Experimental and asymptotic investigation of a rotary wave in a cylindrical container

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Rotary gravity waves in a partially filled vertical cylindrical container excited by a rotating disc at the top of the cylinder are investigated. Analytical results for the growth rate of the waves are reported. Moreover, the development of the wave is shown in an experiment.

1 Introduction

A vertical cylindrical container container with radius $\tilde{R}$ and height $\tilde{h} = \tilde{h}_A + \tilde{h}_W$ is partially filled with water, see Fig 1a. At rest, the water depth is $\tilde{h}_W$. Thus, between the water level and the top lid is a gap of with $\tilde{h}_A$. The top lid rotates with angular velocity $\tilde{\Omega}_A$ around the cylinder axis and thus, induces a rotational flow in the air buffer and in the water. Initially, this flow is axisymmetric, but under certain conditions, a rotary gravity wave may form. The waveform and the angular wave speed for a wave with small amplitude can be found by assuming an inviscid potential flow with flow potential ϕ and surface elevation h. The wave mode n-k is given by:

$$\phi^{(n-k)} = J_n(\mu^{(n-k)} r) \frac{\sinh \mu^{(n-k)}(z + h_W)}{\omega^{(n-k)} \sinh \mu^{(n-k)} h_W} \sin n(\theta - \omega^{(n-k)} t), \qquad h^{(n-k)} = J_n(\mu^{(n-k)} r) \cos n(\theta - \omega^{(n-k)} t). \tag{1}$$

All lengths are referred to the radius $\tilde{R}$ of the container. The time is made dimensionless with $\sqrt{\tilde{g}\tilde{R}}$. All other quantities are made dimensionless accordingly. Here, the constant $\mu^{(n-k)}$ is the k-th zero of the derivative of the n-th Bessel function $J_n'(\mu^{(n-k)}) = 0$. and the dimensionless angular velocity of the mode n-k is $\omega^{(n-k)} = \sqrt{\mu^{(n-k)} \tanh \mu^{(n-k)} h_W}$. The first wave modes are shown in Figure 1b., resembling the modes of sloshing waves in a cylindrical tank, cf. [1]. Due to viscosity these wave are damped. The decay rate can be determined from the boundary-layer along the rigid cylinder wall and is of order $1/\sqrt{\mathrm{Re}}$, where the Reynolds-number for the wave is defined by $\mathrm{Re} = \sqrt{\tilde{g}\tilde{R}^2/\tilde{\nu}_W}$, where $\tilde{\nu}_W$ is the kinematic viscosity of water.

2 Stability considerations

To study the excitation of waves, a stability analysis of the axisymmetric flow has been performed. Using the fact that the density and the dynamic viscosity of air are much smaller than the corresponding values of water, the original two-phase flow problem can be decoupled by first solving the air flow problem and then prescribing the resulting shear stresses at the free water surface.

Here, we assume the dimensionless shear stress distribution $\tau_r = -\mathrm{Fr}^{3/2}\mathrm{Re}^{1/2}\frac{3}{4}r(1-e^{2.5(x-1)})^3$, $\tau_\theta = -\mathrm{Fr}^{3/2}\mathrm{Re}^{1/2}\frac{1}{4}r(1-r)$ at the free surface. It is an approximation to the bottom shear stress distribution of the laminar flow in a cylinder with rotating top lid and height $h_A = 2$ for the Reynolds number $Re_A = \tilde{\Omega}_A \tilde{R}^2/\tilde{\nu}_A = 1000$. The axisymmetric flow in the water is characterized by the Ekman number $\mathrm{Ek_W} = \tilde{\nu}_W \tilde{\Omega}_W^{-1} \tilde{R}^{-2}$, where $\tilde{\Omega}_W$ is a characteristic value for the angular velocity. It is defined by $\tilde{\tau}_W = \tilde{\rho}^{1/2}\tilde{\nu}^{1/2}\tilde{R}\tilde{\Omega}_W^{3/2}$, where $\tilde{\tau}_W$ is a characteristic value for the surface shear stress. We define the Froude number $\mathrm{Fr} = 1/\mathrm{ReEk_W}$ as the ratio of $\tilde{\Omega}_W$ and the reference angular wave speed $\sqrt{\tilde{g}/\tilde{R}}$.

We perform a stability analysis of the axisymmetric flow in the asymptotic limit of large (wave) Reynolds number and small Froude number and expand the growth rate g and angular wave speed ω in terms of the Froude number. Thus, the coefficients g_2 and ω_2 are functions of the Ekman number $\mathrm{Ek_W}$:

$$g^{(n-k)} \sim \frac{1}{\sqrt{\mathrm{Re}}} g_{1/2}^{(n-k)} + \mathrm{Fr}^2\, g_2^{(n-k)}(\mathrm{Ek_W}), \qquad \omega^{(\mathrm{n-k})} \sim \omega_0^{(\mathrm{n-k})} + \frac{1}{\sqrt{\mathrm{Re}}}\omega_{1/2}^{(\mathrm{n-k})} + \mathrm{Fr}\,\omega_1^{(\mathrm{n-k})}(\mathrm{Ek_W}). \tag{2}$$

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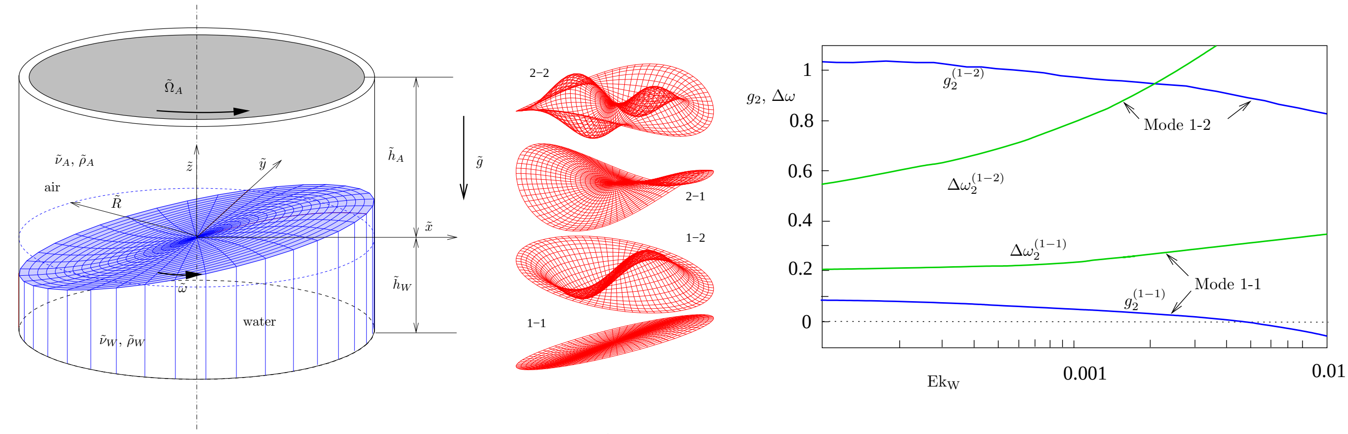
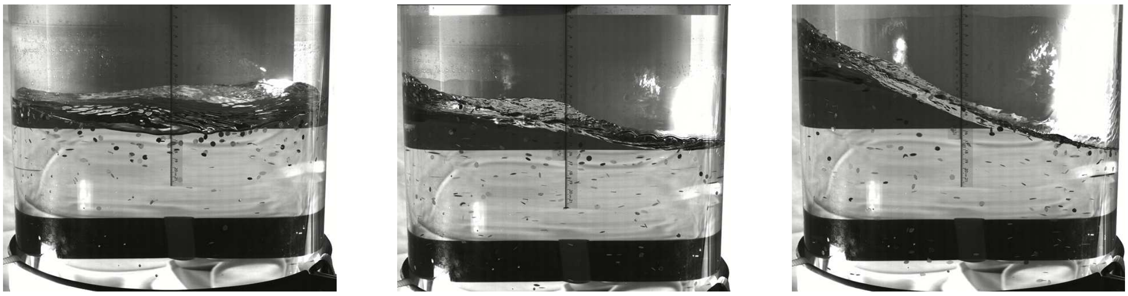


a) b) c)

Fig. 1: Experimental set up a), Surface shape of modes 1-1,1-2,2-1, and 2-2 b), growth rate and perturbation of the angular velocity for the 1-1 and 1-2 mode d)



a) b) c)

Fig. 2: Evolution of rotary gravity wave. 2-1 mode a), 1-1 and 1-2 mode b), finally 1-1 mode c)

It can be shown by partial integration that the $O(\mathrm{Fr})$-term of the expansion of the growth rate is zero. In Figure 1b the growth rate and perturbation of the angular wave speed of the modes 1-1 and 2-1 are shown. We remark, that the viscous damping term $g_{1/2}$ is already known (and negative) from the analysis of sloshing problems, cf. [1]. In the present case, g_2 becomes positive for sufficiently small Ekman numbers. Thus the axisymmetric flow becomes unstable if $g_2 > -\sqrt{\mathrm{Re}}/\mathrm{Fr}^2 g_{1/2}$. Since $g_2^{(1-2)}$ is larger than $g_2^{(1-1)}$, we expect theat the 1-2 mode is, at least initially, much more excited than the 1-1 mode.

3 Experimental results

Experiments have been made in a plexiglass tank with radius $\tilde{R} = 20\,\mathrm{cm}$ and total height of $120\,\mathrm{cm}$. First, the case of a plain rotating disc was investigated. It turned out, that the large amplitude 1-1 mode was only excited when the distance disc water level was 50 mm or less. In the case $\tilde{h}_A = 50\,\mathrm{mm}$ and $\tilde{\Omega}_A = 1100\,\mathrm{rpm}$ it took about 800 s after the start of the disc rotation until a wave on the surface has been observed. After 960 s the peak to peak distance (double amplitude) increased to 50 mm. After that, the amplitude was still growing and finally, the wave hit the disc.

In a second series of the experiments 4 T-shaped profiles with height $20\,\mathrm{mm}$ had been mounted onto the disc. Thus, the rotary flow in the air has been enhanced and a rotary wave could be observed for $\tilde{h}_A = 150\,\mathrm{mm}$ $\tilde{\Omega}_A = 500\,\mathrm{rpm}$. The Reynolds number for the air flow $\mathrm{Re_A} = 1.2 \cdot 10^5$, for the wave $\mathrm{Re} = 2.8 \cdot 10^5$ and the Ekman number for the axisymmetric water flow $1/\mathrm{Ek}_W = 4.8 \cdot 10^4$. It is here estimated by measuring the revolution time of particles floating on the free surface. After about 170 s the peak to peak distance is about 20 mm and the free surface resembles a superposition of the 2-1 and 1-2 modes, see Fig. 2a. After 250 s the peak to peak distance increased to 90 mm and the wave is transformed into a superposition of the 1-1 and 1-2 mode, Fig 2b. Finally, after 400 s the final peak to peak distance of 190 mm is attained. And the waveform is almost a pure 1-1 mode, Fig 2c.

The stability analysis has been performed for laminar flow. But some results can also be carried over to turbulent flow. The experiments reveal in agreement with the analytical result that higher modes are a excited first. With increasing amplitude, the higher modes vanish and the 1-1 base mode persists.

References

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