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# On the existence of Lipschitz continuous optimal feedback control*

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**Abstract**

We consider an optimal control problem governed by a nonlinear ODE with an integral cost functional and a control constraint. We give conditions under which the optimal open-loop control is Lipschitz continuous in time; moreover, we identify the dependence of the Lipschitz constant of the optimal control on the data of the problem. Our main assumptions include a coercivity condition and that the optimal control is an isolated solution of the variational inequality appearing in the first-order optimality system. Then we show the existence of a Lipschitz continuous optimal feedback control. As an application, we establish regularity properties of the optimal value function. A main tool for obtaining these results is the theory around Robinson's strong regularity.

**Key Words:** optimal control, optimal feedback control, Lipschitz continuity, value function

**MSC 2010:** 49N35, 49K40, 49N60, 93B52

# 1 Introduction

In this paper we consider an optimal control problem for a nonlinear control system over a fixed time interval $[0, T]$ with an integral cost functional and control constraints. The feasible controls are elements of $L^\infty$, the space of measurable and essentially bounded functions over $[0, T]$, with values in a given convex and closed set in $\mathbb{R}^m$. We assume twice differentiability (with respect to the state and the control) of the functions involved in the problem and local Lipschitz continuity of these functions and their second derivatives. We also assume the existence of a reference optimal solution. Since the reference optimal control is a function

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in $L^\infty$, its values can be changed in a subset of $[0,T]$ with Lebesque measure zero without violating the optimality. In fact, the optimal control is a class of functions that differ from each other on a set of measure zero.

Our first task is to prove that, under an integral coercivity condition at the reference solution, we can select from the class of optimal controls a function which satisfies the first-order optimality system for all $t \in [0,T]$, instead of for almost every (a.e.) $t \in [0,T]$. Then we show that under a coercivity condition this representative of the optimal controls is Lipschitz continuous with respect to time $t \in [0,T]$ provided that it is an *isolated* solution of the variational inequality in the optimality system. Moreover, we establish that the Lipschitz constant of the optimal control depends only on two constants: the coercivity constant and the Lipschitz constant of all functions defining the problem and their first and second derivatives over a certain bounded set in the space of variables (time, state, control).

The integral coercivity condition is a rather standard assumption in optimal control; the condition we use here goes back to the work of Hager [6], see also [4]. In contrast, the isolatedness condition was introduced only recently in [2, Definition 3.6] in the context of the so-called differential variational inequalities, with the aim to prevent different solution curves from crossing each other. The isolateness assumption is automatically satisfied when the Hamiltonian has a unique minimizer for each $t \in [0,T]$, e.g. when the Hamiltonian is strictly convex. In [2, Theorem 4.1], it was established that if an optimal control $\bar{u}$ is an isolated solution of the Hamiltonian variational inequality and for each $t \in [0,T]$ the mapping defining this variational inequality is strongly metrically regular at $\bar{u}$ for 0, then the optimal control $\bar{u}$ is Lipschitz continuous on $[0,T]$. We also mention the earlier work [3] in that direction for an optimal control problem with linear dynamics and a strongly convex cost for which strong regularity holds automatically; in fact, only continuity of the optimal control is claimed there but the Lipschitz continuity can be gleaned from the proof. We note that integral coercivity implies strong metric regularity in function spaces, see [2, Theorem 4.2].

Our next task is to prove the existence of a Lipschitz continuous optimal feedback control. We show that under the coercivity and isolatedness conditions for the optimal control, there exists an optimal feedback control $(\tau,\xi) \mapsto u^*(\tau,\xi)$ which is a Lipschitz continuous function; here $(\tau,\xi)$ is the parameterizing pair initial time – initial condition.

Our third and last task is to show that the existence of a Lipschitz continuous optimal feedback control implies that the optimal value function $(\tau,\xi) \mapsto V(\tau,\xi)$ is differentiable with respect to $\xi$ and its derivative is Lipschitz continuous.

A main tool for obtaining the results presented in this paper is an enhanced version of Robinson's implicit function theorem stated in [2].

An outline of the paper follows. In Section 2 we introduce the optimal control problem considered and set the stage for the further developments. Section 3 contains preliminary material showing in particular that the optimal control can be redefined on a set of measure zero so that the first-order optimality system holds for all $t \in [0,T]$. Section 4 gives conditions for Lipschitz continuity in time of the optimal open-loop control while Section 5 is devoted to the existence of a Lipschitz continuous optimal feedback control. The last Section 6 applies the latter result to show regularity of the value function.

## 2 The optimal control problem

We consider the following optimal control problem:

$$\min \Big\{ J(u) := g(x(T)) + \int_0^T h(t, x(t), u(t))\,\mathrm{d}t \Big\}, \tag{1}$$

subject to

$$\begin{aligned} \dot{x}(t) = f(t, x(t), u(t)), \qquad & x(0) = x_0, \\ x \in W^{1,\infty}, \quad u \in \mathcal{U} := \{u \in L_\infty \,:\, u(t) \in U & \text{ for a.e. } t \in [0,T]\}, \end{aligned} \tag{2}$$

where the state $x(t) \in \mathbb{R}^n$, the set $U$ of feasible control values is a closed and convex subset of $\mathbb{R}^m$, and the functions $g : \mathbb{R}^n \to \mathbb{R}$, $h : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The final time $T$ and the initial state $x_0$ are fixed.

Throughout we assume that the function $g$ is twice differentiable and its second derivative is locally Lipschitz continuous, the functions $h(t,\cdot,\cdot)$ and $f(t,\cdot,\cdot)$ are two times continuously differentiable (with respect to $(x,u)$), and these functions, together with all their derivatives are locally Lipschitz continuous (with respect to $(t,x,u)$).

We also assume that problem (1)–(2) has a locally optimal solution $(\bar{x}, \bar{u})$. The local optimality is understood in the following way: there exists a number $e_0 > 0$ such that for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_\infty \le e_0$ either there is no solution of (2) over $[0,T]$ or such a solution exists and $J(u) \ge J(\bar{u})$.

In this paper we employ the standard spaces $L^\infty$, $L^2$, $W^{1,\infty}$, $W^{1,2}$, all over $[0,T]$. Specifically, the space of controls $u$ is $L^\infty$, the space of measurable and essentially bounded functions. The state trajectory $x$ is in $W^{1,\infty}$, the space of Lipschitz continuous functions. For the controls we also use the space $L^2$ of measurable square integrable functions, and for the state trajectory $x$ the space $W^{1,2}$ such that both $x$ and its derivative $\dot{x}$ are in $L^2$. Furthermore, for an element $x$ of a metric space we denote by $I\!B_a(x)$ (respectively $\mathring{I\!B}_a(y)$) the closed (respectively open) ball centered at $x$ with radius $a$.

Clearly, any feasible control $u$ is actually a class of functions which differ from each other on a set of Lebesque measure zero. We call any particular function from this class a *representative* and denote it in the same way, by $u$.

Introducing the Hamiltonian $H(t,x,u,\lambda) = h(t,x,u) + \lambda^\top f(t,x,u)$, where $^\top$ means transposition, we employ the known first-order optimality condition, the Pontryagin mimimum principle, according to which there exists a Lipschitz continuous function $\bar{\lambda} : [0,T] \to \mathbb{R}^n$ such that the triple $(\bar{x}, \bar{u}, \bar{\lambda})$ satisfies for a.e. $t \in [0,T]$ the following optimality system:

$$\begin{aligned} & -\dot{x}(t) + f(t, x(t), u(t)) = 0, \quad x(0) - x_0 = 0, \\ & \dot{\lambda}(t) + H_x(t, x(t), u(t), \lambda(t)) = 0, \quad \lambda(T) - g_x(x(T)) = 0, \\ & H_u(t, x(t), u(t), \lambda(t)) + N_U(u(t)) \ni 0, \end{aligned} \tag{3}$$

where $H_x$ denotes the derivative of $H$ with respect to $x$, etc., and $N_U$ is the normal cone mapping to the set $U$ defined as

$$u \mapsto N_U(u) = \begin{cases} \{y \in \mathbb{R}^n \mid \langle y, v - u \rangle \le 0 \text{ for all } v \in U\} & \text{if } u \in U, \\ \emptyset & \text{otherwise.} \end{cases}$$

To put the stage, we first state the following long but important remark, which summarizes various observations that will be used later on.

**Remark 2.1.** It is a standard fact that on our assumptions there exist positive reals $d_0$ and $d$ such that for every $\tilde{u} \in \mathcal{U}$ with $\|\tilde{u} - \bar{u}\|_\infty \le d$ and for every $\xi \in I\!\!B_{d_0}(x_0)$ there exists a unique solution $\tilde{x}$ of the differential equation

$$\dot{x}(t) = f(t, x(t), \tilde{u}(t)) \quad \text{for a.e. } t \in [0, T], \quad x(0) = \xi, \tag{4}$$

which satisfies $\|\tilde{x} - \bar{x}\|_{W^{1,\infty}} \le 1$. Moreover, making $d_0$ and $d$ smaller if necessary, the (unique) solution $\tilde{\lambda}$ of the linear adjoint equation

$$\dot{\lambda}(t) + H_x(t, \tilde{x}(t), \tilde{u}(t), \lambda(t)) = 0 \quad \text{for a.e. } t \in [0, T], \quad \lambda(T) = g_x(\tilde{x}(T)) \tag{5}$$

satisfies $\|\tilde{\lambda} - \bar{\lambda}\|_{W^{1,\infty}} \le 1$. Without loss of generality, we assume that $d \le 1$ and $d \le e_0$, where $e_0$ appears in the definition of local optimality given in the beginning of this section.

Since $\bar{u} \in L_\infty$, there exists a compact set $\bar{U}$ such that $\bar{u}(t) \in \bar{U}$ for a.e. $t \in [0, T]$. Define the set

$$\Omega = \{(t, x, u, \lambda) \,:\, t \in [0, T],\ \mathrm{dist}(u, \bar{U}) \le 1,\ |x - \bar{x}(t)| \le 1,\ |\lambda - \bar{\lambda}(t)| \le 1\}.$$

Denote by $L$ the Lipschitz constant on $\Omega$ of each of the functions $f$, $g$, $h$, $f_x$, $f_u$, $h_x$, $h_u$, $f_{xx}$, $f_{xu}$, $f_{uu}$, $h_{xx}$, $h_{xu}$, $h_{uu}$, as well as of the functions $H$, $H_x$, $H_u$, $H_{xx}$, $H_{xu}$, $H_{uu}$. Since $f$ and $H_x$ are bounded in $\Omega$, then $\dot{\tilde{x}}$ and $\dot{\tilde{\lambda}}$ are also bounded. Make $L$ larger if needed so that for every $\tilde{x}$ and $\tilde{\lambda}$ that satisfy (4) and (5), respectively, the functions $(t, v) \mapsto H_u(t, \tilde{x}(t), v, \tilde{\lambda}(t))$ and $(t, v) \mapsto H_{uu}(t, \tilde{x}(t), v, \tilde{\lambda}(t))$ are Lipschitz continuous with constant $L$ in the set $\{(t, v) :\ t \in [0, T],\ \mathrm{dist}(v, \bar{U}) \le 1\}$. This concludes Remark 2.1.

To shorten the notations we skip arguments with "bar", shifting the "bar" to the functions, e.g. $\bar{H}(t) := H(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t))$, $\bar{H}(t, u) := H(t, \bar{x}(t), u, \bar{\lambda}(t))$, $\bar{f}(t) := f(t, \bar{x}(t), \bar{u}(t))$, $\bar{g}_{xx} := g_{xx}(\bar{x}(T))$, etc. Define the matrices

$$A(t) = \bar{f}_x(t),\ B(t) = \bar{f}_u(t),\ Q(t) = \bar{H}_{xx}(t),\ S(t) = \bar{H}_{xu}(t),\ R(t) = \bar{H}_{uu}(t).$$

Our first main assumption is the following:

– Coercivity: there exists a constant $\rho > 0$ such that

$$y(T)^\top \bar{g}_{xx} y(T) + \int_0^T (y(t)^\top Q(t) y(t) + w(t)^\top R(t) w(t) + 2y(t)^\top S(t) w(t))\,\mathrm{d}t \ge$$

$$\ge \rho \int_0^T |w(t)|^2\,\mathrm{d}t \tag{6}$$

for all $y \in W^{1,2}, y(0) = 0, w \in L^2$ such that $\dot{y}(t) = A(t)y(t) + B(t)w(t),\ y(0) = 0$, and $w(t) \in U - U$ for a.e. $t \in [0, T]$.

In the next section we present some preparatory material. In particular, we show that the coercivity condition implies a pointwise in time coercivity property which plays an important role in further analysis.

# 3 Preliminaries

Denote by $\mathrm{meas}(E)$ the Lebesgue measure of a set $E$. Let $\Delta \subset [0,T]$ be a measurable set with $\mathrm{meas}(\Delta) > 0$, and let $v : \Delta \to \mathbb{R}^m$ be a measurable and bounded function. For $t \in \Delta$ denote by $V_\Delta(v;t)$ the set of points $w \in \mathbb{R}^m$ with the following property: there is a sequence of measurable sets $E_k \subset \Delta$ such that

$$\mathrm{meas}(E_k) > 0, \quad E_k \subset [t-1/k, t+1/k], \quad \lim_{k\to\infty} \sup_{s\in E_k} |v(s) - w| = 0.$$

We abbreviate $V(v;t) := V_{[0,T]}(v;t)$.

A point $t \in \Delta$ is said to be *essentially non-isolated* if for every $\varepsilon > 0$ the set $[t-\varepsilon, t+\varepsilon] \cap \Delta$ is of positive measure.

**Lemma 3.1.** *Let $\Delta \subset [0,T]$ be a measurable set and let $v : \Delta \to \mathbb{R}^m$ be a measurable and bounded function. Then for any $t \in \Delta$ the following statements are equivalent:*
*(i) $V_\Delta(v;t) \neq \emptyset$;*
*(ii) $t$ is essentially non-isolated point of $\Delta$.*

*Proof.* If (i) is holds, then the very definition of $V_\Delta(v;t)$ implies that $t$ is essentially non-isolated.

Let us pick an essentially non-isolated point $t$ of $\Delta$. Let $K \subset \mathbb{R}^m$ be a compact set such that $v(s) \in K$ for every $s \in \Delta$. Take an arbitrary $w \in K$. If for every $\varepsilon > 0$ and every natural number $k$ there exists $E_k \subset [t-1/k, t+1/k] \cap \Delta$ such that $\mathrm{meas}(E_k) > 0$ and $\sup_{s\in E_k} |v(s) - w| < \varepsilon$, then $w \in V_\Delta(v;t)$. If this is not the case, then there exist $\varepsilon(w) > 0$ and a natural number $k(w)$ such that $|v(s) - w| \geq \varepsilon(w)$ for a.e. $s \in [t-1/k(w), t+1/k(w)] \cap \Delta$. Then $v(s) \notin \overset{\circ}{I\!\!B}_{\varepsilon(w)}(w)$ for a.e. $s \in [t-1/k(w), t+1/k(w)] \cap \Delta$. If $w \notin V_\Delta(v;t)$ for every $w \in K$, then, due to the compactness of $K$, there exist $w_1, \dots, w_r \in K$ such that $K \subset \cup_{i=1}^r \overset{\circ}{I\!\!B}_{\varepsilon(w_i)}(w_i)$. Denote $\bar{k} := \max\{k(w_1), \dots, k(w_r)\}$. Then $v(s) \notin \cup_{i=1}^r \overset{\circ}{I\!\!B}_{\varepsilon(w_i)}(w_i)$ for a.e. $s \in [t-1/\bar{k}, t+1/\bar{k}] \cap \Delta$. This contradicts the essential non-isolatedness of $t$, since $K \subset \cup_{i=1}^r \overset{\circ}{I\!\!B}_{\varepsilon(w_i)}(w_i)$ and $\mathrm{meas}([t-1/\bar{k}, t+1/\bar{k}] \cap \Delta) > 0$. Hence $V_\Delta(v;t) \neq \emptyset$ and the proof is complete. $\square$

Taking $\Delta = [0,T]$, we obtain that $V(v;t)$ is non-empty for every $t \in [0,T]$.

**Lemma 3.2.** *Let $u$ and $\tilde{u}$ be two measurable and bounded functions acting from $[0,T]$ to $\mathbb{R}^m$, and let $u(t) \in V(u;t)$ for every $t \in [0,T]$. Then the function $\tilde{u}$ can be redefined on a set of measure zero in such a way that $\tilde{u}(t) \in V(\tilde{u};t)$ and $|\tilde{u}(t) - u(t)| \leq \|\tilde{u} - u\|_\infty$ for every $t \in [0,T]$.*

*Proof.* Take an arbitrary $t \in [0,T]$. Consider first the case where both functions $u$ and $\tilde{u}$ are approximately continuous at $t$. We recall that $u$ is approximately continuous at $t \in (0,T)$ if there exists a measurable set $E \subset [0,T]$ containing $t$ such that

$$\lim_{k\to\infty} 2k\, \mathrm{meas}(E \cap [t-1/k, t+1/k]) = 1$$

and the restriction of $u$ to $E$ is continuous. Let $\tilde{E}$ be the set in the definition of approximate continuity of $\tilde{u}$ at $t$. Then the set $E'_k := E \cap \tilde{E} \cap [t-1/k, t+1/k]$ satisfies $\lim_{k\to\infty} 2k \operatorname{meas}(E'_k) = 1$. In particular, $\operatorname{meas}(E'_k) > 0$ for all sufficiently large $k$. Due to the continuity of $u$ and $\tilde{u}$ on $E \cap \tilde{E}$, we have

$$|\tilde{u}(t) - u(t)| \le \lim_k \left| \frac{1}{\operatorname{meas}(E'_k)} \int_{E'_k} (\tilde{u}(s) - u(s))\, \mathrm{d}s \right| \le \|\tilde{u} - u\|_\infty.$$

Moreover, since the sets $E_k$ in the definition of $V$ can be replaced by $E'_k$ we conclude that $\tilde{u}(t) \in V(\tilde{u}; t)$.,

Now, let $t \in [0, T]$ be such that either $u$ or $\tilde{u}$ is not approximately continuous at $t$. We will now redefine $\tilde{u}(t)$ to fit the claim. It is well known (see e.g. [7, Theorem 7.54]) that almost all $t \in [0, T]$ are points of approximate continuity of both $u$ and $\tilde{u}$, therefore we need to redefine $\tilde{u}$ only on a set of measure zero. Note that the sets $V(\tilde{u}; t)$ are invariant with respect to changes of $\tilde{u}$ on a set of measure zero.

Denote $w := u(t) \in V(u; t)$. Let $E_k$ be the sets in the definition of $V$. In particular, $\varepsilon_k := \sup_{s\in E_k} |u(s) - w| \xrightarrow{k} 0$. Since $E_k$ is of positive measure, it contains an essentially non-isolated point $t_k \in E_k$. According to Lemma 3.1, there exists $\tilde{w}_k \in V_{E_k}(\tilde{u}; t_k)$; hence, there exists a sequence $\{E^i_k\}_i$, $E^i_k \subset E_k$, such that

$$\operatorname{meas}(E^i_k) > 0, \quad E^i_k \subset [t_k - 1/i, t_k + 1/i], \quad \varepsilon^i_k := \sup_{s\in E^i_k} |\tilde{u}(s) - \tilde{w}_k| \xrightarrow{i} 0.$$

Let $\tilde{w}$ be a cluster point of the sequence $\{\tilde{w}_k\}$. To show that $\tilde{w} \in V(\tilde{u}; t)$ we employ the following argument involving choosing a diagonal sequence. For an arbitrary natural number $j$ choose $k = k_j$ so large that

$$|t_{k_j} - t| \le \frac{1}{2j} \quad \text{and} \quad |\tilde{w}_{k_j} - \tilde{w}| \le \frac{1}{j}.$$

Then choose $i = i_j$ such that

$$\frac{1}{i_j} \le \frac{1}{2j} \quad \text{and} \quad \varepsilon^{i_j}_{k_j} \le \frac{1}{j}.$$

We have

$$\begin{aligned} \tilde{E}_j &:= E^{i_j}_{k_j} \subset \big[t_{k_j} - 1/i_j, t_{k_j} + 1/i_j\big] \subset \big[t - |t - t_{k_j}| - 1/i_j, t + |t - t_{k_j}| + 1/i_j\big] \\ &\subset [t - 1/j, t + 1/j], \\ &\quad \sup_{s\in\tilde{E}_j} |\tilde{u}(s) - \tilde{w}| \le |\tilde{w}_{k_j} - \tilde{w}| + \sup_{s\in\tilde{E}_j} |\tilde{u}(s) - \tilde{w}_{k_j}| \le \frac{1}{j} + \varepsilon^{i_j}_{k_j} \le \frac{2}{j}. \end{aligned}$$

Taking also into account that $\operatorname{meas}(\tilde{E}_j) > 0$, the above two relations imply that $\tilde{w} \in V(\tilde{u}; t)$.

For every $k$ and $i$ we have $E^i_k \subset E_k$,

$$\left| \frac{1}{\operatorname{meas}(E^i_k)} \int_{E^i_k} u(s)\, \mathrm{d}s \;-\; w \right| = \left| \frac{1}{\operatorname{meas}(E^i_k)} \int_{E^i_k} (u(s) - w)\, \mathrm{d}s \right| \le \varepsilon_k,$$

and

$$\Big|\frac{1}{\mathrm{meas}(E_k^i)}\int_{E_k^i}\tilde{u}(s)\,\mathrm{d}s \;-\; \tilde{w}_k\Big| = \Big|\frac{1}{\mathrm{meas}(E_k^i)}\int_{E_k^i}(\tilde{u}(s)-\tilde{w}_k)\,\mathrm{d}s\Big| \le \varepsilon_k^i.$$

Hence,

$$|\tilde{w}_k - w| \le \Big|\frac{1}{\mathrm{meas}(E_k^i)}\int_{E_k^i}(\tilde{u}(s)-u(s))\,\mathrm{d}s\Big| + \varepsilon_k + \varepsilon_k^i \le \|\tilde{u}-u\|_\infty + \varepsilon_k + \varepsilon_k^i.$$

Passing to the limit with $i$ and then with $k$ we obtain $|\tilde{w}-w| \le \|\tilde{u}-u\|_\infty$. Then we redefine $\tilde{u}(t)$ as $\tilde{u}(t) = \tilde{w}$ and this completes the proof. □

**Corollary 3.3.** *Every $v \in \mathcal{U}$ can be redefined on a set of measure zero in such a way that $v(t) \in V(v;t)$ for every $t \in [0,T]$.*

For a proof, apply Lemma 3.2 with $\tilde{u} = v$ and the constant function $u(t) = u$ for all $t \in [0,T]$.

**Remark 3.4.** From now on, the element $\bar{u} \in L_\infty$ will be identified with a function (denoted again by $\bar{u}$) satisfying $\bar{u}(t) \in V(\bar{u};t)$ for every $t \in [0,T]$.

Observe that the coercivity condition (6) does not depend on the particular representative of $\bar{u}$.

**Lemma 3.5.** *Let the coercivity condition* (6) *hold, where $\bar{u}$ is identified as in Remark 3.4. Then*

$$w^\top R(t) w \ge \rho |w|^2 \quad \textit{for every } t \in [0,T] \textit{ and } w \in U - U. \tag{7}$$

*Proof.* Fix an arbitrary $t \in [0,T]$. Since $\bar{u}(t) \in V(\bar{u};t)$, there exists a sequence $E_k \subset [0,T]$ such that

$$\mathrm{meas}(E_k) > 0, \quad E_k \subset [t-1/k, t+1/k], \quad \varepsilon_k := \sup_{s\in E_k} |\bar{u}(s) - \bar{u}(t)| \to 0. \tag{8}$$

For an arbitrary $w \in U - U$ we define a function $w_k$ as

$$w_k(s) = \begin{cases} w & \text{if } s \in E_k, \\ 0 & \text{if } s \notin E_k. \end{cases}$$

Using the Cauchy formula for the equation

$$\dot{y}_k(s) = A(s) y_k(s) + B(s) w_k(s) \quad \text{for a.e. } s \in [0,T], \quad y(0) = 0,$$

we obtain that $y_k(s) = 0$ for $s \in [0, t-1/k]$ and $|y_k(s)| \le c_1 \mathrm{meas}(E_k)$ for $s \in (t-1/k, T]$, where here and further $c_1$, $c_2, \dots$ are real constants independent of $k$. Then, for the terms involved in (6), we have

$$|y_k(T)^\top g_{xx} y_k(T)| + \Big|\int_0^T y_k(s)^\top Q(s) y_k(s)\,\mathrm{d}t\Big| \le c_2(\mathrm{meas}(E_k))^2,$$

$$\Big|\int_0^T y_k(s)^\top S(s) w_k(s)\,\mathrm{d}t\Big| = \Big|\int_{E_k} y_k(s)^\top S(s)\,\mathrm{d}s\, w\Big| \le c_2(\mathrm{meas}(E_k))^2,$$

$$\int_0^T w_k(s)^\top R(s) w_k(s)\,\mathrm{d}s = w^\top \int_{E_k} R(s)\,\mathrm{d}s\, w,$$

$$\int_0^T |w_k(s)|^2\,\mathrm{d}s = \mathrm{meas}(E_k)|w|^2.$$

Since $R(s) = \bar{H}_{uu}(s, \bar{u}(s))$, using (8), we obtain (see Remark 2.1) that for $s \in E_k$ one has

$$|R(s) - R(t)| \le L(|s-t| + \varepsilon_k) \le L(1/k + \varepsilon_k) =: \tilde{\varepsilon}_k \to 0.$$

Using the above estimated in (6) and the above five displayed formulas, we obtain

$$\mathrm{meas}(E_k)\, w^\top R(t) w \ge \rho\, \mathrm{meas}(E_k)|w|^2 - c_4(\mathrm{meas}(E_k))^2 - c_5\, \mathrm{meas}(E_k)\tilde{\varepsilon}_k.$$

Dividing by $\mathrm{meas}(E_k)$ (here we use the first inequality in (8)) and passing to the limit with $k$ we obtain (7). □

# 4 Lipschitz continuity of the optimal control

Let us rewrite the optimality system (3) as follows:

$$\begin{aligned} -\dot{x}(t) + f(t, x(t), u(t)) &= 0, \\ x(0) - x_0 &= 0, \\ \dot{\lambda}(t) + H_x(t, x(t), u(t), \lambda(t)) &= 0, \\ \lambda(T) - g_x(x(T)) &= 0, \\ H_u(t, x(t), u(t), \lambda(t)) + N_U(u(t)) &\ni 0. \end{aligned} \tag{9}$$

**Lemma 4.1.** *Let the coercivity condition hold. Then the optimal control $\bar{u} \in L_\infty$ has a representative $\bar{u}$ such that the matrix $R(t) = \bar{H}_{uu}(t, \bar{u}(t))$ satisfies (7) and $(\bar{x}(t), \bar{u}(t), \bar{\lambda}(t))$ satisfies (9) for all $t \in [0,T]$. In fact, any representative of the optimal controls that satisfies $\bar{u}(t) \in V(\bar{u}; t)$ for all $t \in [0,T]$ has this property.*

*Proof.* Let us redefine $\bar{u}$ so that $\bar{u}(t) \in V(\bar{u}; t)$ for all $t \in [0,T]$ (see Corollary 3.3 and Remark 3.4). Then, according to Lemma 3.5, pointwise cercivity condition (7) holds for every $t \in [0,T]$.

Fix an arbitrary $t \in [0,T]$. Since $\bar{u}(t) \in V(\bar{u}; t)$, there exists a sequence $\{E_k\}$ of measurable subsets of $[0,T]$ such that (8) holds. Since $\mathrm{meas}(E_k) > 0$ and (9) is satisfied by $(\bar{x}(t), \bar{u}(t), \bar{\lambda}(t))$ almost everywhere, there exists $t_k \in E_k$ such that (9) holds for $t_k$. From (8) we obtain that $t_k \to t$ and $\bar{u}(t_k) \to \bar{u}(t)$. Then, due to the continuity of the function $(t, u) \mapsto H_u(t, \bar{x}(t), u, \bar{\lambda}(t))$ and the upper semi-continuity of the mapping $u \mapsto N_U(u)$, (9) holds for $t$ as well. □ We recall next

the property of *strong metric regularity* of a general set-valued mapping $\mathcal{F} : \mathcal{Y} \rightrightarrows \mathcal{Z}$, where

$\mathcal{Y}$ and $\mathcal{Z}$ are metric spaces (for more on that, see, e.g. [5, Sect 3.7]). The mapping $\mathcal{F}$ is said to be strongly metrically regular at $\hat{y}$ for $\hat{z}$ if there exist constants $\kappa \geq 0$, $a > 0$ and $b > 0$ such that the truncated inverse mapping

$$\mathbb{B}_b(\hat{z}) \ni z \mapsto \mathcal{F}^{-1}(z) \cap \mathbb{B}_a(\hat{y})$$

is single-valued (a function) and Lipschitz continuous. Here $\mathcal{F}^{-1}(z) = \{y \mid z \in \mathcal{F}(y)\}$.

If $(t, u) \in \operatorname{cl} \operatorname{gph}(\bar{u})$ then there exists a sequence $t_k \to t$ such that $\bar{u}(t_k) \to u$. According to (7), we have

$$w^\top \bar{H}_{uu}(t_k, \bar{u}(t_k))w \geq \rho |w|^2 \quad \text{for every } w \in U - U.$$

Passing to the limit, we obtain that

$$w^\top \bar{H}_{uu}(t, u)w \geq \rho |w|^2 \tag{10}$$

for every $(t, u) \in \operatorname{cl} \operatorname{gph}(\bar{u})$ and every $w \in U - U$. It is well known that the property (10) implies that for every $(t, u) \in \operatorname{cl} \operatorname{gph}(\bar{u})$ the mapping

$$v \mapsto \bar{H}_u(t, u) + \bar{H}_{uu}(t, u)(v - u) + N_U(v) \tag{11}$$

is strongly metrically regular at $u$ for 0 with constants $\kappa' = 1/\rho$, $a' = b' = +\infty$ (that is, with any positive $a'$ and $b'$), see, e.g., [6, Lemma 1]. Note that these constants are independent of $t$.

We will now reformulate, adapted to our notations and needs, Theorem 3.5 in [2].

**Theorem 4.2.** *Assume that for every $(t, u) \in \operatorname{cl} \operatorname{gph}(\bar{u})$ the mapping in (11) is strongly metrically regular at $u$ for 0 with constants $a'$, $b'$, $\kappa'$ independent of $(t, u)$. Then for every $t \in [0, T]$ the mapping $t \mapsto \bar{H}_u(t, u) + N_U(u)$ is strongly metrically regular at $\bar{u}(t)$ for 0 with any constants $a$, $b$, $\kappa$ satisfying the inequalities*

$$a \leq \frac{a'}{2}, 2L'a\kappa' < 1, 4L'a^2 < b', \kappa > \frac{\kappa'}{1 - 2L'a\kappa'}, 4L'a^2 + 2b < b', 2\kappa b < a, \tag{12}$$

*where $L'$ is a Lipschitz constant of the mapping $u \mapsto \bar{H}_{uu}(t, u)$ on $\mathbb{B}_{a'}(\bar{u}(t))$, for every $t \in [0, T]$.*

The conditions (12) are not stated in Theorem 3.5 in [2], but are explicitly written in the beginning of its proof there.

Since we apply the above theorem with $a' = 1$, $b' = +\infty$ and $\kappa' = 1/\rho$, inequalities (12) reduce to

$$a \leq \frac{1}{2}, \quad 2La < \rho, \quad \kappa > \frac{1}{\rho - 2La}, \quad 2\kappa b < a, \tag{13}$$

where now $L$ is the constant from Remark 2.1.

**Remark 4.3.** An important consequence of (13) is that the constants $a$, $b$, $\kappa$ of strong metric regularity of $u \mapsto \bar{H}_u(t, u) + N_U(u)$ at $\bar{u}(t)$ for 0 can be chosen to depend only on the constant $\rho$ in the coercivity condition (6) and the constant $L$ in Remark 2.1.

We introduce next our second main assumption:

– Isolatedness: The function $\bar{u}$ (represented as in Lemma 4.1) is an isolated solution of the inclusion $\bar{H}_u(t,u) + N_U(u) \ni 0$ for all $t \in [0,T]$, meaning that there exists a (relatively) open set $\mathcal{O} \subset [0,T] \times \mathbb{R}^m$ such that

$$\{(t,u) \in [0,T] \times \mathbb{R}^m \ : \ \bar{H}_u(t,u) + N_U(u) \ni 0\} \cap \mathcal{O} = \mathrm{gph}\,(\bar{u}). \tag{14}$$

For example, the isolatedness assumption holds if for every $t \in [0,T]$ the inclusion $\bar{H}_u(t,u) + N_U(u) \ni 0$ has a unique solution (which has to be $\bar{u}(t)$). In this case, one can verify the isolatedness condition taking any (relatively) open set $\mathcal{O} \subset [0,T] \times \mathbb{R}^m$ containing $\mathrm{gph}\,(\bar{u})$.

**Theorem 4.4.** *Suppose that the isolatedness assumption* (14) *and condition (7) hold. Then the optimal control $\bar{u}$ is Lipschitz continuous in* $[0,T]$*. Moreover, the Lipschitz constant of $\bar{u}$ depends only on the number $\rho$ in the coercivity condition* (6) *and the constant $L$ in Remark 2.1.*

*Proof.* The proof is somewhat parallel to the proof of Theorem 3.7 in [2]. Here we use Theorem 4.2 and (13) instead of the more general Theorem 3.5 in [2] (used in the proof of Theorem 3.7 in [2]), which does not imply the second claim of Theorem 4.4.

As mentioned around (10), condition (7) implies that for every $(t,u) \in \mathrm{cl}\,\mathrm{gph}\,(\bar{u})$ the mapping in (11) is strongly metrically regular at $u$ for 0. Then we can apply Theorem 4.2. Let the numbers $a$, $b$, $\kappa$ are chosen to satisfy conditions (13), so that for every $t \in [0,T]$ the mapping $u \mapsto \bar{H}_u(t,u) + N_U(u)$ is strongly metrically regular at $\bar{u}(t)$ for 0 (see Theorem 4.2). Let $L$ be the constant in Remark 2.1; then the mappings $(t,u) \mapsto \bar{H}_u(t,u)$ and $(t,u) \mapsto \bar{H}_{uu}(t,u)$ are Lipschitz continuous with constant $L$ in the set $\{(t,u) \ : \ t \in [0,T], \ u \in I\!B_a(\bar{u}(t))\}$. Without loss of generality we consider $\bar{u}$ as taking values in the set $\bar{U}$ in Remark 2.1; we also recall that $a \leq 1$.

Take an arbitrary $t \in [0,T]$. Then pick $\alpha_t < a/2$ and then $\gamma_t \in (0,1)$ such that $(\tau, v) \in \mathcal{O}$ for every $\tau \in [t-\gamma_t, t+\gamma_t] \cap [0,T]$ and $v \in I\!B_{\alpha_t}(\bar{u}(t))$, and also

$$\kappa L \gamma_t < 1/2, \quad L(a+2)\gamma_t \leq b, \quad 4\kappa L \gamma_t \leq \alpha_t(1-\kappa L\gamma_t). \tag{15}$$

For an arbitrary $\tau \in [t-\gamma_t, t+\gamma_t] \cap [0,T]$ define the mapping $g_{\tau,t} : U \to \mathbb{R}^m$ as

$$g_{\tau,t}(u) = \bar{H}_u(\tau,u) - \bar{H}_u(t,u), \quad u \in U.$$

Then we have that

$$|g_{\tau,t}(\bar{u}(t))| \leq L|\tau - t| \leq L\gamma_t, \tag{16}$$

and, for any $u$, $u' \in I\!B_a(\bar{u}(t))$,

$$\begin{aligned} |g_{\tau,t}(u) - g_{\tau,t}(u')| &= |\bar{H}_u(\tau,u) - \bar{H}_u(\tau,u') - \bar{H}_u(t,u) + \bar{H}_u(t,u')| \\ &\leq \int_0^1 |\bar{H}_{uu}(\tau, u' + s(u-u')) - \bar{H}_{uu}(t, u'+s(u-u'))|\,\mathrm{d}s\,|u-u'| \\ &\leq L\gamma_t|u-u'|. \end{aligned}$$

Set

$$\kappa'_t = 2\kappa/(1-\kappa L\gamma_t), \quad \beta_t = \mu_t = L\gamma_t.$$

According to the inequality $\alpha_t < a/2$, the inequalities in (15), and the definitions of $\kappa'_t$, $\alpha_t$, $\beta_t$, the following inequalities are fulfilled (for convenience we skip the subscripts $t$ for a moment):

$$(17) \qquad \mu > 0,\ \kappa\mu < 1,\ \kappa' > \kappa/(1-\kappa\mu),\ \alpha \le a/2,\ 2\mu\alpha + 2\beta \le b,\ 2\kappa'\beta \le \alpha.$$

Now, we apply Theorem 3.2 in [2] (which is a part of [5, Theorem 5G.3][1]). For readers convenience we reproduce next the claism.

**Theorem 4.5.** *Let $\mathcal{F}$ be a set-valued mapping from the Banach space $\mathcal{Y}$ to the Banach space $\mathcal{Z}$, and let $\mathcal{F}$ be strongly metrically regular at the point $\hat{y}$ for $\hat{z}$ with constants $\kappa$, $a$, $b$. Then for every numbers $\kappa'$, $\alpha$, $\beta$ satisfying (17) and for every function $g : \mathcal{Y} \to \mathcal{Z}$ satisfying*

$$\|g(\hat{y})\| \le \beta \quad \textit{and} \quad \|g(y) - g(y')\| \le \mu\|y - y'\| \quad \textit{for every } y,\ y' \in I\!B_{2\alpha}(\hat{y}),$$

*the mapping $I\!B_\beta(0) \ni z \mapsto (g + \mathcal{F})^{-1}(z) \cap I\!B_\alpha(\hat{y})$ is Lipschitz continuous with Lipschitz constant $\kappa'$.*

For short, denote $G_t(u) := \bar{H}_u(t,u) + N_U(u)$. In our context all assumptions of the last theorem are satisfied with $g = g_{\tau,t}$. Thus we obtain that the mapping

$$I\!B_{\beta_t}(0) \ni z \mapsto (g_{\tau,t} + G_t)^{-1}(z) \cap I\!B_{\alpha_t}(\bar{u}(t)) = (G_\tau)^{-1}(z) \cap I\!B_{\alpha_t}(\bar{u}(t))$$

is Lipschitz continuous with Lipschitz constant $\kappa'_t = 2\kappa/(1-\kappa L\gamma_t) \le 4\kappa$ (see the first inequality in (15)). In particular, there exists a unique $v \in I\!B_{\alpha_t}(\bar{u}(t))$ such that $0 \in G_\tau(v)$. Since $\tau \in [t-\gamma_t, t+\gamma_t] \cap [0,T]$ and $v \in I\!B_{\alpha_t}(\bar{u}(t))$, we also have that $(\tau, v) \in \mathcal{O}$. Due to isolatedness condition, we obtain that $v = \bar{u}(\tau)$. From (16) we obtain that $g_{\tau,t}(\bar{u}(t)) \in I\!B_{\beta_t}(0)$. Thus

$$\bar{u}(t) = (g_{\tau,t} + G_t)^{-1}(g_{\tau,t}(\bar{u}(t))) \cap I\!B_\alpha(\bar{u}(t)).$$

Since $\bar{u}(\tau) = (g_{\tau,t} + G_t)^{-1}(0) \cap I\!B_\alpha(\bar{u}(t))$, using (16), we get that

$$|\bar{u}(t) - \bar{u}(\tau)| \le \kappa' |g_{\tau,t}(\bar{u}(t))| \le 4\kappa L|t-\tau|.$$

Summarizing, we obtain that for every $t \in [0,T]$ there exists a neighborhood $(t-\gamma_t, t+\gamma_t) \cap [0,T]$ in $[0,T]$ in which $\bar{u}$ is Lipschitz continuous with the same constant $4\kappa L$. This implies that $\bar{u}$ is Lipschitz continuous with the same constant in the whole interval $[0,T]$.

The second claim of the theorem follows from Remark 4.3 concerning $\kappa$. □

The example displayed in Remark 9 in [4] demonstrates that the isolatedness assumption (14) is essential for the Lipschitz continuity of the optimal control shown in Theorem 4.4. In this example $h = (u^2-1)^2$, $g = 0$, $h = 0$, $U = \mathbb{R}$, $T = 1$. Here, for each measurable set $\Omega \subset [0,1]$ the function defined as $u(t) = -1$ for $t \in \Omega$ and $u(t) = 1$ for $t \in [0,1] \setminus \Omega$ is an optimal control, and the coercivity condition is satisfied. However, the isolatedness condition is satisfied only if the measure of $\Omega$ is either zero or 1. In these two cases the optimal control is (equivalent to) Lipschitz continuous, indeed.

[1]taking into account Errata and Addenda at https://sites.google.com/site/adontchev/

## 5 Lipschitz continuous optimal feedback control

In this section we prove the existence of a Lipschitz continuous locally optimal feedback control for problem (1)–(2). For this purpose we embed the problem into a family of problems by replacing the initial time 0 with any $\tau \in [0, T]$ and the initial condition $x(0) = x_0$ with $x(\tau) = \xi \in \mathbb{R}^n$. Denote this new family of problems by $\mathrm{P}(\tau, \xi)$, so that $\mathrm{P}(0, x_0)$ is (1)–(2). Also, denote by $J(\tau, \xi; u)$ the value of the objective function of $\mathrm{P}(\tau, \xi)$ for a control $u \in \mathcal{U}$ being defined as

$$J(\tau, \xi; u) := g(x(T)) + \int_\tau^T h(t, x(t), u(t))\,\mathrm{d}t,$$

where $x$ is the solution of the initial-value problem

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{for a.e. } t \in [\tau, T], \quad x(\tau) = \xi. \tag{18}$$

The following definition recasts the usual way a locally optimal feedback control is understood. Recall that $(\bar{x}, \bar{u})$ is a locally unique solution of problem (1)–(2).

**Definition 5.1.** The function $u^* : [0, T] \times \mathbb{R}^n \to U$ is said to be a locally optimal feedback control around the reference solution pair $(\bar{x}, \bar{u})$ if there exist positive numbers $\varepsilon_0$ and $\bar{a}$, and a set $\Gamma \subset [0, T] \times \mathbb{R}^n$ such that

(i) $\mathrm{gph}\,(\bar{x}) + \{0\} \times I\!\!B_{\varepsilon_0}(0) \subset \Gamma$;

(ii) for every $(\tau, \xi) \in \Gamma$ the equation

$$\dot{x}(t) = f(t, x(t), u^*(t, x(t))), \quad x(\tau) = \xi, \tag{19}$$

has a unique absolutely continuous solution $\hat{x}[\tau, \xi]$ on $[\tau, T]$ which satisfies $\mathrm{gph}\,(\hat{x}[\tau, \xi]) \subset \Gamma$;

(iii) the function $\hat{u}[\tau, \xi](\cdot) := u^*(\cdot, \hat{x}[\tau, \xi](\cdot))$ is measurable, bounded, and satisfies $\|\hat{u}[\tau, \xi] - \bar{u}\|_\infty \le \bar{a}$, and $J(\tau, \xi; \hat{u}[\tau, \xi]) \le J(\tau, \xi; u)$, where $u$ is any admissible control on $[\tau, T]$ with $\|u - \bar{u}\|_\infty \le \bar{a}$ for which the corresponding solution $x$ of (19) exists on $[\tau, T]$ and satisfies $\mathrm{gph}\,(x) \subset \Gamma$;

(iv) $u^*(\cdot, \bar{x}(\cdot)) = \bar{u}(\cdot)$.

Our main result in this section follows.

**Theorem 5.2.** *Let the coercivity and isolatedness conditions hold. Then there exists a locally optimal feedback control $u^* : [0, T] \times \mathbb{R}^n \to U$ around $(\bar{x}, \bar{u})$ which is Lipschitz continuouson a set $\Gamma$ appearing (together with the positive numbers $\varepsilon_0$ and $\bar{a}$) in Definition 5.1.*

Let us first sketch the idea of the proof. First, we prove that for $\xi$ close to $\bar{x}(\tau)$ a unique solution $(\bar{x}[\tau, \xi], \bar{u}[\tau, \xi])$ exists and it is close to the restriction of $(\bar{x}, \bar{u})$ to $[\tau, T]$. Moreover, $\bar{u}[\tau, \xi]$ depends in a Lipschitz way on $\xi$ (in the space $L^\infty$). Then we show that $\bar{u}[\tau, \xi]$ is Lipschitz continuous.

For any $\tau \in [0, T)$ we define the spaces

$$Y_\tau = W^{1,\infty} \times L^\infty \times W^{1,\infty}, \qquad Z_\tau = L^\infty \times \mathbb{R}^n \times L^\infty \times \mathbb{R}^n \times L^\infty,$$

where the time interval for these functional spaces is $[\tau, T]$. It is convenient to define the norm in $Y_\tau$ as $\|(x, u, \lambda)\| := \max\{\|x\|_{1,\infty},\, \|u\|_\infty,\, \|\lambda\|_{1,\infty}\}$. For any fixed $\tau \in [0, T)$, any (locally) optimal solution-multiplier triple $y := (x, u, \lambda) \in Y_\tau$ for $\mathrm{P}(\tau, \bar{x}(\tau))$ satisfies the inclusion

$$F_\tau(y) + G_\tau(y) \ni 0, \tag{20}$$

where $F_\tau : Y_\tau \to Z_\tau$ and $G_\tau : Y_\tau \rightrightarrows Z_\tau$ are defined as

$$F_\tau(y) = \begin{pmatrix} -\dot{x} + f(\cdot, x, u) \\ x(\tau) - \bar{x}(\tau) \\ \dot{\lambda} + H_x(\cdot, x, u, \lambda) \\ \lambda(T) - g_x(x(T)) \\ H_u(\cdot, x, u, \lambda) \end{pmatrix}, \qquad G_\tau(y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ N^\infty_{\mathcal{U}}(u) \end{pmatrix}.$$

Here

$$N^\infty_{\mathcal{U}}(u) := \{v \in L^\infty \,:\, v(t) \in N_U(u(t)) \text{ for a.e. } t \in [\tau, T]\}.$$

By using the superscript $\infty$ in the notation of the latter set we emphasize that the cone $N^\infty_{\mathcal{U}}(u)$ includes only a part of the normal cone $N_{\mathcal{U}}(u)$ which is a subset of the dual space of $L^\infty$; note that the dependence on $\tau$ is not indicated.

Further we use the numbers $a$, $b$, $\kappa$, and $L$ introduced in the preceding section (see (13) and Remark 4.3).

**Proposition 5.3.** *Let the coercivity and isolatedness conditions hold. Then the mapping $F_\tau + G_\tau$ is strongly metrically regular at the restriction of $\bar{y} := (\bar{x}, \bar{u}, \bar{\lambda})$ to $[\tau, T]$ (denoted in the same way) for $0$. Moreover, the constants of strong metric regularity, call them $\bar{a}$, $\bar{b}$, $\bar{\kappa}$, can be chosen as depending only on the constants $\rho$ and $L$ (thus independent of $\tau$).*

*Proof.* We use the fact that the coercivity condition (6) is fulfilled for problem $\mathrm{P}(\tau, \bar{x}(\tau))$ with the same constant $\rho$ for all $\tau$. To show this, it is enough to take $w(t) = 0$ on $[0, \tau)$ in (6). The next step is to linearize the generalized equation (20) at $(\bar{x}, \bar{u}, \bar{\lambda})$ and to transform the obtained linearization to a linear-quadratic optimal control problem with canonical parameters. What remains is to utilize the coercivity condition (6) to obtain global Lipschitz continuity of the solution of the linearized generalized equation, where the Lipschitz constant depends only on the coercivity constant $\rho$ and the ranges of the norms of $F_\tau$ and its derivatives when $y = (x, u, \lambda)$ varies, say, in a ball with radius one around the reference $(\bar{x}, \bar{u}, \bar{\lambda})$. This could be accomplished by repeating the proof of [6, Lemma 3] with a few minor adjustments. The last step is to apply Theorem 4.5 obtaining the desired Lipschitz localization with constants depending only on $\rho$ and $L$. $\square$

As a consequence of the last proposition, for any $\xi \in \mathbb{B}_{\bar{b}}(\bar{x}(\tau))$ the inclusion $F_\tau(y) + G_\tau(y) + \zeta \ni 0$ with $\zeta = (0, -\xi, 0, 0, 0)$ has a unique solution $(\bar{x}[\tau, \xi], \bar{u}[\tau, \xi], \bar{\lambda}[\tau, \xi])$ in $\mathbb{B}_{\bar{a}}((\bar{x}, \bar{u}, \bar{\lambda}))$ and it is Lipschitz continuous with respect to $\xi \in \mathbb{B}_{\bar{b}}(\bar{x}(\tau))$ with Lipschitz constant $\bar{\kappa}$ in the norms of $\mathbb{R}^n$ and $Y_\tau$.

Clearly, the constant $\bar{b}$ can be decreased, if necessary, without affecting the strong metric regularity property. Then we may assume that $\bar{b} > 0$ is chosen so small that the coercivity assumption (6) adapted to problem $\mathrm{P}(\tau, \xi)$ with $\xi \in \mathbb{B}_{\bar{b}}(\bar{x}(\tau))$ holds with a constant $\rho/2$

(instead of $\rho$). Here and further "adapted" means that the matrices $A$, $B$, $Q$, $R$, $S$ are calculated along $(\bar{x}[\tau,\xi],\bar{u}[\tau,\xi],\bar{\lambda}[\tau,\xi])$ instead of $(\bar{x},\bar{u},\bar{\lambda})$ and the integration in (6) is on $[\tau,T]$. Since on the coercivity assumption, the necessary optimality condition (3) is also sufficient (for local optimality), we obtain the following proposition.

**Proposition 5.4.** *Let the coercivity and isolatedness conditions hold. Then for any $\xi \in I\!B_{\bar{b}}(\bar{x}(\tau))$ the pair $(\bar{x}[\tau,\xi],\bar{u}[\tau,\xi])$ defined in the second to last paragraph is the unique locally optimal solution of problem $P(\tau,\xi)$, in the set $I\!B_{\bar{a}}((\bar{x},\bar{u}))$. Moreover, the function $I\!B_{\bar{b}}(\bar{x}(\tau)) \ni \xi \mapsto \bar{u}[\tau,\xi]$ is Lipschitz continuous with Lipschitz constant $\bar{\kappa}$ in the space $L_\infty$.*

In is important no note that assuming $\bar{b}$ small enough we may guarantee that Remark 2.1 is still valid with $e_0$, $d_0$ and $d$ replaced with $e_0/2$, $d_0/2$ and $d/2$, respectively, and for the interval $[\tau,T]$ and the function $\bar{u}[\tau,\xi]$, $\xi \in I\!B_{\bar{b}}(\bar{x}(\tau))$, instead of $[0,T]$ and $\bar{u}$. The constant $L$ remains the same.

As already mentioned, the coercivity assumption, adapted to problem $\mathrm{P}(\tau,\xi)$, is fulfilled (with $\rho/2$ instead of $\rho$), provided that $\xi \in I\!B_{\bar{b}}(\bar{x}(\tau))$. According to Lemma 4.1, an arbitrary redefinition of $\bar{u}[\tau,\xi]$ on a set of measure zero which satisfies $\bar{u}[\tau,\xi](t) \in V_{[\tau,T]}(\bar{u}[\tau,\xi];t)$ for every $t \in [\tau,T]$ (and such exists due to Corollary 3.3) fulfills the conditions that the matrix $R(t)$ satisfies (7), and $\bar{y}[\tau,\xi]$ satisfies (9) for every $t \in [\tau,T]$, all adapted to problem $\mathrm{P}(\tau,\xi)$. Moreover, according to Lemma 3.2, $\bar{u}[\tau,\xi]$ can be assumed to satisfy $|\bar{u}[\tau,\xi](t)-\bar{u}(t)| \le \|\bar{u}[\tau,\xi]-\bar{u}\|_\infty$ for every $t \in [\tau,T]$.

**Lemma 5.5.** *Let the coercivity and isolatedness conditions hold. Then there exists a number $\varepsilon \in (0,\bar{b}]$ such that for every $\tau \in [0,T)$ and $\xi \in I\!B_\varepsilon(\bar{x}(\tau))$ the control $\bar{u}[\tau,\xi]$ satisfies the isolatedness assumption (on $[\tau,T]$), namely there exists a (relatively) open set $\mathcal{O} \subset [\tau,T]\times\mathbb{R}^m$ such that*

$$\{(t,u) \in [\tau,T]\times\mathbb{R}^m \,:\, H_u(t,\bar{x}[\tau,\xi](t),u,\bar{\lambda}[\tau,\xi](t)) + N_U(u) \ni 0\} \cap \mathcal{O} =$$

$$= \mathrm{gph}\,(\bar{u}[\tau,\xi]). \tag{21}$$

*Proof.* Let us take $\varepsilon > 0$ so small that

$$\varepsilon \le \bar{b}, \qquad 2L\left(2\bar{\kappa}+\bar{L}+1\right)\varepsilon < \rho, \tag{22}$$

where $\bar{L}$ is the Lipschitz constant of $\bar{u}$ (see Theorem 4.4).

For arbitrarily fixed $\tau \in [0,T)$ and $\xi \in I\!B_\varepsilon(\bar{x}(\tau))$ denote for shortness $\tilde{y} = (\tilde{x},\tilde{u},\tilde{\lambda}) := \bar{y}[\tau,\xi]$. Also denote $\tilde{R}(t) = H_{uu}(t,\tilde{y}(t))$, $\tilde{H}_u(t,u) = H_u(t,\tilde{x}(t),u,\tilde{\lambda}(t))$. Then due to the first inequality in (22) and the redefinition of $\tilde{u}$ described above, we know that for every $t \in [\tau,T]$

$$|\tilde{u}(t)-\bar{u}(t)| \le \|\tilde{u}-\bar{u}\|_\infty, \quad \tilde{H}_u(t,\tilde{u}(t)) + N_U(\tilde{u}(t)) \ni 0, \tag{23}$$

$$w^\top \tilde{R}(t) w \ge \frac{\rho}{2}|w|^2 \quad \forall\, w \in U-U. \tag{24}$$

Let us define

$$\mathcal{O} = \left(\mathrm{gph}\,(\tilde{u}) + (-\varepsilon,\varepsilon)\times \mathring{I\!B}_\varepsilon(0)\right) \cap ([\tau,T)\times\mathbb{R}^m),$$

which is relatively open in $[\tau, T] \times \mathbb{R}^m$. We shall prove that the claim of the lemma holds with this set $\mathcal{O}$.

Since the inclusion "$\supset$" in (21) is clear, we assume, targeting a contradiction, that there exists a point

$$(t_0, u_0) \in \{(t,u) \in [\tau, T] \times \mathbb{R}^m \,:\, \tilde{H}_u(t,u) + N_U(u) \ni 0\} \cap \mathcal{O}$$

which is not in $\operatorname{gph}(\tilde{u})$. Then $\tilde{u}(t_0) \neq u_0$. From the above inclusion and the second relation in (23) we have

$$\tilde{H}_u(t_0, u_0) + N_U(u_0) \ni 0, \qquad \tilde{H}_u(t_0, \tilde{u}(t_0)) + N_U(\tilde{u}(t_0)) \ni 0.$$

From here

$$\tilde{H}_u(t_0, u_0)(\tilde{u}(t_0) - u_0) \geq 0, \qquad \tilde{H}_u(t_0, \tilde{u}(t_0))(u_0 - \tilde{u}(t_0)) \geq 0,$$

which implies

$$(\tilde{H}_u(t_0, u_0) - \tilde{H}_u(t_0, \tilde{u}(t_0)))(u_0 - \tilde{u}(t_0)) \leq 0.$$

Then, using (24) (notice that $u_0 \in U$, since otherwise $N_U(u_0) = \emptyset$) we obtain

$$\begin{aligned}
0 &\geq (\tilde{H}_u(t_0, u_0) - \tilde{H}_u(t_0, \tilde{u}(t_0)))(u_0 - \tilde{u}(t_0)) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \tilde{H}_u(t_0, \tilde{u}(t_0) + s(u_0 - \tilde{u}(t_0)))\,\mathrm{d}s\,(u_0 - \tilde{u}(t_0)) \\
&= \int_0^1 (u_0 - \tilde{u}(t_0))^\top \tilde{H}_{uu}(t_0, \tilde{u}(t_0) + s(u_0 - \tilde{u}(t_0)))(u_0 - \tilde{u}(t_0))\,\mathrm{d}s \\
&\geq \int_0^1 (u_0 - \tilde{u}(t_0))^\top \tilde{R}(t_0)(u_0 - \tilde{u}(t_0))^\top\,\mathrm{d}s - \int_0^1 sL|u_0 - \tilde{u}(t_0)|^3\,\mathrm{d}s \\
&\geq \frac{L\rho}{2}|u_0 - \tilde{u}(t_0)|^2 - \frac{L}{2}|u_0 - \tilde{u}(t_0)|^3.
\end{aligned}$$

Hence,

$$\rho \leq L|\tilde{u}(t_0) - u_0|. \tag{25}$$

Due to the inclusion $(t_0, u_0) \in \mathcal{O}$, there exists $(t_1, u_1) \in \operatorname{gph}(\tilde{u})$ such that $|t_1 - t_0| \leq \varepsilon$, $|u_1 - u_0| \leq \varepsilon$. Then continuing the inequality (25), we obtain

$$\begin{aligned}
\rho &\leq L\left(|\tilde{u}(t_0) - \bar{u}(t_0)| + |\bar{u}(t_0) - \bar{u}(t_1)| + |\bar{u}(t_1) - \tilde{u}(t_1)| + |\tilde{u}(t_1) - u_0|\right) \\
&\leq L\left(\|\tilde{u} - \bar{u}\|_\infty + \bar{L}|t_0 - t_1| + \|\tilde{u} - \bar{u}\|_\infty + \varepsilon\right) \\
&\leq L\left(2\bar{\kappa}\varepsilon + \bar{L}\varepsilon + \varepsilon\right) \leq L\left(2\bar{\kappa} + \bar{L} + 1\right)\varepsilon.
\end{aligned}$$

This inequality contradicts (22), which completes the proof. □

Having proved that the isolatedness condition is also fulfilled for problem $\mathrm{P}(\tau, \xi)$, we can apply Theorem 4.4 to this problem and obtain that the (locally) optimal control $\bar{u}[\tau, \xi]$ is Lipschitz continuous. The Lipschitz constant, $\bar{L}$, depends on the problem only through the constant $\rho$ (now $\rho/2$) and the constant $L$, therefore can be chosen independent of $\tau$ and $\xi$, provided that $|\xi - \bar{x}(\tau)| \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small (independent of $\tau$).

*Proof of Theorem 5.2.* Let the number $\varepsilon \in (0, \bar{b}]$ be the one from Lemma 5.5. There exists a number $\varepsilon_0 \in (0, \varepsilon]$ such that for every $\tau \in [0, T]$ and $\xi \in I\!B_{\varepsilon_0}(\bar{x}(\tau))$ the corresponding $(\bar{x}[\tau, \xi], \bar{u}[\tau, \xi])$ is defined, $|\bar{x}[\tau, \xi](t) - \bar{x}(t)| \le \varepsilon$ for every $t \in [\tau, T]$, and $\|\bar{u}[\tau, \xi] - \bar{u}\|_\infty < \bar{a}$. Such $\varepsilon_0$ exists because the mapping $I\!B_{\bar{b}} \ni \xi \mapsto (\bar{x}[\tau, \xi], \bar{u}[\tau, \xi)]$ is Lipschitz continuous in $W^{1,\infty} \times L^\infty$ with constant $\bar{\kappa}$.

Define the set

$$\Gamma = \{\bar{x}[\tau, \xi](t) \ : \ \tau \in [0, T], \ \xi \in I\!B_{\varepsilon_0}(\bar{x}(\tau)), \ t \in [\tau, T]\}. \tag{26}$$

Clearly $\operatorname{gph}(\bar{x}) + \{0\} \times I\!B_{\varepsilon_0}(0) \subset \Gamma$. For $\tau \in [0, T]$ denote

$$\Gamma_\tau := \{\xi \ : \ (\tau, \xi) \in \Gamma\}.$$

Since for every $\tau \in [0, T]$ we have $\Gamma_\tau \subset B_\varepsilon(\bar{x}(\tau))$, the function $\Gamma_\tau \ni \xi \mapsto \bar{u}[\tau, \xi]$ is Lipschitz continuous with Lipschitz constant $\bar{\kappa}$. Moreover, each of these functions $\bar{u}[\tau, \xi]$ is Lipschitz continuous on $[\tau, T]$ with Lipschitz constant $\bar{L}$. In addition, according to Proposition 5.4, for every $\tau \in [0, T)$ and $\xi \in \Gamma_\tau$, the function $\bar{u}[\tau, \xi]$ is the unique locally optimal control for the corresponding problem $\mathrm{P}(\tau, \xi)$ in the set $I\!B_{\bar{a}}(\bar{u})$.

Now define the feedback control $x \mapsto u^*(\cdot, x)$ as

$$u^*(t, x) := \bar{u}[t, x](t) \quad \text{ for } (t, x) \in \Gamma \tag{27}$$

The values $\bar{u}[t, x](t)$ are well defined since $\bar{u}[t, x]$ is a (Lipschitz) continuous function. Clearly, the requirements of Definition 5.1 are fulfilled; the last one follows from the identity $\bar{u}[t, \bar{x}(t)] = \bar{u}(t)$.

Let us consider two arbitrary pairs $(\tau, \xi), (s, \eta) \in \Gamma$. Due to the Dynamic Programming Principle, for every $s \ge \tau$, $\tau, s \in [0, T)$ and every and every $t \in [s, T]$ we have

$$\bar{u}[\tau, \xi](t) = \bar{u}[s, \bar{x}[\tau, \xi](s)](t).$$

Then

$$\begin{aligned}
|u^*(\tau, \xi) - u^*(s, \eta)| &= |\bar{u}[\tau, \xi](\tau) - \bar{u}[s, \eta](s)| \\
&\le |\bar{u}[\tau, \xi](\tau) - \bar{u}[\tau, \xi](s)| + |\bar{u}[\tau, \xi](s) - \bar{u}[s, \eta](s)| \\
&\le \bar{L}|s - \tau| + |\bar{u}[s, \bar{x}[\tau, \xi](s)](s) - \bar{u}[s, \eta](s)| \\
&\le \bar{L}|s - \tau| + \bar{\kappa}|\bar{x}[\tau, \xi](s) - \eta| \\
&\le \bar{L}|s - \tau| + \bar{\kappa}\left(|\bar{x}[\tau, \xi](s) - \xi| + |\xi - \eta|\right) \\
&\le \bar{L}|s - \tau| + \bar{\kappa}\left(M|s - \tau| + |\xi - \eta|\right),
\end{aligned}$$

where $M$ is an upper bound of $|f(t, x, u)|$ in the set $\Omega$ defined in Remark 2.1. This completes the proof of Theorem 5.2. □

**Remark 5.6.** The last part of the proof and the uniqueness claim in Proposition 5.4 imply that the function $\hat{u}[\tau, \xi]$ appearing in Definition 5.1 is the unique locally optimal control for in problem $\mathrm{P}(\tau, \xi)$ in the set $I\!B_{\bar{a}}(\bar{u})$.

# 6 Regularity of the value function

In this section we show that the existence of a Lipschitz continuous optimal feedback established in Theorem 5.2 implies certain regularity properties of the value function. In the preceding sections we assume only local optimality at the reference point, see Definition 5.1. In line with that assumption, we introduce the following definition:

**Definition 6.1.** The function $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is said to be a local value function of problem (1)–(2) around a reference admissible pair $(\bar{x}, \bar{u})$ if there exist positive numbers $\varepsilon_0$ and $\bar{a}$, and a set $\Gamma \subset [0,T] \times \mathbb{R}^n$ such that $\mathrm{gph}\,(\bar{x}) + \{0\} \times I\!B_{\varepsilon_0}(0) \subset \Gamma$ and for every $(\tau, \xi) \in \Gamma$ one has

$$V(\tau, \xi) = \inf_u J(\tau, \xi; u),$$

where the infimum of the objective function $J(\tau, \xi; u)$ in problem $\mathrm{P}(\tau, \xi)$ is taken over all admissible pairs $(x, u)$ for which $\|u - \bar{u}\|_\infty \le \bar{a}$, equation (18) has a unique solution $x$ on $[\tau, T]$, and $\mathrm{gph}\,(x) \subset \Gamma$.

By this definition the local value function, with a set $\Gamma$ and a neighborhood $I\!B_{\bar{a}}(\bar{u})$, is finite if for every $(\tau, \xi) \in \Gamma$ there exists at least one admissible pair $(x, u)$ satisfying $\|u - \bar{u}\|_\infty \le \bar{a}$ and $\mathrm{gph}\,(x) \in \Gamma$. Clearly, in that case $(\bar{x}, \bar{u})$ is a locally optimal solution.

As in Section 5, we denote $\Gamma_\tau := \{\xi \,:\, (\tau, \xi) \in \Gamma\}$. Thus the condition $\mathrm{gph}\,(\bar{x}) + \{0\} \times I\!B_{\varepsilon_0}(0) \subset \Gamma$ in Definition 6.1 means that $I\!B_{\varepsilon_0}(\bar{x}(t)) \subset \Gamma_t$ for every $t \in [0,T]$. We also denote $\overset{\circ}{\Gamma} = \{(\tau, \xi) \,:\, \tau \in [0,T],\, \xi \in \mathrm{int}(\Gamma_\tau)\}$.

**Theorem 6.2.** *Let the coercivity and isolatedness conditions hold. Then problem* (1)–(2) *has a (finite) local value function $V$ around $(\bar{x}, \bar{u})$ (with a set $\Gamma$ and parameters $\varepsilon_0$ and $\bar{a}$); moreover $V(\tau, \cdot)$ is differentiable with respect to $\xi$ whenever $(\tau, \xi) \in \overset{\circ}{\Gamma}$ and the derivative $V_\xi$ is Lipschitz continuous on $\overset{\circ}{\Gamma}$.*

*Proof.* The proof is routine, in principle, but we present it in full, because we deal here with a *local* value function, which requires some attention to detail. We will prove the theorem with $\Gamma$, $\varepsilon_0$ and $\bar{a}$ as in Theorem 5.2. Then there is a locally optimal Lipschitz continuous feedback control $u^*$ in the sense of Definition 5.1, with the corresponding pairs $(\hat{x}[\tau, \xi], \hat{u}(\tau, \xi])$. According to this definition, we have

$$V(\tau, \xi) = g(\hat{x}[\tau, \xi](T)) + \int_\tau^T h(s, \hat{x}[\tau, \xi](s), \hat{u}[\tau, \xi](s))\, \mathrm{d}s. \tag{28}$$

First we prove the following claim.

*Claim A:* for every $(\tau, \xi_0) \in \overset{\circ}{\Gamma}$ there exists a number $\delta > 0$ such that for every $\xi, \xi' \in I\!B_\delta(\xi_0)$ one has $\|\hat{u}[\tau, \xi] - \bar{u}\|_\infty \le \bar{a}$ and the initial value problem

$$\dot{x}(t) = f(t, x(t), \hat{u}[\tau, \xi](t)), \quad x(\tau) = \xi', \tag{29}$$

has a unique solution $x^{\xi, \xi'}$ on $[\tau, T]$ and $\mathrm{gph}\,(x^{\xi, \xi'}) \subset \Gamma$. Note that $x^{\xi, \xi} = \hat{x}[\tau, \xi]$.

We recall (see Definition 5.1) that $\hat{x}[\tau,\xi]$ is the solution of the equation (18) and for the control $u^*(t,\hat{x}[\tau,\xi](t))$, where $u^*$ is defined in (27). Since $x \mapsto u^*(\cdot,x)$ is Lipschitz continuous, we have by a standard argument that for any $(\tau,\xi_0) \in \overset{\circ}{\Gamma}$ there is a number $\delta_0 \in (0,\bar{a}/\bar{L}]$ such that $I\!B_{\delta_0}(\hat{x}[\tau,\xi_0](t)) \subset \Gamma_t$ for every $t \in [\tau,T]$. Indeed, due to the Lipschitz continuity of the right-hand side of (18) every solution starting backwards from a point in a sufficiently small neighborhood of $\hat{x}[\tau,\xi_0](t)$ at time $t > \tau$ takes values only in $I\!B_{\varepsilon_0}(\xi_0)$ at time $\tau$. Due to the definition of $\Gamma$ in (26), the graph of each such trajectory in contained in $\Gamma$ and $\|\hat{u}[\tau,\xi] - \bar{u}\|_\infty \le \bar{a}$. Then Claim A follows from (29) thanks to the Lipschitz continuity of $\hat{u}[\tau,\xi]$ in $\xi$ (Proposition 5.4). A proof of that fact, e.g., by contradiction, is straightforward

In addition, we have the representation

$$x^{\xi,\xi'}(t) - \hat{x}[\tau,\xi](t) = \Phi[\tau,\xi](t,\tau)(\xi'-\xi) + o(|\xi'-\xi|), \quad t \in [\tau,T],$$

where $\Phi[\tau,\xi](t,s)$ is the fundamental matrix solution of the linearization of the differential equation in (29) normalized at $t=s$, that is,

$$\frac{\partial}{\partial t}\Phi(t,s) = f_x(t,\bar{x}[\tau,\xi](t),\bar{u}[\tau,\xi](t))\Phi(t,s), \quad \Phi(s,s) = \text{ the identity.}$$

In particular, there exists a constant $C$ such that for every $\xi,\ \xi' \in I\!B_\delta(\xi_0)$ one has

$$\|x^{\xi,\xi'} - x^{\xi,\xi}\|_\infty = \|x^{\xi,\xi'} - \hat{x}[\tau,\xi]\|_\infty \le C|\xi'-\xi|. \tag{30}$$

Due to Definition 6.1 and Claim A, we have that

$$V(\tau,\xi') \le g(x^{\xi,\xi'}(T)) + \int_{\bar{\tau}}^T h(s,\hat{x}^{\xi,\xi'}(s),\hat{u}[\tau,\xi](s))\,\mathrm{d}s,$$

From this inequality and (28) we obtain that

$$\begin{aligned}
&V(\tau,\xi') - V(\tau,\xi)\\
&\le g(x^{\xi,\xi'}(T)) - g(\hat{x}[\tau,\xi](T))\\
&\qquad\qquad + \int_\tau^T [h(s,x^{\xi,\xi'}(s),\hat{u}[\tau,\xi](s)) - h(s,\hat{x}[\tau,\xi](s),\hat{u}[\tau,\xi](s))]\,\mathrm{d}s\\
&= g_x(\tilde{x}^T)\Phi[\tau,\xi](T,\tau)(\xi'-\xi)\\
&\qquad\qquad + \int_\tau^T h_x(s,\tilde{x}(s),\hat{u}[\tau,\xi](s))\Phi[\tau,\xi](s,\tau)(\xi'-\xi)\,\mathrm{d}s + o(|\xi'-\xi|),
\end{aligned}$$

where $\tilde{x}^T \in \mathrm{co}\{x^{\xi,\xi'}(T),\hat{x}[\tau,\xi](T)\}$ and $\tilde{x}(\cdot)$ is a measurable selection of the set-valued map $s \mapsto \mathrm{co}\{x^{\xi,\xi'}(s),\hat{x}[\tau,\xi](s)\}$. Due to the Lipschitz continuity of $g_x$ and $h_x$, and (30), we obtain

$$\begin{aligned}
V(\tau,\xi') - V(\tau,\xi) \le \Big[&g_x(\hat{x}[\tau,\xi](T))\Phi[\tau,\xi](T,\tau)\\
&+ \int_\tau^T h_x(s,\hat{x}[\tau,\xi](s),\hat{u}[\tau,\xi](s))\Phi[\tau,\xi](s,\tau)\,\mathrm{d}s\Big](\xi'-\xi) + o(|\xi'-\xi|).
\end{aligned}$$

It is well-known that the expression in the brackets equals $(\hat\lambda[\tau,\xi](\tau))^\top$, where, as before, $\hat\lambda[\tau,\xi]$ is the solution of the adjoint equation (9) for the reference pair $(\hat x[\tau,\xi], \hat u[\tau,\xi])$ and end-point condition $\hat\lambda(T) = g_x(\hat x[\tau,\xi](T))$. Hence,

$$V(\tau,\xi') - V(\tau,\xi) \le \langle \hat\lambda[\tau,\xi](\tau), (\xi' - \xi)\rangle + o(|\xi' - \xi|). \tag{31}$$

Using this inequality with $\xi' = \xi_0$ and taking into account the Lipschitz continuity of $\hat\lambda[\tau,\xi]$ with respect to $\xi$ we obtain

$$V(\tau,\xi_0) - V(\tau,\xi) \le \langle \hat\lambda[\tau,\xi](\tau), \xi_0 - \xi\rangle + o(|\xi_0 - \xi|) = \langle \hat\lambda[\tau,\xi_0](\tau), \xi_0 - \xi\rangle + o(|\xi_0 - \xi|).$$

Using (31) with $\xi' := \xi$ and $\xi = \xi_0$ we obtain

$$V(\tau,\xi) - V(\tau,\xi_0) \le \langle \hat\lambda[\tau,\xi_0](\tau), \xi - \xi_0\rangle + o(|\xi - \xi_0|).$$

Combining the last two inequalities we obtain that $V$ is differentiable with respect to $\xi$ at $\xi_0$; furthermore, $V_\xi(\tau,\xi_0) = \hat\lambda[\tau,\xi_0](\tau)$. The Lipschitz continuity of $V_\xi$ follows from the last expression and the Lipschitz continuity of the function $(\tau,\xi) \mapsto \hat\lambda[\tau,\xi_0](\tau)$. □

Observe that that $\Gamma$, $\varepsilon_0$ and $\bar a$ in this theorem can be taken to be those in the proof of Theorem 5.2. Also, observe that at the end of the last proof we obtained the equality $V_\xi(\tau,\xi_0) = \hat\lambda[\tau,\xi_0](\tau)$, which, as well known, holds under various sets of assumptions. Moreover, based on Theorem 6.2 one can verify that the local value function $V$ is a classical solution of the corresponding Hamilton-Jacobi-Bellman equation (see e.g. [1, Chapter III.3]).

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