



Snarks with Special Spanning Trees

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Abstract

Let G be a cubic graph which has a decomposition into a spanning tree T and a 2-regular subgraph C , i.e. $E(T) \cup E(C) = E(G)$ and $E(T) \cap E(C) = \emptyset$. We provide an answer to the following question: which lengths can the cycles of C have if G is a snark? Note that T is a hist (i.e. a spanning tree without a vertex of degree two) and that every cubic graph with a hist has the above decomposition.

Keywords Cubic graph · Snark · Spanning tree · Hist · 3-Edge coloring

1 Introduction

For terminology not defined here, we refer to [4]. All considered graphs are finite and without loops. A *cycle* is a 2-regular connected graph. A *snark* is a cyclically 4-edge connected cubic graph of girth at least 5 admitting no 3-edge coloring. Snarks play a central role for several well known conjectures related to flows and cycle covers in graph theory, see [16].

A *hist* in a graph is a spanning tree without a vertex of degree two (hist is an abbreviation for homeomorphically irreducible spanning tree, see [2]). A *decomposition* of a graph H is a set of edge disjoint subgraphs covering $E(H)$. Note that a cubic graph G has a hist if and only if G has a decomposition into a spanning tree and a 2-regular subgraph. For an example of a cubic graph with a hist, respectively, with the above decomposition, see for instance Figs. 6 or 10 where the dashed edges illustrate the 2-regular subgraph.

In general connected cubic graphs need not have a hist. Even cubic graphs with arbitrarily high cyclic edge-connectivity do not necessarily have a hist, see [12]. We call a snark G a *hist-snark* if G has a hist. At first glance hist-snarks may seem very special. However, using a computer and [6] (see also [5]), we recognize the following,

Theorem 1.1 *Every snark with less than 38 vertices is a hist-snark.*

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Observe that not every snark has a hist. There are at least two snarks with 38 vertices which do not have a hist, see X_1 and X_2 in Appendix A3. The following definition is essential for the entire paper.

Definition 1.2 Let G be a cubic graph with a hist T .

- (i) An outer cycle of G is a cycle of $G - E(T)$.
- (ii) Let $\{C_1, C_2, \dots, C_k\}$ be the set of all outer cycles of G with respect to T , then we denote by $oc(G, T) = \{|V(C_1)|, |V(C_2)|, \dots, |V(C_k)|\}$.

Note that all vertices of the outer cycles in Definition 1.2 are leaves of the hist. Observe also that $oc(G, T)$ is a multiset since several elements of $oc(G, T)$ are possibly the same number, see for instance the hist-snark in Fig. 10. If we refer to the outer cycles of a hist-snark, then we assume that the hist of the snark is given and thus the outer cycles are well defined (a hist-snark may have several hist). This paper answers the following type of problem. For any $m \in \mathbb{N}$, is there a hist-snark with precisely one outer cycle such that additionally the outer cycle has length m ? Corollary 2.10 answers this question and the more general problem is solved by the main result of the paper:

Theorem 1.3 Let $S = \{c_1, c_2, \dots, c_k\}$ be a multiset of k natural numbers. Then there is a snark G with a hist T such that $oc(G, T) = S$ if and only if the following holds:

- (i) $c_1 = 6$ or $c_1 \geq 10$, if $k = 1$.
- (ii) $c_j \geq 5$ for $j = 1, 2, \dots, k$ if $k > 1$.

One of our motivations to study hist-snarks is a conjecture on cycle double covers, see Conjecture 3.1. Note that this conjecture has recently be shown to hold for certain classes of hist-snarks, see [13]. Special hist-snarks with symmetric properties can be found in [10]. For examples of hist-snarks within this paper, see Figs. 6, 7, 8, 10 (dashed edges illustrate outer cycles). For a conjecture which is similar to the claim that every connected cubic graph G has a decomposition into a spanning tree and a 2-regular subgraph, see the 3-Decomposition Conjecture in [11].

2 On the Lengths of Outer Cycles of Hist-snarks

To prove Theorem 1.3, we develop methods to construct hist-snarks. Thereby we use in particular modifications of the dot product and a handful of computer generated snarks. In all drawings of this section, dotted thin edges symbolize removed edges whereas dashed edges illustrate edges of outer cycles. The graphs which we consider may contain multiple edges.

The *neighborhood* $N(v)$ of a vertex v denotes the set of vertices adjacent to v and does not include v itself. The cyclic edge-connectivity of a graph G is denoted by $\lambda_c(G)$. *Subdividing* an edge e means to replace e by a path of length two.

We define the dot product (see Fig. 1) which is a known method to construct snarks, see [1,14]. Let G and H be two cubic graphs. Let $e_1 = a_1b_1$ and $e_2 = a_2b_2$ be two independent edges of G and let e_3 with $e_3 = a_3b_3$ be an edge of H with

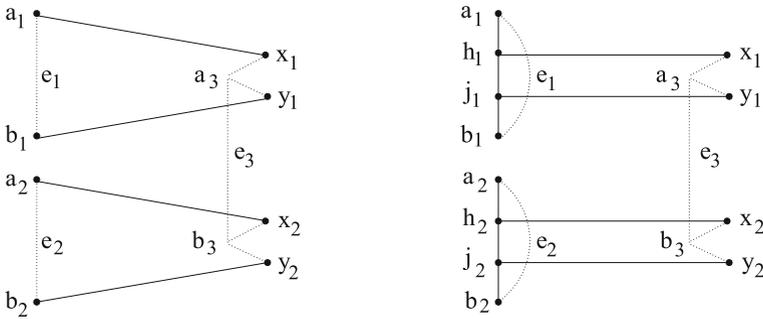


Fig. 1 Constructing the dot product $G \cdot H$ and $G(\underline{e_1}, \underline{e_2}) \bullet H(e_3)$, see Def. 2.2

$N(a_3) - b_3 = \{x_1, y_1\}$ and $N(b_3) - a_3 = \{x_2, y_2\}$. The dot product $G \cdot H$ is the cubic graph

$$(G \cup H - a_3 - b_3 - \{e_1, e_2\}) \cup \{a_1x_1, b_1y_1, a_2x_2, b_2y_2\} .$$

Both results of the next lemma are well known. For a proof of Lemma 2.1(i), see for instance [16, p. 69]. Lemma 2.1(ii) is folklore (a published proof is available in [1]).

Lemma 2.1 *Let G and H be both 2-edge connected cubic graphs, then the following holds*

- (i) $G \cdot H$ is not 3-edge colorable if both G and H are not 3-edge colorable.
- (ii) $G \cdot H$ is cyclically 4-edge connected if both G and H are cyclically 4-edge connected.

We use a modification of a dot product where $\{h_1, j_1, h_2, j_2\}$ is a set of four distinct vertices which is disjoint with $V(G) \cup V(H)$:

Definition 2.2 Set $G(\underline{e_1}, \underline{e_2}) \bullet H(e_3) := (G \cdot H - \{a_1x_1, b_1y_1\}) \cup \{h_1, j_1, a_1h_1, h_1j_1, j_1b_1, h_1x_1, j_1y_1\}$, $G(e_1, \underline{e_2}) \bullet H(e_3) := (G \cdot H - \{a_2x_2, b_2y_2\}) \cup \{h_2, j_2, a_2h_2, h_2j_2, j_2b_2, h_2x_2, j_2y_2\}$ and $G(\underline{e_1}, \underline{e_2}) \bullet H(e_3) := (G(\underline{e_1}, \underline{e_2}) \bullet H(e_3) - \{a_2x_2, b_2y_2\}) \cup \{h_2, j_2, a_2h_2, h_2j_2, j_2b_2, h_2x_2, j_2y_2\}$, see Fig. 1.

The statement of the next proposition is well known, see [15] or [3, proof of Theorem 12] (see also [7,8]).

Proposition 2.3 *Let H be a cubic graph which is constructed from a cubic graph G by subdividing two independent edges in G and by adding an edge joining the two 2-valent vertices. Then $\lambda_c(H) \geq 4$ if $\lambda_c(G) \geq 4$.*

The above result is used for the proofs of the subsequent two lemmas.

Lemma 2.4 *Suppose G and H are snarks. Define the three graphs: $B_1 := G(\underline{e_1}, \underline{e_2}) \bullet H(e_3)$, $B_2 := G(e_1, \underline{e_2}) \bullet H(e_3)$ and $B_3 := G(\underline{e_1}, \underline{e_2}) \bullet H(e_3)$. Then B_i is a snark for $i = 1, 2, 3$.*

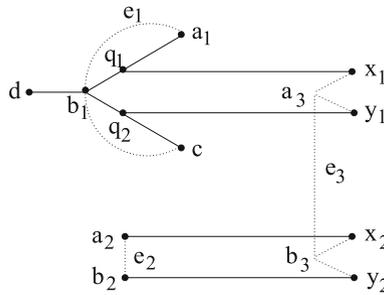


Fig. 2 Constructing the graph $G(e_1, e_2) \blacktriangle H(e_3)$, see Def. 2.5

Proof If B_i with $i = 1, 2, 3$ has a cycle of length less than five, then this cycle must contain precisely two edges of the cyclic 4-edge cut C_i of B_i where C_i is defined by the property that one component of $B_i - C_i$ is $H - a_3 - b_3$. Since the endvertices of the edges of C_i do not induce a 4-cycle, B_i has girth at least 5.

It is not difficult to see that B_i results from subdividing two (in the case $i = 1, 2$) or four (in the case $i = 3$) independent edges of $G \cdot H$ (namely the edges of C_i) and adding one or two new edges. Since $\lambda_c(G \cdot H) \geq 4$ by Lemma 2.1(ii) and since subdividing and adding edges as described keeps by Proposition 2.3 the cyclic 4-edge connectivity, $\lambda_c(B_i) \geq 4$.

It remains to show that B_i is not 3-edge colorable. Let G_1 be the cubic graph which is constructed from G by (i) replacing $e_1 \in E(G)$ by a path of length three (no edge in our graph is now labeled e_1), by (ii) removing the labels a_1, b_1 , by (iii) adding a parallel edge (to make the graph cubic) and calling this new edge e_1 , and by (iv) naming the endvertices of e_1, a_1 and b_1 such that a_1 is adjacent to the vertex whose label a_1 we removed in step (ii). Then $B_1 \cong G_1 \cdot H$. Since G_1 is by construction clearly not 3-edge colorable and since H is a snark, Lemma 2.1(i) implies that B_1 is not 3-edge colorable. The remaining cases B_2 and B_3 can be verified analogously. \square

Let e_1, e_2 in G and e_3 in H be defined as at the beginning of this section. Moreover, suppose that $N(b_1) = \{a_1, c, d\}$ and that all three neighbors of b_1 are distinct. We define another modification of the dot product, see also Fig. 2.

Definition 2.5 Let q_1, q_2 be distinct vertices satisfying $\{q_1, q_2\} \cap V(G \cup H) = \emptyset$. Set $G(e_1, e_2) \blacktriangle H(e_3) := (G \cup H - a_3 - b_3 - \{e_1, e_2, b_1c\}) \cup \{q_1, q_2, a_1q_1, q_1b_1, b_1q_2, q_2c, q_1x_1, q_2y_1, a_2x_2, b_2y_2\}$.

Lemma 2.6 Suppose G and H are snarks, then $G(e_1, e_2) \blacktriangle H(e_3)$ is a snark.

Proof By the same arguments as in the proof of Lemma 2.4, the girth of $G(e_1, e_2) \blacktriangle H(e_3)$ is at least five. Set $X := G \cdot H$. Subdivide in X the edge a_1x_1 and call the obtained 2-valent vertex q_1 , then subdivide the edge b_1d and call this now obtained 2-valent vertex q_2 , then exchange the labels b_1 and q_2 and finally add b_1q_1 . Thus we obtain the graph $Y := G(e_1, e_2) \blacktriangle H(e_3)$. Since $\lambda_c(X) \geq 4$ by Lemma 2.1(ii), and since Y is obtained from X by subdividing two independent edges and by adding an edge joining these vertices, it follows from Proposition 2.3 that $\lambda_c(Y) \geq 4$. It remains

to show that Y is not 3-edge colorable. Let \tilde{G} be the cubic graph which is obtained from G by (i) expanding b_1 to a triangle, by (ii) removing the labels e_1, a_1 and b_1 , by (iii) calling the edge of the triangle which has no endvertex adjacent to d, e_1 , and by (iv) naming the endvertices of e_1, a_1 and b_1 such that a_1 is adjacent to the vertex whose label a_1 we removed in step (ii). Then \tilde{G} is clearly not 3-edge colorable. By Lemma 2.1(i) and since $Y \cong \tilde{G} \cdot H$, Y is not 3-edge colorable. \square

The next result shows that two hist-snarks can be combined to generate new hist-snarks.

Theorem 2.7 *Let G and H be snarks with hist T_G and T_H , then the following holds:*

- (i) *There is a snark G' with a hist T' such that $oc(G', T') = oc(G, T_G) \cup oc(H, T_H)$.*
- (ii) *Suppose $k \in oc(G, T_G)$ and $l \in oc(H, T_H)$, then there is a snark \hat{G} with a hist \hat{T} such that $oc(\hat{G}, \hat{T}) = (oc(G, T_G) \cup oc(H, T_H) \cup \{k + l - 1\}) - \{k, l\}$.*

Proof First we prove (i). Choose two edges $e_1, e_2 \in E(T_G)$ and choose an edge $e_3 \in E(T_H)$ such that all four adjacent edges are part of T_H . Set $G' := G(e_1, e_2) \bullet H(e_3)$. Then G' is a snark by Lemma 2.4. Let T' be the subgraph of G' with $E(T') = (E(T_G) \cup E(T_H) - \{e_1, e_2, e_3, a_3x_1, a_3y_1, b_3x_2, b_3y_2\}) \cup \{a_1h_1, h_1j_1, j_1b_1, a_2h_2, h_2j_2, j_2b_2, h_1x_1, j_1y_1, h_2x_2, j_2y_2\}$.

Since $T_H - a_3 - b_3$ consists of four components, it follows that T' is acyclic. It is straightforward to verify that T' is a hist of G' . Since every outer cycle of G' is an outer cycle of G or H , the proof of (i) is finished.

Using Lemma 2.6, we define a snark $\hat{G} = G(e_1, e_2) \blacktriangle H(e_3)$ where e_1, e_2, e_3 are chosen to satisfy the following properties. In G , let $e_1 \in E(T_G), b_1c \in E(T_G)$ and let $e_2 \in E(G) - E(T_G)$ be part of an outer cycle of length k , see Fig. 2. In H , let b_3 be a leaf of an outer cycle of length l and let $e_3, a_3x_1, a_3y_1 \in E(T_H)$. Let \hat{T} be the subgraph of \hat{G} with $E(\hat{T}) := (E(T_G) \cup E(T_H) - \{e_1, b_1c, e_3, a_3x_1, a_3y_1\}) \cup \{a_1q_1, q_1b_1, b_1q_2, q_2c, q_1x_1, q_2y_1\}$. It is straightforward to verify that \hat{T} is a hist of \hat{G} . Note that a_2x_2, b_2y_2 are contained in an outer cycle of length $k + l - 1$. Hence, $oc(\hat{G}, \hat{T}) = (oc(G, T_G) \cup oc(H, T_H) - \{k\} - \{l\}) \cup \{k + l - 1\}$ which finishes the proof. \square

Lemma 2.8 *Let G be a snark with a hist T_G and let $k \in oc(G, T_G)$, then each of the following statements holds:*

- (i) *there is a snark G' with a hist T' such that $oc(G', T') = (oc(G, T_G) - \{k\}) \cup \{k + 4\}$.*
- (ii) *there is a snark G' with a hist T' such that $oc(G', T') = oc(G, T_G) \cup \{5\}$.*
- (iii) *there is a snark G' with a hist T' such that $oc(G', T') = oc(G, T_G) \cup \{6\}$.*
- (iv) *there is a snark G' with a hist T' such that $oc(G', T') = (oc(G, T_G) - \{k\}) \cup \{k + 2\} \cup \{7\}$.*

Proof The endvertices of the edges e_1, e_2 and e_3 and their neighbors are labeled as defined in the beginning of this section. The Petersen graph is denoted by P_{10} .

(i) Let \hat{C} be the 5-cycle, “the inner star” of the illustrated P_{10} in Fig. 3. Set $U := P_{10} - E(\hat{C})$ and let $e_3 \in E(U)$ with $e_3 = a_3b_3$ where $b_3 \in V(\hat{C})$. In

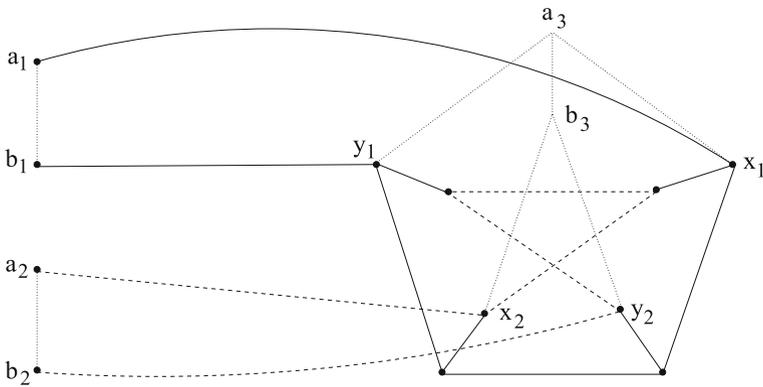


Fig. 3 Constructing the graph G' in the proof of Lemma 2.8 (i)

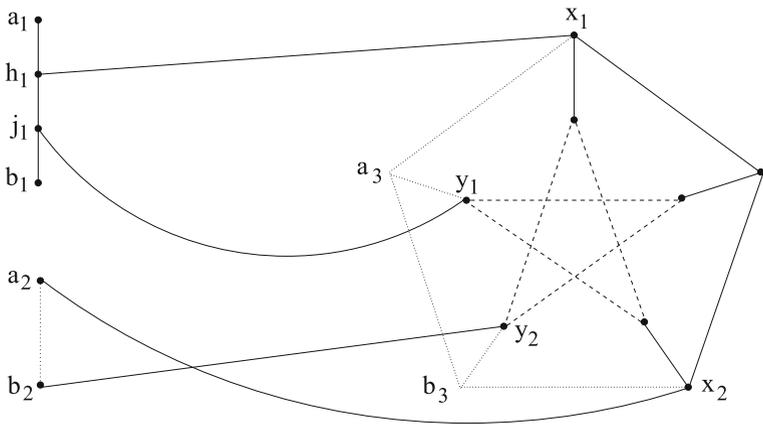


Fig. 4 Constructing the graph G' in the proof of Lemma 2.8 (ii)

G , we choose two independent edges $e_1 \in E(T_G)$ and $e_2 \notin E(T_G)$ where e_2 is part of an outer cycle C_k of length k . Then $G' := G \cdot P_{10}$ is a snark and the subgraph T' of G' with $E(T') := (E(T_G) - e_1) \cup \{a_1x_1, b_1y_1\} \cup E(U - a_3)$ is a hist of G' , see Fig. 3. Since G and G' have the same outer cycles with the only exception that C_k is in G and that $C'_{k+4} := (C_k - e_2) \cup \{a_2x_2, b_2y_2\} \cup (\hat{C} - b_3)$ is in G' , the statement follows.

(ii) Let $e_1, e_2 \in E(T_G)$ with $e_1 = a_1b_1$ and $e_2 = a_2b_2$. Since we could exchange the vertex labels a_2 and b_2 , we can assume that b_1, b_2 are in the same component of $T_G - e_2$. Define the snark $G' := G(e_1, e_2) \bullet P_{10}(e_3)$ with $e_3 \in E(P_{10})$, see Fig. 4. Let C_5 and \hat{C}_5 be two disjoint 5-cycles of P_{10} with $e_3 \in E(C_5)$ and let $x_1, x_2 \in V(C_5)$, see Fig. 4. Then the subgraph T' of G' with $E(T') := (E(T_G) - \{e_1, e_2\}) \cup \{a_1h_1, h_1j_1, j_1b_1, h_1x_1, j_1y_1, a_2x_2, b_2y_2\} \cup E(P_{10}) - (E(\hat{C}_5) \cup \{a_3x_1, a_3y_1, b_3x_2, b_3y_2, e_3\})$ is a hist of G' , see Fig. 4. Since the set of outer cycles of G' consists of \hat{C}_5 and all outer cycles of G , the statement follows.

(iii) P_{10} is a hist-snark with one outer cycle, see Fig. 6. Since this cycle has length 6, statement (iii) follows from applying Theorem 2.7(i) by setting $H := P_{10}$.

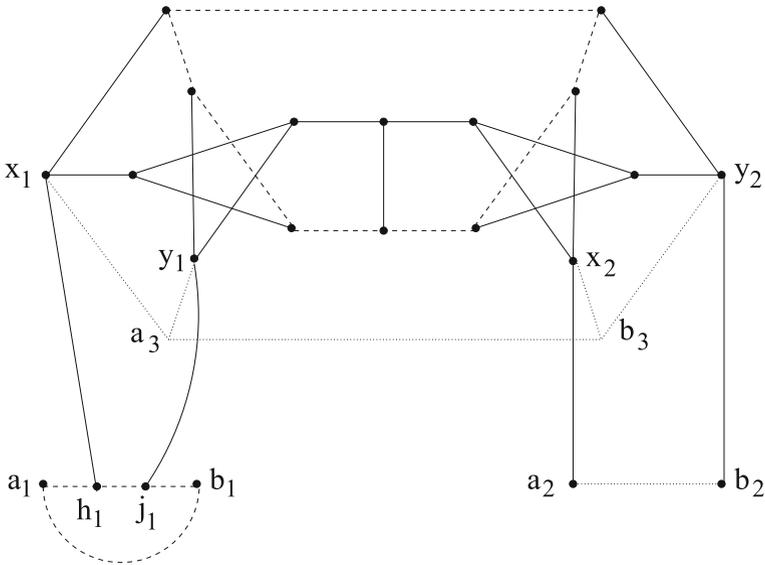


Fig. 5 Constructing the graph G' in the proof of Lemma 2.8 (iv)

(iv) Let $e_1 = a_1b_1$ be an edge of an outer cycle of length k in G and let $e_2 \in E(T_G)$. Let B_{18} denote the Blanusa snark, see Fig. 7. Define the snark $G' := G(e_1, e_2) \bullet B_{18}(e_3)$ as illustrated in Fig. 5. Note that two outer cycles of lengths 7 and $k + 2$ are presented in Fig. 5 by dashed lines. Let B denote the edge set which contains all edges of G' which are shown in bold face in Fig. 5. It is straightforward to verify that the subgraph T' of G' with $E(T') := (E(T_G) - \{e_2\}) \cup B$ is a hist in G' satisfying $oc(G', T') = (oc(G, T_G) - \{k\}) \cup \{k + 2\} \cup \{7\}$. \square

Definition 2.9 Let S be a multiset of natural numbers, then S^* denotes the set of all hist-snarks G which have a hist T_G such that $oc(G, T_G) = S$.

For instance, the Blanusa snark B_{18} satisfies $B_{18} \in \{10\}^*$ (see Fig. 7) but also satisfies $B_{18} \in \{5, 5\}^*$. We leave it to the reader to verify the latter fact. The Petersen graph P_{10} satisfies $P_{10} \in S^*$ if and only if $S = \{6\}$, see Fig. 6.

Corollary 2.10 *There is a snark G having a hist T with $oc(G, T) = \{k\}$ if and only if $k = 6$ or $k \geq 10$.*

Proof As mentioned above $B_{18} \in \{10\}^*$, see Fig. 7. By Theorem 2.7 (ii) and since $P_{10} \in \{6\}^*$, it follows that $\{11\}^* \neq \emptyset$. The second Loupekine snark denoted by L_{22} satisfies $L_{22} \in \{12\}^*$, see Fig. 8. The hist-snark $T(13)$, see Appendix A1 satisfies $T(13) \in \{13\}^*$. Applying Lemma 2.8 (i) to each of these four hist-snarks and proceeding inductively, we obtain for every natural number $k \geq 10$ a hist-snark G satisfying $G \in \{k\}^*$. Since $k = |V(G)|/2 + 1$ (see Theorem 2 in [12]) and since there is no snark with $10 < k < 18$ vertices, $\{l\}^* = \emptyset$ for every $l \in \{1, 2, 3, 4, 5, 7, 8, 9\}$. Finally, since $P_{10} \in \{6\}^*$ the proof is finished. \square

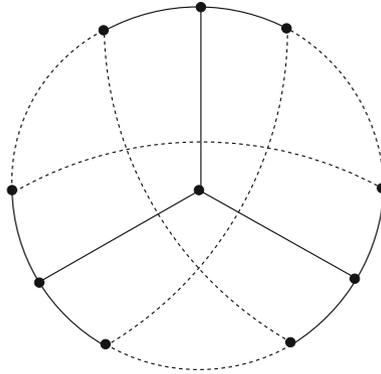


Fig. 6 The Petersen graph with an outer cycle of length 6

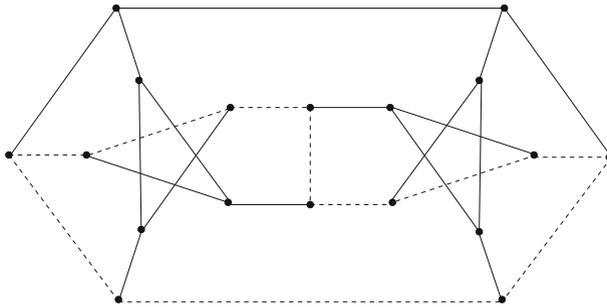


Fig. 7 The Blanusa snark B_{18} with an outer cycle of length 10

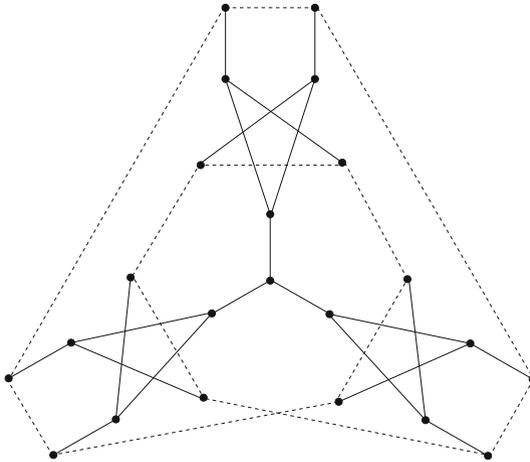


Fig. 8 The second Loupekine snark with an outer cycle of length 12

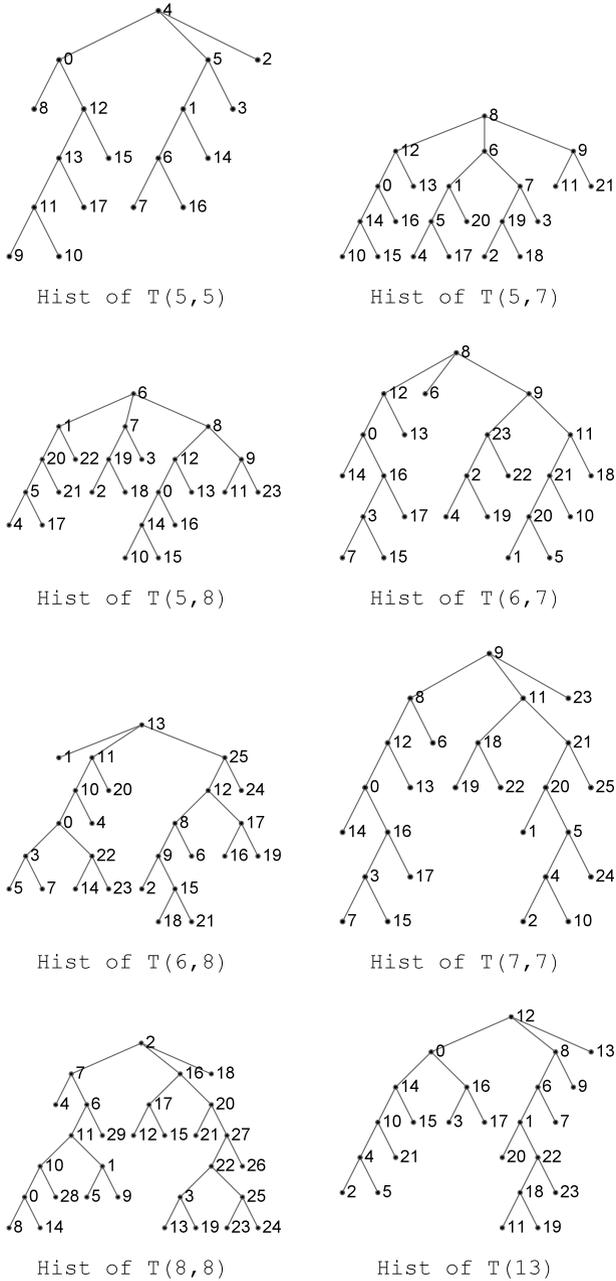


Fig. 9 See Appendix A1, proof of Corollary 2.10 and proof of Lemma 2.11

Lemma 2.11 *Let $S = \{x, y\}$ with $x, y \in \{5, 6, 7, 8\}$, then $S^* \neq \emptyset$.*

Proof By applying Lemma 2.8 (ii),(iii),(iv) by setting $G := P_{10}$, we obtain that $\{5, 6\}^* \neq \emptyset$, $\{6, 6\}^* \neq \emptyset$, $\{7, 8\}^* \neq \emptyset$. To avoid more constructions, we used a computer. We refer to Appendix A1 and Fig. 9, where one member of $\{x, y\}^*$ denoted by $T(x, y)$ is presented for the remaining pairs. \square

Lemma 2.12 *Let $S = \{x, y, z\}$ with $x, y, z \in \{5, 6, 7, 8\}$, then $S^* \neq \emptyset$.*

Proof By Lemma 2.8 (ii), Lemma 2.8 (iii) and Lemma 2.11, $S^* \neq \emptyset$ if $\{5, 6\} \cap S \neq \emptyset$. Hence we assume that $x, y, z \in \{7, 8\}$. Suppose $7 \in S$ and let without loss of generality $x = 7$. By Lemma 2.11, $\{y - 2, z\}^* \neq \emptyset$. Applying Lemma 2.8 (iv), we obtain that $\{7, y, z\}^* \neq \emptyset$. Since there is a snark $T(8, 8, 8)$ (see Fig. 10 in Appendix A2) which is a member of $\{8, 8, 8\}^*$, the lemma follows. \square

Note that the illustrated snark in $\{8, 8, 8\}^*$ has a $2\pi/3$ rotation symmetry and a hist which has equal distance from its central root to every leaf. Such snarks are called *rotation snarks*, for an exact definition see [10]. The Petersen graph and both Loupekine’s snarks (the smallest cyclically 5-edge connected snarks apart from the Petersen graph) are rotation snarks. All rotation snarks with at most 46 vertices are presented in [10].

Proof of Theorem 1.3 Statement (i) is implied by Corollary 2.10. Statement (ii) is obviously a necessary condition, otherwise G has girth less than five. Hence it suffices to show that there is a hist-snark in S^* if S satisfies (ii).

Suppose S is a counterexample which is firstly minimal with respect to $|S|$ and secondly minimal with respect to the largest number in S . Suppose $|S| \geq 4$. Then there is a partition $S = S_2 \cup S_3$ with $|S_2| = 2$ and $|S_3| = |S| - 2 \geq 2$. Since the elements of S_2, S_3 satisfy (ii), there is by minimality a hist-snark $H_i \in S_i^*, i = 2, 3$. By Theorem 2.7, there is a hist-snark in $(S_2 \cup S_3)^* = S^*$ which is a contradiction. Hence $|S| \in \{2, 3\}$. Suppose $m \in S$ and $m > 8$. Set $S_1 = (S - \{m\}) \cup \{m - 4\}$. Obviously the elements of S_1 fulfill (ii) and thus there is by minimality a hist-snark $H_1 \in S_1^*$. Applying Lemma 2.8 (i) to S_1 , we obtain a hist-snark in S^* which is a contradiction. Thus, S consists of two or three elements and each of them is contained in $\{5, 6, 7, 8\}$. By Lemma 2.11 and Lemma 2.12, this is not possible. \square

3 Open Problems

The following conjecture by the first author, was presented firstly at the 9th Workshop on the Matthews-Sumner Conjecture and Related Problems in Pilsen in 2017.

Conjecture 3.1 *Every hist-snark has a cycle double cover which contains all outer cycles.*

The above conjecture is motivated by the following observation on hist-snarks.

Observation 3.2 [9] *Let G be a snark with a hist T . Suppose there is a matching M of G satisfying $M \subseteq E(G) - E(T)$, and suppose the cubic graph homeomorphic to*

$G - M$ is 3-edge colorable. Then G has a cycle double cover containing all outer cycles of G .

We omit here a proof of Observation 3.2 since Theorem 3.2 in [13] implies Observation 3.2. Note that Conjecture 3.1 is already known to hold for all hist-snarks which have at most three outer cycles, see [13].

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Appendix

A1. The following hist-snarks are defined by the corresponding hist-illustrated below and the outer cycles whose vertices are presented within brackets in cyclic order.

- $T(5, 5) := [10, 15, 14, 17, 16] [2, 7, 3, 8, 9]$
- $T(5, 7) := [3, 15, 13, 17, 16] [10, 4, 2, 21, 20, 18, 11]$
- $T(5, 8) := [3, 15, 13, 17, 16] [10, 4, 2, 23, 22, 18, 11, 21]$
- $T(6, 7) := [1, 6, 7, 19, 18, 22] [4, 5, 17, 13, 15, 14, 10]$
- $T(6, 8) := [18, 19, 14, 21, 20, 23] [1, 5, 4, 2, 7, 6, 24, 16]$
- $T(7, 7) := [17, 13, 15, 14, 10, 25, 24] [1, 6, 7, 19, 2, 23, 22]$
- $T(8, 8) := [12, 9, 8, 29, 28, 4, 5, 13] [14, 15, 18, 21, 24, 26, 19, 23]$
- $T(13) := [2, 19, 7, 3, 15, 13, 17, 5, 20, 21, 11, 9, 23]$

The adjacency lists of the above hist-snarks:

- $T(5, 5) : 0(4, 8, 12)1(5, 6, 14)2(4, 7, 9)3(5, 7, 8)4(5)6(7, 16)8(9)9(11)10(11, 15, 16)11(13)12(13, 15)13(17)14(15, 17)16(17)$
- $T(5, 7) : 0(12, 14, 16)1(5, 6, 20)2(4, 19, 21)3(7, 15, 16)4(5, 10)5(17)6(7, 8)7(19)8(9, 12)9(11, 21)10(11, 14)11(18)12(13)13(15, 17)14(15)16(17)18(19, 20)20(21)$
- $T(5, 8) : 0(12, 14, 16)1(6, 20, 22)2(4, 19, 23)3(7, 15, 16)4(5, 10)5(17, 20)6(7, 8)7(19)8(9, 12)9(11, 23)10(14, 21)11(18, 21)12(13)13(15, 17)14(15)16(17)18(19, 22)20(21)22(23)$
- $T(6, 7) : 0(12, 14, 16)1(6, 20, 22)2(4, 19, 23)3(7, 15, 16)4(5, 10)5(17, 20)6(7, 8)8(9, 12)9(11, 23)10(14, 21)11(18, 21)12(13)13(15, 17)14(15)16(17)18(19, 22)20(21)22(23)$
- $T(6, 8) : 0(3, 10, 22)1(5, 13, 16)2(4, 7, 9)3(5, 7)4(5, 10)6(7, 8, 24)8(9, 12)9(15)10(11)11(13, 20)12(17, 25)13(25)14(19, 21, 22)15(18, 21)16(17, 24)17(19)18(19, 23)20(21, 23)22(23)24(25)$

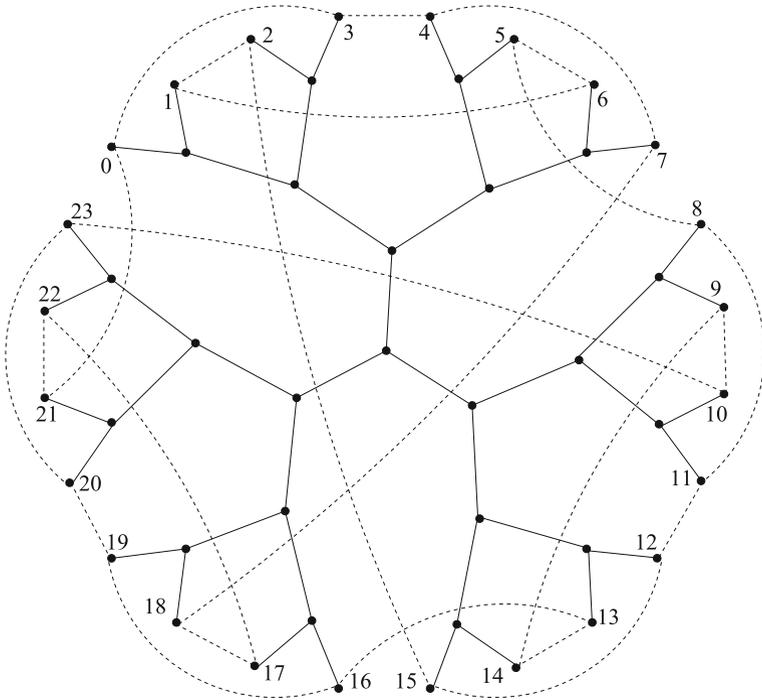


Fig. 10 The hist-snark $T(8, 8, 8)$ with three outer cycles of length 8

$T(7, 7) : 0(12, 14, 16)1(6, 20, 22)2(4, 19, 23)3(7, 15, 16)4(5, 10)5(20, 24)6(7, 8)$
 $7(19)8(9, 12)9(11, 23)10(14, 25)11(18, 21)12(13)13(15, 17)14(15)$
 $16(17)17(24)18(19, 22)20(21)21(25)22(23)24(25)$

$T(8, 8) : 0(8, 10, 14)1(5, 9, 11)2(7, 16, 18)3(13, 19, 22)4(5, 7, 28)5(13)6(7, 11, 29)$
 $8(9, 29)9(12)10(11, 28)12(13, 17)14(15, 23)15(17, 18)16(17, 20)18(21)$
 $19(23, 26)20(21, 27)21(24)22(25, 27)23(25)24(25, 26)26(27)28(29)$

$T(13) : 0(12, 14, 16)1(6, 20, 22)2(4, 19, 23)3(7, 15, 16)4(5, 10)5(17, 20)6(7, 8)$
 $7(19)8(9, 12)9(11, 23)10(14, 21)11(18, 21)12(13)13(15, 17)14(15)$
 $16(17)18(19, 22)20(21)22(23)$

A2. $T(8, 8, 8) := [0, 3, 4, 7, 18, 17, 22, 21] [1, 2, 15, 12, 11, 8, 5, 6]$
 $[9, 10, 23, 20, 19, 16, 13, 14].$

For the sake of completeness we present the adjacency list of $T(8, 8, 8)$.

$T(8, 8, 8) : 0(3, 21, 24)1(2, 6, 24)2(15, 25)3(4, 25)4(7, 26)5(6, 8, 26)6(27)$
 $7(18, 27)8(11, 28)9(10, 14, 28)10(23, 29)11(12, 29)12(15, 30)13(14, 16, 30)$
 $14(31)15(31)16(19, 32)17(18, 22, 32)18(33)19(20, 33)20(23, 34)21(22, 34)$
 $22(35)23(35)24(36)25(36)26(37)27(37)28(38)29(38)30(39)31(39)32(40)$

33(40)34(41)35(41)36(42)37(42)38(43)39(43)40(44)41(44)42(45)43(45)44(45)

A3. The adjacency lists of the hist-free snarks X_1, X_2 with 38 vertices.

X_1 : 0(8, 12, 18)1(5, 9, 13)2(4, 14, 20)3(5, 7, 8)4(5, 12)6(7, 10, 13)7(14)8(15)9(19, 22)
 10(18, 24)11(26, 34, 36)12(16)13(16)14(17)15(17, 19)16(17)18(21)19(21)
 20(25, 36)21(27)22(30, 34)23(25, 28, 31)24(26, 37)25(35)26(32)27(29, 31)
 28(29, 30)29(32)30(33)31(33)32(33)34(35)35(37)36(37)

X_2 : 0(8, 12, 18)1(5, 9, 13)2(4, 14, 20)3(5, 7, 8)4(5, 12)6(7, 10, 13)7(14)8(15)9(19, 22)
 10(18, 24)11(26, 34, 36)12(16)13(16)14(17)15(17, 19)16(17)18(21)19(21)20(28, 34)
 21(27)22(26, 37)23(27, 30, 32)24(25, 36)25(30, 35)26(33)27(29)28(31, 32)29(31, 33)
 30(31)32(33)34(35)35(37)36(37)

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