Source Formulations and Boundary Treatments for Lighthill’s Analogy Applied to Incompressible Flows

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Recently, second-order accurate, nonstationary flow simulations became affordable for low-Mach-number industrial applications. This enables the utilization of Lighthill’s acoustic analogy for flow noise prediction. However, many parameters related to the source term computation such as source formulation and truncation, numerical scheme, and boundary treatment influence the results and are explored in this study. An incompressible flow simulation of a simplified heating, ventilation, and air conditioning outlet geometry is set up using a finite volume solver. Aeroacoustic sources are calculated on a finite volume grid. Finally, the sound propagation is computed by means of a finite element solver. If integral source properties related to the boundary treatment and source truncation technique are satisfied, and the ability of the flow simulation to resolve fine scales according to the numerical scheme and grid resolution is taken into account, good agreement with experimental results could be achieved. In summary, the present work demonstrates how Lighthill’s acoustic analogy can be applied to flow results obtained from usual finite volume solvers. Furthermore, these findings are beneficial for result interpretation and error diagnostics of other similar aeroacoustic simulation cases.

Nomenclature

\begin{itemize}
  \item \(u\) = velocity
  \item \(u_h\) = velocity of hot-wire measurements
  \item \(V\) = volume flow rate
  \item \(v\) = vertices
  \item \(w\) = vertex weighting
  \item \(x, y, z\) = coordinate axis
  \item \(x^+, y^+, z^+\) = dimensionless velocities
  \item \(\Gamma\) = boundary of simulation domain
  \item \(\Delta\) = grid spacing
  \item \(\delta\) = Kronecker delta
  \item \(\zeta\) = damping coordinate axis
  \item \(\nu\) = kinematic viscosity
  \item \(\nu_{\text{sgs}}\) = turbulent kinematic viscosity
  \item \(\rho\) = fluctuation density
  \item \(\rho_0\) = constant/reference density
  \item \(\tau\) = viscous stress tensor
  \item \(\tau_{\text{sg}}\) = subgrid-scale stress tensor
  \item \(\Omega\) = simulation domain
  \item \(\omega\) = vorticity
  \item \(\nabla\) = gradient operator
  \item \(\otimes\) = dyadic product
  \item \(\times\) = cross product
  \item \(*\) = entrywise or Hadamard product
  \item \(\text{—}\) = temporal mean
\end{itemize}

Subscripts

\begin{itemize}
  \item \(t, tt\) = first and second temporal derivative
\end{itemize}

Superscript

\(T\) = transposed

I. Introduction

Although aerospace engineering was the main driving force for the development of tools for the calculation of aeroacoustic sound sources and sound propagation in the past, recent years have seen a rising interest in such tools also in other fields of industries (e.g., the automotive and train industries). The aims are to reduce environmental noise in urban areas and to improve comfort for...
customers. The latter accounts especially for the development of vehicle climate systems, where the aeroacoustic noise directly affects the passengers. In contrast to many aerospace applications, the flows of these applications take place in the low-Mach-number regime. Meanwhile, the increase of computational efficiency made detailed nonstationary flow simulations affordable in a wider range of industries. Owing to this development, several tools have been implemented for the prediction of aeroacoustic noise.

A. Lighthill’s Analogy

Most prediction tools for aeroacoustic noise make use of the acoustic analogy established by Lighthill [1]. Very common approaches for evaluating this analogy are integral formulations derived from Lighthill’s wave equation, such as those of Ffowcs Williams and Hawkings [2] and Curle [3]. Their major advantage is the low computational cost in comparison with solving the inhomogeneous wave equation as described by Lighthill. Their drawbacks are limited applicability to internal flows and the lack of access to the whole acoustic source and propagation field. For these reasons, the inhomogeneous wave equation will be completely resolved in space in this study. For the case of low-Mach-number flows with neglected heat conduction and viscous stresses, Lighthill provided a very popular simplification of his analogy [1]. For the incompressible flow regime, Powell established an analogy [4] mathematically equivalent to Lighthill’s simplified one. Powell’s aim was to describe the acoustic sources by vortex structures to give a more detailed insight into turbulent noise production. Thus, his analogy is often termed “vortex sound theory”. In this context, a version of Powell’s analogy is widely applied, which uses solely the divergence of the Lamb vector as an acoustic source. This method can be applied to high-Reynolds-number and low-Mach-number flows [5]. It makes use of the assumption that the kinetic energy is constant in time. This modification breaks the mathematical equality with Lighthill’s analogy in its simplified formulation. In order not to confuse these two formulations of Powell’s analogy, in this study the latter will be denoted as vortex sound theory and the former as Powell’s analogy. Although the mentioned analogies require solely the velocity field of the flow, Meecham and Ford [6] reformulated Lighthill’s analogy, such that it depends only on the flow pressure field. Powell interpreted this reformulation also as sound produced by vortices [7], but in this case, it is not the vorticity that characterizes the noise sources, but the pressure minimums inside vortices and the pressure maximums of interacting vortices.

A comparison of Lighthill’s analogy, Powell’s analogy, and vortex sound theory was carried out by Martínez-Lera et al. [8]. In contrast to this study, integral solutions by Curle [3] for Lighthill’s analogy and Powell [4] and Möhring [9] for the vortex sound theory were tested. These approaches were tested on a pair of corotating vortices and a cylinder confined in a channel. A higher sensitivity of Powell’s analogy with respect to numerical noise, especially for low-Mach-number flows, was observed. The dipolar character of the used integral formulation accounts for this behavior. Furthermore, a higher sensitivity of Powell’s analogy and vortex sound theory with respect to the spatial source truncation was observed if vortices pass the boundary of the integration volume. Another approach of Lighthill’s analogy and vortex sound theory was carried out by Miyauchi et al. [10], where direct numerical simulation (DNS) data of a compressible flow were compared with the acoustic simulations of a two-dimensional (2-D) and three-dimensional (3-D) turbulent mixing layer. They observed good agreement between Lighthill’s analogy, vortex sound theory and the DNS results for the 2-D case, whereas the vortex sound theory greatly overestimated the acoustic pressure fluctuations of the DNS and Lighthill’s analogy in the 3-D case. Moore et al. [11] compared the results of Lighthill’s analogy and vortex sound theory applied to the data for an incompressible DNS simulation of a low-Mach-number jet. They also used integral formulations. In their case, the acoustic pressures of the vortex sound theory overestimated the results derived from Lighthill’s analogy by approximately 10 dB in a wide range of the frequency spectrum. These results lack a comparison with experiments, and therefore neither of the approaches can be assumed to be more valid than the other. Similar simulations were carried out by Avital et al. [12], who also used integral formulations. They observed good agreement between the Lighthill and the vortex sound theory approaches.

In contrast to previous work, Scheit [13] coupled a finite volume DNS to finite element acoustic simulations similar to the approach in this study. He compared Lighthill’s analogy, its pressure form as described by Meecham and vortex sound theory for a two-dimensional laminar flow around a circular cylinder and turbulent flow over a forward-facing step. The latter was carried out by using a three-dimensional flow simulation and a two-dimensional acoustic simulation. For Lighthill’s analogy and its pressure form, Scheit obtained the same results, whereas the noise pressure levels were overestimated when using vortex sound theory.

Another subject investigated in this study was the contribution of the subgrid-scale stresses of the large-eddy simulation (LES) onto the acoustic spectra. Seror et al. [14] analyzed the contribution of these stresses in an a priori study applied to DNS data for isotropic turbulence. Their results indicated that the subgrid-scale stresses should not be neglected during the source term evaluation.

The aim of this study was to compare Lighthill’s simplified analogy [1], a reformulation of it mentioned later, Powell’s analogy, vortex sound theory [4], the formulation of Meecham and Ford [6], and a reformulation incorporating the LES subgrid-scale stresses.

B. Boundary Conditions and Source Truncation

Lighthill’s analogy [1] was derived directly from the compressible Navier–Stokes equations. As such, it describes all possible aeroacoustic sound production and propagation mechanisms. Common interpretations of these mechanisms were given by Curle [3] and Ffowcs Williams and Hawkings [2] by use of their integral formulations. They applied Green functions to derive integrals describing quadrupole noise due to turbulent flow, dipole noise in the presence of walls (loading noise), and monopole noise in the presence of moving walls (thickness noise). In the test case of our study, only nonmoving walls appeared. Therefore, one can expect only sources of quadrupole and dipole character. Powell, in contrast, argued that the dipole noise emitted from solid walls should be interpreted as the reflection of quadrupoles [15] from the inner flow. This is also consistent with Crighton’s interpretation [16] of Lighthill’s sources. He revealed the double integral vanishing property of Lighthill’s sources, which can only be quadrupole type in the absence of external sources of mass or body forces. He associated this property with the conservation of mass and momentum.

In this paper, we present the importance of fulfilling this double integral vanishing property when the Lighthill equation is solved using a space discrete approach such as finite elements, finite volumes or finite differences. It is shown that the source field should solely consist of discrete quadrupoles that are made of discrete point monopoles; hence, the discrete monopole source field is a superposition of discrete quadrupoles. In this regard, special care has to be taken at the boundaries of the recognized source region. For this purpose, the correct implementation of wall boundary conditions during the source term evaluation is presented. The wall boundary conditions of the incompressible Navier–Stokes equations have been discussed in detail by Gresho and Sani [17]. It is shown that these boundary conditions are important for inhibiting spurious monopoles near wall regions.

Another topic to be revisited in this context is source truncation. Obrist and Kleiser [18] studied this subject by applying various spatial damping functions to the Lighthill stresses of a turbulent jet. They observed high levels of spurious sound for the application of heasides window functions in the streamwise direction of the jet. This spurious sound was significantly reduced by using smooth Hann window functions [19]. They also observed a strong dependence of the radiated sound on the position of the applied window. In contrast to this approach, Caro et al. [20] applied the damping function not to the Lighthill stresses but to the final acoustic sources, namely to the second derivative of the Lighthill stresses. Violato et al. [21] applied the damping function to the Lamb vector while using the vortex sound theory. This equates the application of the damping function to
the first derivative of the Lighthill stresses. As shown in this study, the latter two approaches can lead to spurious monopole and dipole noise. By using the approach of Obrist and Kleiser [18] to damp the Lighthill stresses, even higher order functions are revealed to be appropriate for damping, if the discrete quadrupole behavior is preserved. The only error that appears in this approach is the missing contribution of the neglected sources to the far-field noise. Furthermore, this damping will be used as a tool to suppress sources of insufficient temporal frequency resolution. The resolved damping will be used as a tool to suppress sources of insufficient temporal frequency resolution. The resolved frequency can easily be approximated using the flow velocity and the cell spacing.

C. Parker Mode

The test case used for the simulations in this study combines several flow and acoustic phenomena, namely a jet flow, a Karman vortex shedding, a turbulent channel flow, and the excitation of a Parker mode [22,23]. Although the acoustics of first three phenomena have been numerically investigated by many researchers, this was rarely done for the excitation of Parker modes. Parker [23] and Koch [24] derived analytical solutions to obtain the frequencies of these modes, which were experimentally confirmed by Parker [23]. Dechirep et al. [25] simulated the formation and propagation of Parker modes excited by noise and generic vortices in a two-dimensional case. Yokoyama et al. [26] simulated the Parker modes in a cascade of flat plates using a compressible LES and predicted the far-field noise using the Fowkes Williams and Hawking integral formulation. They observed good agreement of the frequency and amplitude of a Parker mode between simulation and experiment. In contrast to their study, an incompressible flow simulation was used in this work, which is unable to feature Parker modes. By solving the inhomogeneous Lighthill equation, a Parker mode appeared in the last step of the hybrid approach used.

D. Simulation Procedure

For the solution of the incompressible Navier–Stokes equations, a commercial finite volume solver of second-order accuracy in space and time was used. A large-eddy turbulence model and the SIMPLE solver algorithm [27] for coupling of pressure and velocity were applied. This is a common setup applicable to industrial cases; hence, the observations in this study are transferable to practical applications. In a second step, the in-house tool CFS++ was used to evaluate the.aeroacoustic sources at the finite volume grid. Two different numerical schemes were compared in this context. After interpolation of the acoustic sources to a finite element grid, the sound propagation was computed by the in-house simulation tool CFS++. This coupling approach permits the use of adapted grids for each of the solving steps. Other techniques for such flow-acoustic coupling have been proposed by Caro et al. [29] and Martínez-Lera et al. [30], where the calculation is partially or completely performed during the interpolation of the sources onto the finite element mesh. The latter approach has the advantage of implicitly creating discrete quadrupoles from the Lighthill stresses on the finite element grid, but its disadvantage is the requirement of at least second-order finite elements.

In contrast, the method used in this study performs the whole computation of acoustic sources on the finite volume grid of the computational fluid dynamics (CFD). As a consequence, all grid refinements applied for the CFD solution were also used during source evaluation, which resulted in a finer resolution of the acoustic sources than former methods.

II. Basic Equations

In this section, the main equations used are discussed. For this purpose, the formulations used in Lighthill’s analogy are derived in the first step. These formulations differ only in acoustic sources, which were computed from the flowfield obtained from the CFD simulation. In the second step, the boundary conditions and the integral properties of these sources are demonstrated.

A. Formulations

To obtain the nonstationary flowfield, the incompressible Navier–Stokes equations for the Newtonian fluid are solved. These equations describe the conservation of mass [Eq. (1)] and the conservation of momentum [Eq. (2)] with neglected gravity forces:

\[ \nabla \cdot \mathbf{u} = 0 \]  

\[ \rho_0 (\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \rho_0 \nu \nabla^2 \mathbf{u} \]  

where \( \mathbf{u} \) denotes the flow velocity, \( p \) is the flow pressure, \( \rho_0 \) is the constant density, and \( \nu \) is the kinematic viscosity. The subscript \( t \) indicates a partial time derivative. For the test case presented in this paper, the LES–WALE turbulence model [31] was used for solving the Navier–Stokes equations. This turbulence model adds an extra term to Eq. (2), which takes unresolved subgrid-scale stresses into account. This term will for now be neglected. Furthermore, the analogy of Lighthill was used to calculate the sound propagation. In its original formulation [Eq. (3)], it contains the fluctuation pressure \( p' \), the speed of sound \( c_0 \), and the Lighthill stress tensor \( T \). This tensor \( T \) is described by Eq. (4), which introduces the fluctuation density \( \rho' \), the Kronecker delta \( \delta \), and the viscous stresses \( \tau \). The fluctuation pressure \( p' \) contains the acoustic and the aerodynamic pressure fluctuations inside the source region and is equal to the flow pressure in the compressible case. At observer points far from the aerodynamic fluctuations, it consists solely of the acoustic pressure in the incompressible case. The Lighthill stress tensor can be simplified due to the low-Mach-number flow and constant density to obtain \( T \approx \rho_0 \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \). Consequently, the simplified Lighthill (SLH) formulation Eq. (5) is obtained:

\[ p'_{ij} / c_0^2 - \nabla^2 p' = \nabla \cdot (\rho_0 \mathbf{u} \cdot \nabla \mathbf{u}) \]

Powell [4] derived an acoustic analogy in similar manner to Lighthill. His aim was to derive an analogy for incompressible flows, which describes the acoustic sources by the vorticity \( \omega_k = \nabla \times \mathbf{u} \). In its initial formulation (POWELL), Eq. (7) is mathematically identical to the SLH and MOMLH formulations. For high-Reynolds-number and low-Mach-number flows, the second term on the right-hand side of Eq. (7) is often neglected to obtain the vortex source theory formulation; see Eq. (8). This formulation (VORTEX) contains solely the divergence of the Lamb vector \( (\nabla \times \mathbf{u}) \times \mathbf{u} \) as a source term:

\[ p'_{ij} / c_0^2 - \nabla^2 p' = \rho_0 \nabla \cdot (\nabla \times (\nabla \times \mathbf{u}) \times \mathbf{u}) \]

The next reformulation [Eq. (10)] is obtained by substituting the pressure Poisson equation [Eq. (9)] into Eq. (5). This formulation (pressure-based Lighthill analogy, LAPP) was first introduced by Meecham and Ford [6] and is mathematically identical to previous formulations (SLH, MOMLH, and POWELL), if the incompressible
momentum Eq. (2) is solved. In contrast to these, solely the nonstationary pressure field is required instead of the velocity field. This reduces the amount of data to be exchanged between flow simulation and acoustic source calculation by a factor of 3:

\[ \nabla \cdot (\nabla \cdot (\rho_0 \mu \otimes u)) = -\nabla^2 p \]  
(9)

\[ \rho_0 \frac{\partial u}{\partial t} + \nabla \cdot (\rho_0 u \otimes u) = -\nabla p + \rho_0 \nabla^2 u + \nabla \cdot \tau_{sg} \]  
(11)

The inhomogeneous wave equations introduced in the previous section were solved using the finite element solver CFS++ on a finite element grid. For this purpose, the weak formulation Eq. (14) was used, which was more precisely discussed by Kaltenbacher et al. [33]. We multiplied Eq. (3) with the test function \( \delta \), integrated by parts, and replaced the right-hand side by \( S_{FE}^E \). The symbol \( \Omega \) represents the volume of the domain and \( \Gamma \) its boundary:

\[ \int_{\Omega} \left( \rho_0 \frac{\partial u}{\partial t} \right) \delta \, d\Omega + \int_{\Gamma} \nabla q \cdot n \, d\Gamma = \int_{\Omega} \rho_0 \nabla \cdot (u \otimes u - \nu_{sg} (\nabla u + (\nabla u)^T)) \delta \, d\Omega \]  
(13)

\[ \int_{\Omega} q_p \, d\Omega + \int_{\Gamma} \nabla q_p \cdot n \, d\Gamma = S_{FE}^E \]  
(14)

\[ \rho_0 \frac{\partial u}{\partial t} + \nabla \cdot (\rho_0 u \otimes u) = -\nabla p + \rho_0 \nabla^2 u + \nabla \cdot \tau_{sg} \]  
(11)

\[ T = T_{LES} + \tau_{sg} + T'' \]  
(12)

\[ \rho_0 \frac{\partial u}{\partial t} + \nabla \cdot (\rho_0 u \otimes u) = -\nabla p + \rho_0 \nabla^2 u + \nabla \cdot \tau_{sg} \]  
(9)

\[ \rho_0 \frac{\partial u}{\partial t} + \nabla \cdot (\rho_0 u \otimes u) = -\nabla p + \rho_0 \nabla^2 u + \nabla \cdot \tau_{sg} \]  
(11)

\[ T = T_{LES} + \tau_{sg} + T'' \]  
(12)

\[ p''/c_0^2 - \nabla^2 p'' = -\nabla^2 p \]  
(10)

All recent formulations were derived with the assumption that the momentum equation [Eq. (2)] is solved. Actually, in most practical cases, a direct numerical simulation is too costly, and consequently, turbulence models are used. In this study, a large-eddy simulation (LES) was performed using the wall-adapting local eddy-viscosity (WALE) [31] turbulence model. For such an LES model, the momentum equation [Eq. (2)] changes to Eq. (11), which introduces the subgrid-scale stress tensor \( \tau_{sg} \). This tensor represents the stresses acting on the fluid due to the turbulent motions not resolved by the simulation. According to Serer et al. [14], \( T \) can be split up into three terms as written in Eq. (12) for large-eddy simulations. The first tensor \( T_{LES} \) is equal to the simplified Lighthill tensor used in Eq. (5) and contains the stresses resolved by the simulation. The second tensor is \( \tau_{sg} \), which according to Serer et al. [14] should not be neglected. The last tensor \( T'' \) represents stresses that could only be modeled and are not available from the LES. Considering the first two tensors, Eq. (5) can be rewritten as Eq. (13). This formulation is denoted as WALELH in this study. It introduces the turbulent viscosity \( \nu_{sg} \), which is calculated as described elsewhere [31, 32] using a damping of the turbulence length scales, which enforce \( \nu_{sg} = 0 \) at walls [32]. From Eq. (13), it can be observed that the additional sources are of octupole type, similar to the description of Crighton for the viscous stress term in Lighthill’s sources [16]. This indicates that their influence may be neglected. The contribution of the viscous stresses in Lighthill’s tensor [see Eq. (4)] in fact vanishes after its second differentiation due to Eq. (1) for the incompressible flow case:

\[ \rho_0 (u_t + u \cdot \nabla u) = -\nabla p + \rho_0 \nabla^2 u + \nabla \cdot \tau_{sg} \]  
(11)

\[ T = T_{LES} + \tau_{sg} + T'' \]  
(12)

\[ \rho_0 \frac{\partial u}{\partial t} + \nabla \cdot (\rho_0 u \otimes u - \nu_{sg} (\nabla u + (\nabla u)^T)) \]  
(13)

B. Boundary Conditions

The inhomogeneous wave equations introduced in the previous section were solved using the finite element solver CFS++ on a finite element grid. For this purpose, the weak formulation Eq. (14) was used, which was more precisely discussed by Kaltenbacher et al. [33]. We multiplied Eq. (3) with the test function \( \delta \), integrated by parts, and replaced the right-hand side by \( S_{FE}^E \). The symbol \( \Omega \) represents the volume of the domain and \( \Gamma \) its boundary:

\[ \int_{\Omega} q_p \, d\Omega + \int_{\Gamma} \nabla q_p \cdot n \, d\Gamma = \int_{\Omega} \rho_0 \nabla \cdot (u \otimes u - \nu_{sg} (\nabla u + (\nabla u)^T)) \delta \, d\Omega \]  
(13)

Velocities and pressure in- and outlets are the most common boundary conditions for flow simulations beside walls. Usually, the CFD codes offer many different formulations of these boundary conditions to provide stability and realistic physical behavior for as many cases as possible. In contrast, for the acoustic simulation, nonreflecting boundary conditions are often used, such as absorbing boundary condition or perfectly matched layer techniques [34]. Their aim is to damp acoustic waves leaving the simulation domain and therefore inhibit unwanted reflections of the radiating sound. Another aspect to consider is that the flow domain and the domain for the acoustic propagation do not necessarily have to share the same extent. These circumstances enforce the truncation of the source terms, which may lead to nonphysical spurious noise [18]. To handle this issue, we introduce the spatial damping function \( b \) for the sources. This function is also used to suppress sources at the far-field microphones to obtain solely acoustic pressure fluctuations there. Furthermore, sources of coarse grid cells, which have an insufficient frequency resolution for the frequency range of interest, are damped. The damping function \( b \) equals 1 in regions where the acoustic sources are recognized in full and zero where the sources are completely suppressed. Accordingly, it is favorable to set \( b = 0 \) near the flow in- and outlets and near the nonreflecting boundaries of the acoustic simulation domain. As done by Obrist and Kleiser [18], the damping function is applied to the Lighthill stresses to obtain

\[ T = b \rho_0 u \otimes u \]  
(12)

These stresses are derived twice during the source term computation, and therefore also \( \nabla b \cdot \rho_0 u \) should be valid at the mentioned boundaries. Consequently, \( h = 0 \) is also valid there. The formulations used are further modified to contain solely the fluctuating contribution of the sources because the static parts do not contribute to the radiated sound. The resultant formulations are summarized in Appendix A.

Integration of the acoustic sources over the whole domain is performed by Eq. (18). The application of the Green–Gauss theorem and substitution of the condition \( h = 0 \) reveal that the sum of the sources should be zero. As Crighton [16] stated, fulfilling this constraint ensures that no spurious monopoles are present. He further pointed out that also Eq. (19) has to be fulfilled to avoid spurious dipoles. This is in fact the case because \( u = 0 \) at walls and \( b = 0 \) at the other boundaries. By that, the double integral vanishing property described by Crighton [16] for Lighthill’s acoustic sources is linked to the boundary conditions applied in this work. These constraints reflect the conservation of mass and momentum. In Secs. IV.B and
Kulite XCS-093-D pressure sensors were used, two of which were upstream to minimize the influence of sound emitted by the fan used. Furthermore, a tube silencer with core body was placed farther downstream in its orthogonal direction. At the downstream edge of the plate, the frequency of the expected Karman vortex shedding was evaluated. A flow profile consisting of 19 measurement points at a distance of 2.5 mm oriented orthogonal to the plate was measured at HP. The positions of the sensors P1 and P2 and the HWA measurements at HK and HP are shown in Fig. 2. The gray cross indicates the center of origin of the coordinate system, which is relevant for the definition of the damping function $b$. We positioned Bruel & Kjaer 4189-L-001 1/2 in. microphones in the far field to validate the acoustics. Their positions (F1 to F4) are shown in Fig. 3. They were placed far from the jet to inhibit turbulent pressure fluctuations.

**B. Flow Simulation**

We used the commercial CFD code Star-CCM+ version 11.06.010 [32] to obtain the flow solution. For the simulation, the channel length was limited to 300 mm, as illustrated in Fig. 1, to reduce the number of cells needed. Simultaneously, a simulation of an empty channel of 300 mm length was performed. The flow variables at the inlet and outlet of this channel were connected by a fully developed interface [32], which introduced a pressure jump to obtain the desired volume flow. Between the inlet and outlet, a velocity profile was extracted at each time step and set as the inlet boundary condition at the simplified outlet. The size of the whole simulation domain was 6 m in the flow direction and 4.242 m along the other axis to minimize any influence of the pressure outlet boundary condition. The simplified HVAC outlet was placed in the center of the domain.

The trimmed mesher algorithm [32] was used for grid generation, which predominantly creates fully orthogonal hexahedrons. Size transitions are realized by growth rates of a factor of 2 in each direction. The benefit is that the grid is fully aligned with the main flow direction. The cell size inside the channel was set to 0.5 mm, which was further refined to 0.25 mm in the region of the plate and downstream in its orthogonal direction. At the downstream edge of the plate, a refinement to 0.125 mm was applied. In the region of the jet, the cell sizes were successively increased, and a maximum cell size of 64 mm was reached at the outlet. At the inside of the channel and at the plate layers of prism cells were added to achieve $y^+ < 1$, $x^+ < 18$ and $z^+ < 18$ in the first cell layer. Figure 4 shows sections through the grid in the plate and jet regions. The overall HVAC outlet mesh consisted of approximately 22 million cells and the separate channel mesh of approximately 8 million cells.

The LES-WALE turbulence model was used without wall functions. The Green–Gauss scheme [32] using the MinMod limiter was evaluated in this Paper. Furthermore, we measured the flow velocities using a hot-wire anemometer (HWA). At the position HK 5 mm downstream of the plate, the frequency of the expected Karman vortex shedding was evaluated. A flow profile consisting of 19 measurement points at a distance of 2.5 mm oriented orthogonal to the plate was measured at HP. The positions of the sensors P1 and P2 and the HWA measurements at HK and HP are shown in Fig. 2. The gray cross indicates the center of origin of the coordinate system, which is relevant for the definition of the damping function $b$. We positioned Bruel & Kjaer 4189-L-001 1/2 in. microphones in the far field to validate the acoustics. Their positions (F1 to F4) are shown in Fig. 3. They were placed far from the jet to inhibit turbulent pressure fluctuations.

**III. Test Case: Simplified Outlet**

The test case used in our study is a simplified outlet of a car heating, ventilation, and air conditioning (HVAC) system. This outlet consists of a channel with a quadratic cross section of 50 mm edge length. A rectangular plate with a length of 40 mm and a thickness of 1 mm is inserted 50 mm upstream of the outlet of the channel. It is oriented along the mean flow direction. This plate excites a Karman vortex shedding in the flowfield and the β-Parker mode [22,23] in the acoustic field. A schematic section through the outlet geometry is shown in Fig. 1.

**A. Experiments**

We carried out experiments in an anechoic chamber to validate the results of the flow and acoustic simulations. A constant volume flow rate $\dot{V}$ of 0.03 m$^3$/s air was set up. This leads to a Reynolds number $Re$ of 42,756 with respect to the channel diameter. To achieve the expected fully developed turbulent channel flow at the plate and at the outlet, a channel of 2050 mm length was positioned upstream of the plate. Furthermore, a tube silencer with core body was placed farther upstream to minimize the influence of sound emitted by the fan used.

For the validation of pressure fluctuations inside the channel, six Kulite XCS-093-D pressure sensors were used, two of which were positioned Brüel & Kjaer 4189-L-001 1/2 in. microphones in the far field to validate the acoustics. Their positions (F1 to F4) are shown in Fig. 3. They were placed far from the jet to inhibit turbulent pressure fluctuations.
was set up for the gradient computation. The convection terms were calculated using a bounded central difference scheme with 15% upwinding. For solving the incompressible Navier–Stokes equations, the SIMPLE algorithm was chosen, where second-order time discretization and a time step size of $10^{-5}$ s were applied to achieve a maximum convective Courant number of 1. In contrast, the acoustic Courant number is about 20 times higher but irrelevant because no acoustic effects are captured by the CFD. After initialization from a stationary solution, the simulation was run for about 1 s of physical time. Subsequently, flow data of 14,000 time steps were exported for the acoustic simulations.

An exemplary image of the instantaneous flow velocity magnitude is shown in Fig. 5. The main flow features visible are the turbulent channel flow, the Karman vortex shedding downstream of the plate, and vortex shedding at the trailing edge of the channel where a free jet arises.

**C. Source Term Computation**

Two numerical schemes were tested to obtain $S_F$ by evaluating Eq. (16). The aim of the first scheme used is to be applicable to arbitrary polyhedral cells and conservative, which is important to satisfy the Eqs. (18) and (19). The damping should be small and the numerical error low. It is arguable whether a least-squares (LSQ) approach of high order should be used. Such an approach could easily calculate the second derivatives needed for the SLH and LAPP formulations in one run. Disadvantages of LSQ approaches are the lack of conservativeness, which would require a further modification of the stencils, and the challenging implementation of the boundary conditions. Furthermore, spurious derivative values can appear at badly shaped cells (e.g., cells with a high volume change rate or large skewness angles). To circumvent these problems, a scheme was developed that uses two Green–Gauss approaches consecutively.

To calculate the spatial derivatives needed during the evaluation of $h$ in Eq. (16), the face-based dual grid of the CFD grid was created. This was done by inverting the hierarchical structure of the grid elements as shown in Fig. 6. This structure illustrates the relation of the different grid elements from highest (3-D) to lowest (0-D) dimensions for the CFD grid and its dual grid, respectively. The lines with the diamonds indicate associated elements (e.g., cells that consist of faces). CFD grid elements and their dual counterparts are vertically aligned. The vertices of the dual grid cells are associated with the CFD grid cells and located at their centroids, whereas the vertices of the CFD grid are associated with the dual grid cells but are not necessarily located at their centroids.

Star-CCM+ is a collocated cell-centered finite-volume solver, which hence provides flow values $\Psi$ at the centroids of the cells and the boundary faces and therefore also at the vertices of the dual grid. This dual grid is used for the evaluation of the first spatial derivative using the Green–Gauss theorem. In this way, the derivative can be approximated using Eq. (20) by summing over all cell faces $f$. $\Psi_f$ denotes the average value of the faces, $a_f$ the face normal vector, and $\Omega$ the cell volume. The value $\Psi_f$ is approximated by the given values at the vertices $\Psi_v$ and a weighting $w_v$ of each vertex $v$ on the face [see Eq. (21)]. The weighting is calculated using the inverse distance between the vertex coordinates $x_v$ and the face centroid $x_f$ as written in Eq. (22):

$$\nabla \Psi \approx \sum_f \Psi_f \cdot a_f / \Omega$$  \quad (20)
The derivative $\nabla \Psi$ is now available at each dual cell. Each of these cells can be associated with a vertex of the initial CFD grid. Hence, they were interpreted as being located at these vertices during the calculation of $S_{\text{CV}}$. Before this calculation, $h$ was evaluated, whereby Eqs. (21) and (22) were adapted and used for the interpolation of flow variables $\Psi$ from the CFD cell centroids to the vertices. In the case of boundary vertices, only the values of the adjacent boundary faces were recognized. In the final step, Eq. (23) was evaluated on the CFD grid using the Green–Gauss theorem again. The approximation of $a_f$ was performed as described with Eqs. (20–22). To apply the boundary condition $h = 0$, the boundary faces $f_f$ were skipped during the summation. If the boundary condition [Eq. (17)] was applied during the CFD simulation, this may be interpreted as a wall correction to ensure that condition (18) is fulfilled in the case of the LAPP formulation:

$$S_{\text{CV}} \approx \sum_{f_f} h_f \cdot a_f \quad (23)$$

This scheme allows the calculation of second spatial derivatives on polyhedral meshes. The second derivative at each cell depends only on the flow values given in the cell and its direct neighbor cells. Neighbors in this sense are cells that share one vertex. Therefore, this scheme will be called the compact Gauss (CG) scheme.

For comparison, a noncompact or smoothed Gauss (SG) scheme was implemented as well, which makes also use of Eqs. (20) and (23). These formulas were both evaluated on the CFD grid, and the values at the faces $\Psi_f$ and $h_f$ were approximated using the values given at both adjacent cells, by weighting each with the factor $1/2$. In the case of boundary faces in Eq. (20), the given boundary values of the CFD were used. In the equivalent one-dimensional (1-D) case, this scheme would be the same as CG convoluted with a smoothing box filter. The truncation error of CG is $O(\Delta^2)$ and therefore smaller than $O((2\Delta)^2)$ of the SG scheme, with the grid spacing $\Delta$. Both schemes are second-order accurate on regular hexahedral meshes; on irregular meshes, this accuracy is reduced. In Fig. 7, the amplitude response $A$ of the equivalent 1-D cases for both schemes is plotted against the wave number $f_f \Delta$ in dimensionless form. In comparison with the theoretical second derivative, the CG scheme is able to capture high frequencies better than the SG scheme. After the interpolation described in Sec. III.B, the sources appear at the vertices of the acoustic finite element grid. The stability of the solution of Eq. (14) was shown by Hüppe [35] and Kaltenbacher et al. [34] in union with the perfectly matched layer boundaries for arbitrary vertex excitations. Consequently, both schemes are assumed to be stable in the given context, if the flow solution is not divergent.

**D. Aeroacoustic Simulation**

The aeroacoustic simulation was carried out with the finite element solver CFS++ [28]. The acoustic grid was created using Ansys ICEM-CFD [36]. A structured mesh consisting of hexahedrons was created using the resolved refinement technique [36], which creates cell size transitions of factor 3. The outer far-field domain was chosen to be large enough to contain the most intense regions of the jet as well as the microphone positions of the experiment. The smallest cell size chosen was 0.8 mm around the plate and in the region of the Karman vortex street, 2.4 mm inside the channel, 4 mm in the region of the free jet, and 12 mm in the far-field sound propagation region at the microphones. At the inlet of the channel and around the far-field region, perfectly matched layer cells [34] were added to inhibit reflections. Overall, the mesh consisted of approximately 1 million vertices. A section through the mesh is shown in Fig. 8.

For the temporal discretization, the Newmark method was set up. A temporal blending function depending on the simulation time was defined to activate the sources smoothly from 0 to 100% amplitude during the first 500 time steps, meaning the first $5 \cdot 10^{-2}$ s. To recognize this function and the dilatation of the sound waves from the channel to the microphone positions, only the last 10,000 time steps (0.1 s) were evaluated for the comparison with the experiments. The damping functions used and an instantaneous acoustic source field are illustrated in Fig. 9. The function $b_0$ damps in directions orthogonal to the main flow, whereas the other functions vary only in the flow direction $z$. The functions $b_{\text{HH}}$ and $b_f$ are damping functions at the inlet. The functions $b_{\text{HF}}$ and $b_{\text{H}}$ damp near the region of the CFD mesh coarsening from 0.5 to 1 mm, whereas $b_f$ damps near the

![Fig. 7 Amplitude response spectrum of CG and SG schemes in comparison with the theoretical second derivative.](image)

![Fig. 9 Spatial damping functions (top and bottom) and acoustic source distribution (center).](image)
IV. Results and Discussion

A. Validation of the Flowfield

In Fig. 10, a validation of the flowfield is shown by evaluation of the velocity and turbulence intensity profile at HP (see Fig. 2). For this purpose, the velocity $u_b = |u \ast (1, 0, 1)|$ is defined, which is equivalent to the velocity measured by the hot wire. This wire recognizes velocities in the $y$ direction as it is oriented along the $y$ axis. The overbar denotes the temporal average, and $\ast$ is the Hadamard or entrywise product. The turbulence intensity profile was calculated using $I = \overline{(u_b - \overline{u_b})^2}/\overline{u_b}$. Both profiles are in good agreement with the experimental results. At the point HK (see Fig. 2), a vortex shedding frequency of 2.68 kHz was measured, whereas the simulation exhibited 2.44 kHz; hence, the simulation deviated by 9%. This is due to the incompressible flow simulation, which does not capture the locking of the vortex shedding frequency by the acoustic Parker mode [22,23]. This effect was shown experimentally by Welsh and Gibson [37].

For the validation of the pressure fluctuations at P1 and P2, the power spectral density (PSD) is shown in Fig. 11. These plots include the flow simulation results (CFD) as well as computational aeroacoustic (CAA) results obtained using SLH formulation, SG scheme, and $b_0 b_1 b_2$ damping function. In the low-frequency range up to 2.6 kHz, the numerical results agree very well with the experimental results for P2, whereas at P1 the pressure fluctuations were slightly overestimated by the simulation. This may be caused by the assumption of incompressibility. In the measurement, the peak of the $\beta$-Parker mode at 2624 Hz appeared at P1. The peak did not appear at P2 because it is on the node line of this mode. Its theoretical frequency was calculated to be 2617 Hz for a plate of zero thickness using an analytical formula of Koch [24]. For the CAA simulation, this peak appeared at 2588 Hz, whereas it was not visible in the CFD simulation owing to the incompressibility. The good agreement between CFD and CAA indicates the superposition of flow pressure and acoustic pressure in the source region for Lighthill’s analogy. In this region, flow pressure fluctuations are dominant except for the Parker mode. For frequencies larger than 4 kHz, a comparison is not possible because the noise level of the sensors is 40 dB.

B. Importance of Boundary Conditions

In this section, we present the influence and importance of the previously discussed boundary conditions and integral source properties by showing examples of their violation. For this purpose, spectra obtained from different simulations at microphone F4 are shown in Fig. 12. All of these results were obtained using SLH formulation, SG scheme and damping function $b_0 b_1 b_2$, if not stated otherwise. First, we present the importance of the wall boundary conditions. For the result labeled “No Wall BCs”, Eq. (24) was applied instead of Eq. (23). The boundary faces $\Gamma_f$ were not omitted, and the values $h_f$ at the wall faces were taken from the adjacent boundary cells. As a result, discrete monopoles appeared near the wall, and condition (18) was not fulfilled. This simulation overestimated the experiment in the whole frequency range.

To demonstrate the correct application technique of the damping function, the results obtained using Eqs. (25) and (26) are shown. The former applies $b$ to the double divergence of Lighthill’s stress tensor. This scheme creates discrete monopoles because every evaluated face contributes the same value to its adjacent cells with opposite sign in Eq. (23). If consecutively the values of adjacent cells are weighted differently by multiplication with $b$, discrete dipoles are broken and discrete monopoles appear in regions where $Vb \neq 0$. These lead to a strong overestimation in the low-frequency range because Eq. (18) is not fulfilled. Equation (26) applies $b$ to $\nabla \cdot T$, hence, it creates spurious discrete dipoles in the same manner as Eq. (25) creates monopoles. These also lead to an overestimation of the low-frequency noise because Eq. (19) is violated.

The last example shows a simulation using the damping function $b_0 h_3$, which contains no damping at the inlet. Equation (19) is violated, whereas Eq. (18) is fulfilled. The $T$ value of each cell appears in up to two layers of adjacent cells as a discrete quadrupole in the SG scheme. These layers were missing at the inlet, which led to a sharp overestimation of the PSD at about 3.5 kHz:

$$S_{\tilde{V}V} = \sum_f h_f \cdot a_f$$  \hspace{1cm} (24)
The damping function from recent observations, as shown in Fig. 13b for far-field 6 kHz due to the coarsening of the cells in the jet region. The drawback is more prominent noise at frequencies higher than 3.5 kHz significantly in comparison with the microphone F1. The damping function has to be used at the inlet. The same observation applies to the jet damping functions, as shown by a comparison of the damping function from Obrist and Kleiser [18], which indicate that using a smooth Hann window function is preferable to Heaviside damping functions.

The results using the LAPP formulation differed considerably from recent observations, as shown in Fig. 13b for far-field microphone F1. The damping function $b_0b_n$ overestimated the high frequencies above 3.5 kHz significantly in comparison with the damping function $b_0b_Hb_n$, which makes use of a smooth damping at the inlet. This may be a consequence of the velocity inlet used, which permits a varying inlet pressure. The minor differences between damping functions $b_0b_Hb_n$ and $b_0b_Hb_{nf}$ indicate that the use of a smooth damping function is not required in the jet region. In contrast to the SLH formulation, a more significant difference between the near damping function $b_n$ and the far damping function $b_f$ was observed. Using $b_f$, the acoustic pressure was overestimated in the frequency range above 1 kHz. As sources of coarser cells were introduced by $b_f$, their noise became significant.

This can also be shown by considering spectra of the flow simulation results, as shown in Fig. 15 for the PSD spectra of the kinetic energy $k = \rho u^2/2$ and pressure $p$ at two points in the jet with varying cell sizes $\Delta$. The first point is in the refinement area with a cell size of 0.5 mm, and the second is farther downstream in the region of 1.0 mm cell size. Therefore, similar spectra may be expected, whereas those at the $\Delta = 1.0$ mm point are estimated to be lower because of the coarser mesh. For $k$, this was the case, whereas for $p$, some higher amplitudes appeared due to numerical noise. This noise is related to aliasing, and the cutoff frequency $f_{max}$ is assessed using the velocity and the cell size to be $f_{max} = |i|/(4\cdot\Delta)$. As a consequence, this cutoff frequency should be taken into account when defining the damping function, especially for the LAPP formulation, which is done by use of $b_n$. For cells of a size $\Delta \leq 0.5$ mm, the frequency $f_{max}$ was larger than about 6.8 kHz for the given case. This frequency was about seven times smaller than the Nyquist frequency of the simulation, which was 50 kHz. The frequency $f_{max}$ relates to a grid resolution of four cells per wavelength. However, a second-order accurate CFD simulation is not able to resolve a wave properly by four cells. The decay of the amplitude $A$ of the numerical derivatives in comparison with the theoretical derivative for high wave numbers shown in Fig. 7 indicates that a higher number of points per wavelength is required to resolve nondamped hydrodynamic waves.

\[
\frac{p_i}{c_0^2} - \nabla^2 p' = b \nabla \cdot (\nabla \cdot (u \otimes u - \bar{u} \otimes \bar{u})) \quad (25)
\]

\[
\frac{p_i}{c_0^2} - \nabla^2 p' = \nabla \cdot (b \nabla \cdot (u \otimes u - \bar{u} \otimes \bar{u})) \quad (26)
\]

C. Influence of Source Truncation

We now consider the influence of the damping functions used on the results. For this purpose, results obtained using the SLH formulation and SG scheme are shown in Fig. 13a for F1 and in Fig. 14 for F4, respectively. The latter is shown with a logarithmic frequency scale to highlight the low-frequency impact. The damping functions $b_0b_f$ and $b_0b_Hb_n$ vary solely at the inlet, and negligible differences were observed. Consequently, no smooth damping function has to be used at the inlet. The same observation applies to the inlet. This may be a consequence of the velocity inlet used, which requires a varying inlet pressure.

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D. Comparison of Schemes

Figure 16 shows results obtained using the CG scheme in comparison with the SG scheme for the SLH and LAPP formulations. For SLH, the damping function \( b_0 b_1 b_n \) was used, and for LAPP, the function \( b_0 b_1 b_n \) was used. For the SLH formulation, the CG scheme overestimated the experimental results at frequencies higher than 1 kHz. In the case of the LAPP formulation, the high-frequency noise above 4.5 kHz became even more relevant. Similar observations have previously been made by Scheit [13] when comparing three-point and five-point stencils. This effect relates to the wave-number spectra of the schemes in Fig. 7. The CG scheme captures more high-frequency components than the SG scheme and estimates the theoretical derivative more accurately. As a consequence, the amplitudes obtained using the CG scheme are higher. During the evaluation of Eq. (2), a scheme equivalent to SG scheme was used. Consequently, the high-frequency components recognized by the CG scheme may not be interpreted as part of the flow solution. These observations are also in agreement with the spectra shown in Fig. 15, as the cutoff frequency \( f_{\text{max}} \) relates to the wave number \( f_{\omega} = \Delta^{-1/4} \). The CG scheme might be advantageous if higher-order schemes are used in the CFD solution.

E. Comparison of Formulations

Spectra obtained at F1 using different formulations are presented in Fig. 17. For the LAPP formulation, the damping function \( b_0 b_1 b_n \) was used and \( b_0 b_1 b_n \) for all other formulations. The spectra of SLH, MOMLH, and POWELL were almost indistinguishable from each other. In fact, they are mathematical reformulations of SLH in the case of an incompressible flow. The only differences arise due to the numerical evaluation procedure and were shown to be negligible. By considering the derivation of the MOMLH formulation, one can derive Eq. (27) for the difference of the results:

\[
\delta p_{\text{SLH}} - \delta p_{\text{MOMLH}}
\]

should be zero because the residual of the continuity equation \( E_c \) should be zero. Therefore, these results confirm that the continuity equation was adequately converged. WALELH formulation gives comparable results to previously mentioned formulations. Its difference from the SLH

\[
\delta p_{\text{SLH}} - \delta p_{\text{WALELH}}
\]

can be described by Eq. (28), which was evaluated in Fig. 17b. Its amplitudes were 20–40 dB lower than those of the SLH formulation. This can be due to the high amount of turbulent kinetic energy resolved by the CFD simulation. The ratio of subgrid-scale turbulent kinetic energy \( K_{\text{SGS}} \) to resolved turbulent kinetic \( K_R \) in the evaluated source domain was \( K_{\text{SGS}} / K_R \approx 1/16 \), as calculated by Eq. (29). The influence of the subgrid-scale stresses can become significant in the case of higher ratios. The VORTEX formulation overestimated the SLH results by about 6 dB, especially in the low-frequency range. These results confirm similar observations made by Miyauchi et al. [10], Moore et al. [11], and Scheit [13]. The comparison with the experiments indicates that the VORTEX formulation is not suitable for this field of application. As shown already in Sec. IV.C, the LAPP formulation is more sensitive to the damping function used because its amount of numerical noise is higher. This is also visible in Fig. 17b for frequencies higher than 5 kHz. Furthermore, a slight underestimation was visible in comparison with SLH in the range between 1 and 2 kHz.
\[
\frac{\rho_0}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \nabla \cdot (\rho_0 \mathbf{u} \cdot (\nabla \mathbf{u})) = \nabla \cdot (\rho_0 \mathbf{u} \cdot E_c) \quad (27)
\]

\[
\frac{\partial^2 p}{\partial t^2} = -\rho_0 \nabla \cdot (\mathbf{v}_{sgs} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)) \quad (28)
\]

\[
K_{SGS} = \frac{\int_{\Omega} \rho_0 c_s^2 \mathbf{v}_{sgs} \cdot \mathbf{d} \Omega}{\int_{\Omega} \rho_0 (\mathbf{u}^2 - \mathbf{u} \cdot \mathbf{u}) \mathbf{d} \Omega} \quad (29)
\]

**F. Variation of Flow Rate**

To further validate the results, experiments and simulations were also carried out for an increased flow rate of 0.05 m\(^3\)/s. All simulation parameters except the flow rate remained unchanged. The resulting PSD for all far-field microphones are shown in Fig. 18. The SLH formulation and the SG scheme were used while applying the \(b_0\)\(b_{ibn}\) damping function. Good agreement between experiment and simulation could be observed for all microphones and simulations. Nevertheless, the amplitude of the Parker mode was overestimated. The frequency of the broadband peak at about 4.5 kHz for 0.05 m\(^3\)/s was underestimated, which may be due to the increased Courant number and \(y^+\). A second broadband peak appearing at about 5.5 kHz could not be resolved by the simulation. The 0.05 m\(^3\)/s simulation benefited the resolution of high-frequency components due to an increased cutoff frequency \(f_{\text{max}}\), and so it was able to resolve the peak at 8 kHz.

**V. Conclusions**

The application of Lighthill's acoustic analogy to incompressible computational fluid dynamics (CFD) data offers many different formulation possibilities for the source term of the inhomogeneous wave equation. The simplified Lighthill, momentum flow Lighthill equation, and POWELL formulations were all appropriate and gave comparable results. The influence of the subgrid-scale stresses of the large-eddy simulation simulation on the acoustic results was analyzed and found to be negligible. The VORTEX formulation using the divergence of the Lamb vector as acoustic source turned out to be inappropriate for the given application. The LAPP formulation was the least expensive to implement and offers the advantage of requiring less data exchange between CFD simulation and acoustic source calculation. It suffers from the numerical noise, which is much more dominant in the pressure signal than in the velocity signal obtained from the SIMPLE solver CFD simulation. The cutoff frequency of the mesh was easily approximated using the flow velocity and the grid size for a typical flow simulation setup with second-order accurate discretization in time and space. This cutoff frequency should also be considered during the source term truncation.

Two numerical schemes for the acoustic source term calculation were tested. It was shown that the scheme should be chosen to be consistent with the schemes used in the CFD simulation. If the scheme used in the source term calculation overresolves those of the CFD simulation, spurious noise can significantly pollute the results. Care has to be taken with respect to the boundary of the source domain. The wall boundary conditions have to be applied correctly to avoid spurious discrete monopoles. It was shown that damping
functions should be applied directly to the Lighthill stresses to create solely discrete quadrupole sources. In this way, even Heaviside functions can be used for the damping.

The given results show how the evaluation of Lighthill’s acoustic sources should be carried out for incompressible flow simulation results of second-order accuracy. This is a common simulation setup for transient industrial cases because most finite volume solvers deliver such schemes. Therefore, this work enables distinct error diagnostics and troubleshooting, if acoustoelastic results are of insufficient quality, and provides hints for a proper interpretation.

Appendix A: Final Formulations

In this section, the applied formulations of Lighthill’s analogy are presented. This is done by defining the function $h$ used in the inhomogeneous wave Eq. (15). Overbars denote temporal mean fields in this context. For SLH Eq. (A1), for LAPP Eq. (A5) and for WEALH Eq. (A6) were used. In the case of the other formulations, the function $g$ was used, which relates to $h$ by $h = g(u) − g(\bar{u})$. This resulted in Eq. (A2) for the MOMLH, Eq. (A3) for the POWELL, and Eq. (A4) for the VORTEX formulations:

$$h = \rho_0 \nabla \cdot (b(u \otimes u - \bar{u} \otimes \bar{u})) \quad (A1)$$

$$g(u) = \rho_0 (b u \cdot \nabla u + u \otimes u - \nabla b) \quad (A2)$$

$$g(u) = \rho_0 (b \nabla \times u) \times u + (\nabla b \times u) \times u \quad (A3)$$

$$h = \nabla (bp - \bar{b}) \quad (A5)$$

$$h = \rho_0 \nabla \cdot (b(u \otimes u - \bar{u} \otimes \bar{u}) - \nu_{SGS} (\nabla u + (\nabla u)^T)) \quad (A6)$$

Appendix B: Damping Functions

For the description of the damping functions, we first introduce the general semi-Hann window function $H$ [Eq. (B1)] and Heaviside function $b_h$ [Eq. (B2)]. $\zeta$ denotes the damping coordinate axis, $d$ is the extent of the damping region, and $c$ is its position. Using these definitions, the following damping functions are described:

$$b_{H}(\zeta, c, d) = \begin{cases} 
0 & \text{for } \zeta > c + d \\
(1 + \cos (\pi (\zeta - c)/d))/2 & \text{for } c \leq \zeta \leq c + d \\
1 & \text{for } \zeta < c
\end{cases} \quad (B1)$$

$$b_h(\zeta, c) = \begin{cases} 
0 & \text{for } \zeta > c \\
1 & \text{for } \zeta \leq c
\end{cases} \quad (B2)$$

$$b_{xy}(\zeta) = H(\zeta, 0.098 \cdot z + 0.04 m, 0.02 m) \quad (B3)$$

$$b_{xyz} = b_{xy}(|x|) \cdot b_{xy}(|y|) \cdot H(-z, 0, 0.02 m) \quad (B4)$$

$$b_0 = \begin{cases} 
1 & \text{for } |x| \leq 0.025 m \text{ and } |y| \leq 0.025 m \\
b_{xyz} & \text{else}
\end{cases} \quad (B5)$$

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