmetric channels is feasible but less intuitive.

It turns out that asymptotically optimal power allocation does not yield higher achievable sum rates for moderate blocklength and moderate quantization rates. In the example in Fig. 2 with blocklength $n = 2^{10} = 1024$ (left-hand side plots) only feedback quantization rates $R > 5$ result in better performance than coding without any feedback. Even in this case the optimal region allocates no blocklength to the conventional coding component. Note that $n = N\tilde{N}$. Thus, $N = 1$ if the optimum allocates the total blocklength

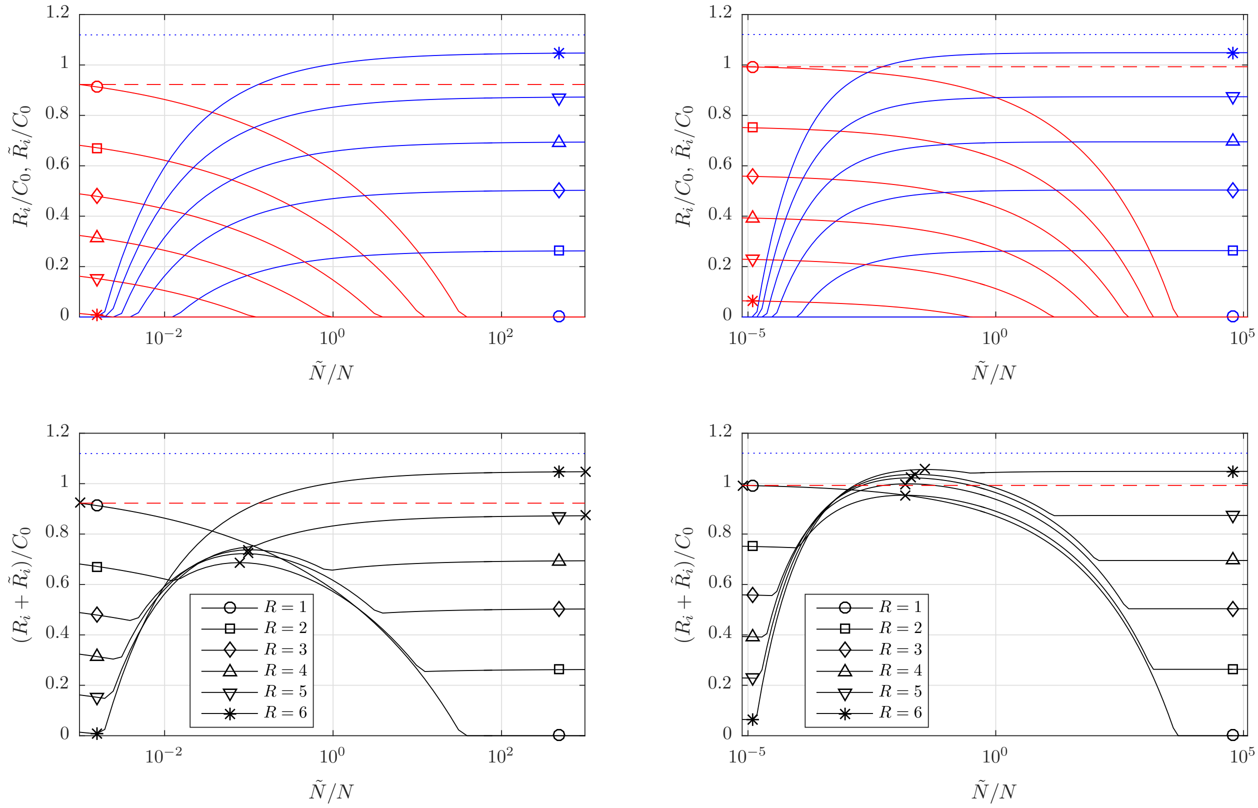

Figure 2: Achievable rates of conventional coding component (top, decreasing curves) and feedback coding component (top, increasing curves) and achievable sum rates (bottom, maxima marked with 'x') versus the blocklength allocation $\tilde{N}/N$ for $n = 2^{10}$ (left) and $n = 2^{17}$ (right). All plots pertain to the symmetric MAC with $P = 2$, $h_1 = h_2 = 1$, $\sigma^2 = 0.01$, $\epsilon \leq 10^{-6}$, and power allocation according to Table I. The dashed and dotted line show the Polyanskiy limit C_ϵ and $\tilde{C}_0$, respectively.

R [bit]	P_i	$\tilde{P}_i$
1	0.999	0.001
2	0.987	0.013
3	0.954	0.045
4	0.884	0.115
5	0.717	0.282
6	0.309	0.691

Table I: Asymptotically optimal power allocation (parameters as in Fig. 2). Details on how to determine this optimal power allocation can be found in [11].

to the feedback based coding component, i.e., $\tilde{N} = n$. For larger blocklength the situation drastically changes. In the example in Fig. 2 with blocklength $n = 2^{17} = 131072$ (right-hand side plots) superposition yields higher achievable sum rates for a wide region of blocklength allocation. Indeed superposition is optimal, except for very low quantization rates, and clearly outperforms pure conventional or feedback coding. We observe that the fraction of blocklength allocated to each component is primarily determined by the channel itself and only weakly depends on the actual power allocations, feedback rates or quantization rates. The next section also confirms this insight.

B. *Impact of Blocklength*

Again, we start with (asymptotically) optimal power allocations and assume a symmetric channel. We observe that even if the asymptotically optimal power allocation splits the power to both constituent schemes, the blocklength might not be allocated in the same proportions. There are essentially three different scenarios.

1) Very low feedback quantization rates: The quantization rate is too low to become beneficial at all. Thus, pure conventional coding is optimal. In Fig. 3 we can observe this behaviour e.g. for $R = 1$ (circle marker).

2) Low feedback quantization rates: In this case only for very small blocklengths pure feedback coding is optimal and then suddenly switches to pure conventional coding. Eventually for large blocklengths superposition is optimal. In Fig. 3 we can observe this behaviour e.g. for $R = 2$ (square marker).

3) Medium to high feedback quantization rates: Here, at first pure feedback coding is optimal and then directly switches to superposition coding. In Fig. 3 we can observe this behaviour e.g. for $R = 3$ (diamond marker).

Fig. 3 (bottom) shows the optimal blocklength allocations. Note that in the region of superposition the growth rates of blocklength are identical and independent of the quantization

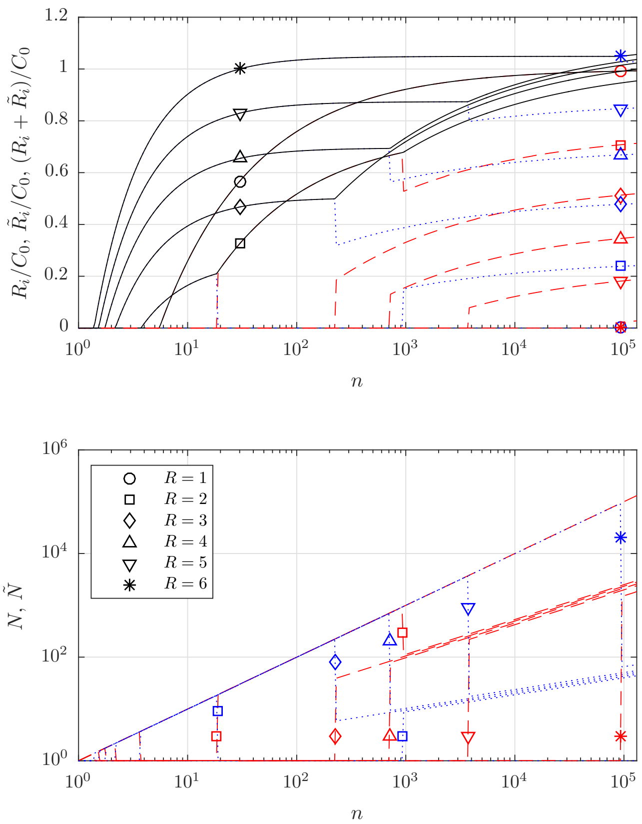

Figure 3: Achievable rates (top) of conventional coding component (top and bottom, dashed curves) and feedback coding component (top and bottom, dotted curves) and achievable sum rates (top, solid curves) with optimal blocklength allocation versus the blocklength n. Optimal blocklength allocations (bottom). All plots pertain to the symmetric MAC with $P = 2$, $h_1 = h_2 = 1$, $\sigma^2 = 0.01$, $\epsilon \leq 10^{-6}$, and power allocation according to Table I.

rates and thus the curves highly overlap. As a consequence, $\tilde{N}/N$ is almost independent of the quantization rates in this region.

VI. Conclusions

We used the information bottleneck principle to model the quantization of the feedback in a two-user asymmetric Gaussian MAC. We proposed a block feedback coding scheme that superimposes a modified version of the Ozarow scheme and a conventional (non-feedback) encoding scheme. Due to the rate limitation on the common feedback link this superposition was useful in the asymptotic regime and we showed that this holds for finite blocklengths as well. This block structure enabled us to separate the constituent schemes by successive cancellation. In addition to the optimization of the power allocation in the asymptotic regime, here we had the blocklength allocation as an additional optimization parameter. Since the optimal power allocation is itself a difference of convex functions problem, the additional combinatorial blocklength optimization renders the resulting problem hard to solve. Thus, using the asymptotically optimal power allocations from our previous work on the Gaussian MAC in the asymptotic regime we showed that the optimization of blocklength allocation generally still yields larger achievable sum rates than any of the two constituent schemes alone. Finally, we numerically assessed the impact of the size of the blocklength on the optimality of the superposition.

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