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Multicomponent proof-theoretic method for proving interpolation properties

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ABSTRACT

Proof-theoretic method has been successfully used almost from the inception of interpolation properties to provide efficient constructive proofs thereof. Until recently, the method was limited to sequent calculi (and their notational variants), despite the richness of generalizations of sequent structures developed in structural proof theory in the meantime. In this paper, we provide a systematic and uniform account of the recent extension of this proof-theoretic method to hypersequents, nested sequents, and labelled sequents for normal modal logic. The method is presented in terms and notation easily adaptable to other similar formalisms, and interpolant transformations are stated for typical rule types rather than for individual rules.

1. Introduction

Interpolation was called "about *the last significant property of first-order logic* that has come to light"[2] (Van Benthem [5]). The property was first formulated and proved by Craig in [17,18] and was inspired by his post-publication review [16] of Beth's paper [6] on definability.

The *Craig interpolation property* for the logic of a given class of models, or *CIP* for short, states roughly that any logical consequence $A \vDash B$ can be supplied with an intermediary statement C, called an *interpolant*,

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[2] Emphasis by Van Benthem.

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that sits between A and B in terms of logical consequence, i.e., satisfies $A \vDash C$ and $C \vDash B$, and uses only the language elements common to A and B. This formulation is not entirely formal as one needs to specify which elements of the language need to be common. Alternatively, if the logic is defined syntactically rather than semantically, one can use $D \vdash E$ instead of $D \vDash E$. Since in most standard logical languages it is possible to define an implication that satisfies modus ponens and the deduction theorem, the usual formulation of the CIP uses $\vdash D \to E$ instead of $D \vdash E$.

Similar to decidability, the interpolation property is a desirable but not necessary property of logics. Similar to decidability, the CIP fails in some reasonably fundamental logics, e.g., the (predicate) intuitionistic logic of constant domains, CD (Mints et al. [51]). On the other hand, many less standard logics often possess the property. As with decidability, for logics that fail the Craig interpolation property, it can sometimes be weakened to a version that holds, while for other logics, it can be strengthened. We do not intend to discuss the complex and intricate hierarchy of interpolation properties, which becomes all the more interesting for weaker logics, where distinctions appear between various formulations of the CIP that are equivalent for classical propositional logic. Those interested are referred to the monograph by Gabbay and Maksimova [26], where the interpolation hierarchy is explored in relation to the corresponding hierarchy of algebraic amalgamation properties.

The only alternative variant of the CIP we will consider in this paper is the *Lyndon interpolation property*, or the *LIP* for short. It was introduced by Lyndon [43] shortly after Craig's original publication.

To explain the difference, we first need to clarify the notion of common language for logics discussed in this paper. As the logics we consider are predominantly monomodal logics based on classical propositional reasoning, with an occasional mention of intermediate logics, the natural definition of the common-language requirement on interpolants of A and B is that they only contain *atomic propositions* common to A and B.

Example 1.1. $P \wedge Q$ is an interpolant of $A = P \wedge Q \wedge S$ and $B = Q \vee \neg P$ for all the logics we consider because

- $\vdash P \wedge Q \wedge S \to P \wedge Q$,
- $\vdash P \wedge Q \to Q \vee \neg P$, and
- both atomic propositions P and Q occurring in the interpolant $P \wedge Q$ occur both in $P \wedge Q \wedge S$ and in $Q \vee \neg P$.

Lyndon suggested discounting $P \wedge Q$ from being an interpolant on the basis that P is not really playing the same role in A and B: P is present positively in A but negatively in B. Indeed, in the example above, it is clear that the transition from A to B has little to do with P and that Q would equally well play the role of a Craig interpolant. Unlike $P \wedge Q$, the formula Q is not only a Craig but also a Lyndon interpolant because the only atomic proposition occurring in Q, Q itself, occurs positively there, as well as in A and B.

The polarity of a subformula occurrence in a given formula is a standard notion related to its monotonicity/antimonotonicity with respect to logical consequence. In all the logics we consider, if any positive occurrence of B in $A(B)$ is replaced with a formula C such that $\vdash B \to C$, then $\vdash A(B) \to A(C)$. Conversely, if any negative occurrence of B in $A(B)$ is replaced with C such that $\vdash B \to C$, then $\vdash A(C) \to A(B)$. Practically, it means that each formula is a positive subformula of itself, the conjunction, disjunction, and modalities $\Box$ and $\Diamond$ do not change the polarity of subformula occurrences, e.g., a positive occurrence of B in $A(B)$ remains positive in $A(B) \wedge D$. The negation flips the polarity of all occurrences within, e.g., a negative occurrence of B in $A(B)$ becomes a positive occurrence in $\neg A(B)$. Finally, the implication preserves the polarity of subformula occurrences in its consequent and flips those in its antecedent, e.g., if B occurs positively in $A(B)$ and C occurs negatively in $D(C)$, both become negative occurrences in $A(B) \to D(C)$.

Note that the equivalency $\equiv$ cannot be used as a primary connective if polarities are to be considered because $P \equiv Q$ is neither monotone nor antimonotone in either of its two arguments.[3]

Definition 1.2 *(Craig and Lyndon interpolation).* Let L be a logic in a language with an implication $\to$ and Boolean constants $\bot$ and $\top$ (primary or defined). We say that L has the *Craig (Lyndon) interpolation property* if, whenever $\mathsf{L} \vdash A \to B$, there exists a formula C, called the *Craig (Lyndon) interpolant* of A and B, such that

- $\mathsf{L} \vdash A \to C$,
- $\mathsf{L} \vdash C \to B$, and
- each atomic proposition occurring in C occurs both in A and in B (in the case of Lyndon interpolation, it must occur in A and in B with the same polarity as in C).

It is clear why the implication should be present in the language to formulate the interpolation properties. The requirement to have Boolean constants is included for the following reason. Consider $\vdash P \wedge \neg P \to Q$. It should have an interpolant but there are no atomic propositions common to $P \wedge \neg P$ and Q. In the language without constants, such implications have to be explicitly excluded from the scope of the CIP, which is more awkward than allowing for Boolean connectives that are natural for all the logics we consider.

Theorem 1.3. *The LIP is strictly stronger than the CIP even for propositional modal logics.*

Proof. A counterexample can be found, e.g., in Maksimova [46]. □

Interpolation has found notable applications in computer science. While these applications are not the focus of this paper, we would like to direct an interested reader to a recent habilitation thesis by Weissenbacher [67] devoted to the use of the CIP for automated verification. In particular, Chapter 4 contains a survey of hardware model checking techniques and their underlying satisfiability checking algorithms, with a special focus on the use of Craig interpolation (see also D'Silva et al. [19], McMillan [48], Vizel et al. [65]).

In this paper, instead of applying interpolation properties, we concentrate on proving them. Multiple methods for doing that have been developed over the years. One has already been mentioned: an interpolation property can be translated to a corresponding amalgamation property of algebraic varieties and proved algebraically (Gabbay and Maksimova [26]). This method is well-developed but not *efficiently constructive*.

Remark 1.4 *(Note on constructivity).* It is common to divide methods of proving interpolation into *constructive* and *non-constructive*. However, as is often pointed out by Baaz, for a recursively enumerable logic, any proof of interpolation is constructive. Indeed, if it is known that an interpolant C of A and B exists, one can enumerate all theorems of the logic in question until, for some formula C satisfying the effectively verifiable common language condition, both $A \to C$ and $C \to B$ occur in the enumeration. The existence of an interpolant guarantees the termination of this algorithm. However, clearly no time or space bound can be extracted from such a procedure. Hence, it is better to speak about *efficiently constructive* methods of proving interpolation, i.e., methods where time/space bounds on the computation can be extracted from the procedure.

[3] For logics enjoying De Morgan dualities, polarity can be defined in a simpler way by restricting the antimonotone connectives to atomic propositions. Formulas in such languages are in the *negation normal form*, where the polarity of each atomic proposition is indicated explicitly. However, this alternative definition is less general: for instance, it cannot be applied to intermediate logics.

Of note among efficiently constructive methods of proving interpolation are those developed in automated reasoning, primarily for first-order theories. Once again, there is vast literature on the subject. Interested readers may consult a recent survey by Bonacina and Johansson [8].

The topic of this paper is another efficiently constructive method of demonstrating interpolation, the proof-theoretic method, i.e., the method of constructing an interpolant using induction on derivations in a given proof formalism. It is almost immediately clear from this description that some form of *analyticity* is required of the proof formalisms in order for the task to be achievable. Indeed, if no restriction is imposed on the kinds of subformulas occurring in the premise(s) of a rule compared to its conclusion, then it is not clear how to process the sudden disappearance of atomic propositions occurring in these subformulas while computing an interpolant for the conclusion from given interpolant(s) of the premise(s). However, even the realm of analytic proof formalisms is too varied to be covered in one paper. We will present a (necessarily incomplete) list of results on interpolation using display calculi, resolution, and tableaus in Sect. 7. The proof formalisms covered in this paper are generalizations of *sequent calculi*, originally introduced by Gentzen in the 1930s and, arguably, one of the most successful and well-studied proof formalisms.

Sequents are typically written $\Gamma \Rightarrow \Delta$ with Γ and Δ being collections of formulas in the underlying object language. These collections can be sequences, sets, or multisets. Γ is called an *antecedent*, and antecedent formulas are understood conjunctively. Δ is called a *consequent*, and consequent formulas are understood disjunctively. The symbol $\Rightarrow$ represents the structural implication. Thus, overall, a sequent $\Gamma \Rightarrow \Delta$ represents the formula $\bigwedge \Gamma \to \bigvee \Delta$.

While sequents are quite popular and can be used to show decidability of a logic, to find its upper complexity bounds, and, as we will discuss presently, to prove interpolation, their major weakness is the limited expressivity. More precisely, it is usually reasonably simple to create a sequent calculus with the *cut rule*:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathrm{cut}\ .$$

However, the (unrestricted) cut rule is exactly the kind of a non-analytic rule precluding the proof of interpolation. Thus, the issue, both for interpolation proofs and in *structural proof theory* in general, is how to describe a logic using a *cut-free* sequent system. And here one meets with significant obstacles. As soon as modalities are added to the language, or even when axioms are added to the intuitionistic propositional logic, a cut-free sequent calculus becomes elusive. Such prominent logics as $\mathsf{S5}$, one of the first modal logics ever to be formally introduced, in Lewis and Langford [42], has not known a cut-free sequent system for more than 80 years. In fact, there are good reasons to believe that no reasonable cut-free sequent system exists (see Lellmann and Pattinson [40]). The situation is similar with the intermediate Gödel–Dummett logic LC, first introduced by Skolem in 1913 (see von Plato [66] on the full history of repeated reintroductions of the logic by Skolem, Gödel, and Dummett).

One of the methods of providing cut-free calculi for such logics suggests combining several sequents into a larger structure. Different ways of combining individual *sequent components* yield different proof formalisms.

The simplest idea of considering an unstructured collection of sequent components, with communication rules shuttling formulas from one component to another, yields the notion of *hypersequents*, which was independently introduced under various names by several researchers, including Mints [49], Pottinger [58], and Avron [1,3]. Both $\mathsf{S5}$ [3,49,58] and LC (Avron [2]; see also a survey by Baaz et al. [4]) possess cut-free hypersequent calculi.

However, hypersequents still do not reach all the logics of interest. The advent of Kripke semantics focused the study of modal logics on so-called *normal* modal logics, and, perhaps, the best studied of these are the 15 logics of what is often described as the *modal cube* (see Garson [27, Sect. 8]). In particular, $\mathsf{S5}$ is the strongest and the smallest normal modal logic K is the weakest of the 15 logics of the cube. All of the logics are quite well-behaved: they have the finite model property, have the LIP, can be axiomatized by finitely many axiom schemes. And still no cut-free hypersequent system is known for logics such as B, the logic of symmetric Kripke frames, named after Brouwer, or K5, the logic corresponding to the epistemic property of negative introspection.

To make these logics amenable to cut-free sequent-like calculi, it turned out to be sufficient to impose a tree structure on the sequent components. Accordingly, the resulting formalism is sometimes called *tree hypersequents*, or simply *nested sequents*. These have also been invented and reinvented by multiple researchers, including Sato [60], Bull [13], Kashima [31], Brünnler [10,11], and Poggiolesi [56,57]. All the logics of the modal cube possess cut-free nested sequent calculi [11]. It might be noted that certain logics in the cube do not require the full unbounded tree depth of nested sequents. As shown by Kuznets and Lellmann [36], K5 can be captured using trees of depth at most 1. (Note that hypersequents are essentially trees of depth exactly 1 with the root node removed.)

Cut-free nested sequents still (seemingly) fail to cover all mainstream modal logics. For instance, the infinite family of *Geach logics*, also known as *Lemmon–Scott* or *Scott–Lemmon logics* (see Lemmon and Scott [41]), contains the modal cube and represents logics of generalized convergence. Already the simplest logic S4.2 of ordinary *convergence* does not have a cut-free nested sequent calculus. Fitting [23] provided cut-free nested sequent systems for all Geach logics using indices to identify certain nodes in a sequent tree. Moreover, the result of Goré and Ramanayake [28] shows that, in general, nested sequent systems can capture exactly those logics that have tree-like Kripke models.

A more general solution to the problem of structural complexity of a proof formalism falling short of the complexity of the logic has been realized in labelled sequent calculi, which once again have a number of creators and developers, including Gabbay [25], Mints [50], Viganò [64], and Negri [52,53]. Labelled calculi drop any preset structure of component sequents, structure often encoded in the notation used to delimit the components, in favor of abstract component labels and explicit description of their hierarchy within the object language of the sequents. Moreover, sequent rules are generally allowed to modify this hierarchy. The resulting dynamic flexible hierarchical structure provides the strongest expressivity among the discussed formalisms. In addition, methods have been provided for automatic generation of cut-free labelled calculi for logics described by various Kripke-frame conditions such as *geometric frame conditions* (see Viganò [64], Negri and von Plato [53]). Dyckhoff and Negri [20] recently extended the method even further by turning first-order frame conditions into geometric ones using additional symbols.

While these cut-free generalized sequent calculi have been used to prove decidability and estimate upper complexity bounds of logics they capture, the proof-theoretic method of proving interpolation has, until recently, evaded extension. In his seminal survey [3], Avron wrote that “in hypersequential calculi cut-elimination usually does not imply the Craig interpolation theorem.” In his habilitation thesis [12] devoted to nested sequents, Brünnler opined that “labelled systems do not seem to be well-suited [for interpolation proofs].”

As we show in this paper, which reports the results of a research program outlined at the Special Session on Proof Theory at the Logic Colloquium 2015 in Helsinki and which supersedes, simplifies, and extends a number of individual results published earlier in Fitting and Kuznets [24], Kuznets [33–35], in fact all these formalisms are well suited for proof-theoretic proof of interpolation properties, if one is given the freedom to entertain a semantic, as well as syntactic point of view.

The paper is structured as follows. In Sect. 2, we recall the underpinnings of the standard proof-theoretic method for sequents. In Sect. 3, we explain the difficulties of extending this well-developed method to more advanced calculi and use the hypersequent formalism to showcase how our method sidesteps these difficulties. Already there we start using general notation applicable to all formalisms considered. In Sect. 4, we generalize the notation and method completely, defining types of rules that are amenable to particular interpolant transformations. Armed with these tools, in Sect. 5, we take a sample of Poggiolesi’s nested sequent rules and categorize most of them according to the interpolant transformations defined in Sect. 4. Two rules, however, that could have been filed into either of two already existing categories, we reveal to be in the intersection of these two categories and, hence, separate them into a special category. In Sect. 6, we turn to the most expressive formalism considered in this paper, that of labelled sequent calculi. We describe the existing method of generating cut-free labelled sequent calculi from frame conditions and outline which of these frame conditions allow the resulting rules to be categorized according to the types defined in Sects. 4 and 5. We prove a general interpolation theorem for logics of Horn-definable frames and apply the developed interpolant transformations to compute a composite transformation for the rules generated by Scott–Lemmon generalized convergence conditions. We provide a detailed comparison of this paper to the already published results on the multisequent interpolation method, and cite the most prominent results for other formalisms in Sect. 7. In the last section we outline the challenges and plans for future work.

2. Proof-theoretic method for sequents

To understand how the proof-theoretic method should work for all these advanced sequent calculi, one should look more closely at the way it works for sequents. While it is common to say that the method proves the (Craig or Lyndon) interpolation statement for $A \to B$ by induction on a given sequent derivation of $A \Rightarrow B$, this is not entirely accurate. Careful treatises on the subject state this clearly, as for instance, in Troelstra and Schwichtenberg [63, Sect. 4.4]: “in order to construct interpolants by induction on the depth of derivations of sequents, we need a more general notion of interpolant.” This more general notion of interpolant, in fact, involves a more general notion of a sequent too. Since a sequent derivation of $A \Rightarrow B$ generally involves sequents $\Gamma \Rightarrow \Delta$ with multiformula antecedents and consequents, it is clear that the induction interpolation statement should not be restricted to singleton Γ and Δ. However, it is not sequents $\Gamma \Rightarrow \Delta$ themselves that are being interpolated. For logics based on classical propositional reasoning, one considers so-called *split sequents*

$$\Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2 \ . \tag{1}$$

More precisely, for any derivable sequent $\Gamma \Rightarrow \Delta$ and for any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$ and of $\Delta = \Delta_1 \sqcup \Delta_2$ into two parts each, the following interpolation statement is proven for the split sequent (1): there exists a formula C such that

- $\vdash \Gamma_1 \Rightarrow \Delta_1, C$,
- $\vdash C, \Gamma_2 \Rightarrow \Delta_2$, and
- each atomic proposition occurring in C occurs both in $\Gamma_1 \Rightarrow \Delta_1$ and in $\Gamma_2 \Rightarrow \Delta_2$.

One obtains the Craig interpolation statement for a theorem $A \to B$ by considering the (derivable) sequent $A \Rightarrow B$ and splitting it as $A; \Rightarrow ; B$.

The idea of split sequents occurs already in Maehara’s proof of the CIP [44]. The necessity of the split comes from the fact that certain rules, e.g., for the propositional negation and implication, move formulas between the antecedent and consequent, thus, disqualifying them from being the stable left and right side to be interpolated.[4]

When Lyndon interpolation is concerned, the common language conditions may seem baffling at first: each atomic proposition occurring in C positively must occur both

- either positively in Γ_1 or negatively in Δ_1 and
- either negatively in Γ_2 or positively in Δ_2

[4] Craig himself considered propositional transitions as trivial and, hence, did not have to deal with rules moving formulas between the antecedent and consequent, avoiding the need for splitting.

(the conditions for negatively occurring atomic propositions are dual). To explain both this convoluted condition and the connection between the CIP/LIP and the property proved by induction, one needs to find an implication between the left sides and the right sides of the split: the standard formula interpretation $\imath(\cdot)$ of the split sequent (1) is

$$\imath(\Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2) = \bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \to \bigvee \Delta_1 \vee \bigvee \Delta_2 \ , \tag{2}$$

but it can be equivalently rewritten as

$$\neg \left(\bigwedge \Gamma_1 \to \bigvee \Delta_1 \right) \to \left(\bigwedge \Gamma_2 \to \bigvee \Delta_2 \right) \ . \tag{3}$$

Thus, the C that is constructed by induction on the sequent derivation is an interpolant of $\neg\imath(\Gamma_1 \Rightarrow \Delta_1)$ and $\imath(\Gamma_2 \Rightarrow \Delta_2)$, and the polarities in the Lyndon formulation can be computed from the negations and implications in (3).

It should, however, be noted that the equivalence between (2) and (3) relies on De Morgan duality, and, thus, only works in classical logics. So how can the method be used for intuitionistic calculi (the first such proof belongs to Schütte [61])? One needs to split sequents differently. (Standard) intuitionistic sequents are single-conclusion, meaning that $\Delta_1 \sqcup \Delta_2$ consists of at most one formula. But the split is performed in such a way that Δ_1 is always empty. In other words, while antecedent formulas are partitioned into left and right ones, the only consequent formula is always on the right. An intuitionistic split sequent

$$\Gamma_1; \Gamma_2 \Rightarrow B$$

is interpolated by a formula C such that

- $\vdash \Gamma_1 \Rightarrow C$,
- $\vdash C, \Gamma_2 \Rightarrow B$, and
- each atomic proposition occurring in C occurs both in Γ_1 and in $\Gamma_2 \Rightarrow B$.

One obtains the Craig interpolation statement for $A \to B$ from the intuitionistic split sequent $A; \Rightarrow B$. This method is based on the representation of $\imath(\Gamma_1; \Gamma_2 \Rightarrow B)$ as

$$\bigwedge \Gamma_1 \to \left(\bigwedge \Gamma_2 \to B \right) \ .$$

Thus, a closer observation of the so-called proof-theoretic methods for sequents uncovers two different methods, operating on different objects, relying on different representations of these objects and, not surprisingly, using different interpolant transformations for corresponding rules, such as the sequent rules introducing implication in the antecedent, intuitionistic and classical. The common core of the methods consists of finding a way of splitting a sequent into the left and right sides so that the sequent can be viewed as an implication from the left side to the right. This basic setup does not yet guarantee the proof of the CIP/LIP, but provides a framework for such a proof.

An additional benefit of the proof-theoretic method, apart from its efficient constructiveness, is its modularity. Suppose a logic L is captured by a cut-free sequent calculus SL, and the CIP/LIP for L has been proved by a proof-theoretic method. If a sequent rule r is added to SL to obtain a logic L', then, to extend the interpolation result to this stronger logic, it is sufficient to provide an interpolant transformation for (all split versions of) the new rule r. Thus, a proof-theoretic proof of interpolation for a sequent calculus is built out of interpolant transformations for the individual rules of the calculus. Since we often deal with logics obtained via different combinations of the same set of rules, it pays off to formalize this modularity.

To prepare the stage for semantic-flavored interpolation statements of the following sections, we formulate modularity using semantic representations of logics.

Let logic L defined by a sequent calculus SL be sound and complete w.r.t. a class $\mathcal{C}$ of models. The second and third columns of the following table contain equivalent definitions of interpolation for (1), while the fourth column for $\mathcal{M} \in \mathcal{C}$ introduces a notion needed for modularity:

	SL-interpolation	$\mathcal{C}$-interpolation	$\mathcal{M}$-interpolation
Left side	$\mathsf{SL} \vdash \Gamma_1 \Rightarrow \Delta_1, C$	$\mathcal{C} \vDash \Gamma_1 \Rightarrow \Delta_1, C$	$\mathcal{M} \vDash \Gamma_1 \Rightarrow \Delta_1, C$
Right side	$\mathsf{SL} \vdash C, \Gamma_2 \Rightarrow \Delta_2$	$\mathcal{C} \vDash C, \Gamma_2 \Rightarrow \Delta_2$	$\mathcal{M} \vDash C, \Gamma_2 \Rightarrow \Delta_2$
		Common language condition	

It is clear that $\mathcal{M}$-interpolation for all $\mathcal{M} \in \mathcal{C}$ is the same as $\mathcal{C}$-interpolation. It is also clear that SL-interpolation or $\mathcal{C}$-interpolation can be demonstrated by induction on a sequent derivation. However, such a proof would not be modular: it cannot be directly transferred to a stronger logic, obtained by adding a rule to SL, as its derivable sequents are not generally valid in all models of $\mathcal{C}$. Thus, to achieve the desired modularity of the method, one needs to prove $\mathcal{M}$-interpolation for any $\mathcal{M} \in \mathcal{C}$ by induction on a sequent derivation.[5] The same principles will be applied later to generalized sequent calculi.

What is now often called *Maehara method* extends well to sequent calculi for modal logics. The first such results can be found in Fitting [21], where the CIP and LIP is demonstrated for many of the logics we will define properly in later sections, such as K, $\mathsf{K4}$, T, $\mathsf{S4}$, D, and $\mathsf{D4}$. (Fitting [21] also used sequent systems with an analytic form of the cut rule to show the CIP for KB, DB, B, and $\mathsf{S5}$, as well as the LIP for $\mathsf{S5}$.[6])

Let us give a short informal description of Maehara method, to be later used for all proof formalisms we encounter. Given a cut-free (generalized) sequent calculus SL for a logic L, complete w.r.t. a class $\mathcal{C}$ of models:

1. An appropriate way of splitting (generalized) sequents into the left and right sides is defined.
2. An interpolation statement is defined for each split (generalized) sequent that puts interpolants in between the left and right sides of the split. To achieve modularity, interpolation is formulated with respect to any model $\mathcal{M} \in \mathcal{C}$ rather than with respect to the class $\mathcal{C}$ itself. For backward compatibility purposes, i.e., for retrieving the Craig/Lyndon interpolation property from the sequent-based proof, it is necessary that the chosen interpolation statement imply the CIP/LIP for the appropriate split of $A \Rightarrow B$, the sequent representing the implication $A \to B$ being interpolated.
3. All rules of SL are split in the following way: for an arbitrary split of the conclusion of the rule, the premise(s) are split in such a way that no subformula changes sides. E.g., the disjunction sequent rule

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee$$

splits into the left and right versions, differing in which side the principal formula $A \vee B$ is on:

$$\frac{\Gamma'; \Gamma'' \Rightarrow \Delta', A, B; \Delta''}{\Gamma'; \Gamma'' \Rightarrow \Delta', A \vee B; \Delta''} \vee_l \qquad \text{and} \qquad \frac{\Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A, B}{\Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A \vee B} \vee_r \ .$$

[5] On the syntactic level, this means that the rule obtaining the interpolation statement for the conclusion from the interpolation statement(s) for the premise(s) must be derivable rather than simply admissible: admissible rules are not generally preserved in extensions of a logic.

[6] Sequent systems have also been used in [21] to show the CIP and LIP for a number of non-normal modal logics we are not planning to discuss in this paper.

$$\dfrac{}{P \xrightarrow{\text{int.}} P; \Rightarrow ; P} \qquad \dfrac{}{\neg P \xrightarrow{\text{int.}} ; P \Rightarrow P;} \qquad \dfrac{}{\top \xrightarrow{\text{int.}} ; P \Rightarrow ; P} \qquad \dfrac{}{\bot \xrightarrow{\text{int.}} P; \Rightarrow P;}$$

$$\dfrac{}{\top \xrightarrow{\text{int.}} ; \Rightarrow ; \top} \qquad \dfrac{}{\bot \xrightarrow{\text{int.}} ; \Rightarrow \top;} \qquad \dfrac{}{\top \xrightarrow{\text{int.}} ; \bot \Rightarrow ;} \qquad \dfrac{}{\bot \xrightarrow{\text{int.}} \bot; \Rightarrow ;}$$

$$\dfrac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma', \Theta'; \Gamma'', \Theta'' \Rightarrow \Delta', \Pi'; \Delta'', \Pi''}$$

$$\dfrac{C \xrightarrow{\text{int.}} \Gamma', A, A; \Gamma'' \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma', A; \Gamma'' \Rightarrow \Delta'; \Delta''} \qquad \dfrac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A, A \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A \Rightarrow \Delta'; \Delta''}$$

$$\dfrac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A, A; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A; \Delta''} \qquad \dfrac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A, A}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A}$$

Fig. 1. Interpolant transformations: initial and structural sequent rules.

4. For each split variant of each rule an appropriate transformation preserving the interpolation statements for each $\mathcal{M} \in \mathcal{C}$ is found. Different split variants of the same rule may require different interpolant transformations.
5. The strongest common language condition achieved by all interpolant transformations determines the type of proved interpolation.

Such interpolation proofs are often accompanied by tables presenting split (generalized) sequent rules along with sufficient interpolant transformations. For sequents, interpolants are often placed above the sequent arrow. However, for more advanced sequent calculi with multiple sequent arrows, such a notation is impossible (let alone misleading), thus, we will denote the fact that C is an interpolant of a split sequent $\Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2$ as follows:

$$C \xrightarrow{\text{int.}} \Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2 \ .$$

To emphasize the scope of an interpolation statement we will also use

$$C \xrightarrow{\mathcal{M}\text{-int.}} \Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2 \ ,$$

$$C \xrightarrow{\mathcal{C}\text{-int.}} \Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2 \ ,$$

$$C \xrightarrow{\mathsf{L}\text{-int.}} \Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2$$

to denote interpolation for a particular model $\mathcal{M}$, class $\mathcal{C}$ of models, or logic L.

As both a reminder and a point of future reference, in Figs. 1 and 2, we present a table with all the interpolant transformations for a multiset-based cut-free sequent calculus for the classical propositional logic with weakening and contraction. This table, in effect, represents both the algorithm for computing interpolants from a given sequent derivation and a skeleton of a proof-theoretic proof of Lyndon interpolation with the only missing piece being the argument demonstrating that the suggested interpolant transformations

$$\frac{C \xrightarrow{\text{int.}} \Gamma', A; \Gamma'' \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', \neg A; \Delta''} \qquad \frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \neg A}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A; \Delta''}{C \xrightarrow{\text{int.}} \Gamma', \neg A; \Gamma'' \Rightarrow \Delta'; \Delta''} \qquad \frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', \neg A \Rightarrow \Delta'; \Delta''}$$

$$i{=}1,2\,\frac{C \xrightarrow{\text{int.}} \Gamma', A_i; \Gamma'' \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma', A_1 \wedge A_2; \Gamma'' \Rightarrow \Delta'; \Delta''} \qquad i{=}1,2\,\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A_i \Rightarrow \Delta'; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A_1 \wedge A_2 \Rightarrow \Delta'; \Delta''}$$

$$i{=}1,2\,\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A_i; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A_1 \vee A_2; \Delta''} \qquad i{=}1,2\,\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A_i}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A_1 \vee A_2}$$

$$i{=}1,2\,\frac{C \xrightarrow{\text{int.}} \Gamma', A; \Gamma'' \Rightarrow \Delta', B; \Delta''}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A \to B; \Delta''} \qquad i{=}1,2\,\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A \Rightarrow \Delta'; \Delta'', B}{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A \to B}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma', A; \Gamma'' \Rightarrow \Delta'; \Delta'' \qquad D \xrightarrow{\text{int.}} \Gamma', B; \Gamma'' \Rightarrow \Delta'; \Delta''}{C \vee D \xrightarrow{\text{int.}} \Gamma', A \vee B; \Gamma'' \Rightarrow \Delta'; \Delta''}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A \Rightarrow \Delta'; \Delta'' \qquad D \xrightarrow{\text{int.}} \Gamma'; \Gamma'', B \Rightarrow \Delta'; \Delta''}{C \wedge D \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A \vee B \Rightarrow \Delta'; \Delta''}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A; \Delta'' \qquad D \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', B; \Delta''}{C \vee D \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A \wedge B; \Delta''}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A \qquad D \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', B}{C \wedge D \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A \wedge B}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta', A; \Delta'' \qquad D \xrightarrow{\text{int.}} \Gamma', B; \Gamma'' \Rightarrow \Delta'; \Delta''}{C \vee D \xrightarrow{\text{int.}} \Gamma', A \to B; \Gamma'' \Rightarrow \Delta'; \Delta''}$$

$$\frac{C \xrightarrow{\text{int.}} \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', A \qquad D \xrightarrow{\text{int.}} \Gamma'; \Gamma'', B \Rightarrow \Delta'; \Delta''}{C \wedge D \xrightarrow{\text{int.}} \Gamma'; \Gamma'', A \to B \Rightarrow \Delta'; \Delta''}$$

Fig. 2. Interpolant transformations: non-structural propositional sequent rules.

preserve the following interpolation statement, a statement we reformulate semantically to be extended in the next section: for a Kripke model $\mathcal{M} = (W, R, V)$ the notation $C \xrightarrow{\mathcal{M}\text{-int.}} \Gamma_1; \Gamma_2 \Rightarrow \Delta_1; \Delta_2$ means that

- for any $w \in W$, if $\mathcal{M}, w \nvDash C$, then $\mathcal{M}, w \vDash \Gamma_1 \Rightarrow \Delta_1$,
- for any $w \in W$, if $\mathcal{M}, w \vDash C$, then $\mathcal{M}, w \vDash \Gamma_2 \Rightarrow \Delta_2$, and
- each atomic proposition occurring in C positively must occur both
  - either positively in Γ_1 or negatively in Δ_1

  and
  - either negatively in Γ_2 or positively in Δ_2;
- each atomic proposition occurring in C negatively must occur both
  - either negatively in Γ_1 or positively in Δ_1

  and
  - either positively in Γ_2 or negatively in Δ_2.

It is worth noting that all structural and single-premise logical rules preserve the interpolation statements for the identity transformation: the premise interpolant can be used for the conclusion. And the six split two-premise classical propositional rules only use two transformations: conjunction or disjunction of the premise interpolants, with the conjunction used for all variants with the principal formula on the right side of the split and the disjunction used for all variants with the principal formula on the left side.

3. Extending the method: hypersequents

So what stood in the way of applying the five steps to interpolation to hypersequents? Negative results are rarely published. But I believe the problem lay in Step 2, namely in finding an appropriate interpolation statement for hypersequents. Being a homogeneous finite collection of sequent components, a hypersequent is typically written as

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \ .$$

The splitting can be performed by splitting each of the sequent components:

$$\Gamma'_1; \Gamma''_1 \Rightarrow \Delta'_1; \Delta''_1 \mid \cdots \mid \Gamma'_n; \Gamma''_n \Rightarrow \Delta'_n; \Delta''_n \ . \tag{4}$$

In modal logic, the corresponding formula interpretation, i.e., the function mapping a hypersequent onto a formula-level statement it is supposed to represent, is

$$\Box\left(\bigwedge \Gamma'_1 \wedge \bigwedge \Gamma''_1 \to \bigvee \Delta'_1 \vee \bigvee \Delta''_1\right) \vee \ldots \vee \Box\left(\bigwedge \Gamma'_n \wedge \bigwedge \Gamma''_n \to \bigvee \Delta'_n \vee \bigvee \Delta''_n\right) \ . \tag{5}$$

Apparently, no one has been able to find an equivalent syntactical representation of this formula as an implication having $\Gamma'_1, \Delta'_1, \ldots, \Gamma'_n, \Delta'_n$ in its antecedent and $\Gamma''_1, \Delta''_1, \ldots, \Gamma''_n, \Delta''_n$ in its consequent.

The intuition behind the method we are now beginning to describe is semantic in nature. The validity for a (split) hypersequent (4) is routinely defined via the validity of its corresponding formula (5). But satisfiability of a hypersequent is a concept we have not been able to find in literature. Thus, this seems the right time to set up the formal definitions.

Definition 3.1 *(Modal language).* The *modal language* $\mathcal{L}$ is defined by the following grammar:

$$A ::= P \mid \bot \mid \top \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \Box A \ ,$$

where P is taken from the set Prop of countably many atomic propositions.

Remark 3.2. We made a choice, perhaps questionable in its asymmetry. Having listed the interpolant transformations for all Boolean connectives, we included all of them into the language, but restricted the primary modalities to $\Box$, only considering $\Diamond$ to be the abbreviation for $\neg\Box\neg$.

Definition 3.3 *(Kripke model).* A *Kripke model* $\mathcal{M} = (W, R, V)$ is a triple of a set of worlds $W \neq \varnothing$, a binary relation $R \subseteq W \times W$, and a valuation function $V\colon \text{Prop} \to 2^W$.

Definition 3.4 *(Satisfaction, validity). Truth* of a formula A at a world $w \in W$ in a model $\mathcal{M} = (W, R, V)$ is denoted by $\mathcal{M}, w \vDash A$ and defined by induction on the construction of the formula:

- $\mathcal{M}, w \vDash P$ iff $w \in V(P)$ for any $P \in \text{Prop}$;
- $\mathcal{M}, w \nvDash \bot$;
- $\mathcal{M}, w \vDash \top$;
- $\mathcal{M}, w \vDash A_1 \wedge A_2$ iff $\mathcal{M}, w \vDash A_i$ for both $i = 1, 2$;
- $\mathcal{M}, w \vDash A_1 \vee A_2$ iff $\mathcal{M}, w \vDash A_i$ for at least one of $i = 1, 2$;
- $\mathcal{M}, w \vDash A \to B$ iff $\mathcal{M}, w \nvDash A$ or $\mathcal{M}, w \vDash B$;
- $\mathcal{M}, w \vDash \Box A$ iff $\mathcal{M}, v \vDash A$ for all $v \in W$ such that wRv.

A sequent $\Gamma \Rightarrow \Delta$ is *true* at the world w of the model $\mathcal{M}$, written $\mathcal{M}, w \vDash \Gamma \Rightarrow \Delta$, iff either $\mathcal{M}, w \nvDash A$ for some $A \in \Gamma$ or $\mathcal{M}, w \vDash B$ for some $B \in \Delta$.

A formula (sequent) is *valid in the model* $\mathcal{M}$ if it is true at all worlds $w \in W$.

A formula (sequent) is *valid in a class* $\mathcal{C}$ of Kripke models if it is valid in all models from the class.

It is easy to see that

Lemma 3.5 *(Validity for sequents). A sequent $\Gamma \Rightarrow \Delta$ is valid in a model (class) iff its formula interpretation $\bigwedge \Gamma \to \bigvee \Delta$ is valid in the same model (class).*

We now define a satisfaction relation for hypersequents with the same property for the hypersequent formula interpretation

$$\imath(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) := \Box\left(\bigwedge \Gamma_1 \to \bigvee \Delta_1\right) \vee \ldots \vee \Box\left(\bigwedge \Gamma_n \to \bigvee \Delta_n\right) . \tag{6}$$

Definition 3.6 *(Rooted tuple).* A *rooted n-tuple* is a tuple $(w_1, \ldots, w_n)$ of worlds $w_i \in W$ of a model $\mathcal{M} = (W, R, V)$ such that there exists a world $v \in W$ with vRw_i for all $i = 1, \ldots, n$.

Definition 3.7 *(Satisfaction, validity for hypersequents).* For a Kripke model $\mathcal{M} = (W, R, V)$, an n-component hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, and a rooted n-tuple $\mathbf{w} = (w_1, \ldots, w_n)$ from $\mathcal{M}$, the hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ is *true* at the tuple $\mathbf{w}$ iff either $\mathcal{M}, w_i \nvDash A$ for some $A \in \Gamma_i$ or $\mathcal{M}, w_i \vDash B$ for some $B \in \Delta_i$.

Splitting does not affect the truth of a hypersequent.

An n-component hypersequent is *valid* in a model (class) iff it is true at all rooted n-tuples from the model (valid in all models from the class).

It is easy to see that

Lemma 3.8 *(Validity for hypersequents). A sequent $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ is valid in a model (class) iff its formula interpretation $\Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n)$ is valid in the same model (class).*

Proof. The negation of validity of the formula interpretation is exactly the existence of a rooted n-tuple that fails the definition of satisfaction for the hypersequent. □

This rather natural definition is sufficient to represent the truth of a split hypersequent (4) as an implication from the left to the right sides: the split hypersequent (4) is true at a rooted n-tuple $\mathbf{w} = (w_1, \dots, w_n)$ iff

$$\mathcal{M}, \mathbf{w} \nvDash \Gamma'_1 \Rightarrow \Delta'_1 \mid \cdots \mid \Gamma'_n \Rightarrow \Delta'_n \qquad \text{implies} \qquad \mathcal{M}, \mathbf{w} \vDash \Gamma''_1 \Rightarrow \Delta''_1 \mid \cdots \mid \Gamma''_n \Rightarrow \Delta''_n \ .$$

Remark 3.9. This definition presupposes a bijection between elements of $\mathbf{w}$ and sequent components. Such a bijection is naturally present for hypersequents that are *sequences* of sequents. However, for the purposes of uniformity within this paper, we will ensure this bijection by assigning labels $1, \dots, n$ to sequent components in the order from left to right.

It is now possible to define interpolants to be intermediaries of this implication. However, this implication is not purely syntactic anymore. Moreover, it generally involves multiple worlds, indicating that interpolants must consist of multiple formulas evaluated at these worlds.

Example 3.10. To illustrate this, consider two hypersequents

$$P; \Rightarrow ; P \mid ; \Rightarrow ; \qquad \text{and} \qquad ; \Rightarrow ; \mid P; \Rightarrow ; P \ .$$

Both are normally initial and should be interpolable. Consider a model $\mathcal{M} = (W, R, V)$. The straightforward extension of the sequent formulation yields the following conditions:

- $\mho$ is an interpolant of $P; \Rightarrow ; P \mid ; \Rightarrow ;$ if for any rooted pair (w_1, w_2) from W:
  if $\mho$ is false, then $\mathcal{M}, w_1 \nvDash P$; if $\mho$ is true, then $\mathcal{M}, w_1 \vDash P$.
- $\mho$ is an interpolant of $; \Rightarrow ; \mid P; \Rightarrow ; P$ if for any rooted pair (w_1, w_2) from W:
  if $\mho$ is false, then $\mathcal{M}, w_2 \nvDash P$; if $\mho$ is true, then $\mathcal{M}, w_2 \vDash P$.

We denote the two requisite interpolants by $1 : P$ and $2 : P$ respectively. The label indicates the world of a rooted tuple where the formula is to be evaluated. In other words, the first interpolant is P evaluated at w_1, while the second is P evaluated at w_2. It is easy to see that they satisfy the interpolation conditions for the same reason that P interpolates $P; \Rightarrow ; P$ for ordinary sequents.

Definition 3.11 *(Multiformula).* A *multiformula* is defined by the following grammar:

$$\mho ::= \ell : A \mid (\mho \owedge \mho) \mid (\mho \ovee \mho) \ ,$$

where ℓ is a label from a countable set $\mathfrak{L}$ of labels, fixed for a particular proof formalism,[7] and $A \in \mathcal{L}$.

To interpret a multiformula $\mho$ in a model $\mathcal{M} = (W, R, V)$, one needs a function from all labels occurring in $\mho$ into W.

Definition 3.12 *(Multiworld interpretation).* A *multiworld interpretation* is a partial function $\mathcal{I} \colon \mathfrak{L} \to W$.

Example 3.13. For hypersequents, we consider $\mathfrak{L} = \{1, 2, \dots, n, \dots\}$ and represent interpretations by tuples of worlds: a tuple $(w_1, \dots, w_n)$ represents an interpretation $\mathcal{I}$ with $\mathcal{I}(i) := w_i$ whenever $1 \leq i \leq n$ and $\mathcal{I}(i)$ undefined whenever $i > n$.

[7] In principle, ω can be used as the set of labels for all formalisms, but we prefer to use more intuitive labels.

While presenting interpolant transformations for hypersequents, we will use the easier to understand tuple notation, keeping the general case in mind.

Definition 3.14 *(Satisfaction for multiformulas).* Given a multiformula $\mho$ and a multiworld interpretation $\mathcal{I}$ into a model $\mathcal{M} = (W, R, V)$, the interpretation $\mathcal{I}$ is called *suitable* for the multiformula $\mho$ iff all labels occurring in $\mho$ belong to the domain of the function $\mathcal{I}$. Note that an interpretation suitable for $\mho$ is also suitable for all multiformulas used in its construction.

If $\mathcal{I}$ is suitable for $\mho$, the *truth* of $\mho$ at $\mathcal{I}$ is defined by induction on the construction of the multiformula:

- $\mathcal{M}, \mathcal{I} \vDash \ell : A$ iff $\mathcal{M}, \mathcal{I}(\ell) \vDash A$;
- $\mathcal{M}, \mathcal{I} \vDash \mho_1 \owedge \mho_2$ iff $\mathcal{M}, \mathcal{I} \vDash \mho_i$ for both $i = 1, 2$;
- $\mathcal{M}, \mathcal{I} \vDash \mho_1 \ovee \mho_2$ iff $\mathcal{M}, \mathcal{I} \vDash \mho_i$ for at least one of $i = 1, 2$.

To incorporate the requirement for all label-interpreting worlds to have a common parent, which was crucial for the hypersequent case, as well as future more complex requirements for other formalisms, we additionally define suitability of an interpretation for a particular split generalized sequent. Here no general definition is possible. This is a requirement custom-tailored to a particular formalism. The one for hypersequents looks as follows:

Definition 3.15 *(Multiworld interpretation suitable for hypersequent).* A multiworld interpretation $\mathcal{I}$ into a model $\mathcal{M} = (W, R, V)$ is *suitable* for a split hypersequent (4) iff the domain of $\mathcal{I}$ is exactly the set $\{1, \dots, n\}$ and there exists $v \in W$ such that $vR\mathcal{I}(i)$ for all $i = 1, \dots, n$.

Now we are ready to give a definition of interpolants for hypersequents that enables one to apply the proof-theoretic method.

Definition 3.16 *(h-interpolant).* Given a split hypersequent (4) and a model $\mathcal{M} = (W, R, V)$, a multiformula $\mho$ is called a *hypersequent interpolant*, or simply *h-interpolant*, of (4) for $\mathcal{M}$, written

$$\mho \xrightarrow{\mathcal{M}\text{-int.}} \quad \Gamma'_1; \Gamma''_1 \Rightarrow \Delta'_1; \Delta''_1 \mid \cdots \mid \Gamma'_n; \Gamma''_n \Rightarrow \Delta'_n; \Delta''_n \ ,$$

iff $\mho$ contains no labels greater than n and, for every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ suitable for (4), i.e., for every rooted n-tuple $\mathbf{w} = (w_1, \dots, w_n)$ of worlds from W,

- if $\mathcal{M}, \mathbf{w} \nvDash \mho$, then $\mathcal{M}, \mathbf{w} \vDash \Gamma'_1 \Rightarrow \Delta'_1 \mid \cdots \mid \Gamma'_n \Rightarrow \Delta'_n$;
- if $\mathcal{M}, \mathbf{w} \vDash \mho$, then $\mathcal{M}, \mathbf{w} \vDash \Gamma''_1 \Rightarrow \Delta''_1 \mid \cdots \mid \Gamma''_n \Rightarrow \Delta''_n$; and

- *Craig condition*: every atomic proposition occurring in $\mho$ occurs both in $\Gamma'_1 \Rightarrow \Delta'_1 \mid \cdots \mid \Gamma'_n \Rightarrow \Delta'_n$ and in $\Gamma''_1 \Rightarrow \Delta''_1 \mid \cdots \mid \Gamma''_n \Rightarrow \Delta''_n$; or
- *Lyndon condition*:
  - every atomic proposition occurring positively[8] in $\mho$ must occur both
    - either positively in some Γ'_i or negatively in some Δ'_i and
    - either negatively in some Γ''_j or positively in some Δ''_j;
  - each atomic proposition occurring in $\mho$ negatively must occur both
    - either negatively in some Γ'_i or positively in some Δ'_i and
    - either positively in some Γ''_j or negatively in some Δ''_j.

[8] The polarity of an occurrence in $\ell : A$ is the same as in A; $\owedge$ and $\ovee$ do not change the polarity of occurrences.

The h-interpolants satisfying the Craig/Lyndon condition are naturally called *Craig/Lyndon h-interpolants.*

We will demonstrate how to prove h-interpolation property presently. But before doing that, let us show how to convert an h-interpolant into an ordinary Craig/Lyndon interpolant.

Definition 3.17 *(Multiformula-to-formula conversion).* We define the following *function* form *from multiformulas to formulas* by recursion on the construction of the multiformula:

- $\mathsf{form}(\ell : A) := A$;
- $\mathsf{form}(\mho_1 \oslash \mho_2) := \mathsf{form}(\mho_1) \wedge \mathsf{form}(\mho_2)$;
- $\mathsf{form}(\mho_1 \otimes \mho_2) := \mathsf{form}(\mho_1) \vee \mathsf{form}(\mho_2)$.

This forgetful conversion is only reasonable in one case: when all labels are the same and this is exactly the case for h-interpolants of the hypersequent $A; \Rightarrow ; B$ representing an implication $A \to B$. The only label that is used in all h-interpolants is 1.

Lemma 3.18 *(Truth for single-label multiformulas). Let ℓ be the only label occurring in a multiformula $\mho$. Then for any model $\mathcal{M}$ and any interpretation $\mathcal{I}$ into $\mathcal{M}$ suitable for $\mho$, i.e., defined at ℓ,*

$$\mathcal{M}, \mathcal{I} \vDash \mho \qquad \Longleftrightarrow \qquad \mathcal{M}, \mathcal{I}(\ell) \vDash \mathsf{form}(\mho) \ .$$

Proof. Easy induction on the construction of $\mho$. □

Definition 3.19 *(Types of models).* A model $\mathcal{M} = (W, R, V)$ is called

- *serial* if for each $w \in W$, there is a $v \in W$ such that wRv;
- *reverse serial* if for each $w \in W$, there is a $v \in W$ such that vRw;
- *reflexive* if for each $w \in W$, we have wRw;
- *transitive* if for arbitrary $w, v, u \in W$ such that wRv and vRu, we have wRu;
- *symmetric* if for arbitrary $w, v \in W$ such that wRv, we have vRw;
- *Euclidean* if for arbitrary $w, v, u \in W$ such that wRv and wRu, we have vRu;
- *linear* if for arbitrary $w, v \in W$, either wRv or vRw;
- *convergent* if for arbitrary $w, v, u \in W$ such that wRv and wRu, there exists a $z \in W$ such that vRz and uRz.

For many of these types additional *shift* variants exist that state the same property under the additional assumption that there exists a world $o \in W$ such that oRw.

The following lemma is formulated for reverse serial models. Hence, the following trivial fact is relevant:

Lemma 3.20. *Any reflexive model is reverse serial.*

Lemma 3.21 *(Reduction from h- to formula interpolation). Let $\mathcal{M} = (W, R, V)$ be a reverse serial Kripke model. If $\mho$ is a Craig (Lyndon) h-interpolant of $A; \Rightarrow ; B$ for $\mathcal{M}$, then $\mathsf{form}(\mho)$ is a Craig (Lyndon) interpolant of $A \to B$ for $\mathcal{M}$.*

Proof. It is easy to see that Craig/Lyndon common language conditions for h-interpolants translate into corresponding conditions for formula interpolants.

Since 1 is the only label used in $\mho$, for any rooted 1-tuple (w_1), by Lemma 3.18, $\mathsf{form}(\mho)$ has the same truth value at the world w_1 as $\mho$ has at the tuple (w_1). The h-interpolation claims the following implications

$$\begin{aligned} \mathcal{M},(w_1) \nvDash \mho \qquad &\Longrightarrow \qquad \mathcal{M},(w_1) \vDash A \Rightarrow\ , \\ \mathcal{M},(w_1) \vDash \mho \qquad &\Longrightarrow \qquad \mathcal{M},(w_1) \vDash\ \Rightarrow B \end{aligned}$$

for any rooted 1-tuple (w_1). Given the truth definition for hypersequents and the already mentioned results of Lemma 3.18, this is equivalent to the requirements that

$$\begin{aligned} \mathcal{M},w_1 \nvDash \mathsf{form}(\mho) \qquad &\Longrightarrow \qquad \mathcal{M},w_1 \nvDash A\ , \\ \mathcal{M},w_1 \vDash \mathsf{form}(\mho) \qquad &\Longrightarrow \qquad \mathcal{M},w_1 \vDash B \end{aligned}$$

for any $w_1 \in W$ such that (w_1) is a rooted 1-tuple. It remains to note that in reverse serial models (w_1) is a rooted 1-tuple for all $w_1 \in W$, thus, $\mathcal{M} \vDash A \to \mathsf{form}(\mho)$ and $\mathcal{M} \vDash \mathsf{form}(\mho) \to B$. □

Corollary 3.22. *If $\mho$ is a Craig/Lyndon h-interpolant of $A; \Rightarrow ; B$ for a class $\mathcal{C}$ of reverse serial Kripke models, i.e., an h-interpolant for each model in the class, then $\mathsf{form}(\mho)$ is a Craig/Lyndon interpolant of $A \to B$ for the logic L sound and complete w.r.t. $\mathcal{C}$.*

Thus, to prove the CIP/LIP of a logic L sound and complete w.r.t. a class $\mathcal{C}$ of reverse serial models, it is sufficient to provide interpolant transformations preserving the h-interpolation property (Definition 3.16) for each model of $\mathcal{C}$ for all the rules of its cut-free hypersequent calculus SL. It should come as no surprise that for propositional rules, the interpolant transformations are essentially the same as in the sequent case (see Figs. 1 and 2). More precisely, the hypersequent rules whose sequent analogs reuse the premise interpolant can reuse it also in the hypersequent case (for any model). The operations of $\wedge$ and $\vee$ on formula interpolants are replaced with $\owedge$ and $\ovee$ respectively for h-interpolants. Finally, whatever interpolant works for initial sequents, would work for initial hypersequents (for any model) when prefixed with the label corresponding to the active sequent component. For instance,

$$i : P \xrightarrow{\mathcal{M}\text{-int.}} \underbrace{\Rightarrow | \cdots | \Rightarrow}_{i-1} | P; \Rightarrow ; P | \Rightarrow | \cdots | \Rightarrow \tag{7}$$

for any Kripke model $\mathcal{M}$. All these properties present no difficulties and are left for the reader to check.

The remaining rules can be divided into hypersequent structural rules and modal rules, which depend on the modal logic in question. We start by stating rather simple interpolant transformations for the external structural rules.

It is common to use $\mathcal{G}$ to denote an arbitrary hypersequent. Since we are dealing mostly with split hypersequents, we will use $\mathcal{G}, \mathcal{H}$ to represent split hypersequents. The left and right sides of $\mathcal{G}$ are denoted by $L(\mathcal{G})$ and $R(\mathcal{G})$ respectively.

The external weakening rule predictably requires no change to the interpolant.

Lemma 3.23 *(Transformation for external weakening). For any Kripke model $\mathcal{M} = (W, R, V)$ and any n-component split hypersequent $\mathcal{G}$, the (trivial) transformation*

$$\frac{\mho \xrightarrow{\mathcal{M}\text{-}int.} \mathcal{G}}{\mho \xrightarrow{\mathcal{M}\text{-}int.} \mathcal{G} \mid \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}\ \mathsf{EW} \tag{8}$$

preserves the h-interpolation for $\mathcal{M}$, i.e., if a multiformula $\mho$ is a Craig/Lyndon h-interpolant of the premise split hypersequent for $\mathcal{M}$, then $\mho$ is also a Craig/Lyndon h-interpolant of the conclusion for the same model.

Proof. Since no atomic proposition disappears going from the premise to the conclusion in either side, the common language condition clearly transfers downwards. Since $\mho$ does not use labels greater than n, it does not use labels greater than $n+1$. Consider any rooted $(n+1)$-tuple $(w_1, \dots, w_n, u)$ from $\mathcal{M}$ and let us denote $(w_1, \dots, w_n)$ by $\mathbf{w}$, which is clearly a rooted n-tuple. In particular, $(w_1, \dots, w_n, u) = \mathbf{w} * u$, where $*$ is the concatenation operation. Since $\mathbf{w}$ is suitable for the premise (split hypersequent), we have

$$\mathcal{M}, \mathbf{w} \nvDash \mho \quad \Longrightarrow \quad \mathcal{M}, \mathbf{w} \vDash L(\mathcal{G}) \qquad \text{and} \qquad \mathcal{M}, \mathbf{w} \vDash \mho \quad \Longrightarrow \quad \mathcal{M}, \mathbf{w} \vDash R(\mathcal{G}) \ .$$

In other words, if $\mho$ is false, then either a left antecedent formula is false or a left consequent formula is true at its corresponding world; if $\mho$ is true, then either a right antecedent formula is false or a right consequent formula is true at its corresponding world. It remains to note that

$$\mathcal{M}, \mathbf{w} \vDash \mho \quad \Longleftrightarrow \quad \mathcal{M}, \mathbf{w} * u \vDash \mho$$

because the label $n+1$ does not occur in $\mho$ and that

$$\mathcal{M}, \mathbf{w} \vDash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \quad \Longrightarrow \quad \mathcal{M}, \mathbf{w} * u \vDash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi \Rightarrow \Sigma$$

because the formula that made the shorter hypersequent true is still present in the longer hypersequent. □

Unlike the external weakening that adds a dummy sequent component, the external contraction merges two components together, so that their labels need to be identified too.

Definition 3.24 *(Label manipulations).* Let $\ell_1, \ell'_1, \dots, \ell_n, \ell'_n \in \mathfrak{L}$ be labels such that $\ell_1, \dots, \ell_n$ are pairwise distinct. Let $\mho$ be a multiformula and $\heartsuit_1, \dots, \heartsuit_n$ be (possibly empty) sequences of unary operators. We define $\mho[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n]$ by induction on the construction of $\mho$:

- $(\ell_i : A)[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n] := \ell'_i : \heartsuit_i A \qquad$ for $i = 1, \dots, n$;
- $(k : A)[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n] := k : A$ for any other label $k \in \mathfrak{L}$, i.e., $k \notin \{\ell_1, \dots, \ell_n\}$;
- $(\mho_1 \owedge \mho_2)[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n] := \mho_1[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n] \owedge \mho_2[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n]$;
- $(\mho_1 \ovee \mho_2)[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n] := \mho_1[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n] \ovee \mho_2[\ell_1 \mapsto \ell'_1\heartsuit_1, \dots, \ell_n \mapsto \ell'_n\heartsuit_n]$.

The most common operations we will use are

- *renaming* a label ℓ into ℓ', i.e., $\mho[\ell \mapsto \ell']$ for $n = 1$ and $\heartsuit_1 = \varepsilon$;
- *swapping* two distinct labels ℓ and ℓ', which we abbreviate as $\mho[\ell \leftrightarrow \ell'] := \mho[\ell \mapsto \ell', \ell' \mapsto \ell]$, with $n = 2$ and $\heartsuit_1 = \heartsuit_2 = \varepsilon$;
- *prefixing* ℓ-labeled formulas with $\heartsuit$, which we abbreviate as $\mho[\ell\heartsuit] := \mho[\ell \mapsto \ell\heartsuit]$, with $n = 1$;
- *moving* a ℓ_c-labeled formulas to a label ℓ_p with prefix $\heartsuit$, i.e., $\mho[\ell_c \mapsto \ell_p\heartsuit]$ with $n = 1$.

It is easy to prove by induction on the construction of $\mho$ that

Lemma 3.25. *For any model $\mathcal{M} = (W, R, V)$, tuple $\mathbf{w}$, world $v \in W$ such that $\mathbf{w} * v$ is a rooted n-tuple, and multiformula $\mho$ with no labels greater than $n+1$, we have that $\mathbf{w} * v * v$ is a rooted $n+1$ tuple and*

$$\mathcal{M}, \mathbf{w} * v * v \vDash \mho \quad \Longleftrightarrow \quad \mathcal{M}, \mathbf{w} * v \vDash \mho[n+1 \mapsto n] \ .$$

Lemma 3.26 *(Transformation for external contraction). For any Kripke model $\mathcal{M} = (W, R, V)$ and any $(n-1)$-component split hypersequent $\mathcal{G}$, the transformation*

$$\frac{\mho \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'' \mid \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mho[n+1 \mapsto n] \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''} \mathsf{EC} \tag{9}$$

preserves the h-interpolation for $\mathcal{M}$.

Proof. Once again, the common language condition is clearly preserved downward. The renaming of $n+1$ into n removes $n+1$ making the interpolant suitable for the conclusion.

Consider any rooted n-tuple $\mathbf{w} * v$. By Lemma 3.25, the truth value of $\mho[n+1 \mapsto n]$ at $\mathbf{w} * v$ is the same as that of $\mho$ at the rooted $n+1$-tuple $\mathbf{w} * v * v$. Thus, from the interpolation statements for the premise, we get the truth of either left or right side of the premise, i.e., we get either an antecedent formula false or a consequent formula true at the corresponding world for the appropriate side. All the formulas from the first n components are still present in the conclusion. If a formula from the $(n+1)$th component is false/true at v, an identical formula is present in the nth component in the conclusion and is also evaluated at v. □

Remark 3.27. Avron [3] suspected that it is the rule of external contraction EC that makes proving interpolation using hypersequents problematic. As can be seen from the lemma above, EC creates no problems for our semantically motivated method.

The situation with the external exchange rule is even simpler than with the contraction, thus, we only provide definitions and statements.

Definition 3.28 *(Label swap).* Let $\ell, \ell' \in \mathfrak{L}$ be two labels and $\mho$ be a multiformula. We define $\mho[\ell \leftrightarrow \ell']$ by induction on the construction of $\mho$:

- $(\ell : A)[\ell \leftrightarrow \ell'] := \ell' : A$;
- $(\ell' : A)[\ell \leftrightarrow \ell'] := \ell : A$;
- $(n : A)[\ell \leftrightarrow \ell'] := n : A$ for any other label $n \in \mathfrak{L}$, i.e., $n \notin \{\ell, \ell'\}$;
- $(\mho_1 \otimes \mho_2)[\ell \leftrightarrow \ell'] := \mho_1[\ell \leftrightarrow \ell'] \otimes \mho_2[\ell \leftrightarrow \ell']$;
- $(\mho_1 \oslash \mho_2)[\ell \leftrightarrow \ell'] := \mho_1[\ell \leftrightarrow \ell'] \oslash \mho_2[\ell \leftrightarrow \ell']$.

It is easy to prove by induction on the construction of $\mho$ that

Lemma 3.29. *For any model $\mathcal{M} = (W, R, V)$, arbitrary tuples $\mathbf{w}$ and $\mathbf{w}'$ of sizes k and l respectively, arbitrary worlds $u, v \in W$ such that $\mathbf{w}, u, v, \mathbf{w}'$ is a rooted n-tuple, and any multiformula $\mho$ with no labels greater than n, we have that $\mathbf{w} * v * u * \mathbf{w}'$ is also a rooted n-tuple and*

$$\mathcal{M}, \mathbf{w} * v * u * \mathbf{w}' \vDash \mho \qquad \Longleftrightarrow \qquad \mathcal{M}, \mathbf{w} * u * v * \mathbf{w}' \vDash \mho[k+1 \leftrightarrow k+2] \; .$$

Lemma 3.30 *(Transformation for external exchange). For any Kripke model $\mathcal{M} = (W, R, V)$ and arbitrary k- and l-component split hypersequents $\mathcal{G}$ and $\mathcal{H}$ respectively, the transformation*

$$\frac{\mho \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'' \mid \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma'' \mid \mathcal{H}}{\mho[k+1 \leftrightarrow k+2] \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma'' \mid \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'' \mid \mathcal{H}} \mathsf{Ex} \tag{10}$$

preserves the h-interpolation for $\mathcal{M}$.

$$\frac{\mathcal{G} \mid \Gamma, A; \Pi \Rightarrow \Delta; \Sigma}{\mathcal{G} \mid \Gamma, \Box A; \Pi \Rightarrow \Delta; \Sigma} \Box^{\mathsf{l}} \Rightarrow \qquad \frac{\mathcal{G} \mid \Gamma; \Pi, A \Rightarrow \Delta; \Sigma}{\mathcal{G} \mid \Gamma; \Pi, \Box A \Rightarrow \Delta; \Sigma} \Box^{\mathsf{r}} \Rightarrow$$

$$\frac{\mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow A;}{\mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow \Box A;} \Rightarrow \Box^{\mathsf{l}} \qquad \frac{\mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow ; A}{\mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow ; \Box A} \Rightarrow \Box^{\mathsf{r}}$$

$$\frac{\mathcal{G} \mid \Box\Lambda, \Gamma; \Box\Theta, \Pi \Rightarrow \Delta; \Sigma}{\mathcal{G} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \Box\Lambda; \Box\Theta \Rightarrow} \mathsf{MS} \qquad \frac{\mathcal{G} \mid \Box\Lambda, \Box\Gamma; \Box\Theta, \Box\Delta \Rightarrow}{\mathcal{G} \mid \Box\Lambda; \Box\Theta \Rightarrow \mid \Box\Gamma; \Box\Delta \Rightarrow} \mathsf{RMS}$$

Fig. 3. Split versions of the modal rules from the cut-free hypersequent systems of Kurokawa [32].

With the structural rules behind, we can finally approach interesting rules that make hypersequents useful, the rules for modalities. We will present two examples sufficient to cover the logics S5 and S4.2. The former has already been discussed. Semantically, it is the logic of models (W, R, V) with R being an *equivalence relation*, i.e., being reflexive, transitive, and symmetric. The latter logic is the logic of reflexive transitive, and convergent models. Note that, being reflexive, all these models are reverse serial (see Definition 3.19). To minimize the number of versions of modal rules used, we take the rules from Kurokawa [32], where hypersequents for S4.2, S4.3 (the logic of reflexive, transitive, and linear models), and S5 are given in a uniform manner.

The split versions of the modal rules that need to be added to propositional and structural hypersequent rules to get a cut-free calculus for S5 can be found in Fig. 3.[9]

Remark 3.31. We are not attempting to prove the interpolation for S4.3 because Maksimova [45] provided a counterexample to the CIP for this logic.

The cut-free split hypersequent calculi for S4.2 and S5 are obtained by adding to the propositional and structural split hypersequent rules all four rules $\Box^{\mathsf{l}} \Rightarrow$, $\Box^{\mathsf{r}} \Rightarrow$, $\Rightarrow \Box^{\mathsf{l}}$, and $\Rightarrow \Box^{\mathsf{r}}$, as well as the *restricted modal splitting rule* RMS for S4.2 or the *modal splitting rule* MS for S5.

In the spirit of modularity, we will provide interpolant transformations for the first four rules, which are common for both logics for the class of reflexive transitive models, which contains both the class of S4.2 and the class of S5 models. In fact, we can achieve an even greater granularity: both antecedent $\Box$ rules can be processed for all reflexive models while both consequent box rules require only transitivity.

Lemma 3.32 *(Identity transformation for $\Box^{\mathsf{l}} \Rightarrow$, $\Box^{\mathsf{r}} \Rightarrow$). For any reflexive Kripke model $\mathcal{M} = (W, R, V)$ and any h-interpolant $\mho$ of the premise of the rule $\Box^{\mathsf{l}} \Rightarrow$ ($\Box^{\mathsf{r}} \Rightarrow$) for $\mathcal{M}$, we have that $\mho$ is also an h-interpolant for $\mathcal{M}$ for the conclusion of the rule.*

Proof. Clearly the common language and suitability conditions transfer downwards. For each of the rules, one side remains unchanged, so the interpolant statements for this side are the same for the premise and conclusion. For the other side, the only change is that A in the antecedent of the premise is replaced with $\Box A$ in the antecedent of the conclusion. Thus, if a truth value of $\mho$ at some rooted tuple $\mathbf{w} * u$ necessitates that this side of the hypersequent be true for the conclusion, we know it to be true for the premise. If in the premise it is true because of any formula other than A, this formula is present in the conclusion in exactly the same capacity, making this side of the conclusion hypersequent true immediately. Otherwise, if in the

[9] To save space we omit the original unsplit rules.

premise the antecedent formula A is false at u, then so is $\Box A$ by reflexivity as uRu. Thus, the relevant side of the conclusion hypersequent is true because $\Box A$ is false at u. $\Box$

These two splits of a sequent rule are also used in sequent calculi for S4 and the identity interpolant transformation is inherited from there. Before we present an interpolant transformation for the consequent $\Box$ rules, we state a rather obvious fact.

Lemma 3.33 *(Equivalent uniform DNF/CNF). For any multiformula $\mho$ there exist multiformulas $\mho_D$ and $\mho_C$ with exactly the same labels and polarities of atomic propositions such that $\mho_D$ is in DNF, $\mho_C$ is in CNF, and for any multiworld interpretation $\mathcal{I}$ into a model $\mathcal{M}$ suitable for one (and, hence, for all) of $\mho$, $\mho_D$, and $\mho_C$,*

$$\mathcal{M}, \mathcal{I} \vDash \mho \qquad \Longleftrightarrow \qquad \mathcal{M}, \mathcal{I} \vDash \mho_D \qquad \Longleftrightarrow \qquad \mathcal{M}, \mathcal{I} \vDash \mho_C \ .$$

Let $\mathfrak{L}' \subset \mathfrak{L}$ contain all labels occurring in $\mho$. W.l.o.g. we can assume that every label from $\mathfrak{L}'$ occurs exactly once in every disjunct of $\mho_D$ and exactly once in every conjunct of $\mho_C$. Formulas satisfying this additional condition are said to be in uniform DNF/CNF.

Proof. Since $\otimes$ and $\oslash$ behave exactly the same way as $\vee$ and $\wedge$, the standard algorithm for converting into DNF/CNF applies.

The uniform DNF/CNF can be achieved simply by exploiting the equivalencies of $\ell : A \oslash \ell : B$ to $\ell : (A \wedge B)$ and $\ell : A \otimes \ell : B$ to $\ell : (A \vee B)$ to remove duplicate occurrences of the same label, as well as equivalencies of $\mho'$ to $\mho' \otimes \ell : \bot$ and $\mho'$ to $\mho' \oslash \ell : \top$ to add occurrences of missing labels. $\Box$

Interpolant transformations we use for some rules, such as consequent $\Box$ rules, require a given interpolant of the premise split sequent to be either in CNF or in DNF. In such cases, we will denote the given interpolants by $\mho_C$ or $\mho_D$ respectively and assume that all labels used in hypersequents occur exactly once within each conjunct (disjunct).

Lemma 3.34 *(Transformation for $\Rightarrow \Box^{\mathsf{r}}$). For any transitive Kripke model $\mathcal{M} = (W, R, V)$, any multiformula $\mho_C$ in uniform CNF, and any $(k-1)$-component split hypersequent $\mathcal{G}$, the transformation*

$$\frac{\mho_C \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow ; A}{\mho_C[k\Box] \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow ; \Box A} \Rightarrow \Box^{\mathsf{r}} \tag{11}$$

preserves the h-interpolation for $\mathcal{M}$.

Proof. Let

$$\mho_C = \bigoslash_{i=1}^{n} \bigotimes_{j=1}^{k} j : C_{ij} \ . \tag{12}$$

Then, modulo commutativity and associativity of $\otimes$,

$$\mho_C[k\Box] = \bigoslash_{i=1}^{n} \left(k : \Box C_{ik} \otimes \bigotimes_{j=1}^{k-1} j : C_{ij} \right) \ . \tag{13}$$

Consider an arbitrary $(k-1)$-tuple $\mathbf{w}$ and world u from $\mathcal{M}$ such that $\mathbf{w} * u$ is a rooted k-tuple. Within this proof we omit the mention of model $\mathcal{M}$ in $\vDash$ statements.

Left side. Let $\mathbf{w} * u \nvDash \mho_C[k\Box]$. Then, for some $1 \le i \le n$, we have $\mathbf{w} \nvDash \bigovee_{j=1}^{k-1} j : C_{ij}$ and $u \nvDash \Box C_{ik}$. From the latter we conclude that $u' \nvDash C_{ik}$ for some $u' \in W$ such that uRu'. Due to the transitivity of R, the tuple $\mathbf{w} * u'$ is also a rooted k-tuple, and $\mathbf{w} * u' \nvDash \mho_C$. Thus, by the left side of interpolation for the premise,

$$\mathbf{w} * u' \vDash L(\mathcal{G}) \mid \Box\Gamma \Rightarrow \quad .$$

If this happens because of some formula within $L(\mathcal{G})$, this formula is also present in the conclusion and is evaluated at the same world for $\mathbf{w} * u$. Otherwise, there must exist a $\Box G \in \Box\Gamma$ such that $u' \nvDash \Box G$. Given the transitivity of R, it follows that $u \nvDash \Box G$, meaning that in this case too we have $\mathbf{w}, u \vDash L(\mathcal{G}) \mid \Box\Gamma \Rightarrow$.

Right side. Let $\mathbf{w} * u \vDash \mho_C[k\Box]$. We need to show

$$\mathbf{w} * u \vDash R(\mathcal{G}) \mid \Box\Pi \Rightarrow \Box A \ .$$

If $\mathbf{w} \vDash R(\mathcal{G})$ or $u \nvDash \Box H$ for some $\Box H \in \Box\Pi$, we are done. If none of these is the case, we need to show that $u \vDash \Box A$. First, we note that $\mathbf{w} * u' \vDash \mho_C$. Indeed, if the ith conjunct of (13) is true at $\mathbf{w} * u$ because of $j : C_{ij}$ with $j \le k-1$, then the ith conjunct of (12) is true at $\mathbf{w} * u'$ for the same reason. And if the ith conjunct of (13) is true at $\mathbf{w} * u$ because $u \vDash \Box C_{ik}$, then the ith conjunct of (12) is true at $\mathbf{w} * u'$ because of $u' \vDash C_{ik}$. Thus,

$$uRu' \qquad \Longrightarrow \qquad \mathbf{w} * u' \vDash R(\mathcal{G}) \mid \Box\Pi \Rightarrow A \ .$$

We assumed that $\mathbf{w} \nvDash R(\mathcal{G})$ and $u \vDash \Box H$ for all $\Box H \in \Box\Pi$. The latter implies by transitivity of R that $u' \vDash \Box H$ for all $\Box H \in \Box\Pi$. Thus, $u' \vDash A$ whenever uRu'. In other words, $u \vDash \Box A$. $\Box$

The interpolant transformation for $\Rightarrow \Box^{\mathsf{l}}$ is easily computable from that for $\Rightarrow \Box^{\mathsf{r}}$ by the usual classical dualities. We only state the requisite transformation leaving the proof to the readers.

Lemma 3.35 *(Transformation for $\Rightarrow \Box^{\mathsf{l}}$). For any transitive Kripke model $\mathcal{M} = (W, R, V)$, any multiformula $\mho_D$ in uniform DNF, and any $(k-1)$-component split hypersequent $\mathcal{G}$, the transformation*

$$\frac{\mho_D \xrightarrow{\mathcal{M}\text{-}int.} \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow A;}{\mho_D[k\Diamond] \xrightarrow{\mathcal{M}\text{-}int.} \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow \Box A;} \Rightarrow \Box^{\mathsf{l}} \tag{14}$$

preserves the h-interpolation for $\mathcal{M}$.

Note that although these two splits of a rule are also used in sequent calculi for S4, the interpolant transformation for h-interpolant is significantly more complicated. The addition of the preceding four split rules yields, in fact, a hypersequent calculus for S4, the logic of reflexive and transitive models. It remains to provide interpolant transformations for the rules that extend it to S4.2 and S5.

Lemma 3.36 *(Transformation for MS). For any transitive and Euclidean Kripke model $\mathcal{M} = (W, R, V)$, any multiformula $\mho$, and any n-component split hypersequent $\mathcal{G}$, the (identity) transformation*

$$\frac{\mho \xrightarrow{\mathcal{M}\text{-}int.} \mathcal{G} \mid \Box\Lambda, \Gamma; \Box\Theta, \Pi \Rightarrow \Delta; \Sigma}{\mho \xrightarrow{\mathcal{M}\text{-}int.} \mathcal{G} \mid \Gamma; \Pi \Rightarrow \Delta; \Sigma \mid \Box\Lambda; \Box\Theta \Rightarrow} \mathsf{MS} \tag{15}$$

preserves the h-interpolation for $\mathcal{M}$.

Proof. As always, the common language and suitability conditions are clearly transferred downwards. Consider any rooted $(n+1)$-tuple $\mathbf{w} * u * v$. Since the $(n+1)$th component does not occur in the premise, there are no occurrences of label $n+1$ in $\mho$. Thus, the truth value of $\mho$ at $\mathbf{w} * u * v$ is the same as at $\mathbf{w} * u$. Depending on the truth value, either left or right side of the premise hypersequent holds. As the two cases are completely analogous, we only consider the right side. Thus, if $\mathbf{w}*u*v \vDash \mho$, then $\mathbf{w}*u \vDash R(\mathcal{G}) \mid \Box\Theta, \Pi \Rightarrow \Sigma$. If $\mathbf{w} \vDash R(\mathcal{G})$ or $u \vDash \Pi \Rightarrow \Sigma$, we are done. The only remaining case is when $u \nvDash \Box B$ for some $\Box B \in \Box\Theta$. Since u and v are both accessible from a common root, it follows that vRu by Euclideanity. Further, as in the previous rules, by transitivity, $v \nvDash \Box B$. Thus, in this case too $\mathbf{w} * u * v \vDash R(\mathcal{G}) \mid \Pi \Rightarrow \Sigma \mid \Box\Theta \Rightarrow$. $\Box$

Corollary 3.37 *(LIP for* S5*). For any reflexive, transitive, and Euclidean model, transformations for all hypersequent rules of* S5 *preserve h-interpolation. Hence,* S5 *has Lyndon interpolation.*

Lemma 3.38 *(Transformation for* RMS*). For any transitive, and convergent Kripke model $\mathcal{M} = (W, R, V)$, any multiformula $\mho_C$ in uniform CNF, and any $(k-1)$-component split hypersequent $\mathcal{G}$, the transformation*

$$\frac{\mho \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Box\Lambda, \Box\Gamma; \Box\Theta, \Box\Delta \Rightarrow}{\mho[k\Diamond\Box] \xrightarrow{\mathcal{M}\text{-}int.} \quad \mathcal{G} \mid \Box\Lambda; \Box\Theta \Rightarrow \mid \Box\Gamma; \Box\Delta \Rightarrow}\ \mathsf{RMS} \tag{16}$$

preserves the h-interpolation for $\mathcal{M}$.

Proof. Let

$$\mho_C = \bigotimes_{i=1}^{n} \bigovee_{j=1}^{k} j : C_{ij} \ . \tag{17}$$

Then, modulo commutativity and associativity of $\ovee$,

$$\mho_C[k\Diamond\Box] = \bigotimes_{i=1}^{n} \left(k : \Diamond\Box C_{ik} \ovee \bigovee_{j=1}^{k-1} j : C_{ij} \right) \ . \tag{18}$$

As always, the common language and suitability conditions are clearly transferred downwards. Consider any rooted $(k+1)$-tuple $\mathbf{w} * u * v$.

Left side. If (18) is false, then for some i, we have $\mathbf{w} \nvDash \bigovee_{j=1}^{k-1} j : C_{ij}$ and $u \nvDash \Diamond\Box C_{ik}$. By convergence, there exists z such that uRz and vRz. Moreover, $z \nvDash \Box C_{ik}$. This implies that there is some z' such that zRz' and $z' \nvDash C_{ik}$. By transitivity, $\mathbf{w} * z'$ is a rooted k-tuple such that uRz' and vRz'. For this tuple (17) is clearly false. Thus, from the premise we conclude that

$$\mathbf{w} * z' \vDash L(\mathcal{G}) \mid \Box\Lambda, \Box\Gamma \Rightarrow \quad .$$

Formulas from $L(\mathcal{G})$ are unchanged and evaluated at the same worlds for the conclusion. If a boxed formula from $\Box\Lambda$ is false at z', it follows by transitivity that it is also false at u. Finally, if a boxed formula from $\Box\Gamma$ is false at z', it follows by transitivity that it is also false at v. Thus, we obtain the desired

$$\mathbf{w} * u * v \vDash L(\mathcal{G}) \mid \Box\Lambda \Rightarrow \mid \Box\Gamma \Rightarrow \quad .$$

Right side. If (18) is true, then each conjunct is true. For each i such that $u \vDash \Diamond\Box C_{ik}$, there exists u_i such that uRu_i and $u_i \vDash \Box C_{ik}$. By convergence and transitivity, there exists a z such that uRz and vRz and, in addition, $z \vDash C_{ik}$ whenever $u \vDash \Diamond\Box C_{ik}$. By transitivity, $\mathbf{w} * z$ is a rooted k-tuple at which (17) is true. Thus, from the premise we conclude that

$$\mathbf{w} * z \vDash R(\mathcal{G}) \mid \Box\Theta, \Box\Delta \Rightarrow \quad .$$

Formulas from $L(\mathcal{G})$ are unchanged and evaluated at the same worlds for the conclusion. If a boxed formula from $\Box\Theta$ is false at z, it follows by transitivity that it is also false at u. Finally, if a boxed formula from $\Box\Delta$ is false at z, it follows by transitivity that it is also false at v. Thus, we obtain the desired

$$\mathbf{w} * u * v \vDash R(\mathcal{G}) \mid \Box\Theta \Rightarrow \mid \Box\Delta \Rightarrow \quad . \qquad \Box$$

Corollary 3.39 *(LIP for* S4.2*). For any reflexive, transitive, and convergent model, transformations for all hypersequent rules of* S4.2 *preserve h-interpolation. Hence,* S4.2 *has Lyndon interpolation.*

4. Multicomponent proof-theoretic method: ingredients for the general case

Before switching to more expressive calculi of nested and labelled sequents, it pays off to stop and observe the general trends already evident in the hypersequent case. A hypersequent can be viewed as a function from $\{1, \dots, n\}$ into the set of sequents. We can generalize it as follows:

Definition 4.1 *(Multisequent).* Let $\mathfrak{L}$ be a fixed countable set of labels and $Str_{\mathfrak{L}} \subseteq 2^{\mathfrak{L}}$ be a fixed set of finite sets of labels. We denote elements of $Str_{\mathfrak{L}}$ by $\mathfrak{S}$ and call them $\mathfrak{L}$-*structures*. An $\mathfrak{S}$-*multisequent* (*split* $\mathfrak{S}$-*multisequent*) is function Φ from $\mathfrak{S} \in Str_{\mathfrak{L}}$ to the set of sequents (split sequents). For $\ell \in \mathfrak{S}$, we call $\Phi(\ell)$ a component of Φ and often denote it by $\Gamma_\ell \Rightarrow \Delta_\ell$ (or $\Gamma'_\ell; \Gamma''_\ell \Rightarrow \Delta'_\ell; \Delta''_\ell$ in the split case). A (split) $Str_{\mathfrak{L}}$-multisequent is a (split) $\mathfrak{S}$-multisequent for some $\mathfrak{S} \in Str_{\mathfrak{L}}$.

Let $\Phi_{\mathfrak{S}}$ be a split multisequent with $\Phi_{\mathfrak{S}}(\ell) = \Gamma'_\ell; \Gamma''_\ell \Rightarrow \Delta'_\ell; \Delta''_\ell$ for each $\ell \in \mathfrak{S}$. Its left and right sides are defined as follows: $L(\Phi_{\mathfrak{S}})$ is a multisequent mapping each label $\ell \in \mathfrak{S}$ into the sequent $\Gamma'_\ell \Rightarrow \Delta'_\ell$ and $R(\Phi_{\mathfrak{S}})$ is a multisequent mapping each label $\ell \in \mathfrak{S}$ into the sequent $\Gamma''_\ell \Rightarrow \Delta''_\ell$.

Example 4.2. Hypersequents can be represented by taking $\mathfrak{L}$ to be the set $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ of positive integers and $Str_{\mathfrak{L}} = \{\{1, \dots, k\} \mid k \in \mathbb{N}\}$. (One can also consider arbitrary non-empty finite subsets of $\mathbb{N}$, but consecutive labels make notation easier.)

Example 4.3. For nested sequents, which will be properly introduced in Sect. 5, $\mathfrak{L}$ is the set $\mathbb{N}^*$ of finite sequences of positive integers (including the empty sequence ε) and $Str_{\mathfrak{L}}$ consists of all non-empty finite subsets of $\mathfrak{L}$ that are closed w.r.t. sequence prefixes, i.e., for any $\mathfrak{S} \in Str_{\mathfrak{L}}$, whenever $\sigma * n \in \mathfrak{S}$, also $\sigma \in \mathfrak{S}$. In particular $\varepsilon \in \mathfrak{S}$ for all $\mathfrak{S} \in Str_{\mathfrak{L}}$.

Example 4.4. For labelled sequents, which will be properly introduced in Sect. 6, $\mathfrak{L}$ is any fixed countable set and $Str_{\mathfrak{L}}$ consists of all non-empty finite subsets of $\mathfrak{L}$.

The notion of multiformulas was initially formulated in Definition 3.11 for an arbitrary set $\mathfrak{L}$ of labels. A multiworld interpretation $\mathcal{I}$ was called suitable for a multiformula if it was defined on all its labels (Definition 3.14) and suitable for a hypersequent if it was defined exactly on its set of labels and an additional condition (of being a rooted tuple) was fulfilled. The latter definition is now extended:

Definition 4.5 *(Multiworld interpretation suitable for multisequent).* Given an $\mathfrak{S}$-multisequent Φ, a multiworld interpretation $\mathcal{I}$, a partial function from $\mathfrak{L}$ to worlds in a Kripke model $\mathcal{M} = (W, R, V)$ is called *suitable for* Φ if the domain of $\mathcal{I}$ is exactly $\mathfrak{S}$ and an additional condition $\mathfrak{S}(\Phi)$ is fulfilled that depends on Φ.

For hypersequents and nested sequents the additional condition is identical for all $\mathfrak{S}$-multisequents:

Example 4.6. A multiworld interpretation $\mathcal{I}$ into a model $\mathcal{M} = (W, R, V)$ is suitable for any $\{1, \dots, k\}$-hypersequent (previously called k-component hypersequent) if

$$\text{there is } w \in W \text{ such that } wR\mathcal{I}(i) \text{ for all } i = 1, \dots, k \ .$$

Example 4.7. A multiworld interpretation $\mathcal{I}$ into a model $\mathcal{M} = (W, R, V)$ is suitable for any $\mathfrak{S}$-nested sequent if

$$\mathcal{I}(\sigma)R\mathcal{I}(\sigma * n) \qquad \text{whenever} \qquad \{\sigma, \sigma * n\} \subseteq \mathfrak{S} \ .$$

For labelled sequents, on the contrary, the conditions $\mathfrak{S}(\Phi)$ do depend on Φ and are determined by its relational atoms (see Sect. 6 for details).

Definition 4.8 *(Satisfaction for multisequents).* Let Φ be an $\mathfrak{S}$-multisequent and $\mathcal{I}$ be a function from $\mathfrak{S}$ into worlds of a model $\mathcal{M} = (W, R, V)$. We define $\mathcal{M}, \mathcal{I} \vDash \Phi_{\mathfrak{S}}$ to hold iff

$$\text{either} \qquad \mathcal{M}, \mathcal{I}(\ell) \nvDash A \text{ for some } \ell \in \mathfrak{S} \text{ and} A \in \Gamma_\ell \qquad \text{or} \qquad \mathcal{M}, \mathcal{I}(\ell) \vDash B \text{ for some } \ell \in \mathfrak{S} \text{ and } B \in \Delta_\ell.$$

For split sequents, $A \in \Gamma'_\ell \cup \Gamma''_\ell$ and $B \in \Delta'_\ell \cup \Delta''_\ell$ instead.

In other words, a multisequent is true if either an antecedent formula is false or a consequent formula is true at the world assigned to the formula's component by $\mathcal{I}$.

Definition 4.9 *(Validity for multisequents).* A multisequent $\Phi_{\mathfrak{S}}$ is *valid* for a class $\mathcal{C}$ of models, written $\mathcal{C} \vDash_{Str_{\mathfrak{L}}} \Phi_{\mathfrak{S}}$, if it is true at all multiworld interpretations $\mathcal{I}$ into models $\mathcal{M} \in \mathcal{C}$ suitable for $\Phi_{\mathfrak{S}}$.

This definition clearly generalizes the definition of validity for hypersequents we gave in the previous section. Using the formula interpretation (5), we showed in Lemma 3.8 that any hypersequent calculus complete w.r.t. a logic in the usual sense is also valid in this sense. The suitability conditions are chosen for other formalisms so as to ensure the same equivalence:

Definition 4.10 *(Multisequent calculus completeness).* Let a modal logic L be complete w.r.t. a class of models $\mathcal{C}$. A (cut-free) multisequent calculus MSL *captures* the logic L if

$$\mathsf{MSL} \vdash \Phi_{\mathfrak{S}} \qquad \Longleftrightarrow \qquad \mathcal{C} \vDash_{Str_{\mathfrak{L}}} \Phi_{\mathfrak{S}}$$

for any multisequent $\Phi_{\mathfrak{S}}$.

The definition of multisequent interpolants is also a straightforward generalization of h-interpolants:

Definition 4.11 *(Multisequent interpolant).* For a given set of labels $\mathfrak{L}$ and set of allowed structures $Str_{\mathfrak{L}}$, a given split multisequent $\Phi_{\mathfrak{S}}$, and a model $\mathcal{M} = (W, R, V)$, an $\mathfrak{L}$-multiformula $\mho$ is called a *multisequent interpolant*, or simply *ms-interpolant*, of $\Phi_{\mathfrak{S}}$ for $\mathcal{M}$, written

$$\mho \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}} \ ,$$

if all labels occurring in $\mho$ belong from $\mathfrak{S}$, for every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ suitable for $\Phi_{\mathfrak{S}}$,

- if $\mathcal{M}, \mathcal{I} \nvDash \mho$, then $\mathcal{M}, \mathcal{I} \vDash L(\Phi_{\mathfrak{S}})$;
- if $\mathcal{M}, \mathcal{I} \vDash \mho$, then $\mathcal{M}, \mathcal{I} \vDash R(\Phi_{\mathfrak{S}})$;

and one of the common language conditions are satisfied:

- *Craig condition*: every atomic proposition occurring in $\mho$ occurs both in some $\Gamma'_{\ell_1} \Rightarrow \Delta'_{\ell_1}$ for some $\ell_1 \in \mathfrak{S}$ and in some $\Gamma''_{\ell_2} \Rightarrow \Delta''_{\ell_2}$ for some $\ell_2 \in \mathfrak{S}$;
- *Lyndon condition*:
  - every atomic proposition occurring positively in $\mho$ must occur both
    - either positively in some Γ'_{ℓ_1} or negatively in some Δ'_{ℓ_1} for some $\ell_1 \in \mathfrak{S}$ and
    - either negatively in some Γ''_{ℓ_2} or positively in some Δ''_{ℓ_2} for some $\ell_2 \in \mathfrak{S}$;
  - each atomic proposition occurring in $\mho$ negatively must occur both
    - either negatively in some Γ'_{ℓ_1} or positively in some Δ'_{ℓ_1} for some $\ell_1 \in \mathfrak{S}$ and
    - either positively in some Γ''_{ℓ_2} or negatively in some Δ''_{ℓ_2} for some $\ell_2 \in \mathfrak{S}$.

The ms-interpolants satisfying the Craig/Lyndon condition are called *Craig/Lyndon ms-interpolants.*

In Lemma 3.21, we reduced h-interpolation to formula interpolation by representing an implication $A \to B$ as a 1-component hypersequent $A; \Rightarrow ; B$. But the reduction only worked for reverse serial models because we needed every singleton sequence (w) to be a suitable multiformula interpretation.

Definition 4.12 *(Characteristic multisequent).* A multisequent $\Phi_{\{\ell\}}$ is called a *characteristic multisequent* of an implication $A \to B$ for model $\mathcal{M} = (W, R, V)$ if $\Phi(\ell) = A \Rightarrow B$ and, for every $w \in W$, the function $\{(\ell, w)\}$ mapping ℓ into w is suitable for $\Phi_{\{\ell\}}$.

$\Phi_{\{\ell\}}$ is called a *characteristic multisequent* of an implication $A \to B$ for a class $\mathcal{C}$ of models if it is characteristic for each model $\mathcal{M} \in \mathcal{C}$. In this case, we denote it by $\ell : A \Rightarrow B$.

Example 4.13.

- For hypersequents, $1 : A \Rightarrow B = A \Rightarrow B$ is the characteristic multisequent for any class of reverse serial models.
- For nested sequents, $\varepsilon : A \Rightarrow B$, usually denoted by $A \Rightarrow B$ is characteristic for any class of models.
- For labelled sequents, we will see that $\ell : A \Rightarrow B = \ell : A \Rightarrow \ell : B$ is characteristic for any class of models.

Lemma 4.14 *(Reduction from ms- to formula interpolation). Let $\mathcal{M} = (W, R, V)$ be a Kripke model and $\ell : A \Rightarrow B$ be a characteristic multisequent for $\mathcal{M}$. If a multiformula $\mho$ is a Craig (Lyndon) ms-interpolant of $\ell : A; \Rightarrow ; B$ for $\mathcal{M}$, then* $\mathsf{form}(\mho)$ *is a Craig (Lyndon) interpolant of $A \to B$ in $\mathcal{M}$.*

Proof. The proof is analogous to that of Lemma 3.21. □

Corollary 4.15. *Let $\ell : A \Rightarrow B$ be a characteristic multisequent for a class $\mathcal{C}$ of models. If a multiformula $\mho$ is a Craig/Lyndon ms-interpolant of $\ell : A; \Rightarrow ; B$ for each model in $\mathcal{C}$, then* $\mathsf{form}(\mho)$ *is a Craig/Lyndon interpolant of $A \to B$ for the logic* L *sound and complete w.r.t. $\mathcal{C}$.*

Thus, to prove the CIP/LIP of a logic L that is

- sound and complete w.r.t. a class $\mathcal{C}$ of models such that each (derivable) implication $A \to B$ has a multisequent $\ell : A \to B$ characteristic for the class $\mathcal{C}$,
- captured by a (cut-free) multisequent calculus MSL,

it is sufficient to provide interpolant transformations preserving ms-interpolation property for each model of $\mathcal{C}$ for all the rules of MSL.

Using this generalized approach, we can now prove Craig/Lyndon interpolation, but what is the benefit of such generality? We argue that this general approach enables one to see similarities in rules transcending different systems. We already transferred interpolant transformations for structural and propositional sequent rules to the hypersequent setting. But we did it by omission, claiming a proof by analogy but not actually performing it. In the general setting just described one formal proof can be given that deals with propositional and internal structural rules for sequents, hypersequents, as well as nested sequents and labelled sequents to be considered in the next sections, simultaneously. For instance, the following lemma is quite trivial:

Lemma 4.16 *(ms-Interpolants for initial multisequents). Let $\Phi_{\mathfrak{S}}$ be a split multisequent such that $\ell \in \mathfrak{S}$ and $\mathcal{M}$ be an arbitrary Kripke model.*

- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi, A; \Lambda \Rightarrow \Sigma; \Theta, A,$ *then* $\ell : A \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi; \Lambda, A \Rightarrow \Sigma, A; \Theta,$ *then* $\ell : \neg A \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi; \Lambda, A \Rightarrow \Sigma; \Theta, A,$ *then* $\ell : \top \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi, A; \Lambda \Rightarrow \Sigma, A; \Theta,$ *then* $\ell : \bot \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi; \Lambda, \bot \Rightarrow \Sigma; \Theta,$ *then* $\ell : \top \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi, \bot; \Lambda \Rightarrow \Sigma; \Theta,$ *then* $\ell : \bot \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi; \Lambda \Rightarrow \Sigma; \Theta, \top,$ *then* $\ell : \top \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*
- *If* $\Phi_{\mathfrak{S}}(\ell) = \Pi; \Lambda \Rightarrow \Sigma, \top; \Theta,$ *then* $\ell : \bot \xrightarrow{\mathcal{M}\text{-int.}} \Phi_{\mathfrak{S}}$*;*

Note that the first two rows of Fig. 1 and (7) are special cases of this lemma.

Further, we observe that for most single-premise rules—structural, propositional, or otherwise—the identity transformation preserves the interpolation statements. Most of them follow from one general statement

Definition 4.17 *(Local rules).* Let a split version of a single-premise multisequent rule

$$\frac{\Phi_{\mathfrak{S}}}{\Phi'_{\mathfrak{S}}}\,\mathsf{sr} \tag{19}$$

be such that every atomic proposition occurring in Φ occurs in Φ' on the same side (and with the same polarity for Lyndon interpolation). Let $\mathcal{M} = (W, R, V)$ be such that every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ that is suitable for Φ' is also suitable for Φ. The rule sr is called *local* for $\mathcal{M}$, or simply $\mathcal{M}$*-local*, if the following two properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$\mathcal{M}, \mathcal{I} \vDash L(\Phi) \quad \Longrightarrow \quad \mathcal{M}, \mathcal{I} \vDash L(\Phi') \qquad \text{and} \qquad \mathcal{M}, \mathcal{I} \vDash R(\Phi) \quad \Longrightarrow \quad \mathcal{M}, \mathcal{I} \vDash R(\Phi')\ .$$

Lemma 4.18 *(Transformation for local rules). Let a single-premise multisequent rule (19) be $\mathcal{M}$-local. Then any ms-interpolant $\mho$ of Φ for $\mathcal{M}$ is also an ms-interpolant of Φ' for $\mathcal{M}$.*

Proof. The proof is rather obvious. The common language conditions are explicitly encoded in the definition of local rules. Both the structure $\mathfrak{S}$ and the interpolant $\mho$ remain unchanged. For any multiworld interpretation $\mathcal{I}$ suitable for Φ', whatever the truth value of $\mho$, it can be used to obtain the corresponding premise interpolation statement because $\mathcal{I}$ is also suitable for Φ. Thus, one of the sides of the premise multisequent must be true at $\mathcal{I}$, and locality ensures that this truth statement can be transferred to the conclusion. □

Example 4.19. All structural rules from Fig. 1, all single-premise rules from Fig. 2 and their hypersequent analogs are $\mathcal{M}$-local for any model $\mathcal{M}$. The hypersequent rules $\Box^{\mathsf{l}} \Rightarrow$ and $\Box^{\mathsf{r}} \Rightarrow$ are $\mathcal{M}$-local for any reflexive model $\mathcal{M}$.

Proof. Let us show this for $\Box^{\mathsf{l}} \Rightarrow$. The structure $\{1, \dots, n\}$ is clearly preserved, and all atomic propositions remain in place. In fact all the formulas remain in place, except for A in the left-side antecedent that becomes $\Box A$. If the left-side of the premise is true at $\mathcal{I}$ because $\mathcal{I}(n) \nvDash A$, then $\mathcal{I}(n) \nvDash \Box A$ because $\mathcal{I}(n) R \mathcal{I}(n)$ by reflexivity. Thus, the left-side of the conclusion is also true at $\mathcal{I}$. □

Similar considerations help outline the situations when interpolants of several premises need to be combined by disjunction or by conjunction. We provide the definitions, leaving most of the proofs to the reader.

Definition 4.20 *(Conjunctive rules).* Let a split version of a multipremise multisequent rule

$$\frac{\Phi^1_{\mathfrak{S}} \quad \dots \quad \Phi^n_{\mathfrak{S}}}{\Phi'_{\mathfrak{S}}}\,\mathsf{sr} \tag{20}$$

be such that every atomic proposition occurring in one of Φ^i occurs in Φ' on the same side (and with the same polarity for Lyndon interpolation). Let $\mathcal{M} = (W, R, V)$ be such that every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ that is suitable for Φ' is also suitable for each of Φ^i. The rule sr is called *conjunctive* for $\mathcal{M}$, or simply $\mathcal{M}$-*conjunctive*, if the following two properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$(\exists i)\left(\mathcal{M}, \mathcal{I} \vDash L\left(\Phi^i\right)\right) \implies \mathcal{M}, \mathcal{I} \vDash L\left(\Phi'\right) \quad \text{and} \quad (\forall i)\left(\mathcal{M}, \mathcal{I} \vDash R\left(\Phi^i\right)\right) \implies \mathcal{M}, \mathcal{I} \vDash R\left(\Phi'\right) .$$

Lemma 4.21 *(Transformation for conjunctive rules). Let rule (20) be $\mathcal{M}$-conjunctive. Then,*

$$(\forall i)\left(\mho^i \xrightarrow{\mathcal{M}\text{-int.}} \Phi^i\right) \implies \mho^1 \owedge \dots \owedge \mho^n \xrightarrow{\mathcal{M}\text{-int.}} \Phi' .$$

Proof. The only non-trivial part is to check the two interpolation statements. If the conjunction of $\mho^i$'s is false, then one of them false, making the left side of one of the premises true, which is sufficient to make the left side of the conclusion true. If the conjunction is true, all conjuncts are true, meaning that the right sides of all premises are true, which is sufficient to make the right side of the conclusion true. □

Example 4.22. A typical conjunctive rule, such as the propositional rules introducing conjunction in the right side of the consequent, disjunction in the right side of the antecedent, or implication in the right side of the antecedent, has the same left side for all the premises and the conjunction: $L\left(\Phi^1\right) = L\left(\Phi^2\right) = L\left(\Phi'\right)$. This makes the conjunctive requirement on the left sides trivial.

Definition 4.23 *(Disjunctive rules).* Let a split version (20) be such that every atomic proposition occurring in one of Φ^i occurs in Φ' on the same side (and with the same polarity for Lyndon interpolation). Let

$\mathcal{M} = (W, R, V)$ be such that every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ that is suitable for Φ' is also suitable for each of Φ^i. The rule sr is called *disjunctive* for $\mathcal{M}$, or simply $\mathcal{M}$-*disjunctive*, if the following two properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$(\forall i)\left(\mathcal{M}, \mathcal{I} \vDash L\left(\Phi^i\right)\right) \implies \mathcal{M}, \mathcal{I} \vDash L\left(\Phi'\right) \quad \text{and} \quad (\exists i)\left(\mathcal{M}, \mathcal{I} \vDash R\left(\Phi^i\right)\right) \implies \mathcal{M}, \mathcal{I} \vDash R\left(\Phi'\right) .$$

Lemma 4.24 *(Transformation for disjunctive rules). Let rule (20) be $\mathcal{M}$-disjunctive. Then,*

$$(\forall i)\left(\mho^i \xrightarrow{\mathcal{M}\text{-}int.} \Phi^i\right) \implies \mho^1 \otimes \ldots \otimes \mho^n \xrightarrow{\mathcal{M}\text{-}int.} \Phi' .$$

Example 4.25. A typical disjunctive rule has the same right side for all the premises and the conclusion.

There is one last type of rules that usually appear only once in each proof formalism but are likely to appear in each proof formalism dealing with modal logics and, hence, worth treating in a general manner.

Definition 4.26 *($\Box$-like and $\Diamond$-like rules).* Let a split version of a single-premise multisequent rule

$$\frac{\Phi_{\mathfrak{S}\sqcup\{\ell_p,\ell_c\}}}{\Phi'_{\mathfrak{S}\sqcup\{\ell_p\}}}\,\mathsf{sr} \tag{21}$$

for some $\ell_p \neq \ell_c$ be such that every atomic proposition occurring in Φ occurs in Φ' on the same side (and with the same polarity for Lyndon interpolation). Let $\mathcal{M} = (W, R, V)$ be such that for every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ that is suitable for Φ' and any world v such that $\mathcal{I}(\ell_p)Rv$, the function $\mathcal{I} \sqcup \{(\ell_c, v)\}$ is suitable for Φ.

We use the following notation: $R(w) := \{v \in W \mid wRv\}$.

The rule sr is called $\Box$-*like* for $\mathcal{M}$, or simply $\mathcal{M}$-$\Box$-*like*, if the following two properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$(\exists v \in R(\mathcal{I}(\ell_p)))\left(\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash L(\Phi)\right) \implies \mathcal{M}, \mathcal{I} \vDash L(\Phi') ,$$
$$(\forall v \in R(\mathcal{I}(\ell_p)))\left(\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash R(\Phi)\right) \implies \mathcal{M}, \mathcal{I} \vDash R(\Phi') .$$

In particular, it is required that $\mathcal{M}, \mathcal{I} \vDash R(\Phi')$ if $R(\mathcal{I}(\ell_p)) = \varnothing$.

The rule sr is called $\Diamond$-*like* for $\mathcal{M}$, or simply $\mathcal{M}$-$\Diamond$-*like*, if the following two properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$(\forall v \in R(\mathcal{I}(\ell_p)))\left(\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash L(\Phi)\right) \implies \mathcal{M}, \mathcal{I} \vDash L(\Phi') ,$$
$$(\exists v \in R(\mathcal{I}(\ell_p)))\left(\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash R(\Phi)\right) \implies \mathcal{M}, \mathcal{I} \vDash R(\Phi') .$$

In particular, it is required that $\mathcal{M}, \mathcal{I} \vDash L(\Phi')$ if $R(\mathcal{I}(\ell_p)) = \varnothing$.

Lemma 4.27 *(Transformation for $\Box$- and $\Diamond$-like rules). Let rule (21) be $\mathcal{M}$-$\Box$-like. Let $\mho_C$ be in uniform CNF (i.e., every label from $\mathfrak{S} \sqcup \{\ell_c, \ell_p\}$ is present in each conjunct exactly once). Then, we have*

$$\mho_C \xrightarrow{\mathcal{M}\text{-}int.} \Phi \implies \mho_C[\ell_c \mapsto \ell_p\Box] \xrightarrow{\mathcal{M}\text{-}int.} \Phi' .$$

Let rule (21) be $\mathcal{M}$-$\Diamond$-like. Let $\mho_D$ be in uniform DNF (i.e., every label from $\mathfrak{S} \sqcup \{\ell_c, \ell_p\}$ is present in each disjunct exactly once). Then, we have

$$\mho_D \xrightarrow{\mathcal{M}\text{-}int.} \Phi \implies \mho_D[\ell_c \mapsto \ell_p\Diamond] \xrightarrow{\mathcal{M}\text{-}int.} \Phi' .$$

Proof. We only prove the statement for $\Box$-like rules as the other one is its dual. The common language condition follows from the preservation of all atomic propositions (and their polarities) for each side encoded into the definition of $\Box$-like rules as well as from the preservation of atomic propositions (and their polarities) by transformation from

$$\mho_C = \bigotimes_{i=1}^{n} \left(\ell_p : A_i \otimes \ell_c : B_i \otimes \bigotimes_{\ell \in \mathfrak{S}} \ell : C_{i\ell} \right) \tag{22}$$

to

$$\mho_C[\ell_c \mapsto \ell_p \Box] = \bigotimes_{i=1}^{n} \left(\ell_p : A_i \otimes \ell_p : \Box B_i \otimes \bigotimes_{\ell \in \mathfrak{S}} \ell : C_{i\ell} \right) . \tag{23}$$

Since this transformation removes the label ℓ_c from $\mho_C$, the suitability condition is also fulfilled. It remains to check the interpolation statements.

Left side. Let (23) be false at some $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ'. For the conjunction (23) to be false, there must be an i such that

$$\mathcal{I} \nvDash \ell_p : A_i \otimes \ell_p : \Box B_i \otimes \bigotimes_{\ell \in \mathfrak{S}} \ell : C_{i\ell} .$$

Thus,

- $\mathcal{I}(\ell_p) \nvDash A_i$,
- $\mathcal{I}(\ell_p) \nvDash \Box B_i$, and
- $(\forall \ell \in \mathfrak{S})(\mathcal{I}(\ell) \nvDash C_{il})$.

In particular, there exists $v \in R(\mathcal{I}(\ell_p))$ such that $v \nvDash B_i$. By $\Box$-likeness of the rule, $\mathcal{I} \sqcup \{(\ell_c, v)\}$ is suitable for Φ and

$$\mathcal{I} \sqcup \{(\ell_c, v)\} \nvDash \ell_p : A_i \otimes \ell_c : B_i \otimes \bigotimes_{\ell \in \mathfrak{S}} \ell : C_{i\ell} .$$

Thus, (22) is false at $\mathcal{I} \sqcup \{(\ell_c, v)\}$, making $L(\Phi)$ true at it. It now follows from the $\Box$-likeness that $L(\Phi')$ is true at $\mathcal{I}$.

Right side. Let (23) be true at some $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ'. Take an arbitrary $v \in R(\mathcal{I}(\ell_p))$. By $\Box$-likeness, $\mathcal{I} \sqcup \{(\ell_c, v)\}$ is suitable for Φ. If the ith conjunct of (23) is true because $\mathcal{I}(\ell_p) \vDash \Box B_i$, then the ith conjunct of (22) is true because $\big(\mathcal{I} \sqcup \{(\ell_c, v)\}\big)(\ell_c) = v \vDash B_i$. If the ith conjunct is true for any other reason, this reason still works for (22) at $\mathcal{I} \sqcup \{(\ell_c, v)\}$. Thus, (22) is true at $\mathcal{I} \sqcup \{(\ell_c, v)\}$. We have shown that

$$\mathcal{I}(\ell_p) R v \qquad \Longrightarrow \qquad \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash R(\Phi) .$$

It now follows from the $\Box$-likeness that $R(\Phi')$ is true at $\mathcal{I}$. $\Box$

In these rules the child ℓ_c-component that disappears from the premise can be interpreted by any world accessible from the interpretation of the parent ℓ_p-component. Not all rules give so much freedom. We now consider a rather important case where such a freedom eventually arrives. This case has connections to the S4.2 hypersequent calculus and will be used later for labelled sequents.

Definition 4.28 *(Convergent rules).* Let a split version of a single-premise multisequent rule

$$\frac{\Phi_{\mathfrak{S}\sqcup\{\ell_{b_1},\ell_{b_2},\ell_c\}}}{\Phi'_{\mathfrak{S}\sqcup\{\ell_{b_1},\ell_{b_2}\}}}\,\mathsf{sr}$$

for some pairwise distinct labels ℓ_{b_1}, ℓ_{b_2}, and ℓ_c be such that every atomic proposition occurring in Φ occurs in Φ' on the same side (and with the same polarity for Lyndon interpolation). Let $\mathcal{M} = (W, R, V)$ be such that for any multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ', the following requirements are satisfied:

- $R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2})) \neq \varnothing$;
- for any world $z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))$ the function $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ for each $y \in R(z)$;
- for arbitrary worlds $z_1, z_2 \in R(\mathcal{I}(\ell_{b_1}))$, there exists a world $z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))$ such that $R(z) \subseteq R(z_1) \cap R(z_2)$.

The rule sr is called *convergent* for $\mathcal{M}$, or simply $\mathcal{M}$-*convergent*, if the following two properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$\begin{aligned}
\big(\forall z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))\big)\, \big(\exists y \in R(z)\big)\, (\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, y)\} \vDash L(\Phi)) \quad &\Longrightarrow \quad \mathcal{M}, \mathcal{I} \vDash L(\Phi')\ ,\\
\big(\exists z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))\big)\, \big(\forall y \in R(z)\big)\, (\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, y)\} \vDash R(\Phi)) \quad &\Longrightarrow \quad \mathcal{M}, \mathcal{I} \vDash R(\Phi')\ .
\end{aligned}$$

Lemma 4.29 *(Transformation for convergent rules). Let rule (21) be $\mathcal{M}$-convergent. Let $\mho_C$ be in uniform CNF (i.e., every label from $\mathfrak{S} \sqcup \{\ell_{b_1}, \ell_{b_2}, \ell_c\}$ is present in each conjunct exactly once). Then,*

$$\mho_C \xrightarrow{\mathcal{M}\text{-}int.} \Phi \qquad \Longrightarrow \qquad \mho_C[\ell_c \mapsto \ell_{b_1}\Diamond\Box] \xrightarrow{\mathcal{M}\text{-}int.} \Phi'\ .$$

Proof. Once again, the common language and suitability conditions are clearly fulfilled. The transformation we consider here is from

$$\mho_C = \bigotimes_{i=1}^{n} \left(\ell_{b_1} : A_i \otimes \ell_c : B_i \otimes \bigovee_{\ell \in \mathfrak{S}\sqcup\{\ell_{b_2}\}} \ell : C_{i\ell} \right) \tag{24}$$

to

$$\mho_C[\ell_c \mapsto \ell_{b_1}\Diamond\Box] = \bigotimes_{i=1}^{n} \left(\ell_{b_1} : A_i \otimes \ell_{b_1} : \Diamond\Box B_i \otimes \bigovee_{\ell \in \mathfrak{S}\sqcup\{\ell_{b_2}\}} \ell : C_{i\ell} \right)\ . \tag{25}$$

Left side. Let (25) be false at some $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ'. Take an arbitrary $z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))$. By the convergeability, $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ for any $y \in R(z)$. For the conjunction (25) to be false, there must be an i such that

$$\mathcal{I} \nvDash \ell_{b_1} : A_i \otimes \ell_{b_1} : \Diamond\Box B_i \otimes \bigovee_{\ell \in \mathfrak{S}\sqcup\{\ell_{b_2}\}} \ell : C_{i\ell}\ .$$

Thus,

- $\mathcal{I}(\ell_{b_1}) \nvDash A_i$,
- $\mathcal{I}(\ell_{b_1}) \nvDash \Diamond\Box B_i$, and
- $(\forall \ell \in \mathfrak{S} \sqcup \{\ell_{b_2}\})(\mathcal{I}(\ell) \nvDash C_{il})$.

In particular, $z \nvDash \Box B_i$. Thus, there exists a $y \in R(z)$ such that $y \nvDash B_i$. For this y, the function $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ, and

$$\mathcal{I} \sqcup \{(\ell_c, y)\} \nvDash \ell_{b_1} : A_i \oslash \ell_c : B_i \oslash \bigoslash_{\ell \in \mathfrak{S} \sqcup \{\ell_{b_2}\}} \ell : C_{i\ell} \ .$$

Thus, (24) is false at $\mathcal{I} \sqcup \{(\ell_c, y)\}$, making $L(\Phi)$ true at it. We showed that for each $z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))$, there is $y \in R(z)$ such that $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ and $L(\Phi)$ is true at it. It now follows from the convergeability that $L(\Phi')$ is true at $\mathcal{I}$.

Before switching to the right side, we prove an auxiliary statement:

$$\Big(\forall u_1 \in R\big(\mathcal{I}(\ell_{b_1})\big)\Big) \ \ldots \ \Big(\forall u_k \in R\big(\mathcal{I}(\ell_{b_1})\big)\Big) \ \Big(\exists z \in R\big(\mathcal{I}(\ell_{b_1})\big) \cap R\big(\mathcal{I}(\ell_{b_2})\big)\Big) \quad R(z) \subseteq R(u_1) \cap \cdots \cap R(u_k) \ .$$

Indeed, for $k = 1$ and $k = 2$, this follows directly from convergeability for $z_1 = z_2 = u_1$ and for $z_1 = u_1$ and $z_2 = u_2$ respectively. We prove the general case by induction on k. Suppose the statement is already proved for k, i.e.,

$$\Big(\exists z' \in R\big(\mathcal{I}(\ell_{b_1})\big) \cap R\big(\mathcal{I}(\ell_{b_2})\big)\Big) \quad R(z') \subseteq R(u_1) \cap \cdots \cap R(u_k) \ .$$

For $z_1 = z'$ and $z_2 = u_{k+1}$, both from $R(\mathcal{I}(\ell_{b_1}))$, there is $z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))$ such that

$$R(z) \subseteq R(z') \cap R(u_{k+1}) \subseteq R(u_1) \cap \cdots \cap R(u_k) \cap R(u_{k+1}) \ .$$

Right side. Let the conjunction (25) be true, meaning that all the conjuncts are true. If the ith conjunct is true because $\mathcal{I}(\ell_{b_1}) \vDash \Diamond\Box B_i$, then there exists $u_i \in R(\mathcal{I}(\ell_{b_1}))$ such that $u_i \vDash \Box B_i$. Let $\{i_1, \ldots, i_k\}$ be all the numbers of conjuncts with $\mathcal{I}(\ell_{b_1}) \vDash \Diamond\Box B_{i_j}$, in particular, $u_{i_j} \vDash \Box B_{i_j}$ for $j = 1, \ldots, k$.

Case 1. $k > 0$. By the auxiliary statement we just proved

$$\Big(\exists z \in R\big(\mathcal{I}(\ell_{b_1})\big) \cap R\big(\mathcal{I}(\ell_{b_2})\big)\Big) \quad R(z) \subseteq R(u_{i_1}) \cap \cdots \cap R(u_{i_k}) \ .$$

By the convergeability, the function $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ for each $y \in R(z)$. Moreover, since $R(z) \subseteq R(u_{i_1}) \cap \cdots \cap R(u_{i_k})$, $y \vDash B_{i_j}$ for each $y \in R(z)$ and each $j = 1, \ldots, k$. We conclude that if the ith conjunct of (25) is true because of $\ell_{b_1} : \Diamond\Box B_{i_j}$, then, for the chosen z, the ith conjunct of (24) is true at each $\mathcal{I} \sqcup \{(\ell_c, y)\}$ for any $y \in R(z)$ because of $\ell_c : B_i$. If the ith conjunct of (25) is true for some other reason, this reason remains valid at $\mathcal{I} \sqcup \{(\ell_c, y)\}$ for any $y \in R(z)$. In other words, we have found a $z \in R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2}))$ such that for any $y \in R(z)$, the function $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ and the conjunction (24) is true at it. Using the right side of the premise, we conclude that $R(\Phi)$ is true at $\mathcal{I} \sqcup \{(\ell_c, y)\}$ for all such y. By the convergeability, it means that $R(\Phi')$ is true at $\mathcal{I}$.

Case 2. $k = 0$. Since $R(\mathcal{I}(\ell_{b_1})) \cap R(\mathcal{I}(\ell_{b_2})) \neq \varnothing$, we can choose some z in it. For any $y \in R(z)$, the function $\mathcal{I} \sqcup \{(\ell_c, y)\}$ is suitable for Φ. Since $\Diamond\Box B_i$ play no role in the truth of (25), we have (24) true at all such $\mathcal{I} \sqcup \{(\ell_c, y)\}$. Using the right side of the premise, we conclude that $R(\Phi)$ is true at all such $\mathcal{I} \sqcup \{(\ell_c, y)\}$. Thus, by convergeability, $R(\Phi')$ is true at $\mathcal{I}$. □

You might have noticed that the general transformations for □-like, ◇-like, and convergent rules are reminiscent of, but not quite identical to the transformations we used for hypersequent rules $\Rightarrow \Box^{\mathsf{r}}$, $\Rightarrow \Box^{\mathsf{l}}$, and RMS respectively. The difference is that there all interpolant parts from the active component receive a □, ◇, or ◇□, whereas in the general case, they are also moved to another component. This is not accidental as the Kleene'd versions of these hypersequent rules can be easily checked to be $\Box$-like, $\Diamond$-like, and convergent respectively for the relevant types of models:

$$\dfrac{\mho_C \xrightarrow{\mathcal{M}\text{-int.}} \quad \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow ; \Box A \mid \Box\Gamma; \Box\Pi \Rightarrow ; A}{\mho_C[k+1 \mapsto k\Box] \xrightarrow{\mathcal{M}\text{-int.}} \quad \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow ; \Box A} \Rightarrow \Box^{\mathsf{r}} \quad \text{is } \Box\text{-like for any transitive model } \mathcal{M};$$

$$\dfrac{\mho_D \xrightarrow{\mathcal{M}\text{-int.}} \quad \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow \Box A; \mid \Box\Gamma; \Box\Pi \Rightarrow A;}{\mho_D[k+1 \mapsto k\Diamond] \xrightarrow{\mathcal{M}\text{-int.}} \quad \mathcal{G} \mid \Box\Gamma; \Box\Pi \Rightarrow \Box A;} \Rightarrow \Box^{\mathsf{l}} \quad \text{is } \Diamond\text{-like for any transitive model } \mathcal{M};$$

finally,

$$\dfrac{\mho \xrightarrow{\mathcal{M}\text{-int.}} \quad \mathcal{G} \mid \Box\Lambda; \Box\Theta \Rightarrow \mid \Box\Gamma; \Box\Delta \Rightarrow \mid \Box\Lambda, \Box\Gamma; \Box\Theta, \Box\Delta \Rightarrow}{\mho[k+2 \mapsto k\Diamond\Box] \xrightarrow{\mathcal{M}\text{-int.}} \quad \mathcal{G} \mid \Box\Lambda; \Box\Theta \Rightarrow \mid \Box\Gamma; \Box\Delta \Rightarrow} \mathsf{RMS}$$

is convergent for any convergent and transitive model $\mathcal{M}$.

In fact, the interpolants for non-Kleene'd versions can be computed from the transformations for $\Box$-like, $\Diamond$-like, and convergent rules because the non-Kleene'd versions are derivable in presence of external weakening and external contraction. It is, therefore, sufficient to generalize Lemmas 3.23 and 3.26:

Lemma 4.30 *(Transformation for external weakening: general case). A multisequent rule*

$$\dfrac{\Phi_{\mathfrak{S}}}{\Phi'_{\mathfrak{S}\sqcup\{\ell\}}} \mathsf{EW}$$

is called external weakening EW *if* $\Phi' \restriction \mathfrak{S} = \Phi$.

Let $\mathcal{M}$ *be a Kripke model. If for any multiworld interpretation* $\mathcal{I}$ *into* $\mathcal{M}$ *suitable for* Φ'*, its restriction* $\mathcal{I} \restriction \mathfrak{S}$ *is suitable for* Φ*, then any interpolant* $\mho$ *of* Φ *for* $\mathcal{M}$ *is also an interpolant of* Φ' *for* $\mathcal{M}$.

Lemma 4.31 *(Transformation for external contraction: general case). A multisequent rule*

$$\dfrac{\Phi_{\mathfrak{S}\sqcup\{\ell,\ell'\}}}{\Phi'_{\mathfrak{S}\sqcup\{\ell\}}} \mathsf{EC}$$

is called external contraction EC *if* $\Phi' \restriction \mathfrak{S} = \Phi$ *and* $\Phi(\ell) = \Phi(\ell') = \Phi'(\ell)$.

Let $\mathcal{M}$ *be a Kripke model. If for any multiworld interpretation* $\mathcal{I}$ *into* $\mathcal{M}$ *suitable for* Φ'*, the function* $\mathcal{I} \sqcup \{(\ell', \mathcal{I}(\ell))\}$ *is suitable for* Φ*, then, for any interpolant* $\mho$ *of* Φ *for* $\mathcal{M}$*, the multiformula* $\mho[\ell' \mapsto \ell]$ *is an interpolant of* Φ' *for* $\mathcal{M}$.

Proofs of Lemmas 4.30 and 4.31. Analogous to those of Lemmas 3.23 and 3.26 respectively. $\Box$

5. Multicomponent proof-theoretic method: nested sequents

We are now in a position to apply the general transformations discussed above to specific calculi. We start with nested sequents. Throughout this section, we use the set of labels $\mathfrak{L}$ from Example 4.3. Thus, each label is a (possibly empty) finite sequence of positive integers. A finite tree is any $\mathfrak{S} \in Str_{\mathfrak{L}}$ as defined in Example 4.3, i.e., any non-empty set of labels that is closed w.r.t. initial prefixes.

Definition 5.1 *(Nested sequent).* A (split) *nested sequent* is any (split) multisequent $\Phi_{\mathfrak{S}}$ with $\mathfrak{S}$ being a finite tree.

You can easily see that this description is equivalent to the more standard definition

Definition 5.2. If $\Gamma \Rightarrow \Delta$ is a sequent, $M \subseteq \mathbb{N}$ is a (possibly empty) set of positive integers, and $\{\Phi_i \mid i \in M\}$ is a set of nested sequents, then $\Gamma \Rightarrow \Delta, [\Phi_{i_1}], \dots, [\Phi_{i_k}]$ is a nested sequent, where $M = \{i_1, \dots, i_k\}$.

Definition 5.3. The standard *formula interpretation for nested sequents* is defined recursively:

$$\imath(\Gamma \Rightarrow \Delta, [\Phi_{i_1}], \dots, [\Phi_{i_k}]) := \bigwedge \Gamma \to \bigvee \Delta \vee \Box\imath(\Phi_{i_1}) \vee \dots \vee \Box\imath(\Phi_{i_k}) \ .$$

Suitable multiworld interpretations for $\Phi_{\mathfrak{S}}$ must satisfy the condition, already mentioned in Example 4.7, namely that $\mathcal{I}(\sigma) R \mathcal{I}(\sigma * n)$ for any $\sigma, \sigma * n \in \mathfrak{S}$. As in the case of hypersequents, it is easy to see that for a given class $\mathcal{C}$ of models, the formula interpretation is valid in $\mathcal{C}$ as a formula iff the nested sequent is valid in $\mathcal{C}$ as a multisequent in the sense of Definition 4.9. Thus, standard completeness theorems for nested sequent calculi directly translate into multisequent calculus completeness. As already mentioned in Example 4.13, the characteristic nested sequent for $A \to B$ is a single-node split multisequent $\Phi_{\{\varepsilon\}}$ with $\Phi(\varepsilon) = A; \Rightarrow; B$, which in the standard nested sequent notation looks just like an ordinary sequent $A; \Rightarrow; B$. Note that for $\mathfrak{S} = \{\varepsilon\}$, the suitability conditions impose no restrictions on the choice of $\mathcal{I}(\varepsilon)$.

Thus, by Lemma 4.14, we can prove Craig/Lyndon interpolation for a logic by finding ms-interpolant transformations for all rules of a cut-free nested sequent calculus for the logic. The treatment of initial sequents, propositional and internal structural rules is the same as for all other formalisms, using Lemmas 4.16, 4.18, 4.21, and 4.24. To demonstrate how the general method can be easily applied, we take several modal logical rules from Poggiolesi [57, Sect. 6.2]. The simplest is the rule t, which in our notation is

$$\frac{\Phi_{\mathfrak{S}} \sqcup \{\sigma \mapsto \Box A, A, \Gamma \Rightarrow \Delta\}}{\Phi_{\mathfrak{S}} \sqcup \{\sigma \mapsto \Box A, \Gamma \Rightarrow \Delta\}}\, t$$

where $\mathfrak{S} \sqcup \{\sigma\} \in Str_{\mathfrak{L}}$. In other words, the rule takes any sequent component with both $\Box A$ and A in the antecedent and removes A. A more familiar notation for this rule is obtained by omitting all mention of labels, which we use purely for bookkeeping:

$$\frac{\Phi\{\Box A, A, \Gamma \Rightarrow \Delta\}}{\Phi\{\Box A, \Gamma \Rightarrow \Delta\}}\, t \ .$$

Since the use of names σ for tree nodes is essential for stating interpolation transformations, we propose an alternative shorthand, which mentions the stable part of the multisequent only once and requires by default that $\mathfrak{S} \sqcup \{\sigma\} \in Str_{\mathfrak{L}}$:

$$\Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Box A, A, \Gamma \Rightarrow \Delta}{\sigma \mapsto \Box A, \Gamma \Rightarrow \Delta}\, t \ .$$

As the name suggests, this rule is used for logics validating the t axiom, i.e., logics, complete w.r.t. reflexive models, e.g., T, $\mathsf{S4}$, or $\mathsf{S5}$.

Lemma 5.4 *(Nested rule t). Both split versions*

$$\Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Box A, A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, t_l \qquad \textit{and} \qquad \Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Gamma'; \Box A, A, \Gamma'' \Rightarrow \Delta'; \Delta''}{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta''}\, t_r$$

of t are local for any reflexive model. Hence, any ms-interpolant for the premise is also an interpolant for the conclusion.

Proof. If $\mathcal{I}(\sigma) \nvDash A$, then $\mathcal{I}(\sigma) \nvDash \Box A$ by reflexivity, and all the other formulas remain intact. $\Box$

Lemma 5.5 *(Nested rule* 4*). The split versions of the rule* 4*, which is used for transitive models, are*

$$\Phi_{\mathfrak{S}} \frac{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Box A, \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''} 4_l \ ,$$

$$\Phi_{\mathfrak{S}} \frac{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Box A, \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''} 4_r \ .$$

They are local for any transitive model. Hence, any ms-interpolant for the premise is also an interpolant for the conclusion.

Proof. The only premise formula that is not preserved downwards is $\Box A$ in the antecedent of the node $\sigma * n$. If the corresponding side of the premise is true because $\mathcal{I}(\sigma * n) \nvDash \Box A$, then, given that $\mathcal{I}(\sigma) R \mathcal{I}(\sigma * n)$ and transitivity of R, it follows that $\mathcal{I}(\sigma) \nvDash \Box A$, which makes the same side of the conclusion true. $\Box$

Lemma 5.6 *(Nested rule* b*). The split versions of the rule* b*, which is used for symmetric models, are*

$$\Phi_{\mathfrak{S}} \frac{\sigma \mapsto A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Box A, \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Box A, \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''} b_l \ ,$$

$$\Phi_{\mathfrak{S}} \frac{\sigma \mapsto \Gamma'; A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Box A, \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Box A, \Pi'' \Rightarrow \Sigma'; \Sigma''} b_r \ .$$

They are local for any symmetric model. Hence, any ms-interpolant for the premise is also an interpolant for the conclusion.

Proof. Indeed, the only premise formula that is not preserved downwards is A in the antecedent of the node σ. If the corresponding side of the premise is true because $\mathcal{I}(\sigma) \nvDash A$, then, given that $\mathcal{I}(\sigma) R \mathcal{I}(\sigma * n)$ and symmetry of R, it follows that $\mathcal{I}(\sigma * n) \nvDash \Box A$, which makes the same side of the conclusion true. $\Box$

Lemma 5.7 *(Nested rule* 5*). The split versions of the rule* 5*, which is used for Euclidean models, are*

$$\Phi_{\mathfrak{S}} \frac{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Box A, \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Box A, \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''} 5_l$$

$$\Phi_{\mathfrak{S}} \frac{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Box A, \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Box A, \Pi'' \Rightarrow \Sigma'; \Sigma''} 5_r$$

They are local for any Euclidean model. Hence, any ms-interpolant for the premise is also an interpolant for the conclusion.

Proof. The only premise formula that is not preserved downwards is $\Box A$ in the antecedent of the node σ. If the corresponding side of the premise is true because $\mathcal{I}(\sigma) \nvDash \Box A$, then, there must exist a world $v \in R(\mathcal{I}(\sigma))$ such that $v \nvDash A$. By Euclideanity, $\mathcal{I}(\sigma * n) R v$, hence $\mathcal{I}(\sigma * n) \nvDash \Box A$, which makes the same side of the conclusion true. $\Box$

By now, the following lemma should be apparent

Lemma 5.8 *(Nested rule $\Box A$). The split versions of the rule $\Box A$ used for all models are*

$$\Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto A, \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}\ \Box A_l\ ,$$

$$\Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; A, \Pi'' \Rightarrow \Sigma'; \Sigma''}{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Pi'; \Pi'' \Rightarrow \Sigma'; \Sigma''}\ \Box A_r\ .$$

They are local for any model. Hence, any ms-interpolant for the premise is also an interpolant for the conclusion.

The only non-local modal rules in Poggiolesi [57, Sect. 6.2] are d and $\Box K$. We start with the latter, which is used for all models.

Lemma 5.9 *(Nested rule $\Box K$). The split versions of $\Box K$ are*

$$\Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Rightarrow A;}{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Box A, \Delta'; \Delta''}\ \Box K_l \quad \text{and} \quad \Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto \Rightarrow ; A}{\sigma \mapsto \Gamma'; \Gamma'' \Rightarrow \Delta'; \Box A, \Delta''}\ \Box K_r\ .$$

The rule $\Box K_r$ is a $\Box$-like rule, and $\Box K_l$ is a $\Diamond$-like rule for $\ell_p = \sigma$ and $\ell_c = \sigma * n$. Hence,

- *for any interpolant $\mho_C$ for the premise of $\Box K_r$ in uniform CNF, $\mho_C[\sigma * n \mapsto \sigma\Box]$ is an interpolant of the conclusion;*
- *for any interpolant $\mho_D$ for the premise of $\Box K_l$ in uniform DNF, $\mho_D[\sigma * n \mapsto \sigma\Diamond]$ is an interpolant of the conclusion.*

Proof. We only show the statement for $\Box K_r$; the other is dual. First of all, we observe that for any multiworld interpretation $\mathcal{I}$ suitable for the conclusion and any world $v \in R(\mathcal{I}(\sigma))$, the function $\mathcal{I} \sqcup \{(\sigma * n, v)\}$ is suitable for the premise. Secondly, the left sides of the premise and conclusion are identical and contain no formulas from the $\sigma * n$-component. Hence, whatever makes the left side of the premise true at $\mathcal{I} \sqcup \{(\sigma * n, v)\}$ for some v, also makes the left side of the conclusion true at $\mathcal{I}$. Let us now assume that the right side of the premise is true at $\mathcal{I} \sqcup \{(\sigma * n, v)\}$ for any $v \in R(\mathcal{I}(\sigma))$. If the right side of the premise is true (for any one v) because of any other formula that A at $\sigma * n$, it similarly transfers downwards and we are done. Otherwise, A must be true at all $v \in R(\mathcal{I}(\sigma))$.[10] This implies that $\mathcal{I}(\sigma) \vDash \Box A$, which also makes the right side of the conclusion true. This completes the proof that $\Box K_r$ is a $\Box$-like rule. □

Finally, we come to an interesting rule. The split versions of the nested d rule, used for serial models, have the form

$$\Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto A; \Rightarrow}{\sigma \mapsto \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\ d_l\ , \qquad \Phi_{\mathfrak{S}}\ \frac{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta'', \quad \sigma * n \mapsto ; A \Rightarrow}{\sigma \mapsto \Gamma'; \Box A, \Gamma'' \Rightarrow \Delta'; \Delta''}\ d_r\ . \tag{26}$$

One can easily prove that for serial models d_l is $\Box$-like and d_r is $\Diamond$-like, so it is clear which interpolant transformations to use. However, it is also possible to prove that for serial models d_l is $\Diamond$-like and d_r is $\Box$-like. We can define such rules as follows:

[10] Note that this includes the case when $R(\mathcal{I}(\sigma)) = \varnothing$.

Definition 5.10 *(Serial rules).* Let a split version of a single-premise multisequent rule

$$\frac{\Phi_{\mathfrak{S}\sqcup\{\ell_p,\ell_c\}}}{\Phi'_{\mathfrak{S}\sqcup\{\ell_p\}}}\,\mathsf{sr} \tag{27}$$

for some $\ell_p \neq \ell_c$ be such that every atomic proposition occurring in Φ occurs in Φ' on the same side (and with the same polarity for Lyndon interpolation). Let $\mathcal{M} = (W, R, V)$ be such that for every multiworld interpretation $\mathcal{I}$ into $\mathcal{M}$ that is suitable for Φ' and any world v such that $\mathcal{I}(\ell_p)Rv$, the function $\mathcal{I} \sqcup \{(\ell_c, v)\}$ is suitable for Φ.

The rule sr is called *serial* for $\mathcal{M}$, or simply $\mathcal{M}$*-serial*, if the following three properties hold for any $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ':

$$\begin{gathered}R(\mathcal{I}(\ell_p)) \neq \varnothing\ ,\\ (\exists v \in R(\mathcal{I}(\ell_p)))\,(\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash L\,(\Phi)) \quad\Longrightarrow\quad \mathcal{M}, \mathcal{I} \vDash L\,(\Phi')\ ,\\ (\exists v \in R(\mathcal{I}(\ell_p)))\,(\mathcal{M}, \mathcal{I} \sqcup \{(\ell_c, v)\} \vDash R\,(\Phi)) \quad\Longrightarrow\quad \mathcal{M}, \mathcal{I} \vDash R\,(\Phi')\ .\end{gathered}$$

Lemma 5.11 *(Transformation for serial rules). For serial rules, one can use either of the transformations for $\Box$-like or for $\Diamond$-like rules. Let rule (27) be $\mathcal{M}$-serial. For $\mho_C$ in uniform CNF and $\mho_D$ in uniform DNF,*

$$\begin{aligned}\mho_C \xrightarrow{\mathcal{M}\text{-}int.} \Phi \quad&\Longrightarrow\quad \mho_C[\ell_c \mapsto \ell_p\Box] \xrightarrow{\mathcal{M}\text{-}int.} \Phi'\ ,\\ \mho_D \xrightarrow{\mathcal{M}\text{-}int.} \Phi \quad&\Longrightarrow\quad \mho_D[\ell_c \mapsto \ell_p\Diamond] \xrightarrow{\mathcal{M}\text{-}int.} \Phi'\ .\end{aligned}$$

Proof. We only prove the first statement as the other one is its dual. The common language condition follows from the preservation of all atomic propositions (and their polarities) for each side encoded into the definition of $\Box$-like rules as well as from the preservation of atomic propositions (and their polarities) by transformation from

$$\mho_C = \bigowedge_{i=1}^{n} \left(\ell_p : A_i \ovee \ell_c : B_i \ovee \bigovee_{\ell\in\mathfrak{S}} \ell : C_{i\ell}\right) \tag{28}$$

to

$$\mho_C[\ell_c \mapsto \ell_p\Box] = \bigowedge_{i=1}^{n} \left(\ell_p : A_i \ovee \ell_p : \Box B_i \ovee \bigovee_{\ell\in\mathfrak{S}} \ell : C_{i\ell}\right)\ . \tag{29}$$

Since this transformation removes the label ℓ_c from $\mho_C$, the suitability condition is also fulfilled. It remains to check the interpolation statements.

Left side. Let (23) be false at some $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ'. For the conjunction (23) to be false, there must be an i such that

$$\mathcal{I} \nvDash \ell_p : A_i \ovee \ell_p : \Box B_i \ovee \bigovee_{\ell\in\mathfrak{S}} \ell : C_{i\ell}\ .$$

Thus,

$$\mathcal{I}(\ell_p) \nvDash A_i\ , \qquad \text{and} \qquad \mathcal{I}(\ell_p) \nvDash \Box B_i\ , \qquad \text{and} \qquad (\forall \ell \in \mathfrak{S})(\mathcal{I}(\ell) \nvDash C_{il})\ .$$

In particular, there exists $v \in R(\mathcal{I}(\ell_p))$ such that $v \nvDash B_i$. By seriality of the rule, $\mathcal{I} \sqcup \{(\ell_c, v)\}$ is suitable for Φ and

$$\mathcal{I} \sqcup \{(\ell_c, v)\} \not\models \ell_p : A_i \oslash \ell_c : B_i \oslash \bigoslash_{\ell \in \mathfrak{S}} \ell : C_{i\ell} \ .$$

Thus, (22) is false at $\mathcal{I} \sqcup \{(\ell_c, v)\}$, making $L(\Phi)$ true at it. It now follows from the seriality that $L(\Phi')$ is true at $\mathcal{I}$.

Right side. Let (23) be true at some $\mathcal{I}$ into $\mathcal{M}$ suitable for Φ'. By seriality, there is a $v \in R(\mathcal{I}(\ell_p))$ and $\mathcal{I} \sqcup \{(\ell_c, v)\}$ is suitable for Φ. If the ith conjunct of (23) is true because $\mathcal{I}(\ell_p) \models \Box B_i$, then the ith conjunct of (22) is true because $\big(\mathcal{I} \sqcup \{(\ell_c, v)\}\big)(\ell_c) = v \models B_i$. If the ith conjunct is true for any other reason, this reason still works for (22) at $\mathcal{I} \sqcup \{(\ell_c, v)\}$. Thus, (22) is true at $\mathcal{I} \sqcup \{(\ell_c, v)\}$, making $R(\Phi)$ true at it. It now follows from the seriality that $R(\Phi')$ is true at $\mathcal{I}$. $\Box$

Lemma 5.12 *(Nested rule d). The split versions* (26) *of the rule d are $\mathcal{M}$-serial for any serial model $\mathcal{M}$ for $\ell_p = \sigma$ and $\ell_c = \sigma * n$. Thus,*

- *for any interpolant $\mho_D$ for the premise of d_r or d_l in uniform DNF, $\mho_D[\sigma * n \mapsto \sigma\Diamond]$ is an interpolant of the conclusion;*
- *for any interpolant $\mho_C$ for the premise of d_l or d_l in uniform CNF, $\mho_C[\sigma * n \mapsto \sigma\Box]$ is an interpolant of the conclusion.*

Proof. We only show seriality for d_l, leaving the case of d_r to the reader. Any world accessible from $\mathcal{I}(\sigma)$ can be used to interpret $\sigma * n$. And the seriality of the model ensures that $R(\mathcal{I}(\sigma)) \neq \varnothing$. So we only need to check the truth descent from premise to conclusion. We first start with the right side, which does not change from premise to conclusion. Hence, the interpretation of $v \in R(\mathcal{I}(\sigma))$ plays no role in the truth of the right side as long as it exists. For the left side, assume that the left side of the premise is true at $\mathcal{I} \sqcup \{(\sigma * n, v)\}$ for some $v \in R(\mathcal{I}(\sigma))$. If it happens because of any formula other than A at $\sigma * n$, then it transfers to the conclusion directly. If $v \not\models A$, then $\mathcal{I}(\sigma) \not\models \Box A$, and the left side of the conclusion is true again. $\Box$

Corollary 5.13. *All modal logics captured by Poggiolesi's rules $\Box K$, $\Box A$, d, b, t, 4, 5 have the CIP and LIP.*

Remark 5.14. The method can be (and in Fitting and Kuznets [24] has been) applied to one-sided nested sequents from Brünnler [11], which cover all 15 modal logics of the modal cube, i.e., all logics complete with respect to any class of models satisfying any subset of the conditions of seriality, reflexivity, symmetry, transitivity, and Euclideanity. However, to keep the notation uniform and avoid overextending the article, we choose to omit the translations from one-sided to two-sided nested sequents and limit ourselves to Poggiolesi's rules that are very similar.

6. Multicomponent proof-theoretic method: labelled sequents

We finally come to the most expressive calculus of those we consider, labelled sequents.

We were hesitant to adapt the standard notation for nested sequents for two reasons: firstly, unlike hypersequents, the naming of the sequent-tree nodes for nested sequents is far from obvious (an accurate account of restoring unique names can be found in [22]); secondly, again unlike hypersequents, the notation for nested sequents has not yet been completely standardized. Of the most recent incarnations, Brünnler's notation is very different from Poggiolesi's. Thus, using a third one seemed more of a compromise than a mindless notation creation.

Labelled sequents, on the other hand, have a rather convenient notation that makes explicit use of the labels. Thus, instead of adapting it to our general framework, we will use it almost the same way it is present in Negri and von Plato [53, Sect. 11.3]. In particular, we will use multiset-based labelled sequents and adapt names for rules from [53].

Definition 6.1 *(Labelled sequent).* A *labelled sequent* Φ has the form

$$w_1 Q_1 v_1, \ldots, w_n Q_n v_n, u_1 : A_1, \ldots, u_k : A_k \Rightarrow z_1 : B_1, \ldots z_l : B_l \ , \tag{30}$$

where each $Q_i \in \{\mathsf{R}, =\}$ for each $i = 1, \ldots, n$ and

$$\mathfrak{S} = \{w_1, \ldots, w_n, v_1, \ldots, v_n, u_1, \ldots, u_k, z_1, \ldots, z_l\} \subseteq \mathfrak{L}$$

are labels (one label may occur more than once in a labelled sequent). Expressions $u_i : A_i$ and $z_j : B_j$ are called *labelled formulas*, $w_s \mathsf{R} v_s$ are called *relational atoms*, and $w_s = v_s$ are called *equational atoms*. Together, relational and equational atoms are called *structural atoms*.

To translate this notation into our framework, for each $\ell \in \mathfrak{S}$ we define $\Phi(\ell) := \Gamma_\ell \Rightarrow \Delta_\ell$ where

$$\Gamma_\ell := \{A_i \mid u_i = \ell\} \qquad \text{and} \qquad \Delta_\ell := \{B_j \mid z_j = \ell\} \ .$$

Note that there may be labels $\ell \in \mathfrak{S}$ with $\Phi(\ell) = \Rightarrow$. This happens if ℓ occurs only in structural atoms.

Satisfaction relation for labelled sequents is originally defined[11] the same way as the one for multisequents with the following suitability condition:

Definition 6.2 *(Suitability for labelled sequents).* Let $\mathcal{M} = (W, R, V)$ be a Kripke model. A multiworld interpretation $\mathcal{I} \colon \mathfrak{S} \to W$ is *suitable* for a labelled sequent Φ from (30) if $\mathcal{I}(w_i) = \mathcal{I}(v_i)$ whenever Q_i is $=$ and $\mathcal{I}(w_i) R \mathcal{I}(v_i)$ whenever Q_i is R.

Remark 6.3. Negri and von Plato [53] also considered labelled sequents with structural atoms in the consequent. However, they do not affect validity of sequents (see also Footnote 11), nor are they needed for completeness of various labelled sequent calculi. Hence, we do not consider them here.

Since validity for labelled sequents is identical to validity of multisequents (modulo trivial equivalent changes), completeness of labelled calculi transfers to labelled sequents understood as multisequent calculi.

Note that structural atoms are not part of the component sequents of Φ. Since only component sequents are split into the left and right sides, structural atoms remain outside of the part being split. (Note that the same happens with vertical lines in the standard hypersequent notation.) To emphasize this visually, we will separate the structural atoms from the rest of the sequent with $\prec$ and write:

$$\mathfrak{R} \prec \Gamma \Rightarrow \Delta$$

where $\mathfrak{R}$ consists of structural atoms and Γ and Δ of labelled formulas. In this notation,

$$\mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''$$

are split labelled sequents. We will omit $\prec$ when $\mathfrak{R}$ is empty.

Definition 6.4 *(Characteristic labelled sequents).* For any label $\ell \in \mathfrak{L}$, the split labelled sequent

$$\ell : A; \Rightarrow ; \ell : B$$

can play the role of a *characteristic sequent* for an implication $A \to B$. Indeed, in the absence of structural atoms, no restrictions are imposed on the interpretation of ℓ.

[11] See *validity* in Negri and von Plato [53, Def. 11.26], where our multiformula interpretation $\mathcal{I}$ is called interpretation $[\![\cdot]\!]$.

$$\frac{w\mathsf{R}o, \mathfrak{R} \prec o : A, w : \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}o, \mathfrak{R} \prec w : \Box A, \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''} L\Box_l \qquad \frac{w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; o : A, w : \Box A, \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; w : \Box A, \Gamma'' \Rightarrow \Delta'; \Delta''} L\Box_r$$

$$\frac{w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow o : A, \Delta'; \Delta''}{\mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow w : \Box A, \Delta'; \Delta''} R\Box_l \qquad \frac{w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; o : A, \Delta''}{\mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; w : \Box A, \Delta''} R\Box_r$$

Fig. 4. Split variants of labelled modal logical rules. For $R\Box_l$ and $R\Box_r$, the *eigenvariable* o does not occur in the conclusion.

Thus, the formalism of labelled sequents is ready for multisequent interpolation.

The situation with initial sequents, propositional, and structural rules is completely standard: they are treated the same way as for all other formalisms.

Where labelled sequent calculi are different and, perhaps, superior to other formalisms is in the method of generating labelled sequent rules. There are only two logical modal rules. Their split versions are presented in Fig. 4. All the other modal rules are purely structural, meaning that they only affect structural atoms.

Lemma 6.5. *Both split versions of the rule $L\Box$ are local for any model. Hence, any ms-interpolant for the premise is also an interpolant for the conclusion.*

Proof. The proof is really a notational variant of that for the nested rule $\Box A$ from Lemma 5.8. The only labelled premise formula missing from the conclusion is $o : A$. If the relevant part of the premise is true because $\mathcal{I}(o) \nvDash A$, then $\mathcal{I}(w) \nvDash \Box A$ because for any $\mathcal{I}$ suitable for the conclusion and premise, $\mathcal{I}(w)R\mathcal{I}(o)$. □

Lemma 6.6. *The rule $R\Box_r$ is a $\Box$-like rule, and $R\Box_l$ is a $\Diamond$-like rule for $\ell_p = w$ and $\ell_c = o$. Hence,*

- *for any interpolant $\mho_C$ for the premise of $R\Box_r$ in uniform CNF, $\mho_C[o \mapsto w\Box]$ is an interpolant of the conclusion;*
- *for any interpolant $\mho_D$ for the premise of $R\Box_l$ in uniform DNF, $\mho_D[o \mapsto w\Diamond]$ is an interpolant of the conclusion.*

Proof. This is a notational variant of the proof of Lemma 5.9, which we will not repeat. We only note that the eigenvariable condition ensures that a suitable multiworld interpretation $\mathcal{I}$ for the conclusion is undefined for o, which makes an extension mapping o to any world accessible from $\mathcal{I}(w)$ possible. □

These rules take care of the labelled sequent system for K. The great advantage of labelled sequents is that labelled sequent calculi for extensions of K described by first-order Kripke-frame conditions of particular types can be computed from the frame conditions, and the resulting labelled sequent calculi are guaranteed to be cut-free. We will now show that for some of these frame conditions, we can also compute interpolant transformations that guarantee the CIP and LIP for the underlying logic.

The simplest types of such frame conditions are

$$\bigwedge \mathfrak{R} \to \bigvee \mathfrak{R}' ,$$

where $\mathfrak{R}$ and $\mathfrak{R}'$ are sets of structural atoms. Keeping in mind the negative example of $\mathsf{S4.3}$, which is described by models satisfying $wRu \wedge wRv \to vRu \vee uRv$, we do not attempt to provide general interpolant transformations for $\mathfrak{R}'$ with two or more structural atoms. This leaves us with two types of frame conditions:

$$\bigwedge \mathfrak{R} \to w\mathsf{R}o , \tag{31}$$

$$\bigwedge \mathfrak{R} \to \bot, \qquad \text{where } \mathfrak{R} \neq \varnothing. \tag{32}$$

The labelled rules and initial sequents corresponding to such frame conditions and guaranteeing cut-elimination for the underlying logic are called *mathematical* and have the form (in the split version)

$$\frac{w\mathsf{R}o, \mathfrak{R}, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R}, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\ \mathsf{Horn}\ , \tag{33}$$

$$\mathfrak{R}, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta'', \qquad \text{where } \mathfrak{R} \neq \varnothing, \tag{34}$$

respectively. Moreover, it is shown in [53] that the completeness of the calculi is not affected if the rule (33) is restricted by requiring both w and o to occur in the conclusion sequent (though not necessarily in $\mathfrak{R}$ or $\mathfrak{R}'$).

Definition 6.7. A labelled (split) rule instance has the *subterm property* if every label occurring in one of the premises also occurs in the conclusion. Any labelled rule r can be restricted to those instances that have the subterm property. The set of those instances will be referred to as the rule $\mathsf{r}^\dagger$.

Lemma 6.8 *(Mathematical rules are local). Let model $\mathcal{M} = (W, R, V)$ satisfy the condition (31). Then the subterm-closed version $\mathsf{Horn}^\dagger$ of the rule (33) is local for $\mathcal{M}$.*

Proof. As most rules yet to be considered in this section, this is a structural rule, meaning that no formula is affected by it. In other words, as long as we manage to extend a given multiworld interpretation $\mathcal{I}$ suitable for the conclusion to one suitable for the premise, we are done. The subterm-closure ensures that no extension is necessary: w and o are already in the domain of $\mathcal{I}$. Finally, since $\mathcal{I}$ is suitable for the conclusion, it satisfies all structural atoms in it, including all structural atoms in $\mathfrak{R}$. It follows by (31) that $\mathcal{I}(w)R\mathcal{I}(o)$. Thus, $\mathcal{I}$ also satisfies the only additional condition $w\mathsf{R}o$ in the premise and is, thus, suitable for the premise. □

Lemma 6.9 *(Interpolants for mathematical initial sequents). Let model $\mathcal{M} = (W, R, V)$ satisfy (32) and w be a label occurring in $\mathfrak{R} \neq \varnothing$. Then*

$$w : \bot \xrightarrow{\mathcal{M}\text{-}int.} \mathfrak{R}, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''\ .$$

Proof. The statement is vacuously true as no $\mathcal{I}$ is suitable for $\mathfrak{R}, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''$ by (32). □

It is time to mention that, in order to be complete, a system with mathematical rules (33) must also contain their contracted instances. In other words, if an instance of (33) has

$$\mathfrak{R} = \mathfrak{R}_0 \cup \{\underbrace{w_1\mathsf{R}o_1, \dots, w_1\mathsf{R}o_1}_{k_1}\} \cup \dots \cup \{\underbrace{w_n\mathsf{R}o_n, \dots, w_n\mathsf{R}o_n}_{k_n}\}\ , \tag{35}$$

there must be another instance in the calculus with

$$\mathfrak{R}_c = \mathfrak{R}_0 \cup \{w_1\mathsf{R}o_1, \dots, w_n\mathsf{R}o_n\}\ . \tag{36}$$

The same applies to initial sequents and to geometrical rules we will consider later. We denote the contracted instances of a rule sr by sr^*.

However, contracted instances present no additional difficulties due to the following trivial observation:

Frame condition		Rule
Reflexivity	wRw	$\dfrac{w\mathsf{R}w, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, Ref^{\dagger}$
Transitivity	$wRo \wedge oRr \to wRr$	$\dfrac{w\mathsf{R}r, w\mathsf{R}o, o\mathsf{R}r, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}o, o\mathsf{R}r, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, Trans$
		$\dfrac{w\mathsf{R}w, w\mathsf{R}w, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}w, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, Trans^{*}$
Euclideanity	$wRo \wedge wRr \to oRr$	$\dfrac{o\mathsf{R}r, w\mathsf{R}o, w\mathsf{R}r, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}o, w\mathsf{R}r, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, Eucl$
		$\dfrac{o\mathsf{R}o, w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, Eucl^{*}$
Symmetry	$wRo \to oRw$	$\dfrac{o\mathsf{R}w, w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w\mathsf{R}o, \mathfrak{R} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\, Sym$

Fig. 5. Split variants of labelled mathematical rules. Each rule is $\mathcal{M}$-local provided the model $\mathcal{M}$ satisfies the corresponding frame condition. For the rule $Ref^{\dagger}$, the label w must occur in the conclusion.

Lemma 6.10. *A multiworld interpretation $\mathcal{I}$ is suitable for a labelled sequent*

$$\mathfrak{R}, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''$$

for $\mathfrak{R}$ from (35) iff $\mathcal{I}$ is suitable for

$$\mathfrak{R}_c, \mathfrak{R}' \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''$$

with $\mathfrak{R}_c$ from (36).

Corollary 6.11. *For any model $\mathcal{M}$ and any interpolant transformation preserving $\mathcal{M}$-ms-interpolation for a rule* sr*, the same interpolant transformation preserves $\mathcal{M}$-ms-interpolation for all contracted instances of* sr^*.

Corollary 6.12. *All contracted instances of a mathematical rule (33) are local for any model satisfying (31).*

Example 6.13. Fig. 5 contains common mathematical rules, their contracted versions, and conditions on models that make them local.

The Craig part of the following claim was first proved by Marx [47, Cor. B.4.1] using a non-(efficiently)-constructive method.

Theorem 6.14 *(Constructive Lyndon interpolation for Horn-definable modal logics). Any logic sound and complete w.r.t. a class of Kripke models defined by Horn statements, i.e., statements of forms (31) and (32), has the CIP and LIP.*

Corollary 6.15. *The LIP holds for logic complete w.r.t. class of models defined by any combination of properties such as the ones mentioned in Fig. 5, as well as*

- *shift reflexivity* $uRw \to wRw$*;*
- *shift versions of other properties from Fig. 5, obtained prepending their frame conditions with* $uRw\wedge$*;*
- *m-transitivity* $wRo_1 \wedge o_1Ro_2 \wedge \dots o_{m-1}Ro_m \to wRo_m$*;*
- *functionality* $wRo \wedge wRu \to o = u$*.*

The second important type of frame conditions that generates so-called *geometric rules* for labelled sequent calculi have the form

$$\bigwedge \mathfrak{R} \to \bigvee_{i=1}^{k} (\exists \mathbf{y}_i) \bigwedge \mathfrak{R}_i$$

where $\mathfrak{R}, \mathfrak{R}_1, \dots, \mathfrak{R}_k$ are sets of structural atoms and $\mathbf{y}_1, \dots, \mathbf{y}_k$ are sequences of labels (without repetitions) such that none of these labels occurs in $\mathfrak{R}$. Once again, as far as hopes for interpolation go, we have to restrict ourselves to the case of $k \leq 1$. Moreover, for $k = 0$, we fall back into the realm of mathematical rules. Hence, we only consider conditions of the form

$$\bigwedge \mathfrak{R} \to (\exists y_1) \dots (\exists y_k) \bigwedge \mathfrak{R}' \tag{37}$$

with none of y_i's occurring in $\mathfrak{R}$. This condition was shown in [53] to be captured by the *geometric rule* with the following split version:

$$\frac{\mathfrak{R}'[y_i \mapsto o_i], \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\ \mathsf{GRS} \tag{38}$$

where none of *eigenvariables* o_i's occurs in the conclusion. As with mathematical rules, it suffices to consider subterm-closed versions of GRS, i.e., versions with every label from the premise being either one of o_i's or occurring in the conclusion. Similarly, for each $\mathsf{GRS}^\dagger$ all contracted versions have to be added to the calculus.

While, having additional eigenvariable labels in the premise, geometric rules cannot be local, interpolant transformations for some important geometric rules can be obtained by using our general definitions for $\Box$-like, $\Diamond$-like, serial, and convergent rules.

Example 6.16. The seriality condition $(\exists y)wRy$ is captured by the geometric rule

$$\frac{wRo, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}\ Ser^\dagger \tag{39}$$

where the eigenvariable o does not occur in the conclusion and w must occur in the conclusion.

Lemma 6.17 *(Labelled rule* $Ser^\dagger$ *is serial for serial models). For any serial model, this rule is serial for* $\ell_p = w$ *and* $\ell_c = o$ *and, hence,*

- *for any interpolant* $\mho_D$ *for the premise of* $Ser^\dagger$ *in uniform DNF,* $\mho_D[o \mapsto w\Diamond]$ *is an interpolant of the conclusion;*
- *for any interpolant* $\mho_C$ *for the premise of* $Ser^\dagger$ *in uniform CNF,* $\mho_C[o \mapsto w\Box]$ *is an interpolant of the conclusion.*

Proof. The proof for such structural rules is almost trivial. First, since o does not occur in the conclusion and w does, for any $\mathcal{I}$ suitable for the conclusion and any $v \in \mathcal{I}(w)$, the function $\mathcal{I} \sqcup \{(o, v)\}$ is suitable for the premise and the truth of a side of the premise at $\mathcal{I} \sqcup \{(o, v)\}$ trivially translates to that for the same side of the conclusion at $\mathcal{I}$ because no labelled formula in the conclusion is labelled o. Finally, seriality of the frame ensures that at least one $\mathcal{I} \sqcup \{(o, v)\}$ exists. □

We recall the standard notation for binary relations R on W: for $k > 1$

$$R^k := \{(w, u) \mid (\exists y_1) \dots (\exists y_{k-1}) wRy_1, y_1Ry_2, \dots, y_{k-2}Ry_{k-1}, y_{k-1}Ru\} \ .$$

$R^1 := R$, and $R^0 = \{(w, w) \mid w \in W\}$.

Example 6.18. The *hijk-convergence properties* defining Geach logics can be compactly stated as

$$wR^h v_h \wedge wR^j u_j \to (\exists o)(v_h R^i o \wedge u_j R^k o)$$

or, written in full, have the form

$$\begin{aligned} &wRv_1 \wedge \dots \wedge v_{h-1}Rv_h \quad \wedge \quad wRu_1 \wedge \dots \wedge u_{j-1}Ru_j \qquad \to \\ &\qquad (\exists z_1) \dots (\exists z_{i-1})(\exists y_1) \dots (\exists y_{k-1})(\exists o)(v_hRz_1 \wedge \dots \wedge z_{i-1}Ro \quad \wedge \quad u_jRy_1 \wedge \dots \wedge y_{k-1}Ro) \end{aligned} \tag{40}$$

where all z_a's, y_b's, and o are pairwise distinct and distinct from any of w, u_c', and v_d's. In this example, for the purposes of uniformity, we only consider the cases of $h, i, j, k \geq 1$. Before presenting these rules, we need some abbreviations to fit things into the page width. Let

$$\mathfrak{R} := \{w'\mathsf{R}v'_1, \dots, v'_{h-1}\mathsf{R}v'_h, w'\mathsf{R}u'_1, \dots, u'_{j-1}\mathsf{R}u'_j\} \tag{41}$$

The split version of the geometric rule corresponding to (40) is

$$\frac{v'_h\mathsf{R}z'_1, \dots, z'_{i-1}\mathsf{R}o', u'_j\mathsf{R}y'_1, \dots, y'_{k-1}\mathsf{R}o', \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''} \, \mathsf{G} \ , \tag{42}$$

where the eigenvariables $z'_1, \dots, z'_{i-1}, y'_1, \dots, y'_{k-1}$, and o' do not occur in the conclusion. Since all types of rules we have considered only remove one label at a time, to compute the interpolant transformation for this rule, we consider an alternative labelled sequent calculus where this rule is derivable (see Fig. 6). The only downside of this method is that, to show that the rules of this other calculus are sound, we will use an additional assumption that all models of the logic are transitive.

Lemma 6.19. *Let $\mathcal{M} = (W, R, V)$ be transitive and hijk-convergent for $h, i, j, k > 0$. Then this model is also $(h + i - 1, j + k - 1, 1, 1)$-convergent, i.e., satisfies the following property*

$$wR^{h+i-1}v_{h+i-1} \wedge wR^{j+k-1}u_{j+k-1} \to (\exists o)(v_{h+i-1}Ro \wedge u_{j+k-1}Ro) \ , \tag{43}$$

and its corresponding labelled sequent rule

$$\frac{v'_{h+i-1}\mathsf{R}o', u'_{j+k-1}\mathsf{R}o', \mathfrak{R}_c, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R}_c, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''} \ , \tag{44}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v'_h \mathsf{R} z'_1, \dots, z'_{i-2} \mathsf{R} z'_{i-1}, \boxed{z'_{i-1} \mathsf{R} o'}, u'_j \mathsf{R} y'_1, \dots, y'_{k-2} \mathsf{R} y'_{k-1}, \boxed{y'_{k-1} \mathsf{R} o'}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{v'_h \mathsf{R} z'_1, z'_1 \mathsf{R} z'_2, \dots, z'_{i-2} \mathsf{R} z'_{i-1}, u'_j \mathsf{R} y'_1, y'_1 \mathsf{R} y'_2, \dots, \boxed{y'_{k-2} \mathsf{R} y'_{k-1}}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}}{\vdots}}{v'_h \mathsf{R} z'_1, z'_1 \mathsf{R} z'_2, \dots, z'_{i-2} \mathsf{R} z'_{i-1}, u'_j \mathsf{R} y'_1, \boxed{y'_1 \mathsf{R} y'_2}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}}{v'_h \mathsf{R} z'_1, z'_1 \mathsf{R} z'_2, \dots, z'_{i-2} \mathsf{R} z'_{i-1}, \boxed{u'_j \mathsf{R} y'_1}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}}{v'_h \mathsf{R} z'_1, z'_1 \mathsf{R} z'_2, \dots, \boxed{z'_{i-2} \mathsf{R} z'_{i-1}}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}}{\vdots}}{v'_h \mathsf{R} z'_1, \boxed{z'_1 \mathsf{R} z'_2}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}}{\boxed{v'_h \mathsf{R} z'_1}, \mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}}{\mathfrak{R}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}
$$

Fig. 6. Geach labelled rule as a sequence of a convergent and several serial rules. Each rule but the topmost removes one relational atom and is serial. The topmost rule removes two relational atoms and is convergent. The relational atom(s) removed in each rule is (are) framed in the premise of the rule.

where $\mathfrak{R}_c = \{w' \mathsf{R} v'_1, \dots, v'_{h+i-1} \mathsf{R} v'_{h+i-1}, w' \mathsf{R} u'_1, \dots, u'_{j+k-2} \mathsf{R} u'_{j+k-1}\}$*,* $\mathfrak{Q}$ *is arbitrary, and* o' *is an eigenvariable, is* $\mathcal{M}$*-convergent (with* $\ell_{b_1} = v'_{h+i-1}$*,* $\ell_{b_2} = u'_{j+k-1}$*, and* $\ell_c = o'$*).*

Thus, for any interpolant $\mho_C$ *for the premise of* (44) *in uniform CNF,* $\mho_C[o' \mapsto v'_{h+i-1}\Diamond\Box]$ *is an interpolant of the conclusion.*

Proof. We first prove (43). Assume the antecedent of the implication holds. By transitivity of R, we have $wR^h v_{h+i-1}$ and $wR^j u_{j+k-1}$ because $i, k \geq 1$. By $hijk$-convergence, there exists o such that $v_{h+i-1} R^i o$ and $u_{j+k-1} R^k o$. Thus, by transitivity, $v_{h+i-1} R o$ and $u_{j+k-1} R o$ because $k, i \geq 1$, as required.

Let us now show that the rule is convergent. Once again, for a structural rule like this, only the bullet points of Definition 4.28 need to be checked. Let a multiworld interpretation $\mathcal{I}$ be suitable for the conclusion of (44). For each label $\ell' \in \mathfrak{L}$ occurring in the conclusion, let $\mathcal{I}(\ell')$ be denoted by $\ell \in W$.

- By (43), there exists $o \in W$ such that $v_{h+i-1} R o$ and $u_{j+k-1} R o$. Thus $R(\mathcal{I}(v'_{h+i-1})) \cap R(\mathcal{I}(u'_{j+k-1})) \neq \varnothing$.
- For any world $o \in R(\mathcal{I}(v'_{h+i-1})) \cap R(\mathcal{I}(u'_{j+k-1}))$, and any $x \in R(o)$, by transitivity we have that $x \in R(\mathcal{I}(v'_{h+i-1})) \cap R(\mathcal{I}(u'_{j+k-1}))$, making $\mathcal{I} \sqcup \{(o', x)\}$ suitable for the premise of (44).
- For arbitrary worlds $x_1, x_2 \in R(\mathcal{I}(v'_{h+i-1}))$, by transitivity $wR^{h+i-1} x_1$. Hence, by (43), there is x_3 such that $x_1 R x_3$ and $u_{j+k-1} R x_3$. By transitivity, $wR^{h+i-1} x_2$ and $wR^{j+k-1} x_3$. Hence, by (43) there exists o such that $x_2 R o$ and $x_3 R o$. By transitivity, $v_{h+i-1} R o$ and $u_{j+k-1} R o$, hence, we have that $o \in R(\mathcal{I}(v'_{h+i-1})) \cap R(\mathcal{I}(u'_{j+k-1}))$. Moreover, since $x_2 R o$ and, by transitivity, $x_1 R o$, therefore, by transitivity, $R(o) \subseteq R(x_1) \cap R(x_2)$. □

Lemma 6.20. *Let* $\mathcal{M} = (W, R, V)$ *be transitive and* $hijk$*-convergent for* $h, i, j, k > 0$*. Then, for any* $a \geq 0$ *and* $b \geq 0$ *this model satisfies the properties*

$$
wR^{h+a} v_{h+a} \wedge wR^{j+b} u_{j+b} \to (\exists o) v_{h+a} R o \qquad \textit{and} \qquad wR^{h+a} v_{h+a} \wedge wR^{j+b} u_{j+b} \to (\exists o) v_{j+b} R o \;, \tag{45}
$$

and their corresponding labelled sequent rules

$$
\frac{v'_{h+a} \mathsf{R} o', \mathfrak{R}_{ab}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R}_{ab}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''} \qquad \textit{and} \qquad \frac{u'_{j+b} \mathsf{R} o', \mathfrak{R}_{ab}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{\mathfrak{R}_{ab}, \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''} \tag{46}
$$

where $\mathfrak{R}_{ab} = \{w'\mathsf{R}v'_1, \ldots, v'_{a-1}\mathsf{R}v'_a, w'\mathsf{R}u'_1, \ldots, u'_{b-1}\mathsf{R}u'_b\}$, $\mathfrak{Q}$ *is arbitrary, and* o' *is an eigenvariable, are* $\mathcal{M}$*-serial (with* $\ell_p = v'_{h+a}$ *for the left rule,* $\ell_p = u'_{j+b}$ *for the right rule, and* $\ell_c = o'$ *for both rules). Thus,*

- *for any interpolant* $\mho_C$ *for the premise of the left* (44) *rule in uniform CNF,* $\mho_C[o' \mapsto v'_{h+a}\Box]$ *is an interpolant of the conclusion;*
- *for any interpolant* $\mho_C$ *for the premise of the right* (44) *rule in uniform CNF,* $\mho_C[o' \mapsto u'_{j+b}\Box]$ *is an interpolant of the conclusion.*

Proof. We first prove (45). Assume the antecedent of the implication holds for either of the rules. By transitivity of R, $wR^h v_{h+a}$ and $wR^j u_{j+b}$ because $a, b \geq 0$. Hence, by $hijk$-convergence, there exists o such that $v_{h+a}R^i o$ and $u_{j+b}R^k o$. Thus, by transitivity, $v_{h+a}Ro$ and $u_{j+b}Ro$ because $k, i \geq 1$, as required. Let us now show that both rules are serial. We only need to show that there are worlds in $R(\mathcal{I}(\ell_p))$ and any of them can be used as an interpretation for o'. The former was just demonstrated, and the latter is immediate. □

Since rule (42) is derivable from (44) and (46), which, together with *Trans* and *Trans** form a cut-free labelled calculus for transitive $hijk$-convergent logics according to the general methods from [53], we can compute the composite interpolant transformation that preserves ms-interpolation for the derivation from Fig. 6.

Corollary 6.21. *Let* $\mathcal{M} = (W, R, V)$ *be transitive and* $hijk$*-convergent for* $h, i, j, k > 0$*. For any* $\mathcal{M}$*-ms-interpolant* $\mho_C$ *for the premise of rule* (42) *in uniform CNF,*

$$\mho_C\left[o' \mapsto z'_{i-1}\Diamond\Box\right]\left[y'_{k-1} \mapsto y'_{k-2}\Box\right]\ldots\left[y'_1 \mapsto u'_j\Box\right]\left[z'_{i-1} \mapsto z'_{i-2}\Box\right]\ldots\left[z'_1 \mapsto v'_h\Box\right]$$

is an $\mathcal{M}$*-ms-interpolant of the conclusion (interpolants are brought to uniform CNF after every replacement).*

For instance, for a transitive $1, 1, 2, 2$-convergent model, the interpolant transformation for the rule

$$\frac{v'\mathsf{R}z', z'\mathsf{R}o', u'\mathsf{R}y', y'\mathsf{R}o', w'\mathsf{R}v', w'\mathsf{R}u', \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}{w'\mathsf{R}v', w'\mathsf{R}u', \mathfrak{Q} \prec \Gamma'; \Gamma'' \Rightarrow \Delta'; \Delta''}$$

with no labels other than w', v', u' in the conclusion proceeds as follows:

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : B_i \otimes u' : C_i \otimes z' : D_i \otimes y' : E_i \otimes o' : F_i\Big)[o' \mapsto z'\Diamond\Box]\,[y' \mapsto u'\Box]\,[z' \mapsto v'\Box] =$$

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : B_i \otimes u' : C_i \otimes z' : D_i \otimes y' : E_i \otimes z' : \Diamond\Box F_i\Big)[y' \mapsto u'\Box]\,[z' \mapsto v'\Box] =$$

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : B_i \otimes u' : C_i \otimes z' : (D_i \vee \Diamond\Box F_i) \otimes y' : E_i\Big)[y' \mapsto u'\Box]\,[z' \mapsto v'\Box] =$$

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : B_i \otimes u' : C_i \otimes z' : (D_i \vee \Diamond\Box F_i) \otimes u' : \Box E_i\Big)[z' \mapsto v'\Box] =$$

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : B_i \otimes u' : (C_i \vee \Box E_i) \otimes z' : (D_i \vee \Diamond\Box F_i)\Big)[z' \mapsto v'\Box] =$$

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : B_i \otimes u' : (C_i \vee \Box E_i) \otimes v' : \Box(D_i \vee \Diamond\Box F_i)\Big) =$$

$$\bigotimes_{i=1}^{n}\Big(w' : A_i \otimes v' : \big(B_i \vee \Box(D_i \vee \Diamond\Box F_i)\big) \otimes u' : (C_i \vee \Box E_i)\Big)\,.$$

One could say that this is interpolation by reaxiomatization: the transitive $hijk$-convergent logic, normally axiomatized by

$$\Diamond^h \Box^i A \to \Box^j \Diamond^k A \qquad \text{and} \qquad \Box A \to \Box\Box A$$

can be equally well axiomatized by

$$\Diamond^{h+i-1} \Box A \to \Box^{j+k-1} \Diamond A\ , \qquad \Box^M \Diamond \top\ , \qquad \Diamond^M \top \to \Box^m \Diamond \top\ , \qquad \text{and} \qquad \Box A \to \Box\Box A\ ,$$

where $M = \max\{h, j\}$ and $m = \min\{h, j\}$, and the latter formulation provides for a straightforward interpolation proof. (This formulation was developed in a discussion with Lellmann.)

7. Related work

7.1. Conclusion and comparison with previous work within the multisequent interpolation project

As already mentioned, this paper follows and summarizes a series of publications Fitting and Kuznets [24], Kuznets [33–35]. Unfortunately, they were published out of order, with [33] written much earlier than [34] but published much later. In addition, the notation and presentation have been going through several stages, oscillating between more general and better adapted to the formalism of the day.

All sequent formalisms in this paper are presented in double-sided format. In [24], the method was also applied to a single-sided formalism of nested sequents. While single-sided sequents necessitate the use of the negation normal form, it can also be used for double-sided sequents. In fact, Fitting [21] noted that the resulting so-called *symmetric sequent calculi* are best suited for proving interpolation. Out method has been applied to symmetric calculi in [34] for the general case of multisequent calculi and in [35] for labelled sequents. However, in this paper, we chose not to restrict the language and deal with ordinary two-sided sequents, like in [33] for hypersequents.

The proof of interpolation using hypersequents for S5 was already present in [33]. The result for S4.2 using hypersequents, originally reported at the Logic Colloquium 2015, has not been published before.[12] The nested calculi results from [24] cover more logics than discussed in this paper. For instance, Poggiolesi's calculi do not include the logic K5. The project of multisequent interpolation started with a collaboration with Melvin Fitting on nested sequents. The labelled sequent results in [35] are more or less the same as in this paper, with, perhaps, more specific logics mentioned there.

The most significant advance of this paper is the simplified and modular presentation of interpolant transformations. While [34] started categorizing rules according to requisite interpolant transformations, only propositional and local rules have been properly delineated there. The definitions for rules requiring the use of $\Box$ or $\Diamond$ for the conclusion interpolant have been significantly improved. For instance, the transformations for the seriality-inducing rules have been completely *ad hoc* in [24], were given a correct informal explanation that, nevertheless, only made half of the connections discussed in this paper, and in [35], they were treated within a family of so-called *telescopic* rules, family so restrictive that the connection to rules for Geach logics could not be understood. These rules have now been separated into their own type of serial rules. The same happened with convergent rules, which for the first time received a general definition. These two new families of rules enabled us to significantly simplify the treatment of Geach logics, which was a *tour de force* in [35] to the point of causing numerous complaints from reviewers as the murkiest part of the paper. We hope, that the modular method of interpolation by reaxiomatization applied in this paper will shed light on the situation.

Another new feature introduced here is a uniform notation $[\ell \mapsto \ell'\heartsuit]$ for various interpolant transformations, allowing concise formulation of interpolation algorithms.

[12] The labelled sequent results for Geach logics [35], however, do cover S4.2.

7.2. Other formalisms

Much of the literature has already been mentioned in previous sections, whenever the discussion warranted it. In this section we collect a (very incomplete) list of results that, although connected to the topic, did not find a particular place in the paper to be mentioned.

Brotherston and Goré [9] developed a method of using display calculi for proving interpolation for displayable substructural logics. Bílková [7] and Herzig and Mengin [29] used nested sequent calculi and resolution respectively to show uniform interpolation, which is stronger than Craig interpolation and incomparable with Lyndon one. Pattinson [55] provided a proof of uniform interpolation for all members of the somewhat restricted class of rank-1 modal logics. Iemhoff [30] connected the existence of ordinary sequent calculi to the property of uniform interpolation, which can be used to show the absence of such sequent calculi, but can only prove uniform interpolation for logics with sequent systems. The description of a big family of tableau rules amenable to interpolation can be found in Rautenberg [59]. Interpolation for few modal calculi, left over from Fitting and Rautenberg, was proved in Nguyen [54].

The Gödel–Löb logic GL is often a source of frustration for various general methods. It possesses many nice properties but often requires specialized approaches to prove them. One such approach was recently developed in Shamkanov [62] to prove syntactically Lyndon interpolation for it.

While labelled sequents possess a well developed method of translating frame conditions to rules, several methods for automatic rule generation have also been developed for hypersequents. A recent paper by Lellmann [39] is, perhaps, the best source for a survey of this literature. However, we would like to mention several influential papers that have helped shape the current transition from *ad hoc* calculi construction to development of general algorithms: Ciabattoni et al. [15], Lahav [38], Lellmann and Pattinson [40].

8. Future work

The method described in this paper is not sufficient to tackle intermediate logics. Together with Lellmann, we are now working on adapting it.[13] The extension of the formalism to first-order logics is also long overdue. Another obvious direction is the extension of this formalism to multimodal logics.

The modular approach to interpolant transformations, which enabled us to pigeonhole every single rule we considered begs the question whether there are other interpolant transformations that can be used for the types of rules we have not yet encountered.

Currently, the method works based on the Kripke semantics. While the semantic component is crucial to our methodology, it seems reasonable to attempt to extend the method to other semantics, the closest of which appears to be Scott–Montague semantics of neighborhood models.

Finally, it should be reasonably straightforward to extend the method to various hybrid formalisms combining features and rules from two or more of the formalisms already treated. This was already done in Kuznets [34] for *grafted hypersequents*, combining hypersequents with nested sequents, introduced in Kuznets and Lellmann [36]. The other obvious candidates for this treatment are *indexed nested sequents* from Fitting [23] and *graph-based tableaux* from Castilho et al. [14], combining nested and labelled sequent features.

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[13] The long wait (more than a year) from the acceptance of this paper till its publication resulted in this particular line of research having already come to fruition by the time this paper is published. The application of this method to intermediate logics, which, additionally, solved positively the open problem of Lyndon interpolation for Gödel–Dummett logic, is published in [37].

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References

[1] A. Avron, A constructive analysis of **RM**, J. Symbolic Logic 52 (1987) 939–951, https://doi.org/10.2307/2273828.
[2] A. Avron, Natural 3-valued logics—characterization and proof theory, J. Symbolic Logic 56 (1991) 276–294, https://doi.org/10.2307/2274919.
[3] A. Avron, The method of hypersequents in the proof theory of propositional non-classical logics, in: W. Hodges, M. Hyland, C. Steinhorn, J. Truss (Eds.), Logic: from Foundations to Applications, European Logic Colloquium, Clarendon Press, 1996, pp. 1–32.
[4] M. Baaz, A. Ciabattoni, C.G. Fermüller, Hypersequent calculi for Gödel logics—a survey, J. Logic Comput. 13 (2003) 835–861, https://doi.org/10.1093/logcom/13.6.835.
[5] J. van Benthem, The many faces of interpolation, Synthese 164 (2008) 451–460, https://doi.org/10.1007/s11229-008-9351-5.
[6] E.W. Beth, On Padoa's method in the theory of definition, Indag. Math. (Proceedings) 56 (1953) 330–339, https://doi.org/10.1016/S1385-7258(53)50042-3.
[7] M. Bílková, A note on uniform interpolation proofs in modal deep inference calculi, in: N. Bezhanishvili, S. Löbner, K. Schwabe, L. Spada (Eds.), Logic, Language, and Computation, 8th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2009, Bakuriani, Georgia, September 21–25, 2009, Revised Selected Papers, in: Lecture Notes in Artificial Intelligence, vol. 6618, Springer, 2011, pp. 30–45, https://doi.org/10.1007/978-3-642-22303-7_3.
[8] M.P. Bonacina, M. Johansson, Interpolation systems for ground proofs in automated deduction: a survey, J. Automat. Reason. 54 (2015) 353–390, https://doi.org/10.1007/s10817-015-9325-5.
[9] J. Brotherston, R. Goré, Craig interpolation in displayable logics, in: K. Brünnler, G. Metcalfe (Eds.), Automated Reasoning with Analytic Tableaux and Related Methods, 20th International Conference, TABLEAUX 2011, Bern, Switzerland, July 4–8, 2011, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 6793, Springer, 2011, pp. 88–103, https://doi.org/10.1007/978-3-642-22119-4_9.
[10] K. Brünnler, Deep sequent systems for modal logic, in: G. Governatori, I. Hodkinson, Y. Venema (Eds.), Advances in Modal Logic, vol. 6, College Publications, 2006, pp. 107–119, http://www.aiml.net/volumes/volume6/Bruennler.ps.
[11] K. Brünnler, Deep sequent systems for modal logic, Arch. Math. Logic 48 (2009) 551–577, https://doi.org/10.1007/s00153-009-0137-3.
[12] K. Brünnler, Nested Sequents, Habilitation thesis, Universität Bern, 2010, https://arxiv.org/abs/1004.1845.
[13] R.A. Bull, Cut elimination for propositional dynamic logic without *, Z. Math. Logik Grundlagen Math. 38 (1992) 85–100, https://doi.org/10.1002/malq.19920380107.
[14] M.A. Castilho, L.F. del Cerro, O. Gasquet, A. Herzig, Modal tableaux with propagation rules and structural rules, Fund. Inform. 32 (1997) 281–297, https://doi.org/10.3233/FI-1997-323404.
[15] A. Ciabattoni, N. Galatos, K. Terui, From axioms to analytic rules in nonclassical logics, in: Proceedings, Twenty-Third Annual IEEE Symposium on Logic in Computer Science, 24–27 June 2008, Pittsburgh, Pennsylvania, IEEE, 2008, pp. 229–240, https://doi.org/10.1109/LICS.2008.39.
[16] W. Craig, Review: On Padoa's method in the theory of definition by E.W. Beth, J. Symbolic Logic 21 (1956) 194–195, https://doi.org/10.2307/2268764.
[17] W. Craig, Linear reasoning. A new form of the Herbrand–Gentzen theorem, J. Symbolic Logic 22 (1957) 250–268, https://doi.org/10.2307/2963593.
[18] W. Craig, Three uses of the Herbrand–Gentzen theorem in relating model theory and proof theory, J. Symbolic Logic 22 (1957) 269–285, https://doi.org/10.2307/2963594.
[19] V. D'Silva, D. Kroening, G. Weissenbacher, A survey of automated techniques for formal software verification, IEEE Trans. Comput.-Aided Design Integr. Circuits Syst. 27 (2008) 1165–1178, https://doi.org/10.1109/TCAD.2008.923410.
[20] R. Dyckhoff, S. Negri, Geometrisation of first-order logic, Bull. Symbolic Logic 21 (2015) 123–163, https://doi.org/10.1017/bsl.2015.7.
[21] M. Fitting, Proof Methods for Modal and Intuitionistic Logics, Synth. Libr., vol. 169, D. Reidel Publishing Company, 1983, https://doi.org/10.1007/978-94-017-2794-5.
[22] M. Fitting, Prefixed tableaus and nested sequents, Ann. Pure Appl. Logic 163 (2012) 291–313, https://doi.org/10.1016/j.apal.2011.09.004.
[23] M. Fitting, Cut-free proof systems for Geach logics, IfCoLog Journal of Logics and their Applications 2 (2015) 17–64, http://www.collegepublications.co.uk/admin/download.php?ID=ifcolog00004.
[24] M. Fitting, R. Kuznets, Modal interpolation via nested sequents, Ann. Pure Appl. Logic 166 (2015) 274–305, https://doi.org/10.1016/j.apal.2014.11.002.
[25] D.M. Gabbay, Labelled deductive systems: a position paper, in: J. Oikkonen, J. Väänänen (Eds.), Logic Colloquium '90, in: Lect. Notes Log., vol. 2, Springer, 1993, pp. 66–88, https://doi.org/10.1017/9781316718254.008.
[26] D.M. Gabbay, L. Maksimova, Interpolation and Definability: Modal and Intuitionistic Logics, Oxford Logic Guides, vol. 46, Clarendon Press, 2005, https://doi.org/10.1093/acprof:oso/9780198511748.001.0001.
[27] J. Garson, Modal logic, in: E.N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, Spring 2016 ed., Metaphysics Research Lab, Stanford University, 2016, https://plato.stanford.edu/archives/spr2016/entries/logic-modal/.

[28] R. Goré, R. Ramanayake, Labelled tree sequents, tree hypersequents and nested (deep) sequents, in: T. Bolander, T. Braüner, S. Ghilardi, L. Moss (Eds.), Advances in Modal Logic, vol. 9, College Publications, 2012, pp. 279–299, http://www.aiml.net/volumes/volume9/Gore-Ramanayake.pdf.

[29] A. Herzig, J. Mengin, Uniform interpolation by resolution in modal logic, in: S. Hölldobler, C. Lutz, H. Wansing (Eds.), Logics in Artificial Intelligence, 11th European Conference, JELIA 2008, Dresden, Germany, September 28–October 1, 2008, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 5293, Springer, 2008, pp. 219–231, https://doi.org/10.1007/978-3-540-87803-2_19.

[30] R. Iemhoff, Uniform interpolation and sequent calculi in modal logic, Arch. Math. Logic Online First (2018), https://doi.org/10.1007/s00153-018-0629-0.

[31] R. Kashima, Cut-free sequent calculi for some tense logics, Studia Logica 53 (1994) 119–135, https://doi.org/10.1007/BF01053026.

[32] H. Kurokawa, Hypersequent calculi for modal logics extending S4, in: Y. Nakano, K. Satoh, D. Bekki (Eds.), New Frontiers in Artificial Intelligence, JSAI-isAI 2013 Workshops, LENLS, JURISIN, MiMI, AAA, and DDS, Kanagawa, Japan, October 27–28, 2013, Revised Selected Papers, in: Lecture Notes in Artificial Intelligence, vol. 8417, Springer, 2014, pp. 51–68, https://doi.org/10.1007/978-3-319-10061-6_4.

[33] R. Kuznets, Craig interpolation via hypersequents, in: D. Probst, P. Schuster (Eds.), Concepts of Proof in Mathematics, Philosophy, and Computer Science, in: Ontos Mathematical Logic, vol. 6, De Gruyter, 2016, pp. 193–214, https://doi.org/10.1515/9781501502620-012.

[34] R. Kuznets, Interpolation method for multicomponent sequent calculi, in: S. Artemov, A. Nerode (Eds.), Logical Foundations of Computer Science, International Symposium, LFCS 2016, Deerfield Beach, FL, USA, January 4–7, 2016, Proceedings, in: Lecture Notes in Comput. Sci., vol. 9537, Springer, 2016, pp. 202–218, https://doi.org/10.1007/978-3-319-27683-0_15.

[35] R. Kuznets, Proving Craig and Lyndon interpolation using labelled sequent calculi, in: L. Michael, A. Kakas (Eds.), Logics in Artificial Intelligence, 15th European Conference, JELIA 2016, Larnaca, Cyprus, November 9–11, 2016, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 10021, Springer, 2016, pp. 320–335, https://doi.org/10.1007/978-3-319-48758-8_21.

[36] R. Kuznets, B. Lellmann, Grafting hypersequents onto nested sequents, Log. J. IGPL 24 (2016) 375–423, https://doi.org/10.1093/jigpal/jzw005.

[37] R. Kuznets, B. Lellmann, Interpolation for intermediate logics via hyper- and linear nested sequents, in: G. Bezhanishvili, G. D'Agostino, G. Metcalfe, T. Studer (Eds.), Advances in Modal Logic, vol. 12, College Publications, 2018, pp. 473–492.

[38] O. Lahav, From frame properties to hypersequent rules in modal logics, in: Proceedings, 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, 25–28 June 2013, New Orleans, Louisiana, IEEE, 2013, pp. 408–417, https://doi.org/10.1109/LICS.2013.47.

[39] B. Lellmann, Hypersequent rules with restricted contexts for propositional modal logics, Theoret. Comput. Sci. 656 (2016) 76–105, https://doi.org/10.1016/j.tcs.2016.10.004.

[40] B. Lellmann, D. Pattinson, Correspondence between modal Hilbert axioms and sequent rules with an application to S5, in: D. Galmiche, D. Larchey-Wendling (Eds.), Automated Reasoning with Analytic Tableaux and Related Methods, 22nd International Conference, TABLEAUX 2013, Nancy, France, September 16–19, 2013, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 8123, Springer, 2013, pp. 219–233, https://doi.org/10.1007/978-3-642-40537-2_19.

[41] E.J. Lemmon, D. Scott, The "Lemmon Notes": An Introduction to Modal Logic, American Philosophical Quarterly Monograph Series, vol. 11, Basil Blackwell, 1977.

[42] C.I. Lewis, C.H. Langford, Symbolic Logic, The Century Co., 1932.

[43] R.C. Lyndon, An interpolation theorem in the predicate calculus, Pacific J. Math. 9 (1959) 129–142, https://doi.org/10.2140/pjm.1959.9.129.

[44] S. Maehara, On the interpolation theorem of Craig, Sūgaku 12 (1961) 235–237, https://doi.org/10.11429/sugaku1947.12.235, in Japanese.

[45] L.L. Maksimova, Absence of the interpolation property in the consistent normal modal extensions of the Dummett logic, Algebra Logic 21 (1982) 460–463, https://doi.org/10.1007/BF01982206.

[46] L. Maksimova, Amalgamation and interpolation in normal modal logics, Studia Logica 50 (1991) 457–471, https://doi.org/10.1007/BF00370682.

[47] M. Marx, Y. Venema, Multi-Dimensional Modal Logic, Appl. Log. Ser., vol. 4, Springer, 1997, https://doi.org/10.1007/978-94-011-5694-3.

[48] K.L. McMillan, Interpolation and SAT-based model checking, in: W.A. Hunt Jr., F. Somenzi (Eds.), Computer Aided Verification, 15th International Conference, CAV 2003, Boulder, CO, USA, July 8–12, 2003, Proceedings, in: Lecture Notes in Comput. Sci., vol. 2725, Springer, 2003, pp. 1–13, https://doi.org/10.1007/978-3-540-45069-6_1.

[49] G.E. Minc, On some calculi of modal logic, in: V.P. Orevkov (Ed.), The Calculi of Symbolic Logic. I, in: Proc. Steklov Inst. Math., vol. 98, AMS, 1971, pp. 97–124, originally published in Russian in 1968, http://mi.mathnet.ru/eng/tm/v98/p88.

[50] G. Mints, Indexed systems of sequents and cut-elimination, J. Philos. Logic 26 (1997) 671–696, https://doi.org/10.1023/A:1017948105274.

[51] G. Mints, G. Olkhovikov, A. Urquhart, Failure of interpolation in constant domain intuitionistic logic, J. Symbolic Logic 78 (2013) 937–950, https://doi.org/10.2178/jsl.7803120.

[52] S. Negri, Proof analysis in modal logic, J. Philos. Logic 34 (2005) 507–544, https://doi.org/10.1007/s10992-005-2267-3.

[53] S. Negri, J. von Plato, Proof Analysis: A Contribution to Hilbert's Last Problem, Cambridge University Press, 2011, https://doi.org/10.1017/CBO9781139003513.

[54] L.A. Nguyen, Analytic tableau systems and interpolation for the modal logics KB, KDB, K5, KD5, Studia Logica 69 (2001) 41–57, https://doi.org/10.1023/A:1013834410884.

[55] D. Pattinson, The logic of exact covers: Completeness and uniform interpolation, in: Proceedings, 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, 25–28 June 2013, New Orleans, Louisiana, IEEE, 2013, pp. 418–427, https://doi.org/10.1109/LICS.2013.48.

[56] F. Poggiolesi, Sequent Calculi for Modal Logic, Ph.D. thesis, Università degli Studi di Firenze, 2008.

[57] F. Poggiolesi, in: Gentzen Calculi for Modal Propositional Logic, in: Trends Log. Stud. Log. Libr., vol. 32, Springer, 2011, https://doi.org/10.1007/978-90-481-9670-8.

[58] G. Pottinger, Uniform, cut-free formulations of T, S_4, and S_5, J. Symbolic Logic 48 (1983) 900, https://doi.org/10.2307/2273495. Abstract.

[59] W. Rautenberg, Modal tableau calculi and interpolation, J. Philos. Logic 12 (1983) 403–423, https://doi.org/10.1007/BF00249258.

[60] M. Sato, A study of Kripke-type models for some modal logics by Gentzen's sequential method, Publ. Res. Inst. Math. Sci. 13 (1977) 381–468, https://doi.org/10.2977/prims/1195189814.

[61] K. Schütte, Der Interpolationssatz der intuitionistischen Prädikatenlogik, Math. Ann. 148 (1962) 192–200, https://doi.org/10.1007/BF01470747.

[62] D.S. Shamkanov, Circular proofs for the Gödel–Löb provability logic, Math. Notes 96 (2014) 575–585, https://doi.org/10.1134/S0001434614090326.

[63] A.S. Troelstra, H. Schwichtenberg, Basic Proof Theory, second ed., Cambridge Tracts Theoret. Comput. Sci., vol. 43, Cambridge University Press, 2000, https://doi.org/10.1017/CBO9781139168717.

[64] L. Viganò, Labelled Non-Classical Logics, Springer, 2000, https://doi.org/10.1007/978-1-4757-3208-5.

[65] Y. Vizel, G. Weissenbacher, S. Malik, Boolean satisfiability solvers and their applications in model checking, Proc. IEEE 103 (2015) 2021–2035, https://doi.org/10.1109/JPROC.2015.2455034.

[66] J. von Plato, Skolem's discovery of Gödel–Dummett logic, Studia Logica 73 (2003) 153–157, https://doi.org/10.1023/A:1022997524909.

[67] G. Weissenbacher, Logical Methods in Automated Hardware and Software Verification, Habilitation thesis, Technische Universität Wien, 2016, http://www.georg.weissenbacher.name/papers/habilitation.pdf.