Variational Inequality Approach to Spectrum Balancing in Vectoring xDSL Networks

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***Abstract*—We study the multi-user and multi-carrier power allocation (spectrum balancing) problem in digital subscriber line (DSL) networks under inter-carrier and inter-user interference as well as self-interference. The key assumption of this work is that we do *not* have knowledge of the interference coefficients, but only have access to the total per-line interference noise power. Furthermore, lines may support different technologies or be part of linear crosstalk cancellation (Vectoring) groups, which implies coupling among lines through *per-transceiver* sum-power and spectral mask constraints.**

We propose a novel generalized Nash equilibrium formulation for spectrum balancing in a virtual network with modified self-interference coefficients and an associated variational inequality problem. The latter allows for intuitive analytical criteria for algorithms' convergence to a unique solution in the virtual network as well as in the real network, related to the sum of interference coefficients. Exemplarily a fully autonomous projected gradient algorithm is proposed which requires no information exchange among Vectoring groups and obeys transmit power limitations throughout its run-time. The applicability of the proposed framework is demonstrated by testing this algorithm on two previously studied power allocation problems in Vectoring G.fast networks.

I. Introduction

Linear crosstalk cancelation (Vectoring) is applied in the latest multi-carrier digital subscriber line (DSL) generations, including Very-high-speed DSL 2 (VDSL2) [1] and G.fast [2]. It is well known that uncanceled interference inside or among groups of Vectoring DSL lines is highly detrimental to their achievable data-rates [3], [4]. This includes inter-carrier interference (ICI) among carriers of a single user ("self-interference") and among users (i.e., transceivers). This work builds on the assumption that the interference channel coefficients are unknown, and only the interference noise power is accessible. This information shall be used by the users for the allocation of power over carriers ("spectrum balancing") with the goal to improve their data-rates *individually*, while at all times remaining compliant with transmit power limitations.

Numerous autonomous spectrum balancing algorithms have been proposed in the DSL literature [5]–[12]. In the most narrow sense these let each user decide individually on its power allocation and do neither require run-time information exchange among users nor information on interference power gains. The latter may not be available in many practical situations, such as when multiple DSL access multiplexers (DSLAMs) are deployed on the same cable but at different locations / by different operators, or when interference is the result of unknown errors in channel state information [4]. As the transmit power of users in a Vectoring group is controlled by a single entity, our goal is the design of autonomous algorithms which control the transmit power in each Vectoring group individually and do not require information exchange among groups. This problem is in general NP-hard, which follows from the fact that even the special case with no linear precoding coupling among power constraints, no ICI, and no self-interference was shown to be NP-hard in [13]. In [14] the authors deal with distributed successive convex approximation algorithms for difference of convex functions (DC) programming with coupling constraints, which covers our rate-maximization problem. However, the applied linearization requires the knowledge of interference channel gains, which we assume are not available. This in fact motivates our game-theoretical model where each user aims to maximize its data-rate irrespective of its generated interference.

Autonomous spectrum balancing in the form of an iterative application of single-user waterfilling has for instance been proposed and analyzed a) for multi-user and multi-carrier rate-maximization under co-channel interference and decoupled sum-power constraints [5], under additional spectral mask constraints [6], and under synchronous as well as asynchronous power updates [10]; b) for single-user multi-carrier systems with ICI [15] ("self-interference"); or c) for multi-user multi-carrier networks with ICI [7], [8]. The addition of static pricing in the form of "virtual lines" led to further autonomous algorithms for the latter problem in [9]. Projected gradient algorithms for a rate-maximization interference game under *decoupled* power constraint were proposed in [16] based on the equivalence of the corresponding Nash equilibrium problem with a variational inequality problem. However, these algorithms and analyses are not directly extendable to the case where there is linear coupling among per-transceiver power constraints.

The *interference-free* power allocation problem in a single Vectoring group is convex and has been solved using dual relaxation and projected gradient updates in [17]. Multi-user spectrum balancing heuristics for imperfect Vectoring multi-carrier DSL under co-channel interference, self-interference, and linear precoding power constraints have been proposed in [12], including an autonomous "bounded iterative waterfilling" algorithm. However, the latter has been derived without an underlying problem formulation and lacks a convergence

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analysis. In fact, we will see that it can be interpreted as a heuristic for solving the game model proposed in this work, a generalized Nash equilibrium problem (GNEP) [18].

A GNEP formulation of sum-power minimization over Gaussian parallel interference channels with *coupling per-user rate constraints* has been analyzed in [11]. Differently, we target sum-rate maximization subject to linear coupling constraints on users' power allocation based on given per-carrier precoding matrices. A resembling problem has in fact been analyzed in [19] in the context of vector power allocation for secondary users in cognitive radio networks with linear *coupling interference temperature* constraints and later extended in [20] to the multiple-input multiple-output interference channel under uncertain coupling constraint coefficients. In this problem the secondary users are assumed to be distributed, which motivates a decomposed pricing-based approach for handling the coupling constraints. However, this analysis does not capture self-interference and ICI, and besides an additional price-control-loop entails potentially infeasible power allocations during the run-time of the algorithm. Differently, we consider a more general interference model and a fully autonomous primal projected gradient algorithm. It is motivated by the specific structure of our constraint set with coupling among users being restricted to Vectoring groups where a joint projection is feasible without incurring communication overhead. This has the advantage that in each iteration the power allocation remains primal feasible and therefore compliant with mandatory transmit power limitations.

Our contributions are summarized as follows: Based on the system model in Section II, in Section III we propose a fully distributed and autonomous projected gradient algorithm obeying coupling power constraints throughout its run-time. The central results of this work are stated in Section IV which contains a) a game-theoretical model for multi-user rate-maximizing power allocation which is specifically designed to capture our assumption of unknown interference coefficients, linear coupling power constraints (for given linear precoding matrices), ICI, as well as self-interference (on the same carrier or among carriers); b) an associated variational inequality (VI) problem formulation which recovers specific solutions of the game model; and c) sufficient criteria for the uniqueness of a VI solution and for the convergence of the projected gradient algorithm. Lastly, the algorithm is exemplarily tested in Section V on specific problems studied in the recent DSL literature where our sufficient conditions hold: power allocation for Vectoring DSL with residual interference [12] and power allocation for coexistence among legacy Vectoring VDSL2 and G.fast networks [21], [22].

II. System Model

We consider a DSL network with U active (upstream and/or downstream) transceivers (referred to as "users" in the following), indexed by $u \in \mathcal{U} = \{1, \ldots, U\}$, each based on discrete multitone (DMT) modulation with individual parameters (e.g., carrier-spacing, number of carriers), and potentially belonging to one of several Vectoring groups in the network applying (imperfect) linear crosstalk cancelation. Our objective is the maximization of the users' data rates by optimizing the power allocation over DMT carriers over Vectoring groups and users. In general these rates deteriorate due to far-end crosstalk (FEXT) as well as near-end crosstalk (NEXT) [23]. Users belonging to a Vectoring group may suffer from residual uncanceled crosstalk [12]. Insufficient cyclic prefix results in ICI among carriers of a single user, and coexistence among asynchronous or mixed-technology users results in inter-user ICI. Transmit symbols of user u on carrier c are zero-mean and scaled to variance p_c^u. We study the *continuous* spectrum balancing problem of allocating power levels p_c^u, for all users $u \in \mathcal{U}$ and usable carriers indexed by $c \in \mathcal{C}^u$, resulting in $C := \sum_{u\in\mathcal{U}} |\mathcal{C}^u|$ variables. The joint power allocation over all users will be denoted as $\boldsymbol{p} \in \mathbb{R}_+^C$, and the allocation of all users except user u as $\boldsymbol{p}^{\setminus u}$. For better distinction between the users' power allocation and the power variables generating the interference we define an auxiliary vector of the same dimension, $\tilde{\boldsymbol{p}} \in \mathbb{R}_+^C$. The (strictly positive) normalized background noise power on carrier c of user u is denoted as $\sigma_c^u > 0$, and the effective normalized squared channel magnitude from disturber $j \in \mathcal{U}$ and carrier $k \in \mathcal{C}^j$ to victim $u \in \mathcal{U}$ and carrier $c \in \mathcal{C}^u$ as $H_{ck}^{uj} \geq 0$. In both cases the *normalization* is with respect to the victim's squared direct channel gain on carrier c. Furthermore, the term "effective" means that various effects in the transmission chain are already incorporated (e.g., linear crosstalk cancelation matrices [21] or transmit and receive filters). The (normalized) interference-plus-noise power $m_c^u(\tilde{\boldsymbol{p}})$ is expressed as

$$m_c^u(\tilde{\boldsymbol{p}}) := \sigma_c^u + \underbrace{\sum_{k\in\mathcal{C}^u} H_{ck}^{uu}\tilde{p}_k^u}_{Self-Xtalk} + \underbrace{\sum_{j\in\mathcal{U}\setminus u}\sum_{k\in\mathcal{C}^j} H_{ck}^{uj}\tilde{p}_k^j}_{Inter-user\ Xtalk}, \quad (1)$$

where we explicitly separate a user's self-interference from inter-user interference. This leads us to a signal-to-noise ratio (SNR) defined as

$$\gamma_c^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}}) := p_c^u / m_c^u(\tilde{\boldsymbol{p}}), \quad \forall u \in \mathcal{U}, c \in \mathcal{C}. \quad (2)$$

Using the gap approximation to the scalar Gaussian interference channel capacity, we obtain the per-carrier rates as $f_c^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}}) := \log_2\left(1 + \gamma_c^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}})/\Gamma^u\right)$, where Γ^u is the SNR gap to capacity [23]. The vector $\boldsymbol{f}(\boldsymbol{p}^u, \tilde{\boldsymbol{p}}) \in \mathbb{R}_+^U$ of all users' objective functions (negative per-user data rates) is defined as

$$f^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}}) := -r^u \cdot \sum_{c\in\mathcal{C}^u} f_c^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}}), \quad \forall u \in \mathcal{U}, \quad (3)$$

where r^u captures the symbol-rate (including cyclic extension overhead), Reed-Solomon coding overhead, as well as the effective transmission time (as in time-division duplexing DSL technologies [2]) of user $u \in \mathcal{U}$. A feasible power allocation has to have non-negative elements and it must not violate the per-user spectral power mask and aggregate transmit power (ATP) constraints. These are subject to the power scaling introduced by linear precoding (downstream Vectoring) [12], [17]. We write the (nonnegative) Vectoring power gain of user j onto user u on carrier c as $B_c^{(u,j)} \geq 0$, having the understanding that this gain is zero if users $u, j \in \mathcal{U}$, are not in the same Vectoring group, and that $B_c^{(u,u)} = 1, B_c^{(u,j)} = 0, \forall j \in \mathcal{U} \setminus \{u\}, c \in \mathcal{C}^u$, for an *upstream* user $u \in \mathcal{U}$. Furthermore, we assume that $\sum_{u \in \mathcal{U}} B_c^{(u,j)} > 0, \forall j \in \mathcal{U}, c \in \mathcal{C}$, which means that the variable p_c^j impacts the transmitted power on at least one transceiver. In fact, if this does not hold we must automatically have that the corresponding direct channel gain is zero, and we can safely remove the variable from the formulation.[1] The joint constraint set for all users is formally written as the product set

$$\mathcal{Q} \coloneqq \prod_{u \in \mathcal{U}} \mathcal{Q}^u(\boldsymbol{p}^{\setminus u}), \tag{4}$$

where the constraint set of user u is written as

$$\mathcal{Q}^u(\boldsymbol{p}^{\setminus u}) \coloneqq \{\boldsymbol{p}^u \succeq \boldsymbol{0} | \text{Constraints in } (6)\}, \tag{5}$$

based on the linear constraints

$$\sum_{j \in \mathcal{U}} B_c^{(u,j)} p_c^j \le \hat{p}_c^u, \qquad \forall u \in \mathcal{U}, c \in \mathcal{C}, \tag{6a}$$

$$\sum_{c \in \mathcal{C}^u} \sum_{j \in \mathcal{U}} B_c^{(u,j)} p_c^j \le \hat{P}^u, \qquad \forall u \in \mathcal{U}, \tag{6b}$$

where $\hat{p}_c^u > 0$ is the spectral power mask of user u on carrier c, and where $\hat{P}^u > 0$ is the maximum ATP of user u. The transmit power constraints in (6) are formally shared among *all* users. However, effectively they are only coupled over users in the same downstream Vectoring group where information exchange is commonplace.

III. Autonomous Spectrum Balancing Algorithm

In this section we define a simple projected-gradient algorithm for finding equilibrium solutions where no user is able to increase its data-rate by unilaterally adjusting its power allocation, under the assumptions that the users are not aware of the interference coefficients and each user only measures its total noise. We will state the underlying game model in a precise form in the next section. Let us write the elements of the stacked gradient vector $\boldsymbol{F}(\cdot) \in \mathbb{R}^U$ of all users *for given total interference power* by $F_c^u(\cdot)$, $c \in \mathcal{C}, u \in \mathcal{U}$, given as

$$F_c^u(\boldsymbol{p}) \coloneqq \frac{\partial f^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}})}{\partial p_c^u}\Big|_{\tilde{\boldsymbol{p}}=\boldsymbol{p}}, \tag{7a}$$

$$= -r^u \cdot \frac{\partial f_c^u(\boldsymbol{p}^u, \tilde{\boldsymbol{p}})}{\partial p_c^u}\Big|_{\tilde{\boldsymbol{p}}=\boldsymbol{p}}, \tag{7b}$$

$$= -\frac{r^u}{\log(2)} \cdot (\Gamma^u m_c^u(\boldsymbol{p}^u, \boldsymbol{p}) + p_c^u)^{-1}. \tag{7c}$$

Furthermore, let us define the projection of an allocation $\boldsymbol{p}$ onto the constraint set $\mathcal{Q}$ by

$$\Pi_{\mathcal{Q}}(\boldsymbol{p}) \coloneqq \operatorname*{argmin}_{\tilde{\boldsymbol{p}} \in \mathcal{Q}} \|\tilde{\boldsymbol{p}} - \boldsymbol{p}\|_2^2. \tag{8}$$

Note that this projection is a separable convex quadratic optimization problem which can be independently solved by each Vectoring group for its own power allocation variables.

[1]A situation where this may apply is when the precoder is designed to use inactive lines to increase the rates on active lines [24].

Algorithm 1 Autonomous Projected Gradient Algorithm

1: Initialize $\boldsymbol{p}_0 \in \mathcal{Q}$, $l = 0$, and step-size $\tau_0 > 0$
2: **while** $\boldsymbol{p}_l \neq \Pi_{\mathcal{Q}}(\boldsymbol{p}_l - \boldsymbol{F}(\boldsymbol{p}_l))$ **do**
3:     Update $\tau_l > 0$ according to a non-summable, diminishing step-size rule (e.g., $\tau_l = \tau_0/\sqrt{l+1}$)
4:     Set $\boldsymbol{p}_{l+1} = \Pi_{\mathcal{Q}}(\boldsymbol{p}_l - \tau_l \boldsymbol{F}(\boldsymbol{p}_l))$ and $l = l + 1$

Algorithm 1 summarizes the projected gradient algorithm where in Line 4 the power allocation is *individually* updated by all users in parallel based on the gradients of their objective functions in the previous iteration (calculated based on the noise measured in the previous iteration as defined in (7)), and in Line 3 the step-size is updated based on some non-summable, positive, and diminishing step-size rule. Note that this algorithm requires only noise measurements for the calculation of the per-user gradient vectors, and *individual* projections of a Vectoring group's power allocation onto its own power constraints. Based on our Vectoring-specific definition in Section I this algorithm is labeled fully autonomous.

We end this section by noting that there are numerous further algorithms which fall into our theoretical framework presented in the next section (i.e., primal feasible algorithms for solving VI problems). For instance, the hyperplane projection method in [25, Sec. 12.1.3] and the scaled projected gradient algorithm in [26] are *semi-autonomous* and require the exchange of two numbers / one number among Vectoring groups per iteration. We proceed in the following section by designing a game-theoretical framework for the convergence analysis of Algorithm 1.

IV. Convergence Analysis

The following analysis of conditions for convergence of Algorithm 1 to a unique allocation is based on the following steps: a) the introduction of virtual users which simplifies analysis in case of self-interference, b) the formulation of a GNEP based on our initial assumption on the absence of interference channel gain knowledge, c) the focus on a subset of the solution set of this GNEP which is the solution set of an associated VI problem, d) the derivation of existence and uniqueness conditions for solutions of this VI problem based on VI theory, e) the derivation of convergence conditions for Algorithm 1 applied in the network with virtual users, and f) the critical observation that under solution uniqueness the algorithm in the virtual network can be simplified to Algorithm 1, thereby establishing the result.

A. Introduction of a Virtual Network

Let us start by introducing U additional "virtual" users. The set of real and virtual users is indexed by $u \in \bar{\mathcal{U}} = \{1, \ldots, U, U+1, \ldots, 2U\}$. We define the one-to-one index-map between real and virtual users as

$$v(u) \coloneqq \begin{cases} u + U, & \forall u \in \mathcal{U}, \\ u - U, & \forall u \in \bar{\mathcal{U}} \setminus \mathcal{U}, \end{cases} \tag{9}$$

and the modified channel, $\forall u, j \in \bar{\mathcal{U}}$, $\forall c \in \mathcal{C}^u, k \in \mathcal{C}^j$, as

$$\bar{H}_{ck}^{uj} := \begin{cases} H_{ck}^{uj}, & \text{if } (u \neq j) \wedge (u, j \in \mathcal{U}), \\ H_{ck}^{v(u)v(j)}, & \text{if } (u \neq j) \wedge (u, j \in \bar{\mathcal{U}} \setminus \mathcal{U}), \\ H_{ck}^{uu}, & \text{if } (j = v(u)), \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Hence, the modified network consists of two subnetworks. Each of them is identical to the original network, with the exception that *self-interference is eliminated*. Instead, the self-interference appears as interference *between* the subnetworks. Similarly the constraint sets are replicated in the sense that

$$\bar{B}_c^{(u,j)} := \begin{cases} B_c^{(u,j)}, & \text{if } (u, j \in \mathcal{U}), \\ B_c^{(v(u),v(j))}, & \text{if } (u, j \in \bar{\mathcal{U}} \setminus \mathcal{U}), \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

Furthermore, users $u \in \mathcal{U}$ lend all remaining transmission parameters to the virtual users $v(u)$, e.g., $\sigma_c^{v(u)} = \sigma_c^u, \hat{p}_c^{v(u)} = \hat{p}_c^u, \hat{P}^{v(u)} = \hat{P}^u, \Gamma^{v(u)} = \Gamma^u$. The functions/sets $\bar{m}_c^u(\cdot)$, $\bar{\gamma}_c^u(\cdot)$, $\bar{f}_c^u(\cdot)$, $\bar{f}^u(\cdot)$, $\bar{\mathcal{Q}}^u(\cdot)$ and $\bar{\mathcal{Q}}$ are defined analogously to those in (1)–(6) for the modified network with $2U$ users $u \in \bar{\mathcal{U}}$. For notational simplicity we will write the extended power allocation vector in the modified network as $\boldsymbol{s} \in \mathbb{R}_+^{2\cdot C}$.

B. Generalized Nash Equilibrium Problem Formulation (GNEP) with "Shared Constraints"

In order to cast our algorithmic requirements (autonomous power allocation and feasibility with respect to power constraints throughout the optimization) in an analytically tractable framework, we formulate the spectrum balancing problem as a noncooperative game, defined, $\forall u \in \bar{\mathcal{U}}$, by the following GNEP *with shared constraints*,

$$\text{(GNEP):} \qquad \underset{\boldsymbol{s}^u}{\text{minimize}} \ \bar{f}^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u}), \tag{12a}$$

$$\text{subject to } \ \boldsymbol{s}^u \in \bar{\mathcal{Q}}^u(\boldsymbol{s}^{\setminus u}). \tag{12b}$$

Denoting the solution set to (12) by $\mathcal{S}^u(\boldsymbol{s}^{\setminus u})$, a solution $\bar{\boldsymbol{s}}$ to the GNEP is formally defined [18] as $\bar{\boldsymbol{s}}^u \in \mathcal{S}^u(\bar{\boldsymbol{s}}^{\setminus u}), \forall u \in \bar{\mathcal{U}}$.

The constraints in (12b) are described by nonstrict linear inequalities, bounded by the power allocation's nonnegativity and mask constraints, and always contain the all-zeros power allocation, resulting in the following observation.

Observation 1: The constraint sets $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}^u(\boldsymbol{s}^{\setminus u}), u \in \bar{\mathcal{U}}$, are convex, nonempty, and compact.

The users' objectives in (12a) are the negative sum of per-carrier functions $\bar{f}_c^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u})$ which only depend on the corresponding user's power allocation $\boldsymbol{s}^u$ in the concave form $g_c^u(\boldsymbol{s}^u) = \log_2(1 + s_c^u / a_c^u(\boldsymbol{s}^{\setminus u}))$ for some denominator $a_c^u(\boldsymbol{s}^{\setminus u}) > 0$ that does *not* depend on the user's own power allocation, leading to the following observation.

Observation 2: The objective function $\bar{f}^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u})$ is continuous, (twice) continuously differentiable, and convex in the user's own power allocation $\boldsymbol{s}^u$, $\forall u \in \bar{\mathcal{U}}$.

Remark 1: Practical limitations on maximum spectral efficiency, which are occasionally covered in previous power allocation models [12], [21], are not considered in our model. However, these are less stringent compared to the mandatory coupling power constraints. Not enforcing maximum spectral efficiency restrictions may result in lower rates and higher noise margins for the same power allocation compared to our model. However, their inclusion complicates the following analysis which is based on (formally) shared constraints among all users and further would imply information exchange and full knowledge of the magnitudes of channel coefficients. We will demonstrate the handling of spectral efficiency constraints in the context of a specific application in Section V-B.

C. Variational Inequality (VI) Problem Formulation

By $\mathrm{VI}(\bar{\mathcal{Q}}, \bar{\boldsymbol{F}})$ we denote the VI problem of finding a spectrum balancing solution $\bar{\boldsymbol{s}}$ such that

$$\text{(VI):} \qquad (\boldsymbol{s} - \bar{\boldsymbol{s}})^T \bar{\boldsymbol{F}}(\bar{\boldsymbol{s}}) \geq 0, \quad \forall \boldsymbol{s} \in \bar{\mathcal{Q}}, \tag{13}$$

where $\bar{\boldsymbol{F}}(\cdot) \in \mathbb{R}^{2C}$ denotes the gradient vector in the virtual network, calculable analogously to that in (7) as

$$\bar{F}_c^u(\boldsymbol{s}) = -\frac{r^u}{\log(2)} \cdot (\Gamma^u \bar{m}_c^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u}) + s_c^u)^{-1}. \tag{14}$$

The following well-known result connects the VI formulation to the GNEP with shared constraints in (12).

Proposition 1: Every solution to the VI problem $\mathrm{VI}(\bar{\mathcal{Q}}, \bar{\boldsymbol{F}})$ is a ("variational") solution of the GNEP in (12).

Proof: This follows from the corresponding general result in [18, Thm. 5] by noting that (12) is a "jointly convex GNEP" according to the definition in [18, Def.2] as the users' coupling constraints are closed and convex (cf. Observation 1) and shared by all users, and that the objective functions $\bar{f}^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u}), u \in \bar{\mathcal{U}}$, are convex in $\boldsymbol{s}^u$ and continuously differentiable (cf. Observation 2), concluding the proof. ■

A condition required for later analysis is the Lipschitz continuity of the gradient vector $\bar{\boldsymbol{F}}(\cdot)$. Before stating the corresponding result, we introduce the modified normalized crosstalk coefficients $\breve{\boldsymbol{H}} \in \mathbb{R}^{2C \times 2C}$, defined as

$$\breve{H}_{ck}^{uj} := \begin{cases} 1, & \text{if } k = c, j = u, \\ \Gamma^u \bar{H}_{ck}^{uj}, & \text{otherwise,} \end{cases} \tag{15}$$

and the modified normalized background noise $\tilde{\sigma}_c^u := \Gamma^u \sigma_c^u$.

Proposition 2: The gradient vector $\bar{\boldsymbol{F}}(\cdot)$ is Lipschitz continuous with constant $L = \sigma_{\max}(\boldsymbol{\Upsilon})$, where $\sigma_{\max}(\cdot)$ denotes the maximum singular value and $\boldsymbol{\Upsilon} \in \mathbb{R}^{2C \times 2C}$ is defined as

$$\Upsilon_{(u,c),(j,k)} := \frac{r^u \breve{H}_{ck}^{uj}}{\log(2)(\tilde{\sigma}_c^u)^2}, \forall u, j \in \bar{\mathcal{U}}, c \in \mathcal{C}^u, k \in \mathcal{C}^j. \tag{16}$$

Proof: Analogously to (1) we define $\tilde{m}_c^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u}) = \tilde{\sigma}_c^u + \sum_{j \in \bar{\mathcal{U}}} \sum_{k \in \mathcal{C}^j} \breve{H}_{ck}^{uj} p_k^j$, as well as $\phi_c^u(\boldsymbol{s}, \bar{\boldsymbol{s}}) := \sqrt{\tilde{m}_c^u(\boldsymbol{s}^u, \boldsymbol{s}^{\setminus u})} \cdot \sqrt{\tilde{m}_c^u(\bar{\boldsymbol{s}}^u, \bar{\boldsymbol{s}}^{\setminus u})}$. Lipschitz continuity is defined as $\|\bar{\boldsymbol{F}}(\boldsymbol{s}) -$

$\bar{\boldsymbol{F}}(\bar{\boldsymbol{s}})\|_2 \leq L\|\boldsymbol{s}-\bar{\boldsymbol{s}}\|_2$. Bounding the left hand side in this definition we obtain

$$\|\boldsymbol{F}(\boldsymbol{s})-\boldsymbol{F}(\bar{\boldsymbol{s}})\|_2$$

$$= \sqrt{\sum_{u\in\bar{\mathcal{U}},c\in\mathcal{C}^u}\left(r^u\frac{\sum_{j\in\bar{\mathcal{U}}}\sum_{k\in\mathcal{C}^j}\breve{H}_{ck}^{uj}(s_k^j-\bar{s}_k^j)}{\log(2)(\phi_c^u(\boldsymbol{s},\bar{\boldsymbol{s}}))^2}\right)^2}, \quad (17a)$$

$$\leq \sqrt{\sum_{u\in\bar{\mathcal{U}},c\in\mathcal{C}^u}\left(r^u\frac{\sum_{j\in\mathcal{U}}\sum_{k\in\mathcal{C}^j}\breve{H}_{ck}^{uj}(s_k^j-\bar{s}_k^j)}{\log(2)(\tilde{\sigma}_c^u)^2}\right)^2}, \quad (17b)$$

$$\leq \sigma_{\max}(\boldsymbol{\Upsilon})\|\boldsymbol{s}-\bar{\boldsymbol{s}}\|_2, \quad (17c)$$

where $\boldsymbol{\Upsilon}$ is defined in (16). This concludes the proof. ■

The solution set of the VI problem is only a subset of the GNEP's solution set. However, VI theory provides a rich mathematical toolbox for the analysis of GNEPs, cf. [27] for an introductory survey. An example is given in the following.

Proposition 3: The solution set of VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ is nonempty and compact.

Proof: The result follows directly from the elementary result in [28, Part-II/Thm.A.2] requiring convexity and compactness of $\bar{\mathcal{Q}}$ (cf. Observation 1) as well as continuity of $\bar{\boldsymbol{F}}(\boldsymbol{s})$ (cf. Observation 2). ■

By Proposition 1 this also implies the existence of solutions to the GNEP in (12). Note however that this does not show existence of "symmetric" solutions ($\boldsymbol{s}^u=\boldsymbol{s}^{v(u)},\forall u\in\mathcal{U}$).

Remark 2: At this point it is possible to extend the corresponding analysis and proof in [19] to the studied virtual network with ICI, obtaining similar sufficient conditions for strong monotonicity of $\bar{\boldsymbol{F}}(\cdot)$ as in [19, Prop. 2] and hence for uniqueness of a solution to VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ [28, Part II / Thm. A.3(c)] (details omitted). However, this result depends on spectral mask constraints and background noise levels. The following, more intuitive sufficient conditions are derivable by focusing on the weaker strict monotonicity property instead, and only depend on the channel gains.

Proposition 4: The solution to the spectrum balancing problem VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ in (13) is unique if

$$\sum_{j\in\bar{\mathcal{U}}\setminus u,k\in\mathcal{C}^j}\breve{H}_{ck}^{uj}<1, \quad \forall u\in\bar{\mathcal{U}},c\in\mathcal{C}^u. \quad (18)$$

Proof: This follows from the elementary result in [28, Part II / Thm. A.3(b)] which states that there is at most one solution to VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ provided that $\bar{\mathcal{Q}}$ is closed and convex (cf. Observation 1), and $\bar{\boldsymbol{F}}(\boldsymbol{s})$ is continuous (cf. Observation 2) as well as strictly monotone on $\bar{\mathcal{Q}}$. Furthermore, by Proposition 3 the solution set is nonempty, implying that under these conditions there exists exactly one solution. In the remaining we prove sufficient conditions for strict monotonicity of $\bar{\boldsymbol{F}}(\boldsymbol{s})$. As $\bar{\mathcal{Q}}$ is convex and the partial derivatives of $\bar{\boldsymbol{F}}(\boldsymbol{s})$ are continuous (cf. Observations 1 and 2), strict monotonicity follows from positive definiteness of the Jacobian $\boldsymbol{J}\in\mathbb{R}^{2C\times 2C}$ of $\bar{\boldsymbol{F}}(\boldsymbol{s})$ [29, Thm. 2.3], given as

$$J_{(u,c),(j,k)}=\frac{\partial^2\bar{f}_c^u(\boldsymbol{s})}{\partial s_c^u\partial s_k^j}, \quad (19a)$$

$$=\frac{\partial}{\partial s_k^j}\Big(-\frac{r^u}{\log(2)}\cdot(\tilde{m}_c^u(\boldsymbol{s}^u,\boldsymbol{s}^{\setminus u}))^{-1}\Big), \quad (19b)$$

$$=\frac{r^u}{\log(2)\left(\tilde{m}_c^u(\boldsymbol{s}^u,\boldsymbol{s}^{\setminus u})\right)^2}\cdot\breve{H}_{ck}^{uj}, \quad (19c)$$

for all $u,j\in\bar{\mathcal{U}},c\in\mathcal{C}^u,k\in\mathcal{C}^j$. We note that all elements of the Jacobian are non-negative, that only the last term in (19c) depends on the column (j,k), and that the remaining terms in (19c) multiply the whole row (u,c). Furthermore, on the diagonal we have terms including $\breve{H}_{cc}^{uu}=1$, cf. the definition in (15). Applying the Gerschgorin circle theorem [30, p. 498] we can bound the eigenvalues of the Jacobian away from zero, leading to the result in (18). ■

We have now collected all necessary insights to derive sufficient conditions for convergence of Algorithm 1 to a unique power allocation based on the general result in [31].

Theorem 1: Under the conditions in (18) Algorithm 1 converges to a unique power allocation $\bar{\boldsymbol{p}}$, related to the unique solution $\bar{\boldsymbol{s}}$ of VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ by $\bar{\boldsymbol{s}}^u=\bar{\boldsymbol{s}}^{v(u)}=\bar{\boldsymbol{p}}^u,\forall u\in\mathcal{U}$.

Proof: From Assumptions 2.1 and 4.1, Proposition 4.1, Theorem 4.1, as well as the subsequent remarks in [31] it follows that the discrete update $\boldsymbol{s}_{l+1}=\Pi_{\bar{\mathcal{Q}}}\left(\boldsymbol{s}_l-\tau_l\bar{\boldsymbol{F}}(\boldsymbol{s}_l)\right)$ results in allocations $\boldsymbol{s}_l$ whose distance to the solution set of VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ converges in the limit to zero as $l\to\infty$ if a) the step-size is positive, decreasing, and non-summable; b) $\bar{\mathcal{Q}}$ is compact; c) $\bar{\boldsymbol{F}}(\cdot)$ is Lipschitz continuous; and d) $\bar{\boldsymbol{F}}(\cdot)$ is strictly monotone at a solution to VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$. Condition a) is stated in Algorithm 1, b) follows from Observation 1, c) follows from Proposition 2, and d) follows under condition (18) as shown in the proof to Proposition 4. Furthermore, the result in [31, Corollary 4.1] says that the limit power allocation exists and equals a solution if the solution set is finite. This in fact follows from Proposition 4 under condition (18).

The final point to note is that if the solution $\bar{\boldsymbol{s}}$ to VI$(\bar{\mathcal{Q}},\bar{\boldsymbol{F}})$ is unique, then it must be a symmetric solution in the sense that $\bar{\boldsymbol{s}}^u=\bar{\boldsymbol{s}}^{v(u)},\forall u\in\mathcal{U}$. Otherwise, due to symmetry of the virtual network, we could flip the power allocations between real users $u\in\mathcal{U}$ and the corresponding virtual users $v(u)$, resulting in another solution. However, this would contradict uniqueness of the solution $\bar{\boldsymbol{s}}$. Furthermore, whenever the sequence of parallel updates starts at a symmetric allocation, the power updates of real and virtual users remain identical in every iteration, and it therefore suffices to calculate the updates of real users, cf. Algorithm 1. This concludes the proof. ■

V. Simulation Results

The simulation assumptions include a background noise of -140 dBm/Hz, dropping to -145 dBm/Hz above 30 MHz, spectral power masks according to [1] (VDSL2, bandplan 998/ADE/17a) and [32] (G.fast, 106 MHz profile), an SNR gap of 10.75 dB, and an upstream/downstream asymmetry ratio for

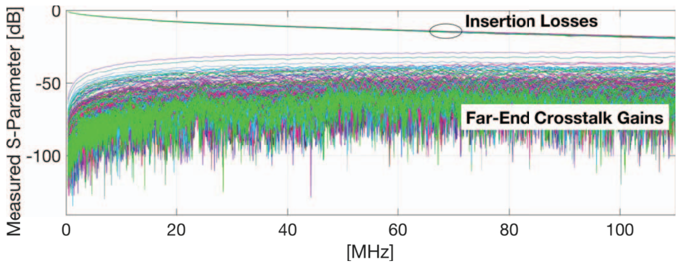

Fig. 1. Measurements of an unshielded 100 m installation cable with 25 individually twisted wire-pairs.

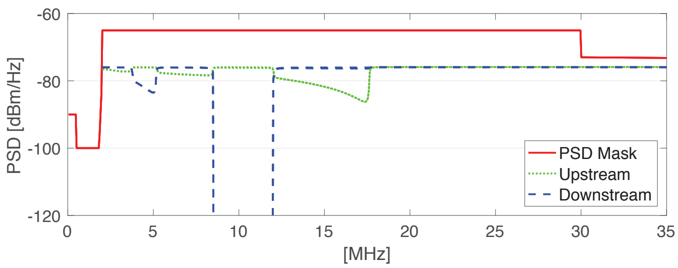

Fig. 2. Upstream/Downstream G.fast power allocation at the VI solution under coexistence with VDSL2/17a and colocated line ends (100 m loop-length).

G.fast of 1:4. The used insertion loss and FEXT channel data are based on the 24-pair "CAD55" wideband cable model in [33] (excluding dual-slope and time-variation effects) or on measurements of an unshielded 100 m 25-pair installation cable with individually twisted wire-pairs, cf. Figure 1. The NEXT channel gains are based on the ETSI NEXT model [23]. We assume a zero-forcing diagonalizing precoder in the downstream, and a zero-forcing linear crosstalk canceler in the upstream. The initial step-size in Algorithm 1 is selected as $\tau_0 = 10^{-11}$. For all simulations we confirmed that the uniqueness/convergence conditions in (18) hold.

A. *VDSL2-G.fast Coexistence*

The first test scenario is a network with 8 users: 2 lines in a VDSL2 Vectoring group and 2 lines in a G.fast Vectoring group (upstream and downstream transceivers are treated as independent users), with colocated line ends on both, DSLAM and customer side, and a 100 m joint cable section. Both, the (error-free) linear crosstalk cancelation in upstream/downstream as well as the ICI among the groups influences the effective direct and crosstalk channels [21]. We obtain 6 simulation scenarios using the 24-pair cable model by sequentially selecting 4 out of the 24 lines. Bit-cap limitations and Trellis-coding overhead are only taken into account for the calculation of the final data-rates [21].

Figure 2 shows the obtained upstream/downstream G.fast power spectra (results for other pair selections are similar). As intuitively expected, due to the high SNR the allocation on carriers without interference noise is flat. Furthermore, when G.fast downstream transmission is subject to strong NEXT from VDSL2 upstream bands the transmit power in these bands is reduced. This in turn reduces the NEXT at

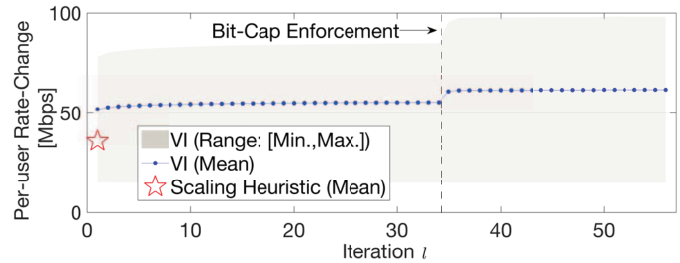

Fig. 3. Change in per-user rates (with bit-cap limitation and Phy-layer overhead) over iterations compared to an initial power allocation (−80 dBm/Hz on all carriers and for all users) and a power scaling heuristic [12] for the imperfect Vectoring example with a 200 m cable section and measured cable data – after iteration 34 the bit-cap enforcement algorithm reduces the SNR margin and thereby frees power which can be reallocated.

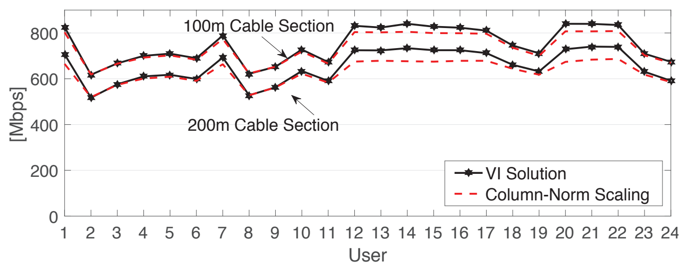

Fig. 4. Simulated data-rates (with bit-cap limitation and Phy-layer overhead) under the VI solution and column-norm scaling [12] for Vectoring xDSL based on measured installation cable data.

the VDSL2 upstream receiver. A similar intuition holds for G.fast upstream transmission as can be seen from the notches in the upstream G.fast spectra in downstream VDSL2 bands. Note that the used ICI model is not symmetric between G.fast and VDSL2 technologies [21] and the notch-depth depends on the interference noise level seen at the G.fast receiver. As an example of the impact of this protection (based on mutual disturbance) on VDSL2 we find that the simulated VDSL2 rates improve on average (over line-selections and users) by 9 % (downstream) and 12 % (upstream) under the VI solution compared to an initial allocation (parameter $\boldsymbol{p}_0$ in Algorithm 1) based on interference-free column-norm scaling (without spectral efficiency limits).

B. *Imperfect linear Vectoring in G.fast*

The second test scenario is a G.fast network with 24 users (downstream transceivers) suffering from residual interference due to imperfect precoding [12]. More precisely, we assume 1 % error in crosstalk channel magnitudes relative to the lines' direct channel gains according to the model in [4]. Line ends are once more colocated and the loop-length is either 100 m or 200 m. The initial power allocation $\boldsymbol{p}_0$ in Algorithm 1 is set to −80 dBm/Hz on all carriers and for all users. In order to exemplify how bit-cap constraints can be enforced by subsequent heuristics we apply the bit-cap enforcement step of the power scaling algorithm in [12] to our VI solution when the maximum change in power on all tones / for all users falls below 0.01 dB. Subsequently we exclude carriers where the bit-load is capped from the carrier set we optimize, and repeat solving the VI problem under a restricted carrier set. This loop repeats a finite number of times until no per-carrier bit-load exceeds the bit-cap limitation. Under the CAD55 cable model the comparison to column-norm scaling [12] shows negligible differences in line rates (results omitted). Differently, rate differences are observable under the measured cable data, cf. Figure 3 where the convergence of our algorithm is illustrated. Under the given initialization the largest rate-step is already made in the algorithm's first iteration. We find that the average rate-difference over users compared to column-norm scaling for a $100\,\mathrm{m}$ cable amounts to $14\,\mathrm{Mbps}$ (between $-1.8\,\mathrm{Mbps}$ and $35\,\mathrm{Mbps}$), and increases to $26\,\mathrm{Mbps}$ (between $-2.4\,\mathrm{Mbps}$ and $57\,\mathrm{Mbps}$) for a $200\,\mathrm{m}$ cable, cf. Figure 4.

VI. Conclusions

A general power spectral resource allocation problem under inter-user, inter-carrier, and self-interference as well as coupled per-transceiver power constraints in Vectoring digital subscriber line (DSL) networks is investigated. Based on the assumption that interference channel gains are not accessible we propose an autonomous spectrum balancing algorithm which maintains primal feasibility throughout run-time. Sufficient conditions for its convergence to a unique allocation are derived based on a variational inequality problem formulation in a virtual network with modified self-interference. These conditions only depend on the sum of disturbing interference gains per user and carrier. Two test problems from the recent DSL literature demonstrate the wide applicability of the proposed framework and that significant gains in rates can be leveraged for certain lines by the proposed algorithm.

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