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Note

Defensive alliances in graphs of bounded treewidth

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ABSTRACT

The DEFENSIVE ALLIANCE problem has been studied extensively during the last fifteen years, but the question whether it is FPT when parameterized by treewidth has still remained open. We show that this problem is W[1]-hard. This puts it among the few problems that are FPT when parameterized by solution size but not when parameterized by treewidth (unless FPT = W[1]).

1. Introduction

In many applications, we are interested in finding groups of vertices of a graph so that group members collaborate in a certain way. This is formalized in a natural way by the notion of a *defensive alliance* [7,8]. A set S of vertices is a defensive alliance in an undirected graph if each $x \in S$ has at least as many neighbors (including itself) in S as it has neighbors not in S. We also call the former set of neighbors the *defenders* of x and the latter set the *attackers* of x. This gives rise to the following computational problem.

DEFENSIVE ALLIANCE
Input: An undirected graph G and an integer k with $1 \leqslant k \leqslant |V(G)|$
Question: Is there a defensive alliance $S \subseteq V(G)$ such that $1 \leqslant |S| \leqslant k$?

This problem is known to be NP-complete and fixed-parameter tractable (FPT) when parameterized by the solution size. If the input graphs are trees, the problem is trivial, and a natural NP-complete variant that is not trivial on trees becomes tractable [3]. In fact DEFENSIVE ALLIANCE is solvable in polynomial time on graphs of bounded treewidth [6] and even in linear time if in addition the maximum degree is bounded [5]. However, the question of whether it is FPT when parameterized by treewidth alone has remained open so far. In this work, we give a negative answer (unless FPT = W[1]) by proving that DEFENSIVE ALLIANCE parameterized by treewidth is W[1]-hard.

The reduction that we use in our proof is similar to one used in a recent paper [1], which shows W[1]-hardness for a related problem called "Secure Set". Due to the different nature of DEFENSIVE ALLIANCE, several nontrivial changes were necessary to obtain our present result. Some proofs in the current work are quite compact and we refer to [2] for full elaborations.

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2. Hardness of defensive alliance parameterized by treewidth

In this work, we prove the following theorem:

Theorem 1. *The* Defensive Alliance *problem is* W[1]*-hard when parameterized by the treewidth of the graph.*

To prove this, we reduce from the following problem, which is known to be W[1]-hard when parameterized by the treewidth of the input graph [10]:

> Minimum Maximum Outdegree
>
> Input: An undirected graph G, an edge weighting $w : E(G) \to \mathbb{N}^+$ given in unary and a positive integer r
>
> Question: Is there an orientation of the edges such that, for each vertex, the sum of the weights of outgoing edges is at most r?

We reduce this problem to the following generalization of Defensive Alliance, where some vertices are necessary (i.e., they must appear in every solution), some are forbidden (i.e., they must not appear in any solution) and some pairs of vertices are complementary in a certain sense:

> Defensive Alliance$^{\text{FNC}}$
>
> Input: An undirected graph G, an integer k, a set $V_\square \subseteq V(G)$, a set $V_\triangle \subseteq V(G)$ and a set $C \subseteq V(G)^2$
>
> Question: Is there a defensive alliance $S \subseteq V(G)$ such that (i) $1 \leqslant |S| \leqslant k$, (ii) $S \cap V_\square = \emptyset$, (iii) $V_\triangle \subseteq S$, and (iv) for all $(a, b) \in C$, the set S contains either a or b but not both?

We also define the intermediate problem Defensive Alliance$^{\text{FN}}$ by dropping condition (iv) and the problem Defensive Alliance$^{\text{F}}$ by dropping both (iii) and (iv).

To measure the treewidth of a Defensive Alliance$^{\text{FNC}}$ instance, we use the following graph representation:

Definition 2. Let I be a Defensive Alliance$^{\text{FNC}}$ instance, let G be the graph in I and let C be the set of complementary vertex pairs in I. By the *primal graph* of I we mean the graph G' with $V(G') = V(G)$ and $E(G') = E(G) \cup C$.

To prove Theorem 1, we now reduce Minimum Maximum Outdegree to Defensive Alliance$^{\text{FNC}}$ and then show how we can successively reduce the latter problem to Defensive Alliance$^{\text{FN}}$, Defensive Alliance$^{\text{F}}$ and finally Defensive Alliance. We assume some familiarity with parameterized complexity and treewidth; for an introduction, we refer to [9].

Lemma 3. Defensive Alliance$^{\text{FNC}}$ *is* W[1]*-hard when parameterized by the treewidth of the primal graph.*

Proof. Let an instance of Minimum Maximum Outdegree be given by a graph G, an edge weighting $w : \mathrm{E}(G) \to \mathbb{N}^+$ in unary and a positive integer r. From this we construct an instance of Defensive Alliance$^{\text{FNC}}$. An example is given in Fig. 1. For each $v \in \mathrm{V}(G)$, we define the set of new vertices $H_v = \{h_1^v, \dots, h_{2r-1}^v\}$, and for each $(u, v) \in \mathrm{E}(G)$, we define the sets of new vertices $V_{uv} = \{u_1^v, \dots, u_{w(u,v)}^v\}$, $V_{uv}^\square = \{u_1^{v\square}, \dots, u_{w(u,v)}^{v\square}\}$, $V_{vu} = \{v_1^u, \dots, v_{w(u,v)}^u\}$ and $V_{vu}^\square = \{v_1^{u\square}, \dots, v_{w(u,v)}^{u\square}\}$. We now define the graph G' with

$$\begin{aligned} \mathrm{V}(G') =& \mathrm{V}(G) \cup \bigcup_{v \in \mathrm{V}(G)} H_v \cup \bigcup_{(u,v) \in \mathrm{E}(G)} (V_{uv} \cup V_{uv}^\square \cup V_{vu} \cup V_{vu}^\square), \\ \mathrm{E}(G') =& \{(v, h) \mid v \in \mathrm{V}(G),\ h \in H_v\} \\ & \cup \{(u, x) \mid (u, v) \in \mathrm{E}(G),\ x \in V_{uv} \cup V_{uv}^\square\} \\ & \cup \{(x, v) \mid (u, v) \in \mathrm{E}(G),\ x \in V_{vu} \cup V_{vu}^\square\}. \end{aligned}$$

We also define the set of complementary vertex pairs $C = \{(u_i^v, v_i^u) \mid (u, v) \in \mathrm{E}(G),\ 1 \leqslant i \leqslant w(u, v)\} \cup \{(v_i^u, u_{i+1}^v) \mid (u, v) \in \mathrm{E}(G),\ 1 \leqslant i < w(u, v)\}$. Finally, we define the set of necessary vertices $V_\triangle = \mathrm{V}(G) \cup \bigcup_{v \in \mathrm{V}(G)} H_v$, the set of forbidden vertices $V_\square = \bigcup_{(u,v) \in \mathrm{E}(G)} (V_{uv}^\square \cup V_{vu}^\square)$ and $k = |V_\triangle| + \sum_{(u,v) \in \mathrm{E}(G)} w(u, v)$. We use I to denote $(G', k, C, V_\triangle, V_\square)$, which is an instance of Defensive Alliance$^{\text{FNC}}$.

Clearly I can be computed in polynomial time. We now show that the treewidth of the primal graph of I depends only on the treewidth of G. We do so by modifying an optimal tree decomposition $\mathcal{T}$ of G as follows:

1. For each $(u, v) \in \mathrm{E}(G)$, we take an arbitrary node whose bag B contains both u and v and add to its children a chain of nodes $N_1, \dots, N_{w(u,v)-1}$ such that the bag of N_i is $B \cup \{u_i^v, u_{i+1}^v, v_i^u, v_{i+1}^u\}$.
2. For each $(u, v) \in E(G)$, we take an arbitrary node whose bag B contains u and add to its children a chain of nodes $N_1, \dots, N_{w(u,v)}$ such that the bag of N_i is $B \cup \{u_i^{v\square}\}$.

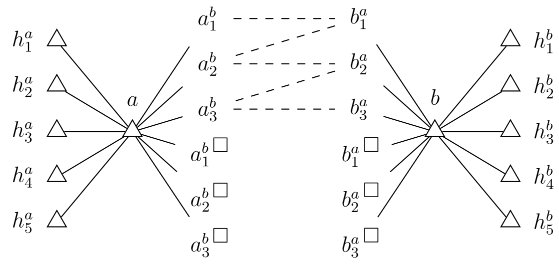

Fig. 1. Result of our transformation on a sample Minimum Maximum Outdegree instance with $r = 3$ and two vertices a, b that are connected by an edge of weight 3. Complementary vertex pairs are shown via dashed lines. Necessary and forbidden vertices have a $\triangle$ and $\square$ symbol next to their name, respectively.

3. For each $(u, v) \in \mathrm{E}(G)$, we take an arbitrary node whose bag B contains v and add to its children a chain of nodes $N_1, \ldots, N_{w(u,v)}$ such that the bag of N_i is $B \cup \{v_i^{u\square}\}$.
4. For each $v \in \mathrm{V}(G)$, we take an arbitrary node whose bag B contains v and add to its children a chain of nodes $N_1, \ldots, N_{r-1}$ such that the bag of N_i is $B \cup \{h_i^v\}$.

It is easy to verify that the result is a valid tree decomposition of the primal graph of I and its width is at most the treewidth of G plus four.

The intention is that for each orientation of G we have a solution candidate S in I such that an edge orientation from u to v entails $V_{vu} \subseteq S$ and $V_{uv} \cap S = \emptyset$, and the other orientation entails $V_{uv} \subseteq S$ and $V_{vu} \cap S = \emptyset$. For each vertex $v \in \mathrm{V}(G)$ and every incident edge $(v, u) \in \mathrm{E}(G)$ regardless of its orientation, the vertex v is attacked by the forbidden vertices $V_{vu}^{\square}$. So every vertex $v \in \mathrm{V}(G)$ has as least as many attackers as the sum of the weights of all incident edges. If in the orientation of G all edges incident to v are incoming edges, then each attack on v from $V_{vu}^{\square}$ can be repelled by V_{vu}, since $V_{vu} \subseteq S$. Due to the fact that the helper vertices H_v consist of exactly $2r - 1$ elements, v can afford to have outgoing edges of total weight at most r.

We claim that (G, w, r) is a positive instance of Minimum Maximum Outdegree iff I is a positive instance of Defensive Alliance$^{\mathrm{FNC}}$.

"Only if" direction. Let D be the directed graph given by an orientation of the edges of G such that for each vertex the sum of weights of outgoing edges is at most r. The set $S = V_{\triangle} \cup \{v_1^u, \ldots, v_{w(u,v)}^u \mid (u, v) \in \mathrm{E}(D)\}$ is a defensive alliance in G': Let x be an arbitrary element of S. If x is an element of a set H_v or V_{uv}, then the only neighbor of x in G' is a necessary vertex, so x can trivially defend itself; so suppose $x \in \mathrm{V}(G)$. Let the sum of the weights of outgoing and incoming edges be denoted by w_{out}^x and w_{in}^x, respectively. The neighbors of x that are also in S consist of the elements of H_x and all elements of sets V_{xv} such that $(v, x) \in \mathrm{E}(D)$. Hence, including itself, x has $2r + w_{\mathrm{in}}^x$ defenders in G'. The attackers of x consist of all elements of sets V_{xv} such that $(x, v) \in \mathrm{E}(D)$ (in total w_{out}^x) and all elements of sets $V_{xv}^{\square}$ such that either $(v, x) \in \mathrm{E}(D)$ or $(x, v) \in \mathrm{E}(D)$ (in total $w_{\mathrm{in}}^x + w_{\mathrm{out}}^x$). Hence x has $w_{\mathrm{in}}^x + 2w_{\mathrm{out}}^x$ attackers in G'. This shows that x has at least as many defenders as attackers, as by assumption $w_{\mathrm{out}}^x \leqslant r$. Finally, it is easy to verify that $|S| = k$, $V_{\square} \cap S = \emptyset$, $V_{\triangle} \subseteq S$, and exactly one element of each pair of complementary vertices is in S.

"If" direction. Let S be a solution of I. For every $(u, v) \in \mathrm{E}(G)$, either $V_{uv} \subseteq S$ or $V_{vu} \subseteq S$ due to the complementary vertex pairs. We define a directed graph D by $\mathrm{V}(D) = \mathrm{V}(G)$ and $\mathrm{E}(D) = \{(u, v) \mid V_{vu} \subseteq S\} \cup \{(v, u) \mid V_{uv} \subseteq S\}$. Suppose there is a vertex x in D whose sum of weights of outgoing edges is greater than r. Clearly $x \in S$. Let the sum of the weights of outgoing and incoming edges be denoted by w_{out}^x and w_{in}^x, respectively. The defenders of x in G' beside itself consist of the elements of H_x and of w_{in}^x neighbors due to incoming edges in D. These are in total $2r + w_{\mathrm{in}}^x$ defenders. The attackers of x in G' consist of $2w_{\mathrm{out}}^x$ elements (of the form x_i^v as well as $x_i^{v\square}$) due to outgoing edges in D and w_{in}^x elements (of the form $x_i^{v\square}$) due to incoming edges. These are in total $2w_{\mathrm{out}}^x + w_{\mathrm{in}}^x$ attackers. But then x has more attackers than defenders, as by assumption $w_{\mathrm{out}}^x > r$. $\square$

Next we present an FPT reduction that eliminates complementary pairs.

Lemma 4. Defensive Alliance$^{\mathrm{FN}}$, *parameterized by the treewidth of the graph, is* W[1]*-hard.*

Proof. Let $I = (G, k, V_{\square}, V_{\triangle}, C)$ be a Defensive Alliance$^{\mathrm{FNC}}$ instance and let n denote $|\mathrm{V}(G)|$.

For each $(a, b) \in C$, we introduce new vertices a^{ab}, b^{ab} and $\triangle^{ab}$ as well as, for any $x \in \{a, b\}$, the following sets of new vertices.

$$Y_{x\circ}^{ab} = \{x_1^{ab}, \ldots, x_{n+1}^{ab}\} \qquad Z_{x\circ}^{ab} = \{x_{n+2}^{ab}, x_{n+3}^{ab}, x_{n+4}^{ab}\}$$
$$Y_{x\square}^{ab} = \{x_1^{ab\square}, \ldots, x_{n+1}^{ab\square}\} \qquad Z_{x\square}^{ab} = \{x_{n+2}^{ab\square}, x_{n+3}^{ab\square}, x_{n+4}^{ab\square}\}$$

We write $u \oplus v$ to denote the set of edges $\{(u, v), (u, u^{\square}), (v, v^{\square}), (u, v^{\square}), (v, u^{\square})\}$. Now we construct the Defensive Alliance$^{\mathrm{FN}}$ instance $I' = (G', k', V'_{\square}, V'_{\triangle})$, where $k' = k + |C| \cdot (n+6)$, $V'_{\square} = V_{\square} \cup \bigcup_{(a,b)\in C}(Y_{a\square}^{ab} \cup Y_{b\square}^{ab} \cup Z_{a\square}^{ab} \cup Z_{b\square}^{ab})$, $V'_{\triangle} = V_{\triangle} \cup \bigcup_{(a,b)\in C}\{\triangle^{ab}\}$

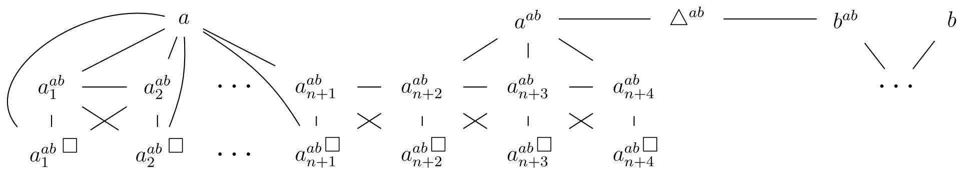

Fig. 2. Gadget for a pair of complementary vertices (a, b) in the reduction from Defensive Alliance$^{\mathrm{FNC}}$ to Defensive Alliance$^{\mathrm{FN}}$. The vertices a and b may have additional neighbors from the original graph. The omitted parts attached to b and b^{ab} are symmetric to the respective parts for a and a^{ab}.

and G' is the graph defined by

$$\mathrm{V}(G') = \mathrm{V}(G) \cup \bigcup_{(a,b)\in C} \big(\{\triangle^{ab}, a^{ab}, b^{ab}\} \cup Y^{ab}_{a\circ} \cup Y^{ab}_{b\circ} \cup Y^{ab}_{a\Box} \cup Y^{ab}_{b\Box} \cup Z^{ab}_{a\circ} \cup Z^{ab}_{b\circ} \cup Z^{ab}_{a\Box} \cup Z^{ab}_{b\Box}\big),$$

$$\mathrm{E}(G') = \mathrm{E}(G) \cup \bigcup_{(a,b)\in C} \bigcup_{x\in\{a,b\}} \Big(\{(\triangle^{ab}, x^{ab})\} \cup (\{x\} \times Y^{ab}_{x\circ}) \cup (\{x\} \times Y^{ab}_{x\Box}) \cup (\{x^{ab}\} \times Z^{ab}_{x\circ}) \cup \bigcup_{1\leqslant i\leqslant n+3} x^{ab}_i \oplus x^{ab}_{i+1}\Big).$$

We illustrate our construction in Fig. 2. The following holds for every solution S' of I': For each complementary pair $(a, b) \in C$ and every $x \in \{a, b\}$, either all or none of $\{x, x^{ab}\} \cup Y^{ab}_{x\circ} \cup Z^{ab}_{x\circ}$ are contained in S'. Moreover, S' contains $\triangle^{ab}$ for every $(a, b) \in C$, so it must also contain a^{ab} or b^{ab}. It cannot contain both a^{ab} and b^{ab} for any $(a, b) \in C$ because its size would then exceed k'. Restricting S' to the original vertices thus is a solution to I. Conversely, for every solution S of I, the set $S \cup \{\triangle^{ab}, x^{ab}, x^{ab}_1, \ldots, x^{ab}_{n+4} \mid (a, b) \in C,\ x \in \{a, b\} \cap S\}$ is a solution of I'. Finally, it is easy to see that we can compute I' in polynomial time and that its treewidth is linear in the treewidth of I. $\Box$

Next we give an FPT reduction that eliminates necessary vertices.

Lemma 5. Defensive Alliance$^{\mathrm{F}}$ *is* W[1]*-hard when parameterized by the treewidth of the graph.*

Proof. Let $I = (G, k, V_\Box, V_\triangle)$ be a Defensive Alliance$^{\mathrm{FN}}$ instance, let $V_\circ$ denote $\mathrm{V}(G) \setminus (V_\Box \cup V_\triangle)$ and let n denote $|\mathrm{V}(G)|$. First we obtain an ordering $\preceq$ of the non-forbidden elements of $\mathrm{V}(G)$: We compute an optimal tree decomposition $\mathcal{T}$ of G, which is well known to be possible in FPT time. Then we do a post-order traversal of $\mathcal{T}$ and sequentially record the elements that occur for the last time in the current bag. Now that we have $\preceq$, we use it together with I to construct a Defensive Alliance$^{\mathrm{F}}$ instance in polynomial time. We use H to denote the set of new vertices $\{v', g_v, h_v, g^\Box_v, h^\Box_v \mid v \in V_\circ\}$. The intention is for each $g^\Box_v$ and $h^\Box_v$ to be forbidden, for each v' and h_v to be in every defensive alliance, and for g_v to be in a defensive alliance iff v is in it at the same time. We write V^+ to denote $V_\triangle \cup V_\circ \cup \{v' \mid v \in V_\circ\}$; for each $v \in V^+$, we use A_v to denote the set of new vertices $\{v_1, \ldots, v_{n+1}, v^\Box_1, \ldots, v^\Box_{n+1}\}$, and we use shorthand notation $A^\circ_v = \{v_1, \ldots, v_{n+1}\}$ and $A^\Box_v = \{v^\Box_1, \ldots, v^\Box_{n+1}\}$. The intention is for each $v^\Box_i$ to be forbidden and for each v_i to be in a defensive alliance iff v is in it at the same time. We use the notation $u \oplus v$ to denote the set of edges $\{(u, v), (u, u^\Box), (v, v^\Box), (u, v^\Box), (v, u^\Box)\}$. For any vertex $v \in V_\circ \cup V_\triangle$, we define $p(v) = v$ if $v \in V_\triangle$ and $p(v) = v'$ if $v \in V_\circ$. Let P be the set consisting of all pairs $(p(u), p(v))$ such that v is the direct successor of u according to $\preceq$. Now we define the Defensive Alliance$^{\mathrm{F}}$ instance $I' = (G', k', V'_\Box)$, where $V'_\Box = V_\Box \cup \{g^\Box_v, h^\Box_v \mid v \in V_\circ\} \cup \bigcup_{v\in V^+} A^\Box_v$, $k' = k \mapsto (n+3) \cdot (k + |V_\circ|) - |V_\triangle|$, and G' is the graph defined by

$$\mathrm{V}(G') = \mathrm{V}(G) \cup H \cup \bigcup_{v\in V^+} A_v,$$

$$\begin{aligned}\mathrm{E}(G') = {} & \mathrm{E}(G) \cup \{(v, v_i), (v, v^\Box_i) \mid v \in V^+,\ 1 \leqslant i \leqslant n+1\} \\ & \cup \bigcup_{v\in V^+,\ 1\leqslant i\leqslant n} v_i \oplus v_{i+1} \cup \bigcup_{(u,v)\in P} u_{n+1} \oplus v_1 \\ & \cup \bigcup_{v\in V_\circ} v_{n+1} \oplus g_v \cup \{(v', g_v), (v', h_v), (g_v, h_v), (g_v, h^\Box_v) \mid v \in V_\circ\}.\end{aligned}$$

We illustrate our construction in Figs. 3 and 4. The treewidth of I' is linear in the treewidth of I: The way we obtained our ordering $\preceq$ makes sure that for any two vertices a, b whose corresponding parts in Fig. 4 are next to each other, we can add new bags to $\mathcal{T}$ that account for the neighbors of a and b in Fig. 4 in such a way that bags do not become arbitrarily large. Next we show correctness. The following holds for every solution S' of I': For every non-necessary, non-forbidden vertex a,

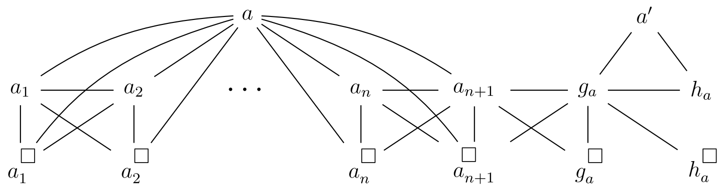

Fig. 3. Illustration of the gadget that makes sure that every solution containing a also contains f_a, g_a and a'. The vertex a is a non-necessary, non-forbidden vertex from the Defensive Alliance$^{\text{FN}}$ instance and may have other neighbors from this instance. The vertex a' additionally has the neighbors depicted in Fig. 4.

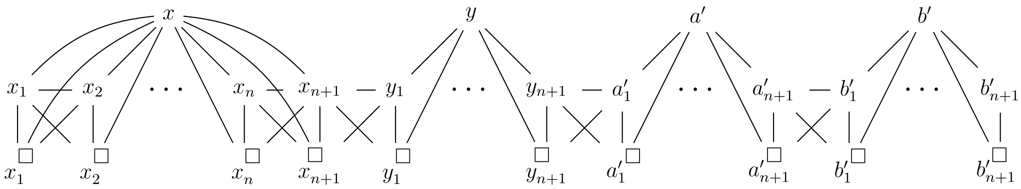

Fig. 4. Illustration of the gadget that makes sure that every solution contains all necessary vertices if it contains some necessary vertex or if it contains v' for some non-necessary vertex v. Here we assume there are the four vertices a, b, x, y, among which x and y are necessary, and we use the ordering $x \preceq y \preceq a \preceq b$.

the gadget in Fig. 3 ensures that $a \in S'$ implies $a' \in S'$. Since S' is nonempty, the gadget in Fig. 4 then causes that S' contains every necessary vertex and all primed copies of vertices. Hence $S' \cap V(G)$ is a solution to I. For each solution S of I, we can obtain a solution of I' by adding $v_1, \dots, v_{n+1}$ for every $v \in S$, adding h_v, v' for every non-necessary vertex v, and adding g_v for every non-necessary vertex $v \in S$. $\square$

Finally we define a reduction that eliminates forbidden vertices.

Lemma 6. Defensive Alliance, *parameterized by the treewidth of the graph, is* W[1]-*hard.*

Proof. Let $I = (G, k, V_\square)$ be a Defensive Alliance$^{\text{F}}$ instance. We construct a graph G' from G: For each $f \in V_\square$, we introduce new vertices $f', f_1, \dots, f_{2k}$ and add edges from f and f' to every f_i. Clearly the Defensive Alliance instance $I' = (G', k)$ is equivalent to the Defensive Alliance$^{\text{F}}$ instance I: For every $f \in V_\square$, no solution of I' contains any of $f, f', f_1, \dots, f_{2k}$ as its size would then need to exceed k. Hence the solutions of I and I' are the same because the subgraphs induced by their neighborhoods in G and G' are equal. Furthermore, it is easy to turn a tree decomposition of G into one of G' and only increase the width by two. $\square$

This proves Theorem 1.

3. Conclusion

In this work we proved that the Defensive Alliance problem parameterized by treewidth is W[1]-hard and thus not FPT (unless FPT = W[1]). This is especially interesting because most "subset problems" that are FPT when parameterized by solution size turned out to be FPT for the parameter treewidth [4], and moreover Defensive Alliance is easy on trees. By the construction of our proofs, it is clear that hardness also holds for problem variants that ask for defensive alliances *exactly* of a given size. In the future it may be interesting to study if our ideas can be useful for different kinds of alliances from the literature such as offensive or powerful alliances.

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