THROUGH AN INFERENCE RULE, DARKLY

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Abstract. Mathematical logic provides a formal language to describe complex abstract phenomena whereby a finite formula written in a finite alphabet states a property of an object that may even be infinite. Thus, the complexity of the underlying objects is abstracted away to give way for a simple syntactic description, a kind of *mathesis universalis*. The complexity, however, continues affecting which ways of reasoning are valid.

Structural proof theory reasons using proof objects more complex than individual formulas. One of its goals is to find minimal additional structures, depending on the complexity of the underlying objects, sufficient for efficient and modular reasoning about them.

In this paper, we are primarily interested in both the global structure used for reasoning and the local part of this structure employed to justify single inference steps. We provide recently developed examples where adapting the global structure of the sequent to the local structure of (potentially infinite) Kripke models yielded both quantitative and qualitative benefits in establishing fundamental logical properties such as complexity and interpolation.

1. Preface

This is an unusual paper for me to write. It will probably be an unusual paper to read. Thus, I start by undertaking to explain the genesis of this paper and preempt the expectations one would usually have of a paper written by a mathematical logician working in structural proof theory. This is not a technical paper. There will be no teeth-grinding proofs of syntactic cut elimination with multi-layered induction. There will be few (if any) new formalisms or new results about existing formalisms, though plenty of very fresh results will be discussed. This is a survey paper, a taxonomy paper, a stop-and-smell-the-proof-theoretical-roses paper.

Proof-theorists are very good at crunching formulas and, especially in recent years, have been producing a steady stream of impressive and increasingly general results. Arguably, the very benchmark of what should be considered a respectable result in the field is moving more and more from providing proof systems for individual logics towards general automated methods for designing analytic calculi. As Kazushige Terui often explains, if there are infinitely many logics of a certain type, one could engage in writing infinitely many papers about cut elimination for each of them, but he chooses to develop one algorithm to treat them all instead.

It is interesting to examine the general attitude that seems to underlie and fuel much of recent (and not so recent) work. When proof theorists discuss some famous logic, e.g., $\mathsf{S5}$ or $\mathsf{K5}$, that still lacks a cut-free sequent system, they rarely frame

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it as a limitation of the proof-theoretic method. The belief is that all reasonable[1] logics *should* possess simple analytic[2] representations.

This belief in proof theory being a(n almost) universal tool for scrutinizing logics is of the same ilk as the *mathesis universalis* thesis. It is also usually ingrained on such a deep level that a working proof theorist, especially a young and ambitious one, would rarely stop to acknowledge that the cornerstone of their worldview is, so to say, *apodicticus universalis*, the universality of proving—certainly not when there are still so many things to actually prove.

I was granted a chance to ponder this question thanks to the wonderful Humboldt-Kolleg on *Proof Theory as Mathesis Universalis* organized by Stefania Centrone, Sara Negri, Denis Sarikaya, and Peter Schuster on the picturesque banks of Lake Como. With the announced topic and an almost equal mixture of proof theorists and philosophers, with some historians of science thrown in for good measure, one had no choice but to start talking the talk instead of walking the walk or, in this case, thinking the thought instead of proving the proof.

What follows is a free-form essay describing these thoughts, as seen through the prism of recent results obtained by the author in several collaborations, notably with Björn Lellmann. Each of the papers cited provides technical details and local motivation, local in the sense that it motivates this one paper. Thus, here, in thinking about the general impetus behind all these results, I purport to explain the big picture created by these seemingly patchwork results.

I have no doubt that these musings would exhibit some level of dilettantish philosophizing. Indeed, I am no trained philosopher. But I strongly believe the exchange of ideas and worldviews facilitated by Stefania, Sara, Denis, and Peter to be of mutual value to both communities. In this paper, I humbly present my attempts to see green pastures and forests of semantics through proof-theoretic trees.

2. Sequents Provide a Window into a Model

The question I started asking myself is: why do my colleagues, me included, believe so strongly that proof theory will overcome all barriers in the end? While Gentzen's sequents [22, 23][3] was a visionary and elegant invention, facilitating greatly the study of two logics of paramount contemporaneous interest—classical and intuitionistic—could one expect at the time that sequents will become a *de facto* standard in efficient logic representation? Perhaps, part of the success was thanks to the idea that the meaning of a formula must be fully determined by its parts. The *analyticity* condition demands that nothing but the parts of a given formula be used in its analysis. Indeed, a typical sequent-style proof reduces every valid statement, however complex, to several self-evident statements $p \Rightarrow p$ about *atoms* p. It is well-known that this deconstruction into atoms, which, incidentally, is not

[1]Needless to say, this is not a cultist dogma claiming all logics to have analytic descriptions. Exceptions are made for degenerate logics, for monstrosities artificially created to defy proof-theoretic analysis, and for logics involving objectively complex notions, e.g., fixpoints (though the latter are still pursued and not entirely without success (cf., e.g., a recent survey of proof systems for common knowledge [42]). But an idea that adding one axiom may rob a logic of its cut-free representation is often met with disbelief and resistance.

[2]Which is almost inevitably interpreted to be cut-free.

[3]Bibliographic note: These two articles constituted Gentzen's dissertation and were originally published in separate *Hefte* (issues) of *Mathematische Zeitschrift* in German. Their translation into English, combined into one article as originally intended, can be found in [24].

entirely unlike *atomism*, is the reason analytic sequent calculi typically suffice to prove decidability. But it is also a paradigm otherwise fitting the way modern science dissects various phenomena. After all, we all agree that matter consists of atoms, light consists of photons, and even pressure can be broken into the force of individual gas molecules hitting the wall of a container.

It is, perhaps, little wonder then that a logician trained as a mathematician rather than philosopher would generally prefer extensional logics where the total is completely determined by the parts. Yes, the axiomatic method of deriving theorems from postulates by *modus ponens* (or *modus tolens* and similar inference methods) provides an alternative, equally pertinent description for logics. But it is the do-it-yourself LEGO simplicity of sequents that occupies minds and captures hearts.

The main difference is that the axiomatic method works with unstructured formulas, only making an exception for implication (to apply modus ponens, A is matched to a subformula of $A \to B$), whereas Gentzen imbued the basic objects of inquiry with *structure*, hence, the term *structural proof theory*. He made the main Boolean operations—conjunction and disjunction—structural. In other words, the basic object of reasoning was not one formula but finitely many formulas connected by conjunctions, disjunctions, and one implication, the object

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \ ,$$

collectively called a *sequent*, interpreted via an *interpretation function* ι:

$$\iota(A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m) \quad = \quad A_1 \wedge \cdots \wedge A_n \to B_1 \vee \cdots \vee B_m \ .$$

Here $A_1, \ldots, A_n$ form the *antecedent* and $B_1, \ldots, B_m$ the *succedent* of a sequent. With the power of hindsight, knowing that classical logic can be equally well represented by one-sided sequents, using the formula interpretation

$$\iota(\overline{A_1}, \ldots, \overline{A_n}, B_1, \ldots, B_m) \quad = \quad \neg A_1 \vee \cdots \vee \neg A_n \vee B_1, \ldots, \vee B_m \ ,$$

one could even argue that it is sufficient to make structural only two primary connectives $\vee$ and $\neg$.

All[4] systems discussed in this paper are based on the same semantic intuition, on the much celebrated isomorphism between sequents and tableaus popularized by Raymond Smullyan and Melvin Fitting. The classical propositional case is the most transparent one: the sequent

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \qquad (\text{equivalently, } \overline{A_1}, \ldots, \overline{A_n}, B_1, \ldots, B_m)$$

is valid iff a tableau beginning with

$$TA_1, \ldots, TA_n, FB_1, \ldots, FB_m$$

can be closed, i.e., these particular truth values—T and F standing for "true" and "false" respectively—for the formulas lead to a contradiction. Under this view, sequent rules lose their mysterious veneer and become simple truth-table automata. For instance, the sequent rule

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$$

[4]Exceptions such as Gentzen's sequents for intuitionistic logic only prove the rule.

corresponds to the tableau rule

$$\frac{T A \vee B}{T A \mid T B}$$

and states no more and no less than the truth condition for the disjunction:

$$A \vee B \text{ is true} \qquad \text{iff} \qquad \text{either } A \text{ or } B \text{ is true.}$$

Disclaimer. Not all proof theorists support this correspondence. There is a strong movement for expelling all signs and mentions of semantics from proof-theoretic practice. In particular, proof-theoretic formalisms are divided into *external* and *internal* ones: the latter must possess a formula interpretation ι that gives proof objects a meaning without resorting to any kind of models. Internal proof calculi never say anything outside of the object language. This restriction on the expressivity of proof objects is considered a desired property. While this is a respectable position, the purpose of this paper is exactly the opposite. Our goal is to ensure that semantics provides an inspiration for sequent-like formalisms we discuss. Ideally, sequent rules can be read off the definition of truth in models. For this purpose, throughout the paper, we will use *Kripke models*, the main tenet of which is the ability to evaluate the same formula differently at different possible worlds.

Let me illustrate such a process of sequent design using intuitionistic propositional logic as an example. Using the standard propositional language with primary connectives $\bot$, $\wedge$, $\vee$, and $\to$ and a countably infinite set Prop of *propositional atoms*, we recall the definition of intuitionistic Kripke semantics.

Definition 1 (Kripke frames and models)**.** An *intuitionistic Kripke frame*, or *i-frame* for short, is a pair $(W, \leq)$ of a set $W \neq \varnothing$ of *worlds* and preorder $\leq$ on W.[5] If $u \leq v$, we say that the world v is a *future* of the world u.

An *intuitionistic Kripke model*, or *i-model* for short, is a triple $(W, \leq, V)$ where $(W, \leq)$ is an i-frame and $V \colon W \to 2^{\mathrm{Prop}}$ is a *valuation* that is monotone with respect to $\leq$, i.e., $w \leq u$ implies $V(w) \subseteq V(u)$.

Definition 2 (Truth in i-models)**.** For an i-model $\mathcal{M} = (W, \leq, V)$, the *forcing relation* $\Vdash$ for $w \in W$ is defined by

- $\mathcal{M}, w \Vdash p \quad$ iff $\quad p \in V(w) \qquad$ for each $p \in \mathrm{Prop}$;
- $\mathcal{M}, w \nVdash \bot$;
- $\mathcal{M}, w \Vdash A \wedge B \quad$ iff $\quad \mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
- $\mathcal{M}, w \Vdash A \vee B \quad$ iff $\quad \mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
- $\mathcal{M}, w \Vdash A \to B \quad$ iff $\quad$ for all $v \geq w$, we have $\mathcal{M}, v \nVdash A$ or $\mathcal{M}, v \Vdash B$.

A formula A is *valid in* $\mathcal{M}$, written $\mathcal{M} \Vdash A$, if $(\forall w \in W)\, \mathcal{M}, w \Vdash A$. A formula A is *valid in a class* $\mathcal{C}$ *of i-models*, written $\mathcal{C} \Vdash A$, if $(\forall \mathcal{M} \in \mathcal{C})\, \mathcal{M} \Vdash A$.

It is well-known that monotonicity explicitly postulated for propositional atoms extends to all formulas:

Lemma 3 (Monotonicity [47])**.** *$\mathcal{M}, w \Vdash A$ implies $\mathcal{M}, v \Vdash A$ whenever $w \leq v$ for any i-model $\mathcal{M} = (W, \leq, V)$.*

Theorem 4 (Completeness [47])**.** *Intuitionistic logic* Int *is sound and complete with respect to the class of all i-models.*

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[5] A preorder is a reflexive and transitive binary relation.

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge\Rightarrow \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \Rightarrow\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee\Rightarrow \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \Rightarrow\vee_c$$

Figure 1. Propositional rules for conjunction and disjunction

Remark. Strictly speaking, the way presented in this paper, the above theorem should have been a definition of Int since we provide no alternative description of the logic. But these descriptions are well-known and predate Kripke semantics.

Reading sequent rules from the semantics given in Definition 2 in a tableau way, it is clear that the sequent rules for conjunction and disjunction should be the same as in the classical case (Figure 1), e.g., for conjunction:

- notwithstanding other true (Γ) and false (Δ) formulas, if $A \wedge B$ is true (in the antecedent of the rule's conclusion), then both A and B must be true (in the antecedent of the rule's premise);
- notwithstanding other true (Γ) and false (Δ) formulas, if $A \wedge B$ is false (in the succedent of the rule's conclusion), then either A must be false (in the succedent of the left premise) or B must be false (in the succedent of the right premise);

Note that these semantic conditions are, in fact, bidirectional, e.g., both A and B being true suffices to conclude that $A \wedge B$ is true. Proof-theoretically, this means that all four rules should be *invertible*, i.e., if the conclusion is a derivable sequent, so should be each of the premises. Invertibility of rules is important for proof search because invertible rules can be applied greedily, without the need for backtracking. In the standard sequent calculus for classical propositional logic, all rules are invertible because truth tables can be fully encoded by sequent rules, which must be of the form

The rules for propositional atoms and $\bot$ are standard and state that $\bot$ is never true and p cannot be both true and false (recall that reading a sequent proof in a tableau way means starting with the endsequent and arriving at contradictions in the leaves) of the form

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\,\mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta} \bot\Rightarrow \tag{1}$$

Let us now generate the rules for implication. The invertible rules for classical propositional logic are

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta} \to\Rightarrow_c \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \Rightarrow\to_c \tag{2}$$

The rule ($\to\Rightarrow_c$) is still sound intuitionistically: if $A \to B$ is true (at a world w in an i-model $\mathcal{M}$), then either A is false or B is true at w. But the converse does not hold because an implication also affects all futures of w. To avoid the loss of information in this rule, ensuring both the completeness of the system and invertibility of the rule, $A \to B$ should be retained in the left premise (it would have been redundant in the right premise because the truth of B intuitionistically entails the truth of $A \to B$), as in the rule ($\to\Rightarrow_{im}$):

$$\frac{\Gamma, A \to B \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\,\to\Rightarrow_{im} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}\,\Rightarrow\to_{im} \tag{3}$$

The succedent implication rule ($\Rightarrow\to_{im}$), on the other hand, introduces us to a new phenomenon. The falsity of $A \to B$ at a world w need not have any effect on w itself. It must be witnessed by some future world $v \geq w$ where A is true and B is false. This yields the premise $A \Rightarrow B$, however, the monotonicity property (Lemma 3) enables us to add all true formulas from Γ to the premise antecedent, resulting in the rule ($\Rightarrow\to_{im}$) from (3). Note that false formulas from Δ do not survive the transition from w to v, making this rule inherently non-invertible.

Thus, (1), (3), and Figure 1 form a standard cut-free sequent calculus for Int.

Theorem 5 (Completeness)**.** *A sequent $\Gamma \Rightarrow \Delta$ can be derived using rules from* (1)*,* (3)*, and Figure 1 iff* $\mathsf{Int} \vdash \iota(\Gamma \Rightarrow \Delta)$*.*

This standard calculus is, however, not the initial intuitionistic calculus by Gentzen [22]. Instead, this is the multi-conclusion calculus by Maehara [38], who is also responsible for the first proof of the above completeness theorem. It is worth noting that it matches what Fitting calls semantic Beth tableaus [15] the same way classical sequents match tableaus.[6]

If classical sequents can be viewed as a microscope into a model with inference rules serving as scalpels dissecting the formula into ever smaller pieces until indivisible atoms are reached, then intuitionistic sequents are more like a probe sent down into the model. While some inference rules still act like a scalpel, there is a new type of rules that ups and moves the whole probe to a new site, while retaining some of the already collected samples.

Remark. I make no claim that this semantic analysis guarantees a (cut-free) complete system, i.e., a system complete *without* the rule

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Pi, A \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{cut} \tag{4}$$

Cut elimination still needs to be demonstrated. But this yields a robust tool for both designing and verifying potential rules.

This leaves the mystery of how Gentzen's original calculus [22] relates to the underlying semantics. Unlike classical and Maehara's intuitionistic sequents, Gentzen used single-conclusion sequents, meaning that Δ in $\Gamma \Rightarrow \Delta$ contains no more than one formula. The motivation for that comes from constructive thinking purporting that having more than one formula in the succedent amounts to a non-deterministic choice and should be disallowed. While this idea works spectacularly well in both determining appropriate modifications for all classical rules and in the resulting system being complete, one can't help but spot a certain amount of luck involved.

[6]According to a private communication with Melvin Fitting, at the time he did not know of Maehara's results, which were published in German. In fact, I had the pleasure of translating them for him to confirm the connection as recently as in 2015. Given that Beth did not reference Maehara either, it is likely that the two systems have been developed independently, which often happens with fruitful ideas in proof theory.

After all, semantically classical and intuitionistic disjunctions do not differ[7]: it is the intuitionistic implication that deviates from the classical reading. Moreover, the structural disjunction $\quad A, B \quad$ in classical sequents is supposed to faithfully represent the formula level disjunction $A \vee B$. So why, intuitionistically, is the latter allowed but not the former? It is true that each (essential) disjunction in a succedent can be traced to one of its disjuncts, seemingly providing the determinizing information, demanded by constructivist thinking. But $A \vee B \Rightarrow A \vee B$ is also derivable, and there the succedent disjunction can be traced to both A and B, depending on which branch of the derivation tree is considered.

Whatever the original inspiration, two things are definite: it works and it was not based on Kripke models, which were invented much later. The main difference of Gentzen's sequent calculus is in the rules

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \Rightarrow\vee_{ig} \qquad \frac{\Gamma, A \to B \Rightarrow A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta} \to\Rightarrow_{ig} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \Rightarrow\to_{ig}$$

In exact reversal of Maehara's variant, the first two rules are not invertible, despite the premise referring to the same world while the last rule is invertible despite a move to another world. This is an example of a successful purely syntactical calculus development that is outside of the scope of this paper.

While not our first choice, this calculus can still be explained semantically. I would like to thank Melvin Fitting for steering me towards this explanation. As immediately follows from the monotonicity property (Lemma 3):

Corollary 6. *The following definitions of truth for intuitionistic conjunction and disjunction are equivalent to the ones given in Definition 2:*

- $\mathcal{M}, w \Vdash A \wedge B$ *iff* *for all* $v \geq w$, *we have* $\mathcal{M}, v \Vdash A$ *and* $\mathcal{M}, v \Vdash B$*;*
- $\mathcal{M}, w \Vdash A \vee B$ *iff* *for all* $v \geq w$, *we have* $\mathcal{M}, v \Vdash A$ *or* $\mathcal{M}, v \Vdash B$*.*

Hence, these definitions can be used as an alternative basis for sequent rules (more in line with the implication rules):

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge\Rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B, \Delta} \Rightarrow\wedge_{if}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee\Rightarrow \qquad \frac{\Gamma \Rightarrow A, B}{\Gamma \Rightarrow A \vee B, \Delta} \Rightarrow\vee_{if}$$

Note that the antecedent rules remain classical for the same reason as the right premise of the rules ($\to\Rightarrow_{im}$) and ($\to\Rightarrow_{ig}$) is the same as that of ($\to\Rightarrow_{c}$). On the other hand, the resulting succedent rules for conjunction and disjunction are non-invertible, on par with the one for implication. While nominally multi-conclusion, this system is single-conclusion morally. Indeed, whenever a formula in the succedent is used in any way, exactly one succedent formula in the premise of a rule lives on to itself become principal in another rule up the proof tree, at which point other succedent formulas will not be simply ignored but summarily erased in the premises of that rule. Thus, Gentzen's sequent calculus can be seen as a variant of this system that pre-selects the succedent formula to become principal already at the moment of its first appearance rather than at the moment of its exploitation. We illustrate

[7]Some intuitionistic semantics insist on the proof of a disjunction having to carry information about which disjunct was proved, but the official BHK interpretation does not (see [47]).

this by presenting side by side three derivations of $p \wedge q \Rightarrow (((p' \to p) \wedge q) \vee r) \vee r'$ in Maehara's, the intermediary, and Gentzen's sequent calculi (the middle one is based on Fitting's suggestion):

$$\text{Maehara}: \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p,q,p' \Rightarrow p}{p,q \Rightarrow p' \to p, r, r'}\Rightarrow\to_{im} \qquad p,q \Rightarrow q,r,r'}{p,q \Rightarrow (p' \to p) \wedge q, r, r'}\Rightarrow\wedge}{p \wedge q \Rightarrow (p' \to p) \wedge q, r, r'}\wedge\Rightarrow}{p \wedge q \Rightarrow ((p' \to p) \wedge q) \vee r, r'}\Rightarrow\vee_{im}}{p \wedge q \Rightarrow (((p' \to p) \wedge q) \vee r) \vee r'}\Rightarrow\vee_{im}}{}$$

$$\text{Fitting}: \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p,q,p' \Rightarrow p}{p,q \Rightarrow p' \to p}\Rightarrow\to_{if} \qquad p,q \Rightarrow q}{p,q \Rightarrow (p' \to p) \wedge q, r}\Rightarrow\wedge}{p \wedge q \Rightarrow (p' \to p) \wedge q, r}\wedge\Rightarrow}{p \wedge q \Rightarrow ((p' \to p) \wedge q) \vee r, r'}\Rightarrow\vee_{if}}{p \wedge q \Rightarrow (((p' \to p) \wedge q) \vee r) \vee r'}\Rightarrow\vee_{if}$$

$$\text{Gentzen}: \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p,q,p' \Rightarrow p}{p,q \Rightarrow p' \to p}\Rightarrow\to_{ig} \qquad p,q \Rightarrow q}{p,q \Rightarrow (p' \to p) \wedge q}\Rightarrow\wedge}{p \wedge q \Rightarrow (p' \to p) \wedge q}\wedge\Rightarrow}{p \wedge q \Rightarrow ((p' \to p) \wedge q) \vee r}\Rightarrow\vee_{ig}}{p \wedge q \Rightarrow (((p' \to p) \wedge q) \vee r) \vee r'}\Rightarrow\vee_{ig}$$

As can be seen from these three derivations,

- Maehara keeps as many succedent formulas as long as possible, allowing, in particular, to unfold one of them without affecting the others in all cases except for implications,
- Gentzen removes them at first opportunity, always keeping at most one,
- the intermediate variant can keep more than one but only as a decoration because the moment one of them is to be unfolded the others are removed.

Despite being semantically grounded, the intermediate variant is the worst from the point of view of proof search as all succedent rules in it are non-invertible, i.e., more backtrack points need to be created in the decision algorithm. Both Maehara and Gentzen systems improve on this. However, the invertibility status of Gentzen sequent rules is somewhat worse because of the antecedent implication rule $(\to\Rightarrow_{ig})$, which is invertible with respect to the right premise but not the left one. The resulting asymmetry can create serious problems for proof-theoretical analysis.

One important thing must be noted with regard to potential success or failure of a calculus based on intuitionistic Kripke semantics. When the probe leaves a world, which in Maehara calculus happens only in the rule $(\Rightarrow\to_{im})$, this departure is final because the information about other formulas from Δ is lost. If Kripke models relied on loops, this forgetfulness could have become a serious impediment. Fortunately, Int is known to be complete with respect to trees.

3. Hypersequents as an Army of Probes

It has soon become apparent that one probe exploring the model, as afforded by sequents, does not always suffice. Historically, the first work in the direction of extending the probing concerned the modal logic $\mathsf{S5}$ [45], but it is not our intention to give a chronological account. The resulting formalism, which was discovered independently multiple times, took its ultimate shape both in terms of name and notation as *hypersequents* [1, 2] and has become a successful tool for both intermediate[8] and modal logics [9, 10, 34, 36]. We can hardly hope to shed new light on hypersequent design. Thus, we simply explain how hypersequents can be interpreted semantically, thereby providing a robust intuition for understanding hypersequent inference rules.

A (single-conclusion) *hypersequent* is a figure

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

where each $\Gamma_i \Rightarrow \Delta_i$ is a (single-conclusion) sequent. Naturally, if one sequent can probe one world of the model, in effect, a hypersequent represents n parallel probes into n worlds. The crucial point to understand about these worlds is that they are implicitly assumed to belong to the same (at least undirected) component and that no specific relationship is presupposed among these worlds, in other words, any two of these worlds are interchangeable as far as the inference rules are concerned. Since each world is treated as a generic i-model world, sequent rules discussed above can freely be applied to each sequent within a hypersequent, to each *sequent component* individually, notwithstanding the rest of the hypersequent. The system is then appended by one or several additional rules describing the global frame conditions. For instance Jankov logic LQ [26] is known to be sound and complete with respect to confluent i-models, i.e., models such that $w \leq v$ and $w \leq u$ imply that there is a world z such that $v \leq z$ and $u \leq z$. Accordingly, the standard hypersequent rule

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi, \Lambda \Rightarrow}{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi \Rightarrow \mid \Lambda \Rightarrow}\ \mathsf{lq}$$

used for this logic states that whenever (all) formulas (from) Π are true at some world v and formulas Λ are true at some world u, at their common future z that exists by confluence both Π and Λ are going to be true simultaneously, by monotonicity. (As mentioned earlier, the common past w is implicit.)

As we see, hypersequents, permit the use of rules that move the focus from *several* worlds in the conclusion to one new world in the premise that is in a particular relationship with these worlds.

Another freedom afforded by hypersequents is the ability to make non-deterministic choices depending on how the observed worlds are related. A good illustration of this is the standard hypersequent system for Gödel logic G, i.e., the logic of linear frames (see [3]) obtained by adding the rule

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi, \Lambda' \Rightarrow \Delta \quad \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi', \Lambda \Rightarrow \Delta'}{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi, \Lambda \Rightarrow \Delta \mid \Pi', \Lambda' \Rightarrow \Delta'}\ \mathsf{com}$$

to the hypersequent variant of the sequent calculus for Int. The branching of this two-premise rule acts as a non-deterministic choice on the structure of the underlying model. Given two worlds observed in the conclusion—w corresponding to

[8] An intermediate logic is a logic between Int and classical propositional logic.

$\Pi, \Lambda \Rightarrow \Delta$ and v corresponding to $\Pi', \Lambda' \Rightarrow \Delta'$—the decision is made to move the probe to the most "futuristic" of the two worlds, which must be comparable ($v \leq w$ or $w \leq v$) by linearity. No advance knowledge of their relationship is assumed. If $v \leq w$, the two probes are moved to w and monotonicity allows to add Λ' to the antecedent. If $w \leq v$, the situation is symmetric, the probe is moved to v and Λ can be lifted to it from w by monotonicity.

The parallel probes of a hypersequent can, thus, be merged, exchange information. Separate arguments can be employed depending on comparative positions of the probes at a given time, but no hierarchy is imposed on the probes. Rules like external contraction

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi \Rightarrow \Sigma \mid \Pi \Rightarrow \Sigma}{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid \Pi \Rightarrow \Sigma}\ \mathsf{EC}$$

enable us to duplicate a probe so that alternative arguments can be tried in parallel. The two initially duplicate probes can also diverge later because they are moved independently of each other. It should be noted that this parallel processing was one of the initial intuitions behind hypersequents [2].

In general, whichever the genesis of two sequent components, no assumptions are made regarding the relative positions of their corresponding semantic probes. Generally speaking, any two components can look into the same world, into two incomparable worlds, etc. This non-committal, egalitarian attitude adds a certain flexibility but turns out to have its downsides too. It turns out that non-deterministic branching based on varying possible structure of the underlying model impedes the proof of interpolation. Alongside efficient proof search, demonstrating interpolation is one of the primary benefits of having an analytic proof system. Moreover, proof-theoretic interpolation arguments are constructive in that an interpolant is constructed by induction on the depth of an analytic derivation. Further, the syntactic nature of the construction facilitates the proof of Lyndon interpolation, a syntactic strengthening of Craig interpolation. To explain all this, we need to make a short detour into what interpolation is and how it is proved using sequents and their generalizations.

4. Interpolation Proof-Theoretically

The property of interpolation was first formulated by Craig [11, 12] and was once called the last fundamental property of logics to be discovered [4]. A logic L enjoys the *Craig interpolation property* (CIP) if, given a theorem $A \to B$ of L, one can always find its *interpolant* C, i.e., a formula using only propositional atoms common to both A and B such that $A \to C$ and $C \to B$ are both theorems of L. Lyndon's strengthening [37] (LIP) additionally requires propositional atoms occurring in C positively (negatively) to occur positively (negatively) in both A and B, thus, demanding the commonality not only of language but also of polarity.

The first fully proof-theoretic proof of CIP is due to Maehara [39] and has been successfully extended to many different sequent calculi. I wrote earlier that the proof is by induction on the depth of a given sequent derivation. This is not entirely accurate. The interpolation connects A and B, which is generalized for sequents as connecting the left and right sides of the sequent $A \Rightarrow B$. But antecedent and succedent are poorly suited to represent the left and right sides despite giving a good calligraphic impression of being on the left and on the right. The issue is that the induction proof has to keep the common languages between the left and right sides stable throughout the sequent derivation. But implication rules, both classical (2) and intuitionistic, e.g., (3), move formulas between the antecedent and succedent. What Maehara proposed and what has become a *de facto* standard[9] is additionally splitting the sequent, be it one- or two-sided, single- or multi-conclusion, into the left and right sides, more precisely the antecedent and succedent are each split into the left and right sides independently from each other, whereas the sequent structure, i.e., $\Rightarrow$, remains on the global level and, hence, unsplit. For a derivable sequent, all splits are considered and each split is interpolated. Sequent rules are turned into their split versions, whereby, for each possible split of the rule's conclusion, the premise(s) are split so as to ensure each subformula remains on the same side. For instance, $(\to\Rightarrow_{ig})$ produces two versions:

$$\frac{\Gamma, A \to B; \Gamma' \Rightarrow A; \quad \Gamma, B; \Gamma' \Rightarrow \Delta; \Delta'}{\Gamma, A \to B; \Gamma' \Rightarrow \Delta; \Delta'} \to\Rightarrow^{l}_{ig}$$

$$\frac{\Gamma; A \to B, \Gamma' \Rightarrow; A \quad \Gamma; B, \Gamma' \Rightarrow \Delta; \Delta'}{\Gamma; A \to B, \Gamma' \Rightarrow \Delta; \Delta'} \to\Rightarrow^{r}_{ig}$$

where left and right sides are separated from each other by a semicolon ; in each antecedent and each succedent. Most sequent rules have two split variants, depending on which side the principal formula is (naturally, active formulas in the premises then have to go on the same side). By virtue of having two principal formulas, initial sequents $\Gamma, p \Rightarrow p, \Delta$ can be split in four different ways.

The proof-theoretic method of proving interpolation consists of an algorithm that provides interpolants for each split of an initial sequent and explains how to modify/combine interpolants of the premise/premises to obtain an interpolant for the conclusion for each split variant of each rule. Note that the `cut` rule (4) would have been problematic because the cut formula A generally contributes to the common language in the premises but disappears in the conclusion without a clear strategy how to cleanse the premise interpolants from its residue.

It remains to clarify what it means to interpolate a split sequent $\Gamma; \Gamma' \Rightarrow \Delta; \Delta'$ between its left side $\Gamma \Rightarrow \Delta$ and its right side $\Gamma' \Rightarrow \Delta'$. While many equivalent definitions exist, we opt for the semantic one, which is the only one that has been generalized to extensions of sequent calculi so far. The comparison of these definitions and discussion of their equivalence can be found in [30].

Definition 7 (Truth for sequents)**.** A sequent $\Gamma \Rightarrow \Delta$ is *true at a world* w of a Kripke model $\mathcal{M}$, written $\mathcal{M}, w \vDash \Gamma \Rightarrow \Delta$, iff some antecedent formula is false at w or some succedent formula is true at w, i.e., iff

$$\mathcal{M}, w \nVdash A \text{ for some } A \in \Gamma \qquad \text{or} \qquad \mathcal{M}, w \Vdash B \text{ for some } B \in \Delta \ .$$

A sequent is *valid in a model* $\mathcal{M}$ iff it is true at all worlds of $\mathcal{M}$.

Needless to say, this definition of truth matches the standard formula interpretation:

[9]Alternatively, Fitting suggested working in a language without implication, where formulas never reach the other side of the sequent arrow $\Rightarrow$. He called such sequent systems *symmetric* [15]. However, the symmetric option is only available for classical-based logics since implication is not definable through other connectives intuitionistically.

Theorem 8 (Equivalence to formula interpretation)**.** *A sequent $\Gamma \Rightarrow \Delta$ is valid in a model $\mathcal{M}$ iff its formula interpretation $\iota(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$ is valid in $\mathcal{M}$.*

Remark. This equivalence also holds locally at a world, i.e., $\mathcal{M}, w \vDash \Gamma \Rightarrow \Delta$ iff $\mathcal{M}, w \Vdash \bigwedge \Gamma \to \bigvee \Delta$ for classical-based logics, such as modal logics, but not for intuitionistic-based logics because intuitionistic implication is not a local operation.

Definition 9 (Sequent interpolant)**.** A formula C is an *interpolant of a split sequent*

$$\Gamma; \Gamma' \Rightarrow \Delta; \Delta'$$

with respect to a class $\mathcal{C}$ of models if C is in the common language between $\Gamma \Rightarrow \Delta$ and $\Gamma' \Rightarrow \Delta'$ and for every model $\mathcal{M} \in \mathcal{C}$ and every world w from $\mathcal{M}$,

$$\mathcal{M}, w \nVdash C \implies \mathcal{M}, w \vDash \Gamma \Rightarrow \Delta \quad \text{and} \quad \mathcal{M}, w \Vdash C \implies \mathcal{M}, w \vDash \Gamma' \Rightarrow \Delta' \ . \tag{5}$$

In other words, the left side of the split sequent holds whenever the interpolant is false, and the right side holds whenever the interpolant it true.

Returning to the distinction between internal and external calculi, many an attempt was made and failed to extend this method of proving interpolation to hypersequents and other extensions of sequent calculi while staying within strict confines of object language. In other words, the requirement that each interpolant and each interpolation statement be representable by a formula has not yet seen a successful implementation outside of sequents proper.

The situation only improved when Melvin Fitting's tableau intuitions came to fruition in the form of an interpolation algorithm for nested sequents (to be discussed later) in the joint paper [18]. The application of the method to hypersequents was published in a volume of an earlier Humboldt-Kolleg [28] and recently extended to intuitionistic hypersequents in [32, 33]. Here is how the semantic definitions of truth and interpolant are extended from sequents to hypersequents.

Definition 10 (Truth for hypersequents)**.** A hypersequent

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

with n components is *true at a sequence* $\vec{w} = w_1, \dots, w_n$ of n worlds of a model $\mathcal{M}$ iff some formula from the ith antecedent is false at the ith world or some formula from the ith succedent is true at the ith world, i.e., iff there exists i such that

$$\mathcal{M}, w_i \nVdash A \text{ for some } A \in \Gamma_i \qquad \text{or} \qquad \mathcal{M}, w_i \Vdash B \text{ for some } B \in \Delta_i \ .$$

A hypersequent is *valid in a model* $\mathcal{M}$ iff it is true at all *rooted* sequences $\vec{w}$ of worlds of $\mathcal{M}$ of the same length as the hypersequent. A sequence $w_1, \dots, w_n$ is called *rooted* if it has a *root* v such that $v \leq w_i$ for all w_i.

Once again, this definition of validity is equivalent to the standard formula-interpretation one:

Theorem 11 (Equivalence to formula interpretation)**.** *A hypersequent*

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

is valid in a model $\mathcal{M}$ iff its formula interpretation

$$\iota(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) = \bigvee_{i=1}^{n} \left(\bigwedge \Gamma_i \to \bigvee \Delta_i \right)$$

is valid in $\mathcal{M}$.

Since formulas in a hypersequent are evaluated at different worlds, an interpolant cannot be a formula, which would anchor it to just one world.

Definition 12 (Multiformula)**.** *Multiformulas* are defined by the grammar

$$\mho ::= C^{(k)} \mid \overline{C}^{(k)} \mid (\mho \owedge \mho) \mid (\mho \ovee \mho) \ .$$

Atomic multiformulas $C^{(k)}$ and $\overline{C}^{(k)}$ are also called *uniformulas*. The *arity* $\|\mho\|$ of $\mho$ is the largest k such that $C^{(k)}$ or $\overline{C}^{(k)}$ occurs in $\mho$.[10]

We also denote the number of sequent components of a hypersequent $\mathcal{G}$ by $\|\mathcal{G}\|$ and the length of a sequence $\vec{w}$ by $\|\vec{w}\|$.

Definition 13 (Truth for multiformulas)**.** The *truth of a multiformula* $\mho$ *at a sequence* $\vec{w} = w_1, \dots, w_n$ of worlds from a model $\mathcal{M}$ of length $n = \|\vec{w}\| \geq \|\mho\|$ is defined recursively:

- $\mathcal{M}, \vec{w} \vDash C^{(k)}$ iff $\mathcal{M}, w_k \Vdash C$;
- $\mathcal{M}, \vec{w} \vDash \overline{C}^{(k)}$ iff $\mathcal{M}, w_k \nVdash C$;
- $\mathcal{M}, \vec{w} \vDash \mho_1 \owedge \mho_2$ iff $\mathcal{M}, \vec{w} \vDash \mho_1$ and $\mathcal{M}, \vec{w} \vDash \mho_2$;
- $\mathcal{M}, \vec{w} \vDash \mho_1 \ovee \mho_2$ iff $\mathcal{M}, \vec{w} \vDash \mho_1$ or $\mathcal{M}, \vec{w} \vDash \mho_2$.

Definition 14 (Hypersequent interpolant)**.** A multiformula $\mho$ is a *componentwise interpolant of a split hypersequent*

$$\widetilde{\mathcal{G}} = \Gamma_1; \Gamma'_1 \Rightarrow \Delta_1; \Delta'_1 \mid \cdots \mid \Gamma_n; \Gamma'_n \Rightarrow \Delta_n; \Delta'_n$$

with respect to a class $\mathcal{C}$ of models iff $\|\mho\| \leq n$, and $\mho$ is in the common language between $L\widetilde{\mathcal{G}} = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ and $R\widetilde{\mathcal{G}} = \Gamma'_1 \Rightarrow \Delta'_1 \mid \cdots \mid \Gamma'_n \Rightarrow \Delta'_n$, and for every model $\mathcal{M} \in \mathcal{C}$ and every rooted sequence $\vec{w}$ of worlds from $\mathcal{M}$ of length $\|\vec{w}\| = \|\widetilde{\mathcal{G}}\| = n$,

$$\text{(6)} \qquad \mathcal{M}, \vec{w} \nvDash \mho \implies \mathcal{M}, \vec{w} \vDash L\widetilde{\mathcal{G}} \quad \text{and} \quad \mathcal{M}, \vec{w} \vDash \mho \implies \mathcal{M}, \vec{w} \vDash R\widetilde{\mathcal{G}} \ .$$

In other words, the left side of the split sequent holds whenever the interpolant is false, and the right side holds whenever the interpolant it true.

We omit the technical details, which can be found in [28, 32, 33]. Suffice it to say that

Theorem 15 (Reducing Craig/Lyndon to componentwise interpolation)**.** *If a multiformula*

$$\mho = \bigowedge_{j=1}^{m} \left(\overline{C_j}^{(1)} \ovee D_j^{(1)} \right)$$

is a componentwise interpolant of a hypersequent $A; \Rightarrow; B$ *with a single sequent component, then*

$$C_\mho := \bigwedge_{j=1}^{m} (C_j \to D_j)$$

is a Craig interpolant of $A \to B$*. Additionally, if all propositional atoms occurring positively (negatively)*[11] *in* $\mho$ *occur positively (negatively) in both* A *and* B*, then* $C_\mho$ *is a Lyndon interpolant of* $A \to B$*.*

[10] The symbol $\mho$ is used as a unit in physics. Due to its reciprocal connection to ohm, denoted Ω, the symbol $\mho$ represents an upside down Ω and its pronunciation, *mho* is the word *ohm* read backwards.

[11] The polarity of p occurring in $\overline{E}^{(k)}$ (in $E^{(k)}$) is opposite to (the same as) the polarity of p in E. The operations $\ovee$ and $\owedge$ preserve polarity.

5. Incorporating Model Structure into Sequent Structure

5.1. **Incorporating linearity.** With the formal definitions of truth for sequents (Definition 7) and hypersequents (Definition 10) in place, it should be evident now that the falsification of each sequent component in a hypersequent that we discussed in Section 3 means exactly making the hypersequent false at a sequence of worlds. (The formal definition for hypersequents did introduce an additional restriction of rootedness, i.e., for all the worlds in a sequence to have a common past. This condition is needed to prove the equivalence with the formula interpretation, and one can imagine it being dropped for calculi that are not internal.)

Thus, the central idea of using multiformula interpolants is that each uniformula $C^{(k)}$ or $\overline{C}^{(k)}$ within is associated with the kth component of a hypersequent in that C is to be evaluated at the same world as the kth component. Allowing for different parts of an interpolant to be evaluated at different worlds can (for interpolable logics) provide sufficient expressive power to describe interpolants for all derivable hypersequents. And the final result can be converted into a formula as stated in Theorem 15.

Remark. In Section 4, we used the term model instead of i-model because the discussion is fully applicable to intuitionistic i-models and classical Kripke models, *m-models*, for modal logics.

Now we are ready to explain why non-deterministic model structure can stand in the way of proving interpolation. Gentzen rules such as com, in effect serve as a case analysis. For each specific sequence of worlds, the validity of the conclusion of the rule follows from the validity of just one of the premises, but no one premise would work for all sequences. Why is such information flow problematic in the interpolation proof? Because it makes arguments for different model configurations completely disjoint, and disjoint arguments can be separated by both the object language and the split.

Example 16. Consider, e.g., the following split of a derivation of $q \Rightarrow p \mid p \Rightarrow q$:

$$\dfrac{p; \Rightarrow p; \quad ; q \Rightarrow ; q}{; q \Rightarrow p; \mid p; \Rightarrow ; q}\,\mathsf{com}$$

Since, one side of the split is empty in each of the premises, both accept trivial interpolants $\bot^{(1)}$ and $\top^{(1)}$ respectively. But the split of the conclusion has non-trivial left and right sides, and the common language requirement still precludes the use of any propositional atoms to interpolate the conclusion. It is easy to show by induction on multiformula construction that no multiformula constructed from $\bot^{(k)}$ and $\top^{(k)} = \overline{\bot}^{(k)}$ for $k = 1, 2$ can interpolate the conclusion. In effect, different (trivial) interpolants are needed depending on the i-model configuration, and there is no way to encode the choice into the interpolant.

If observing multiple worlds at once without differentiating them has disadvantages, the next idea is to enshrine the structure of the underlying model in the structure of sequents. After all, structural proof theory is called *structural* for a reason.

The natural solution for Gödel logic G of linear i-frames is to use the linearity directly in the sequent. In modal setting, the idea goes back to [43] and was recently streamlined and expanded on by Lellmann [35] under the name of *linear* *nested sequents*. Its application to the Gödel logic was worked out in [32]. There is surprisingly little that needs to be modified in our semantic interpretation of a multi-component sequent. Like hypersequents, linear nested sequents

$$\Gamma_1 \Rightarrow \Delta_1 /\!/ \dots /\!/ \Gamma_n \Rightarrow \Delta_n \tag{7}$$

are to be interpreted in a sequence $\vec{w} = w_1, \dots, w_n$ of worlds. But instead of being rooted, the sequence must be monotone, i.e., $w_1 \leq \cdots \leq w_n$. To emphasize this distinction, we use the delimiter $/\!/$ instead of $|$ to separate sequent components. Needless to say, a system inspired by semantics uses multi-conclusion-style (i.e., classical) rules for conjunction and disjunction for sequent components, as well as the classical version $(\rightarrow\Rightarrow_c)$ of the antecedent implication rule. It should also be clear that the external exchange rule that is either explicitly present or hidden in the multiset structure of hypersequents must be dispensed with. Components of a linear nested sequent are not interchangeable. (In fact, tellingly, adding the external exchange rule to the system yields classical logic [35].)

The main innovation is the succedent implication rule, which has two versions, depending on whether the implication is in the last component. Let us start with the case it is:

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ A \Rightarrow B}{\mathcal{G} /\!/ \Gamma \Rightarrow A \rightarrow B, \Delta} \Rightarrow\rightarrow^t_{lns}$$

where $\mathcal{G}$ (and later $\mathcal{H}$) represents an arbitrary linear nested sequent. The intuition should be transparent: for $A \rightarrow B$ to be false at w, there must exist a future $u \geq w$ with A true and B false. But what if there are already futures in the linear structure?

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ A \Rightarrow B /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H} \quad \mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Sigma \Rightarrow A \rightarrow B, \Pi /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma \Rightarrow A \rightarrow B, \Delta /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H}} \Rightarrow\rightarrow^h_{lns}$$

Let v be the nearest known future, which corresponds to the component $\Sigma \Rightarrow \Pi$. Due to the linearity, the witness u for the falsity of $A \rightarrow B$ is either before v or after, i.e., $u < v$ or $u \geq v$. In the former case, the left premise works the same way as the only premise in $(\Rightarrow\rightarrow^t_{lns})$. In the latter case, A being true and B being false at u also witnesses the falsity of $A \rightarrow B$ at v.

Note that, unlike the Maehara's rule $(\Rightarrow\rightarrow_{im})$ from (3), succedent neighbors of $A \rightarrow B$ from Δ need not be discarded because, instead of moving the focus from w to u, the focus forks, creating a new component for u, while keeping the w-evaluated component intact. The benefit of this is, as shown in [32], the invertibility of both rules $(\Rightarrow\rightarrow^t_{lns})$ and $(\Rightarrow\rightarrow^h_{lns})$.

We do not present the full calculus here, instead referring the reader to [32]. But we do discuss the main changes dictated by the semantics. First, in addition to initial sequents of the type $\mathcal{G} /\!/ \Gamma, p \Rightarrow p, \Delta /\!/ \mathcal{H}$, we postulate initial sequents based on the monotonicity property

$$\mathcal{G} /\!/ \Gamma, p \Rightarrow \Delta /\!/ \mathcal{H} /\!/ \Pi \Rightarrow p, \Sigma /\!/ \mathcal{I} \ .$$

Indeed, it is impossible to both make p true at a world and false at some future of this world. Moreover, we codify monotonicity in its own rule

$$\frac{\mathcal{G} /\!/ \Gamma, A \Rightarrow \Delta /\!/ \Sigma, A \Rightarrow \Pi /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma, A \Rightarrow \Delta /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H}} \mathsf{lift}$$

With the help of this calculus, we were able to establish

Theorem 17 (Lyndon interpolation for Gödel logic)**.** *There is a linear nested sequent calculus for Gödel logic* G *that enjoys componentwise interpolation. Consequently,* G *has the Craig (and, indeed, also the Lyndon) interpolation.*

Remark. Incidentally, Lyndon interpolation for G remained an open problem [8, 21, 41] until it was demonstrated using this method in [32].

We should also mention that linear nested sequents are an internal calculus. The formula interpretation for the sequent (7) is

$$\bigwedge \Gamma_1 \to \bigvee \Delta_1 \vee \left(\bigwedge \Gamma_2 \to \bigvee \Delta_2 \vee \left(\cdots \vee \left(\bigwedge \Gamma_n \to \bigvee \Delta_n\right)\right)\right) \quad .$$

5.2. **Incorporating finite model structure.** Hypersequents and linear nested sequents treat worlds as homogeneous, i.e., each component can be interpreted at any world of a model. Thus, rules applicable to different components can differ only based on their relationship to other components. For instance, the hypersequent rule com requires the presence of two components, whereas the choice between $(\Rightarrow\to^t_{lns})$ and $(\Rightarrow\to^h_{lns})$ is dictated by whether the principle[12] component is the last one. However, a component with two implications in the succedent can prompt the application of $(\Rightarrow\to^t_{lns})$ first, followed by $(\Rightarrow\to^h_{lns})$, which becomes applicable after $(\Rightarrow\to^t_{lns})$ added a component after it.

Note that we have smuggled in another idea. In both sequents and hypersequents, the individual components served as mobile probes traversing the model. Linear nested sequents was the first example we gave of sequent components forming a "stationary research base" with a rigid structure. Each segment of the base remains stationary but new segments can be added both around and in the middle of existing ones. Thus, we switched from history-less algorithms to perfect-recall ones, where all the facts discovered earlier can be reused whenever needed. In other words, the underlying model is implicitly assumed to be a linear order, infinite towards the future and dense. This view does not, however, preclude the use of finite models because two components are allowed to be interpreted in the same world.

There are, however, intermediate logics whose model structure is much more varied and strict. In addition, models can be restricted quantitatively, in their depth and/or the number of children per world. It is well known, for instance, that leaves of an i-model behave like classical worlds. Unlike linear i-models for Gödel logic, models of bounded depth must necessarily have such classical worlds, and it stands to reason that inference rules for components representing these classical worlds should be different from those for other components. The ability to distinguish components, however, has its price: the reflexivity of the future relation cannot be accommodated anymore. Indeed, we just saw that the said reflexivity and the ability to interpret several sequent components in the same world makes it possible to unfold a finite model into a potentially infinite one, contravening the ability to control depth.

This led Björn Lellmann [33] to suggest a new formalism, which can be described as *injective nested sequents*. It is applicable to logics whose i-models correspond to

[12] We apply the term *principle* to components of a hypersequent by analogy with its usage for formulas: a principle component is the one in the succedent of the rule, to which the rule is applied, i.e., the one without which this rule would not have been applicable.

classes $\mathcal{T}$ of tree-like i-frames. The systems are remarkably simple and largely classical. In particular, sequent components are multi-conclusion and most rules are versions of classical rules for connectives. Once again, it is not the goal of this paper to advertise the systems introduced in [33]. We are mostly interested in how rules can be designed based on the semantic structure of the underlying model. The main insight of Lellmann in this direction was to demand a faithful embedding of the nested sequent structure, which is a tree, into an i-model, which led to the following rule:

$$(8) \qquad \dfrac{\begin{array}{l}\{\ \nabla\{\Gamma, A \Rightarrow B, \Delta^*\}\ \} \quad \cup \quad \{\ \nabla\{\Gamma \Rightarrow \Delta^*, [A \Rightarrow B]\}\ \}\restriction_{\mathcal{T}} \\ \cup \{\ \nabla\{\Gamma \Rightarrow \Upsilon^*, [A \Rightarrow B, [\Sigma \Rightarrow \Pi^*]]\} \ \mid\ \Delta^* = \Upsilon^*, [\Sigma \Rightarrow \Pi^*]\ \}\restriction_{\mathcal{T}} \\ \cup \{\ \nabla\{\Gamma \Rightarrow \Upsilon^*, [\Sigma \Rightarrow A \to B, \Pi^*]\} \ \mid\ \Delta^* = \Upsilon^*, [\Sigma \Rightarrow \Pi^*]\ \}\end{array}}{\nabla\{\Gamma \Rightarrow A \to B, \Delta^*\}} \Rightarrow\!\to_{inj}$$

(for better readability, we present the version of the rule without the embedded contraction, which can be used for more efficient decision procedures). In accordance with the postulated tree-like models, sequents here also have a tree structure, with brackets $[\dots]$ marking a subtree rooted at a child of the current node. $\nabla\{\Gamma \Rightarrow \Delta^*\}$ means that a rule is applied to the principle component $\Gamma \Rightarrow \Delta$ somewhere inside the bigger tree ∇, whereas $*$ signals that this component may have its own subtree rooted in it, i.e., $\Delta^* = \Delta, [\nabla_1], \dots, [\nabla_n]$. There are four types of premises in (8), with potentially several premises of the same type: these types depend on how the conclusion $A \to B$, which is evaluated, say, at a world w, is falsified in the underlying i-model, i.e., where the witness u that makes A true and B false is situated relative to the worlds representing existing sequent components:

- $\nabla\{\Gamma, A \Rightarrow B, \Delta^*\}$ corresponds to the situation when A is true and B is false at w itself (there is only one such premise);
- $\nabla\{\Gamma \Rightarrow \Upsilon^*, [\Sigma \Rightarrow A \to B, \Pi^*]\}$ where $\Delta^* = \Upsilon^*, [\Sigma \Rightarrow \Pi^*]$, represents the case when u is at or beyond a world v where the child component $\Sigma \Rightarrow \Pi$ is evaluated because then $A \to B$ is also false at v (this case is analogous to the right premise of $(\Rightarrow\!\to^h_{lns})$ for G);
- $\nabla\{\Gamma \Rightarrow \Delta^*, [A \Rightarrow B]\}$ means that u is a strict future of w but is incomparable with all v's representing existing children of the active component $\Gamma \Rightarrow \Delta$ (in particular, if there are no children yet). This rule creates a new sequent component, but only if frames with such extra child belong to $\mathcal{T}$, which is recorded in the restriction $\restriction_{\mathcal{T}}$ (there is at most one premise of this type);
- finally, $\nabla\{\Gamma \Rightarrow \Upsilon^*, [A \Rightarrow B, [\Sigma \Rightarrow \Pi^*]]\}$ where $\Delta^* = \Upsilon^*, [\Sigma \Rightarrow \Pi^*]\}$ corresponds to $w < u < v$ for the world v representing the child with $\Sigma \Rightarrow \Pi$. Once again, these premises are only used if inserting an extra world between w and v does not lead outside the class $\mathcal{T}$ (this case is analogous to the left premise of $(\Rightarrow\!\to^h_{lns})$ for G).

Example 18. For instance, Smetanich logic Sm is sound and complete with respect to linear frames of depth at most 2. Accordingly, borrowing the simplified linear notation from Gödel logic, its right implication rules are as follows:

$$\frac{\Gamma, A \Rightarrow B, \Delta \quad \Gamma \Rightarrow \Delta /\!/ A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta} \Rightarrow\to^{r}_{\mathsf{Sm}} \qquad \frac{\Gamma \Rightarrow \Delta /\!/ \Sigma, A \Rightarrow B, \Pi}{\Gamma \Rightarrow \Delta /\!/ \Sigma \Rightarrow A \to B, \Pi} \Rightarrow\to^{rl}_{\mathsf{Sm}}$$

$$\frac{\Gamma, A \Rightarrow B, \Delta /\!/ \Sigma \Rightarrow \Pi \quad \Gamma \Rightarrow \Delta /\!/ \Sigma \Rightarrow A \to B, \Pi}{\Gamma \Rightarrow A \to B, \Delta /\!/ \Sigma \Rightarrow \Pi} \Rightarrow\to^{l}_{\mathsf{Sm}}$$

The rule $(\Rightarrow\to^{r}_{\mathsf{Sm}})$ corresponds to an implication at the root of the model and creates a sequent component for the unique leaf; the rule $(\Rightarrow\to^{l}_{\mathsf{Sm}})$ is almost the same situation except the leaf has already been created. Since the model is linear, no other children can appear on other branches; since the depth of the model is at most 2, there cannot be a child in between the root and the leaf on the same branch. Finally, $(\Rightarrow\to^{rl}_{\mathsf{Sm}})$ describes the classical behavior at the leaf.

The injective nested sequents like these are suitable for a wide variety of intermediate logics and provide for efficient decision procedures (after certain amount of standard proof-theoretic tinkering, e.g., embedding contraction into the rules). Using them, we have also been able to show Craig interpolation for all interpolable strictly intermediate logics [33] other than Jankov, which is not complete with respect to tree-like models. Interestingly, to ensure the injectivity, we were forced to employ interpolant transformations that change polarity of propositional atoms. Thus, for logics such as Smetanich logic Sm, greatest semi-constructive logic GSc, and logic Bd_2 of models of depth at most 2 the proof-theoretic method only yielded Craig but not Lyndon interpolation. Note that this does not mean Lyndon interpolation fails for them. In fact, for all but one, it has been demonstrated (for Sm in [40] and for Bd_2 in [41]). The only logic, for which Lyndon interpolation remains open is GSc, the logic of three-world frames with two maximal worlds.

It should also be noted that, unlike all the calculi discussed earlier, injective nested sequents seem to be an external calculus. There is little hope of being able to represent the injectivity of the faithful embedding of components to worlds in the object language. For instance, in Sm, the injective nested sequent $\Rightarrow \neg p /\!/ \Rightarrow p$ is valid because $w \nVdash \neg p$ implies that there is a world with p true, in which case $v \Vdash p$ in the leaf v by monotonicity. However, the standard formula interpretation for linear nested sequents yields $\top \to \neg p \vee (\top \to p)$, i.e., the *tertium non datur*, which would have turned the logic into classical were it valid.

6. Comparison with Labelled Sequents

Injective nested sequents can be compared with labelled nested sequents [13] in that both rely on information about the model so detailed it generally cannot be represented by a formula anymore. This discussion would have been incomplete if we did not elaborate on this connection.

Naturally, all types of calculi discussed so far can also be represented in the labelled sequent paradigm [49, 53], where each formula $w\colon A$ in a sequent is assigned an explicit label w representing a sequent component/world of a Kripke model and the accessibility relations $w \leq v$ among labels/worlds are explicitly present in (the antecedent of) a sequent. Minor modifications, such as the restriction on the number of succedent formulas with the same label for single-conclusion hypersequents or the presence of equality/inequality among labels for injective nested sequents, may be necessary but it is fairly clear that labelled sequents are more expressive [25] that the calculi discussed above.

It is, therefore, natural to ask whether all these weaker calculi are useful? Why should one tinker with minute details when there is one formalism to label them all? The answer is quite straightforward. It is a general rule of thumb that the more expressive a theory, the less efficient it is. Proof search in labelled sequents is more difficult though not impossible [53]. Making proof search efficient to obtain optimal upper complexity bounds is an even more daunting task. Perhaps, it is even fair to say that a heuristics for efficient optimal proof search for labelled sequents for a particular logic can also be used as a blueprint for constructing a tailor-made calculus.

But there is a bigger problem underscoring that labelled sequents cannot be a one-size-fits-all approach. The problem of proving interpolation using hypersequents for Gödel logic transfers to the labelled sequent setting too. The corresponding labelled rule

$$\frac{w \leq v, \Gamma \Rightarrow \Delta \quad v \leq w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathsf{lin}$$

where w and v are labels occurring in the conclusion exhibits the same kind of case analysis with one premise applicable to models of one type and the other premise to those of the other type. Thus, to prove interpolation of Gödel logic, one can use neither hypersequents nor labelled sequents.

7. Non-Standard Calculi for Modal Logics

As further proof that calculi tailor-made for a particular class of models can yield better results than generic ones, we would like to briefly discuss one more example, this time for modal logic. Kripke models for modal logics, which we will call *m-models* to distinguish from i-models, are essentially the same as i-models, only the monotonicity for propositional atoms and the minimum preorder requirement are dropped, though individual modal logics can reinstate the latter or impose other restrictions on the accessibility relation. Since the accessibility relation is not necessarily a preorder, it is common to use R instead of $\leq$ (although $\approx$ is also used for logics complete with respect to models with equivalence accessibility relation).

Sequents, hypersequents, and labelled sequents [19, 20, 46, 48, 49, 53] have all been successfully used to describe modal logics.[13] In addition, nested sequents [5, 6, 7, 27, 50, 51, 52] turned out to be suitable for a wide range of modal logics. Like hypersequents, nested sequents impose a uniform predefined structure on the sequent components. Where the structure in hypersequents is empty, i.e., components are treated homogeneously, nested sequents view all components as forming a tree. Since many modal logics are complete with respect to tree-like models, the success of nested sequents was to be expected. (Of course, nested sequents can also be used for intuitionistic logic [17].) Incidentally, as the name suggests, linear nested sequents discussed earlier can be viewed as a subset of nested sequents with no branching.

Curiously, as predicted in [6] and established in [16], the tableaus corresponding to nested sequents have been developed even earlier than nested sequents themselves. These tableaus are called *prefixed tableaus* and have been introduced and

[13]Bibliographic note: Both papers [45, 46] belong to the same person, whose preferred spelling was *Mints*. The spelling *Minc* originated from an alternative transliteration system from the Cyrillic alphabet. The original Cyrillic spelling *Минц* can be found, e.g., in [55].

further developed in [14, 15, 44]. At this point, it should come as no surprise that prefixes used in prefixed tableaus form a tree-like structure.

Being less expressive than labelled sequents [25], nested sequents are, nevertheless, expressive enough to suffer from inefficiency. In particular, a complete proof search tends to result in even individual nested sequents of exponential size, thus, making it difficult to prove the common upper complexity bound of PSPACE, let alone NP.

Paying closer attention to specific type of models helps alleviate this problem too. For instance, models for the modal logic $\mathsf{K5}$ of Euclidean frames, i.e., frames where wRv and wRu imply vRu, are rather simple in structure. They would generally have a root ρ, a clique of pairwise accessible worlds Cl and accessibility arrows leading from ρ to worlds in Cl. If Cl is not empty, there must be at least one such arrow but there need not be arrows leading to all worlds in Cl. Brünnler was able to provide cut-free nested sequents for $\mathsf{K5}$ because a tree rooted in ρ can be homomorphically mapped into a Euclidean model. But one could argue that a tree structure is quite antithetical to a balloon-like structure of Euclidean models. In fact, the clique part Cl can be viewed as an $\mathsf{S5}$ model, i.e., Kripke model with the equivalence accessibility relation. The homogeneous structure of $\mathsf{S5}$ models made them a perfect candidate for the first hypersequent system [45]. From this intuition was born the idea of *grafting* hypersequent rules for the sequent components corresponding to the clique onto the trunk or root, that is treated in a nested sequent manner [31]. The experiment was a success and yielded complexity-optimal proof search. Moreover, it has been shown in [29] that these *grafted hypersequents* can also be used to show Craig/Lyndon interpolation for $\mathsf{K5}$.

8. Conclusion

Proof theory is often considered a purely syntactical endeavor to the point of abhorrence of semantics by part of the community. This paper is an argument in favor of harnessing semantic intuitions in designing sequent-like calculi. Similar to modality $\Box$ having multiple interpretations ranging from necessity to knowledge, syntactic constructs can and should be interpreted in multiple ways enriching each view with new angles. Much of proof theory is developed in a hammer-seeks-nail manner, with a syntactic construct settled on first, followed by a search of logics it can apply to. As a matter of a personal anecdote, Brünnler spent quite a lot of effort trying to capture the logic $\mathsf{S4.2}$ of confluent frames, before Goré and Ramanayake showed that nested sequents correspond exclusively to tree-like models.

Thus, a good fit between structures employed by structural proof theory and those creating models can help facilitate impossibility results, as well as yield the applications proof theory tends to provide, such as efficient proof search and constructive interpolation.

The examples in this paper showcase some of the tricks one may employ in designing custom-made calculi for a particular logic. We show that chasing the most general and expressive system is not always the reasonable path as it may preclude the above-mentioned applications from materializing.

We believe that using semantic structures as an inspiration can combine the best of the two worlds and create new efficient proof formalisms. The history of structural proof theory is a story of liberation from restraints—from one component to many, from homogeneous to heterogeneous components, etc. But between the hubs of hypersequents, nested sequents, etc., there are many smaller stops whose inimitable charm is well worth exploring.

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References

[1] Arnon Avron. A constructive analysis of **RM**. *Journal of Symbolic Logic*, 52(4):939–951, December 1987. `doi:10.2307/2273828`.

[2] Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In Wilfrid Hodges, Martin Hyland, Charles Steinhorn, and John Truss, editors, *Logic: from Foundations to Applications: European Logic Colloquium*, pages 1–32. Clarendon Press, 1996.

[3] Matthias Baaz, Agata Ciabattoni, and Christian G. Fermüller. Hypersequent calculi for Gödel logics—a survey. *Journal of Logic and Computation*, 13(6):835–861, December 2003. `doi:10.1093/logcom/13.6.835`.

[4] Johan van Benthem. The many faces of interpolation. *Synthese*, 164(3):451–460, October 2008. `doi:10.1007/s11229-008-9351-5`.

[5] Kai Brünnler. Deep sequent systems for modal logic. In Guido Governatori, Ian Hodkinson, and Yde Venema, editors, *Advances in Modal Logic, volume 6*, pages 107–119. College Publications, 2006. Available from: `http://www.aiml.net/volumes/volume6/Bruennler.ps`.

[6] Kai Brünnler. *Nested Sequents*. Habilitation thesis, Institut für Informatik und angewandte Mathematik, Universität Bern, April 2010. Available from: `https://arxiv.org/abs/1004.1845`.

[7] Robert A. Bull. Cut elimination for propositional dynamic logic without *. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 38:85–100, 1992. `doi:10.1002/malq.19920380107`.

[8] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Clarendon Press, 1997. Available from: `https://global.oup.com/academic/product/modal-logic-9780198537793`.

[9] Agata Ciabattoni and Mauro Ferrari. Hypersequent calculi for some intermediate logics with bounded Kripke models. *Journal of Logic and Computation*, 11(2):283–294, April 2001. `doi:10.1093/logcom/11.2.283`.

[10] Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. From axioms to analytic rules in nonclassical logics. In *Proceedings, Twenty-Third Annual IEEE Symposium on Logic in Computer Science*, pages 229–240. IEEE, 2008. `doi:10.1109/LICS.2008.39`.

[11] William Craig. Linear reasoning. A new form of the Herbrand–Gentzen Theorem. *Journal of Symbolic Logic*, 22(3):250–268, September 1957. `doi:10.2307/2963593`.

[12] William Craig. Three uses of the Herbrand–Gentzen Theorem in relating model theory and proof theory. *Journal of Symbolic Logic*, 22(3):269–285, September 1957. `doi:10.2307/2963594`.

[13] Roy Dyckhoff and Sara Negri. Proof analysis in intermediate logics. *Archive for Mathematical Logic*, 51(1–2):71–92, February 2012. `doi:10.1007/s00153-011-0254-7`.

[14] Melvin Fitting. Tableau methods of proof for modal logics. *Notre Dame Journal of Formal Logic*, XIII(2):237–247, April 1972. `doi:10.1305/ndjfl/1093894722`.

[15] Melvin Fitting. *Proof Methods for Modal and Intuitionistic Logics*, volume 169 of *Synthese Library*. D. Reidel Publishing Company, 1983. `doi:10.1007/978-94-017-2794-5`.

[16] Melvin Fitting. Prefixed tableaus and nested sequents. *Annals of Pure and Applied Logic*, 163(3):291–313, March 2012. `doi:10.1016/j.apal.2011.09.004`.

[17] Melvin Fitting. Nested sequents for intuitionistic logics. *Notre Dame Journal of Formal Logic*, 55(1):41–61, 2014. `doi:10.1215/00294527-2377869`.

[18] Melvin Fitting and Roman Kuznets. Modal interpolation via nested sequents. *Annals of Pure and Applied Logic*, 166(3):274–305, March 2015. `doi:10.1016/j.apal.2014.11.002`.

[19] D. M. Gabbay. Labelled deductive systems: A position paper. In Juha Oikkonen and Jouko Väänänen, editors, *Logic Colloquium '90*, volume 2 of *Lecture Notes in Logic*, pages 66–88. Springer, 1993. `doi:10.1017/9781316718254.008`.

[20] Dov M. Gabbay. *Labelled Deductive Systems, Volume 1*, volume 33 of *Oxford Logic Guides*. Clarendon Press, 1996.

[21] Dov M. Gabbay and Larisa Maksimova. *Interpolation and Definability: Modal and Intuitionistic Logics*, volume 46 of *Oxford Logic Guides*. Clarendon Press, 2005. `doi:10.1093/acprof:oso/9780198511748.001.0001`.

[22] Gerhard Gentzen. Untersuchungen über das logische Schließen. I. *Mathematische Zeitschrift*, 39:176–210, 1935. In German. `doi:10.1007/BF01201353`.

[23] Gerhard Gentzen. Untersuchungen über das logische Schließen. II. *Mathematische Zeitschrift*, 39:405–431, 1935. In German. `doi:10.1007/BF01201363`.

[24] Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, volume 55 of *Studies in Logic and the Foundations of Mathematics*, pages 68–131. North-Holland, 1969. Originally published in German in 1935. `doi:10.1016/S0049-237X(08)70822-X`.

[25] Rajeev Goré and Revantha Ramanayake. Labelled tree sequents, tree hypersequents and nested (deep) sequents. In Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence Moss, editors, *Advances in Modal Logic, Volume 9*, pages 279–299. College Publications, 2012. Available from: `http://www.aiml.net/volumes/volume9/Gore-Ramanayake.pdf`.

[26] V. A. Jankov. The calculus of the weak "law of excluded middle". *Mathematics of the USSR–Izvestiya*, 2(5):997–1004, 1968. Originally published in Russian as [56]. `doi:10.1070/IM1968v002n05ABEH000690`.

[27] Ryo Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–135, March 1994. `doi:10.1007/BF01053026`.

[28] Roman Kuznets. Craig interpolation via hypersequents. In Dieter Probst and Peter Schuster, editors, *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, volume 6 of *Ontos Mathematical Logic*, pages 193–214. De Gruyter, 2016. `doi:10.1515/9781501502620-012`.

[29] Roman Kuznets. Interpolation method for multicomponent sequent calculi. In Sergei Artemov and Anil Nerode, editors, *Logical Foundations of Computer Science, International Symposium, LFCS 2016, Deerfield Beach, FL, USA, January 4–7, 2016, Proceedings*, volume 9537 of *LNCS*, pages 202–218. Springer, 2016. `doi:10.1007/978-3-319-27683-0_15`.

[30] Roman Kuznets. Multicomponent proof-theoretic method for proving interpolation property. *Annals of Pure and Applied Logic*, to appear, 2018.

[31] Roman Kuznets and Björn Lellmann. Grafting hypersequents onto nested sequents. *Logic Journal of the IGPL*, 24(3):375–423, June 2016. `doi:10.1093/jigpal/jzw005`.

[32] Roman Kuznets and Björn Lellmann. Interpolation for intermediate logics via hyper- and linear nested sequents. In *Advances in Modal Logic, Volume 12*. College Publications, 2018. In Press.

[33] Roman Kuznets and Björn Lellmann. Interpolation for intermediate logics via injective nested sequents. Submitted, 2018.

[34] Ori Lahav. From frame properties to hypersequent rules in modal logics. In *Proceedings, 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013*, pages 408–417. IEEE, 2013. `doi:10.1109/LICS.2013.47`.

[35] Björn Lellmann. Linear nested sequents, 2-sequents and hypersequents. In Hans De Nivelle, editor, *Automated Reasoning with Analytic Tableaux and Related Methods, 24th International Conference, TABLEAUX 2015, Wrosław, Poland, September 21–24, 2015, Proceedings*, volume 9323 of *Lecture Notes in Artificial Intelligence*, pages 135–150. Springer, 2015. `doi:10.1007/978-3-319-24312-2_10`.

[36] Björn Lellmann. Hypersequent rules with restricted contexts for propositional modal logics. *Theoretical Computer Science*, 656(A):76–105, December 2016. `doi:10.1016/j.tcs.2016.10.004`.

[37] Roger C. Lyndon. An interpolation theorem in the predicate calculus. *Pacific Journal of Mathematics*, 9(1):129–142, May 1959. `doi:10.2140/pjm.1959.9.129`.

[38] Shôji Maehara. Eine Darstellung der intuitionistischen Logik in der Klassischen. *Nagoya Mathematical Journal*, 7:45–64, June 1954. In German. `doi:10.1017/S0027763000018055`.

[39] Shôji Maehara. On the interpolation theorem of Craig. *Sugaku*, 12(4):235–237, 1961. In Japanese. `doi:10.11429/sugaku1947.12.235`.

[40] L. L. Maksimova. The Lyndon interpolation theorem for modal logics. In *Mathematical Logic and Algorithm Theory*, volume 2 of *Trudy Instituta matematiki*, pages 45–55. Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, 1982. In Russian. Available from: `http://math.nsc.ru/journals/ti/02/ti_02_0003.pdf`.

[41] L. L. Maksimova. The Lyndon property and uniform interplolation over the Grzegorczyk logic. *Siberian Mathematical Journal*, 55(1):118–124, January 2014. Originally published in Russian as [54]. `doi:10.1134/S0037446614010145`.

[42] Michel Marti and Thomas Studer. The proof theory of common knowledge. In Hans van Ditmarsch and Gabriel Sandu, editors, *Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*, volume 12 of *Outstanding Contributions to Logic*, pages 433–455. Springer, 2018. `doi:10.1007/978-3-319-62864-6_18`.

[43] Andrea Masini. 2-sequent calculus: a proof theory of modalities. *Annals of Pure and Applied Logic*, 58(3):229–246, November 1992. `doi:10.1016/0168-0072(92)90029-Y`.

[44] Fabio Massacci. Strongly analytic tableaux for normal modal logics. In Alan Bundy, editor, *Automated Deduction—CADE-12, 12th International Conference on Automated Deduction, Nancy, France, June 26–July 1, 1994, Proceedings*, volume 814 of *Lecture Notes in Artificial Intelligence*, pages 723–737. Springer, 1994. `doi:10.1007/3-540-58156-1_52`.

[45] G. E. Minc. On some calculi of modal logic. In V. P. Orevkov, editor, *The Calculi of Symbolic Logic. I*, volume 98 (1968) of *Proceedings of the Steklov Institute of Mathematics*, pages 97–124. AMS, 1971. Originally published in Russian in 1968 as [55]. Available from: `http://mi.mathnet.ru/eng/tm/v98/p88`.

[46] Grigori Mints. Indexed systems of sequents and cut-elimination. *Journal of Philosophical Logic*, 26(6):671–696, December 1997. `doi:10.1023/A:1017948105274`.

[47] Joan Moschovakis. Intuitionistic logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2015 edition, 2015. Available from: `http://plato.stanford.edu/archives/spr2015/entries/logic-intuitionistic/`.

[48] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5–6):507–544, October–December 2005. `doi:10.1007/s10992-005-2267-3`.

[49] Sara Negri and Jan von Plato. *Proof Analysis: A Contribution to Hilbert's Last Problem*. Cambridge University Press, 2011. `doi:10.1017/CBO9781139003513`.

[50] Francesca Poggiolesi. *Sequent Calculi for Modal Logic*. PhD thesis, University of Florence, 2008.

[51] Francesca Poggiolesi. *Gentzen Calculi for Modal Propositional Logic*, volume 32 of *Trends in Logic*. Springer, 2011. `doi:10.1007/978-90-481-9670-8`.

[52] Masahiko Sato. A study of Kripke-type models for some modal logics by Gentzen's sequential method. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 13(2):381–468, 1977. `doi:10.2977/prims/1195189814`.

[53] Luca Viganò. *Labelled Non-Classical Logics*. Springer, 2000. `doi:10.1007/978-1-4757-3208-5`.

[54] Л. Л. Максимова. Свойство Линдона и униформная интерполяция над логикой Гжегорчика. *Сибирский математический журнал*, 55(1):147–155, 2014. Available from: `http://mi.mathnet.ru/eng/smj2520`.

[55] Г. Е. Минц. О некоторых исчислениях модальной логики. In В. П. Оревков, editor, *Логические и логико-математические исчисления. I*, volume 98 of *Труды МИАН СССР*, pages 88–111. Nauka, 1968. Available from: `http://mi.mathnet.ru/tm2927`.

[56] В. А. Янков. Об исчислении слабого закона исключенного третьего. *Известия Академии наук СССР, Серия математическая*, 32(5):1044–1051, 1968. Available from: `http://mi.mathnet.ru/eng/izv2504`.

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