Small noise spectral gap asymptotics for a large system of nonlinear diffusions

Giacomo Di Gesù*, Dorian Le Peutrec†

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Abstract

We study the L^2 spectral gap of a large system of strongly coupled diffusions on unbounded state space and subject to a double-well potential. This system can be seen as a spatially discrete approximation of the stochastic Allen-Cahn equation on the one-dimensional torus. We prove upper and lower bounds for the leading term of the spectral gap in the small temperature regime with uniform control in the system size. The upper bound is given by an Eyring-Kramers-type formula. The lower bound is proven to hold also for the logarithmic Sobolev constant. We establish a sufficient condition for the asymptotic optimality of the upper bound and show that this condition is fulfilled under suitable assumptions on the growth of the system size. Our results can be reformulated in terms of a semiclassical Witten Laplacian in large dimension.

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1 Introduction

This paper concerns the rate of convergence to equilibrium at low temperature of a stochastic interacting particle system, which may be described as

*CERMICS – École des Ponts ParisTech, 6-8 avenue Blaise Pascal, F-77455 Marne-la-Vallée Cedex 2. (giacomo.di-gesu@enpc.fr)

†Département de Mathématiques, UMR-CNRS 8628, Bât. 425, Université Paris Sud, F-91405 Orsay Cedex. (dorian.lepeutrec@math.u-psud.fr)

follows. There are N particles, at each time $t \geq 0$ the state of the k-th particle is a real random number $\xi_k(t)$ and the trajectory $\xi_k = (\xi_k(t))_{t\geq 0}$ satisfies for some fixed $\mu > 1$ the stochastic differential equation

$$d\xi_k \;=\; \Big[\;\mu\,\frac{\xi_{k+1}+\xi_{k-1}-2\xi_k}{4\sin^2\frac{\pi}{N}} \;+\; \xi_k \;-\; \xi_k^3\;\Big]\,dt \;+\; \sqrt{2hN}\,dB_k \;\;. \tag{1.1}$$

Here $B_1 = (B_1(t))_{t\geq 0}, \ldots, B_N = (B_N(t))_{t\geq 0}$ are N independent standard Brownian motions, h is a positive constant and $\xi_{N+1} := \xi_1$, i.e. periodic boundary conditions are assumed. When $h > 0$ is kept fixed and N is large, system (1.1) can be seen as a discrete space approximation of the stochastically perturbed Allen-Cahn equation on the interval $(0, \frac{2\pi}{\sqrt{\mu}})$:

$$du(x,t) \;=\; \big[\,\partial_x^2 u(x,t) \;+\; u(x,t) \;-\; u^3(x,t)\,\big]\,dt \;+\; \sqrt{2h}\,dW(x,t)\;\;, \tag{1.2}$$

where now $(x,t) \in (0, \frac{2\pi}{\sqrt{\mu}}) \times (0,\infty)$, the boundary condition $u(0,t) = u(\frac{2\pi}{\sqrt{\mu}}, t)$ has to be satisfied for every $t \geq 0$, and dW is a space-time white noise. Thus, for large N, one might think of $\xi_k(t) \sim u\big(\frac{k}{N}\frac{2\pi}{\sqrt{\mu}}, t\big)$, and of the chain $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ as giving the position at time t of an elastic ring of length $\frac{2\pi}{\sqrt{\mu}}$ moving in a highly viscous, noisy environment and subject to a simple bistable external force.

Equation (1.2) is a basic and widely studied stochastic partial differential equation, see e.g. [FaJo, Fun, BDP, GoMa, KORV, Hai2, BeGe, OWW, DaZa, Bar] and references therein. For a more general background on the particle system (1.1) we refer to [BFG1, BFG2]. See also [BBM] for aspects closely related to this work. The convergence of (1.1) to (1.2) for $N \to \infty$ is discussed in [Bar].

Relaxation properties: heuristics and previous results.

For each fixed $h > 0$ and number of particles N, the long time behaviour of (1.1) is described by its unique equilibrium distribution, explicitly given by the probability measure on $\mathbb{R}^N$

$$m_{h,N}(dx) \;:=\; \frac{e^{-\frac{V(x)}{hN}}\,dx}{\int_{\mathbb{R}^N} e^{-\frac{V(x)}{hN}}\,dx}\;,$$

where the energy function $V : \mathbb{R}^N \to \mathbb{R}$ is defined as

$$V(x) \;:=\; \sum_{k=1}^{N}\Big(\,\frac{1}{4}x_k^4 \;-\; \frac{1}{2}x_k^2\,\Big) \;+\; \mu\sum_{k=1}^{N}\frac{(x_k - x_{k+1})^2}{8\sin^2(\frac{\pi}{N})} \;+\frac{N}{4}\;, \tag{1.3}$$

with $x_{N+1} := x_1$. This follows from the observation that the drift term in (1.1) is the gradient of V and from general facts about gradient-type diffusions. Similarly, for fixed $h > 0$, there exists a unique equilibrium distribution $m_{h,\infty}$ for the infinite-dimensional system (1.2), see [DaZa, ReVe]. One might say that at equilibrium no "phase transition" occurs in the thermodynamic limit $N \to \infty$. On the contrary, since for each N the energy V admits two local minima, given by

$$I_+ := (1, \dots, 1) \quad \text{and} \quad I_- := (-1, \dots, -1) ,$$

the deterministic dynamics $d\xi = -\nabla V(\xi)dt$, obtained from (1.1) by setting $h = 0$, admits two stable equilibrium points. Thus, when h is positive but small, the typical picture of a so-called metastable dynamics emerges [FrWe, FaJo, BBM]: the system quickly reaches a local equilibrium in the basin of attraction of I_+ or I_-, depending on its initial condition; this local equilibrium endures for a long time, since, in order to be able to explore the whole state space and distribute according to the global equilibrium $m_{h,N}$, the system has to wait for a sufficiently large stochastic fluctuation allowing to overcome the energetic barrier separating I_+ and I_-. The critical time scale at which such transitions between minima typically occur is exponentially large in the parameter h. Thus, for $h \to 0$, one observes a significant slowdown in the relaxation towards $m_{h,N}$.

The aim of this paper is to quantify the mentioned slowdown in the approach to equilibrium of (1.1) when at the same time h is small and N is large. More specifically we shall study for $h \to 0$ and $N \to \infty$ the behaviour of the Poincaré constant $\lambda(h, N)$ and the logarithmic Sobolev constant $\rho(h, N)$ of (1.1). These are defined as the largest constants satisfying respectively, for every $\varphi \in H^1(\mathbb{R}^N, m_{h,N})$, the weighted Poincaré inequality

$$\lambda(h, N) \operatorname{Var}_{m_{h,N}}(\varphi) \;\leq\; hN \int |\nabla\varphi|^2 \, dm_{h,N} \;, \tag{1.4}$$

and the Gross inequality (or logarithmic Sobolev inequality)

$$\rho(h, N) \operatorname{Ent}_{m_{h,N}}(\varphi^2) \;\leq\; 2\, hN \int |\nabla\varphi|^2 \, dm_{h,N} \;. \tag{1.5}$$

Here $\operatorname{Var}_{m_{h,N}}$ and $\operatorname{Ent}_{m_{h,N}}$ denote the variance and entropy with respect to $m_{h,N}$, i.e. $\operatorname{Var}_{m_{h,N}}(\varphi) := \int \varphi^2 dm_{h,N} - \big(\int \varphi \; dm_{h,N}\big)^2$ and, for $\varphi \geq 0$, $\operatorname{Ent}_{m_{h,N}}(\varphi) := \int \varphi \; \log\varphi \; dm_{h,N} - \int \varphi \; dm_{h,N} \log\big(\int \varphi \; dm_{h,N}\big)$. The right hand side of (1.4) is also called the Dirichlet form associated with the Markov

process defined by (1.1).
It is well-known that the Poincaré constant and logarithmic Sobolev constant give the exponential rate of convergence to equilibrium, respectively in variance and in entropy. We refer e.g. to Theorem 4.2.5 and 5.2.1 in [BGL], which also gives a general overview of the interplay between functional inequalities and Markov processes. We stress that, from the point of view of spin systems in statistical mechanics, we are dealing here with the problem of relaxation to equilibrium in a case of continuous unbounded single-spin state space and nonconvex energy function (see e.g. [Led, Zeg, BoHe1, BoHe2] in this context). Concerning exponential convergence of stochastic equations in infinite dimensions with fixed noise parameter h we point e.g. to [GoMa, Hai1, Hai2, DaZa].

If N is kept fixed it is known that the leading asymptotic behaviour of $\lambda(h,N)$ in the limit $h \to 0$ is given by an Eyring-Kramers-type formula (see [BEGK, BGK, HKN], treating generic multiwell-diffusions in the small noise regime, and also the recent [MeSc, Mic]). More specifically, it follows for example from [HKN] and some straightforward adaptations of their arguments, that

$$\lambda(h,N) \;=\; \frac{1}{\pi}\left|\frac{\det \mathrm{Hess}\, V(I_-)}{\det \mathrm{Hess}\, V(0)}\right|^{\frac{1}{2}} \; e^{-\frac{1}{4h}}\;\big(\,1\;+\;\epsilon(h,N)\,\big)\;, \tag{1.6}$$

where the error $\epsilon(h,N)$ satisfies, for $h>0$ sufficiently small, $|\epsilon(h,N)| \le C_N\, h$. Here C_N is some positive constant which may a priori explode in N. On the other hand, as was already observed in [Ste], the prefactor in (1.6) is convergent in the limit $N \to \infty$:

$$p(N) \;:=\; \frac{1}{\pi}\left|\frac{\det \mathrm{Hess}\, V(I_-)}{\det \mathrm{Hess}\, V(0)}\right|^{\frac{1}{2}} \;\underset{N\to+\infty}{\longrightarrow}\; \frac{\sinh(\pi\sqrt{2\mu^{-1}})}{\pi\sin(\pi\sqrt{\mu^{-1}})}\;. \tag{1.7}$$

Similarly, regarding the log-Sobolev constant $\rho(h,N)$, it follows again from general results (see [MeSc]), that for fixed N the leading term of $\rho(h,N)$ is again given by $p(N)e^{-\frac{1}{4h}}$. We stress that also here, as for the error in (1.6), there is no control in N on the error term. Thus no rigorous conclusion in the limit $N \to \infty$ can be directly inferred from these results.

On the other hand, rather strong results have been obtained in the analysis of the mean time needed for system (1.1) to go from I_+ to I_-: indeed it has been shown that an Eyring-Kramers-type formula holds for this transition time, with an error which is uniform in N (see in particular [BBM] and [Bar, BeGe], which extend the results to the infinite-dimensional system (1.2) and even to more general situations). Nevertheless, while the asymptotic relation between stochastically defined mean transition times and analytic objects as $\lambda(h, N)$ is well-established in very general situations for fixed N (see again [BGK]), to the best of our knowledge there are no rigorous results on how it might behave in the regime of large N, even in the specific model we are considering. In this paper we do not rely on the mentioned results on mean transition times and rather use purely analytical arguments, partly inspired by the semiclassical spectral-theoretic approach developped in [HKN].

Statement of the main results

Our first main result below shows that the Eyring-Kramers formula (1.6) provides an upper bound on $\lambda(h, N)$ with an error term which can indeed be uniformly controlled in the system size N. Moreover it provides a quantitative lower bound at logarithmic scale on $\rho(h, N)$ which is independent of N. In particular it ensures that $\rho(h, N)$ and $\lambda(h, N)$ do not degenerate for any fixed h. One might say that no "dynamical phase transition" occurs in the thermodynamic limit $N \to \infty$ (see also [GoMa]).

Theorem 1.1. *For every $\delta > 0$ there exists a constant $C_\delta > 0$ such that for every $h > 0$ and every $N \in \mathbb{N}$*

$$C_\delta \; e^{-\frac{3+2\sqrt{2}+\delta}{24h}} \; e^{-\frac{1}{4h}} \;\leq\; \rho(h, N) \;\leq\; \lambda(h, N) \;\leq\; p(N) \; e^{-\frac{1}{4h}} \; \big(\, 1 \;+\; \epsilon(h, N) \, \big) \; ,$$

where the prefactor $p(N)$ is given by (1.7) *and the error term $\epsilon(h, N)$ satisfies*

$$\exists C > 0 \;\; s.t. \;\; \forall h \in (0, 1] \; , \; \forall N \in \mathbb{N} \; , \qquad |\epsilon(h, N)| \;\leq\; C \; h \; .$$

The exponential decay in h given by the lower bound in Theorem 1.1 appears to be rather rough, but unfortunately, when insisting to get bounds with uniform control in N, it is for the moment not clear how one could obtain a substantial improvement, even when focusing only on $\lambda(h, N)$. For the latter one can exploit the spectral theory of self-adjoint operators: the generator of the Markovian semigroup giving the evolution of (1.1) is indeed the differential operator

$$L_h \;:=\; -hN\Delta \;+\; \nabla V \cdot \nabla \; . \tag{1.8}$$

The closure in $L^2(m_{h,N})$ of L_h acting on $C_c^\infty(\mathbb{R}^N)$, which we still denote by L_h, is self-adjoint and nonnegative, admits 0 as eigenvalue and has purely discrete spectrum for each h, N fixed (see Section 2.2 for more details). As a consequence of the Max-Min principle, its spectral gap, defined as its first nonzero eigenvalue, coincides with $\lambda_{h,N}$.

According to our second main result below, the problem of obtaining the Eyring-Kramers formula as lower bound for $\lambda(h,N)$ can then be reduced to the problem of proving a suitable separation between $\lambda(h,N)$ and the next eigenvalue of L_h. More precisley, the existence of a uniform lower bound on the "second spectral gap" in a certain regime in which N possibly grows to infinity, turns out to be sufficient for the validity of the Eyring-Kramers formula in the same regime:

Theorem 1.2. *Assume there exist constants $h_0, \delta > 0$ and, for each $h \in (0,h_0]$, a set $\mathcal{N}(h) \subset \mathbb{N}$ such that*

$$\forall h \in (0,h_0]\ ,\ \forall N \in \mathcal{N}(h)\ , \quad \mathrm{Spec}(L_h) \cap \,]\lambda(h,N), \lambda(h,N)+\delta[\, = \emptyset\ . \quad (1.9)$$

Then

$$\lambda(h,N) \;=\; p(N)\; e^{-\frac{1}{4h}}\, \big(\, 1 \;+\; \epsilon(h,N)\, \big)\ ,$$

where the prefactor $p(N)$ is given by (1.7) *and the error term $\epsilon(h,N)$ satisfies*

$$\exists C > 0 \;\; s.t. \;\; \forall h \in (0,h_0]\ ,\ \forall N \in \mathcal{N}(h)\ , \quad |\epsilon(h,N)| \;\le\; C\, h\ .$$

Our last main theorem implies that there exist regimes with unbounded N under which the Eyring-Kramers formula (1.6) holds with bounded error $\epsilon(h,N)$. Indeed, in order to be in the situation of Theorem 1.2, it is enough that N grows slower than $h^{-\frac{3}{4}}$:

Theorem 1.3. *Let $C > 0$ and $\alpha \in (0, \frac{3}{4})$. Then there exist constants $h_0, \delta > 0$ such that condition* (1.9) *in Theorem 1.2 is fulfilled with*

$$\mathcal{N}(h) \;=\; \left\{\ N \in \mathbb{N} : N \le C h^{-\alpha}\ \right\}\ .$$

The above results concerning $\lambda_{h,N}$ can be equivalently reformulated in terms of splitting properties of the ground state of a specific semiclassical Schrödinger operator in large dimension. This is a consequence of the well-known ground state transformation, see e.g. [JMS]: up to conjugation with $e^{-\frac{V}{2hN}}$ and some N-dependent dilatation (see Subsection 2.2 for more details), the operator $h\,L_h$ turns out to be unitarily equivalent to the operator acting in the flat space $L^2(dx)$ and defined through

$$\Delta^{(0)}_{f,h} \;:=\; -h^2\Delta \;+\; |\nabla f|^2 \;-\; h\Delta f\ , \quad \text{where} \quad f(x) \;:=\; \frac{V(\sqrt{N}x)}{2N}\ . \quad (1.10)$$

We like to mention that a semiclassical Schrödinger operator having the form of $\Delta^{(0)}_{f,h}$, with f a generic smooth function, is also called semiclassical Witten Laplacian associated to f. The superscript (0) stresses that we consider only operators on functions, the Witten Laplacian being more generally defined through a supersymmetric extension on the full algebra of differential forms. The operator acting on p-forms is commonly denoted by $\Delta^{(p)}_{f,h}$ and connects in the semiclassical limit $h \to 0$ topological properties of the underlying manifold to the topology of the energy landscape induced by f [Wit, HeSj, CFKS].

We stress that, even if one focuses only on the operator $\Delta^{(0)}_{f,h}$ acting on functions (that is, equivalently on the diffusion operator L_h, as in the present paper), the enlarged supersymmetric point of view may provide further insights and a powerful technical tool. We refer especially to [Sjö3, Joh, Hel2, Hel3, HKN, KuTă, HeNi1, HeNi2, Lep, Dig, BHM, LeNi] for works in this spirit and the links between statistical mechanics and Witten Laplacians. In particular, as was recognized in [HKN], the operator $\Delta^{(1)}_{f,h}$ acting on 1-forms, being related for $h \to 0$ to the energetic bottlenecks responsible for the slowdown of the underlying stochastic process, appears rather naturally when analyzing the low-lying eigenvalues of $\Delta^{(0)}_{f,h}$.

We emphasize that semiclassical techniques as WKB expansions, Agmon estimates and harmonic approximation for Schrödinger operators, used e.g. in [HKN], are generally not uniformly controlled in the limit $N \to \infty$ (see however [MaMø] for previous work on $\Delta^{(p)}_{f,h}$ in large dimension under convexity assumptions on f and [Sjö1, Sjö2, Hel1]). Also for the specific model we consider here, the arguments of [HKN] do not carry over with uniform bounds in N.

Comments on the techniques used in this paper

Though inspired by the supersymmetric approach of [HKN], in this paper we do not make explicit use of $\Delta^{(1)}_{f,h}$. Indeed, a careful analysis of the energy V permits to construct a very efficient global quasimode passing through the bottleneck and connecting the two minima of V (see Definition 3.10). This construction, together with a precise analysis of Laplace integrals in large dimension, enables us to give the upper bound of Theorem 1.1.

For the lower bound in Theorem 1.1, we depart from the semiclassical approach and rather exploit perturbation techniques for fixed h. These permit, even though for general $\mu > 1$ the function V is not convex outside a compact set, to reduce to the case of a convex energy and then to apply the well-known Bakry-Émery criterion. We use here that the interaction part in the energy V is strong enough to ensure good relaxation properties for large N. Thus, roughly speaking, we regard the energy coming from the single particle double-well potential as a perturbation of the interaction part.

This is opposed to the perturbative regime considered in previous works as [BoHe1, BoHe2]: in these references the interaction constant μ is tuned in a way that it is rather the interaction part to become a perturbation of the single particle potential.

The relevant quantity naturally appearing in the estimates leading to Theorem 1.2 is the quadratic form

$$\mathcal{E}(\varphi) \;:=\; \frac{\int_{\mathbb{R}^N} |L_h\varphi|^2\; dm_{h,N}}{hN\int_{\mathbb{R}^N} |\nabla\varphi|^2\; dm_{h,N}}\,.$$

To connect to the existing literature, we point out that, after integration by parts, this quantity can be equivalently rewritten in the two forms

$$\mathcal{E}(\varphi) \;=\; \frac{\int_{\mathbb{R}^N} \Gamma^2(\varphi)\; dm_{h,N}}{\int_{\mathbb{R}^N} \Gamma(\varphi)\; dm_{h,N}} \;=\; \frac{\int_{\mathbb{R}^N} \left(L_h^{(1)}\nabla\varphi\right)\cdot\nabla\varphi\;\; dm_{h,N}}{hN\int_{\mathbb{R}^N} |\nabla\varphi|^2\; dm_{h,N}}\,. \qquad (1.11)$$

Here Γ, Γ^2 are respectively the carré du champ operator and its iteration (see for example [BGL] for more details about this notion) and $L_h^{(1)} := L_h \otimes \mathrm{Id} + hN\,\mathrm{Hess}\,V$ is an operator acting on vector fields (i.e. 1-forms), related to $\Delta_{f,h}^{(1)}$ via ground state transformation. The last expression in (1.11) can be generalized by allowing, instead of $\nabla\varphi$, more general, non-gradient vector fields. This is one of the main advantages of the supersymmetric approach and is crucially exploited in works as [HKN, HeNi1, Lep, LeNi], or [Dig] in a discrete setting. In the arguments we give here we do not use this additional freedom since we can work with the gradient of the quasimode already exploited in the proof of Theorem 1.1 and thus streamline both the results and the presentation.

For the proof of Theorem 1.3 we shall adopt the Schrödinger point of view and thus work with $\Delta_{f,h}^{(0)}$. We combine here standard localization techniques for the analysis of semiclassical Schrödinger operators [CFKS] and a two-scale analysis naturally adapted to the structure of the energy V.

Plan of the paper

The rest of the paper is organized as follows. In Section 2 we discuss basic properties of the model and the precise relation between the diffusion operator L_h and the Schrödinger operator $\Delta_{f,h}^{(0)}$. Sections 3, 4 and 5 are respectively devoted to the proofs of Theorems 1.1, 1.2 and 1.3.

Subsection 3.2, which might also be of independent interest, provides a sharp Laplace-type asymptotics for the normalization constant $Z_{h,N} := \int_{\mathbb{R}^N} e^{-\frac{V(x)}{hN}}dx$ when $h \to 0$ with uniform control in N.

2 Basic properties of the model

2.1 Properties of V and related Gaussian estimates

The aim of this section is to fix our notation and to provide some basic background information on our model which we shall use throughout in the rest of the analysis.

We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^N$, by $\| \cdot \|$, or $| \cdot |$ when no ambiguity is possible, the corresponding Hilbert norm and, more generally, for $p \in \mathbb{N}$ we write

$$\|x\|_p \ := \ \Big(\sum_{k=1}^{N} |x_k|^p \Big)^{\frac{1}{p}} \ .$$

The gradient, Hessian and (negative) Laplacian acting on functions in $\mathbb{R}^N$ are denoted respectively by ∇, Hess and Δ.

Throughout the paper we fix a $\mu > 1$. The energy function V, defined in (1.3), can be rewritten in a more compact notation as

$$V(x) \ = \ \frac{1}{4}\|x\|_4^4 \ + \ \frac{1}{2}\langle x, (K-1)x \rangle \ + \ \frac{N}{4} \ , \tag{2.1}$$

where $K : \mathbb{R}^N \to \mathbb{R}^N$ is a normalised discrete Laplacian, defined by setting for $x \in \mathbb{R}^N$ and $k \in \{1, \dots, N\}$,

$$(Kx)_k \ := \ \frac{\mu}{4\sin^2(\frac{\pi}{N})} \big(\ 2x_k \ - \ x_{k+1} \ - \ x_{k-1} \ \big) \ . \tag{2.2}$$

It is understood that $x_{N+1} := x_1$ and $x_0 := x_N$, which corresponds to periodic boundary conditions. It holds $\langle x, Kx \rangle = \langle Kx, x \rangle$ and, according to our choice of sign, $\langle x, Kx \rangle \geq 0$. The operator K is diagonalised through the discrete Fourier transform $\hat{x} \in \mathbb{R}^N$ of $x \in \mathbb{R}^N$, defined by

$$\hat{x}_k \ := \ \frac{1}{\sqrt{N}} \sum_{j=1}^{N} x_j \ e^{-\mathrm{i}2\pi \frac{j}{N} k} \ .$$

More precisely we have for every $k \in \{0, \dots, N-1\}$,

$$\widehat{(Kx)}_k \ = \ \nu_k \ \hat{x}_k \ , \quad \text{where} \ \ \nu_k \ := \ \mu \, \frac{\sin^2(k\frac{\pi}{N})}{\sin^2(\frac{\pi}{N})} \ . \tag{2.3}$$

Note that $\nu_0 = 0$ is a simple eigenvalue of K and that its smallest non-zero eigenvalue equals μ for every $N \in \mathbb{N}, N \geq 2$. We shall denote by

$P : \mathbb{R}^N \to \mathbb{R}^N$ the projection onto the eigenspace of K corresponding to the eigenvalue 0 and by $P^\perp := 1 - P$ the projection onto its orthogonal complement. Note that $\operatorname{Ran} P = \operatorname{Span}(1, \ldots, 1)$ so that P associates to $x \in \mathbb{R}^N$ the constant vector with components the mean of x: for every $k \in \{1, \ldots, N\}$,

$$(Px)_k \;=\; \overline{x} \;:=\; \frac{1}{N}\sum_{j=1}^{N} x_j \;=\; \frac{\hat{x}_0}{\sqrt{N}} \; . \tag{2.4}$$

For shortness the range $\operatorname{Ran} P$ of P will sometimes also be denoted by $\mathcal{C}$, and we refer to it as the space of constant states, or the "diagonal" of $\mathbb{R}^N$. Similarly we write $\mathcal{C}^\perp := \operatorname{Ran} P^\perp$ for the space of states orthogonal to the constants.

We mention here explicitly the following simple identities, which we shall frequently use in the sequel:

$$\forall x \in \mathbb{R}^N \;, \quad \sum_{k=1}^{N} (P^\perp x)_k \;=\; 0 \quad \text{and} \quad \|Px\| \;=\; \sqrt{N}\overline{x} \;=\; \hat{x}_0 \; .$$

The fact that the first non-zero eigenvalue of K equals μ implies the following discrete Poincaré-type inequality:

$$\forall \rho \in [0, \mu] \;, \; \forall x \in \mathbb{R}^N \;, \quad \langle x, Kx \rangle \geq \; \rho \; \big(\, \|x\|^2 \; - \; \langle x, Px \rangle \, \big) \; . \tag{2.5}$$

For more information on the discrete Fourier transform and discrete Laplacian, see for example [Ter].

Some basic features of the energy landscape determined by V are the following. First, it is straightforward to check that the constant states given by

$$I_+ \;:=\; (1, \ldots, 1) \;\;, \;\; I_- \;:=\; (-1, \ldots, -1) \quad \text{and} \quad O \;:=\; (0, \ldots, 0) \; ,$$

are critical points of V, i.e. satisfy $\nabla V(x) = 0$, for every $N \in \mathbb{N}$. Moreover

$$\operatorname{Hess} V(I_\pm) \;=\; K + 2 \quad \text{and} \quad \operatorname{Hess} V(O) \;=\; K - 1 \; . \tag{2.6}$$

It follows from (2.3) that $K + 2$ admits only strictly positive eigenvalues, while $K - 1$ has one simple eigenvalue -1 and, since $\mu > 1$, all the others are strictly positive. The identities (2.6) imply therefore in particular that

$I_\pm$ are local minima and O is a saddle point, i.e. a critical point of index 1. The additive constant $\frac{N}{4}$ appearing in (2.1) is chosen such that

$$V(I_\pm) \;=\; 0 \quad \text{and} \quad V(O) \;=\; \frac{N}{4}\;. \tag{2.7}$$

A crucial feature of the model, implied by the assumption $\mu > 1$, is the following. When restricted to the $N-1$ dimensional subspace $\mathcal{C}^\perp = \operatorname{Ran} P^\perp$, the quadratic form $\operatorname{Hess} V$ is strictly convex, uniformly in N and $x \in \mathbb{R}^N$. Indeed, according to the discrete Poincaré inequality given in (2.5), for every x and $w \in \mathbb{R}^N$,

$$\langle w, \operatorname{Hess} V(x) w\rangle \;\geq\; \langle w, (K-1)w\rangle \;\geq\; (\mu-1)\,\|w\|^2 \;-\; \mu\langle w, Pw\rangle\;.$$

In particular one gets the lower bound

$$\forall x \in \mathbb{R}^N\;,\;\forall w \in \mathcal{C}^\perp\;,\quad \langle w, \operatorname{Hess} V(x) w\rangle \;\geq\; (\mu-1)\,\|w\|^2\;. \tag{2.8}$$

The latter inequality can be used to show that I_+, I_- and O are the only critical points of V (see also [BFG1]):

Lemma 2.1. *Fix $N \in \mathbb{N}$ and let $x \in \mathbb{R}^N \setminus \{O, I_+, I_-\}$. Then $\nabla V(x) \neq 0$.*

Proof. Estimate (2.8) implies that for each $c \in \mathbb{R}$ there can be at most one critical point of the restriction $V|_{H_c}$ of V to the hyperplane $H_c := \{x \in \mathbb{R}^N : \overline{x} = c\}$. Since for every $c \in \mathbb{R}$ the constant vector $\mathbf{c} := (c, \dots, c) \in \mathcal{C}$ satisfies

$$\forall w \in \mathcal{C}^\perp\;,\quad \langle \nabla V(\mathbf{c}), w\rangle \;=\; (c^3 - c)\sum_{k=1}^N w_k \;=\; 0\;,\;\; \text{i.e}\; \nabla\,(V|_{H_c})(\mathbf{c}) \;=\; 0\;,$$

the critical points of V necessarily have to be on the diagonal $\mathcal{C}$. The statement of the lemma follows now by noting that for $\mathbf{c} := (c, \dots, c)$

$$V(\mathbf{c}) \;=\; \frac{1}{4}c^4 \;-\; \frac{1}{2}c^2 \;+\; \frac{1}{4}\;.$$

∎

Since, according to (2.6), the quadratic part of V around its critical points is essentially given by the discrete Laplacian K, part of our analysis will rely on a good control in large dimension of Gaussian integrals, whose covariances are given by the resolvent of K or slight perturbations thereof. To be specific, we shall consider for suitable $\alpha, \beta \in \mathbb{R}$ operators $Q : \mathbb{R}^N \to \mathbb{R}^N$ of the form

$$Q \;:=\; \big(\,\alpha P \;+\; K \;+\beta\,\big)^{-1}\;, \tag{2.9}$$

where P is the projection given by (2.4). Note that the particular case $\alpha = 0, \beta = 2$ corresponds to $Q = (K+2)^{-1}$, which according to (2.6) equals the inverse of the Hessian of V at the minima. Taking instead $\alpha = 2$, $\beta = -1$, one obtains $Q = (2P + K - 1)^{-1}$, which is the inverse of the Hessian of V at the saddle point, modulo inverting sign of its unique negative eigenvalue.
In general, for any choice of α, β such that Q is well-defined, it follows from (2.3) that for each $k \in \{0, \dots, N-1\}$ it holds $(\widehat{Qx})_k = \sigma_k \, \hat{x}_k$, where the eigenvalues are now given by

$$\sigma_k = \frac{1}{\nu_k + \beta} \quad \text{for } k \in \{1, \dots, N-1\} \quad \text{and} \quad \sigma_0 = \frac{1}{\alpha + \beta} \, .$$

In particular, Q is positive in the sense of quadratic forms if and only if $\alpha + \beta > 0$ and $\mu + \beta > 0$, which is assumed from now on. A crucial property of Q is that it is of trace class, uniformly in the dimension:

$$\exists C > 0 : \ \forall N \in \mathbb{N} \, , \quad \operatorname{Tr} Q \ := \ \sum_{k=0}^{N-1} \sigma_k \ < \ C \, . \tag{2.10}$$

The latter estimate is obtained by straightforward estimates on the ν_k's (essentially $\nu_k \sim \mu k^2$, see their expression in (2.3)). We remark en passant that (2.10) fails to hold in the case of higher-dimensional single particle state, i.e. $x_k \in \mathbb{R}^d$ with $d > 1$. This is linked to well-known difficulties in the analytical treatment of the Stochastic Allen-Cahn equation in higher spatial dimension. A straightforward consequence of (2.10) is a uniform control in N of moments of the centered Gaussian distribution with covariance Q whose density is given by

$$d\mu = \frac{e^{-\frac{\langle x, Q^{-1} x \rangle}{2}}}{\big((2\pi)^N \det Q\big)^{\frac{1}{2}}} \, dx \, .$$

In particular we will repeatedly exploit in this paper the following uniform bound, which we state here for later reference.

Lemma 2.2. *Let Q be defined as in* (2.9) *with $\alpha + \beta > 0$ and $\mu + \beta > 0$. Then there exists a constant $C > 0$ such that for $p \in \{4, 6\}$ and every $h > 0$, $N \in \mathbb{N}$,*

$$\frac{1}{\big((2\pi h N)^N \det Q\big)^{\frac{1}{2}}} \int_{\mathbb{R}^N} N^{-1} \|x\|_p^p \, e^{-\frac{\langle x, (hNQ)^{-1} x \rangle}{2}} \, dx \ \leq \ C h^{\frac{p}{2}} \, . \tag{2.11}$$

Proof. Differentiating suitably the moment generating function of a Gaussian with covariance Q, given by

$$\forall\, \xi \in \mathbb{R}^N \ , \qquad \frac{1}{\big((2\pi)^N \det Q\big)^{\frac{1}{2}}} \int_{\mathbb{R}^N} e^{-\frac{\langle x, Q^{-1}x\rangle}{2}} e^{\langle x,\xi\rangle}\, dx \;=\; e^{\frac{\langle \xi, Q\xi\rangle}{2}} \ ,$$

yields for the left hand side of (2.11) the expression

$$\frac{C_p}{N}\sum_{k=1}^{N}(hNQ_{k,k})^{\frac{p}{2}} \ , \quad \text{with} \quad C_4 = 3,\ C_6 = 15 \ . \tag{2.12}$$

Using the "Fourier integral representation" of Q we get for its diagonal terms the expression $Q_{k,k} = \frac{1}{N}\sum_{j=0}^{N-1}\sigma_j$ for every k. It follows that (2.12) equals $h^{\frac{p}{2}}C_p(\sum_{k=0}^{N-1}\sigma_k)^{\frac{p}{2}}$. This yields the desired result thanks to (2.10). ∎

For convenience of the reader, we also state explicitly the following simple tail estimate, which will be exploited throughout the paper. Recall that $\overline{x}$ denotes the mean of $x \in \mathbb{R}^N$ as defined in (2.4).

Lemma 2.3. *Let Q be defined as in (2.9) with $\alpha+\beta > 0$ and $\mu+\beta > 0$. Then for every $r > 0$, the following estimate holds for $C = C(r) = \frac{(\alpha+\beta)r^2}{2}$ and for every $h \in (0,1]$ and $N \in \mathbb{N}$,*

$$\frac{1}{\big((2\pi hN)^N \det Q\big)^{\frac{1}{2}}} \int_{\{|\overline{x}|>r\}} e^{-\frac{\langle x,(hNQ)^{-1}x\rangle}{2}}\, dx \;\le\; \left(\frac{h}{\pi C}\right)^{\frac{1}{2}} e^{-\frac{C}{h}} \ .$$

Proof. Diagonalising Q via the Fourier transform and recalling that $\overline{x} = \frac{\hat{x}_0}{\sqrt{N}}$, we obtain

$$\frac{1}{\big((2\pi hN)^N \det Q\big)^{\frac{1}{2}}} \int_{\{|\overline{x}|>r\}} e^{-\frac{\langle x,(hNQ)^{-1}x\rangle}{2}}\, dx \;=\; \frac{2}{\sqrt{2\pi}} \int_{\{y_0 > r\sqrt{\frac{\alpha+\beta}{h}}\}} e^{-\frac{y_0^2}{2}}\, dy_0$$

and the statement boils down to the standard Gaussian tail-estimate:

$$\forall \eta > 0 \ , \quad \int_{\eta}^{+\infty} e^{-\frac{t^2}{2}}\, dt \quad\le\quad \frac{1}{\eta} e^{-\frac{\eta^2}{2}} \ . \tag{2.13}$$

∎

Lastly, the ratio of the determinants of Hess $V(I_+)$ and Hess $V(0)$ converges, as already observed in [Ste] (see also [BBM, BeGe]). More precisely the following statement holds true:

Lemma 2.4. *The relation* (1.7) *mentioned in the introduction holds true:*

$$\sqrt{\frac{\det \operatorname{Hess} V(I_+)}{|\det \operatorname{Hess} V(0)|}} = \sqrt{\frac{\det \operatorname{Hess} V(I_-)}{|\det \operatorname{Hess} V(0)|}} \underset{N\to+\infty}{\longrightarrow} \frac{\sinh(\pi\sqrt{2\mu^{-1}})}{\sin(\pi\sqrt{\mu^{-1}})} . \tag{2.14}$$

Proof. According to (2.6) and to (2.3), we have for $2 \leq N \in \mathbb{N}$,

$$\frac{\det \operatorname{Hess} V(I_\pm)}{|\det \operatorname{Hess} V(0)|} = \frac{\det(K+2)}{|\det(K-1)|} = \prod_{k=0}^{N-1} \frac{\nu_k+2}{|\nu_k-1|} = 2\prod_{k=1}^{N-1}\frac{\nu_k+2}{\nu_k-1} ,$$

so we want to show that

$$\sqrt{2}\sqrt{\prod_{k=1}^{N-1}\frac{\nu_k+2}{\nu_k-1}} = \sqrt{2}\prod_{k=1}^{N-1}\left(1+\frac{3}{\nu_k-1}\right)^{\frac{1}{2}} \underset{N\to+\infty}{\longrightarrow} c_\mu ,$$

where c_μ is given by

$$c_\mu := \frac{\sinh(\pi\sqrt{2\mu^{-1}})}{\sin(\pi\sqrt{\mu^{-1}})} = \sqrt{2}\prod_{k=1}^{+\infty}\frac{\mu k^2+2}{\mu k^2-1} ,$$

the last equality being a direct consequence of Euler's product formula

$$\forall\, z \in \mathbb{C} , \quad \sin(\pi z) = \pi z \prod_{k=1}^{+\infty}\left(1-\frac{z^2}{k^2}\right) .$$

Noticing now the relation

$$\prod_{k=1}^{N-1}\left(1+\frac{3}{\nu_k-1}\right)^{\frac{1}{2}} = \left(1+\frac{3}{\mu\frac{1}{\sin^2(\frac{\pi}{N})}-1}\right)^{\frac{\mathbf{1}_{2\mathbb{N}}(N)}{2}} \prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left(1+\frac{3}{\nu_k-1}\right) ,$$

we are then lead to prove that

$$\prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left(1+\frac{3}{\nu_k-1}\right) \underset{N\to+\infty}{\longrightarrow} \prod_{k=1}^{+\infty}\frac{\mu k^2+2}{\mu k^2-1} = \lim_{N\to+\infty}\prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left(1+\frac{3}{\mu k^2-1}\right) ,$$

and it is therefore sufficient to show that

$$\prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1+\frac{3}{\nu_k-1}}{1+\frac{3}{\mu k^2-1}} = \prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left(1+3\frac{\nu_k-\mu k^2}{(\nu_k-1)(\mu k^2+2)}\right) \underset{N\to+\infty}{\longrightarrow} 1 . \tag{2.15}$$

The end of the proof follows from the computations done in [BBM] pp. 331-332 but we give the details for the sake of completeness. From the inequalities

$$\forall\, x \in [0, \frac{\pi}{2}]\,, \quad 0 \,\leq\, x^2(1-\frac{x^2}{3}) \,=\, x^2 - \frac{x^4}{3} \,\leq\, \sin^2 x \,\leq\, x^2\,,$$

we deduce that for every $2 \leq N \in \mathbb{N}$ and $k \in \{1, \dots, \lfloor \frac{N-1}{2} \rfloor\}$,

$$\mu\, k^2(1-\frac{\pi^2}{12}) \,\leq\, \mu\, k^2 \left(1 - \frac{\pi^2 k^2}{3N^2}\right) \,\leq\, \nu_k \,\leq\, \frac{3\mu N^2 k^2}{3N^2 - \pi^2}\,, \tag{2.16}$$

and therefore

$$-\frac{\mu\pi^2 k^4}{3N^2} \,\leq\, \nu_k - \mu\, k^2 \,\leq\, \frac{\mu\pi^2 k^2}{3N^2 - \pi^2}\,,$$

from which we obtain that for every $N \geq 2$ and $k \in \{1, \dots, \lfloor \frac{N-1}{2} \rfloor\}$,

$$|\nu_k - \mu\, k^2| \,\leq\, \frac{2\mu\pi^2 k^4}{N^2}\,. \tag{2.17}$$

It follows from (2.16) and (2.17) that there exist $2 \leq N_0 \in \mathbb{N}$ and a positive constant C such that for every $N \geq N_0$ and $k \in \{\lfloor \frac{N_0-1}{2} \rfloor, \dots, \lfloor \frac{N-1}{2} \rfloor\}$,

$$\left| 3\frac{\nu_k - \mu\, k^2}{(\nu_k - 1)(\mu k^2 + 2)} \right| \,\leq\, \frac{C}{N^2}\,.$$

Using the inequality $|\ln(1+x)| \,\leq\, \frac{|x|}{1-|x|}$ valid on $(-1,1)$, we get that for every $\mathbb{N} \ni N \geq \max\{N_0, \sqrt{C}+1\}$,

$$\left| \ln \prod_{k=\lfloor \frac{N_0-1}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} \left(1 + 3\frac{\nu_k - \mu k^2}{(\nu_k - 1)(\mu k^2 + 2)} \right) \right| \,\leq\, \sum_{k=\lfloor \frac{N_0-1}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} \frac{C}{N^2 - C} \underset{N\to+\infty}{\longrightarrow} 0\,,$$

and the equation (2.15) we were looking for follows, since for any fixed k, $1 + 3\frac{\nu_k - \mu k^2}{(\nu_k - 1)(\mu k^2 + 2)}$ goes to 1 when $N \to +\infty$. ∎

For additional background on Gaussian measures and perturbations thereof in large and infinite dimensions we point to [GlJa, Sim, Dap].

2.2 Relation between L_h and the Witten Laplacian

As already mentioned in the introduction, all the results stated there can equivalently be reformulated in terms of Witten Laplacians using the ground state transformation. More precisely, up to a multiplicative factor hN^{-1}, the operator $L_h := -hN\Delta + \nabla V\cdot\nabla$ acting on $L^2(e^{-\frac{V}{hN}}dx)$ is unitarily equivalent to a semiclassical Witten Laplacian acting on the flat $L^2(dx)$:

$$e^{-\frac{\tilde f}{h}}\, h\, L_h\, e^{\frac{\tilde f}{h}} \;=\; N\big(-h^2\Delta \;+\; |\nabla\tilde f|^2 \;-\; h\Delta\tilde f\big) \;=:\; N\,\Delta^{(0)}_{\tilde f,h}\,, \qquad (2.18)$$

where

$$\tilde f(x) \;:=\; \frac{V(x)}{2N}\,.$$

Using in addition the unitary dilatations Dil_λ on $L^2(dx)$, which are defined, for any $\lambda>0$ and any $g\in L^2(dx)$, by $\mathrm{Dil}_\lambda g := \lambda^{\frac{N}{2}} g(\lambda\,\cdot)$, we have also the unitary equivalence

$$\mathrm{Dil}_{\sqrt N}\; N\Delta^{(0)}_{\tilde f,h}\;\mathrm{Dil}_{\frac{1}{\sqrt N}} \;=\; -h^2\Delta \;+\; |\nabla f|^2 \;-\; h\Delta f \;=:\; \Delta^{(0)}_{f,h}\,, \qquad (2.19)$$

where

$$f(x) \;:=\; \tilde f(\sqrt N x) \;=\; \frac{V(\sqrt N x)}{2N}\,.$$

Note that $\Delta^{(0)}_{f,h} = \sum_j (\partial_j + h\partial_j f)^*\,(\partial_j + h\partial_j f)$ with domain $C_c^\infty(\mathbb{R}^N)$ is symmetric and nonnegative in $L^2(dx)$, that its Schrödinger potential $|\nabla f|^2 - h\Delta f$ is smooth and, for fixed h and N, tending to infinity as $|x|\to\infty$. It follows then from standard arguments of the theory of Schrödinger operators (see e.g. [Hel4, Proposition 7.10 and Theorem 9.15]) that $\Delta^{(0)}_{f,h}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^N)$, and that its closure, which we still denote by $\Delta^{(0)}_{f,h}$, has compact resolvent and therefore purely discrete spectrum. Notice moreover that $\ker\Delta^{(0)}_{f,h} \;=\; \mathrm{Span}(e^{-\frac{f}{h}})$.

Due to (2.18) and (2.19), these properties can be immediately transferred from $\Delta^{(0)}_{f,h}$ to L_h and it holds

$$\mathrm{Spec}(\Delta^{(0)}_{f,h}) \;=\; h\,\mathrm{Spec}(L_h)\,. \qquad (2.20)$$

3 Uniform bounds in the dimension

This section is devoted to the establishment of lower bounds on the log-Sobolev constant $\rho_{h,N}$ (defined through (1.5)) and upper bounds on the spectral gap $\lambda_{h,N}$ (defined through (1.4)), which are uniform in the system size N. The main results here are the following.

Theorem 3.1 (Lower Bound)**.** *For every $\delta > 0$, there exists a positive constant C_δ such that the log-Sobolev constant $\rho(h,N)$ satisfies*

$$\forall h > 0 \ , \forall N \in \mathbb{N} \ , \quad C_\delta \ e^{-\frac{3+2\sqrt{2}+\delta}{24h}} \ e^{-\frac{1}{4h}} \ \leq \ \rho(h,N) \ .$$

Theorem 3.2 (Upper Bound)**.** *The spectral gap $\lambda(h,N)$ satisfies for every $h > 0$ and $N \in \mathbb{N}$ the inequality*

$$\lambda(h,N) \ \leq \ p(N) \ e^{-\frac{1}{4h}} \ \big(\ 1 \ + \ \epsilon(h,N) \ \big) \ ,$$

where the prefactor $p(N)$ is given by (1.7) *and the error term $\epsilon(h,N)$ satisfies*

$$\exists \, C > 0 \ \ s.t. \ \ \forall h \in (0,1] \ , \ \forall N \in \mathbb{N} \ , \quad |\epsilon(h,N)| \ \leq \ C \ h \ .$$

Note that, together with the well-known inequality $\rho(h,N) \leq \lambda(h,N)$, which is easily obtained by applying (1.5) to $1+\varepsilon u$ and letting $\varepsilon \to 0$, Theorem 3.1 and Theorem 3.2 yield Theorem 1.1.

The proof of Theorem 3.1 is based on a careful perturbation of the energy V and a combination of the Holley-Stroock perturbation principle and the Bakry-Émery criterion for log-Sobolev constants (c.f. Proposition 3.3 and Proposition 3.4 below). The proof of Theorem 3.2 relies on a suitable choice of a test function (or "quasimode") and exploits a good control on the normalisation constant $Z_{h,N} := \int_{\mathbb{R}^N} e^{-\frac{V(x)}{hN}} dx$.

This section is organised as follows. Subsection 3.1 contains the proof of Theorem 3.1. Subsection 3.2, which might also be of independent interest, provides a sharp Laplace-type asymptotics for $Z_{h,N}$ when $h \to 0$ with uniform control in N. Finally, Subsection 3.3 contains the proof of Theorem 3.2.

3.1 Proof of Theorem 3.1 (Lower Bound on $\rho_{h,N}$)

Our proof is based on a combination of the following two well-known criteria for establishing lower bounds on the log-Sobolev constant (see for example [Roy, Prop. 3.1.18 and Theorem 3.1.29] or the original papers [HoSt], [BaÉm]).

We fix here $N \in \mathbb{N}$ and use the following standard notation: for a measurable function $U : \mathbb{R}^N \to \mathbb{R}$ such that $e^{-U} \in L^1(\mathbb{R}^N, dx)$, we define the probability measure $dm_U := Z_U^{-1} e^{-U} dx$, where $Z_U := \int_{\mathbb{R}^N} e^{-U} \, dx$ is the normalisation constant. Moreover we write for nonnegative $u \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathbb{R})$,

$$\mathrm{Ent}_{\,m_U}(u) \ := \ \int_{\mathbb{R}^N} u \log u \ dm_U \ - \ \int_{\mathbb{R}^N} u \log \left(\int_{\mathbb{R}^N} u \ dm_U \right) \ dm_U \ ,$$

and define ρ_U as the largest positive constant such that $\forall u \in \mathcal{C}_c^\infty(\mathbb{R}^N;\mathbb{R})$,

$$\rho_U \ \mathrm{Ent}\,(u^2) \ \leq \ 2 \int_{\mathbb{R}^N} |\nabla u|^2 \ dm_U \ . \tag{3.1}$$

Proposition 3.3 (Holley-Stroock perturbation principle)**.**
Let $U : \mathbb{R}^N \to \mathbb{R}$ s.t. $e^{-U} \in L^1(\mathbb{R}^N, dx)$ and let $W : \mathbb{R}^N \to \mathbb{R}$ be a bounded measurable function. Then

$$\rho_U \ \geq \ e^{-(\sup W - \inf W)} \ \rho_{U+W} \ .$$

Proposition 3.4 (Bakry-Émery criterion)**.** *Let $U \in \mathcal{C}^2(\mathbb{R}^N)$ such that (in the sense of quadratic forms)*

$$\exists\, C > 0 \ : \ \forall x \in \mathbb{R}^N \ , \quad \mathrm{Hess}\, U(x) \ \geq \ C \ .$$

Then the log-Sobolev constant satisfies

$$\rho_U \ \geq \ C.$$

In order to prove Theorem 3.1, we construct a suitable perturbation W which added to our energy V produces a strictly convex function. This is done as follows. First, for each $n \in \mathbb{N}$, we take some even $\theta_n \in \mathcal{C}^2(\mathbb{R};[0,1])$ satisfying

$$\theta_n(r) \ = \ \begin{cases} 1 & \text{if } |r| \leq 1 \\ 0 & \text{if } |r| \geq \sqrt{2} \end{cases} \ , \qquad \theta_n'(r) \leq 0 \quad \text{if} \quad r \geq 0\,, \tag{3.2}$$

and, for every $r \in \mathbb{R}$,

$$\theta_n''(r) \geq -\frac{2}{(\sqrt{2}-1)^2}(1+\frac{1}{n}) \ . \tag{3.3}$$

This is indeed possible since, by elementary arguments, one can check that

$$\sup\left\{ \min_{r\in\mathbb{R}} f''(r)\,;\ f \in \mathcal{C}^2(\mathbb{R};[0,1]) \text{ is even and satisfies (3.2)} \right\} = -\frac{2}{(\sqrt{2}-1)^2} \ .$$

Next, in order to "convexify" V with some as small as possible perturbation, we consider for every $n \in \mathbb{N}$, $\alpha \in (0,1)$ and $\beta > 0$ the family of perturbations

$$W_{\alpha,\beta,n}(x) \ := \ \sum_k \theta_n(c_{\alpha,\beta} x_k) \left(\ - \ \frac{1-\alpha}{4} x_k^4 \ + \ \frac{1+\beta}{2} x_k^2 \right) \ - \ \frac{N}{4} \ , \tag{3.4}$$

where $c_{\alpha,\beta} := \sqrt{\frac{1-\alpha}{1+\beta}}$. Since the polynomial part of (3.4) is nonnegative on $\operatorname{supp}\theta_n(c_{\alpha,\beta}\,\cdot)$, one gets from $0 \leq \theta_n \leq 1$ the two bounds

$$-\frac{N}{4} \;\leq\; W_{\alpha,\beta,n}(x) \;\leq\; \frac{N}{4}\left(\frac{(1+\beta)^2}{1-\alpha} \;-\; 1\right), \tag{3.5}$$

valid for every $n \in \mathbb{N}, \alpha \in (0,1), \beta > 0$ and every $x \in \mathbb{R}^N$. Moreover, for a suitable choice of the parameters α, β and n, the $W_{\alpha,\beta,n}$–perturbation of the original energy V becomes a uniformly strictly convex function:

Lemma 3.5. *Let $\alpha \in (\frac{1}{3(2-\sqrt{2})^2+1}, 1)$. Then there exists $n_0 \in \mathbb{N}$ such that for any $\beta > 0$ and $n \geq n_0$, we have in the sense of quadratic forms:*

$$\exists C_{\alpha,\beta,n} > 0 \textit{ s.t. } \;\forall x \in \mathbb{R}^N\,,\;\forall N \in \mathbb{N}\,,\quad \operatorname{Hess}\big(V + W_{\alpha,\beta,n}\big)(x) \;\geq\; C_{\alpha,\beta,n}\;.$$

Proof. Recalling the definition (1.3) of V and that the discrete Laplacian μK is nonnegative we get the estimate

$$\forall x \in \mathbb{R}^N\,,\;\forall \alpha,\beta > 0,\;\forall n \in \mathbb{N}\,,\quad \operatorname{Hess}\big(V + W_{\alpha,\beta,n}\big)(x) \;\geq\; \operatorname{Hess} U_{\alpha,\beta,n}(x)\;,$$

where

$$U_{\alpha,\beta,n} \;:=\; \frac{1}{4}\sum_k \big(\,1-(1-\alpha)\theta_n(c_{\alpha,\beta}x_k)\,\big)\,x_k^4 - \frac{1}{2}\sum_k \big(\,1-(1+\beta)\theta_n(c_{\alpha,\beta}x_k)\,\big)\,x_k^2\,.$$

The Hessian of $U_{\alpha,\beta,n}$ is diagonal and we have, for any $k \in \{1,\dots,N\}$:

$$\begin{aligned}\partial_k^2 U_{\alpha,\beta,n}\,(x) \;=\;& \underbrace{c_{\alpha,\beta}^2\theta_n''(c_{\alpha,\beta}x_k)\left(\;-\;\frac{1-\alpha}{4}x_k^4 \;+\; \frac{1+\beta}{2}x_k^2\right)}_{\text{I}} \;+ \\ &+\; \underbrace{2c_{\alpha,\beta}\theta_n'(c_{\alpha,\beta}x_k)\;\Big(-\;(1-\alpha)x_k^3 \;+\; (1+\beta)x_k\Big)}_{\text{II}} \;+ \\ &+\; \underbrace{\theta_n(c_{\alpha,\beta}x_k)\Big(-\;3(1-\alpha)x_k^2 \;+\; (1+\beta)\Big) \;+\; 3x_k^2 \;-\; 1}_{\text{III}}\;.\end{aligned}$$

Case 1: $|c_{\alpha,\beta}x_k| > \sqrt{2}$.
Then $\theta_n = \theta_n' = \theta_n'' = 0$ for every $n \in \mathbb{N}$ and we obtain

$$\forall \alpha,\beta > 0\,,\quad \partial_k^2 U_{\alpha,\beta,n}(x) \;=\; 3x_k^2 \;-\; 1 \;\geq\; \frac{6(1+\beta)}{1-\alpha} \;-1 \;\geq\; 5\;.$$

Case 2: $|c_{\alpha,\beta,n}x_k| < 1$.
Then $\theta_n' = \theta_n'' = 0$, $\theta_n = 1$ for every $n \in \mathbb{N}$ and we obtain

$$\forall \alpha, \beta > 0\ , \quad \partial_k^2 U_{\alpha,\beta,n}(x) \ = \ 3\alpha x_k^2 \ + \beta \ \geq \ \beta\ .$$

Case 3: $c_{\alpha,\beta}x_k \in [1, \sqrt{2}]$.
First, for every $\beta > 0, \alpha \in (0,1)$, $n \in \mathbb{N}$, we have, using $\theta_n'(c_{\alpha,\beta}x_k) \leq 0$ (see indeed (3.2)) and $\big(-(1-\alpha)x_k^3 + (1+\beta)x_k\big) \leq 0$, that II ≥ 0.
Moreover, we deduce from $\big(3(1-\alpha)x_k^2 - (1+\beta)\big) \geq 0$ that the term III satisfies

$$\text{III} \ = \ (1-\theta_n(c_{\alpha,\beta}x_k)) \left(\ 3(1-\alpha)x_k^2 \ - \ (1+\beta) \ \right) \ + \ 3\alpha x_k^2 + \beta \ \geq \ 3\alpha x_k^2 + \beta\ .$$

Let us lastly look at the term I. Since $\big(-\frac{1-\alpha}{4}x_k^4 + \frac{1+\beta}{2}x_k^2\big) \geq 0$, we have

$$\begin{aligned} 0 \ \leq \ c_{\alpha,\beta}^2 \left(\ - \ \frac{1-\alpha}{4}x_k^4 \ + \ \frac{1+\beta}{2}x_k^2 \ \right) \ &= \ x_k^2\, c_{\alpha,\beta}^2 \left(\ - \ \frac{1-\alpha}{4}x_k^2 \ + \ \frac{1+\beta}{2} \ \right) \\ &\leq \ x_k^2 \left(\ - \ \frac{1-\alpha}{4} \ + \ \frac{1-\alpha}{2} \ \right) \ = \ \frac{1}{4}(1-\alpha)\ x_k^2\ , \end{aligned}$$

and so, since $\theta_n''(c_{\alpha,\beta}x_k) \geq \frac{-2}{(\sqrt{2}-1)^2}(1+\frac{1}{n})$ according to (3.3),

$$\text{I} \ \geq \ -(1+\frac{1}{n})\frac{1-\alpha}{(2-\sqrt{2})^2}\ x_k^2\,.$$

Summing up, we then have in *Case 3*:

$$\forall \alpha, \beta > 0\ , \quad \partial_k^2 U_{\alpha,\beta,n}(x) \geq \ \frac{\alpha\big(3(2-\sqrt{2})^2 + 1 + \frac{1}{n}\big) - 1 - \frac{1}{n}}{(2-\sqrt{2})^2}\ x_k^2 + \beta\,. \quad (3.6)$$

If $\alpha > \frac{1}{3(2-\sqrt{2})^2+1}$ as in the assumption, there exists $n_0 \in \mathbb{N}$ such that the right hand side of (3.6) is bigger than β and hence strictly positive for any $n \geq n_0$.

The case $c_{\alpha,\beta}x_k \in [-2,-1]$ can be treaten in an analogous way and thus the lemma is proven. ∎

The proof of Theorem 3.1 can now be easily concluded: for any $\delta > 0$, taking some fixed $\alpha \in (\frac{1}{3(2-\sqrt{2})^2+1}, 1)$, we have for $\beta > 0$ sufficiently small,

$$\frac{1+\delta}{4} + \frac{3+2\sqrt{2}}{24} \ = \ \frac{3(2-\sqrt{2})^2(1+\delta)+1}{12(2-\sqrt{2})^2} \ \geq \ \frac{(1+\beta)^2}{4(1-\alpha)}\ . \qquad (3.7)$$

According to Lemma 3.5, fixing n sufficiently large, there exists a $C_\delta > 0$ such that the perturbation $W_{\alpha,\beta,n}$ defined in (3.4) satisfies uniformly with respect to the dimension $N \in \mathbb{N}$ and to $x \in \mathbb{R}^N$:

$$\mathrm{Hess}(V + W_{\alpha,\beta,n})(x) \ \geq\ C_\delta\ .$$

Moreover, by estimates (3.5) and (3.7),

$$\sup_x \frac{W_{\alpha,\beta,n}(x)}{Nh} \ - \ \inf_x \frac{W_{\alpha,\beta,n}(x)}{Nh} \ \leq\ \frac{1+\delta}{4} + \frac{3+2\sqrt{2}}{24}\ .$$

Applying the perturbation principle as stated in Proposition 3.3 with $U = \frac{V}{hN}$ and $W = (W_{\alpha,\beta,n})/hN$ yields therefore

$$\rho_{V/hN} \ \geq\ \frac{C}{hN}\, e^{-\frac{1+\delta}{4h} - \frac{3+2\sqrt{2}}{24h}}\ .$$

Noting that the rescaled log-Sobolev constant $\rho(h, N)$ as defined in (1.5) satisfies

$$\forall h > 0\ ,\ \forall N \in \mathbb{N}\ , \quad \rho(h, N) \ =\ hN\rho_{V/hN}\ ,$$

we get the statement of Theorem 3.1.

3.2 Computation of the normalisation constant $Z_{h,N}$

To obtain a good quantitative upper bound, we are lead to compute precise Laplace asymptotics. Similar computations are done in [BBM] exploiting the Hausdorff-Young inequality. We follow a different route based on a comparison with a suitable quadratic form (see (3.11) below) and giving better error estimates.

Note first the expressions for V shifted to the minima,

$$V(x + I_\pm) \ =\ \frac{1}{4}\|x\|_4^4 \ \pm\ \sum_k x_k^3 \ +\ \frac{1}{2}\langle x, (K+2)x\rangle\ , \tag{3.8}$$

and let Q be the following operator that will be used to control V from below in the rest of this subsection:

$$Q \ :=\ \Big(\ \frac{3}{2}P \ +\ K\ - 1\ \Big)^{-1}\ . \tag{3.9}$$

This linear operator then satisfies in particular

$$\det Q^{-1} \ =\ \frac{1}{2}|\det(K-1)| \ =\ \frac{1}{2}|\det \mathrm{Hess}\, V(0)|\ . \tag{3.10}$$

Lemma 3.6. *Let $Q : \mathbb{R}^N \to \mathbb{R}^N$ be the linear operator defined through equation* (3.9)*. Then the following two estimates hold:*

$$\forall\, \overline{x} \geq -1 \ , \quad V(x+I_+) \ \geq \ \frac{1}{2}\langle x, Q^{-1}x\rangle \ , \tag{3.11}$$

and

$$\forall\, |\overline{x}| \leq 1 \ , \quad \frac{1}{4}\|x\|_4^4 \ - \ \big|\sum_k x_k^3\big| \ + \ \frac{1}{2}\langle x, (K+2)x\rangle \ \geq \ \frac{1}{2}\langle x, Q^{-1}x\rangle \ . \tag{3.12}$$

Proof. Note first the following estimate implied by Hölder's inequality:

$$\forall x \in \mathbb{R}^N \ , \quad \|x\|_4^4 \ \geq \ N\overline{x}^4 \ . \tag{3.13}$$

It follows that

$$\frac{1}{4}\|x\|_4^4 \ - \ \frac{N}{2}\overline{x}^2 \ + \ \frac{N}{4} \ \geq \ \frac{N}{4}\overline{x}^4 \ - \ \frac{N}{2}\overline{x}^2 \ + \frac{N}{4} \ = \ \frac{N}{4}(\overline{x}-1)^2(\overline{x}+1)^2 \ ,$$

and therefore

$$\begin{aligned} V(x) \ &\geq \ \frac{N}{4}(\overline{x}-1)^2(\overline{x}+1)^2 \ + \ \frac{1}{2}\langle x, (K-1)x\rangle \ + \ \frac{N}{2}\overline{x}^2 \\ &= \ \frac{N}{4}(\overline{x}-1)^2(\overline{x}+1)^2 \ + \ \frac{1}{2}\langle x, (P+K-1)x\rangle \ , \end{aligned} \tag{3.14}$$

where the last inequality follows from the relation $N\overline{x}^2 = \langle x, Px\rangle$.
From (3.14), since $P+K-1$ annihilates constants, we get for the shifted potential the estimate

$$V(x+I_+) \ \geq \ \frac{N}{4}\overline{x}^2 \ + \ \frac{1}{2}\langle x, (P+K-1)\ x\rangle \quad \text{for } \overline{x} \geq -1 \ , \tag{3.15}$$

which proves (3.11). Note moreover that (3.14) also gives

$$\forall\, \overline{x} \leq 1 \ , \quad V(x+I_-) \ \geq \ \frac{1}{2}\langle x, Q^{-1}x\rangle \ , \tag{3.16}$$

which is actually equivalent to (3.11) due to the symmetry of V. Estimate (3.12) is then an immediate consequence of the expressions for $V(x+I_+)$ and $V(x+I_-)$ given in (3.8) and (3.11), (3.16). ∎

Proposition 3.7. *For every $r > 0$ there exists a constant $C > 0$ such that for each $h \in (0,1]$ and $N \in \mathbb{N}$,*

$$\int_{\{\overline{x}\geq 0\,;\,|\overline{x}-1|\geq r\}} e^{-\frac{V(x)}{hN}}\ dx \ \leq \ C^{-1}\ \frac{(2\pi hN)^{\frac{N}{2}}}{|\det \mathrm{Hess}\, V(I_+)|^{\frac{1}{2}}}\ e^{-\frac{C}{h}} \ . \tag{3.17}$$

Proof. Fix $r > 0$. Shifting the origin to the minimum I_+ and using the quadratic lower bound given in (3.11) of Lemma 3.6, we get

$$I \;:=\; \int_{\{\overline{x}\geq 0\,;\,|\overline{x}-1|\geq r\}} e^{-\frac{V(x)}{hN}}\, dx \;\leq\; \int_{\{\overline{x}\geq -1\,;\,|\overline{x}|\geq r\}} e^{-\frac{\langle x,(hNQ)^{-1}x\rangle}{2}}\, dx\ ,$$

where $Q : \mathbb{R}^N \to \mathbb{R}^N$ is the positive operator defined in (3.9). According to the Gaussian tail estimate of Lemma 2.3, there exists a constant $C > 0$ such that for every $h \in (0, 1]$ and $N \in \mathbb{N}$,

$$I \;\leq\; C^{-1} \frac{(2\pi hN)^{\frac{N}{2}}}{\big(\det Q^{-1}\big)^{\frac{1}{2}}} e^{-\frac{C}{h}}\ .$$

The desired result follows now from (3.10) and the convergence of the ratio of determinants given by (2.14) of Lemma 2.4. ∎

Proposition 3.8. *Let* $r \in (0, 1]$*. Then*

$$\int_{\{|\overline{x}-1|\leq r\}} e^{-\frac{V(x)}{hN}}\, dx \;=\; \frac{(2\pi hN)^{\frac{N}{2}}}{|\det \mathrm{Hess}\, V(I_+)|^{\frac{1}{2}}}\ \big(\,1 +\ \epsilon_r(h, N)\,\big)\ , \qquad (3.18)$$

where the error term $\epsilon_r(h, N)$ *satisfies*

$$\exists\, C = C(r) > 0 \quad s.t. \quad \forall h \in (0, 1]\ ,\ \forall N \in \mathbb{N}\ , \qquad |\epsilon_r(h, N)| \;\leq\; C\ h\ .$$

Proof. Fix $r \in (0, 1]$. Shifting the origin to the minimum I_+ (recall (3.8)) and isolating the contribution of the integral given by the non-quadratic part of V from the rest, we write

$$\begin{aligned}\int_{\{|\overline{x}-1|\leq r\}} e^{-\frac{V(x)}{hN}}\, dx \;&=\; \int_{\{|\overline{x}|\leq r\}} e^{-\frac{V(x+I_+)}{hN}}\, dx \\ &=\; \underbrace{\int_{\{|\overline{x}|\leq r\}} e^{-\frac{\langle x,(K+2)x\rangle}{2hN}}\, dx}_{=:I} \;+\; \underbrace{\int_{\{|\overline{x}|\leq r\}} a(x)\, e^{-\frac{\langle x,(K+2)x\rangle}{2hN}}\, dx}_{=:II}\ , \end{aligned} \qquad (3.19)$$

with

$$a(x) \;:=\; \exp\Big(\,-\,\frac{\frac{1}{4}\|x\|_4^4 \;+\; \sum_k x_k^3}{hN}\,\Big)\; - 1\ .$$

Computation of I:
For the integral I appearing in (3.19), recalling that $\operatorname{Hess} V(I_+) = K+2$ and using the Gaussian tail estimate of Lemma 2.3, we obtain the existence of $C > 0$, such that for every $h \in (0,1]$, $N \in \mathbb{N}$,

$$I \;=\; \frac{(2\pi hN)^{\frac{N}{2}}}{|\det \operatorname{Hess} V(I_+)|^{\frac{1}{2}}}\ \big(\,1 + \ \epsilon_r(h,N)\,\big)\ , \tag{3.20}$$

where the error term $\epsilon_r(h,N)$ satisfies

$$\exists\, C = C(r) > 0 \ \text{ s.t. } \ \forall h \in (0,1]\ ,\ \forall N \in \mathbb{N}\ , \quad |\epsilon_r(h,N)| \;\le\; C^{-1}\, e^{-\frac{C}{h}}\ .$$

Estimate of II:
For the integral II appearing in (3.19), we proceed as follows: evaluating the estimate

$$\forall\, t \in \mathbb{R}\ , \quad |e^t - 1 - t| \;\le\; \frac{1}{2} t^2 e^{|t|}$$

at $t = -(hN)^{-1}\sum_k x_k^3$, we get

$$a(x) \;=\; \underbrace{e^{-\frac{\|x\|_4^4}{4hN}} - 1}_{=:A} \;-\; \underbrace{(hN)^{-1}\sum_k x_k^3\, e^{-\frac{\|x\|_4^4}{4hN}}}_{=:B} \;+\; \epsilon(h,N,x)\ , \tag{3.21}$$

with

$$|\epsilon(h,N,x)| \;\le\; \frac{1}{2(hN)^2}\|x\|_3^6\ \exp\Big(-\ \frac{\frac{1}{4}\|x\|_4^4 - \big|\sum_k x_k^3\big|}{hN}\Big)\ . \tag{3.22}$$

Using $\|x\|_3^6 \le N\|x\|_6^6$ and (3.12) in Lemma 3.6, it follows from (3.22) that

$$e^{-\frac{\langle x,(K+2)x\rangle}{2hN}}|\epsilon(h,N,x)| \;\le\; \frac{1}{2h^2N}\|x\|_6^6\ e^{-\frac{\langle x,(hNQ)^{-1}x\rangle}{2}}\ , \tag{3.23}$$

where $Q : \mathbb{R}^N \to \mathbb{R}^N$ is defined in (3.9). Using for the term A appearing in (3.21) the inequality $0 \le 1 - e^{-|t|} \le |t|$, the antisymmetry of the term B and estimate (3.23) to control $\epsilon(h,N,x)$, we get

$$|II| \;\le\; \frac{1}{4hN}\int_{\mathbb{R}^N}\|x\|_4^4\ e^{-\frac{\langle x,(K+2)x\rangle}{2hN}}\ dx \;+\; \frac{1}{2h^2N}\int_{\mathbb{R}^N}\|x\|_6^6\ e^{-\frac{\langle x,(hNQ)^{-1}x\rangle}{2}}\ dx\ .$$

The statement of Lemma 2.2 about the control of moments of Gaussian integrals together with the expression of $\det Q^{-1}$ given in (3.10) yield the existence of a $C > 0$ such that for every $h > 0$, $N \in \mathbb{N}$,

$$|II| \;\le\; C\,h\,\frac{(2\pi h)^{\frac{N}{2}}}{|\det \operatorname{Hess} V(I_+)|^{\frac{1}{2}}} \;+\; C\,h\,\frac{(2\pi h)^{\frac{N}{2}}}{|\det \operatorname{Hess} V(0)|^{\frac{1}{2}}}\ .$$

Recalling the convergence of the ratio of determinants given by (2.14) of Lemma 2.14, we finally obtain

$$\exists\, C>0 \;:\; \forall\, h>0\,,\; \forall\, N\in\mathbb{N}\,,\quad |II| \;\leq\; C\,h\,\frac{(2\pi h)^{\frac{N}{2}}}{|\det \mathrm{Hess}\, V(I_+)|^{\frac{1}{2}}}\;. \tag{3.24}$$

Putting together (3.20) and (3.24) gives the statement of the proposition.

∎

According to the symmetry of V, Propositions 3.7 and 3.8 finally lead to the precise computation of the normalisation constant $Z_{h,N}$:

Corollary 3.9. *For the normalisation constant $Z_{h,N}$ we have*

$$Z_{h,N} \;:=\; \int_{\mathbb{R}^N} e^{-\frac{V(x)}{hN}}\; dx \;=\; 2\frac{(2\pi hN)^{\frac{N}{2}}}{|\det \mathrm{Hess}\, V(I_+)|^{\frac{1}{2}}}\;\big(\,1+\;\epsilon(h,N)\,\big)\quad, \tag{3.25}$$

where the error term $\epsilon(h,N)$ satisfies

$$\exists\, C>0 \quad s.t. \quad \forall h\in(0,1]\;,\; \forall N\in\mathbb{N}\;,\qquad |\epsilon(h,N)| \;\leq\; C\;h\;.$$

3.3 Upper Bound on $\lambda_{h,N}$

We give in this section the proof of Theorem 3.2. We recall that for $x\in\mathbb{R}^N$,

$$\overline{x} \;:=\; \frac{1}{N}\sum_{k=1}^{N} x_k\;,$$

and we consider in the rest of this subsection the following operator Q, whose inverse is $\mathrm{Hess}\, V(O)$ modulo inverting sign of its unique negative eigenvalue,

$$Q \;:=\; (2P+K-1)^{-1}\;. \tag{3.26}$$

We have then in particular the relation

$$\det Q^{-1} \;=\; |\det(K-1)| \;=\; |\det \mathrm{Hess}\, V(0)|\;. \tag{3.27}$$

Definition 3.10. *Let $\chi=\chi_{h,N}:\mathbb{R}^N\to[-1,1]$ be the function defined by*

$$\chi(x) \;:=\; \frac{2}{\sqrt{2\pi hN}}\int_0^{\sqrt{N}\overline{x}} e^{-\frac{t^2}{2hN}}\; dt \;=\; \frac{2}{\sqrt{2\pi h}}\int_0^{\overline{x}} e^{-\frac{t^2}{2h}}\; dt\;.$$

For $h>0$ let $\psi=\psi_{h,N}:\mathbb{R}^N\to\mathbb{R}$ be given by

$$\psi(x) \;:=\; \frac{\chi(x)}{\big(\;\int_{\mathbb{R}^N}\chi^2(x)\; e^{-\frac{V(x)}{hN}}\; dx\;\big)^{\frac{1}{2}}}\;.$$

Remark 3.11. *Note that by antisymmetry, the quasimode* ψ *has mean zero:*

$$\int_{\mathbb{R}^N} \psi(x) e^{-\frac{V(x)}{hN}}\ dx \ = \ 0\ .$$

Lemma 3.12. *The square of the weighted* L^2*-norm of* χ *satisfies*

$$\int_{\mathbb{R}^N} \chi^2(x)\ e^{-\frac{V(x)}{hN}}\ dx \ = \ 2\ \frac{(2\pi hN)^{\frac{N}{2}}}{|\det \operatorname{Hess} V(I_+)|^{\frac{1}{2}}}\ \big(\ 1+\ \epsilon(h,N)\ \big)\quad ,$$

where the error term $\epsilon(h,N)$ *satisfies*

$$\exists C>0\ \ s.t.\ \ \forall h\in(0,1]\ ,\ \forall N\in\mathbb{N}\ ,\qquad |\epsilon(h,N)|\ \le\ C\ h\ . \tag{3.28}$$

Proof. By the symmetry of V and χ^2 and splitting the integral we get

$$\begin{aligned}\int_{\mathbb{R}^N} \chi^2(x)\ e^{-\frac{V(x)}{hN}}\ dx \ &= \ 2\int_{\{\overline{x}\ge 0\}} \chi^2(x)\ e^{-\frac{V(x)}{hN}}\ dx\\ &= \ \underbrace{2\int_{\{|\overline{x}-1|\le\frac{1}{2}\}} \chi^2(x)\ e^{-\frac{V(x)}{hN}}\ dx}_{=:I}\ +\ \underbrace{2\int_{\{\overline{x}\ge 0\,;\,|\overline{x}-1|\ge\frac{1}{2}\}} \chi^2(x)\ e^{-\frac{V(x)}{hN}}\ dx}_{=:II}\ .\end{aligned}$$

Using for the term I the simple estimate

$$\exists\, C>0\ :\ \forall x\in\{|\overline{x}-1|\le\frac{1}{2}\}\ ,\ \forall\, h\in(0,1]\ ,\qquad |\chi(x)-1|\ \le C^{-1}\ e^{-\frac{C}{h}}\ ,$$

and Proposition 3.8, and for the term II the bound $|\chi|\le 1$ and Proposition 3.7, we get

$$\int_{\mathbb{R}^N} \chi^2(x)\ e^{-\frac{V(x)}{hN}}\ dx \ = \ 2\ \frac{(2\pi hN)^{\frac{N}{2}}}{|\det \operatorname{Hess} V(I_+)|^{\frac{1}{2}}}\ \big(\ 1+\ \epsilon(h,N)\ \big)\ , \tag{3.29}$$

where the error term $\epsilon(h,N)$ satisfies (3.28). ∎

Theorem 3.2 is then a direct consequence of the following proposition:

Proposition 3.13. *The function* ψ *from Definition 3.10 satisfies for every* $h>0$ *and every* $N\in\mathbb{N}$*,*

$$hN\int_{\mathbb{R}^N} |\nabla\psi|^2\ e^{-\frac{V(x)}{hN}}\ dx \ = \ \frac{1}{\pi}\ \left|\frac{\det \operatorname{Hess} V(I_-)}{\det \operatorname{Hess} V(0)}\right|^{\frac{1}{2}}\ e^{-\frac{1}{4h}}\ \big(\ 1\ +\ \epsilon(h,N)\ \big)\ ,$$

where the error term $\epsilon(h,N)$ *satisfies*

$$\exists\, C>0\ \ s.t.\ \ \forall h\in(0,1]\ ,\ \forall N\in\mathbb{N}\ ,\qquad |\epsilon(h,N)|\ \le\ C\ h\ .$$

Proof. Since for every $x \in \mathbb{R}^N$,

$$hN\, |\nabla \chi|^2(x) \;=\; \frac{2}{\pi}\, e^{-\frac{\overline{x}^2}{h}} \;=\; \frac{2}{\pi}\, e^{-\frac{\langle x, 2Px\rangle}{2hN}} \;,$$

we get with Q as defined in (3.26),

$$\begin{aligned} hN \int_{\mathbb{R}^N} |\nabla \chi|^2\, e^{-\frac{V(x)}{hN}}\, dx \;&=\; \int_{\mathbb{R}^N} e^{-\frac{\langle x, (hNQ)^{-1}x\rangle}{2}}\, dx \\ &\quad + \int_{\mathbb{R}^N} e^{-\frac{\langle x, (hNQ)^{-1}x\rangle}{2}} \left(e^{-\frac{\|x\|_4^4}{4hN}} - 1 \right) dx \;. \end{aligned}$$

From this equality, the expression of the determinant of Q given in (3.27), and from the inequality $0 \le 1 - e^{-|t|} \le |t|$, we obtain the following estimate also using the uniform bounds on Gaussian moments provided by Lemma 2.2,

$$hN \int_{\mathbb{R}^N} |\nabla \chi|^2\, e^{-\frac{V(x)}{hN}}\, dx \;=\; \frac{2}{\pi}\, \frac{(2\pi hN)^{\frac{N}{2}}}{|\det \operatorname{Hess} V(0)|^{\frac{1}{2}}} \left(1 \,+\, \epsilon(h,N) \right) , \quad (3.30)$$

where the error term $\epsilon(h,N)$ satisfies (3.13). Combining this with Lemma 3.12 finishes the proof. ∎

4 Sharp spectral gap asymptotics

In this section, we prove Theorem 1.2. To do so, we will again use the test function ψ introduced in Definition 3.10 in order to show that it asymptotically saturates the inequality

$$\forall\, \varphi \in D(L_h) \quad \text{s.t.} \quad \|\varphi\|_{L^2(e^{-\frac{V}{hN}})} = 1 \;, \quad \lambda(h,N) \;\le\; hN \int_{\mathbb{R}^N} |\nabla \varphi|^2 e^{-\frac{V(x)}{hN}} dx$$

under a further assumption on the separation between the second and the third eigenvalues of L_h. Under this condition, we can indeed reverse this inequality up to an error term involving the quadratic form

$$\mathcal{E}(\varphi) \;:=\; \frac{\int_{\mathbb{R}^N} |L_h \varphi|^2\, e^{-\frac{V(x)}{hN}}\, dx}{hN \int_{\mathbb{R}^N} |\nabla \varphi|^2 e^{-\frac{V(x)}{hN}}\, dx} \,,$$

which was already mentioned in the introduction.

Proposition 4.1. *Let $\delta, h_0 > 0$ and, for every $h \in (0, h_0]$, $\mathcal{N}(h) \subset \mathbb{N}$ s.t.*

$$\forall h \in (0, h_0] \;,\; \forall N \in \mathcal{N}(h) \;, \quad \operatorname{Spec}(L_h) \,\cap\, [0, \delta) \;=\; \{0, \lambda(h,N)\} \;. \quad (4.1)$$

Then, for all $h \in (0, h_0]$, $N \in \mathcal{N}(h)$ and $\varphi := \varphi_{h,N} \in D(L_h)$ satisfying

$$\int_{\mathbb{R}^N} \varphi^2 e^{-\frac{V(x)}{hN}} dx = 1\,,\ \int_{\mathbb{R}^N} \varphi\, e^{-\frac{V(x)}{hN}} dx = 0\ ,\ \int_{\mathbb{R}^N} \varphi(L_h\varphi) e^{-\frac{V(x)}{hN}} dx \ <\ \frac{\delta}{2}\,, \quad (4.2)$$

we have the lower bound

$$\lambda(h,N)\ \geq\ hN \int_{\mathbb{R}^N} |\nabla\varphi|^2 e^{-\frac{V(x)}{hN}} dx\ \big(\ 1\ -\ \epsilon(h,N)\ \big)\ ,$$

where the error term $\epsilon(h,N)$ satisfies

$$0\ \leq\ \epsilon(h,N)\ \leq\ \min\Big\{\,1\ ,\ \frac{2hN}{\delta} \int_{\mathbb{R}^N} |\nabla\varphi|^2 e^{-\frac{V(x)}{hN}} dx + \frac{2}{\sqrt{\delta}}\ \sqrt{\mathcal{E}(\varphi)}\,\Big\}\ .$$

The proof is a simple application of the following standard Markov-type inequality, which is a consequence of the spectral theory for self-adjoint operators.

Lemma 4.2. *Let T be a nonnegative self-adjoint operator on a Hilbert space H with domain D. Then for every $u \in D$ and every $b > 0$,*

$$\|1_{[b,\infty)}(T)u\|^2\ \leq\ \frac{\langle Tu, u\rangle}{b}\quad .$$

Proof of Proposition 4.1. We denote respectively by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the scalar product and the Hilbert norm in $L^2(e^{-\frac{V}{hN}})$, and by $P := \mathbf{1}_{[0,\delta)}(L_h)$ the spectral projector of L_h onto the interval $[0,\delta)$. From

$$\int_{\mathbb{R}^N} \varphi(L_h\varphi) e^{-\frac{V(x)}{hN}}\ dx\ =\ hN \int_{\mathbb{R}^N} |\nabla\varphi|^2 e^{-\frac{V(x)}{hN}}\ dx\,,$$

using the third point of the property (4.2) together with Lemma 4.2, we get

$$\|(1-P)\varphi\|^2\ =\ \|\mathbf{1}_{[\delta,+\infty)}(L_h)\varphi\|\ \leq\ \frac{hN}{\delta}\|\nabla\varphi\|^2\ <\ \frac{1}{2}\ . \qquad (4.3)$$

In particular, since $\|\varphi\| = 1$, we have also

$$\|P\varphi\|^2 = \|\varphi\|^2 - \|(1-P)\varphi\|^2 \geq \frac{1}{2}\ , \qquad (4.4)$$

and so $P\varphi \neq 0$. We can therefore define $u := \frac{P\varphi}{\|P\varphi\|}$. Since moreover $\langle\varphi, 1\rangle = 0$, we have $\langle P\varphi, 1\rangle = \langle\varphi, P1\rangle = 0$. Thus, using also (4.1), u is necessarily a normalised eigenfunction of L_h associated with the eigenvalue $\lambda(h,N)$.

Consequently, it follows from $L_h P = P L_h$ on $D(L_h)$, the self-adjointness of $P = P^2$, and elementary rearrangements of terms, that

$$\lambda(h,N) \;=\; \langle u, L_h u\rangle \;=\; \frac{\langle P\varphi, L_h P\varphi\rangle}{\|P\varphi\|^2} \;=\; \frac{\langle \varphi, L_h\varphi\rangle}{\|P\varphi\|^2} \;+\; \frac{\langle P\varphi-\varphi, L_h\varphi\rangle}{\|P\varphi\|^2}$$

$$=\; hN\|\nabla\varphi\|^2\Big[1 \;+\; \underbrace{\frac{\|(1-P)\varphi\|^2}{\|P\varphi\|^2}}_{=:I} \;+\; \underbrace{\frac{\langle (P-1)\varphi, L_h\varphi\rangle}{hN\|\nabla\varphi\|^2}}_{=:II}\underbrace{\Big(1 \;+\; \frac{\|(1-P)\varphi\|^2}{\|P\varphi\|^2}\Big)}_{=:III}\Big].$$

The statement of the proposition follows now by observing that, according to (4.3) and (4.4),

$$I \;\le\; \frac{2hN}{\delta}\|\nabla\varphi\|^2\;, \quad |II| \;\le\; \frac{\|L_h\varphi\|}{\sqrt{\delta\, hN}\|\nabla\varphi\|}\;, \;\text{ and }\quad III \;\le\; 2\;,$$

and $\lambda(h,N)$ is nonnegative. ∎

Applying Proposition 4.1 with the test function ψ as defined in Definition 3.10, the statement of Theorem 1.2 is then a direct consequence of the quasimodal estimates given in Proposition 3.13 and in the following proposition:

Proposition 4.3. *Let ψ be the test function introduced in Definition 3.10. Then there exists $C>0$ such that for every $h\in(0,1]$ and every $N\in\mathbb{N}$,*

$$\int_{\mathbb{R}^N} |L_h\psi|^2\, e^{-\frac{V(x)}{hN}}\, dx \;\le\; C\,h^2\left|\frac{\det \operatorname{Hess} V(I_-)}{\det \operatorname{Hess} V(0)}\right|^{\frac{1}{2}}\; e^{-\frac{1}{4h}}\;. \tag{4.5}$$

Proof. A straightforward computation, whose details are given below for the sake of completeness, leads to the identity

$$\int_{\mathbb{R}^N} |L_h\chi|^2\, e^{-\frac{V(x)}{hN}}\, dx \;=\; \frac{2}{\pi h N^2}\int_{\mathbb{R}^N}\Big(\sum_{k=1}^N x_k^3\Big)^2\, e^{-\frac{V(x)+\langle x,Px\rangle}{hN}}\, dx\;. \tag{4.6}$$

Hence, using the estimate $\left(\sum_{k=1}^N x_k^3\right)^2 \le N\|x\|_6^6$ implied by the Cauchy-Schwarz inequality, we obtain the bound

$$0 \;\le\; \int_{\mathbb{R}^N} |L_h\chi|^2\, e^{-\frac{V(x)}{hN}}\, dx \;\le\; \frac{2\,e^{-\frac{1}{4h}}}{\pi h N}\int_{\mathbb{R}^N}\|x\|_6^6\, e^{-\frac{\langle x,(hNQ)^{-1}x\rangle}{2}}\, dx\;, \tag{4.7}$$

where Q is defined in (3.26). The estimate (4.5) follows by applying the uniform moment bound of Lemma 2.2 to the right hand side of (4.7), recalling the expression (3.27) of the determinant of Q and finally invoking Lemma 3.12 for $\int_{\mathbb{R}^N} \chi^2 e^{-\frac{V(x)}{hN}} dx$.

To show (4.6) and thus completing the proof, we note that for $k \in \{1, \ldots, N\}$,

$$\partial_k \chi(x) \;=\; \frac{1}{N} \frac{2}{\sqrt{2\pi h}} e^{-\frac{\overline{x}^2}{2h}} \;=\; \frac{1}{N} \frac{2}{\sqrt{2\pi h}} e^{-\frac{\langle x, Px \rangle}{2hN}} \;,$$

and compute

$$\begin{aligned} L_h \chi \;&=\; -hN\, e^{\frac{V(x)}{hN}} \sum_{k=1}^{N} \partial_k \big(\, e^{-\frac{V(x)}{hN}} \partial_k \chi \,\big) \\ &=\; -\frac{2}{\sqrt{2\pi h} N}\, e^{-\frac{\langle x, Px \rangle}{2hN}} \sum_{k=1}^{N} \big(-\partial_k V(x) \;+\; \overline{x} \big) \\ &=\; \frac{2}{\sqrt{2\pi h} N}\, e^{-\frac{\langle x, Px \rangle}{2hN}} \sum_{k=1}^{N} x_k^3 \quad , \end{aligned}$$

where for the last inequality we used $\sum_{k=1}^{N} (Kx)_k = 0$. Taking the square we get (4.6). ∎

5 Lower bound on the second spectral gap

The aim of this section is to prove Theorem 1.3. Instead of working directly with the diffusion operator L_h, we switch to the Schrödinger operator point of view and consider the semiclassical Witten Laplacian on functions acting in the flat $L^2(dx)$ and given by

$$\Delta_{f,h}^{(0)} \;=\; -h^2 \Delta \;+\; |\nabla f|^2 \;-h\Delta f \;,$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is defined as

$$f(x) \;:=\; \frac{V(\sqrt{N}x)}{2N} \;=\; \frac{N}{8} \|x\|_4^4 \;+\; \frac{1}{4} \langle x, (K-1)x \rangle \;+\; \frac{1}{8} \;. \qquad (5.1)$$

Note that due to the rescaling of variables, the two minima of f are rescaled by a factor $\sqrt{N}$ with respect to the minima of V. More precisely they are given by

$$J_+ \;:=\; \frac{I_+}{\sqrt{N}} \;=\; \frac{1}{\sqrt{N}}(1, \ldots, 1) \quad , \quad J_- \;:=\; \frac{I_-}{\sqrt{N}} \;=\; \frac{1}{\sqrt{N}}(-1, \ldots, -1) \;.$$

Since in this proof we deal only with the Witten Laplacian acting on functions we drop in the sequel the superscript (0) and write for short $\Delta_{f,h} := \Delta^{(0)}_{f,h}$. Moreover, note also that from the relation (2.20) between L_h and $\Delta_{f,h}$, Theorem 1.3 is implied by the following.

Theorem 5.1. *Let $C > 0$ and $\alpha \in (0, \frac{3}{4})$. Then there exist two positive constants h_0 and ℓ such that*

$$\forall h \in (0, h_0] \;\; \text{and} \;\; \forall N \leq C\, h^{-\alpha} \;, \qquad \dim \big(\operatorname{Ran} \mathbf{1}_{[0,\ell h)}(\Delta_{f,h}) \big) \;\leq\; 2 \;.$$

According to the Max-Min principle (see for example [Hel4, Theorem 11.7]), in order to prove Theorem 5.1, it is sufficient to show that there exist $h_0, \ell > 0$ such that for every $h \in (0, h_0]$ and $N \leq Ch^{-\alpha}$, there exist $E_+, E_- \in L^2(\mathbb{R}^N)$ s.t. for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathbb{R})$,

$$\langle \psi, \Delta_{f,h}\psi \rangle \;\geq\; \ell\, h \left(\; \|\psi\|^2_{L^2(\mathbb{R}^N)} \;-\; \langle \psi, E_+ \rangle^2_{L^2(\mathbb{R}^N)} \;-\; \langle \psi, E_- \rangle^2_{L^2(\mathbb{R}^N)} \right) . \quad (5.2)$$

To obtain estimate (5.2) we first follow a standard "decoupling" approach by introducing a suitable partition of unity allowing to split the integral on the left hand side of (5.2) into integrals over almost disjoint sets. These will be localized respectively around the two minima of f, around the diagonal $\mathcal{C}$ but far from the minima, and far from the diagonal. The main tool here is the so-called IMS localization formula (see [CFKS]).

Proposition 5.2. *(IMS Localization Formula)*
Let $d \in \mathbb{N}$ and $\{\eta_k\}_{1,\ldots,d}$ be a quadratic partition of unity of $\mathbb{R}^N$, i.e. such that $\eta_k \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ for every k and $\sum_{k=1}^d \eta_k^2 \equiv 1$. Then for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$,

$$\langle \psi, \Delta_{f,h}\psi \rangle_{L^2(\mathbb{R}^N)} \;=\; \sum_{k=1}^{d} \langle\, \eta_k\psi,\, \Delta_{f,h}(\eta_k\psi)\, \rangle_{L^2(\mathbb{R}^N)} - h^2 \, \|\, |\nabla\eta_k|\, \psi\|^2_{L^2(\mathbb{R}^N)} \;. \quad (5.3)$$

The second main ingredient to obtain estimate (5.2) relies on the decomposition $\mathbb{R}^N = \mathcal{C} \oplus \mathcal{C}^\perp$ and on a two-scale approach. We recall that $\mathcal{C} = \operatorname{Ran} P$ is one-dimensional where P has been defined in (2.4). For any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathbb{R})$, we then have the decomposition

$$\Delta_{f,h}\psi \;=\; \Delta^{\mathcal{C}}_{f,h}\psi \;+\; \Delta^{\mathcal{C}^\perp}_{f,h}\psi \;, \quad (5.4)$$

where

$$\Delta^{\mathcal{C}}_{f,h} := -h^2\Delta^{\mathcal{C}} + |\nabla^{\mathcal{C}} f|^2 - h\Delta^{\mathcal{C}} f \quad \text{and} \quad \Delta^{\mathcal{C}^\perp}_{f,h} := -h^2\Delta^{\mathcal{C}^\perp} + |\nabla^{\mathcal{C}^\perp} f|^2 - h\Delta^{\mathcal{C}^\perp} f.$$

Here, the superscripts $\mathcal{C}, \mathcal{C}^{\perp}$ on a differential operator mean that differentiation is restricted to the corresponding subspace. Thus, chosing some normalized coordinate y_0 on $\mathcal{C}$ and orthonormal coordinates $(z_1, \ldots, z_{N-1})$ on $\mathcal{C}^{\perp}$ we have for example for every $\psi \in \mathcal{C}^{\infty}(\mathbb{R}^N)$,

$$\Delta^{\mathcal{C}}\psi \;=\; \frac{\partial^2 \psi}{\partial y_0^2} \quad , \quad \Delta^{\mathcal{C}^{\perp}}\psi \;=\; \sum_{k=1}^{N-1} \frac{\partial^2 \psi}{\partial z_k^2} \, .$$

Note in particular that the orthogonal decomposition

$$x \;=\; Px \,+\, P^{\perp}x \;=\; \hat{x}_0 \left(\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}} \right) \,+\, P^{\perp}x$$

leads to

$$\nabla^{\mathcal{C}} f(x) \;=\; \frac{1}{2}\sqrt{N} \sum_{k=1}^{N} \big(\frac{\hat{x}_0}{\sqrt{N}} + P^{\perp}x \big)_k^3 \,-\, \frac{1}{2}\hat{x}_0 \tag{5.5}$$

and

$$\Delta^{\mathcal{C}} f(x) \;=\; \frac{1}{2}\big(3\hat{x}_0^2 \,+\, 3\|P^{\perp}x\|^2 \,-\, 1 \big) \, . \tag{5.6}$$

Given $\psi : \mathbb{R}^N \to \mathbb{R}$ and $y \in \mathcal{C}$, we denote by ψ_y the partial application

$$\psi_y \;:\; z \,\in\, \mathcal{C}^{\perp} \quad \longmapsto \quad \psi(y+z) \tag{5.7}$$

and hence satisfying $\psi_y(\mathcal{C}^{\perp}) = \psi(\{x \in \mathbb{R}^N \,:\, Px = y\})$.
Roughly speaking, we shall exploit decomposition (5.4) as follows. Away from the diagonal, we use $\Delta^{\mathcal{C}}_{f,h} \geq 0$ and exploit (2.8), namely that $\mathrm{Hess}^{\mathcal{C}^{\perp}}$ is strictly convex, uniformly in x and $N \in \mathbb{N}$. This leads for every fixed $y \in \mathcal{C}$ to spectral gap lower bounds for the operator $\psi_y \mapsto (\Delta^{\mathcal{C}^{\perp}}_{f,h}\psi)_y$ (see Lemma 5.3 below). The dependence on y of these estimates is controlled in Lemma 5.4 below. Around the diagonal $\mathcal{C}$ but away from the critical points we use $\Delta^{\mathcal{C}^{\perp}}_{f,h} \geq 0$ and work with $\Delta^{\mathcal{C}}_{f,h}$, which, when restricted to sufficiently small neighbourhoods of $\mathcal{C}$, behaves essentially like the 1-dimensional Witten Laplacian associated with $f|_{\mathcal{C}}$ (see the discussion after (5.34) below). Around the minima J_+ and J_- we work directly with $\Delta_{f,h}$. Here we use that the restriction of f to sufficiently small neighbourhoods around J_+ and J_- is uniformly convex, and thus, locally, good spectral gap lower bounds can be obtained (see Lemma 5.5 and Proposition 5.7 below).

The rest of the section is organized as follows. In Subsection 5.1 we make precise and prove some aforementioned preliminary results which are needed for the proof of Theorem 5.1. In Subsection 5.2 we introduce a suitable quadratic partition of unity of $\mathbb{R}^N$ and give the proof of Theorem 5.1.

5.1 Preliminary estimates

In the sequel we write y and z to denote generic elements of $\mathcal{C}$ and $\mathcal{C}^\perp$ respectively and dy, dz for the Lebesgue measures on $\mathcal{C}$ and $\mathcal{C}^\perp$. We recall also the notation defined in (5.7).

The combination of the following two lemmata allows to control the quadratic form $\langle \Delta_{f,h}\psi, \psi\rangle_{L^2(\mathbb{R}^N)}$ away from the diagonal $\mathcal{C}$.

Lemma 5.3 (Poincaré inequality for fixed $y \in \mathcal{C}$)**.** *The following inequality holds true for every $h > 0, N \in \mathbb{N}$, $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ and $y \in \mathcal{C}$:*

$$\langle\, \psi_y,\, (\Delta^{\mathcal{C}^\perp}_{f,h}\psi)_y\,\rangle_{L^2(\mathcal{C}^\perp)} \;\geq\; h\,(\mu-1)\,\Big(\, \|\psi_y\|^2_{L^2(\mathcal{C}^\perp)} \;-\; \langle\, \psi_y,\, E_y\,\rangle^2_{L^2(\mathcal{C}^\perp)}\,\Big)\;, \tag{5.8}$$

where $E : \mathbb{R}^N \to \mathbb{R}$ is given by

$$E(x) \;:=\; \frac{e^{-\frac{f(x)}{h}}}{\Big(\; \int_{\mathcal{C}^\perp} e^{-2\frac{f(Px+z)}{h}}\; dz\;\Big)^{\frac{1}{2}}}\;. \tag{5.9}$$

Proof. Note that (5.8) is equivalent to the inequality

$$\int_{\mathcal{C}^\perp} \|\nabla^{\mathcal{C}^\perp}(E^{-1}\psi)_y\|^2 E_y^2 dz \;\geq\; \frac{\mu-1}{h}\,\Big(\int_{\mathcal{C}^\perp}(E^{-1}\psi)^2_y E^2_y dz - \Big(\int_{\mathcal{C}^\perp}(E^{-1}\psi)_y E^2_y dz\Big)^2\Big)$$

which follows from the uniform convexity estimate

$$\frac{1}{h}\,\mathrm{Hess}^{\mathcal{C}^\perp}\, 2f(x) \;=\; \frac{1}{h}\,\mathrm{Hess}^{\mathcal{C}^\perp}\, V(\sqrt{N}x) \;\geq\; \frac{\mu-1}{h}$$

implied by (2.8), and standard criteria for the spectral gap of strictly log-concave measures (see for example [Led, Corollary 11.4] or the already used Bakry-Émery criterion of Proposition 3.4 which gives an even stronger result). ∎

Note that by integrating the relation (5.8) of Lemma 5.3 in y and using the Cauchy-Schwarz inequality, we get that for every $h > 0$, $N \in \mathbb{N}$ and $\psi \in C^\infty_c(\mathbb{R}^N)$,

$$\langle\, \psi,\, \Delta^{\mathcal{C}^\perp}_{f,h}\psi\,\rangle_{L^2(\mathbb{R}^N)} \;\geq\; h(\mu-1)\,\|\psi\|^2_{L^2(\mathbb{R}^N)}\Big(1 - \sup_{y\in\mathrm{supp}\,\psi}\int_{\mathrm{supp}\,\psi_y} E^2_y(z)\; dz\Big). \tag{5.10}$$

In order to fully exploit Estimate (5.10), we need a control on the integral appearing on its right hand side when ψ is localised far from the diagonal. The following rough tail estimate will be enough for our purposes.

Lemma 5.4 (Concentration Lemma)**.**
Let h, R_h and ρ be three positive numbers. Then there exist $h_0 > 0$ and $\gamma > 0$ such that for every $h \in (0, h_0]$ and $N \in \mathbb{N}$, the function E defined in (5.9) *satisfies*

$$\sup_{\|y\|\le\rho} \int_{\{\|z\|\ge R_h\}} E_y^2(z)\ dz \ \le\ \min\{\gamma\, e^{-\frac{R_h^2}{\gamma h}}\ , 1\}\ , \tag{5.11}$$

and

$$\sup_{y\in\mathcal{C}} \int_{\{\|z\|\ge R_h\}} E_y^2(z)\ dz \ \le\ \min\{e^{\gamma N}\, e^{-\frac{R_h^2}{\gamma h}}\ , 1\}\ . \tag{5.12}$$

Proof. For every $\tau \in \mathbb{R}^+$, $h > 0$, $N \in \mathbb{N}$ and $y \in \mathcal{C}$, we have the upper bound

$$\int_{\|z\|\ge R_h} E_y^2(z)\ dz \ \le\ e^{-\tau\frac{R_h^2}{h}} \int_{\mathcal{C}^\perp} e^{\tau\frac{\|z\|^2}{h}}\ E_y^2(z)\ dz\ . \tag{5.13}$$

To estimate the integral on the right hand side of (5.13), we shall use the following two bounds on f:

$$2f(x) - 2f(Px) \ \le\ \frac{3N}{4}\|P^\perp x\|_4^4 + \frac{1}{2}\langle\, P^\perp x\,, \big(K-1+4\|Px\|^2\big)\, P^\perp x\,\rangle\ , \tag{5.14}$$

and

$$2f(x)\ -\ 2f(Px)\ \ge\ \frac{1}{2}\langle\, P^\perp x\,, (K-1+\|Px\|^2)\, P^\perp x\,\rangle\ . \tag{5.15}$$

Estimate (5.15) follows immediately from the definition (5.1) of f and from the inequalities

$$\frac{N}{4}\|x\|_4^4\ \ge\ \frac{1}{4}\|x\|^4\ =\ \frac{1}{4}(\|Px\|^2+\|P^\perp x\|^2)^2\ \ge\ \frac{1}{2}\|Px\|^2\,\|P^\perp x\|^2\ +\ \frac{1}{4}\|Px\|^4$$

together with $\|Px\|^4 = N\|Px\|_4^4 = N^2\overline{x}^4$. To see (5.14), note first that from the definition of f,

$$\begin{aligned} 2f(x)\ -\ 2f(Px)\ =\ \frac{N}{4}\|P^\perp x\|_4^4\ \ +\ \ N\overline{x}\sum_{k=1}^{N}(P^\perp x)_k^3 \\ +\ \frac{1}{2}\langle\, P^\perp x, \big(K-1+3\|Px\|^2\big)\, P^\perp x\,\rangle\ , \end{aligned}$$

so (5.14) is a consequence of the elementary inequalities

$$\big|N\overline{x}\sum_{k=1}^{N}(P^\perp x)_k^3\big| \le \sqrt{N}\|Px\|\|P^\perp x\|_4^2\|P^\perp x\| \le \frac{1}{2}\|Px\|^2\|P^\perp x\|^2 + \frac{N}{2}\|P^\perp x\|_4^4\,.$$

From (5.13), together with (5.14), (5.15) and computations of Gaussian integrals, we obtain for every $h > 0$, $N \in \mathbb{N}$ and for every $\tau \in (0, \frac{\mu-1}{2})$,

$$\int_{\|z\| \geq R_h} E_y^2(z)\, dz \;\leq\; e^{-\tau \frac{R_h^2}{h}}\, \frac{\Theta(\tau, N, y)}{1 - \epsilon(h, N, y)}\ , \tag{5.16}$$

where

$$\Theta(\tau, N, y) \;:=\; \Big(\frac{\det(K - 1 + 4\|y\|^2)}{\det(K - 1 + \|y\|^2 - 2\tau)}\Big)^{\frac{1}{2}}\ , \tag{5.17}$$

and

$$\epsilon(h, N, y) \;:=\; \frac{\big(\det(K - 1 + 4\|y\|^2)\big)^{\frac{1}{2}}}{(2\pi h)^{\frac{N}{2}}} \int_{\mathcal{C}^\perp} \big(1 - e^{-\frac{3N\|z\|_4^4}{4h}}\big) e^{-\frac{\langle z, (K-1+4\|y\|^2) z\rangle}{2h}}\, dz.$$

As in the proof of Proposition 3.13, we use the simple estimate

$$|\epsilon(h, N, y)| \;\leq\; \frac{\big(\det(K - 1 + 4\|y\|^2)\big)^{\frac{1}{2}}}{(2\pi h)^{\frac{N}{2}}} \frac{1}{4h} \int_{\mathcal{C}^\perp} 3N\|z\|_4^4\, e^{-\frac{\langle z, (K-1+4\|y\|^2) z\rangle}{2h}}\, dz\ ,$$

and conclude, by applying a straightforward modification of Lemma 2.2, that there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$, $h \in (0, 1]$ and $y \in \mathbb{R}^N$,

$$|\epsilon(h, N, y)| \;\leq\; C\, h\ . \tag{5.18}$$

In order to control $\Theta(\tau, N, y)$, we fix $\tau \in (0, \frac{3}{8}(\mu - 1))$ so that $\Theta(\tau, N, y)$ increases with $\|y\|$ (for any fixed N), and observe that, arguing as in the proof of Lemma 2.4, for every $\rho > 0$ there exists a constant $C > 0$ such that

$$\forall\, N \,\in\, \mathbb{N}\ , \quad \sup_{\|y\| \leq \rho} \Theta(\tau, N, y) \;\leq\; C\ . \tag{5.19}$$

If y is not constrained to a compact set, we get the existence of a constant $C > 0$ such that for every $N \in \mathbb{N}$, $y \in \mathbb{R}^N$,

$$\Theta(\tau, N, y) \;\leq\; 4^{\frac{N}{2}} \;\leq e^{CN}\ . \tag{5.20}$$

Thus, from (5.16), taking h_0 and γ^{-1} sufficiently small, one obtains (5.11) according to (5.18), (5.19) and one obtains (5.12) according to (5.18), (5.20). ∎

The following lemma shows the existence of a suitable neighbourhood of the minimum J_+ on which f is uniformly convex. Note that by symmetry arguments, the analogous statement holds with J_- instead of J_+.

Lemma 5.5 (Uniform Convexity around the minima). *There exist constants $r, \rho > 0$ such that*

$$\forall N \in \mathbb{N} \ , \ \forall x \in \Omega_r \ , \quad \operatorname{Hess} f(x) \ \geq \ \rho \ \ ,$$

where the set Ω_r is given by

$$\Omega_r \ := \ \left\{ \ x \in \mathbb{R}^N \ : \ \|Px - J_+\| \ \leq \ r \ , \ \|P^\perp x\| \ \leq \ rN^{-\frac{1}{4}} \ \right\} .$$

Proof. For $N \in \mathbb{N}$ take $x, w \in \mathbb{R}^N$ with $\|w\| = 1$. Then, recalling the expression of f given in (5.1), we get

$$2\langle w, \operatorname{Hess} f(x) w\rangle \ = \ \langle w, (K-1)w\rangle \ + \ 3N \sum_{k=1}^{N-1} x_k^2 w_k^2 \ . \tag{5.21}$$

For the first term in (5.21), the discrete Poincaré inequality (2.5) gives with $\rho := \frac{1}{4}\min\{\mu - 1, 1\}$ the lower bound

$$\langle w, (K-1)w\rangle \ \geq \ 4\rho\|w\|^2 \ - \ (1+4\rho)\|Pw\|^2 \ \geq \ 4\rho \ - \ 2\|Pw\|^2 \ . \tag{5.22}$$

To estimate the second term in (5.21), we use the decomposition $\mathrm{Id} = P + P^\perp$ and a straightforward computation yields

$$\begin{aligned} 3N \sum_{k=1}^{N} x_k^2 w_k^2 \ &\geq \ 3N \sum_{k=1}^{N} \left(\overline{x}^2 + 2\overline{x}(P^\perp x)_k + (P^\perp x)_k^2\right)\left(\overline{w}^2 + 2\overline{w}(P^\perp w)_k\right) \\ &\geq \ 3\|Px\|^2\|Pw\|^2 - 12\|Px\|\|P^\perp x\| - 6\sqrt{N}\|P^\perp x\|^2 \,. \end{aligned} \tag{5.23}$$

Note that by the triangular inequality, we have the two uniform bounds for every $r > 0$,

$$\forall N \in \mathbb{N}, \ \forall\, x \in \Omega_r \ , \qquad 1 - r \ \leq \ \|Px\| \ \leq \ 1 + r \ .$$

Thus, estimate (5.23) gives for every $r > 0$, $N \in \mathbb{N}$ and $x \in \Omega_r$,

$$3N \sum_{k=1}^{N} x_k^2 w_k^2 \ \geq \ 3(1-r)^2\|Pw\|^2 \ - \ 12(1+r)rN^{-\frac{1}{4}} \ - \ 6r^2 \ .$$

Taking $r > 0$ sufficiently small we get $3N\sum_k x_k^2 w_k^2 \geq 2\|Pw\|^2 - 2\rho$, which together with (5.21) and (5.22) finishes the proof.

∎

The preceding Lemma 5.5 is used to establish the localized spectral gap estimate of Proposition 5.7 below. As in the proof of Lemma 5.3, we argue by means of standard results for strictly log-concave measures. To reduce to the standard situation, we use the following general result on convex extensions, for which we provide a proof for the sake of completeness.

Lemma 5.6. *Fix $d \in \mathbb{N}$. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and A be a compact and convex subset of $\mathbb{R}^d$ such that*

$$\exists\, \varepsilon > 0 \ , \ \exists\, C > 0 \quad s.t. \quad \mathrm{Hess}\,\varphi \ \geq \ C \ \text{on} \ A_\varepsilon := \{x + y \ ; \ x \in A \, , \ \|y\| \leq \varepsilon\} \ .$$

Then there exists $\tilde{\varphi} \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\forall x \in A \ , \quad \tilde{\varphi}(x) = \varphi(x) \quad \text{and} \quad \forall x \in \mathbb{R}^d \ , \quad \mathrm{Hess}\,\tilde{\varphi}(x) \ \geq \ C \ .$$

Proof. The proof consists in smoothly cutting φ outside A and adding a function g vanishing on A and sufficiently convex outside A. To easily construct such a function g it is convenient to reduce to radial cut-off's as follows (see [Yan]): first, since A is convex and compact, $A_\varepsilon \setminus \mathring{A}_{\frac{\varepsilon}{2}}$ is compact and there exist $\ell \in \mathbb{N}$, $(x_i)_{i\in\{1,\dots,\ell\}} \subset (\mathbb{R}^N)^\ell$ and $(r_i)_{i\in\{1,\dots,\ell\}} \subset (0,\infty)^\ell$ such that, denoting by $\overline{B}(x_i, r_i)$ the closed ball of radius r_i centered at x_i,

$$A \ \subset \ \cap_{i=1}^{\ell} \overline{B}(x_i, r_i) \ \subset \ \mathring{A}_{\frac{\varepsilon}{2}} \ \subset \ A_\varepsilon \ . \tag{5.24}$$

We shall consider $\theta \in C^\infty(\mathbb{R})$ defined by

$$\theta(t) := \begin{cases} 0 & \text{if} \quad t \in (-\infty, 1] \\ t^2 e^{-\frac{1}{t-1}} & \text{if} \quad t \in (1, +\infty) \, . \end{cases}$$

This function is strictly increasing on $(1, +\infty)$, as well as $t \mapsto \frac{\theta'(t)}{t}$, since we have for any $t > 1$,

$$\theta'(t) \ = \ \left(2t + \frac{t^2}{(t-1)^2}\right) e^{-\frac{1}{t-1}} \quad \text{and} \quad \left(\frac{\theta'(t)}{t}\right)' \ = \ \frac{t^2 - 3t + 3}{(t-1)^4} e^{-\frac{1}{t-1}} \, .$$

Moreover notice that $\mathrm{Hess}\,\big(x \mapsto \theta(\|x\|)\big)$ is given by

$$\mathrm{Hess}\,\big(x \mapsto \theta(\|x\|)\big) \ = \ \frac{\theta'(\|x\|)}{\|x\|} \, Id + \frac{1}{\|x\|} \left(\frac{\theta'(t)}{t}\right)'_{t=\|x\|} \big(x_i x_j\big)_{1 \leq i,j \leq N} \, ,$$

and so is positive definite for $\|x\| > 1$ (and zero for $\|x\| \leq 1$). We then define

$$g(x) \ := \ \sum_{i=1}^{\ell} \theta\left(\frac{\|x - x_i\|}{r_i}\right) \, .$$

Note that g is smooth, $\mathrm{Hess}\, g \geq 0$ and that, according to (5.24), $g \equiv 0$ on A and $\mathrm{Hess}\, g > 0$ on the complementary of $\mathring{A}_{\frac{\varepsilon}{2}}$. Finally we define the following extension of $\varphi|_A$,

$$\tilde{\varphi} \;:=\; \chi\varphi \;+\; \alpha g \;,$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfies $\chi \equiv 1$ on $A_{\frac{\varepsilon}{2}}$ and $\mathrm{supp}\,\chi \subset A_\varepsilon$, and $\alpha > 0$ is chosen large enough so that $\mathrm{Hess}\,\tilde{\varphi} \geq C$. This is indeed possible since $\mathrm{Hess}\,\tilde{\varphi} \geq \mathrm{Hess}\,\varphi \geq C$ on $\mathring{A}_{\frac{\varepsilon}{2}}$, $\mathrm{Hess}\,\tilde{\varphi} = \alpha\,\mathrm{Hess}\, g$ on $\mathbb{R}^N \setminus A_\varepsilon$, $\mathrm{Hess}\,\tilde{\varphi} = \alpha\,\mathrm{Hess}\, g + \mathrm{Hess}\,(\chi f)$ on $A_\varepsilon \setminus \mathring{A}_{\frac{\varepsilon}{2}}$ and $\min\{\mathrm{Hess}\, g(x)\,,\; x \in \mathbb{R}^N \setminus \mathring{A}_{\frac{\varepsilon}{2}}\} > 0$. ∎

As a corollary of Lemmata 5.5 and 5.6, the following spectral gap estimate for a suitably localized problem around the minimum J_+ holds true. Note that the analogous version around the other minimum J_- holds true by symmetry.

Proposition 5.7. *Let $r > 0$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that*

$$\mathrm{supp}\,\varphi \;\subset\; \Omega_r := \left\{\, x \in \mathbb{R}^N \;:\; \|Px - J_+\| \;\leq\; r\;,\; \|P^\perp x\| \;\leq\; rN^{-\frac{1}{4}} \,\right\}. \quad (5.25)$$

If r is suffiently small, then there exists a constant $\rho > 0$ such that for all $h > 0$ and $N \in \mathbb{N}$, there exists $\mathcal{E}_h^+ \in L^2(\mathbb{R}^N)$ such that

$$\langle\, \varphi,\, \Delta_{f,h}\varphi \,\rangle_{L^2(\mathbb{R}^N)} \;\geq\; \rho\, h\, \Big(\; \|\varphi\|^2_{L^2(\mathbb{R}^N)} \;-\; \langle \varphi, \mathcal{E}_h^+ \rangle^2_{L^2(\mathbb{R}^N)} \Big)\;. \quad (5.26)$$

Proof. According to Lemma 5.5, by taking $r > 0$ sufficiently small there exists $\rho > 0$ such that

$$\forall N \in \mathbb{N}\;,\; \forall x \in \Omega_{2r}\;, \quad \mathrm{Hess}\, f(x) \;\geq\; \rho \;\; .$$

By Lemma 5.6, there exists for each $N \in \mathbb{N}$ a function $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\tilde{f}|_{\Omega_r} \equiv f|_{\Omega_r}$ and

$$\forall N \in \mathbb{N}\;,\; \forall x \in \mathbb{R}^N\;, \quad \mathrm{Hess}\, \tilde{f}(x) \;\geq\; \rho \;\; . \quad (5.27)$$

As in the proof of Lemma 5.3, by standard results for the spectral gap of strictly log-concave measures (see for example [Led, Corollary 11.4]), Property (5.27) implies (5.26), with the differential operator $\Delta_{\tilde{f},h}$ instead of $\Delta_{f,h}$ and with

$$\mathcal{E}_h^+(x) \;:=\; \frac{e^{-\frac{\tilde{f}(x)}{h}}}{\int_{\mathbb{R}^N} e^{-2\frac{\tilde{f}(x)}{h}}\; dx}\;.$$

Noting that $\langle\, \varphi,\, \Delta_{\tilde{f},h}\varphi \,\rangle_{L^2(\mathbb{R}^N)} \;=\; \langle\, \varphi,\, \Delta_{f,h}\varphi \,\rangle_{L^2(\mathbb{R}^N)}$ for any smooth φ with support in Ω_r, which follows from $\tilde{f}|_{\Omega_r} \equiv f|_{\Omega_r}$, finishes the proof.

∎

5.2 Proof of Theorem 5.1

We fix from the outset $C > 0$ and $\alpha \in (0, \frac{3}{4})$. As we already mentioned, it is sufficient to prove (5.2). For this we introduce as follows a quadratic partition of unity $\{\eta_k\}$ depending on the given α and on a parameter $r > 0$, independent of N and h, which will be chosen sufficiently small so that the estimates required for the proof hold true.
We start with $\theta := \theta_r \in \mathcal{C}_c^\infty(\mathbb{R}; [0,1])$ such that $\theta(x) = \theta(-x)$, $\theta \equiv 1$ in $[-r, r]$ and $\theta \equiv 0$ in $[2r, +\infty)$ and define $\kappa_{\min}, \kappa_0, \kappa_\infty : \mathbb{R}^N \to [0,1]$ by setting

$$\kappa_{\min}(x) \;:=\; \theta(\|Px - J_+\|) \;+\; \theta(\|Px - J_-\|)\;, \tag{5.28}$$

$$\kappa_0 \;:=\; \left(1 - \kappa_{\min}^2\right)^{\frac{1}{2}} \mathbf{1}_{\{\|Px\| \le 1\}} \quad , \quad \kappa_\infty := \left(1 - \kappa_{\min}^2\right)^{\frac{1}{2}} \mathbf{1}_{\{\|Px\| \ge 1\}}\;. \tag{5.29}$$

Moreover we define for $p \in \{4, 6\}$ the functions $\chi_{0,p}, \chi_{\infty,p} : \mathbb{R}^N \to [0,1]$ as

$$\chi_{0,p}(x) := \theta(h^{-\alpha/p}\|P^\perp x\|) \quad \text{and} \quad \chi_{\infty,p} := \left(1 - \chi_{0,p}^2\right)^{\frac{1}{2}}\;. \tag{5.30}$$

Note that the χ's depend on h, while the κ's do not. Note also that

$$\left(\,\kappa_{\min}^2 \;+\; \kappa_0^2\,\right)\left(\,\chi_{0,4}^2 \;+\; \chi_{\infty,4}^2\,\right) \;+\; \kappa_\infty^2\left(\,\chi_{0,6}^2 \;+\; \chi_{\infty,6}^2\,\right) \;\equiv\; 1\;. \tag{5.31}$$

We shall consider in the sequel the partition of unity $\{\eta_1, \dots, \eta_6\}$, where η_k^2 is given by one of the six products $\kappa_j^2 \chi_{j'}^2$ appearing when multiplying out the left hand side of (5.31). Observe that for each $k \in \{1, \dots, 6\}$,

$$\forall N \in \mathbb{N}\,,\; \forall x \in \mathbb{R}^N\,, \quad h^2|\nabla \eta_k(x)|^2 \;\lesssim\; h^{2-\frac{\alpha}{2}}\;.$$

Thus the IMS localization formula of Proposition 5.2 implies that for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and h sufficiently small,

$$\langle \psi, \Delta_{f,h}\psi \rangle_{L^2(\mathbb{R}^N)} \;+\; h^{2-\frac{\alpha}{2}} \;\gtrsim\; \sum_{k=1}^{6} \langle\, \eta_k\psi,\, \Delta_{f,h}(\eta_k\psi)\,\rangle_{L^2(\mathbb{R}^N)}\;. \tag{5.32}$$

Here and in the sequel we shall use for short the notation $\gtrsim$ and $\lesssim$ to denote inequalities which hold true up to multiplication of (say) the right hand side by a positive constant which is independent of h and N.

In the rest of the proof, we fix a $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and discuss separately the addends on the right hand side of (5.32).

Analysis around the diagonal

a) Analysis on supp $\big(\kappa_{\min}\,\chi_{0,4}\big)$*:* According to the definitions given in (5.28) and (5.30), we have for every $h>0$ and $N\in\mathbb{N}$ such that $N\leq Ch^{-\alpha}$,

$$\mathrm{supp}(\,\psi\,\kappa_{\min}\,\chi_{0,4}\,)\ \subset\ \Omega_{+,r}\ \cup\ \Omega_{-,r}\ ,$$

$$\text{where}\quad \Omega_{\pm,r}\ :=\ \big\{x\in\mathbb{R}^N:\|Px\pm J_+\|\ \leq\ 2r\ ,\ \|P^{\perp}x\|\leq 2rC^{\frac14}N^{-\frac14}\ \big\}\ .$$

Then it follows from Proposition 5.7 (and its analogous version around J_-) that, chosing r sufficiently small, for all $h>0$ and $N\leq Ch^{-\alpha}$ there exist $\mathcal{E}_h^+,\mathcal{E}_h^-\in L^2(\mathbb{R}^N)$ such that, denoting for short $\varphi:=\kappa_{\min}\chi_{0,4}\,\psi$,

$$\langle\,\varphi,\,\Delta_{f,h}\,\varphi\,\rangle_{L^2(\mathbb{R}^N)}\gtrsim h\left(\|\varphi\|^2_{L^2(\mathbb{R}^N)}-\langle\varphi,\mathcal{E}_h^+\rangle^2_{L^2(\mathbb{R}^N)}-\langle\varphi,\mathcal{E}_h^-\rangle^2_{L^2(\mathbb{R}^N)}\right).\quad(5.33)$$

b) Analysis on supp $\big(\kappa_0^2\,\chi_{0,4}^2+\kappa_\infty^2\,\chi_{0,6}^2\big)$*:* Here we shall use that, in the sense of quadratic forms,

$$\Delta_{f,h}\ \geq\ \Delta^{\mathcal{C}}_{f,h}\ \geq\ |\nabla^{\mathcal{C}}f|^2\ -\ h\,\Delta^{\mathcal{C}}f\ .\quad(5.34)$$

The final estimate (5.40) given below follows then by elementary inequalities which we spell out for completeness. Note first that the definitions (5.29) and (5.30) imply in particular that for every $h>0$ and $N\in\mathbb{N}$,

$$\mathrm{supp}\,\big(\kappa_0^2\,\chi_{0,4}^2+\kappa_\infty^2\,\chi_{0,6}^2\big)\ \subset\ \Omega_0\ \cup\ \Omega\ ,\quad(5.35)$$

where Ω and Ω_0 are defined as

$$\Omega\ :=\ \big\{x\in\mathbb{R}^N:\|Px-J_\pm\|\geq r\ ,\ \|Px\|\geq r\ ,\ \|P^{\perp}x\|\leq 2rh^{\frac{\alpha}{6}}\ \big\}\ ,$$

and

$$\Omega_0\ :=\ \big\{x\in\mathbb{R}^N:\|Px\|\leq r\ ,\ \|P^{\perp}x\|\leq 2rh^{\frac{\alpha}{6}}\ \big\}\ .$$

On Ω_0 one can immediately give a lower bound for the right hand side in (5.34). Indeed, chosing r sufficiently small, we have from (5.6) for every $x\in\Omega_0$, $h\in(0,1]$, and $N\in\mathbb{N}$,

$$|\nabla^{\mathcal{C}}f|^2\ -\ h\,\Delta^{\mathcal{C}}f\ \geq\ -\,h\,\Delta^{\mathcal{C}}f\ =\ \frac{h}{2}\,(1-3\hat{x}_0^2-3\|P^{\perp}x\|^2)\ \geq\ \frac{h}{4}\ .\quad(5.36)$$

To deal with Ω, we develop the expression of $\nabla^{\mathcal{C}}f(x)$ given in (5.5),

$$\nabla^{\mathcal{C}}f(x)\ =\ \frac12\hat{x}_0^3\ -\ \frac12\hat{x}_0\ +\ \frac32\hat{x}_0\|P^{\perp}x\|^2\ +\ \frac12\sqrt{N}\sum_{k=1}^{N}(P^{\perp}x)_k^3\ ,$$

from which we get, using that for all $h > 0$ we have $\|P^{\perp}x\| \leq 2rh^{\frac{\alpha}{6}}$,

$$\forall N \leq Ch^{-\alpha}\ , \quad |\sqrt{N}\sum_{k=1}^{N}(P^{\perp}x)_k^3| \ \leq \ 8\sqrt{C}r^3\ .$$

Then, we obtain for sufficiently small h the lower bound

$$|\nabla^{\mathcal{C}}f(x)| \ \geq \ \left|\frac{1}{2}\hat{x}_0^3 \ - \ \frac{1}{2}\hat{x}_0 \ + \ \frac{3}{2}\hat{x}_0\|P^{\perp}x\|^2\right| \ - \ 4\sqrt{C}r^3\ , \tag{5.37}$$

from where it follows, choosing r sufficiently small, that

$$\begin{cases} |\nabla^{\mathcal{C}}f(x)| \gtrsim 1 & \text{for } x \in \Omega \ \text{ s.t. } \ r \leq \|Px\| \leq 1-r \\ |\nabla^{\mathcal{C}}f(x)| \gtrsim \|Px\|^3 & \text{for } x \in \Omega \ \text{ s.t. } \ 1+r \leq \|Px\|\ . \end{cases} \tag{5.38}$$

Combining (5.38) with the estimate

$$\forall\, x \, \in \, \Omega\,,\ h\,|\Delta^{\mathcal{C}}f(x)| \ = \ \frac{h}{2}\,\big|\,1-3\|Px\|^2-3\|P^{\perp}x\|^2\,\big| \ \leq \ \frac{h}{2}\max\{1\,,\,3\|Px\|^2\}$$

valid for h sufficiently small, we finally get the existence of $h_0 > 0$ such that

$$\forall x \in \Omega\ , \forall h \in (0,h_0]\ , \forall N \leq Ch^{-\alpha}\ , \qquad |\nabla^{\mathcal{C}}f|^2 \ - \ h\,\Delta^{\mathcal{C}}f \ \gtrsim \ 1\ . \tag{5.39}$$

Summing up this part, setting for short $\varphi := (\,\kappa_0^2\,\chi_{0,4}^2+\kappa_\infty^2\,\chi_{0,6}^2\,)^{\frac{1}{2}}\,\psi$, it follows from (5.34)–(5.36) and (5.39) that there exist $h_0 > 0$ and r sufficiently small such that, for every $h \in (0,h_0]$ and every $N \in \mathbb{N}$ satisfying $N \leq Ch^{-\alpha}$,

$$\langle\, \varphi,\, \Delta_{f,h}\,\varphi\,\rangle_{L^2(\mathbb{R}^N)} \ \gtrsim \ \|\varphi\|^2_{L^2(\mathbb{R}^N)}\ . \tag{5.40}$$

Analysis away from the diagonal

Here it is convenient to work with $\Delta^{\mathcal{C}^{\perp}}_{f,h}$, which is sufficient due to the inequality $\Delta_{f,h} \geq \Delta^{\mathcal{C}^{\perp}}_{f,h}$.

a) Analysis on $\operatorname{supp}\big((\kappa_{\min}^2+\kappa_0^2)\,\chi_{\infty,4}\big)$: Let for short $\varphi := (\kappa_{\min}^2+\kappa_0^2)^{\frac{1}{2}}\chi_{\infty,4}\,\psi$. Note that by the definitions (5.28), (5.29), and (5.30), we have

$$\operatorname{supp}\varphi \ \subset \ \{x \in \mathbb{R}^N : \|Px\| \leq 1+2r\ ,\ \|P^{\perp}x\| \geq rh^{\frac{\alpha}{4}}\}\ . \tag{5.41}$$

It follows from the Poincaré inequality (5.10), the concentration estimate (5.11) of Lemma 5.4 and (5.41) that there exists a constant $\gamma > 0$ such that

$$\langle\, \varphi,\, \Delta_{f,h}\varphi\,\rangle_{L^2(\mathbb{R}^N)} \ \geq \ h\ (\mu-1)\ \|\varphi\|^2_{L^2(\mathbb{R}^N)}\ \Big(1-\gamma e^{-\frac{r^2h^{\frac{\alpha}{2}}}{\gamma h}}\Big)\ . \tag{5.42}$$

Since $0 < \alpha < \frac{3}{4}$, estimate (5.42) implies that there exists $h_0 > 0$ such that

$$\forall h \in (0, h_0]\,,\ \forall N \in \mathbb{N}\,, \quad \langle\, \varphi,\, \Delta_{f,h}\varphi\,\rangle_{L^2(\mathbb{R}^N)} \ \geq\ h\, \frac{(\mu-1)}{2}\, \|\varphi\|^2_{L^2(\mathbb{R}^N)}\,. \quad (5.43)$$

b) Analysis on $\operatorname{supp}\big(\kappa_\infty\, \chi_{\infty,6}\big)$*:* Let $\varphi := \kappa_\infty\, \chi_{\infty,6}\, \psi$. By the definitions (5.29) and (5.30), we have

$$\operatorname{supp}\varphi \ \subset\ \{x \in \mathbb{R}^N :\ \|P^\perp x\| \geq r h^{\frac{\alpha}{6}}\}\ . \quad (5.44)$$

As for the point a) above, we use the Poincaré inequality (5.10) but we can only use here the concentration estimate (5.12) of Lemma 5.4 since $\operatorname{supp}\psi$ is arbitrary. This leads to the existence of a constant $\gamma > 0$ such that

$$\langle\, \varphi,\, \Delta_{f,h}\varphi\,\rangle_{L^2(\mathbb{R}^N)} \ \geq\ h(\mu-1)\ \|\varphi\|^2_{L^2(\mathbb{R}^N)} \left(1 - e^{\gamma N - \frac{r^2 h^{\frac{\alpha}{3}}}{\gamma h}}\right)\,. \quad (5.45)$$

Since $0 < \alpha < \frac{3}{4}$, estimate (5.45) implies that there exists $h_0 > 0$ such that

$$\forall h \in (0, h_0]\,,\ \forall N \leq C h^{-\alpha}\,,\ \langle\, \varphi,\, \Delta_{f,h}\varphi\,\rangle_{L^2(\mathbb{R}^N)} \ \geq\ h\frac{(\mu-1)}{2}\|\varphi\|^2_{L^2(\mathbb{R}^N)}. \quad (5.46)$$

End of the proof

Chosing the parameter $r > 0$ of the partition of unity $\{\eta_k\}$ sufficiently small and putting together (5.32), (5.33), (5.40), (5.43), and (5.46), we obtain estimate (5.2) with $E_\pm := \mathcal{E}_h^\pm\, \kappa_{\min}\, \chi_{0,4}$.

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