Habilitation

Evolution Equations with Fractional Diffusion

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von

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... dedicated to my family.

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Introduction

This Habilitationsschrift provides an overview about my research on non-local nonlinear evolution equations. The focus of my research is the analysis of models in applied mathematics which involve anomalous diffusion. The presentation is done in a cumulative style of my results which I published in refereed scientific journals. In this introduction I will explain the overarching theme of this line of my research.

The Habilitationsschrift is structured as follows: The Section 'Scientific Overview' is divided into subsections where I give a short introduction of the results, whereas the original articles are included in the chapter of the same number. For example, I explain in Section 1 results concerning viscous conservation laws, whereas the articles in which these results have been published are collected in Chapter 1. The front matter ends with its own bibliography collecting only the most essential references. If a cited article can be found in this Habilitationsschrift, the citation will be extended by an additional '`HS`', for example [9, `HS`].

Scientific Overview

I am interested in applied mathematics, especially in evolution equations appearing in the natural and social sciences. In my analysis I aim for rigorous mathematical results which help to assess if a given model is suitable to reproduce ascribed phenomena. As a Post-doc I started to investigate various nonlocal nonlinear partial differential equations. In the models under consideration the nonlocal effects arise either by direct modeling or in an asymptotic analysis of multi-dimensional partial differential equations.

The main focus have been equations with nonlocal diffusion. The heat equation

$$\partial_t u = \Delta_x u\,, \qquad t > 0\,, \quad x \in \mathbb{R}\,, \tag{1}$$

is a simple model for diffusion of a quantity in space over time. The Laplacian Δ_x generates via the initial-value problem a positivity preserving semigroup. Lévy operators $\mathcal{L}$ are those (nonlocal) linear operators with space and time independent coefficients that generate again such a (positivity preserving) semigroup. Therefore, Lévy operators appear naturally in evolution equations modeling diffusion.

A first systematic study of the erratic movements of particles suspended in a fluid at rest has been conducted by Robert Brown at the beginning of the 19th century. A century later, Albert Einstein modeled such a movement as a random walk of individual particles and derived that the evolution of the probability density of such a particle ensemble is governed by the classical heat equation (1). Finally, the theory of stochastic processes allowed to describe the movement of individual particles, which are nowadays known as Brownian motion or Wiener process. Brownian motion is a diffusion process whose mean squared displacement depends linearly on time. However, examples of **anomalous diffusion** have been observed in natural and social sciences where ensembles of particles spread slower (subdiffusion) or faster (superdiffusion) than Brownian motion. Loosely speaking, evolution equations where the Laplacian (the infinitesimal generator of standard Brownian motion) is replaced by Riesz-Feller operators (the infinitesimal generator of strictly-stable Levy processes) are models for superdiffusion. Such evolution equations can be derived from microscopic models via continuous time random walks.

A classical **reaction-diffusion equation** is a partial differential equation where the Laplacian models diffusion and a (nonlinear) function models reaction of some quantities over time. Reaction-diffusion equations are important models in chemistry, biology, ecology, physics and material science, and their analysis has a long history. In the last decades reaction-diffusion processes with anomalous diffusion have been identified, e.g. in the dynamics of fronts in magnetically confined plasmas, the spreading of epidemics due to complex mobility patterns of individuals, examples of step-flow growth of a crystal surface. A modified continuous time random walk approach allows to derive reaction-diffusion equations with anomalous diffusion from microscopic models. For example, we consider reaction-diffusion equations

$$\partial_t u = \mathcal{L}u + r(u)\,, \qquad t > 0\,, \quad x \in \mathbb{R}^d\,, \tag{2}$$

for some (nonlocal) Lévy operator $\mathcal{L}$ and nonlinear reaction function $r(u)$.

In some **models of fluid dynamics**, the asymptotic analysis of boundary layers leads naturally to evolution equations with nonlocal operators, e.g. fractional derivatives. Although the original multi-dimensional equations are posed on a bounded domain, the resolution of finer scales yields evolution equations on the real line/whole space. For example, in the analysis of a shallow water flow in a channel, a nonlocal Korteweg-de Vries-Burgers equation,

$$\partial_t u + \partial_x f(u) = \varepsilon \mathcal{L}u + \delta \partial_x^3 u\,, \qquad t > 0\,, \quad x \in \mathbb{R}\,, \tag{3}$$

for some nonlinear flux function f, a nonlocal Lévy operator $\mathcal{L}$ and constants $\varepsilon > 0$ and $\delta \in \mathbb{R}$ has been derived.

Traveling wave solutions are solutions with a spatial profile that keeps its rigid shape as it is translated in the spatial domain over the course of time. These solutions are of interest in applications, e.g. modeling phase transitions in material science or water waves in fluid dynamics. In the following, we use the definition

Definition 1. A Traveling Wave Solution (TWS) of (2) resp. (3) is a solution of the form $u(x,t) = \overline{u}(\xi)$ with $\xi := x - st$, for some *wave speed* $s \in \mathbb{R}$ and a function $\overline{u} : \mathbb{R} \to \mathbb{R}$ connecting distinct endstates $\lim_{\xi \to \pm\infty} \overline{u}(\xi) = u_\pm$.

In this *Habilitationsschrift*, I present one line of my research focusing on the Traveling Wave Problem (TWP), i.e. the existence, uniqueness and asymptotic stability of traveling wave solutions. The spatial domain of the models under consideration is the real line $\mathbb{R}$, although, in a multidimensional spatial setting, the ansatz of a *planar* traveling wave solution yields again a Traveling Wave Equation (TWE) on $\mathbb{R}$.

The collaborations on numerical methods are not at the center of my research, but in an ongoing research project I contribute to devise structure preserving numerical schemes.

Next, I provide some background on **Lévy operators**: The heat equation

$$\partial_t u = \Delta_x u\,, \qquad t > 0\,, \quad x \in \mathbb{R}^d\,, \tag{4}$$

is a simple model to describe the diffusion of a quantity in space over time. Its fundamental solution is a Gaussian distribution

$$G(x,t) = \frac{1}{\sqrt{4\pi t}^d}\, \mathrm{e}^{-|x|^2/4\pi}\,,$$

which is interpreted as the evolution of the quantity being initially concentrated at the origin. The solution of the Cauchy problem for (4) with initial condition $u(\cdot, 0) = u_0$ is given by $u(x,t) = (G(\cdot,t) * u_0)(x)$. Then, the solution operator

$$S_t : u_0 \mapsto u(x,t) = (G(\cdot,t) * u_0)(x)\,,$$

defines a semigroup $(S_t)_{t\geq 0}$, i.e. $S_0 = \mathrm{Id}$ and $S_s S_t = S_{s+t}$ for $s, t \geq 0$. *The Laplacian Δ_x is said to generate a semigroup $(S_t)_{t\geq 0}$.*

In fact, the Laplacian generates a positivity preserving semigroup. Lévy operators are those (nonlocal) linear operators with time and space independent coefficients that again generate a positivity preserving semigroup. Lévy operators are of the form

$$\begin{aligned} \mathcal{L}f(x) &= \tfrac{1}{2}\nabla \cdot (A\nabla f)(x) + \gamma \cdot \nabla f(x) \\ &\quad + \int_{\mathbb{R}^d} \big(f(x+y) - f(x) - y \cdot \nabla f(x)\; c(y)\big)\, \nu(\mathrm{d}y) \end{aligned} \tag{5}$$

for functions $f(x)$ decaying sufficiently fast to 0 as $|x| \to \infty$, a symmetric positive semi-definite matrix $A \in \mathbb{R}^{d\times d}$ and a vector $\gamma \in \mathbb{R}^d$ with constant coefficients, as well as a measure ν on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \min(1, |y|^2)\, \nu(\mathrm{d}y) \; < \infty\,,$$

and $c(y) = 1_B(y)$ with $B := \{\, x \in \mathbb{R}^d \,|\, |x| \leq 1\,\}$. [1] A Lévy operator may consist of a transport operator $\gamma \cdot \nabla f$, a local diffusion operator $\nabla \cdot (A\nabla f)$ and a nonlocal (jump) operator. In classical models diffusion is most often represented by the local operator $\nabla \cdot (A\nabla f + \gamma f)$. Evolution equations involving a nonlocal Lévy operator modeling diffusion have emerged as useful models in the applied sciences.

Examples. In the following, we restrict to the $d = 1$ dimensional situation. Convolution operators. The Lévy operators

$$\mathcal{L}f(x) = \int_{\mathbb{R}} \big(f(x+y) - f(x)\big)\, \nu(\mathrm{d}y) \tag{6}$$

are infinitesimal generators associated to a compound Poisson process with finite Lévy measure ν with $c \equiv 0$ [1]. The special case of $\nu(\mathrm{d}y) = \phi(-y)\,\mathrm{d}y$ for some function $\phi \in L^1(\mathbb{R})$ yields

$$\mathcal{L}f(x) = \Big(\phi * f - \int_{\mathbb{R}} \phi\; \mathrm{d}y\;\; f\Big)(x)\;. \tag{7}$$

[1] For fixed x, the function $f(x+y) - f(x) - y \cdot \nabla f(x)c(y)$ is integrable with respect to $\nu(\mathrm{d}y)$, because it is bounded and $f(x+y) - f(x) - y \cdot \nabla f(x)c(y) = O(|y|^2)$ as $|y| \to 0$. The indicator function $c(y) = 1_B(y)$ is only one possible choice to obtain an integrable integrand. Whereas γ depends on c, the coefficients A and ν are independent of c.

Riesz-Feller operators. The Riesz-Feller operators of order a and asymmetry θ are defined as Fourier multiplier operators [2]

$$\mathcal{F}[D^a_\theta f](k) = \psi^a_\theta(k)\ \mathcal{F}[f](k)\ , \qquad k \in \mathbb{R}\ , \tag{8}$$

with symbol $\psi^a_\theta(k) = -|k|^a \exp\left[\mathrm{i}\ \operatorname{sgn}(k)\ \theta\pi/2\right]$ such that $(a,\theta) \in \mathfrak{D}_{a,\theta}$ and

$$\mathfrak{D}_{a,\theta} := \{\,(a,\theta) \in \mathbb{R}^2 \,|\, 0 < a \le 2\ , \quad |\theta| \le \min\{a, 2-a\}\,\}\ .$$

Nonlocal Riesz-Feller D^a_θ operators are those with parameters

$$(a,\theta) \in \mathfrak{D}^\bullet_{a,\theta} := \{\,(a,\theta) \in \mathfrak{D}_{a,\theta} \,|\, 0 < a < 2\ , \quad |\theta| < 1\,\}\,.$$

If $(a,\theta) \in \mathfrak{D}^\bullet_{a,\theta}$ with $a \neq 1$, then for all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$

$$\begin{aligned} D^a_\theta f(x) =& \frac{c_+(\theta) - c_-(\theta)}{1-a} f'(x) \\ &+ c_+(\theta) \int_0^\infty \frac{f(x+y) - f(x) - f'(x)\, y 1_{(-1,1)}(y)}{y^{1+a}}\, \mathrm{d}y \\ &+ c_-(\theta) \int_0^\infty \frac{f(x-y) - f(x) + f'(x)\, y 1_{(-1,1)}(y)}{y^{1+a}}\, \mathrm{d}y \end{aligned} \tag{9}$$

with $c_\pm(\theta) = \Gamma(1+a)\sin((a \pm \theta)\pi/2)/\pi$, see e.g. [14, HS].

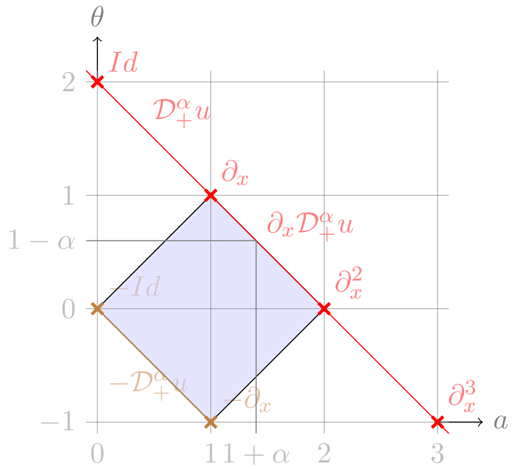


FIGURE 1. The family of Fourier multipliers $\psi^a_\theta(k) = -|k|^a \exp\left[\mathrm{i}\operatorname{sgn}(k)\theta\pi/2)\right]$ has two parameters a and θ. Some examples for the associated Fourier multiplier operators $\mathcal{F}[Tf](k) = \psi^a_\theta(k)\ \mathcal{F}[f](k)$ are partial derivatives ∂^k_x $(k \in \mathbb{N}_0)$ and Caputo derivatives $\mathcal{D}^\alpha_+$ with $0 < \alpha < 1$. The Riesz-Feller operators D^a_θ are those operators with parameters $(a,\theta) \in \mathfrak{D}_{a,\theta}$. The set $\mathfrak{D}_{a,\theta}$ is also called *Feller-Takayasu diamond* and depicted as a shaded region, see also [38].

[2]We use the conventions in probability theory, and define the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ for $g \in L^1(\mathbb{R}^d)$ and $x, k \in \mathbb{R}^d$ as $\mathcal{F}[g](k) := \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}k\cdot x}\, g(x)\, \mathrm{d}x$ and $\mathcal{F}^{-1}[g](x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}k\cdot x}\, g(k)\, \mathrm{d}k$. In the following, $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote also their respective extensions to $L^2(\mathbb{R}^d)$.

Special cases of Riesz-Feller operators are

- Fractional Laplacians $-(-\Delta)^{a/2}$ on $\mathbb{R}$ with Fourier symbol $-|k|^a$ for $0 < a \le 2$. In particular, fractional Laplacians are the only symmetric Riesz-Feller operators where $-(-\Delta)^{a/2} = D^a_0$ and $\theta \equiv 0$.
- Caputo derivatives $-\mathcal{D}^\alpha_+$ with $0 < \alpha < 1$,
$$\mathcal{D}^\alpha_+[f](x) = \tfrac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} (x-y)^{-\alpha} f'(y)\, \mathrm{d}y \ ,$$
are Riesz-Feller operators $-\mathcal{D}^\alpha_+ = D^\alpha_{-\alpha}$ with $a = \alpha$ and $\theta = -\alpha$.
- Derivatives of Caputo derivatives $\partial_x \mathcal{D}^\alpha_+$ with $0 < \alpha < 1$ are Riesz-Feller operators $\partial_x \mathcal{D}^\alpha_+ = D^{1+\alpha}_{1-\alpha}$ with $a = 1+\alpha$ and $\theta = 1-\alpha$.

Nonlocal Lévy operators appear in different ways in a model and/or its analysis:

- <u>as a technical tool</u>, see e.g. Riesz [43].
- <u>via a model reduction</u>. From an analytical point of view, the square root of the Laplacian $(-\Delta)^{1/2}$ can be realized as a Dirichlet-to-Neumann operator for smooth bounded functions $f : \mathbb{R}^n \to \mathbb{R}$: If u is the solution of
$$\begin{cases} \Delta u(x,y) = 0 & \text{for } x \in \mathbb{R}^n \,,\ y > 0 \,, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}^n \,, \end{cases} \tag{10}$$
then $-\partial_y u(x,0) = (-\Delta)^{1/2} f(x)$. Thus, for a given Dirichlet datum f, the solution u of the extension problem (10) carries the information $-(-\Delta)^{1/2} f(x)$ as its Neumann datum on the boundary. Caffarelli and Silvestre [25] obtained a similar characterization for all fractional Laplacians $-(-\Delta)^{\alpha/2}$ of order $0 < \alpha < 2$. This connection is useful in the study of nonlinear integro-differential equations involving the fractional Laplacian on $\mathbb{R}^n$, since the associated partial differential equations on an extended half-space are better understood.
- <u>direct modeling of particle systems subject to anomalous diffusion</u>. One approach to model diffusion processes is via continuous time random walks (CTRW); particles are considered to jump where the jump length and the waiting time between two successive jumps are distributed according to a joint probability distribution [41, 42]. A simplifying assumption is that the jump length and the waiting time are independent random variables; in particular Einstein considered deterministic waiting times between successive jumps in a regular lattice to derive the heat equation. Considering a jump length distribution with diverging variance and a waiting time distribution with finite mean one can derive a fractional diffusion equation for a pure jump process which is Markovian, such as
$$\partial_t u = D^a_\theta[u] \quad \text{for } x \in \mathbb{R} \,, \quad t > 0 \,, \tag{11}$$
with a Riesz-Feller operator in space. [3]

[3] In contrast, considering a jump length distribution with finite variance and a waiting time distribution with diverging mean, one can derive a fractional diffusion equation for

However, the process exhibits a diverging mean squared displacement such that the model is not applicable for massive particles with finite propagation velocity. Nonetheless it has found applications describing the diffusion in energy space encountered in single molecule spectroscopy, the diffusion on a polymer chain in chemical space, amongst others.

At the center of the continuous time random walks is the classical central limit theorem. It asserts for a sequence of independent and identically distributed random numbers with finite variance that the limit of its normalized partial sums will have a Gaussian probability distribution. Relaxing the assumption on the variance, Lévy α-stable probability distributions are the general class of "attractors" for properly normed sums of independent and identically-distributed random variables.

A Lévy process is a stochastic process with independent and stationary increments which is continuous in probability [20, 33, 44]. Therefore a Lévy process is characterized by its transition probabilities $p(t, x)$, which evolve according to an evolution equation

$$\partial_t p = \mathcal{L} p \tag{12}$$

for some Lévy operator $\mathcal{L}$, see (5). A Lévy operator is also called the infinitesimal generator of its Lévy process.

Whereas, the Laplacian is the infinitesimal generator of Brownian motion, Riesz-Feller operators D^a_θ are infinitesimal generators of a-stable Lévy processes. Reflecting the decomposition of the Lévy operator (5), every Lévy process can be decomposed into the sum of a Brownian motion, a deterministic motion and a pure jump process. In particular, a-stable Lévy process – also called Lévy flights – are examples of pure jump Lévy processes. In Table 1 we give examples for the relation between a stochastic process, its probability distribution, and its infinitesimal generator $\mathcal{L}$ such that (12) governs the evolution of the probability distribution p.

generator	stochastic process	distribution
$\mathcal{L} = \Delta_x = D^2_0$	Brownian motion	Gaussian
Riesz-Feller operator $\mathcal{L} = D^a_\theta$	a-stable Lévy process	a-stable
Lévy operator $\mathcal{L}$	Lévy process	infinitely divisible

TABLE 1. Examples for the relation between a stochastic processes, its probability distribution, and its infinitesimal generator. We recall that a Laplacian is a Riesz-Feller operator and Riesz-Feller operators are Lévy operators.

In conclusion, integro-differential equations with Riesz-Feller operator modeling diffusion are a generic class of macroscopic equations modeling many-particle systems for diffusing particles.

a subdiffusive process, such as $\mathcal{D}^\beta_t[u] = \partial^2_x u$ for $t > 0$, $x \in \mathbb{R}$, with a Caputo derivative in time of order $0 < \beta < 1$.

1. Viscous conservation laws

We consider scalar viscous conservation laws

$$\partial_t u + \partial_x f(u) = D^a_\theta u \quad \text{for } x \in \mathbb{R}\,,\ t > 0\,, \tag{13}$$

with nonlocal diffusion of Riesz-Feller type where $1 < a < 2$ and $|\theta| \leq 2 - a$.

The subclass of scalar viscous conservation laws

$$\partial_t u + \partial_x f(u) = D^a_0 u \quad \text{for } x \in \mathbb{R}\,,\ t > 0\,, \tag{14}$$

with fractional Laplacian $D^a_0 = -(-\partial_x^2)^{a/2}$ $(0 < a \leq 2)$ are used to model overdriven detonation in gas and anomalous diffusion in semiconductor growth [51]. The parameter a of a fractional Laplacian/Riesz derivative D^a_0 corresponds to its order of derivatives. Roughly speaking, for $1 < a < 2$, the diffusion dominates the transport term. Thus, for the subcritical case $1 < a < 2$, the Cauchy problem with essentially bounded initial datum is globally well posed. In fact solutions are smooth for positive times, see also Droniou et al. [30, 31]. Also, in the critical case $a = 1$, the well-posedness of the Cauchy problem has been established [29, 40, 26]. In contrast, for the supercritical case $0 < a < 1$ and Burgers flux, weak solutions of the Cauchy problem may not be unique [19, 18]. In this case, discontinuities can appear even for smooth initial datum [19], and there exists essentially bounded initial data such that uniqueness of a weak solution to the Cauchy problem fails [18]. For uniqueness, an entropy condition [17] is necessary to select admissible solutions (just like for inviscid conservation laws). Finally, Biler et al. [23] showed that no continuous traveling wave solutions can exist for $a \in (0, 1]$, however they provide no existence result for the case $a \in (1, 2)$ (which was proven recently by Chmaj [27]).

Sugimoto and Kakutani [47, 46] derived

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha_+ u \quad \text{for } x \in \mathbb{R}\,,\ t > 0\,, \tag{15}$$

with Burgers flux function $f(u) = u^2$ and extremal Riesz-Feller operator $D^a_\theta = \mathcal{D}^\alpha_+$ with $0 < \alpha < 1$ as a model for the far-field evolution of unidirectional viscoelastic waves in polymers. The profile $\overline{u}(\xi)$ of a traveling wave solution $u(x,t) = \overline{u}(\xi)$ with $\xi := x - st$ is governed by a fractional differential equation

$$-s\partial_\xi \overline{u} + \partial_\xi \overline{u}^2 = \partial_\xi \mathcal{D}^\alpha_+[\overline{u}] \quad \text{for } \xi \in \mathbb{R}\,. \tag{16}$$

They indicate that bounded continuous traveling wave solution may exist, but give no analytical proof, instead they construct numerical solutions and study the asymptotic behavior of $\overline{u}(\xi)$ as $\xi \to \pm\infty$.

[9, HS]: F. Achleitner, S. Hittmeir, and C. Schmeiser. "On nonlinear conservation laws with a nonlocal diffusion term". In: *J. Differential Equations* 250.4 (2011), pp. 2177–2196

Cauchy problem: global well-posedness for initial data in $L^\infty(\mathbb{R})$. Droniou et al. [30] studied the well-posedness for (14) with $1 < a < 2$ in a mild formulation using that D^a_0 generates a positivity preserving convolution semigroup. The associated kernel is a smooth probability density function, satisfies a scaling property and has integrable spatial derivatives (among other useful properties). This allows to prove the existence of a mild solution and its smoothness for positive times. Finally, the mild solution turns out to be a classical solution of (14).

We realized that the diffusion operator in (13) is a non-symmetric Riesz-Feller operator of order between 1 and 2, whose semigroup shares important properties with the fractional Laplacian – i.e. the symmetric Riesz-Feller operator - of the same order. Therefore, we can use the same approach to show that the Cauchy problem of (13) with essentially bounded initial data is globally well-posed and solutions will be smooth for positive times [8, HS].

TWP: existence of TWS associated to Lax shock waves. We use a dynamical systems approach to prove the existence of traveling wave solutions of (15) with convex flux functions, as long as the shock triple $(u_-, u_+; s)$ satisfies the Rankine-Hugoniot condition and the Lax entropy condition $u_- > u_+$. To this end, we integrate (16) on $(-\infty, \xi)$ and use the properties of a TWS to deduce

$$h(\overline{u}) = f(\overline{u}) - f(u_-) - s(\overline{u} - u_-) = \mathcal{D}_+^\alpha[\overline{u}] \quad \text{for } \xi \in \mathbb{R}\,. \tag{17}$$

The approach relies on the causality of the operator

$$\mathcal{D}_+^\alpha[u](x) = \tfrac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha}\, \mathrm{d}y\,, \tag{18}$$

i.e. the value of $\mathcal{D}_+^\alpha[\overline{u}](\xi)$ at a point ξ depends only on the values of the profile $\overline{u}$ on the interval $(-\infty, \xi]$. The traveling wave equation is translation invariant. First we prove the existence of a unique solution $u_{down} : I_\varepsilon \to \mathbb{R}$ of (16) satisfying

$$\lim_{\xi \to -\infty} u_{down}(\xi) = u_- \quad \text{and} \quad u_{down}(\xi_\varepsilon) = u_- - \varepsilon\,. \tag{19}$$

on the interval $I_\varepsilon = (-\infty, \xi_\varepsilon]$ for $\xi_\varepsilon := \log \varepsilon/\lambda$ and some sufficiently small $\varepsilon > 0$. Then we establish that this solution $u_{down} : (-\infty, I_\varepsilon] \to \mathbb{R}$ can be extended locally and remains monotone and bounded, such that the argument for local existence can be iterated to imply the existence of a solution $u_{down} \in C_b^1(\mathbb{R})$ with $\lim_{\xi\to-\infty} u_{down}(\xi) = u_-$. Finally, the proof is completed by proving $\lim_{\xi\to\infty} u(\xi) = u_+$.

In fact, the TWE (17) is a fractional differential equation on the real line. Therefore we provided the first proof for the existence of heteroclinic orbits in fractional differential equations with Caputo derivative. Moreover, this result proves the existence of a TWS for a reaction-diffusion equation with monostable reaction function, see [1, HS].

TWP: Stability of TWS associated to Lax shock waves. The asymptotic stability of TWS under zero-mass perturbation is proven by constructing a Lyapunov functional which is equivalent to some Sobolev norm.

[8, HS]: F. Achleitner, S. Hittmeir, and C. Schmeiser. "On nonlinear conservation laws regularized by a Riesz-Feller operator". In: *Hyperbolic Problems: Theory, Numerics, Applications*. Ed. by F. Ancona et al. Vol. 8. AIMS on Applied Mathematics. AIMS, 2014, pp. 241–248

We prove the global well-posedness and instantaneous smoothing for the Cauchy problem associated to (13) with essentially bounded initial data for *all* Riesz-Feller operators D^a_θ with $1 < a < 2$ and $|\theta| \le 2 - a$; and not just $D^a_\theta = \partial_x \mathcal{D}^\alpha_+$ as in [9, HS].

[15, HS]: F. Achleitner and Y. Ueda. "Asymptotic stability of traveling wave solutions for nonlocal viscous conservation laws with explicit decay rates". In: *Journal of Evolution Equations* (Feb. 2018), pp. 1–24

In our main result [15, HS Theorem 2], we prove the asymptotic stability with explicit algebraic-in-time decay rate for traveling wave solutions of (13) with monotone decreasing profiles.

For classical viscous conservation laws, the weighted energy method allows to prove that initial perturbations in a weighted Lebesgue space with polynomial weights induce an algebraic-in-time decay of the L^∞-norm of the perturbation, see e.g. [34]. Due to the nonlocal diffusion operator, this method is difficult to apply to (13). Instead, we employ another technique which focuses on the interpolation property in Sobolev space. In this way, optimal decay estimates for the asymptotic stability of viscous *rarefaction* waves in scalar viscous conservation laws (13) with $D^a_0 = \partial_x^2$ have been derived by Yoshihiro Ueda and collaborators.

2. Korteweg-de Vries-Burgers equations

Kluwick and collaborators were interested to identify a physical meaningful model which supports the existence of non-classical shock waves, i.e. shock waves that do not satisfy Lax' entropy condition. Therefore, they considered a shallow water flow in a channel and studied the internal structure of weakly nonlinear bores in laminar flow at the high Reynolds number limit. Via matched asymptotic expansion, they obtain an equation for the pressure in the form

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x \mathcal{D}^\alpha_+ u + \delta \partial_x^3 u \quad \text{for } x \in \mathbb{R}\,,\ t > 0\,, \tag{20}$$

with $0 < \alpha < 1$, $\epsilon > 0$, and $\delta \in \mathbb{R}$. The flux function is convex for a single layer flow and non-convex for a two-layer flow [48, 35, 37, 36].

For $\alpha = 1$, equation (20) reduces formally to a classical Korteweg-de Vries-Burgers (KdVB) equation

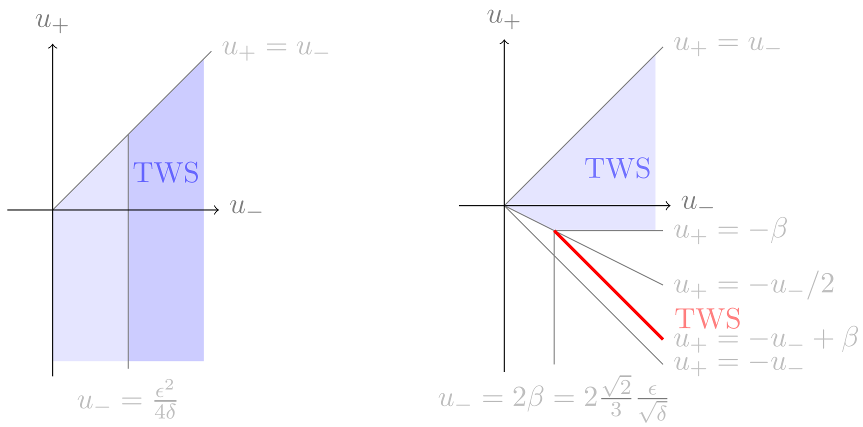
$$\partial_t u + \partial_x f(u) = \epsilon \partial_x^2 u + \delta \partial_x^3 u, \quad x \in \mathbb{R}, \quad t > 0, \tag{21}$$

for constants $\epsilon > 0$ and $\delta \in \mathbb{R}$. A TWS $u(t,x) = \overline{u}(x - st)$ for (21) satisfies the (integrated) TWE

$$h(\overline{u}) := f(\overline{u}) - s\overline{u} - \big(f(u_-) - su_-\big) = \epsilon \overline{u}' + \delta \overline{u}'' \quad \text{for } \xi \in \mathbb{R}\,, \tag{22}$$

which can be analyzed by phase plane analysis. The existence of TWS for given endstates is fully understood, see Figure 2.

In case of Burgers flux function, a linearization of the TWE around the endstate u_+ reveals that the monotonicity of the profile depends on the relation between the diffusion and dispersion parameters (i.e. for $u_- < \epsilon^2/4\delta$ the profile of a TWS has an oscillatory tail, see Figure 2A. In case of a cubic flux function, TWS appear which do not satisfy Lax' entropy condition, see Figure 2B.



(A) $f(u) = u^2/2$ (B) $f(u) = u^3$

FIGURE 2. We consider KdVB equation (21) with (A) Burgers' flux function $f(u) = u^2/2$, and (B) cubic flux function $f(u) = u^3$. For a given pair of endstates $(u_-, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, the (blue) shaded region and the (red) thick line indicate the existence of a TWS associated to a Lax shock and a non-classical shock wave, respectively.

[7, HS]: F. Achleitner, C. M. Cuesta, and S. Hittmeir. "Travelling waves for a non-local Korteweg–de Vries–Burgers equation". In: *J. Differential Equations* 257.3 (2014), pp. 720–758

We studied equation (20) for the Burgers flux and established well-posedness of Cauchy problem in Sobolev spaces. We proved the existence of traveling wave solutions associated to classical Lax shock waves again via a dynamical system approach like in our previous work [9, HS]. However, the profile of TWS of the nonlocal KdV–Burgers equation may not be monotone (if dispersive effects dominate), just like in the classical KdV–Burgers equation. Moreover, the integrated TWE is a fractional differential equation of order 2 (compared to TWE (17) which is of order $0 < \alpha < 1$). The TWP for the classical KdV–Burgers equation can be analyzed via a phase-plane analysis, which is no longer useful due to the presence of the nonlocal operator $\mathcal{D}^\alpha_+$. To overcome these challenges, we need to modify the existence proof in our previous work [9, HS]. The existence of a unique solution $u_{down} : I_\varepsilon \to \mathbb{R}$ of the TWE satisfying (19) follows again by a fixed point argument. Whereas, an energy estimate shows that the (extended) profile is bounded from below, allowing for the possibility that $\overline{u}$ takes values below u_+. The extension of the profile $\overline{u}$ follows by rewriting the TWE as a system of non-autonomous fractional differential equation on bounded time intervals, the Lipschitz continuity of the nonlinearity, and the uniform boundedness of the profile. Finally, the convergence $\lim_{\xi\to\infty} \overline{u}(\xi) = u_+$ is proven.

In addition, we discuss the monotonicity of profiles for sufficiently small dispersion and the asymptotic stability of monotone traveling wave solutions for zero-mass perturbations.

[28, HS]: C. M. Cuesta and F. Achleitner. "Addendum to "Travelling waves for a non-local Korteweg–de Vries-Burgers equation" [J. Differential Equations 257 (3) (2014) 720–758] [MR3208089]". In: *J. Differential Equations* 262.2 (2017), pp. 1155–1160

We add a theorem to [9, 7, `HS`]. There the proof of existence and uniqueness of traveling wave solutions relies on the assumption that the exponentially decaying functions are the only bounded solutions of the linearized traveling wave equation. In [9, 7, `HS`], we prove it in suitable Sobolev spaces with exponential weights. We show this by writing the linearized traveling wave equation as a Wiener–Hopf equation and using results by Krein. In this addendum, we prove the assumption in unweighted Sobolev spaces, hence, we close the existence and uniqueness proof of traveling wave solutions.

[1, HS]: F. Achleitner. "Two Classes of Nonlocal Evolution Equations Related by a Shared Traveling Wave Problem". In: *From Particle Systems to Partial Differential Equations.* Ed. by P. Gonçalves and A. J. Soares. Springer International Publishing, 2017, pp. 47–72

We consider nonlocal reaction-diffusion equations, and nonlocal Korteweg-de Vries-Burgers (KdVB) equations, i.e. scalar conservation laws with diffusive-dispersive regularization. We review the existence of traveling wave solutions for these two classes of evolution equations. For classical equations the traveling wave problem (TWP) for a local KdVB equation (21) can be identified with the TWP for a reaction-diffusion equation $\partial_t v = -h(v) + \delta\partial_x^2 v$, since a TWS $v(t,x) = \overline{u}(x - \epsilon t)$ for this reaction-diffusion equation solves again TWE (22). In this article we study this relationship for these two classes of evolution equations with nonlocal diffusion/dispersion. This connection is especially useful, if the TWE is not studied directly, but the existence of a TWS is proven using one of the evolution equations instead, e.g. as in [14, HS]. Finally, we present three models from fluid dynamics and discuss the TWP via its link to associated reaction-diffusion equations.

3. Reaction-diffusion equations

A scalar reaction-diffusion equation is a partial differential equation

$$\partial_t u = \partial_x^2 u + f(u) \quad \text{for } x \in \mathbb{R}\,, \quad t > 0\,, \tag{23}$$

where the spatial derivative models diffusion and a (nonlinear) function f models reaction of some quantity $u = u(x,t)$ over time. Reaction-diffusion equations are important models in chemistry, biology, ecology, physics and material science, and their analysis has a long history [21, 45, 49].

In the last decades reaction-diffusion equations with nonlocal diffusion have emerged as useful models, e.g. for the dynamics of fronts in magnetically confined plasmas, the spreading of epidemics due to complex mobility patterns of individuals, or the step-flow growth of a crystal surface, see the references in [14, 11, `HS`]. Starting with a modified continuous time random walk (CTRW) for a single particle, one can derive a reaction-diffusion equation

$$\partial_t u = D_\theta^a u + f(u) \quad \text{for } x \in \mathbb{R}\,, \quad t > 0\,, \tag{24}$$

which governs the evolution of the particle ensemble density (where D^a_θ is a Riesz-Feller operator), see e.g. [39]. The (a-stable) Lévy process associated to a Riesz-Feller operator is Markovian, hence, the effects of diffusion and reaction are separable, and the Laplacian in (23) gets simply replaced by the Riesz-Feller operator in (24). In contrast, subdiffusion is a non-Markovian stochastic process and the resulting reaction-diffusion equations are much more involved.

Next, we discuss equations with reaction functions of the following type:

- *monostable* reaction functions, i.e.,

$$\exists u_1 < u_2 : \qquad f(u) \begin{cases} = 0 & \text{for } u \in \{u_1, u_2\}\,, \\ > 0 & \text{for } u \in (u_1, u_2)\,, \end{cases}$$

- *bistable* reaction functions $f \in \mathcal{C}^1$, i.e.,

$$\exists u_1 < u_2 < u_3 : \qquad f(u) \begin{cases} = 0 & \text{for } u \in \{u_1, u_2, u_3\}\,, \\ < 0 & \text{for } u \in (u_1, u_2)\,, \\ > 0 & \text{for } u \in (u_2, u_3)\,, \end{cases}$$

  and $f'(u_1) < 0$, $f'(u_3) < 0$.

In ecology, the Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation is a reaction-diffusion equation (23) with monostable reaction function $f(u) = u(1-u)$ describing the competition of species; in particular the traveling wave solution of (23) connecting the steady states 0 and 1 shows that the stable state 1 will invade the unstable state 0 at a constant speed. An important difference between the FKPP equation with normal diffusion (23) and anomalous diffusion (24) is that fronts connecting the stable state with the unstable state will travel with constant speed in the former case and with exponential speed in the latter case [24, 32].

[12]: F. Achleitner and C. Kuehn. “On bounded positive stationary solutions for a nonlocal Fisher-KPP equation”. In: *Nonlinear Anal.* 112 (2015), pp. 15–29

A FKPP equation with nonlocal reaction term of the form

$$\partial_t u = \Delta u + \mu u(1 - \phi * u) \quad \text{for } x \in \mathbb{R}^d\,, \quad t > 0\,, \tag{25}$$

for $\mu > 0$ and some non-negative integrable convolution kernel $\phi \in C^1_b(\mathbb{R}^d;\mathbb{R})$ is used in ecology to model nonlocal saturation and nonlocal competition effects. In one space dimension ($d = 1$) and for sufficiently ”‘small”’ nonlocality, it was shown that there are only two bounded non-negative stationary solutions [22]. We proved this result in general space dimensions ($d \geq 1$) using a different approach. Our approach combines bifurcation theory in weighted Sobolev spaces with direct geometric analysis of the degenerate steady state at zero.

Reaction-diffusion equations (23) with a bistable reaction term have been derived as Nagumo’s equation to model propagation of signals, as one-dimensional real Ginzburg-Landau equation to model long-wave amplitudes, as well as Allen-Cahn’s equation to model phase transitions in solids. For applications of reaction-diffusion equations (24) with Riesz-Feller diffusion and bistable reaction term we refer to [50] and references therein.

[14, HS]: F. Achleitner and C. Kuehn. "Traveling waves for a bistable equation with nonlocal diffusion". In: *Adv. Differential Equations* 20.9-10 (2015), pp. 887–936

To our knowledge, we established the first rigorous result on existence, uniqueness (up to translations) and stability of traveling wave solutions for reaction-diffusion equations with Riesz-Feller D^a_θ operators of order $1 < a < 2$ and bistable reaction functions. Previous results were restricted to fractional Laplacians and/or balanced potentials and focused on the (static) properties of the profile, see our discussion in the introduction of [14, HS]. In essence, we modify the approach of Xinfu Chen for convolution models which relies on a strict comparison principle and the construction of sub- and supersolutions. To prove the existence of a TWS, one considers the initial-value problem for (24) with prepared initial datum and proves that the asymptotic limit of the solution is the profile of a TWS. This approach is very robust and its extension to reaction-diffusion equations (2) with the largest possible subclass of Levy operators $\mathcal{L}$ is a work in progress [13].

[11, HS]: F. Achleitner and C. Kuehn. "Analysis and numerics of traveling waves for asymmetric fractional reaction-diffusion equations". In: *Commun. Appl. Ind. Math.* 6.2 (2015), e–532, 25

In this work we discuss our analytical result (proven in [14, HS]) on the existence, uniqueness and stability of traveling waves for nonlocal reaction-diffusion equations with Riesz-Feller operators. Then we survey numerical schemes for symmetric anomalous diffusion and suggest a new scheme for the anomalous case based upon discretization of the integral representation of Riesz-Feller operators (9). This scheme is used with projection boundary conditions to numerically compute the stable traveling wave solution.

Additional material

Hypocoercive Equations. Another line of research is concerned with hypocoercive equations such as kinetic Fokker-Planck equations and BGK equations.

[**5**]: F. Achleitner, A. Arnold, and D. Stürzer. "Large-time behavior in non-symmetric Fokker-Planck equations". In: *Riv. Math. Univ. Parma (N.S.)* 6.1 (2015), pp. 1–68

We considered distinct classes of linear non-symmetric **Fokker-Planck equations** having a unique steady state and established exponential convergence of solutions towards the steady state with explicit (estimates of) decay rates. First, "hypocoercive" Fokker-Planck equations are degenerate parabolic equations such that the entropy method to study large-time behavior of solutions had to be modified. We reviewed a recent modified entropy method (for non-symmetric Fokker-Planck equations with drift terms that are linear in the position variable). Second, kinetic Fokker-Planck equations with non-quadratic potentials are another example of non-symmetric Fokker-Planck equations. Their drift term is nonlinear in the position variable. In case of potentials with bounded second-order derivatives, the modified entropy method allows to prove exponential convergence of solutions to the steady state. In this application of the modified entropy method symmetric positive definite matrices solving a matrix inequality are needed. We determined all such matrices achieving the optimal decay rate in the modified entropy method. In this way we proved the optimality of previous results.

[**2**]: F. Achleitner, A. Arnold, and E. A. Carlen. "On linear hypocoercive BGK models". In: *From Particle Systems to Partial Differential Equations III*. Springer, 2016, pp. 1–37

Together with Anton Arnold and Eric Carlen, I studied hypocoercivity for a class of linear and linearized BGK models for discrete and continuous phase spaces. We developed methods for constructing entropy functionals that prove exponential rates of relaxation to equilibrium. Our strategies are based on the entropy and spectral methods, adapting Lyapunov's direct method (even for "infinite matrices" appearing for continuous phase spaces) to construct appropriate entropy functionals. Finally, we also proved local asymptotic stability of a nonlinear BGK model.

[**3**]: F. Achleitner, A. Arnold, and E. A. Carlen. "On multi-dimensional hypocoercive BGK models". In: *Kinetic and Related Models* (2018). accepted, arXiv preprint arXiv:1711.07360

We study hypocoercivity for a class of linearized BGK models for continuous phase spaces. We develop methods for constructing entropy functionals that enable us to prove exponential relaxation to equilibrium with explicit and physically meaningful rates. In fact, we not only estimate the exponential rate, but also the second time scale governing the time one must wait before one begins to see the exponential relaxation in the L^1 distance. This waiting time phenomenon, with a long plateau before the exponential decay "kicks in" when starting from initial data that is well-concentrated in phase space, is familiar from work of Aldous and Diaconis on Markov chains, but is new in our continuous phase space setting. Our strategies are based on the entropy and spectral methods, and we introduce a new "index of hypocoercivity" that is relevant to models of our type involving jump processes and not only diffusion. At the heart of our method is a decomposition technique that allows us to adapt Lyapunov's direct method to our continuous phase space setting in order to construct our entropy functionals. These are used to obtain precise information on linearized BGK models. Finally, we also prove local asymptotic stability of a nonlinear BGK model.

[**4**]: F. Achleitner, A. Arnold, and B. Signorello. "On optimal decay estimates for ODEs and PDEs with modal decomposition". accepted, arXiv preprint arXiv:1802.00767

We consider the Goldstein-Taylor model, which is a 2-velocity BGK model, and construct the "optimal" Lyapunov functional to quantify the convergence to the unique normalized steady state. The Lyapunov functional is optimal in the sense that it yields decay estimates in L^2-norm with the sharp exponential decay rate and minimal multiplicative constant. The modal decomposition of the Goldstein-Taylor model leads to the study of a family of 2-dimensional ODE systems. Therefore we discuss the characterization of "optimal" Lyapunov functionals for linear ODE systems with positive stable diagonalizable matrices. We give a complete answer for 2-dimensional ODE systems, and a partial answer for higher dimensional ODE systems.

Perspectives

The focus of my research are nonlinear nonlocal partial integro-differential equations in applied sciences. Building on my expertise, I will continue to study nonlocal Fokker-Planck equations, nonlocal models in hydrodynamics and nonlocal reaction-diffusion equations. In the future, I also want to consider nonlocal models with memory effects, e.g. models with subdiffusion. Moreover, I am interested in numerical simulations of these models by collaborating with experts in the field of numerics.

[10]: F. Achleitner, A. Jüngel, and M. Yamamoto. "Large-time asymptotics of a fractional drift-diffusion-Poisson system via the entropy method". submitted, arXiv preprint arXiv:1802.10272

The self-similar asymptotics for solutions to the drift-diffusion equation with fractional dissipation, coupled to the Poisson equation, is analyzed in the whole space. It is shown that in the subcritical and supercritical cases, the solutions converge to the fractional heat kernel with algebraic rate. The proof is based on the entropy method and leads to a decay rate in the $L^1(\mathbb{R}^d)$ norm. The technique is applied to other semilinear equations with fractional dissipation.

[13]: F. Achleitner and C. Kuehn. "Traveling wave solutions for bistable reaction-diffusion equations with nonlocal diffusion of Lévy type". work in progress

[16]: F. Achleitner and Y. Ueda. "L^1-stability of nonlinear waves in scalar viscous conservation laws with nonlocal diffusion of Lévy type". work in progress

[6]: F. Achleitner and C. M. Cuesta. "Non-classical shocks in a non-local Korteweg-de Vries-Burgers equation with cubic flux function". work in progress

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CHAPTER 1

Viscous conservation laws

ON NONLINEAR CONSERVATION LAWS WITH A NONLOCAL DIFFUSION TERM

FRANZ ACHLEITNER, SABINE HITTMEIR, AND CHRISTIAN SCHMEISER

Abstract. Scalar one-dimensional conservation laws with a nonlocal diffusion term corresponding to a Riesz-Feller differential operator are considered. Solvability results for the Cauchy problem in L^∞ are adapted from the case of a fractional derivative with homogeneous symbol. The main interest of this work is the investigation of smooth shock profiles. In case of a genuinely nonlinear smooth flux function we prove the existence of such travelling waves, which are monotone and satisfy the standard entropy condition. Moreover, the dynamic nonlinear stability of the travelling waves under small perturbations is proven, similarly to the case of the standard diffusive regularization, by constructing a Lyapunov functional.

1. Introduction

We consider one-dimensional conservation laws for a density $u(t,x)$, $t>0$, $x\in\mathbb{R}$, of the form

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u\,, \tag{1}$$

where $\mathcal{D}^\alpha$ is the non-local operator

$$(\mathcal{D}^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha}dy\,, \tag{2}$$

with $0<\alpha<1$. The flux function $f(u)$ is smooth and satisfies $f(0)=0$.

We shall analyse the local and global solvability of the Cauchy problem for (1), as well as the existence and stability of travelling wave solutions. In particular, we shall show that smooth travelling wave solutions exist, which are asymptotically stable. These waves are shock profiles satisfying the standard entropy conditions like those derived from the standard parabolic regularization with $\mathcal{D}^\alpha$ replaced by ∂_x.

Since $\mathcal{D}^\alpha u$ can be written as the convolution of the derivative u' with $\Gamma(1-\alpha)^{-1}\theta(x)x^{-\alpha}$ (with the Heaviside function θ), $\mathcal{D}^\alpha$ is a pseudo-differential operator with symbol

$$\frac{ik\sqrt{2\pi}}{\Gamma(1-\alpha)}\mathcal{F}\left(\frac{\theta(x)}{x^\alpha}\right)(k) = ik\left(a_\alpha - ib_\alpha\, sgn(k)\right)|k|^{\alpha-1} = \left(b_\alpha + ia_\alpha\, sgn(k)\right)|k|^\alpha\,,$$

i.e. $\mathcal{F}(\mathcal{D}^\alpha u)(k) = (b_\alpha + ia_\alpha\, sgn(k))\,|k|^\alpha \widehat{u}(k)$. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}\varphi(k) = \widehat{\varphi}(k) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx}\varphi(x)dx\,,$$

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Key words and phrases. nonlocal evolution equation, fractional derivative, travelling wave.

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and

$$a_\alpha = \sin\left(\frac{\alpha\pi}{2}\right) > 0\,, \qquad b_\alpha = \cos\left(\frac{\alpha\pi}{2}\right) > 0\,,$$

(see [2] for the details of the computation). Obviously, the operator on the right hand side of (1) also is a pseudo-differential operator with symbol

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha) = -\left(a_\alpha - ib_\alpha\, sgn(k)\right)|k|^{\alpha+1}\,. \tag{3}$$

Due to the negativity of its real part, it is dissipative.

Remark 1. *For $s \in \mathbb{R}$, we use the Sobolev space*

$$H^s := \{u :\, \|u\|_{H^s} < \infty\}\,, \qquad \|u\|_{H^s} := \|(1+|k|)^s \widehat{u}\|_{L^2(\mathbb{R})}\,,$$

and the corresponding homogeneous norm

$$\|u\|_{\dot{H}^s} := \||k|^s \widehat{u}\|_{L^2(\mathbb{R})}\,.$$

The fact $\|\mathcal{D}^\alpha u\|_{\dot{H}^s} = \sqrt{a_\alpha^2 + b_\alpha^2}\,\|u\|_{\dot{H}^{s+\alpha}}$ justifies to interpret $\mathcal{D}^\alpha$ as a differentiation operator of order α. It is bounded as a map from H^s to $H^{s-\alpha}$.

Denoting by C_b^m, $m \geq 0$, the set of functions, whose derivatives up to order m are continuous and bounded on $\mathbb{R}$, $\mathcal{D}^\alpha u : C_b^1 \to C_b$ is bounded. This can be easily seen by splitting the domain of integration in (2) into $(-\infty, x-\delta]$ and $[x-\delta, x]$ for some positive $\delta > 0$. Then integration by parts in the first integral shows the boundedness of $\mathcal{D}^\alpha u$.

The operator $\partial_x \mathcal{D}^{1/3}$ occurs in applications. It has been derived as the physically correct viscosity term in two layer shallow water flows by performing formal asymptotic expansions associated to the triple-deck regularization used in fluid mechanics (see, e.g., [15]). Moreover $\mathcal{D}^{1/3}$ appears in the work of Fowler [10] in an equation for dune formation:

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}^{1/3} u\,. \tag{4}$$

Here the fractional derivative appears with the negative sign, but this instability is regularized by the second order derivative. Alibaud et al. showed the well-posedness of (4) in L^2 as well as the violation of the maximum principle, which is intuitive in the context of the application due to underlying erosions [1]. Travelling wave solutions of (4) have been analysed by Alvarez-Samaniego and Azerad in [2].

Fractal conservation laws of the form

$$\partial_t u + \partial_x f(u) = D^{\alpha+1} u\,, \tag{5}$$

where $D^{\alpha+1}$ is the pseudo-differential operator with symbol $-|k|^{\alpha+1}$ (meaning $D^{\alpha+1}u = \mathcal{F}^{-1}(-|k|^{\alpha+1}\hat{u})$) have been investigated in several works, see e.g. Biler et al. [4] and Droniou et al. [8].

This work is organized as follows. In the remainder of this section we present an existence result for the Cauchy problem in L^∞. The crucial property here is the nonnegativity of the semigroup generated by $\partial_x \mathcal{D}^\alpha$, which is a consequence of its interpretation as a Riesz-Feller derivative [9, 11]. This allows to prove a maximum principle for solutions of (1) as in [8].

Section 2 is devoted to the analysis of travelling wave solutions connecting different far-field values. Such wave profiles are typically smooth. Working with the original representation (2) of $\mathcal{D}^\alpha$, we obtain a nonlinear Volterra integral equation as the travelling wave version of (1). Assuming (even a

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bit less than) convexity of the flux function and that the solutions of the associated linear Volterra integral equation form a one-dimensional subspace of $H^2(\mathbb{R}_-)$, we can show the existence and uniqueness of monotone solutions satisfying the entropy condition for classical shock waves of the inviscid conservation law underlying (1). This essentially requires to extend the well known results for the existence of viscous shock profiles, which solve (local) ordinary differential equations.

Biler et al. [4] showed that no travelling wave solutions of (5) can exist for $\alpha \in (-1, 0]$. For the case $\alpha \in (0, 1)$ also no existence result is available.

To show the asymptotic stability of the travelling waves, we use the antiderivative method typically applied in the case of the classical viscous regularisation and derive a Lyapunov functional. This allows to deduce the decay of initially small perturbations.

In the appendix we consider linear Volterra integral equations and prove the assumption on the dimension of the solution space with respect to subspaces of $H^2(\mathbb{R}_-)$.

The Cauchy Problem. In the following, we verify the applicability of the work of Droniou et al. [8] on the Cauchy problem of (5) in L^∞ to

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u, \quad u(0,x) = u_0(x). \tag{6}$$

Applying the Fourier transform to the linear evolution equation $\partial_t u = \partial_x \mathcal{D}^\alpha u$, we see that the semigroup generated by the fractional derivative is formally given by the convolution with the kernel

$$K(t,x) = \mathcal{F}^{-1} e^{-\Lambda(k)t}(x), \quad \text{where } \Lambda(k) = (a_\alpha - i b_\alpha sgn(k))|k|^{\alpha+1}. \tag{7}$$

To analyse the well-posedness, we use the mild formulation of (6),

$$u(t,x) = K(t,.) * u_0(x) - \int_0^t K(t-\tau,.) * \partial_x f(u(\tau,.))(x) d\tau. \tag{8}$$

As a main ingredient in [8], Droniou et al. used the non-negativity of the kernel associated to the semigroup generated by $D^{\alpha+1}$. To make use of their methods in the analysis of the Cauchy problem (6), we need to investigate the properties of the kernel K associated to the operator $\partial_x \mathcal{D}^\alpha$.

Lemma 1. *For $0 < \alpha < 1$, the kernel K given by (7) is non-negative:*

$$K(t,x) \geq 0, \qquad \text{for all } \ t > 0, x \in \mathbb{R}.$$

Additionally, the kernel K satisfies the properties:

(i) *For all $t > 0$ and $x \in \mathbb{R}$, $K(t,x) = \frac{1}{t^{1/(1+\alpha)}} K\left(1, \frac{x}{t^{1/(1+\alpha)}}\right)$.*

(ii) *For all $t > 0$, $\|K(t,.)\|_{L^1(\mathbb{R})} = 1$.*

(iii) *$K(t,x)$ is C^∞ on $(0,\infty) \times \mathbb{R}$ and for all $m \geq 0$ there exists a B_m such that*

$$\forall (t,x) \in (0,\infty) \times \mathbb{R}, \qquad |\partial_x^m K(t,x)| \leq \frac{1}{t^{(1+m)/(1+\alpha)}} \frac{B_m}{(1 + t^{-2/(1+\alpha)}|x|^2)}. \tag{9}$$

(iv) *There exists a C_0 such that for all $t > 0$: $\|\partial_x K(t,.)\|_{L^1(\mathbb{R})} = \frac{C_0}{t^{1/(1+\alpha)}}$.*

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Proof. We already mentioned that the operator $\partial_x \mathcal{D}^\alpha$ is a Riesz-Feller differential operator, see also Gorenflo and Mainardi [11]. Due to Feller [9], the symbol of $\partial_x \mathcal{D}^\alpha$ is the characteristic exponent of a random variable with Lévy stable distribution. Hence the kernel K is the scaled probability density function of a Lévy stable distribution and is non-negative.

The additional properties of the kernel K are verified in the same manner as in [8]: (i) follows from the change of variable $\eta = t^{1/(1+\alpha)}k$ under the integral sign. Since the kernel K is non-negative, we deduce $\|K(1,.)\|_{L^1(\mathbb{R})} = \int K(1,x)dx = \mathcal{F}(K(1,.))(0) = 1$, which together with (i) implies (ii). To show (iii), we write $\partial_x^m K(1,x) = \frac{1}{\sqrt{2\pi}} \int (ik)^m e^{ikx} e^{-\Lambda(k)} dk$. Since $\alpha > 0$, we can integrate by parts twice and obtain $\partial_x^m K(1,x) = O(1/x^2)$. Together with the boundedness of $\partial_x^m K(1,x)$, we get the estimate for $t = 1$ and deduce the estimate for arbitrary $t > 0$ from (i). Finally, (iv) follows from (i) and (iii). $\square$

Hence the kernel associated to $\partial_x \mathcal{D}^\alpha$ satisfies the same properties as the one for $D^{\alpha+1}$ required in the work of Droniou et al. [8]. Thus their analysis carries over to our problem and we obtain the analogous result:

Theorem 1. *If $u_0 \in L^\infty$, then there exists a unique solution $u \in L^\infty((0,\infty)\times\mathbb{R})$ of (6) satisfying the mild formulation (8) almost everywhere. In particular*

$$\|u(t,.)\|_\infty \le \|u_0\|_\infty, \qquad \text{for } t > 0.$$

Moreover, the solution has the following properties:

(1) *$u \in C^\infty((0,\infty)\times\mathbb{R})$ and $u \in C_b^\infty((t_0,\infty)\times\mathbb{R})$ for all $t_0 > 0$.*
(2) *u satisfies equation (1) in the classical sense.*
(3) *$u(t) \to u_0$, as $t \to 0$, in $L^\infty(\mathbb{R})$ weak-$*$ and in $L^p_{loc}(\mathbb{R})$ for all $p \in [1,\infty)$.*

To motivate the well-posedness, we estimate the terms in (8) for $t > 0$, with the help of the properties of the kernel K, as follows: $|K(t,.)*u_0(x)| \le \|u_0\|_\infty$ and

$$\left| \int_0^t \partial_x K(t-s,.) * f(u(s,.))ds \right| \le C \|f(u)\|_{L^\infty((0,t)\times\mathbb{R})} t^{1-\frac{1}{1+\alpha}} .$$

Due to the Lipschitz continuity of f, we get a contraction for small times t_0 on $L^\infty((0,t_0)\times\mathbb{R})$ and therefore the well-posedness.

To show the global existence as well as the maximum principle, Droniou et al. [8] constructed an approximate solution by a splitting method and used a compactness argument to pass to the limit.

We shall also mention that an alternative L^2-based existence theory of (1) can be obtained by standard approaches such as contraction arguments and Lyapunov functionals. Here the main ingredient is the a priori decay of the L^2-norm. Testing (1) with u and assuming vanishing far-field values of u, the flux term vanishes

$$\int_{\mathbb{R}} u\partial_x f(u)dx = \int_{\mathbb{R}} uf'(u)\partial_x u dx = \int_{\mathbb{R}} \partial_x G(u)dx = 0, \quad G(u) = \int_0^u vf'(v)dv,$$

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since G is smooth and $G(0) = 0$. We obtain the L^2-estimate:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 dx = -a_\alpha \int_{\mathbb{R}} |k|^{1+\alpha}|\hat{u}|^2 dk \le 0\,.$$

Here we have used Plancherel's theorem together with $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$, implying

$$\int_{\mathbb{R}} \mathrm{sgn} k |k|^j |\hat{u}(k,t)|^2 dk = 0\,.$$

This relation shows that in an L^2-framework the operator $\partial_x \mathcal{D}^\alpha$ behaves similarly to $D^{\alpha+1}$. Due to the decay of the L^2-norm of the solution to (1), one would hope for well-posedness of the Cauchy problem with initial data in L^2 allowing us to deduce the global existence. Using a contraction argument similar to the one by Dix [6] for the classical viscous Burgers equation, we can show the well-posedness in L^2 for the quadratic flux $f(u) = u^2$ in the case $\alpha > 1/2$. This critical value was already mentioned by Biler, Funaki and Woyczynski [4] for (5). For the general flux and $\alpha \in (0,1)$ we have to require higher regularity of the initial data: $u_0 \in H^1$. To deduce global existence of solutions in H^1, a Lyapunov functional can be derived under an additional smallness assumption on $\|u_0\|_{H^1}$. These results follow from the proofs we carry out in Section 2.2. Since obviously the assumptions on the initial data are much more restrictive as in the L^∞-based existence result, we do not go into more details here.

2. Travelling wave solutions

2.1. **Existence of travelling wave solutions.** We introduce the travelling wave variable $\xi = x - st$ with the wave speed s and look for solutions $u(x,t) = u(\xi)$ of (1), which are connecting the different far-field values u_- and u_+. A straightforward calculation shows that if u depends on x and t only through the travelling wave variable ξ, then so does $\mathcal{D}^\alpha u$, and we arrive at

$$-su' + f(u)' = (\mathcal{D}^\alpha u)'\,, \qquad u(-\infty) = u_-\,, \quad u(\infty) = u_+\,,$$

where the prime denotes differentiation with respect to ξ. Integration gives the travelling wave equation

$$h(u) := -s(u - u_-) + f(u) - f(u_-) = \mathcal{D}^\alpha u = d_\alpha \int_0^\infty \frac{u'(\xi - y)}{y^\alpha} dy\,, \tag{10}$$

with $d_\alpha = 1/\Gamma(1-\alpha)$. If the derivative u' decays to zero fast enough as $\xi \to \pm\infty$, then we obtain, at least formally, the Rankine-Hugoniot conditions, which correspond to shock solutions of the inviscid conservation law and relate the far-field values and the wave speed via

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}\,. \tag{11}$$

If the flux function $f(u)$ is convex between the far-field values u_- and u_+, then the left hand side $h(u)$ of (10) is negative between its zeroes u_- and u_+. If $u(\xi)$ is monotone, the right hand side in (10) has the same sign as u'. Therefore if a monotone solution exists, it has to be nonincreasing, leading to the standard entropy condition

$$u_- > u_+\,,$$

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derived by replacing $\mathcal{D}^\alpha u$ by u'. Under this assumption, the existence of a smooth monotone travelling wave will be proved. The precise assumptions on the flux function will be formulated in terms of $h(u)$: We require

$$\begin{aligned}(12)\quad & h \in C^\infty([u_+, u_-])\,, \quad h(u_+) = h(u_-) = 0\,, \quad h < 0 \text{ in } (u_+, u_-)\,, \\ & \exists u_m \in (u_+, u_-) \text{ such that } \quad h' < 0 \text{ in } (u_+, u_m)\,, \quad h' > 0 \text{ in } (u_m, u_-]\,.\end{aligned}$$

Note that this is a little less than asking for convexity of f, and it allows for the slightly weakened form $f'(u_+) \le s < f'(u_-)$ of the Lax entropy condition.

The integral operator

$$\mathcal{D}^\alpha u(\xi) = d_\alpha \int_{-\infty}^{\xi} \frac{u'(y)}{(\xi - y)^\alpha} dy$$

in the travelling wave problem

$$(13) \qquad h(u) = \mathcal{D}^\alpha u\,, \qquad u(-\infty) = u_-\,, \quad u(\infty) = u_+\,,$$

is of the Abel type. It is well known that it can be inverted by multiplying (13) with $(z - \xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z. This leads to

$$(14) \qquad u(\xi) - u_- = \mathcal{D}^{-\alpha}(h(u))(\xi) := d_{1-\alpha} \int_{-\infty}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} dy\,.$$

Equations (13) and (14) are equivalent if $u \in C_b^1(\mathbb{R})$ and $u' \in L^1(\mathbb{R}_-)$, hence in particular if $u \in C_b^1(\mathbb{R})$ is monotone. We will use both formulations to deduce the existence result. An important property of both integral equations is their translation invariance, which will be used several times below.

The equation (14) is a nonlinear Volterra integral equation with a locally integrable kernel, where a well developed theory exists for problems on bounded intervals. Therefore we shall start our investigations by proving a 'local' existence result around $\xi = -\infty$. The subsequent monotonicity and boundedness results will lead to global existence for $\xi \in \mathbb{R}$.

The local existence result is based on linearisation at $\xi = -\infty$ (or, equivalently, at $u = u_-$). This can be done for either (13) or (14) with the same result. As could be expected for ordinary differential equations, the linearisations

$$(15) \qquad h'(u_-)v = \mathcal{D}^\alpha v\,, \qquad v = h'(u_-)\mathcal{D}^{-\alpha} v\,,$$

have solutions of the form $v(\xi) = be^{\lambda\xi}$, $b \in \mathbb{R}$, where a straightforward computation gives $\lambda = h'(u_-)^{1/\alpha}$, see also [5]. We will need that these are the only non-trivial solutions of (15) in the space $H^2(-\infty, \xi_0]$ for some $\xi_0 \le 0$. In particular, we assume that

$$(16) \quad \mathcal{N}\big(id - h'(u_-)\mathcal{D}^{-\alpha}\big) = \operatorname{span}\{\exp(\lambda\xi)\} \qquad \text{with} \qquad \lambda = h'(u_-)^{1/\alpha}\,,$$

which is reasonable due to our analysis in the appendix A. The main result of this section is the following.

Theorem 2. *Let* (12) *and* (16) *hold. Then there exists a decreasing solution* $u \in C_b^1(\mathbb{R})$ *of the travelling wave problem* (13)*. It is unique (up to a shift) among all* $u \in u_- + H^2((-\infty, 0)) \cap C_b^1(\mathbb{R})$.

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The following local existence result shows that the nonlinear problem has, up to translations, only two nontrivial solutions, which can be approximated by $u_- \pm e^{\lambda\xi}$ for large negative ξ. The choice 1 of the modulus of the coefficient of the exponential is irrelevant due to the translation invariance of the solution.

Lemma 2. *(Local existence) Let* (16) *hold. Then, for every small enough* $\varepsilon > 0$*, the equation* (13) *has solutions* $u_{up}, u_{down} \in u_- + H^2(I_\varepsilon)$*,* $I_\varepsilon = (-\infty, \xi_\varepsilon]$*,* $\xi_\varepsilon = \log \varepsilon/\lambda$*, satisfying*

$$u_{up}(\xi_\varepsilon) = u_- + \varepsilon \,, \qquad u_{down}(\xi_\varepsilon) = u_- - \varepsilon \,. \tag{17}$$

These are unique among all functions u satisfying $\|u - u_-\|_{H^2(I_\varepsilon)} \le \delta$, with δ small enough, but independently from ε. They satisfy (with an ε-independent constant C)

$$\|u_{up} - u_- - e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \le C\varepsilon^2 \,, \qquad \|u_{down} - u_- + e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \le C\varepsilon^2 \,.$$

Proof. The proof will only be given for existence and uniqueness of u_{down}, which will be of greater interest below, but the proof for u_{up} is analogous.

We start by writing (13) and the initial condition (17) in terms of the perturbation $\bar{u}(\xi) = u_{down}(\xi) - u_- + e^{\lambda\xi}$:

$$(\mathcal{D}^\alpha - h'(u_-))\bar{u} = F(\bar{u}, \xi) \,, \quad \bar{u}(\xi_\varepsilon) = 0 \,, \tag{18}$$

where we denote

$$F(\bar{u}, \xi) = h(u_- - e^{\lambda\xi} + \bar{u}) + h'(u_-)(e^{\lambda\xi} - \bar{u}).$$

The idea is to write (18) as a fixed point problem considering the right hand side as given. Since we shall use the Fourier transform for constructing a particular solution, we need a smooth enough extension to $\xi \in \mathbb{R}$, although we are only interested in $\xi < \xi_\varepsilon$. For $f \in H^2(I_\varepsilon)$, let the extension $\mathcal{E}(f) \in H^2(\mathbb{R})$ satisfy

$$\mathcal{E}(f)\Big|_{I_\varepsilon} = f \,, \quad \|\mathcal{E}(f)\|_{H^2(\mathbb{R})} \le \gamma \|f\|_{H^2(I_\varepsilon)} \,.$$

The bounded solution of the equation

$$(\mathcal{D}^\alpha - h'(u_-))u_{part} = \mathcal{E}(f) \,,$$

and of its derivatives with respect to ξ can be written as

$$u_{part}^{(m)} = \mathcal{F}^{-1}\left[\left(b_\alpha |k|^\alpha - h'(u_-) + i a_\alpha \mathrm{sgn}(k)|k|^\alpha\right)^{-1} \mathcal{F}\mathcal{E}(f)^{(m)}\right] \,, \quad m = 0, 1, 2 \,.$$

The coefficient can easily be seen to be bounded uniformly in k, leading to the estimate

$$\|u_{part}\|_{H^2(I_\varepsilon)} \le \|u_{part}\|_{H^2(\mathbb{R})} \le C\|\mathcal{E}(f)\|_{H^2(\mathbb{R})} \le C\gamma\|f\|_{H^2(I_\varepsilon)} \,.$$

By the assumption (16), $U[f](\xi) = u_{part}(\xi) - u_{part}(\xi_\varepsilon)e^{\lambda(\xi - \xi_\varepsilon)}$ is the unique solution of

$$(\mathcal{D}^\alpha - h'(u_-))U = f \quad \text{in } I_\varepsilon \,, \qquad U(\xi_\varepsilon) = 0 \,,$$

satisfying by the Sobolev imbedding of $H^2(I_\varepsilon)$ in $C_b(I_\varepsilon)$ the estimate

$$\begin{aligned} \|U[f]\|_{H^2(I_\varepsilon)} &\le \|u_{part}\|_{H^2(I_\varepsilon)} + \|u_{part}\|_{L^\infty(I_\varepsilon)} \|e^{\lambda(\xi-\xi_\varepsilon)}\|_{H^2(I_\varepsilon)} \\ &\le C\gamma\|f\|_{H^2(I_\varepsilon)} + C\|u_{part}\|_{H^2(I_\varepsilon)} \le K\|f\|_{H^2(I_\varepsilon)} \end{aligned}$$

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for some $K > 0$. This allows to write (18) as a fixed point problem:

$$\bar{u} = U\left[F(\bar{u}, \xi)\right] .$$

In order to estimate $F(\bar{u}, \xi)$, we first rewrite it as follows:

$$F(\bar{u}, \xi) = \frac{h''(\tilde{u})}{2}\left(e^{\lambda\xi} - \bar{u}\right)^2 = \frac{h''(\tilde{u})}{2}\left(\varepsilon^2 e^{2\lambda(\xi-\xi_\varepsilon)} - 2\varepsilon e^{\lambda(\xi-\xi_\varepsilon)}\bar{u} + \bar{u}^2\right) .$$

We recall that f is smooth and hence $\|h''(u)\|_{L^\infty} \le L_1(\|u\|_{L^\infty})$ for some positive nondecreasing function L_1. Using moreover the continuous imbedding of $H^2(I_\varepsilon)$ in $C_b(I_\varepsilon)$, it can easily be seen that

$$\begin{aligned} \|F(\bar{u}, \xi)\|_{H^2(I_\varepsilon)} &\le C\|h''(\tilde{u})\|_{L^\infty(I_\varepsilon)}\left(\varepsilon^2 + \varepsilon\|\bar{u}\|_{H^2(I_\varepsilon)} + \|\bar{u}\|_{L^\infty(I_\varepsilon)}\|\bar{u}\|_{H^2(I_\varepsilon)}\right) \\ &\le L(\|\bar{u}\|_{H^2(I_\varepsilon)})\left(\varepsilon^2 + \varepsilon\|\bar{u}\|_{H^2(I_\varepsilon)} + \|\bar{u}\|^2_{H^2(I_\varepsilon)}\right) , \end{aligned}$$

where L is a positive nondecreasing function. The fixed point map is now bounded by

$$\|U[F(\bar{u}, \xi)]\|_{H^2(I_\varepsilon)} \le KL(\|\bar{u}\|_{H^2(I_\varepsilon)})\left(\varepsilon^2 + \varepsilon\|\bar{u}\|_{H^2(I_\varepsilon)} + \|\bar{u}\|^2_{H^2(I_\varepsilon)}\right) .$$

We assume for simplicity that $R = KL(1) > 1$. It is easily seen that the fixed point map is a contraction on the ball with radius $(2R)^{-1}$, which is independent of ε. Moreover the ball with radius $\varepsilon^2 2R$ is mapped into itself. Hence we conclude that there exists a solution $\bar{u}$ bounded in $H^2(I_\varepsilon)$ by a constant of $O(\varepsilon^2)$, which is unique in a ball with a radius of $O(1)$. □

Lemma 3. *(Local monotonicity) Let the assumptions of Lemma 2 hold. Then, in I_ε,*

$$u_{up} > u_- , \quad u'_{up} > 0 , \qquad u_{down} < u_- , \quad u'_{down} < 0 .$$

Proof. Again we restrict our attention to u_{down} and skip the analogous proof for u_{up}. As a consequence of Lemma 2 and of Sobolev imbedding

$$|u_{down}(\xi) - u_- + e^{\lambda\xi}| \le C\varepsilon^2 , \quad \xi \le \xi_\varepsilon .$$

Thus, there exists ξ^* satisfying

$$u_{down}(\xi^*) = u_- - 2C\varepsilon^2 , \qquad \xi_{C\varepsilon^2} \le \xi^* \le \xi_{3C\varepsilon^2} .$$

Since $u_{down}(\xi) < u_-$ for $\xi \ge \xi^*$, we may restrict our attention in the following to $\xi \le \xi^*$. Thus, we eliminated a subinterval of length $d_1 \ge \xi_\varepsilon - \xi_{3C\varepsilon^2}$. Now we set $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 2C\varepsilon_1^2$, and, by a shift in ξ, replace ξ^* by ξ_{ε_2}. This means that the shifted solution becomes the unique u_{down} from Lemma 2, where ε_1 has been replaced by ε_2. Of course, the argument can be iterated to produce a sequence $\{\varepsilon_n\}$, determined by $\varepsilon_{n+1} = 2C\varepsilon_n^2$, and in each step a subinterval of length $d_n \ge \xi_{\varepsilon_n} - \xi_{3C\varepsilon_n^2}$ can be eliminated, where $u_{down} < u_-$ holds. It is easily seen that, for $\varepsilon_1 = \varepsilon$ small enough, $\sum_{n=1}^\infty d_n = \infty$ completing the proof of $u_{down} < u_-$ in I_ε.

The proof of the second property of u_{down} is completely analogous noting that, again by Sobolev imbedding,

$$|u'_{down}(\xi) + \lambda e^{\lambda\xi}| \le C\varepsilon^2 \quad \text{for } \xi \le \xi_\varepsilon .$$

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Remark 2. *Together with $u_{up} - u_-, u_{down} - u_- \in L^2(I_\varepsilon)$, the result of the lemma implies*

$$\lim_{\xi\to-\infty} u_{up}(\xi) = \lim_{\xi\to-\infty} u_{down}(\xi) = u_- \,.$$

Together the two solutions constitute the 'unstable manifold' of the point u_-.

The Lemmata 2 and 3 show the existence of a solution u of (13), which satisfies $u \in C_b^1$ and is monotone. Thus u is also a solution of equation (14).

Lemma 4. *(Continuation principle) Let $u \in C_b^1((-\infty,\xi_0])$ be a (continuation of a) solution of (14) as constructed in Lemma 2. Then there exists a $\delta > 0$, such that it can be extended uniquely to $C_b^1((-\infty,\xi_0+\delta))$.*

Proof. Defining

$$f(\xi) = u_- + d_{1-\alpha}\int_{-\infty}^{\xi_0} \frac{h(u(y))}{(\xi-y)^{1-\alpha}}dy\,,$$

which can be considered as given and smooth by the assumptions, (14) can be written as

$$u(\xi) = f(\xi) + d_{1-\alpha}\int_{\xi_0}^{\xi} \frac{h(u(y))}{(\xi-y)^{1-\alpha}}dy\,.$$

Local existence of a smooth solution for ξ close to ξ_0 is a standard result for Volterra integral equations, see e.g. Linz [14]. $\square$

It is now obvious that, as for ordinary differential equations, boundedness will be enough for global existence.

Lemma 5. *(Global uniqueness) Let $u \in u_- + H^2((-\infty,\xi_0))$ be a solution of (14). Then, up to a shift in ξ, it is the continuation of u_{up} or of u_{down}, or $u \equiv u_-$.*

Proof. For every $\delta > 0$ there exists a $\xi^* \le \xi_0$, such that $\|u-u_-\|_{H^2((-\infty,\xi^*))} < \delta$, and therefore, by Sobolev imbedding, also $|u(\xi^*) - u_-| < \delta$. Choosing δ small enough, there are only the options $u(\xi^*) = u_-$ (implying $u \equiv u_-$) or $u(\xi^*) \ne u_-$ whence, by local uniqueness, u is up to a shift either equal to u_{up} or to u_{down}, depending on the sign of $u(\xi^*) - u_-$. $\square$

This result already implies the uniqueness of the travelling wave, if it exists.

Lemma 6. *(Global monotonicity) Let $u \in C_b^1(-\infty,\xi_0]$ be (a continuation of) the solution u_{down} of (14) as constructed in Lemma 2. Then u is non-increasing.*

Proof. We recall the properties of h given in (12). We shall use both formulations (13) and (14). First we prove that the derivative of u remains negative as long as $u \ge u_m$. Assume to the contrary that

$$u(\xi_*) \ge u_m\,, \quad u'(\xi_*) = 0\,, \quad u' < 0 \text{ in } (-\infty,\xi_*)\,.$$

Then we obtain from the derivative of (14), evaluated at $\xi = \xi_*$, the contradiction

$$0 = u'(\xi_*) = d_{1-\alpha}\int_{-\infty}^{\xi_*} \frac{h'(u(y))u'(y)}{(\xi_*-y)^{1-\alpha}}dy < 0\,.$$

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Now we show that u cannot become increasing for $u < u_m$. Again, assume the contrary

$$u(\xi_*) < u_m\,, \quad u' > 0 \text{ in } (\xi_*, \xi_* + \delta)\,, \quad u' \le 0 \text{ in } (-\infty, \xi_*]\,,$$

where we assume additionally that δ is small enough such that $u(\xi_* + \delta) < u_m$. This implies

$$\begin{aligned}\int_{-\infty}^{\xi_*+\delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy &= \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy + \int_{\xi_*}^{\xi_*+\delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy \\ &> \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy\,.\end{aligned}$$

But on the other hand we know

$$\begin{aligned} 0 &> h(u(\xi_* + \delta)) - h(u(\xi_*)) \\ &= d_\alpha \int_{-\infty}^{\xi_*+\delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy - d_\alpha \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy \; > \; 0\,,\end{aligned}$$

leading again to a contradiction. Therefore u' cannot get positive. □

Lemma 7. *(Boundedness) Let $u \in C_b^1(-\infty, \xi_0]$ be (a continuation of) the solution u_{down} of* (14) *as constructed in Lemma 2. Then $u_+ < u < u_-$.*

Proof. Suppose the solution would reach the value u_+ in finite time, i.e. there exists a ξ_*, such that $u(\xi_*) = u_+$. Since u is nonincreasing and, by Lemma 3, strictly decreasing at least close to $\xi = -\infty$, we obtain the contradiction

$$0 = h(u_+) = d_\alpha \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy < 0\,.$$

□

The proof of Theorem 2 is completed by proving $\lim_{\xi\to\infty} u(\xi) = u_+$. Assuming to the contrary $\lim_{\xi\to\infty} u(\xi) > u_+$, would imply $\lim_{\xi\to\infty} h(u(\xi)) < 0$. Then, however, $-\mathcal{D}^{-\alpha} h(u) = u_- - u$ would increase above all bounds, which is impossible by Lemma 7.

2.2. **Asymptotic stability of travelling waves for convex fluxes.** We change to the moving coordinate $\xi = x - st$ in (1),

$$\partial_t u + \partial_\xi (f(u) - su) = \partial_\xi \mathcal{D}^\alpha u\,, \tag{19}$$

and look for solutions of (19), which are small perturbations of travelling wave solutions and in particular share the same far-field values. Let $u_0(\xi)$ be an initial datum and $\phi(\xi)$ a travelling wave solution as constructed in the previous section, with the shift chosen such that

$$\int_{\mathbb{R}} (u_0(\xi) - \phi(\xi)) d\xi = 0\,. \tag{20}$$

Due to the conservation property of the equation (19) we see that (formally)

$$\int_{\mathbb{R}} (u(t,\xi) - \phi(\xi)) d\xi = 0\,, \qquad \text{for all } t \ge 0\,.$$

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The flux function will be assumed to be convex between the far-field values of the travelling wave, i.e.

$$f''(\phi(\xi)) \geq 0\,, \qquad \text{for all } \xi \in \mathbb{R}\,.$$

The perturbation $U = u - \phi$ satisfies the equation

$$\partial_t U + \partial_\xi((f'(\phi) - s)U) + \frac{1}{2}\partial_\xi\left(f''(\phi + \vartheta U)U^2\right) = \partial_\xi \mathcal{D}^\alpha U\,, \tag{21}$$

for some $\vartheta \in (0,1)$. The aim is to show local stability of travelling waves, i.e. the decay of U for small initial perturbations $U_0 = u_0 - \phi$. Testing (21) with U, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} f''(\phi)\phi' U^2 d\xi - \frac{1}{2}\int_{\mathbb{R}} f''(\phi + \vartheta U)U^2 \partial_\xi U\, d\xi \\ = -a_\alpha \|U\|^2_{\dot{H}^{(1+\alpha)/2}}\,,\end{aligned} \tag{22}$$

where several integrations by parts have been carried out. Recalling $\phi' \leq 0$, we see that the second term has the unfavourable sign. As one would do for the conservation law with the classical viscous regularisation, we introduce the primitive of the perturbation:

$$W(t,\xi) = \int_{-\infty}^{\xi} U(t,\eta)d\eta\,, \qquad W_0(\xi) = \int_{-\infty}^{\xi} U_0(\eta)d\eta\,.$$

Integration of (21) gives the equation for W,

$$\partial_t W + (f'(\phi) - s)\partial_\xi W + \frac{1}{2}f''(\phi + \vartheta U)(\partial_\xi W)^2 = \partial_\xi \mathcal{D}^\alpha W\,, \tag{23}$$

which we test with W to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} f''(\phi)\phi' W^2 d\xi + \frac{1}{2}\int_{\mathbb{R}} f''(\phi + \vartheta U)(\partial_\xi W)^2 W\, d\xi \\ = -a_\alpha \|W\|^2_{\dot{H}^{(1+\alpha)/2}}\,.\end{aligned} \tag{24}$$

This equation has the crucial property that the quadratic terms have the favour-able sign. From the cubic term (arising from the nonlinearity) we pull out the L^∞-norm of W (and of U if f'' is not constant), which we shall control by Sobolev imbedding.

Well-posedness of the perturbation equation. Before deriving decay estimates, we have to guarantee the well-posedness of the Cauchy problem for (23),

$$\partial_t W + (f'(\phi) - s)\partial_\xi W + \frac{1}{2}f''(\phi + \vartheta U)(\partial_\xi W)^2 = \partial_\xi \mathcal{D}^\alpha W\,, \quad W(0,x) = W_0(x). \tag{25}$$

Therefore we use a contraction argument. Assuming $f(u) = u^2$ and $\alpha > 1/2$ allows to estimate the nonlinearity in the fashion of Dix [6] implying the well-posedness in H^1. For the general flux and $\alpha \in (0,1)$ we have to require more regularity of the initial data, $W_0 \in H^2$.

We recall the definition (7) of the kernel K associated to the linear evolution equation and rewrite (25) in its mild formulation

$$\begin{aligned}W(t,x) &= K(t,.) * W_0(x) \\ &\int_0^t K(t-\tau,.) * \left((f'(\phi) - s)U(\tau,.) + \frac{f''(\phi + \vartheta U))}{2}(U(\tau,.))^2\right)(x)d\tau.\end{aligned} \tag{26}$$

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Before proceeding with the contraction arguments, we note that for any $W_0 \in H^s$ we have $K(t,.) * W_0 \to W_0$ as $t \to 0$ in H^s. In particular, the integral

$$\|K(t,.) * W_0 - W_0\|_{H^s}^2 = \int (1+|k|)^{2s} |e^{-\Lambda(k)t} - 1|^2 |\widehat{W}_0(k)|^2 dk$$

is bounded by $4\|W_0\|_{H^s}^2$ and we can apply the Dominated Convergence Theorem to pass to the limit under the integral sign. Moreover $\|K(t,.)*W_0\|_{H^s} \leq \|W_0\|_{H^s}$.

Proposition 1. *Let $f(u) = u^2$ and $\alpha > \frac{1}{2}$. Then for any $W_0 \in H^1$ there exists a $T > 0$ such that (25) has a unique solution $W \in H^1$ for $t \in [0,T)$.*

Proof. Denoting the right hand side of (26) with $\mathcal{G}W$ the mild formulation gives a fixed point problem $W = \mathcal{G}W$. We note that $f'' = 2$ and briefly explain how to carry out the contraction argument. Let $T > 0$ and denote $\|W\|_{H^s}^* = \sup_{t\in[0,t_0]} \|W\|_{H^s}$. Applying Plancherel's Theorem we can bound the H^1 norm of $\mathcal{G}W$ by

$$\begin{aligned}
\|\mathcal{G}W\|_{H^1}^* &\leq \|W_0\|_{H^1} + \int_0^T \left\| (1+|k|)e^{-\Lambda(k)(t-\tau)} \mathcal{F}((2\phi - s)U + U^2)(\tau,k) \right\|_{L^2} d\tau \\
&\leq \|W_0\|_{H^1} + C\int_0^T \sup_{k\in\mathbb{R}} \left| (1+|k|)e^{-\Lambda(k)(t-\tau)} \right| \|U(\tau,.)\|_{L^2} d\tau \\
&\qquad + \int_0^T \left\| (1+|k|)e^{-\Lambda(k)(t-\tau)} \right\|_{L^2} \sup_{k\in\mathbb{R}} |(U(\tau,.)^2)\hat{}\,| d\tau
\end{aligned}$$

Using Cauchy-Schwarz inequality it is easy to see that $\|(gh)\hat{}\,\|_\infty \leq \|g\|_{L^2}\|h\|_{L^2}$, hence $\sup_{k\in\mathbb{R}} |(U(\tau,.)^2)\hat{}\,| \leq \|U\|_{L^2}^{*\,2}$. We then bound

$$\begin{aligned}
\sup_{k\in\mathbb{R}} \left| (1+|k|)e^{-\Lambda(k)(T-\tau)} \right| &\leq 1 + \frac{\left\| y e^{-a_\alpha |y|^{\alpha+1}} \right\|_\infty}{(T-\tau)^{\frac{1}{1+\alpha}}} \\
&\leq C\left(1 + (T-\tau)^{-\frac{1}{1+\alpha}}\right), \qquad (27)\\
\|(1+|k|)e^{-\Lambda(k)(T-\tau)}\|_{L^2} &\leq C\left((T-\tau)^{-\frac{1}{2(1+\alpha)}} + (T-\tau)^{-\frac{3}{2(1+\alpha)}}\right),
\end{aligned}$$

where we have performed the substitution $k \mapsto k(t-\tau)^{\frac{1}{\alpha+1}}$ in the integrand. For $\alpha > 1/2$, the terms on the right hand side are integrable from 0 to T and the operator $\mathcal{G}$ is a contraction for small times T: There exists a constant $C_0 > 0$, such that

$$\|\mathcal{G}W\|_{H^1}^* \leq C_0\left(1 + (T + T^{1-\frac{1}{1+\alpha}})\|W\|_{H^1}^* + (T^{1-\frac{1}{2(1+\alpha)}} + T^{1-\frac{3}{2(\alpha+1)}})\|W\|_{H^1}^{*\,2}\right),$$

Then for T small enough, $\mathcal{G}$ maps the ball $B_{2C_0}(T) = \{W \in C([0,T],H^1) : \|W\|_{H^1}^* \leq 2C_0\}$ into itself. With Banach's fixed point argument we can conclude the existence of a solution $W \in B_{2C_0}(T)$ of (26), which is therefore the solution of (25) on $[0,T)$. The uniqueness result is only local in B_{2C_0}. Hence let us now assume $W, V \in C([0,T],H^1)$ are two solutions of (26) and let $M = \max\{\|W\|_{H^1}^*, \|V\|_{H^1}^*\}$. Then $W - V$ solves a fixed point equation,

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where for a small enough $T_0 > 0$ the fixed point operator is again a contraction on $B_{2M}(T_0)$. Therefore $W = V$ on $[0, T_0]$. Repetition of this argument provides uniqueness on the whole time interval of existence. □

Proposition 2. *Let $W_0 \in H^2$. Then there exists a $T > 0$ such that the Cauchy problem* (25) *has a unique solution $W \in H^2$ for $t \in [0, T)$.*

Proof. We again consider the fix point operator $\mathcal{G}W$ associated to the right hand side of (26), where now f'' is not constant. This requires to pull out the L^∞-norm of U and therefore, by Sobolev-Imbedding, we shall control W in H^2. We estimate the nonlinearity as follows:

$$\begin{aligned}
&\left\| K(T-\tau,.) * f''(\phi + \vartheta U) U^2(\tau,.) \right\|_{H^2} \\
&= \left\| (1+|k|)\widehat{K}\ \ (1+|k|)\mathcal{F}(f''(\phi+\vartheta U)U^2) \right\|_{L^2} \\
&\le C\left(1 + (T-\tau)^{-\frac{1}{1+\alpha}}\right) \|f''(\phi+\vartheta U)U^2\|_{H^1} \\
&\le L(\|U\|_{H^1})\|U\|_{H^1}^2 \left(1 + (T-\tau)^{-\frac{1}{1+\alpha}}\right),
\end{aligned}$$

where we have used (27) and Sobolev Imbedding. L is a positive non-decreasing function. The linear terms are estimated in a similar fashion as above, such that for a $C_0 > 0$

$$\|\mathcal{G}W\|_{H^2}^* \le C_0 \left(1 + (T + T^{1-\frac{1}{1+\alpha}})\left(1 + L(\|W\|_{H^2}^*)\|W\|_{H^2}^*\right)\|W\|_{H^2}^*\right).$$

The proof can be concluded in a similar way as before. □

Global existence will be the consequence of the existence of a Lyapunov functional, which also allows to deduce the asymptotic stability of travelling waves. The Lyapunov functional is also easier to derive in the case of the Burgers flux. Mainly for pedagogical reasons we first derive the result in this simplified situation and then proceed with the stability for the general convex flux function.

Stability of travelling waves for the quadratic flux. Assuming $f(u) = u^2$ and $\alpha > 1/2$, the Cauchy problem for (23) is well-posed in H^1. Since $f'' = 2$, the nonlinear term in (22) vanishes. Therefore to derive the global existence as well as asymptotic stability it suffices to construct a Lyapunov-functional controlling the H^1-norm of W.

Theorem 3. *Let $f(u) = u^2$ and $\alpha > 1/2$. Let ϕ be a travelling wave solution as in Theorem 2, and let $u_0(\xi)$ be an initial datum for* (19)*, such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))d\eta$ satisfies $W_0 \in H^1$. If $\|W_0\|_{H^1}$ is small enough, the Cauchy problem for equation* (19) *with initial datum u_0 has a unique global solution converging to the travelling wave in the sense that*

$$\lim_{t\to\infty} \int_t^\infty \|u(\tau,\cdot) - \phi\|_{L^2} d\tau = 0\,.$$

Remark 3. *Note that the condition* (20)*, which can be translated to $W_0(\pm\infty) = 0$, is incorporated in the condition $W_0 \in H^1$.*

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Proof. Equations (22) and (24) imply the estimates

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 \le -a_\alpha \|U\|^2_{\dot{H}^{(1+\alpha)/2}}\,, \tag{28}$$

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \le -a_\alpha \|W\|^2_{\dot{H}^{(1+\alpha)/2}}\,, \tag{29}$$

with $C_0 = \|\phi'\|_{L^\infty}$. We shall construct a Lyapunov functional by a linear combination of these estimates. For $\gamma > 0$, we denote $\gamma_* = \min\{1,\gamma\}$ and $\gamma^* = \max\{1,\gamma\}$. Then

$$J(t) = \frac{1}{2}\left(\|W\|_{L^2}^2 + \gamma\|U\|_{L^2}^2\right)$$

is bounded from above and below by

$$\frac{\gamma_*}{2}\|W\|_{H^1}^2 \le J \le \frac{\gamma^*}{2}\|W\|_{H^1}^2\,. \tag{30}$$

The combined estimate reads

$$\frac{dJ}{dt} - \left(\gamma C_0 + \|W\|_{L^\infty}\right)\|W\|_{\dot{H}^1}^2 + a_\alpha\left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma\|W\|^2_{\dot{H}^{(3+\alpha)/2}}\right) \le 0\,.$$

The idea is to control the second term by the third, which seems plausible, since the interpolation inequality

$$\|W\|_{\dot{H}^1}^2 \le \|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \|W\|^2_{\dot{H}^{(3+\alpha)/2}}\,, \tag{31}$$

holds as a consequence of $k^2 \le |k|^{1+\alpha} + |k|^{3+\alpha}$, $k \in \mathbb{R}$. The same inequality with k replaced by $k(a_\alpha/(2C_0))^{1/(1+\alpha)}$ implies

$$\gamma C_0\|W\|_{\dot{H}^1}^2 \le \frac{a_\alpha}{2}\left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma\|W\|^2_{\dot{H}^{(3+\alpha)/2}}\right)\,,$$

with $\gamma = (a_\alpha/(2C_0))^{2/(1+\alpha)}$. For the term arising from the nonlinearity we use the consequence $\|W\|_{\dot{H}^1}^2 \le \frac{1}{\gamma_*}(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma\|W\|^2_{\dot{H}^{(3+\alpha)/2}})$ of (31), which leads to

$$\frac{dJ}{dt} + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}\|W\|_{L^\infty}\right)\left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma\|W\|^2_{\dot{H}^{(3+\alpha)/2}}\right) \le 0\,.$$

By Sobolev imbedding and (30) we have

$$\|W\|_{L^\infty} \le \|W\|_{H^1} \le \sqrt{\frac{2}{\gamma_*}J}\,.$$

We now let the initial data be small enough such that $J(0) < (\gamma_*)^3 a_\alpha^2/8$. This immediately implies the existence of a $\lambda > 0$, such that

$$\frac{dJ}{dt} \le -\lambda\left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma\|W\|^2_{\dot{H}^{(3+\alpha)/2}}\right) \le -\lambda\gamma_*\|U\|_{L^2}^2\,, \qquad \text{for all } t > 0\,.$$

Integration with respect to time concludes the proof. ∎

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Stability for a general convex flux function. In contrary to the quadratic flux, now the nonlinearity in estimate (22) does not vanish:

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 - L(\|U\|_{L^\infty})\|U\|_{L^\infty}\|U\|_{H^1}^2 \le -a_\alpha \|U\|_{\dot H^{(1+\alpha)/2}}^2 , \tag{32}$$

with a positive nondecreasing function L and, similarly to above, $C_0 = \|f''(\phi)\phi'\|_{L^\infty}/2$. The estimate for W reads

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2} - L(\|U\|_{L^\infty})\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \le -a_\alpha \|W\|_{\dot H^{(1+\alpha)/2}}^2 , \tag{33}$$

We see that now we have to control U and W in $H^1 \subset L^\infty$, and therefore also need to derive an estimate for $\partial_\xi U$. As we have mentioned above, the Cauchy problem for (23) is well-posed in H^2. Hence the decay of W in H^2 is needed to repeat the local existence as well as to control the nonlinearities. We differentiate (21) and test it with $\partial_\xi U$. After several integrations by parts, we can estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_\xi U\|_{L^2}^2 - C_1\|U\|_{H^1}^2 - L(\|U\|_{L^\infty})\left(\|U\|_{L^\infty}\|\partial_\xi U\|_{L^2}^2 + \|\partial_\xi U\|_{L^3}^3\right)& \\ \le -a_\alpha\|\partial_\xi U\|_{\dot H^{(1+\alpha)/2}}^2 ,&\end{aligned} \tag{34}$$

where C_1 depends on the travelling wave and its derivatives up to order 2. We now apply a generalisation of the celebrated Gagliardo-Nirenberg inequalities (see e.g. [12]) to Sobolev spaces with fractional order, which was proven by Amann [3] (Proposition 4.1):

$$\|\partial_\xi U\|_{L^3}^3 \le C\|\partial_\xi U\|_{H^{\frac{\alpha+1}{4}}}^2 \|\partial_\xi U\|_{L^2} \le C\|U\|_{H^1}\|U\|_{H^{\frac{5+\alpha}{4}}}^2 \tag{35}$$

We are now ready to prove a stability result similar to Theorem 3 for the general convex flux function:

Theorem 4. *Let* (12) *hold and let ϕ be a travelling wave solution as in Theorem 2. Let u_0 be an initial datum for* (19) *such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))d\eta$ satisfies $W_0 \in H^2$. If $\|W_0\|_{H^2}$ is small enough, then the Cauchy problem for equation* (19) *with initial datum u_0 has a unique global solution converging to the travelling wave in the sense that*

$$\lim_{t\to\infty}\int_t^\infty \|u(\tau,\cdot) - \phi\|_{H^1} d\tau = 0 .$$

Proof. We proceed similarly to above and define

$$J(t) = \frac{1}{2}(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2 + \gamma_2\|\partial_\xi U\|_{L^2}^2) ,$$

with positive constants $\gamma_1, \gamma_2 > 0$. We denote $\gamma_* = \min\{1, \gamma_1, \gamma_2\}$ and $\gamma^* = \max\{1, \gamma_1, \gamma_2\}$. Then, as a functional of W, J is equivalent to the square of the H^2-norm. Combining (33), (32) and (34) together with (35) gives the complete estimate

$$\begin{aligned}&\frac{d}{dt}J + a_\alpha\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1\|W\|_{\dot H^{(3+\alpha)/2}}^2 + \gamma_2\|W\|_{\dot H^{(5+\alpha)/2}}^2\right)\\ &-\gamma_1 C_0\|U\|_{L^2}^2 - \gamma_2 C_1\|U\|_{H^1}^2 - L(\|W\|_{H^2})\|W\|_{H^2}\|U\|_{H^{(5+\alpha)/4}}^2 \le 0 .\end{aligned}$$

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Similarly to above we now choose $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 C_0 \|U\|_{L^2}^2 + \gamma_2 C_1 \|U\|_{H^1}^2 \leq \frac{a_\alpha}{2}\left(\|W\|^2_{\dot H^{(1+\alpha)/2}} + \gamma_1 \|W\|^2_{\dot H^{(3+\alpha)/2}} + \gamma_2 \|W\|^2_{\dot H^{(5+\alpha)/2}}\right),$$

and get the final estimate

$$\frac{d}{dt}J + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}L(\|W\|_{H^2})\|W\|_{H^2}\right)\left(\|W\|^2_{\dot H^{(1+\alpha)/2}} + \gamma_1 \|W\|^2_{\dot H^{(3+\alpha)/2}}\right) + \gamma_2\left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}L(\|W\|_{H^2})\|W\|_{H^2}\right)\|W\|^2_{\dot H^{(5+\alpha)/2}} \leq 0\,.$$

Letting again the initial data be such that $J(0)$ is small enough, we can deduce that J is nonincreasing for all times and moreover

$$\int_0^\infty \|U(t,\cdot)\|^2_{H^1} dt < \infty\,.$$

□

Appendix A. Linear Integral Equation

In this appendix we analyse the assumption (16) in more detail. We will show that all continuous and bounded solutions on $\mathbb{R}_-$ of the linear equation (36)

$$v(\xi) = C_0 \int_{-\infty}^{\xi} \frac{v(y)}{(\xi - y)^{1-\alpha}} dy, \qquad v(-\infty) = 0, \qquad C_0 = h'(u_-)/\Gamma(\alpha),$$

are given by the one-parameter family $\{be^{\lambda\xi} : b \in \mathbb{R}\}$ with $\lambda = h'(u_-)^{1/\alpha}$. A proof for the space $C_b(\mathbb{R}_-)$ cannot be carried out directly, since the kernel is only locally integrable. Therefore we first derive the uniqueness result in the space of continuous functions with exponential decay as $\xi \to -\infty$. We also present a less direct, but more general approach, which gives a similar result for the underlying space $L^\infty(\mathbb{R}_-)$. In addition we show that no continuous solutions with polynomial decay can exist.

We start by analysing solutions of (36) in $C_b(-\infty, \xi_0]$ for a $\xi_0 < 0$. Since it is easier to work with integral operators acting on a finite domain, we perform the transformation

$$w(\eta) = u(\xi), \qquad \text{where} \quad \eta = -\frac{1}{\xi} \in [0, \eta_0], \quad \text{for an} \quad \eta_0 > 0,$$

leading to the following equation for w

$$w(\eta) = C_0\, \eta^{1-\alpha} \int_0^\eta \frac{w(s)}{(\eta - s)^{1-\alpha} s^{1+\alpha}} ds, \qquad w(0) = 0. \tag{37}$$

To prove that the only non-trivial solutions with exponential decay are $w(\eta) = be^{-\frac{\lambda}{\eta}}$, we adapt the approach of Wolfersdorf for another integral equation (see the Appendix in [17]):

Lemma 8. *All solutions of* (36) *within the space*

$$C_w(\mathbb{R}_-) = \{f \in C_b(\mathbb{R}_-) :\ f(\xi) = e^{\mu\xi} g(\xi) \text{ for a } 0 < \mu < \lambda, \text{ where } g \in C_b(\mathbb{R}_-)\}$$

are given by the one-parameter family $\{be^{\lambda\xi} : b \in \mathbb{R}\}$ *with* $\lambda = h'(u_-)^{1/\alpha}$.

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Proof. Let $w(\eta) = e^{-\frac{\mu}{\eta}} z(\eta)$ be a solution of (37), where $0 < \mu < \lambda$. For $z \in C_b[0, \eta_0]$ we assume w.l.o.g. $z(0) = 0$ (otherwise we can shift some decay of the exponential function onto z). We shall show that $z = be^{-\frac{\lambda-\mu}{\eta}}$. Therefore we introduce

$$\phi(\eta) = z(\eta) - C_1 e^{-\frac{\lambda-\mu}{\eta}} \int_0^{\eta_0} z(s)ds, \qquad 1 = C_1 \int_0^{\eta_0} e^{-\frac{\lambda-\mu}{s}} ds$$

and note that $\phi(0) = 0$. Its primitive $\Phi(\xi) = \int_0^\eta \phi(s)ds$ satisfies $\Phi(0) = \Phi(\eta_0) = 0$. Due to Rolle's Theorem there exists an $\eta_1 > 0$ such that $\Phi'(\eta_1) = \phi(\eta_1) = 0$. If $\phi \equiv 0$, the proof is finished. Let now $\phi \not\equiv 0$. W.l.o.g. we assume that $\eta_1 > 0$ is the smallest value with $\phi(\eta_1) = 0$ and that $\phi(\eta) \geq 0$ in $[0, \eta_1]$ with $\phi(\eta) > 0$ in (η_2, η_1) for an $\eta_2 \in [0, \eta_1)$. Then we obtain

$$\begin{aligned} z(\eta_1) &= C_0\, \eta_1^{1-\alpha} \int_0^{\eta_1} \frac{e^{\mu\left(\frac{1}{\eta_1} - \frac{1}{s}\right)} z(s)}{(\eta_1 - s)^{1-\alpha} s^{1+\alpha}} ds \\ &> \underbrace{C_0\, \eta_1^{1-\alpha} \int_0^{\eta_1} \frac{e^{\lambda\left(\frac{1}{\eta_1} - \frac{1}{s}\right)}}{(\eta_1 - s)^{1-\alpha} s^{1+\alpha}} ds}_{=1}\; C_1\, e^{-\frac{\lambda-\mu}{\eta_1}} \int_0^{\eta_0} z(s)ds \;=\; z(\eta_1), \end{aligned}$$

leading again to a contradiction, and thus $\phi \equiv 0$. □

We shall also mention a more general approach, which was introduced for integral equations of Fredholm type. A similar result to Lemma 8 with the underlying space being $L^\infty(\mathbb{R}_-)$, can also be deduced from results on the *Wiener-Hopf equation*, which has the standard form

$$W(\xi) - \int_0^\infty K(\xi - y) W(y) dy = 0, \qquad \xi \geq 0. \tag{38}$$

Wiener and Hopf related the Fredholm property of the associated operator in (38) to conditions on its symbol [16]. Krein extended the Wiener-Hopf method to equations with L^1-integrable kernels [13]. We only state the part of his result which we will use in the following:

Let $K \in L^1(\mathbb{R})$. If the symbol $a(z) := 1 - \sqrt{2\pi}\mathcal{F}(K)(z)$ is elliptic, i.e. $\inf_{z\in\mathbb{R}} |a(z)| > 0$, and the winding number of the curve $\{a_\mu(z) : z \in (-\infty, \infty)\}$ around the origin is a non-positive number r. Then equation (38) *has exactly $|r|$ linearly independent solutions in any of the Lebesgue spaces $L^p(\mathbb{R}_+)$, $1 \leq p \leq \infty$.*

Since the kernel in (36) is only locally integrable we introduce as above exponential weights, which will allow to apply this result.

For a generalization of the Wiener-Hopf method to other spaces than the Lebesgue ones, we refer to the work of Duduchava [7], in which also the Theorem of Krein is given more detailed.

Lemma 9. *All solutions of* (36) *within the space*

$$L^\infty_w(\mathbb{R}_-) = \{f \in L^\infty(\mathbb{R}_-) : \ f(\xi) = e^{\mu\xi} g(\xi) \text{ for a } 0 < \mu < \lambda \text{ and } g \in L^\infty(\mathbb{R}_-)\}$$

are given by the one-parameter family $\{be^{\lambda\xi} : b \in \mathbb{R}\}$ with $\lambda = h'(u_-)^{1/\alpha}$.

Proof. Consider solutions v of (36) of the form $v(\xi) = e^{\mu\xi} w(\xi)$ for some $0 < \mu < \lambda$ and $w \in L^\infty(\mathbb{R}_-)$. Setting $W(\xi) = w(-\xi)$ and $K(\xi) = e^{-\mu\xi}\theta(\xi)\xi^{\alpha-1}$,

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equation (36) becomes a Wiener Hopf equation in the form (38). The kernel K is integrable, since $\mu > 0$. Thus, to apply the result of Krein, it remains to investigate the properties of the symbol

$$\begin{aligned} a_\mu(z) &= 1 - \frac{h'(u_-)\sqrt{2\pi}}{\Gamma(\alpha)}\mathcal{F}\left(\frac{\theta(\xi)}{\xi^{1-\alpha}}\right)(z - i\mu) \;=\; 1 - h'(u_-)(\mu + iz)^{-\alpha} \\ &= 1 - h'(u_-)(\mu^2 + z^2)^{-\alpha/2}(\cos(\alpha\,\varphi_{\mu,z}) - i\sin(\alpha\,\varphi_{\mu,z}))\,, \end{aligned}$$

where $\varphi_{\mu,z} = \arctan\frac{z}{\mu}$ and $\frac{\sqrt{2\pi}}{\Gamma(\alpha)}\mathcal{F}\big(\frac{\theta(\xi)}{\xi^{1-\alpha}}\big)(z) = (iz)^{-\alpha}$ for $z \in \mathbb{C}$. To check the ellipticity of the symbol, rewrite $|a_\mu(z)|^2$ as follows

$$|a_\mu(z)|^2 = \big(1 - h'(u_-)(\mu^2+z^2)^{-\alpha/2}\big)^2 + 2h'(u_-)(\mu^2+z^2)^{-\alpha/2}\big(1 - \cos(\alpha\,\varphi_{\mu,z})\big),$$

which attains its minimum with respect to z at $z = 0$ and does not vanish if $0 < \mu < \lambda$. Thus the symbol a_μ is elliptic and forms a closed curve $\{a_\mu(z) : z \in (-\infty,\infty)\}$, since $a_\mu(\pm\infty) = 1$. Thus the winding number of the closed curve is a well-defined integer, which remains to be computed. We note that $Re(a_\mu)$ is an even function and $Re(a_\mu(0)) < 0$ for $0 < \mu < \lambda$. Moreover $Im(a_\mu)$ is an odd function and $Im(a_\mu(z)) = 0$ only if $z = 0$ or $z = \pm\infty$. Hence the parametrization of the closed curve runs once around the origin in the counter clockwise sense. Thus the winding number is -1 and the result of Krein implies the statement. □

Finally, we show that no bounded solutions with polynomial decay can exist.

Lemma 10. *(i) If $v \in C_b(\mathbb{R}_-)$ is a solution of* (36)*, then v cannot change the sign.*
(ii) Equation (36) has no solution $v \in C_b(\mathbb{R}_-)$ with polynomial decay as $\xi \to -\infty$.

Proof. Again it easier to consider equation (37) instead. Solutions cannot change sign due to the nonlocality: If a smooth solution w is positive (negative) on $(0,\eta_*)$ for some $\eta_* > 0$, then the solution remains positive (negative). In contrast, if $w = 0$ on $[0,\eta_*)$, then $w(\eta)$ is a solution of equation (37) where the integration starts at η_* instead of $s = 0$. Therefore, we avoid the singularity of the kernel at $s = 0$ and are left with the integrable singularity at $s = \eta$. Given the initial value $w(\eta_*) = 0$, we conclude from standard theory that there exists only the trivial solution.

We prove statement (ii) by contradiction. Suppose that there exists a solution with polynomial decay $w(\eta) = \eta^\beta z(\eta)$ for some $\beta > 0$ and $z \in C_b(-\infty,\eta_0]$ which satisfies w.l.o.g. $z(\eta) \ge z_* > 0$. Then

$$z(\eta) \ge z_*\,\frac{h'(u_-)}{\Gamma(\alpha)}\,\eta^{1-\alpha-\beta}\int_0^\eta \frac{1}{(\eta-s)^{1-\alpha}s^{1+\alpha-\beta}}ds = \frac{h'(u_-)}{\Gamma(\alpha)}\,z_*\,B(\alpha,\beta-\alpha)\,\eta^{-\alpha},$$

where B denotes the Beta function. We see that for any β the right hand side grows unbounded as $\eta \to 0$, which contradicts our assumption $z \in C_b(-\infty,\eta_0]$. □

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ON NONLINEAR CONSERVATION LAWS REGULARIZED BY A RIESZ-FELLER OPERATOR

FRANZ ACHLEITNER, SABINE HITTMEIR, AND CHRISTIAN SCHMEISER

ABSTRACT. Scalar one-dimensional conservation laws with nonlocal diffusion term are considered. The wellposedness result of the initial-value problem with essentially bounded initial data for scalar one-dimensional conservation laws with fractional Laplacian is extended to a family of Riesz-Feller operators.

The main interest of this work is the investigation of smooth traveling wave solutions. In case of a genuinely nonlinear smooth flux function we prove the existence of such traveling waves, which are monotone and satisfy the standard entropy condition. Moreover, the dynamic nonlinear stability of the traveling waves under small perturbations is proven, similarly to the case of the standard diffusive regularization, by constructing a Lyapunov functional.

Apart from summarizing our results in the article Achleitner et al. (2011), we provide the wellposedness of the initial-value problem for a larger class of Riesz-Feller operators.

1. INTRODUCTION

We consider one-dimensional conservation laws with nonlocal diffusion term

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u \tag{1}$$

for a scalar quantity $u : \mathbb{R}_+ \times \mathbb{R}$, $(t, x) \mapsto u(t, x)$, a smooth flux function $f : \mathbb{R} \to \mathbb{R}$ and a non-local operator

$$(\mathcal{D}^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha} dy\,, \tag{2}$$

with $0 < \alpha < 1$.

1.1. **Motivation.** Conservation laws with nonlocal diffusion term of the form (1) appear in viscoelasticity - modeling the far-field behavior of uni-directional viscoelastic waves [1] - as well as in fluid mechanics - modeling the internal structure of hydraulic jumps in near-critical single-layer flows [3]. Moreover the nonlocal operator $\mathcal{D}^{1/3}$ appears in Fowler's equation

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}^{1/3} u\,, \tag{3}$$

which models the uni-directional evolution of sand dune profiles [4].

Equation (1) is closely related to

$$\partial_t u + \partial_x f(u) = D^{\alpha+1} u \tag{4}$$

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with a fractional Laplacian $D^{\alpha+1} = (-\frac{\partial^2 u}{\partial x^2})^{(\alpha+1)/2}$, $0 < \alpha < 1$. This kind of nonlinear conservation law with nonlocal regularization has been studied e.g. in [5, 6].

Remark 1. The nonlocal operators $\partial_x \mathcal{D}^\alpha$, $0 < \alpha < 1$, and the fractional Laplacian $D^{\alpha+1}$, $0 < \alpha < 1$, are Fourier multiplier operators, i.e.

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha u)(\xi) = -(\sin(\alpha\pi/2) - i\cos(\alpha\pi/2)\,\mathrm{sgn}(\xi))|\xi|^{\alpha+1}\mathcal{F}u(\xi)$$

and

$$\mathcal{F}(D^{\alpha+1}u)(\xi) = -|\xi|^{\alpha+1}\mathcal{F}u(\xi)\,,$$

whereat the Fourier transform $\mathcal{F}$ is defined as $\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\varphi(\xi)d\xi$.

1.2. **Riesz-Feller operators.** Riesz-Feller operators [7, 8, 9] are Fourier multiplier operators

$$(\mathcal{F}D_{a,\theta}f)(\xi) = -\psi_{a,\theta}(-\xi)(\mathcal{F}f)(\xi)$$

whose multiplier $\psi_{a,\theta}(\xi) = |\xi|^a e^{(i\,\mathrm{sgn}(\xi)\theta\pi/2)}$ is the logarithm of the characteristic function of a general Lévy strictly stable probability density with *index of stability* $0 < a \leq 2$ and asymmetry parameter $|\theta| \leq \min(a, 2-a)$. The nonlocal operators $\partial_x\mathcal{D}^\alpha$, $0 < \alpha < 1$, and the fractional Laplacian $D^{\alpha+1}$, $0 < \alpha < 1$, are Riesz-Feller operators, see also Remark 1 and Figure 1.

Theorem 1.1. *For $0 < a \leq 2$, $|\theta| \leq \min\{a, 2-a\}$ and $|\theta| < 1$, the Riesz-Feller operator $D_{a,\theta}$ generates a strongly continuous, convolution semigroup*

$$T(t) : L^p(\mathbb{R}) \to L^p(\mathbb{R})\,, \quad u_0 \mapsto T(t)u_0 = K(t,\cdot) * u_0\,,$$

with $1 \leq p < \infty$ and a convolution kernel $K(t,x) = \mathcal{F}^{-1}\exp(-t\psi(-.))(x)$ satisfying - for all $x \in \mathbb{R}$, $t > 0$ and $m \in \mathbb{N}$ - the properties

- *(non-negative) $K(t,x) \geq 0$,*
- *(integrable) $\|K(t,.)\|_{L^1(\mathbb{R})} = 1$,*
- *(scaling) $K(t,x) = t^{-\frac{1}{1+\alpha}}K(1, xt^{-\frac{1}{1+\alpha}})$,*
- *(smooth) $K(t,x)$ is $\mathcal{C}^\infty$ smooth,*
- *(bounded) there exists $B_m \in \mathbb{R}_+$ such that*

$$\left|\frac{\partial^m K}{\partial x^m}\right|(t,x) \leq t^{-\frac{1+m}{1+\alpha}}\frac{B_m}{1 + t^{-\frac{2}{1+\alpha}}|x|^2}\,.$$

The initial-value problem

$$\partial_t u + \partial_x f(u) = D_{a,\theta}u\,, \quad u(0,x) = u_0(x)\,, \tag{5}$$

for Riesz-Feller operators $D_{a,\theta}$ with *index of stability* $1 < a \leq 2$ and asymmetry parameter $a - 2 \leq \theta \leq 2 - a$ covers the special cases (1) and (4).

Theorem 1.2. *Suppose $1 < a \leq 2$ and $a - 2 \leq \theta \leq 2 - a$. If $u_0 \in L^\infty$, then there exists a unique solution $u \in L^\infty((0,\infty)\times\mathbb{R})$ of (5) satisfying the mild formulation*

$$u(t,x) = K(t,.) * u_0(x) - \int_0^t \left[\frac{\partial K}{\partial x}(t-\tau,.) * f(u(\tau,.))\right](x)\,\mathrm{d}\tau \tag{6}$$

almost everywhere. In particular

$$\|u(t,.)\|_\infty \leq \|u_0\|_\infty, \qquad \text{for } t > 0\,,$$

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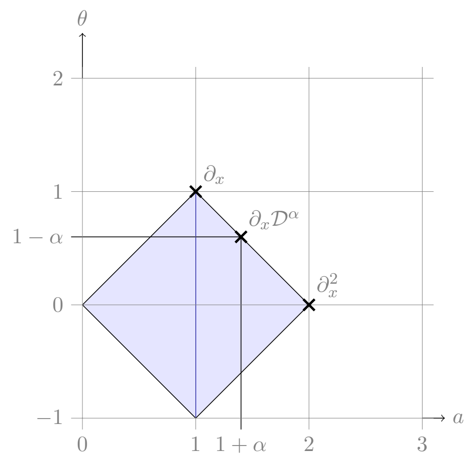

FIGURE 1. The family of Fourier multipliers $\psi_{a,\theta}(\xi) = |\xi|^a e^{(i\,\mathrm{sgn}(\xi)\theta\pi/2)}$ has two parameters a and θ. Some associated Fourier multiplier operators $(\mathcal{F}Tf)(\xi) = -\psi_{a,\theta}(-\xi)(\mathcal{F}f)(\xi)$ are displayed in the parameter space (a,θ). The Riesz-Feller operators $D_{a,\theta}$ are those operators, that take their parameters in the blue set $\{(a,\theta) \in \mathbb{R}^2 \,|\, 0 < a \le 2\,,\, |\theta| \le \min(a, 2-a)\}$, also known as Feller-Takayasu diamond. The family of operators $\partial_x \mathcal{D}^\alpha$, $0 < \alpha < 1$, interpolates formally between the first derivative ∂_x and second derivative ∂_x^2. Thus the limiting cases of equation (1) are a hyperbolic conservation law (for $\alpha = 0$) and a viscous conservation law (for $\alpha = 1$) [1].

and, in fact, u takes its values between the essential lower and upper bounds of u_0. Moreover, the solution has the following properties:

(i) *$u \in C^\infty((0,\infty)\times\mathbb{R})$ and $u \in C_b^\infty((t_0,\infty)\times\mathbb{R})$ for all $t_0 > 0$.*
(ii) *u satisfies equation (5) in the classical sense.*
(iii) *$u(t) \to u_0$, as $t \to 0$, in $L^\infty(\mathbb{R})$ weak-$*$ and in $L^p_{loc}(\mathbb{R})$ for all $p \in [1,\infty)$.*

Sketch of proof. The analysis of the initial-value problem for (4) by Droniou, Gallouët and Vovelle [6] depends on the properties in Theorem 1.1 of the semigroup (and its convolution kernel $K(t,x)$) generated by the fractional Laplacian $D^{\alpha+1}$ for $0 < \alpha < 1$. However all Riesz-Feller operators $D_{a,\theta}$ with *index of stability* $1 < a \le 2$ and asymmetry parameter $a - 2 \le \theta \le 2 - a$ share these properties. Thus the analysis in [6] carries over to the initial-value problem (5). □

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2. Traveling wave solutions

Definition 2.1. Suppose $(u_-, u_+, s) \in \mathbb{R}^3$. A traveling wave solution of (1) is a solution of the form $u(t,x) = \bar{u}(\xi)$ with $\xi := x - st$ and some function $\bar{u} : \mathbb{R} \to \mathbb{R}$ that connects the distinct endstates $\lim_{\xi\to\pm\infty} \bar{u}(\xi) = u_\pm$.

Inserting a traveling wave ansatz in (1) and integrating with respect to ξ yields the traveling wave equation

(7)
$$h(u) := f(u) - su - \big(f(u_-) - su_-\big) = \mathcal{D}^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha}\, \mathrm{d}y\,,$$

which is translation invariant.

If a smooth profile $\bar{u}$ approaches the endstates sufficiently fast, then the formal limit $\xi \to \infty$ in (7) leads to the Rankine-Hugoniot condition $f(u_+) - f(u_-) = s(u_+ - u_-)$.

If f is a convex flux function, then the vector field h is non-positive for values between u_- and u_+. Thus and due to the right-hand side of (7), a monotone traveling wave solution has to be monotone decreasing and the standard entropy condition $u_- > u_+$ has to hold.

The profile $\bar{u}$ of a traveling wave solution is governed by (7), whence its value at $\xi \in \mathbb{R}$ depends (only) on its values on the interval $(-\infty, \xi)$. Therefore, first the existence of a profile on an interval $(-\infty, \xi_\varepsilon]$ is established, subsequently its monotonicity and boundedness are verified and finally its global existence is deduced from an continuation argument.

The integral operator

$$\mathcal{D}^\alpha u(\xi) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\xi} \frac{u'(y)}{(\xi-y)^\alpha}\, \mathrm{d}y$$

is of Abel type and can be inverted by multiplying it with $(z-\xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z. Thus the traveling wave problem

$$h(u) = \mathcal{D}^\alpha u\,, \qquad \lim_{\xi\to-\infty} \bar{u}(\xi) = u_-\,, \quad \lim_{\xi\to+\infty} \bar{u}(\xi) = u_+\,, \tag{8}$$

and

$$u(\xi) - u_- = \mathcal{D}^{-\alpha}(h(u))(\xi) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\xi} \frac{h(u(y))}{(\xi-y)^{1-\alpha}}\, \mathrm{d}y \tag{9}$$

are equivalent if $u \in C_b^1(\mathbb{R})$ and $u' \in L^1(\mathbb{R}_-)$, and in particular if $u \in C_b^1(\mathbb{R})$ is monotone. Equation (9) is a nonlinear Volterra integral equation with a locally integrable kernel, where a well developed theory exists for problems on bounded intervals.

The linearizations of (8) and (9) at $\xi = -\infty$ (or, equivalently, at $u = u_-$) are

$$h'(u_-)v = \mathcal{D}^\alpha v \quad \text{and} \quad v = h'(u_-)\mathcal{D}^{-\alpha} v\,, \tag{10}$$

respectively. Both linearizations have solutions of the form $v(\xi) = be^{\lambda\xi}$ with $\lambda = h'(u_-)^{1/\alpha}$ and arbitrary $b \in \mathbb{R}$, see also [10]. We will need that

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these are the only non-trivial solutions of (10) in the space $H^2(-\infty,\xi_0]$ for some $\xi_0 \le 0$. In particular, we assume that

$$\mathcal{N}\big(id - h'(u_-)\mathcal{D}^{-\alpha}\big) = \operatorname{span}\{\exp(\lambda\xi)\} \qquad \text{with} \qquad \lambda = h'(u_-)^{1/\alpha}\,, \tag{11}$$

which is reasonable due to our analysis in [11, Appendix A].

In the existence result both formulations (8) and (9) will be used.

Theorem 2.2 ([11, Theorem 2])**.** *Suppose $f \in C^\infty(\mathbb{R})$ is a convex flux function, the shock triple (u_-,u_+,s) satisfies the Rankine-Hugoniot condition $f(u_+) - f(u_-) = s(u_+ - u_-)$ as well as the entropy condition $u_- > u_+$, and condition (11) holds. Then there exists a decreasing solution $u \in C_b^1(\mathbb{R})$ of the traveling wave problem (8). It is unique (up to a shift) among all $u \in u_- + H^2((-\infty,0)) \cap C_b^1(\mathbb{R})$.*

Remark 2 (Extensions)**.** In [11] we prove the result assuming only

$$\begin{aligned} &h \in C^\infty([u_+,u_-])\,, \quad h(u_+) = h(u_-) = 0\,, \quad h < 0 \text{ in } (u_+,u_-)\,, \\ &\exists\, u_m \in (u_+,u_-) \text{ such that } \quad h' < 0 \text{ in } (u_+,u_m) \text{ and } h' > 0 \text{ in } (u_m,u_-]\,. \end{aligned} \tag{12}$$

This is a little less than asking for convexity of f and the Lax entropy condition, since it covers the case $f'(u_+) \le s < f'(u_-)$.

The case of an concave flux function f can be analyzed in a similar way.

Idea of proof. The nonlinear problem has, up to translations, only two non-trivial solutions u_{down} and u_{up}, which can be approximated for large negative ξ by $u_- - e^{\lambda\xi}$ and large positive ξ by $u_- + e^{\lambda\xi}$, respectively. The choice 1 of the modulus of the coefficient of the exponential is irrelevant due to the translation invariance of the traveling wave equations (7) and (9).

The traveling wave equation (7) involves a causal integral operator, i.e. to evaluate $\mathcal{D}^\alpha \bar{u}(\xi)$ at a point ξ the profile $\bar{u}$ on the interval $(-\infty,\xi]$ is needed. Thus, for $\varepsilon > 0$ and $\xi_\varepsilon := \log\varepsilon/\lambda$, we investigate the existence of solution $u_{down} : I_\varepsilon \to \mathbb{R}$ of (7) on the interval $I_\varepsilon = (-\infty,\xi_\varepsilon]$

$$\lim_{\xi\to-\infty} u_{down}(\xi) = u_- \quad \text{and} \quad u_{down}(\xi_\varepsilon) = u_- - \varepsilon\,. \tag{13}$$

Due to the analysis of the linearized equation (10) and assumption (11), the solution is written as $u_{down}(\xi) = u_- - \exp(\lambda\xi) + v$. Thus the perturbation v satisfies a boundary value problem (BVP)

$$\big(\mathcal{D}^\alpha - h'(u_-)\big)v = h(u_- - \exp(\lambda\xi) + v) + h'(u_-)\big(\exp(\lambda\xi) - v\big)\,, \quad v(\xi_\varepsilon) = 0\,.$$

This can be formulated as a fixed point problem for a given right-hand side in $H^2(I_\varepsilon)$ and an application of Banach's fixed point theorem yields the existence of u_{down} which is unique among all functions u satisfying (13) and $\|u - u_-\|_{H^2(I_\varepsilon)} \le \delta$ for some sufficiently small δ, which is independent of ε. Moreover

$$\|u_{down} - u_- + e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \le C\varepsilon^2 \tag{14}$$

for some ε-independent constant C. The boundedness and monotonicity of u_{down},

$$u_{down}(\xi) < u_- \quad \text{and} \quad u'_{down}(\xi) < 0 \qquad \forall \xi \in I_\varepsilon\,,$$

follows from (14), a Sobolev imbedding $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ and the properties of $u_- - \exp(\lambda\xi)$.

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Next, the continuation of the solution $u_{down} : (-\infty, I_\varepsilon] \to \mathbb{R}$ is proven. The boundedness and monotonicity of u_{down} imply that u_{down} is also a solution of (9). Due to the causality of the integral operator, (9) can be written as a Volterra integral equation on a bounded interval $[I_\varepsilon, I_\varepsilon + \delta)$ for some $\delta > 0$

$$u(\xi) = f(\xi) + \frac{1}{\Gamma(\alpha)} \int_{\xi_\varepsilon}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} \, \mathrm{d}y \,.$$

with a well-defined inhomogeneity $f(\xi) = u_- + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\xi_\varepsilon} \frac{h(u(y))}{(\xi-y)^{1-\alpha}} \, \mathrm{d}y$. The (local) existence of a smooth solution for sufficiently small δ is a standard result in the theory of Volterra integral equations on bounded intervals, see e.g. Linz [13].

Then, the boundedness and monotonicity of these continued solutions is proven, such that the argument for local existence can be iterated to imply the existence of a solution

$$u_{down} \in C_b^1(\mathbb{R}) \quad \text{with} \quad \lim_{\xi \to \infty} u_{down}(\xi) = u_- \,.$$

Finally, the proof of Theorem 2.2 is completed by proving $\lim_{\xi\to\infty} u(\xi) = u_+$. Assuming to the contrary $\lim_{\xi\to\infty} u(\xi) > u_+$, would imply $\lim_{\xi\to\infty} h(u(\xi)) < 0$. Then, however, $-\mathcal{D}^{-\alpha} h(u) = u_- - u$ would increase above all bounds, which is impossible by the boundedness of the solution. □

Remark 3 (Discussion of previous results)**.** Sugimoto and Kakutani [1, 2] studied the existence of traveling wave solutions of (1). They prove that bounded continuous traveling wave solution may exist, but give no analytical proof of existence, instead they construct numerical solutions and study the asymptotic behavior analytically.

In case of Burgers' equation with fractional Laplacian (4), Biler et al. [5] showed that no continuous traveling wave solutions can exist for $\alpha \in (-1, 0]$, however they provide no existence result for the case $\alpha \in (0, 1)$.

Alvarez-Samaniego and Azerad [12] proved the existence of traveling wave solutions of (3) with perturbation methods.

Remark 4 (Comparison with previous results)**.** The dynamical systems approach to prove the existence of traveling wave solutions in [11, Theorem 2], parallels the one in case of viscous conservation laws. This approach is possible due to the causality of the operator $\mathcal{D}^\alpha$ in (7) and the monotonicity of the profiles.

In contrast in case of a conservation law with fractional Laplacian (4) the traveling wave equation for traveling wave solutions $u(t, x) = \bar{u}(\xi)$ with $\bar{u} \in C_b^2(\mathbb{R})$ can be written as

$$h(u) := f(u) - su - \big(f(u_-) - su_-\big) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{u'(y)}{(x-y)^\alpha} \, \mathrm{d}y \,.$$

Thus the value of a profile $\bar{u}$ at $\xi \in \mathbb{R}$ depends on the entire profile $\bar{u}$, such that a different approach is needed.

Whereas in case of Fowler's equation (3) the profile of a traveling wave solution is not necessarily monotone, such that the boundedness of a profile is difficult to establish.

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2.1. **Asymptotic stability of traveling wave solutions.** To study the asymptotic stability of traveling wave solutions ϕ of (1), equation (1) is cast in a moving coordinate frame $(t,x) \to (t, \xi = x - st)$,

$$\partial_t u + \partial_\xi (f(u) - su) = \partial_\xi \mathcal{D}^\alpha u\,, \tag{15}$$

such that a traveling wave solution becomes a stationary solution of (15). Analogous to viscous conservation laws asymptotic stability of ϕ is only to be expected for integrable zero-mass perturbations $U_0 := u_0 - \phi$, i.e.

$$\int_{\mathbb{R}} U_0(\xi)\,\mathrm{d}\xi = 0\,. \tag{16}$$

The evolution of a perturbation $U := u - \phi$ is governed by

$$\partial_t U + \partial_\xi (f(\phi + U) - f(\phi) - sU) = \partial_\xi \mathcal{D}^\alpha U\,. \tag{17}$$

However the L^2-norms of the perturbation U and its derivative are not enough to construct a Lyapunov functional. Therefore the primitive

$$W(t,\xi) = \int_{-\infty}^{\xi} U(t,\eta)\,\mathrm{d}\eta$$

of the perturbation U has to be considered.

The flux function will be assumed to be convex between the far-field values $u_\pm$ of the traveling wave solution ϕ, i.e.

$$f''(\phi(\xi)) \geq 0 \quad \text{for all} \quad \xi \in \mathbb{R}\,. \tag{18}$$

Theorem 2.3 ([11, Theorem 4]). *Suppose $f \in C^\infty(\mathbb{R})$, the conditions (12) and (18) hold and ϕ is a traveling wave solution of (1) as in Theorem 2.2. Let u_0 be such that $W_0(\xi) = \int_{-\infty}^{\xi} (u_0(\eta) - \phi(\eta))\,\mathrm{d}\eta$ satisfies $W_0 \in H^2(\mathbb{R})$. If $\|W_0\|_{H^2}$ is small enough, then the initial-value problem for equation (15) with initial datum u_0 has a unique global solution converging to the traveling wave solution ϕ in the sense that*

$$\lim_{t\to\infty} \int_t^\infty \|u(\tau,.) - \phi\|_{H^1}\,\mathrm{d}\tau = 0\,. \tag{19}$$

Proof. First, the local-in-time wellposedness of the initial-value problem

$$\partial_t W + (f(U+\phi) - f(\phi) - sU) = \partial_\xi \mathcal{D}^\alpha W\,, \quad W(0,x) = W_0(x)\,, \tag{20}$$

is established by an fixed point argument [11, Proposition 2].

Then a (Lyapunov) functional

$$J(t) = \frac{1}{2}(\|W\|_{L^2}^2 + \gamma_1 \|U\|_{L^2}^2 + \gamma_2 \|\partial_\xi U\|_{L^2}^2)$$

is defined with positive constants $\gamma_1, \gamma_2 > 0$. The functional $J : H^2(\mathbb{R}) \to \mathbb{R}$, $W(t) \mapsto J(t)$, is equivalent to $\|W(t)\|_{H^2}^2$, since $\gamma_* \|W(t)\|_{H^2}^2 \leq 2J(t) \leq \gamma^* \|W(t)\|_{H^2}^2$ with $\gamma_* = \min\{1, \gamma_1, \gamma_2\}$ and $\gamma^* = \max\{1, \gamma_1, \gamma_2\}$. Combining the energy estimates of the perturbation U, its primitive W and its derivative $\partial_\xi U$, and using a Gagliardo-Nirenberg inequality yields

$$\begin{aligned}\frac{d}{dt} J + a_\alpha \Big(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1 \|W\|_{\dot H^{(3+\alpha)/2}}^2 + \gamma_2 \|W\|_{\dot H^{(5+\alpha)/2}}^2\Big) \\ - \gamma_1 C_0 \|U\|_{L^2}^2 - \gamma_2 C_1 \|U\|_{H^1}^2 - L(\|W\|_{H^2}) \|W\|_{H^2} \|U\|_{\dot H^{(5+\alpha)/4}}^2 \leq 0\,,\end{aligned}$$

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where $a_\alpha = \sin(\alpha\pi/2) > 0$ and $\dot{H}^s$ denotes the homogeneous Sobolev space of order s. Finally, the constants $\gamma_1, \gamma_2 > 0$ are chosen such that

$$\gamma_1 C_0 \|U\|^2_{L^2} + \gamma_2 C_1 \|U\|^2_{H^1} \leq \frac{a_\alpha}{2} \left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma_1 \|W\|^2_{\dot{H}^{(3+\alpha)/2}} + \gamma_2 \|W\|^2_{\dot{H}^{(5+\alpha)/2}} \right) ,$$

which implies the final estimate

$$\frac{d}{dt} J + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*} L(\|W\|_{H^2}) \|W\|_{H^2} \right) \left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma_1 \|W\|^2_{\dot{H}^{(3+\alpha)/2}} \right) + \gamma_2 \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*} L(\|W\|_{H^2}) \|W\|_{H^2} \right) \|W\|^2_{\dot{H}^{(5+\alpha)/2}} \leq 0 .$$

For initial data such that $J(0)$ is sufficiently small, the functional $J(t)$ - being equivalent to $\|W(t)\|^2_{H^2}$ - is non-increasing for all times. This implies the global-in-time existence of $W(t)$ as a solution of (20) and moreover (19). □

Remark 5. In case of Burgers' flux $f(u) = u^2$ and $\alpha > 1/2$, asymptotic stability of a traveling wave solution ϕ is established in case of $W_0 \in H^1(\mathbb{R})$, see also [11, Theorem 3].

Due to a Sobolev imbedding $H^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$, the asymptotic stability result $\lim_{t\to\infty} \|U(t)\|_{H^1} = 0$ implies also $\lim_{t\to\infty} \|U(t)\|_{L^\infty} = 0$.

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ASYMPTOTIC STABILITY OF TRAVELING WAVE SOLUTIONS FOR NONLOCAL VISCOUS CONSERVATION LAWS WITH EXPLICIT DECAY RATES

FRANZ ACHLEITNER AND YOSHIHIRO UEDA

ABSTRACT. We consider scalar conservation laws with nonlocal diffusion of Riesz-Feller type such as the fractal Burgers equation. The existence of traveling wave solutions with monotone decreasing profile has been established recently (in special cases). We show the local asymptotic stability of these traveling wave solutions in a Sobolev space setting by constructing a Lyapunov functional. Most importantly, we derive the algebraic-in-time decay of the norm of such perturbations with explicit algebraic-in-time decay rates.

1. INTRODUCTION

We consider the evolution of a scalar quantity $u : \mathbb{R} \times (0, \infty) \to U \subset \mathbb{R}$, $(x, t) \mapsto u(x, t)$, which is governed by the Cauchy problem

$$\begin{aligned} &\partial_t u + \partial_x f(u) = D^\alpha_\theta u && \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ &u(0,x) = u_0(x) && \text{for } x \in \mathbb{R}, \end{aligned} \tag{1}$$

with an initial datum $u_0 : \mathbb{R} \to U \subset \mathbb{R}$, a flux function $f : U \subset \mathbb{R} \to \mathbb{R}$ and a Riesz-Feller operator D^α_θ for some $1 < \alpha \leq 2$ and $|\theta| \leq 2 - \alpha$. Equation (1) models nonlinear transport and nonlocal diffusion of a quantity $u(x, t)$ in space over time. The flux function f is assumed to be smooth and convex as well as to satisfy w.l.o.g. $f(0) = 0$. The Riesz-Feller operator can be defined as a Fourier multiplier operator, see also [23]. Precisely, the Riesz-Feller operator D^α_θ of order α and skewness θ is defined as

$$\mathcal{F}[D^\alpha_\theta v](k) = \psi^\alpha_\theta(k) \mathcal{F}[v](k) , \qquad k \in \mathbb{R} , \tag{2}$$

with symbol

$$\psi^\alpha_\theta(k) = -|k|^\alpha \exp\big(\mathrm{i}\ \mathrm{sgn}(k)\, \theta \tfrac{\pi}{2}\big) = -|k|^\alpha \big(\cos(\theta \tfrac{\pi}{2}) + \mathrm{i}\ \mathrm{sgn}(k)\ \sin(\theta \tfrac{\pi}{2})\big) \tag{3}$$

and parameters $0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2 - \alpha\}$, where $\mathcal{F}$ denotes the Fourier transform.

Remark 1. (i) Riesz-Feller operators D^α_θ with $\theta = 0$ are also known as fractional Laplacians $D^\alpha_0 = -(-\partial_x^2 u)^{\alpha/2}$ with $0 < \alpha \leq 2$ and Fourier symbol $-|k|^\alpha$. In particular, the Laplacian $D^2_0 = \partial_x^2$ is a special case with $\alpha = 2$ and $\theta = 0$.

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(ii) For $0 < \gamma < 1$, Riesz-Feller operators D^α_θ with $\alpha = \gamma$ and $\theta = -\gamma$, can be identified with fractional Caputo derivatives of order $0 < \gamma < 1$:

$$-(\mathcal{D}^\gamma u)(x) = -\frac{1}{\Gamma(1-\gamma)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\gamma} \,\mathrm{d}y \quad \text{for } x \in \mathbb{R}\,, \tag{4}$$

which have Fourier symbol $-(-\mathrm{i}k)^\gamma$. The symbol $(-\mathrm{i}k)^\gamma$ is multi-valued, however (only) the choice $(-\mathrm{i}k)^\gamma = \big(|k| \exp(-\mathrm{i}\,\mathrm{sgn}(k)\,\frac{\pi}{2})\big)^\gamma = |k|^\gamma \exp(-\mathrm{i}\,\mathrm{sgn}(k)\,\gamma\frac{\pi}{2})$ yields a causal operator. For details, see [20]. Moreover, its derivative $\partial_x(\mathcal{D}^\gamma u)$ is a Riesz-Feller operator with $\alpha = 1+\gamma$ and $\theta = 2-\alpha$.

Taking $\alpha = 2$ and $\theta = 0$ in (1), we formally obtain a classical viscous conservation law:

$$\partial_t u + \partial_x f(u) = \partial_x^2 u \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty). \tag{5}$$

The existence and asymptotic stability of traveling wave solutions of equation (5) has been studied thoroughly. A first example of equation (1) with nonlocal diffusion is

$$\partial_t u + \partial_x f(u) = D^\alpha_0 u \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty)\,, \tag{6}$$

with a fractional Laplacian D^α_0, $0 < \alpha \le 2$, which has been studied e.g. in [6, 11]. For $1 < \alpha \le 2$, the Cauchy problem for (6) with $f \in C^\infty(\mathbb{R})$ and essentially bounded initial data has a global-in-time mild solution which becomes smooth for positive times, see [11] and its extension to (1) in [2].

Other examples of equation (1) with nonlocal diffusion appear in viscoelasticity [27] and fluid dynamics [21]. In particular,

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\gamma u \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty)\,, \tag{7}$$

with $0 < \gamma < 1$ is used as a model for the far-field behavior of uni-directional viscoelastic waves [27], and derived as a model for the internal structure of hydraulic jumps in near-critical single-layer flows [21]. Moreover the nonlocal operator $\mathcal{D}^{1/3}$ appears in Fowler's equation

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}^{1/3} u\,, \tag{8}$$

which models the uni-directional evolution of sand dune profiles [13]. In the theory of water waves similar models $\partial_t u + \partial_x u^2 = \mathcal{N}[u]$ with different (nonlocal) Fourier multiplier operators $\mathcal{N}$ are studied, see the book [25] and references therein.

To explain our main results, we introduce traveling wave solutions for equation (1). Traveling wave solutions (TWS) are of the form $u(x,t) = \overline{u}(\xi)$ for some profile $\overline{u}$ with $\xi = x - st$ and (constant) wave speed $s \in \mathbb{R}$. We are interested in TWS with profiles $\overline{u}$ connecting distinct endstates $u_\pm$ such that

$$\lim_{x \to \pm\infty} \overline{u}(x) = u_\pm\ . \tag{9}$$

Using this ansatz in equation (1) and assumption (9), we find that the wave speed s has to satisfy the Rankine-Hugoniot condition

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{10}$$

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Here, an extension of Riesz-Feller operators to non-integrable functions is needed, see Appendix A. Due to translational invariance of equation (1), traveling wave solutions are only unique up to a shift.

For classical viscous conservation laws (5), the profile of a TWS satisfies an ordinary differential equation $\overline{u}' = f(\overline{u}) - s\overline{u} - (f(u_-) - su_-)$. In fact, TWS exist only for parameters $(u_-, u_+; s)$ satisfying (10) and $u_+ < u_-$. In case of equation (7), the existence and asymptotic stability (without decay rates) of traveling wave solutions for parameters $(u_-, u_+; s)$ satisfying (10) and $u_+ < u_-$ has been shown [1, 8]. Here, a profile satisfies a fractional differential equation $\mathcal{D}^\gamma \overline{u} = f(\overline{u}) - s\overline{u} - (f(u_-) - su_-)$. The proof of existence relies on the causality of the Caputo derivative $\mathcal{D}^\gamma$, i.e. to evaluate $\mathcal{D}^\gamma \overline{u}$ at x the profile $\overline{u}$ on $(-\infty, x)$ is needed. In contrast, the profile for a TWS of a nonlocal conservation law (6) for $1 < \alpha < 2$ has to satisfy

$$D_0^\alpha \overline{u}(x) = \int_{\mathbb{R}} \frac{\overline{u}(x+\xi) - \overline{u}(x) - \overline{u}'(x)\,\xi}{\xi^{1+\alpha}}\,\mathrm{d}\xi = \partial_x\big(f(\overline{u}) - s\overline{u} - (f(u_-) - su_-)\,.$$

Thus $D_0^\alpha \overline{u}(x)$ depends on the whole profile $\overline{u}$. For fractal Burgers equation, i.e. equation (6) with $1 < \alpha < 2$ and Burgers flux function $f(u) = u^2$, the existence of traveling wave solutions has been proven recently [7]. The idea is to approximate the operators D_0^α by convolution operators $\mathcal{K}_\epsilon[u] = K_\epsilon * u - u$ for suitable convolution kernels $K_\epsilon \in L^1(\mathbb{R})$. The existence of TWS for the approximate equations is known and the TWS is established as the limit of this family. It is conceivable to use this approach to prove the existence of traveling wave solutions in the general case (1) for convex flux functions f with $1 < \alpha < 2$ and $|\theta| \leq 2 - \alpha$.

For fractal Burgers equation (6) results in the complementary cases $\alpha \in (0,1)$ and/or $u_- \leq u_+$ are also available: For example, for $\alpha \in (0,1)$ and (9) no traveling wave solutions of (6) with smooth profile exists [6]. Whereas under the assumption $u_- < u_+$ the solution of (6) converges as $t \to \infty$ to a rarefaction wave of the underlying Burgers equation if $\alpha \in (1,2)$ and to a self-similar solution if $\alpha = 1$; see [17] and [4], respectively.

The asymptotic stability of traveling wave solutions of classical viscous conservation laws (5) has been studied thoroughly. At first, historically, Il'in and Oleinik [16] proved the asymptotic stability of nonlinear waves for viscous conservation laws (5) by making use of the maximum principle for linear parabolic equations. For Burgers' equation, i.e. equation (5) with Burgers' flux function $f(u) = u^2$, Nishihara [26] obtained the decay estimates toward traveling wave solutions by making use of the explicit solution formula. And, Kawashima and Matsumura [18] generalized Nishihara's time decay result to a class of viscous conservation laws. They considered weighted L^2 spaces and used a weighted energy method. Furthermore, Kawashima, Nishibata and Nishikawa [19] extended the L^2 energy method to general L^p spaces. Their techniques have been applied to a model system for compressible viscous gas in [24] and a hyperbolic system with relaxation in [28].

Assuming the existence of a traveling wave solution of (1) with monotone decreasing profile, we show that asymptotic stability of a traveling wave solution in a Sobolev space setting follows from a standard Lyapunov functional argument: To investigate the stability of the traveling wave solution

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with profile $\overline{u}$, we consider initial data u_0 such that $u_0 - \overline{u}$ is integrable and determine the unique shift x_0 which yields $\int_{-\infty}^{\infty} (u_0(\xi) - \overline{u}(\xi + x_0))\,\mathrm{d}\xi = 0$. Moreover, we restrict the domain of initial data u_0 further such that $W_0(\xi) = \int_{-\infty}^{\xi} (u_0(\eta) - \overline{u}(\eta))\,\mathrm{d}\eta$ exists (using a suitable shifted profile $\overline{u}$) and satisfies $W_0 \in H^2$. (For details, we refer to [28].) More precisely, we can derive the following theorem.

Theorem 1. *Suppose $1 < \alpha \le 2$ and $\theta \le \min\{\alpha, 2-\alpha\}$. Let the flux function $f \in C^2(\mathbb{R})$ be convex and let $u(x,t) = \overline{u}(x - st)$ be a traveling wave solution of* (1) *with monotone decreasing profile $\overline{u}$. Let u_0 be an initial datum for* (1) *such that $W_0(\xi) = \int_{-\infty}^{\xi} (u_0(\eta) - \overline{u}(\eta))\,\mathrm{d}\eta$ satisfies $W_0 \in H^2(\mathbb{R})$. Then there exists a positive constant δ_0 such that if $\|W_0\|_{H^2} \le \delta_0$, then the Cauchy problem* (1) *has a unique global solution converging to the traveling wave in the sense that*

$$\|(u - \overline{u})(t)\|_{L^\infty} \longrightarrow 0 \qquad \text{for} \quad t \to \infty.$$

The proof of Theorem 1 for the general equation (1) is similar to the one of [1, Theorem 4] for the special case (7) without decay rates.

Our main result is to prove the asymptotic stability with algebraic-in-time decay rate for traveling wave solutions of (1) with monotone decreasing profiles.

Theorem 2. *Suppose the same assumptions as in Theorem 1 hold and $f \in C^\infty(\mathbb{R})$. For all $W_0 \in W^{1,\infty}(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, the Cauchy problem* (1) *has a unique global solution. Moreover, there exists a positive constant δ_1 such that if $\|W_0\|_{W^{1,1}} \le \delta_1$ then the unique global solution u satisfies*

$$\|(u - \overline{u})(t)\|_{L^2} \le C E_1 (1+t)^{-1/(2\alpha)} \tag{11}$$

for $t \ge 0$, where $E_1 := \|W_0\|_{H^1} + \|W_0\|_{W^{1,1}}$ and C is a constant which is independent of time t.

Remark 2. We employ sharp interpolation inequalities in Sobolev spaces to derive (11). In this way optimal decay estimates for the asymptotic stability of viscous rarefaction waves in scalar viscous conservation laws (5) have been derived in [14].

Remark 3. We want to explain the functional setting in Theorem 2: We considered the function spaces $H^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R}) \subset W^{1,\infty}(\mathbb{R}) \cap W^{1,1}(\mathbb{R}) \subset H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ in variants of Theorem 2. The choice $H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ leads to the restriction $\alpha \in (3/2, 2)$ if we use an estimate of the nonlinearity like Dix [9, 10] to establish the existence of solutions for the Cauchy problem. Assuming higher regularity of the initial data removes the need for this restriction: Under the assumptions of Theorem 1 with $W_0 \in H^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$, the solution constructed in Theorem 1 satisfies

$$\|(u - \overline{u})(t)\|_{H^1} \le C \widetilde{E}_1 (1+t)^{-1/(2\alpha)}$$

for $t \ge 0$, where $\widetilde{E}_1 := \|W_0\|_{H^2} + \|W_0\|_{W^{2,1}}$ and a constant C independent of time t. Our choice $W_0 \in W^{1,\infty}(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ in Theorem 2 leads to the technical assumption $f \in C^\infty(\mathbb{R})$, since we use a result on the existence of global-in-time solutions for the Cauchy problem with essentially bounded initial data [11, 2]. The assumption $f \in C^2(\mathbb{R})$ in Theorem 1 could be retained by aiming for less regularity in their approach.

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Unfortunately it is difficult to apply the weighted energy method in [18] to our problem (to derive the convergence rate). Instead of this method, we employ another technique which focuses on the interpolation property in Sobolev space. For example, this argument is utilized in [14].

The contents of this paper are as follows. In Section 2, we reformulate our problem and consider the well-posedness of the new one. In Section 3, we derive the asymptotic stability result by uniform energy estimates as *a-priori* estimates of solutions in the Sobolev space H^2. Furthermore, our main result on the asymptotic stability with explicit algebraic decay rate in Theorem 2 is proved in Section 4, by using the energy method with an L^2–L^1 interpolation argument. In Appendix A, we collect results on the singular integral representation of Riesz-Feller operators.

Notation. Before closing this section, we give some notations used in this paper. We define the Fourier transform for $v \in \mathcal{S}$ in the Schwartz space $\mathcal{S}$ as

$$\hat{v}(k) = \mathcal{F}[v](k) := \int_{\mathbb{R}} e^{-\mathrm{i}kx} v(x)\,\mathrm{d}x \quad \text{for } k \in \mathbb{R}\,,$$

and the inverse Fourier transform as

$$\mathcal{F}^{-1}[v](x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}kx} v(k)\,\mathrm{d}k \quad \text{for } x \in \mathbb{R}\,.$$

The Fourier transform and its inverse are linear operators and $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote also their respective extensions to $L^2(\mathbb{R})$.

For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R})$ the usual Lebesgue space over $\mathbb{R}$ with norm $\|\cdot\|_{L^p}$, and $W^{s,p} = W^{s,p}(\mathbb{R})$ the usual Sobolev space over $\mathbb{R}$ with norm $\|\cdot\|_{W^{s,p}}$. Using the short-hand notation $H^s(\mathbb{R}) := W^{s,2}(\mathbb{R})$ with norm $\|\cdot\|_{H^s}$. Moreover, we set $\|W(t)\|_{W^{1,\infty}} = \max\{\|W(t)\|_{L^\infty},\ \|\partial_\xi W(t)\|_{L^\infty}\}$ and its analog in case of $\|W(t)\|_{W^{\ell,\infty}}$ for all $\ell \in \mathbb{N}$. Finally, for nonnegative integer ℓ, $C^\ell(I;X)$ (respectively $C^\ell_b(I;X)$) denotes the space of ℓ-times continuously differentiable functions (respectively with bounded derivatives) on the interval I with values in the Banach space X.

The constants in our estimates may change their value from line to line.

2. Reformulation for the problem

In the special case (7), the existence and asymptotic stability of traveling wave solutions $u(x,t) = \overline{u}(x - st)$ with monotone decreasing profile $\overline{u}$ has been proven without rates of decay [1, 8]. However, assuming in the general case (1) the existence of a traveling wave solution $u(x,t) = \overline{u}(x - st)$ with monotone decreasing profile $\overline{u}$, then the proof of asymptotic stability generalizes with obvious modifications:

To prove the asymptotic stability of a traveling wave solution $\overline{u}$ of (1), one can follow the standard approach called the anti-derivative method introduced in [18] for viscous conservation laws. It is convenient to cast (1) in a moving coordinate frame $(x,t) \mapsto (\xi,t)$, such that

$$\partial_t u + \partial_\xi (f(u) - su) = D^\alpha_\theta u\,, \tag{12}$$

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and $\overline{u}$ is a stationary solution of (12). The Cauchy problem for (12) with initial datum u_0 governs the evolution of u_0. If its solution u is considered as a perturbation of the traveling wave solution $\overline{u}$, then this perturbation $U(\xi,t) := u(\xi,t) - \overline{u}(\xi)$ satisfies the Cauchy problem

$$\begin{aligned} \partial_t U + \partial_\xi(f(\overline{u}+U) - f(\overline{u})) - s\partial_\xi U &= D^\alpha_\theta U, \\ U(\xi,0) &= U_0(\xi), \end{aligned} \tag{13}$$

where $U_0(\xi) := u_0(\xi) - \overline{u}(\xi)$. To obtain the desired result, we try to construct the L^2-energy estimate for U by employing the energy method. However, because of the decreasing property of traveling wave solutions, it is hard to construct the L^2-energy estimate. To overcome this difficulty, we apply the anti-derivative method.

Precisely, we introduce the new function $W(\xi,t)$ which satisfies $\partial_\xi W = U$. Then we can formally rewrite (13) as

$$\begin{aligned} \partial_t W + f(\overline{u}+\partial_\xi W) - f(\overline{u}) - s\partial_\xi W &= D^\alpha_\theta W, \\ W(\xi,0) &= W_0(\xi). \end{aligned} \tag{14}$$

If a global-in-time solution of (14) with $W_0(\xi) = \int_{-\infty}^{\xi} U_0(\eta)\,\mathrm{d}\eta$ is sufficiently smooth, then its derivative $\partial_\xi W$ satisfies Cauchy problem (13). Therefore, we try to construct a global-in-time solution of (14), instead of (13). For this purpose, we discuss the well-posedness of problem (14) in this section.

The well-posedness of the Cauchy problem for (14), will follow from a contraction argument. Assuming $f(u) = u^2$ and $\alpha > 3/2$ allows to estimate the nonlinearity in the fashion of Dix [9, 10] implying the well-posedness in H^1. For general flux functions and $\alpha \in (1,2]$, we have to require more regularity of the initial data, e.g. $W_0 \in H^2$.

Proposition 1. *Let $f \in C^2(\mathbb{R})$, $1 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\} = 2-\alpha$. Suppose M is an arbitrary positive constant and suppose $W_0 \in H^2(\mathbb{R})$ such that $\|W_0\|_{H^2} \leq M$. Then there exists a positive constant T, which depends on M, such that the Cauchy problem (14) has a unique mild solution $W \in C([0,T]; H^2)$ with $\|W(t)\|_{H^2} \leq 2M$ for $t \in [0,T]$.*

To prove Proposition 1, we first present some properties of the fundamental solution of $\partial_t u = D^\alpha_\theta u$.

Lemma 1 ([3, Lemma 2.1])**.** *For $1 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\} = 2-\alpha$, $G^\alpha_\theta(x,t) := \mathcal{F}^{-1}[e^{t\psi^\alpha_\theta(\cdot)}](x)$ with ψ^α_θ defined in (3) is the fundamental solution of $\partial_t u = D^\alpha_\theta u$. Moreover, G^α_θ satisfies for all $(x,t) \in \mathbb{R}\times(0,\infty)$ the properties*

- *(G1)* $G^\alpha_\theta(x,t) \geq 0$,
- *(G2)* $G^\alpha_\theta(x,t) = t^{-1/\alpha} G^\alpha_\theta(xt^{-1/\alpha}, 1)$,
- *(G3)* $\|G^\alpha_\theta(\cdot,t)\|_{L^1(\mathbb{R})} = 1$,
- *(G4)* $G^\alpha_\theta(\cdot,s) * G^\alpha_\theta(\cdot,t) = G^\alpha_\theta(\cdot,s+t)$ *for all* $s,t \in (0,\infty)$,
- *(G5)* $\|G^\alpha_\theta(\cdot,t)\|_{L^p(\mathbb{R})} \leq \|G^\alpha_\theta(\cdot,1)\|_{L^p(\mathbb{R})} t^{-\frac{1}{\alpha}(1-\frac{1}{p})}$ *for all* $1 \leq p < \infty$,
- *(G6)* $G^\alpha_\theta \in C^\infty_0(\mathbb{R}\times(0,\infty))$,
- *(G7) For all $t > 0$, there exists a constant $\mathcal{K}$ such that* $\|\partial_x G(\cdot,t)\|_{L^1(\mathbb{R})} \leq \mathcal{K} t^{-1/\alpha}$.

Due to the properties of G^α_θ, it is easy to show that D^α_θ generates a semi-group.

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Lemma 2. *For $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\} = 2-\alpha$, the Riesz-Feller operator D_θ^α generates a strongly continuous, convolution semigroup*

$$S_t : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,, \quad u_0 \mapsto S_t u_0 = G_\theta^\alpha(\cdot, t) * u_0$$

with G_θ^α defined in Lemma 1. Moreover, the semigroup satisfies the dispersion property for $u \in L^1(\mathbb{R})$

$$\|S_t u\|_{L^p(\mathbb{R})} \leq C_p \, t^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u\|_{L^1(\mathbb{R})} \tag{15}$$

for all $1 \leq p < \infty$ and some $C_p > 0$.

Proof. Due to (G3) and Young's inequality for convolutions,

$$\|S_t u\|_{L^p} \leq \|G_\theta^\alpha(\cdot, t)\|_{L^1} \|u\|_{L^p} = \|u\|_{L^p}$$

for all $u \in L^p(\mathbb{R}^n)$. Therefore $S_t : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ are well-defined bounded linear operators for all $t \geq 0$. $(S_t)_{t \geq 0}$ is a semigroup, since $S_{t+s} = S_t S_s$ for all $s, t \geq 0$ holds due to (G4) and $S_0 := \mathrm{Id}$. Strong continuity of $(S_t)_{t\geq 0}$ follows from a standard result about convolutions [22, p.64] and (G2). The dispersion property

$$\forall 1 \leq p < \infty \quad \exists C_p > 0 : \quad \|S_t u\|_{L^p(\mathbb{R})} \leq C_p \, t^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u\|_{L^1(\mathbb{R})} \quad \forall u \in L^1(\mathbb{R})$$

can be proved using (G5) and Young's inequality [22, p.98-99]. □

Lemma 3. *Let $1 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. The fundamental solution G_θ^α defined in Lemma 1 satisfies for all $\ell \in \mathbb{N}_0$ and $0 \leq r \leq \ell$ the following estimates:*

$$\|\partial_x^\ell \big(G_\theta^\alpha(t) * \phi\big)\|_{L^2} \leq C t^{-(\ell - r)/\alpha} \|\partial_x^r \phi\|_{L^2}, \qquad t > 0 \,, \tag{16}$$

where C is a certain positive constant. If $r = \ell$, then inequality (16) *with $C = 1$ is optimal.*

Proof. By using Plancherel's theorem, we compute that

$$\begin{aligned}\|\partial_x^\ell \big(G_\theta^\alpha(t) * \phi\big)\|_{L^2} &= \|(\mathrm{i}k)^\ell e^{t\psi_\theta^\alpha(k)} \hat{\phi}\|_{L^2} \\ &\leq \|(\mathrm{i}k)^{\ell - r} e^{t\psi_\theta^\alpha(k)}\|_{L^\infty} \|(\mathrm{i}k)^r \hat{\phi}\|_{L^2} \leq C t^{-(\ell - r)/\alpha} \|\partial_x^r \phi\|_{L^2};\end{aligned}$$

since $\|(\mathrm{i}k)^{\ell-r} e^{t\psi_\theta^\alpha(k)}\|_{L^\infty} = \sup_{k\in\mathbb{R}} |k|^{\ell - r} e^{-t|k|^\alpha \cos(\theta\pi/2)} \leq C t^{-(\ell-r)/\alpha}$, due to the positivity of $\cos(\theta\pi/2)$ under the assumption in Lemma 3. If $r = \ell$, then we obtain $\|\partial_x^\ell \big(G_\theta^\alpha(t) * \phi\big)\|_{L^2} \leq \|G_\theta^\alpha\|_{L^1} \|\partial_x^\ell \phi\|_{L^2} = \|\partial_x^\ell \phi\|_{L^2}$, by using the fact that G_θ^α is a non-negative integrable function with mass one. □

Lemma 4. *Suppose that the same assumption as in Lemma 3 holds, and $\phi \in H^\sigma$ for $\sigma \geq 0$. Then the fundamental solution satisfies $G_\theta^\alpha * \phi \in C([0,\infty); H^\sigma)$.*

Proof. For arbitrary constants $t_1, t_2 \in [0, \infty)$, we have

$$\|G_\theta^\alpha(t_1) * \phi - G_\theta^\alpha(t_2) * \phi\|_{H^\sigma}^2 \leq \int_{\mathbb{R}} (1 + |k|)^{2\sigma} |e^{t_1 \psi_\theta^\alpha(k)} - e^{t_2 \psi_\theta^\alpha(k)}|^2 |\hat{\phi}(k)|^2 \, \mathrm{d}k,$$

where the integral is bounded by $4\|\phi\|_{H^\sigma}^2$. Thus, the Dominated Convergence Theorem allows to pass to the limit under the integral sign, which completes the proof. □

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Proof of Proposition 1. Using the fundamental solution G^α_θ of the linear evolution equation $\partial_t u = D^\alpha_\theta u$, the mild formulation of (14) reads

$$W(t) = G^\alpha_\theta(t) * W_0 - \int_0^t G^\alpha_\theta(t-\tau) * F(\overline{u}, \partial_\xi W)\,\mathrm{d}\tau, \tag{17}$$

where $F(\overline{u}, \partial_\xi W) := f(\overline{u} + \partial_\xi W) - f(\overline{u}) - s\partial_\xi W$. To employ a fix point argument, we consider the mapping $\mathcal{G}[W]$ defined by

$$\mathcal{G}[W](t) := G^\alpha_\theta(t) * W_0 - \int_0^t G^\alpha_\theta(t-\tau) * F(\overline{u}, \partial_\xi W)\,\mathrm{d}\tau, \tag{18}$$

on the Banach space $X := C([0,T]; H^2)$ with norm $\|W\|_X := \sup_{t\in[0,T]} \|W(t)\|_{H^2}$. Then we show that $\mathcal{G}$ is a contraction mapping on a closed convex subset S_R of X, where $S_R := \{W \in X; \|W\|_X \le R\}$ for some parameter $R > 0$ which will be determined later.

Due to a Sobolev embedding, $\|W\|_X \le R$ implies that $\|W(t)\|_{W^{1,\infty}} \le R$ for $t \in [0,T]$. Thus, if $\|W\|_X \le R$ and $\ell = 0, 1$, then we compute that

$$\begin{aligned}
&\|\partial_\xi^\ell(\mathcal{G}[W] - \mathcal{G}[V])(t)\|_{L^2} \\
&\quad \le \int_0^t \|\partial_\xi^\ell G^\alpha_\theta(t-\tau) * \{F(\overline{u}, \partial_\xi W) - F(\overline{u}, \partial_\xi V)\}\|_{L^2}\,\mathrm{d}\tau \\
&\quad \le C \int_0^t (t-\tau)^{-\ell/\alpha} \|\{F(\overline{u}, \partial_\xi W) - F(\overline{u}, \partial_\xi V)\}(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\quad \le C(C(R) + |s|) \int_0^t (t-\tau)^{-\ell/\alpha} \|\partial_\xi(W - V)(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\quad \le C_\ell(R)\ t^{1-\ell/\alpha}\ \|W - V\|_X
\end{aligned}$$

where we used Lemma 3 and the identity

$$\begin{aligned}
F(\overline{u}, \partial_\xi W) - F(\overline{u}, \partial_\xi V) &= f(\overline{u} + \partial_\xi W) - f(\overline{u} + \partial_\xi V) - s\partial_\xi(W - V) \\
&= \int_0^1 \left[f'(\overline{u} + \sigma\partial_\xi W + (1-\sigma)\partial_\xi V)) - s\right] \partial_\xi(W - V)\,\mathrm{d}\sigma.
\end{aligned}$$

Similarly, we can calculate that

$$\begin{aligned}
&\|\partial_\xi^2(\mathcal{G}[W] - \mathcal{G}[V])(t)\|_{L^2} \\
&\quad \le \int_0^t \|\partial_\xi G^\alpha_\theta(t-\tau) * \partial_\xi\{F(\overline{u}, \partial_\xi W) - F(\overline{u}, \partial_\xi V)\}\|_{L^2}\,\mathrm{d}\tau \\
&\quad \le C \int_0^t (t-\tau)^{-1/\alpha} \|\partial_\xi\{F(\overline{u}, \partial_\xi W) - F(\overline{u}, \partial_\xi V)\}(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\quad \le C(C(R) + |s|) \int_0^t (t-\tau)^{-1/\alpha} \|(W - V)(\tau)\|_{H^2}\,\mathrm{d}\tau \\
&\quad \le C_2(R)\ t^{1-1/\alpha}\ \|W - V\|_X.
\end{aligned}$$

Combining the above estimates, we obtain

$$\|\mathcal{G}[W] - \mathcal{G}[V]\|_X \le \{C_0(R)T^{1/\alpha} + C_1(R) + C_2(R)\}T^{1-1/\alpha}\|W - V\|_X.$$

Therefore, letting $T = \min\{1, (2C_*(R))^{-\alpha/(\alpha-1)}\}$, we deduce

$$\|\mathcal{G}[W] - \mathcal{G}[V]\|_X \le \frac{1}{2}\|W - V\|_X, \tag{19}$$

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where $C_*(R) := C_0(R) + C_1(R) + C_2(R)$. On the other hand, letting $V \equiv 0$ in (19), we get

$$\|\mathcal{G}[W]\|_X \le \|\mathcal{G}[0]\|_X + \frac{1}{2}\|W\|_X \le \|W_0\|_{H^2} + \frac{1}{2}\|W\|_X \le M + \frac{1}{2}R,$$

where we used (16) with $\ell = r$. Therefore, choosing $R = 2M$, we obtain $\|\mathcal{G}[W]\|_X \le 2M$.

Finally we discuss the continuity of $\mathcal{G}[W]$ in time t. It follows from the continuity at time 0 and the semigroup property (G4) of G^α_θ. Due to Lemma 4, for $W_0 \in H^\sigma(\mathbb{R})$ with $\sigma \ge 0$, the convergence $\lim_{t\searrow 0} G^\alpha_\theta(\cdot, t) * W_0 = W_0$ in H^σ holds. Moreover, for $t \in [0, T]$ and $s \ge 0$ the identity

$$\begin{aligned}\mathcal{G}[W](s+t) &= G^\alpha_\theta(\cdot, s+t) * W_0(x) - \int_0^{s+t} G^\alpha_\theta(\cdot, s+t-\tau) * F(\overline{u}, \partial_\xi W(\tau))\,\mathrm{d}\tau \\ &= G^\alpha_\theta(\cdot, s) * \left(\mathcal{G}[W](t) - \int_t^{s+t} G^\alpha_\theta(\cdot, t-\tau) * F(\overline{u}, \partial_\xi W(\tau))\,\mathrm{d}\tau\right)\end{aligned}$$

holds, where the last integral converges to zero for $s \to 0$. Thus, for $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ (without loss of generality), we have

$$\begin{aligned}&\mathcal{G}[W](t_1) - \mathcal{G}[W](t_2) = \mathcal{G}[W](t_1) - \mathcal{G}[W]((t_2 - t_1) + t_1) \\ &= \mathcal{G}[W](t_1) - G^\alpha_\theta(\cdot, t_2 - t_1) * \left(\mathcal{G}[W](t_1) - \int_{t_1}^{t_2} G^\alpha_\theta(\cdot, t_1 - \tau) * F(\overline{u}, \partial_\xi W(\tau))\,\mathrm{d}\tau\right).\end{aligned} \tag{20}$$

Therefore, by the fact that $W_0 \in H^2$, $W \in X$ and Lemma 4, we find that the right hand side of (20) tends to zero in H^2 as $t_1 \to t_2$. Hence, we deduce the continuity of $\mathcal{G}[W]$ in t and that $\mathcal{G}[W] \in S_{2M}$ for $W \in S_{2M}$.

Consequently, we conclude that there exist $T = T(M)$ such that $\mathcal{G}$ is a contraction mapping of S_{2M}. This means that the mapping $\mathcal{G}$ admits a unique fixed point W in S_{2M}, such that $W = \mathcal{G}[W]$. Hence the proof of Proposition 1 is complete. □

3. Asymptotic stability of traveling waves

In this section, we consider the asymptotic stability of traveling wave solutions with monotone decreasing profile in (1). To this end we derive the existence of global-in-time solutions for evolution equation (14) and that these perturbations decay. Precisely we prove the following theorem.

Theorem 3. *Suppose that the same assumptions as in Theorem 1 hold. Then the Cauchy problem* (14) *has a unique global solution* $W(\xi, t)$ *satisfying* $W \in C([0,\infty); H^2) \cap C^1([0,\infty); H^1)$ *and*

$$\|W(t)\|^2_{H^2} + C\sum_{\ell=0}^{2}\int_0^t \|W(\tau)\|^2_{\dot{H}^{\alpha/2+\ell}}\,\mathrm{d}\tau - \int_0^t\int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2\,\mathrm{d}\xi\,\mathrm{d}\tau \le \|W_0\|^2_{H^2} \tag{21}$$

for some positive constant C *and for all* $t \ge 0$. *Furthermore, the solution* $W(\xi, t)$ *converges to zero in the sense that*

$$\|W(t)\|_{W^{1,\infty}} \longrightarrow 0 \qquad \textit{for} \quad t \to \infty. \tag{22}$$

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We note that the third integral of the left hand side in (21) is non-negative, since the flux function $f \in C^2$ is convex such that $f'' \geq 0$ and the profile $\overline{u}$ is monotone decreasing, i.e. $\overline{u}' \leq 0$. For the solution W constructed in Theorem 3, it is easy to check that $\partial_\xi W$ satisfies Cauchy problem (13). Consequently we obtain Theorem 1. Global existence will be the consequence of the existence of a Lyapunov functional, which also allows to deduce the asymptotic stability of traveling waves, see also [1, Theorem 4] for the special case $\theta = 2 - \alpha$.

Lemma 5. *Suppose that the same assumptions as in Theorem 1 hold. Let W be a solution to* (14) *satisfying $W \in C([0,T]; H^2)$ for some $T > 0$. Then there exists some positive constant δ_1 independent of T such that if $\sup_{0 \leq t \leq T} \|W(t)\|_{H^2} \leq \delta_1$, the* a-priori *estimate expressed in* (21) *holds for $t \in [0,T]$.*

Proof. We rewrite the first equation of (14),

$$\partial_t W + (f(\overline{u} + \partial_\xi W) - f(\overline{u}) - f'(\overline{u})\partial_\xi W) + (f'(\overline{u}) - s)\partial_\xi W = D_\theta^\alpha W,$$

and test it with W,

$$\frac{1}{2}\partial_t(W^2) + \frac{1}{2}\partial_\xi\{(f'(\overline{u}) - s)\, W^2\} - \frac{1}{2} f''(\overline{u})\overline{u}' W^2 - W D_\theta^\alpha W \\ = -\big(f(\overline{u} + \partial_\xi W) - f(\overline{u}) - f'(\overline{u})\partial_\xi W\big)\, W.$$

Integrating with respect to $\xi \in \mathbb{R}$, we obtain

$$\frac{1}{2}\partial_t \|W\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi + \cos\left(\theta\tfrac{\pi}{2}\right) \|W\|_{\dot{H}^{\alpha/2}}^2 \\ = -\int_{\mathbb{R}}\int_0^1 \int_0^\sigma f''(\overline{u} + \gamma\partial_\xi W)(\partial_\xi W)^2 \,\mathrm{d}\gamma\,\mathrm{d}\sigma W \,\mathrm{d}\xi \\ \leq L(\|\partial_\xi W\|_{L^\infty}) \|W\|_{L^\infty} \|\partial_\xi W\|_{L^2}^2$$

where L is a positive non-decreasing function. Due to a Sobolev embedding and the assumption on W, we deduce $\|W(t)\|_{W^{1,\infty}} \leq \|W(t)\|_{H^2} \leq \delta_1$ for all $t \in [0,T]$. Thus the energy estimate becomes

(23)
$$\frac{1}{2}\partial_t \|W\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi + \cos\left(\theta\tfrac{\pi}{2}\right) \|W\|_{\dot{H}^{\alpha/2}}^2 \leq 2C_{\delta_1} \|W\|_{L^\infty} \|\partial_\xi W\|_{L^2}^2$$

for some positive constant C_{δ_1} depending on δ_1. Note that we keep $\|W\|_{L^\infty}$ for further reference. Here we used that

$$\int_{\mathbb{R}} W D_\theta^\alpha W \,\mathrm{d}\xi = \int_{\mathbb{R}} \psi_\theta^\alpha(k) |\hat{W}(k)|^2 \,\mathrm{d}k = -\cos\left(\theta\tfrac{\pi}{2}\right) \|W\|_{\dot{H}^{\alpha/2}}^2$$

due to Plancherel's theorem and $\mathrm{sgn}(k)|\hat{W}(k)|^2$ being an odd function. Similarly, we multiply the first equation of (13) by U, obtaining

$$\frac{1}{2}\partial_t(U^2) + \partial_\xi\Big\{(f(\overline{u} + U) - f(\overline{u}))U - \int_0^U (f(\overline{u} + \eta) - f(\overline{u}))\,\mathrm{d}\eta - \frac{1}{2} s U^2\Big\} \\ + \overline{u}' \int_0^U (f'(\overline{u} + \eta) - f'(\overline{u}))\,\mathrm{d}\eta - U D_\theta^\alpha U = 0.$$

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Thus, integrating with respect to $\xi \in \mathbb{R}$, we have

(24)
$$\frac{1}{2}\partial_t\|U\|_{L^2}^2 + \cos\big(\theta\tfrac{\pi}{2}\big)\|U\|_{\dot{H}^{\alpha/2}}^2 \le \tfrac{1}{2}\|\overline{u}'\|_{L^\infty} L(\|U\|_{L^\infty})\ \|U\|_{L^2}^2 \le \breve{C}_{\delta_1}\|U\|_{L^2}^2$$

with a positive constant $\breve{C}_{\delta_1}$ depending on δ_1. Next, we differentiate (13), obtaining $\partial_t\partial_\xi U + \partial_\xi^2\{f(\overline{u}+U) - f(\overline{u})\} - s\partial_\xi^2 U = D_\theta^\alpha \partial_\xi U$. Testing this equation by $\partial_\xi U$ yields

$$\begin{aligned}\frac{1}{2}\partial_t(|\partial_\xi U|^2) + \frac{1}{2}\partial_\xi\{(f'(\overline{u}+U) - s)(\partial_\xi U)^2\} - \partial_\xi U D_\theta^\alpha \partial_\xi U \\ = -\frac{1}{2}\partial_\xi f'(\overline{u}+U)\,(\partial_\xi U)^2 - \partial_\xi\big((f'(\overline{u}+U) - f'(\overline{u}))\,\overline{u}'\big)\,\partial_\xi U.\end{aligned}$$

Integrating with respect to $\xi \in \mathbb{R}$, we get

$$\begin{aligned}&\frac{1}{2}\partial_t\|\partial_\xi U\|_{L^2}^2 + \cos\big(\theta\tfrac{\pi}{2}\big)\|\partial_\xi U\|_{\dot{H}^{\alpha/2}}^2 \\ &= -\frac{1}{2}\int_{\mathbb{R}} \partial_\xi f'(\overline{u}+U)\,(\partial_\xi U)^2\,\mathrm{d}\xi - \int_{\mathbb{R}} \partial_\xi\big((f'(\overline{u}+U) - f'(\overline{u}))\,\overline{u}'\big)\,\partial_\xi U\,\mathrm{d}\xi,\end{aligned}$$

and hence

$$\frac{1}{2}\partial_t\|\partial_\xi U\|_{L^2}^2 + \cos\big(\theta\tfrac{\pi}{2}\big)\|\partial_\xi U\|_{\dot{H}^{\alpha/2}}^2 \le \widetilde{C}_{\delta_1}\big(\|U\|_{H^1}^2 + \|\partial_\xi U\|_{L^3}^3\big), \tag{25}$$

where $\widetilde{C}_{\delta_1}$ is a positive constant depending on δ_1.

By combining (23), (24) and (25), we construct the good energy estimate. For this purpose, we prepare some useful interpolation inequalities. For $0 \le \sigma \le 2$ and $\varepsilon > 0$, we obtain

$$\|v\|_{\dot{H}^1}^2 \le \varepsilon^{\sigma-2}\|v\|_{\dot{H}^{\sigma/2}}^2 + \varepsilon^\sigma\|v\|_{\dot{H}^{\sigma/2+1}}^2. \tag{26}$$

The inequality (26) is proved as follows. For arbitrary constants $\varepsilon > 0$ and $k \in \mathbb{R}$, we put $h = \varepsilon k$. Then, by the fact that $h^2 \le |h|^\sigma + |h|^{2+\sigma}$ for all $h \in \mathbb{R}$ and $0 \le \sigma \le 2$, we obtain $k^2 \le \varepsilon^{\sigma-2}|k|^\sigma + \varepsilon^\sigma|k|^{2+\sigma}$. Thus, by using this inequality and Plancherel's theorem, we arrive at (26). On the other hand, for $\sigma > 1/4$, we have

$$\|v\|_{L^3}^3 \le C_0\|v\|_{L^2}\|v\|_{H^\sigma}^2 \le 2^\sigma C_0\|v\|_{L^2}(\|v\|_{L^2}^2 + \|v\|_{\dot{H}^\sigma}^2), \tag{27}$$

where C_0 is a certain positive constant. The first interpolation inequality of (27) is a generalization of the celebrated Gagliardo-Nirenberg inequalities (see e.g. [15]) to Sobolev spaces with fractional order, which was proven by Amann [5, Proposition 4.1]. The second inequality holds as a consequence of $(1 + |k|^2)^\sigma \le 2^{2\sigma}(1 + |k|^{2\sigma})$ for all $k \in \mathbb{R}$.

We multiply (24) by γ_1 and combine the resultant inequality with (23), obtaining

$$\begin{aligned}&\frac{1}{2}\partial_t(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2) - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2\,\mathrm{d}\xi \\ &+ \cos\big(\theta\tfrac{\pi}{2}\big)(\|W\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1\|U\|_{\dot{H}^{\alpha/2}}^2) \\ \le\ &\gamma_1\breve{C}_{\delta_1}\|U\|_{L^2}^2 + 2C_{\delta_1}\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2,\end{aligned}$$

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where γ_1 is a positive constant to be determined later. By the fact that $\partial_\xi W = U$, we can apply (26) with $v = W$ and $\sigma = \alpha$ to the above inequality, and get

$$\begin{aligned}&\frac{1}{2}\partial_t(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2) - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}'W^2\,\mathrm{d}\xi\\&+\{\cos\left(\theta\tfrac{\pi}{2}\right) - \varepsilon_1^{\alpha-2}\gamma_1\breve{C}_{\delta_1}\}\|W\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1\{\cos\left(\theta\tfrac{\pi}{2}\right) - \varepsilon_1^{\alpha}\breve{C}_{\delta_1}\}\|U\|_{\dot{H}^{\alpha/2}}^2\\&\le 2C_{\delta_1}\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2.\end{aligned}$$

Therefore, we choose ε_1 satisfying $4\varepsilon_1^{\alpha}\breve{C}_{\delta_1} = \cos(\theta\pi/2)$, and $\gamma_1 = \varepsilon_1^2$ to get

$$\begin{aligned}&\frac{1}{2}\partial_t(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2) - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}'W^2\,\mathrm{d}\xi\\&+\frac{3}{4}\cos\left(\theta\tfrac{\pi}{2}\right)(\|W\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1\|U\|_{\dot{H}^{\alpha/2}}^2)\\&\le 2C_{\delta_1}\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2.\end{aligned}\tag{28}$$

Similarly we multiply (25) by γ_2 and combine the resultant inequality with (28). Furthermore, applying (26) to the resultant inequality, we have

$$\begin{aligned}&\frac{1}{2}\partial_t(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2 + \gamma_2\|\partial_\xi U\|_{L^2}^2) - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}'W^2\,\mathrm{d}\xi\\&+\{\frac{3}{4}\cos\left(\theta\tfrac{\pi}{2}\right) - \varepsilon_2^{\alpha-2}\gamma_2\widetilde{C}_{\delta_1}\}\|W\|_{\dot{H}^{\alpha/2}}^2\\&+\{\frac{3}{4}\gamma_1\cos\left(\theta\tfrac{\pi}{2}\right) - (1+\varepsilon_2^{-2})\varepsilon_2^{\alpha}\gamma_2\widetilde{C}_{\delta_1}\}\|U\|_{\dot{H}^{\alpha/2}}^2\\&+\gamma_2\{\cos\left(\theta\tfrac{\pi}{2}\right) - \varepsilon_2^{\alpha}\widetilde{C}_{\delta_1}\}\|\partial_\xi U\|_{\dot{H}^{\alpha/2}}^2\\&\le 2C_{\delta_1}\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 + \gamma_2\widetilde{C}_{\delta_1}\|\partial_\xi U\|_{L^3}^3.\end{aligned}$$

Then, choosing ε_2 such that $4\varepsilon_2^{\alpha}\widetilde{C}_{\delta_1} = \cos(\theta\pi/2)$, and $\gamma_2 = \min\{\varepsilon_2^2, \gamma_1(1+\varepsilon_2^{-2})^{-1}\}$, yields

$$\begin{aligned}&\frac{1}{2}\partial_t(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2 + \gamma_2\|\partial_\xi U\|_{L^2}^2) - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}'W^2\,\mathrm{d}\xi\\&+\frac{1}{2}\cos\left(\theta\tfrac{\pi}{2}\right)(\|W\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1\|U\|_{\dot{H}^{\alpha/2}}^2 + \gamma_2\|\partial_\xi U\|_{\dot{H}^{\alpha/2}}^2)\\&\le 2C_{\delta_1}\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 + \gamma_2\widetilde{C}_{\delta_1}\|\partial_\xi U\|_{L^3}^3.\end{aligned}\tag{29}$$

We introduce the energy and dissipation norms as follows.

$$\begin{aligned}E(t)^2 &:= \sup_{0\le\tau\le t}(\|W(\tau)\|_{L^2}^2 + \gamma_1\|U(\tau)\|_{L^2}^2 + \gamma_2\|\partial_\xi U(\tau)\|_{L^2}^2),\\D(t)^2 &:= \int_0^t(\|W(\tau)\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1\|U(\tau)\|_{\dot{H}^{\alpha/2}}^2 + \gamma_2\|\partial_\xi U(\tau)\|_{\dot{H}^{\alpha/2}}^2)\,\mathrm{d}\tau.\end{aligned}$$

Then, integrating (29) with respect to t, we have

$$\begin{aligned}\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2 + \gamma_2\|\partial_\xi U\|_{L^2}^2 + \cos\left(\theta\tfrac{\pi}{2}\right)D(t)^2 - \int_0^t\int_{\mathbb{R}} f''(\overline{u})\overline{u}'W^2\,\mathrm{d}\xi\,\mathrm{d}\tau\\\le E_0^2 + \int_0^t\left(4C_{\delta_1}\|W\|_{L^\infty}\|U\|_{L^2}^2 + 2\gamma_2\widetilde{C}_{\delta_1}\|\partial_\xi U\|_{L^3}^3\right)\mathrm{d}\tau,\end{aligned}$$

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where we define $E_0^2 := \|W_0\|_{L^2}^2 + \gamma_1 \|U_0\|_{L^2}^2 + \gamma_2 \|\partial_\xi U_0\|_{L^2}^2$. Thus, by employing (26), and (27) with $v = \partial_\xi U$ and $\sigma = \alpha/2$, we arrive at

$$E(t)^2 + \cos\left(\theta \tfrac{\pi}{2}\right) D(t)^2 - \int_0^t \int_{\mathbb{R}} f''(\overline{u}) \overline{u}' W^2 \, \mathrm{d}\xi \, \mathrm{d}\tau \le E_0^2 + C E(t) D(t)^2$$

for some positive constant C. Finally, using the fact that $E(t) \le \delta_1^2 C$, we arrive at the desired a-priori estimate. □

Proof of Theorem 3. The existence of global-in-time solutions to the initial value problem (14) can be obtained by the continuation argument based on a local existence result in Proposition 1 combined with the *a-priori* estimate in Lemma 5. Because the argument is standard, we may omit the details here. In the rest of this proof, we prove only the asymptotic stability result (22).

To this end, we prepare the following interpolation inequality. For $0 \le \sigma \le 2$, we have

$$\|v\|_{\dot{H}^\sigma} \le 2(\|v\|_{\dot{H}^{\sigma/2}} + \|v\|_{\dot{H}^{\sigma/2+1}}),$$

by using the fact that $k^{2\sigma} \le 2(|k|^\sigma + |k|^{2+\sigma})$. By virtue of this interpolation inequality, (26), and the first equation of (13), we have

$$\begin{aligned}\|\partial_t U\|_{L^2} &\le \|D_\theta^\alpha U\|_{L^2} + \|\{f'(\overline{u}+U) - f'(\overline{u})\}\overline{u}'\|_{L^2} + \|\{f'(\overline{u}+U) - s\}\partial_\xi U\|_{L^2} \\ &\le \|U\|_{\dot{H}^\alpha} + C\|U\|_{H^1} \le C \sum_{\ell=0}^{2} \|W\|_{\dot{H}^{\alpha/2+\ell}}.\end{aligned}$$

Thus, by the above estimate, we compute that

$$\left|\partial_t \|U\|_{L^2}^2\right| \le \|U\|_{L^2}^2 + \|\partial_t U\|_{L^2}^2 \le C \sum_{\ell=0}^{2} \|W\|_{\dot{H}^{\alpha/2+\ell}}^2.$$

This estimate and (26) with (21) tell us that $\|U(\cdot)\|_{L^2}^2 \in W^{1,1}(0,\infty)$, and hence $\|U(t)\|_{L^2} \to 0$ as $t \to \infty$. Finally, employing the Sobolev inequality that $\|v\|_{L^\infty} \le \sqrt{2}\|v\|_{L^2}^{1/2} \|\partial_\xi v\|_{L^2}^{1/2}$, we arrive at the desired result. □

4. Convergence rate toward traveling waves

We consider the convergence rate of the solution toward the corresponding traveling waves. Kawashima, Nishibata and Nishikawa [19] proposed an L^p energy method to study the asymptotic stability and the associated convergence rates of planar viscous rarefaction waves of multi-dimensional viscous conservation laws. When the authors obtain the convergence estimate, they derived the L^1 estimate by using the energy method associated with the sign function. This approach is useful. It is however difficult to apply this method because of a Riesz-Feller operator. To overcome this difficulty, we employ not only the energy method but also the representation of the mild solution. Precisely, our purpose in this section is to derive the following theorem.

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Theorem 4. *Suppose that the same assumptions as in Theorem 1 and* $f \in C^\infty(\mathbb{R})$ *hold. Then the Cauchy problem* (14) *with* $W_0 \in W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ *has a unique global solution* $W(\xi, t)$ *satisfying*

$$W \in C([0,\infty); W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R})) \cap L^\infty(0,\infty; W^{1,\infty}(\mathbb{R}))$$

with estimates (37) *and* (38). *Moreover, there exists a positive constant* δ_1 *such that if* $\|W_0\|_{W^{1,1}} \leq \delta_1$ *then*

$$\|W(t)\|_{H^1} \leq C E_1 \ (1+t)^{-1/(2\alpha)} \tag{30}$$

for $t \geq 0$, *where* $E_1 := \|W_0\|_{H^1} + \|W_0\|_{W^{1,1}}$ *and* C *is a certain positive constant independent of* t.

The proof of the existence of global-in-time solutions is based on results for the Cauchy problem (1) with fractional Laplacian [11] and its extension to the Cauchy problem (1) with Riesz-Feller operators [2]. There the assumption $f \in C^\infty(\mathbb{R})$ is made to simplify the presentation. The method is applicable also in case of $f \in C^k(\mathbb{R})$, $k \geq 2$, but yields a lower regularity for the unique solution u.

Lemma 6. *Suppose that* $f \in C^\infty(\mathbb{R})$ *and* $W_0 \in W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. *Then Cauchy problem* (14) *has a unique mild solution* $W \in C([0,T]; W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R})) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$ *for any* $T > 0$ *with*

$$\|W(t)\|_{L^1} \leq \|W_0\|_{L^1} + L(\sup_{\tau\in[0,t]} \|\partial_\xi W(\tau)\|_{L^\infty}) \|\partial_\xi W_0\|_{L^1} \ t \ , \tag{31}$$

$$\|\partial_\xi W(t)\|_{L^1} \leq \|\partial_\xi W_0\|_{L^1} \ , \tag{32}$$

$$\|W(t)\|_{L^\infty} \leq \|\partial_\xi W_0\|_{L^1} \ , \tag{33}$$

$$\|\partial_\xi W(t)\|_{L^\infty} \leq \|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty} \ , \tag{34}$$

for $0 \leq t \leq T$, *where* L *is a positive non-decreasing function. Moreover, for any positive time* $t_0 > 0$, $W \in C_b^\infty(\mathbb{R} \times (t_0,\infty))$ *and it is a classical solution of the first equation of* (14).

Proof. We use again $U = \partial_\xi W$ and analyze the Cauchy problem (13) with initial datum $U_0 := \partial_\xi W_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ first. We recall $U = u - \overline{u}$ where u and $\overline{u}$ solve equation (12), and $\overline{u}$ is a monotone decreasing function satisfying $\lim_{\xi\to\pm\infty} \overline{u}(\xi) = u_\pm$. Thus, $u_0 := U_0 + \overline{u}$ is essentially bounded. Due to [11, Theorem 1] and its extension to equations with Riesz-Feller operators in [2], the Cauchy problem for (12) with initial datum $u_0 \in L^\infty(\mathbb{R})$ has a (unique) solution which satisfies $\|u(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$ for all $t \geq 0$; in fact, the solution u takes values between the essential lower and upper bounds of u_0. Therefore, $U(t) = u(t) - \overline{u} \in L^\infty(\mathbb{R}_\xi)$ for all $t \geq 0$ and estimate (34) follows.

Due to [11, Remark 1.2] and its extension to equations with Riesz-Feller operators, equation (12) supports an L^1 contraction principle: If u_0, $v_0 \in L^\infty(\mathbb{R})$ satisfy $u_0 - v_0 \in L^1(\mathbb{R})$, then the associated solutions u and v of the Cauchy problem for (12) satisfy $\|u(t) - v(t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}$ for all $t \geq 0$. Therefore, $U(t) = u(t) - \overline{u} \in L^1(\mathbb{R}_\xi)$ with $\|U(t)\|_{L^1} \leq \|u_0 - \overline{u}\|_{L^1} = \|U_0\|_{L^1}$ for all $t \geq 0$, which implies estimate (32). Moreover, its primitive $W(t) \in L^\infty(\mathbb{R}_\xi)$ for all $t \geq 0$, since

$$\|W(t)\|_{L^\infty} = \left\| \int_{-\infty}^{\xi} \partial_y W(y,t) \, \mathrm{d}y \right\|_{L^\infty} \leq \int_{-\infty}^{\infty} |\partial_y W(y,t)| \, \mathrm{d}y = \|\partial_\xi W(t)\|_{L^1} \ .$$

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Then, we are left to prove that $W(t) \in L^1(\mathbb{R}_\xi)$ for all $t \geq 0$ and the stated continuity in time. Considering the mild formulation (17), we obtain the estimate

$$\begin{aligned}\|W(t)\|_{L^1} &\leq \|G_\theta^\alpha(t) * W_0\|_{L^1} + \int_0^t \|G_\theta^\alpha(t-\tau) * \{f(\overline{u}+U) - f(\overline{u}) - sU\}\|_{L^1} \,\mathrm{d}\tau \\ &\leq \|W_0\|_{L^1} + \int_0^t \|f(\overline{u}+U) - f(\overline{u}) - sU\|_{L^1} \,\mathrm{d}\tau \\ &\leq \|W_0\|_{L^1} + \int_0^t \left(\widetilde{L}(\|U(\tau)\|_{L^\infty})\ \|U(\tau)\|_{L^1}\right) \mathrm{d}\tau \\ &\leq \|W_0\|_{L^1} + \widetilde{L}(\|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty})\ \|U_0\|_{L^1}\ t\ , \end{aligned} \tag{35}$$

for $t \geq 0$ by using the local Lipschitz continuity of f and the previous estimates on $U = \partial_\xi W$; again, $\widetilde{L}$ is a positive non-decreasing function. Moreover, for any positive time $t_0 > 0$, $U \in C_b^\infty(\mathbb{R}\times(t_0,\infty))$ and $U = \partial_\xi W$ satisfies the first equation of (13) in the classical sense, see [11, 1]. Due to integrability of U, also W is a global-in-time solution of (14), and $W \in C_b^\infty(\mathbb{R}\times(t_0,\infty))$ is a classical solution of the first equation of (14) for all $t \geq t_0 > 0$.

To prove that $W \in C([0,T]; W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}))$, we will use the mild formulation

$$W(t) = G_\theta^\alpha(t) * W_0 - \int_0^t G_\theta^\alpha(t-\tau) * F(\overline{u}, \partial_\xi W) \,\mathrm{d}\tau, \tag{36}$$

where $F(\overline{u}, \partial_\xi W) := f(\overline{u} + \partial_\xi W) - f(\overline{u}) - s\partial_\xi W$. The first summand on the right hand side satisfies $G_\theta^\alpha(\cdot) * W_0 \in C([0,T]; W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}))$, due to the assumptions on W_0 and the strong continuity of the semigroup in Lemma 2. To prove continuity of the second summand,

$$\mathcal{G}_2[W](t) := \int_0^t G_\theta^\alpha(t-\tau) * F(\overline{u}, \partial_\xi W) \,\mathrm{d}\tau\ ,$$

we use the estimates (31)–(34) and the strong continuity of the semigroup in Lemma 2. In particular, we assume w.l.o.g. $0 < t_1 < t_2$ and rewrite

$$\begin{aligned}&\mathcal{G}_2[W](t_1) - \mathcal{G}_2[W](t_2) \\ &= \int_0^{t_1} (G_\theta^\alpha(t_1-\tau) - G_\theta^\alpha(t_2-\tau)) * F(\overline{u}, \partial_\xi W) \,\mathrm{d}\tau \\ &\quad + \int_{t_1}^{t_2} G_\theta^\alpha(t_2-\tau) * F(\overline{u}, \partial_\xi W) \,\mathrm{d}\tau \\ &= \int_0^{t_1} \left[G_\theta^\alpha(t_1-\tau) * F(\overline{u}, \partial_\xi W) - G_\theta^\alpha(t_2-t_1) * \left(G_\theta^\alpha(t_1-\tau) * F(\overline{u}, \partial_\xi W)\right)\right] \mathrm{d}\tau \\ &\quad + \int_{t_1}^{t_2} G_\theta^\alpha(t_2-\tau) * F(\overline{u}, \partial_\xi W) \,\mathrm{d}\tau\end{aligned}$$

using the semigroup property (G4). The first summand converges to zero as $t_2 \to t_1$ in the $W^{1,p}$-norms, $p = 1, 2$, due to the Dominated Convergence Theorem, the strong continuity of the semigroup in Lemma 2 and that $\int_0^{t_1} \left(G_\theta^\alpha(t_1-\tau) * F(\overline{u}, \partial_\xi W)\right) \mathrm{d}\tau \in W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R})$. Similarly, the second summand converges to zero as $t_2 \to t_1$ in the $W^{1,p}$-norms, $p = 1, 2$, since $G_\theta^\alpha(t_2 - \cdot) * F(\overline{u}, \partial_\xi W) \in L^1((t_2, t_1); W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$. Thus, the

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right hand side of (36) is continuous in time with respect to the $W^{1,p}$-norms, $p = 1, 2$, hence $W \in C([0,T]; W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}))$. Finally, $W \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$ follows from the estimates (33)-(34). ☐

Next we prove the following *a-priori* estimate obtained by Lemma 6.

Lemma 7. *Suppose that the same assumptions as in Theorem 4 hold. Let $W(\xi,t)$ be a solution to (14) satisfying $W \in C([0,T]; W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R})) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$ for any $T > 0$. Then there exists some positive constants δ_1 independent of T such that if $\|W_0\|_{W^{1,1}} \le \delta_1$, the a-priori estimates*

$$
\begin{aligned}
&\|W(t)\|_{H^1}^2 + C\int_0^t (\|W(\tau)\|_{\dot{H}^{\alpha/2}}^2 + \|W(\tau)\|_{\dot{H}^{\alpha/2+1}}^2)\,\mathrm{d}\tau \\
&- \int_0^t \int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi\,\mathrm{d}\tau \\
\le&\|W_0\|_{H^1}^2 \ ,
\end{aligned} \tag{37}
$$

$$
\|W(t)\|_{W^{1,1}} \le C(\|W_0\|_{W^{1,1}} + \|W_0\|_{H^1}^2) \ , \tag{38}
$$

hold for $t \in [0,T]$, where C is a constant independent of time t.

Proof. Following the proof of Lemma 5, we deduce again estimate (28), i.e.

$$
\begin{aligned}
&\frac{1}{2}\partial_t(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2) - \frac{1}{2}\int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi \\
&+ \frac{3}{4}\cos\left(\theta\tfrac{\pi}{2}\right)(\|W\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1\|U\|_{\dot{H}^{\alpha/2}}^2) \\
\le& L(\|\partial_\xi W\|_{L^\infty})\ \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2
\end{aligned}
$$

for some positive non-decreasing function L. Integrating this inequality with respect to time and using (26), the estimates (33)–(34) as well as the smallness of $\|W_0\|_{W^{1,1}}$, we arrive at (37).

Thus it remains to prove (38). Due to Lemma 6, for all $t_0 > 0$, $W \in C_b^\infty(\mathbb{R} \times (t_0,\infty))$ and it is a classical solution of the first equation of (14). Therefore we can adapt the L^1 energy method introduced by Kawashima, Nishibata and Nishikawa [19]. For a non-negative function $\rho : \mathbb{R} \to \mathbb{R}$ satisfying $\rho \in \mathbb{C}_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1$, the convolution operator $\rho_\delta *$ with $\rho_\delta(x) = \delta^{-1}\rho(x/\delta)$ is a Friedrichs' mollifier. We introduce the functions

$$
s_\delta(x) := (\rho_\delta * \operatorname{sgn})(x) \quad \text{and} \quad S_\delta(x) := \int_0^x s_\delta(\xi)\,\mathrm{d}\xi\,,
$$

in which the signature function $\operatorname{sgn}(x)$ is defined by

$$
\operatorname{sgn}(x) := \begin{cases} -1 & \text{for } x < 0\,, \\ 0 & \text{for } x = 0\,, \\ 1 & \text{for } x > 0\,. \end{cases}
$$

Note that the convergence of $s_\delta(x) \to \operatorname{sgn}(x)$ as $\delta \to 0$ is in the sense of a weak $\star$ convergence in $L^\infty(\mathbb{R})$, respectively, a strong convergence in $L^q_{loc}(\mathbb{R})$, $1 \le q < \infty$. The function $s_\delta(x)$ satisfies $s_\delta'(x) = 2\rho_\delta(x) \ge 0$ and $s_\delta(0) = 0$ by choosing ρ to be an even function. Moreover $S_\delta(x) \to |x|$ converges strongly in $L^1(\mathbb{R})$ as $\delta \to 0$.

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To estimate $\|W(t)\|_{W^{1,1}}$, we recall that $\|U(t)\|_{L^1} \leq \|U_0\|_{L^1}$ for all $t \in [0,T]$, due to estimate (32) in Lemma 6. Next we show that

$$\|W(t)\|_{L^1} \leq C\|W_0\|_{W^{1,1}} + C\|W_0\|_{H^1}^2 \tag{39}$$

for $t \in [0,T]$. We will use estimate (31) for small times $t \leq 1$, and derive (39) for large times $t \geq 1$: We multiply the first equation of (14) by $s_\delta(W) = (\rho_\delta * \operatorname{sgn})(W)$ and obtain

$$\partial_t S_\delta(W) + s_\delta(W)\{h(\overline{u}+U) - h(\overline{u})\} = s_\delta(W) D_\theta^\alpha W \ , \tag{40}$$

where $h(v) := f(v) - sv$ is a convex function. We integrate equation (40) over $\mathbb{R} \times [t_0, t]$ and derive

$$\begin{aligned} &\int_{t_0}^t \int_{\mathbb{R}} \partial_t S_\delta(W) \,\mathrm{d}x \,\mathrm{d}\tau + \int_{t_0}^t \int_{\mathbb{R}} s_\delta(W)\{h(\overline{u}+U) - h(\overline{u})\} \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_{t_0}^t \int_{\mathbb{R}} s_\delta(W) D_\theta^\alpha W \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{41}$$

The first integral satisfies, due to Fubini's theorem and the strong convergence of S_δ in L^1,

$$\begin{aligned} \int_{t_0}^t \int_{\mathbb{R}} \partial_t S_\delta(W) \,\mathrm{d}x \,\mathrm{d}\tau &= \int_{\mathbb{R}} \{S_\delta(W(x,t)) - S_\delta(W(x,t_0))\} \,\mathrm{d}x \\ \to& \|W(t)\|_{L^1} - \|W(t_0)\|_{L^1} \end{aligned} \tag{42}$$

as $\delta \to 0$. Next, we prove that the integral on the right-hand side of (41) is non-positive,

$$\int_{t_0}^t \int_{\mathbb{R}} s_\delta(W) D_\theta^\alpha [W] \,\mathrm{d}x \,\mathrm{d}\tau \leq 0. \tag{43}$$

Indeed, $S_\delta \in C^2(\mathbb{R})$ is a convex function with $S_\delta' = s_\delta$ and $S_\delta'' = s_\delta' = 2\rho_\delta \geq 0$. Moreover, under our assumptions, $W(\cdot,t) \in H^1(\mathbb{R})$ for $t \geq 0$ and $W \in C_b^\infty(\mathbb{R} \times (t_0,\infty))$ for $t_0 > 0$. Thus, $\lim_{\xi \to \pm\infty} W(\xi,t) = 0$ and $S_\delta(W) \in C_b^2$ with

$$s_\delta(W)\, D_\theta^\alpha[W] = S_\delta'(W)\, D_\theta^\alpha[W] \leq D_\theta^\alpha[S_\delta(W)] \, ,$$

due to Lemma 8. Consequently,

$$\int_{\mathbb{R}} s_\delta(W)\, D_\theta^\alpha[W] \,\mathrm{d}x \leq \int_{\mathbb{R}} D_\theta^\alpha[S_\delta(W)] \,\mathrm{d}x = 0 \, ,$$

due to Proposition 3. We estimate the second term on the left-hand side of (41) as follows. Using the fact that $|s_\delta(W)| \leq 1$ and $h(\overline{u}+U) - h(\overline{u}) = h'(\overline{u})U + O(|U|^2)$, we have

$$\int_{\mathbb{R}} s_\delta(W)\{h(\overline{u}+U) - h(\overline{u})\} \,\mathrm{d}\xi = \int_{\mathbb{R}} s_\delta(W) h'(\overline{u}) U \,\mathrm{d}\xi + R$$

with $|R| \leq L(\|U\|_{L^\infty})\ \|U\|_{L^2}^2/2$. Furthermore, we compute from the fact $U = \partial_\xi W$ that

$$\int_{\mathbb{R}} s_\delta(W) h'(\overline{u}) U \,\mathrm{d}\xi = -\int_{\mathbb{R}} S_\delta(W) h''(\overline{u}) \overline{u}' \,\mathrm{d}\xi \geq 0,$$

since the function S_δ is non-negative with $S_\delta(0) = 0$, $h \in C^2(\mathbb{R})$ is a convex function, and $\overline{u}$ is a monotone decreasing traveling wave profile. Therefore,

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employing the previous estimates and taking the limit $\delta \to 0$ in equation (41) yields

$$\begin{aligned} \|W(t)\|_{L^1} &\le \|W(t_0)\|_{L^1} + L(\|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty}) \int_{t_0}^t \|U(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\ &\le \|W(t_0)\|_{L^1} + C\|W_0\|_{H^1}^2 \end{aligned} \tag{44}$$

for $t \ge t_0 > 0$ and some positive constant C; here we used (37) and (26). The estimate (44) is valid for an arbitrary positive constant t_0. Thus we can estimate from (44) and (35) that

$$\|W(t)\|_{L^1} \le \|W(1)\|_{L^1} + C\|W_0\|_{H^1}^2 \le \|W_0\|_{L^1} + C\|U_0\|_{L^1} + C\|W_0\|_{H^1}^2$$

for $t \ge 1$. Eventually, combining this estimate and (35) again, we arrive at the desired estimate (39). □

Proof of Theorem 4. The existence of the global solution follows from Lemma 6 and the a-priori estimates in Lemma 7. We derive just the decay estimate (30). To this end, we first introduce the following Nash inequality:

$$\|v\|_{L^2}^{2(1+2\sigma)} \le C_\sigma \|v\|_{L^1}^{4\sigma} \|v\|_{\dot{H}^\sigma}^2 \tag{45}$$

for $\sigma > 0$ and $v \in L^1(\mathbb{R}) \cap H^\sigma(\mathbb{R})$, where C_σ is a positive constant which depends on σ. Following the proof of Lemma 5, we deduce again estimate (28). Multiplying this inequality with $(1+\tau)^\beta$ for $\beta \in \mathbb{R}$ and integrating over $\tau \in [0, t]$, we obtain

$$\begin{aligned} &\mathcal{E}_\beta(t)^2 - \int_0^t (1+\tau)^\beta \int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi \,\mathrm{d}\tau + \frac{3}{2}\cos\left(\theta\tfrac{\pi}{2}\right) \int_0^t \mathcal{D}_\beta(\tau)^2 \,\mathrm{d}\tau \\ &\quad \le \|W_0\|_{L^2}^2 + \gamma_1 \|U_0\|_{L^2}^2 + \beta \int_0^t \mathcal{E}_{\beta-1}(\tau)^2 \,\mathrm{d}\tau \\ &\qquad + L(\|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty}) \int_0^t (1+\tau)^\beta \|W\|_{L^\infty} \|\partial_\xi W\|_{L^2}^2 \,\mathrm{d}\tau \end{aligned}$$

where $\mathcal{E}_\beta(t)^2 := (1+t)^\beta (\|W(t)\|_{L^2}^2 + \gamma_1 \|U(t)\|_{L^2}^2)$, and

$$\mathcal{D}_\beta(t)^2 := (1+t)^\beta (\|W(t)\|_{\dot{H}^{\alpha/2}}^2 + \gamma_1 \|U(t)\|_{\dot{H}^{\alpha/2}}^2).$$

We compute via Nash's inequality (45) with $\sigma = \alpha/2$ and Young's inequality that

$$\begin{aligned} (1+t)^{\beta-1} \|v\|_{L^2}^2 &\le C(1+t)^{\beta-1} \|v\|_{\dot{H}^{\alpha/2}}^{\frac{2}{1+\alpha}} \|v\|_{L^1}^{\frac{2\alpha}{1+\alpha}} \\ &= C\{(1+t)^\beta \|v\|_{\dot{H}^{\alpha/2}}^2\}^{\frac{1}{1+\alpha}} \{(1+t)^{\beta - \frac{1+\alpha}{\alpha}} \|v\|_{L^1}^2\}^{\frac{\alpha}{1+\alpha}} \\ &\le \epsilon (1+t)^\beta \|v\|_{\dot{H}^{\alpha/2}}^2 + C_\epsilon (1+t)^{\beta - \frac{1+\alpha}{\alpha}} \|v\|_{L^1}^2 \ , \end{aligned}$$

for all $\epsilon > 0$ and some positive constant C_ϵ. Thus we get $\mathcal{E}_{\beta-1}(t)^2 \le \epsilon \mathcal{D}_\beta(t)^2 + C_\epsilon (1+t)^{\beta - \frac{1+\alpha}{\alpha}} (\|W\|_{L^1}^2 + \gamma_1 \|U\|_{L^1}^2)$. Therefore, employing this estimate and

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(38), we obtain

$$
\begin{aligned}
&\mathcal{E}_\beta(t)^2 - \int_0^t (1+\tau)^\beta \int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi\,\mathrm{d}\tau + \left\{\tfrac{3}{2}\cos\left(\theta\tfrac{\pi}{2}\right) - \epsilon\beta\right\}\int_0^t \mathcal{D}_\beta(\tau)^2\,\mathrm{d}\tau \\
\leq& \|W_0\|_{L^2}^2 + \gamma_1\|U_0\|_{L^2}^2 + \beta C_\epsilon \int_0^t (1+\tau)^{\beta-\frac{1+\alpha}{\alpha}} (\|W\|_{L^1}^2 + \gamma_1\|U\|_{L^1}^2)\,\mathrm{d}\tau \\
&+ L(\|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty}) \int_0^t (1+\tau)^\beta \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2\,\mathrm{d}\tau \\
\leq& C\|W_0\|_{H^1}^2 + \beta C_\epsilon (\|W_0\|_{H^1}^2 + \|W_0\|_{W^{1,1}})^2 \int_0^t (1+\tau)^{\beta-\frac{1+\alpha}{\alpha}}\,\mathrm{d}\tau \\
&+ L(\|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty}) \int_0^t (1+\tau)^\beta \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2\,\mathrm{d}\tau.
\end{aligned}
$$

For this inequality, we take β and ϵ which satisfy

$$
\beta - \frac{1+\alpha}{\alpha} > 1, \qquad \frac{3}{2}\cos\left(\theta\tfrac{\pi}{2}\right) - \epsilon\beta > 0,
$$

obtaining

$$
\begin{aligned}
&\mathcal{E}_\beta(t)^2 - \int_0^t (1+\tau)^\beta \int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi\,\mathrm{d}\tau + c\int_0^t \mathcal{D}_\beta(\tau)^2\,\mathrm{d}\tau \\
\leq& C(\|W_0\|_{H^1}^2 + \|W_0\|_{W^{1,1}})^2\ (1+t)^{\beta-\frac{1}{\alpha}} \\
&+ L(\|\partial_\xi W_0\|_{L^\infty} + 2\|\overline{u}\|_{L^\infty}) \int_0^t (1+\tau)^\beta \|W\|_{L^\infty}\|U\|_{L^2}^2\,\mathrm{d}\tau\ ,
\end{aligned}
$$

for some positive constant c. Finally, using (26), the estimates (33)–(34) and the smallness of $\|W_0\|_{W^{1,1}}$, we arrive at

$$
\begin{aligned}
&\mathcal{E}_\beta(t)^2 - \int_0^t (1+\tau)^\beta \int_{\mathbb{R}} f''(\overline{u})\overline{u}' W^2 \,\mathrm{d}\xi\,\mathrm{d}\tau + c\int_0^t \mathcal{D}_\beta(\tau)^2\,\mathrm{d}\tau \\
\leq& C(\|W_0\|_{H^1}^2 + \|W_0\|_{W^{1,1}})^2 (1+t)^{\beta-1/\alpha} \leq CE_1^2\ (1+t)^{\beta-1/\alpha}
\end{aligned}
$$

and the desired estimate (30). □

Appendix A. Riesz-Feller operators

To study the existence of traveling wave solutions with smooth profiles, we need the singular integral representation of Riesz-Feller operators D^α_θ.

Proposition 2 ([3, Proposition 2.3]). *If $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, then for all $v \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$*

$$
\begin{aligned}
D^\alpha_\theta v(x) =& c_1 \int_0^\infty \frac{v(x+\xi) - v(x) - v'(x)\,\xi}{\xi^{1+\alpha}}\,\mathrm{d}\xi \\
&+ c_2 \int_0^\infty \frac{v(x-\xi) - v(x) + v'(x)\,\xi}{\xi^{1+\alpha}}\,\mathrm{d}\xi\,,
\end{aligned} \tag{46}
$$

for some constants $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$.

The singular integral representation (46) for Riesz-Feller operators D^α_θ is well-defined for C^2_b functions such that $D^\alpha_\theta C^2_b(\mathbb{R}) \subset C_b(\mathbb{R})$.

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Proposition 3. *The integral representation* (46) *of* D^α_θ *with* $1<\alpha<2$ *and* $|\theta| \le \min\{\alpha, 2-\alpha\}$ *is well-defined for functions* $v \in C^2_b(\mathbb{R})$ *with*

$$\sup_{x\in\mathbb{R}} |D^\alpha_\theta v(x)| \le \tfrac{1}{2}(c_1+c_2)\|v''\|_{C_b(\mathbb{R})} \frac{M^{2-\alpha}}{2-\alpha} + 2(c_1+c_2)\|v'\|_{C_b(\mathbb{R})} \frac{M^{1-\alpha}}{\alpha-1} < \infty \tag{47}$$

for some positive constant M *and the positive constants* c_1 *and* c_2 *in Proposition 2.*

Moreover, if $v \in C^2_b(\mathbb{R})$ *is a function such that the limits* $\lim_{x\to\pm\infty} v(x)$ *exist, then* $\int_{\mathbb{R}} D^\alpha_\theta v(x)\,\mathrm{d}x = 0$.

Proof. The first statement follows by direct estimates on the extension of Riesz-Feller operators in (46), see [3, Proposition 2.4]. To prove the second statement, we consider the two summands in (46) separately, starting with $\int_0^\infty \frac{v(x+\xi)-v(x)-v'(x)\xi}{\xi^{1+\alpha}}\,\mathrm{d}\xi$ for any $v \in C^2_b(\mathbb{R})$. Like before, we rewrite the integral

$$\begin{aligned}
\int_0^\infty \frac{v(x+\xi)-v(x)-v'(x)\xi}{\xi^{1+\alpha}}\,\mathrm{d}\xi &= \int_0^\infty \frac{1}{\xi^{1+\alpha}} \left[\int_0^1 v'(x+\theta\xi)\,\xi\,\mathrm{d}\theta - v'(x)\xi \right] \mathrm{d}\xi \\
&= \int_0^\infty \frac{1}{\xi^\alpha} \int_0^1 \big[v'(x+\theta\xi) - v'(x)\big]\,\mathrm{d}\theta\,\mathrm{d}\xi \\
&= \int_0^\infty \frac{1}{\xi^\alpha} \partial_x \int_0^1 \big[v(x+\theta\xi) - v(x)\big]\,\mathrm{d}\theta\,\mathrm{d}\xi \\
&= \partial_x \int_0^\infty \frac{1}{\xi^\alpha} \int_0^1 \big[v(x+\theta\xi) - v(x)\big]\,\mathrm{d}\theta\,\mathrm{d}\xi\,,
\end{aligned}$$

where exchanging integration and taking derivatives is possible, since in each step the integrands are absolutely integrable uniformly with respect to x. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty \frac{v(x+\xi)-v(x)-v'(x)\xi}{\xi^{1+\alpha}}\,\mathrm{d}\xi\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \partial_x \int_0^\infty \frac{1}{\xi^\alpha} \int_0^1 \big[v(x+\theta\xi) - v(x)\big]\,\mathrm{d}\theta\,\mathrm{d}\xi\,\mathrm{d}x
\end{aligned}$$

and the primitive satisfies

$$\begin{aligned}
&\lim_{x\to\pm\infty} \int_0^\infty \frac{1}{\xi^\alpha} \int_0^1 \big[v(x+\theta\xi) - v(x)\big]\,\mathrm{d}\theta\,\mathrm{d}\xi \\
&= \int_0^\infty \frac{1}{\xi^\alpha} \int_0^1 \lim_{x\to\pm\infty} \big[v(x+\theta\xi) - v(x)\big]\,\mathrm{d}\theta\,\mathrm{d}\xi = 0\,,
\end{aligned}$$

where exchanging integration and taking limits is possible, since in each step the integrands are absolutely integrable and $\lim_{x\to\pm\infty} \big[v(x+\theta\xi)-v(x)\big] = 0$ due to the assumptions on v. ☐

Using the singular integral representation of D^α_θ and [12, Lemma 1], we deduce the following result:

Lemma 8. *Let* $1<\alpha<2$, $u \in C^2_b(\mathbb{R})$ *and* $\eta \in C^2(\mathbb{R})$ *be a convex function. Then* $\eta'(u)(D^\alpha_\theta u) \le D^\alpha_\theta \eta(u)$.

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Proof. Since η is convex, we have $\eta'(a)(b-a) \leq \eta(b) - \eta(a)$. Hence,

$$\eta'(u(x))(u(x+z) - u(x)) \leq \eta(u(x+z)) - \eta(u(x))$$

and $\eta'(u(x))(u(x+z)-u(x)-u'(x)\cdot z) \leq \eta(u(x+z))-\eta(u(x))-(\eta(u))'(x)\cdot z$. The conclusion follows from these inequalities and Equation (46). □

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CHAPTER 2

Korteweg-de Vries-Burgers equations

TRAVELLING WAVES FOR A NON-LOCAL KORTEWEG-DE VRIES-BURGERS EQUATION

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Abstract. We study travelling wave solutions of a Korteweg-de Vries-Burgers equation with a non-local diffusion term. This model equation arises in the analysis of a shallow water flow by performing formal asymptotic expansions associated to the triple-deck regularisation (which is an extension of classical boundary layer theory). The resulting non-local operator is of fractional type with order between 1 and 2. Travelling wave solutions are typically analysed in relation to shock formation in the full shallow water problem. We show rigorously the existence of these waves. In absence of the dispersive term, the existence of travelling waves and their monotonicity was established previously by two of the authors. In contrast, travelling waves of the non-local KdV-Burgers equation are not in general monotone, as is the case for the corresponding classical (or local) KdV-Burgers equation. This requires a more complicated existence proof compared to the previous work. Moreover, the travelling wave problem for the classical KdV-Burgers equation is usually analysed via a phase-plane analysis, which is not applicable here due to the presence of the non-local diffusion operator. Instead, we apply fractional calculus results available in the literature and a Lyapunov functional. In addition we discuss the monotonicity of the waves in terms of a control parameter and prove their dynamic stability in case they are monotone.

1. Introduction

In this paper we study existence and stability of travelling waves of the following one-dimensional evolution equation:

$$\partial_t u + \partial_x u^2 = \partial_x \mathcal{D}^\alpha u + \tau \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0 \tag{1.1}$$

with $\tau > 0$ and $\mathcal{D}^\alpha$ denotes the non-local operator

$$\mathcal{D}^\alpha u(x) = d_\alpha \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha} dy\,, \quad \text{with} \quad 0 < \alpha < 1\,, \quad d_\alpha = \frac{1}{\Gamma(1-\alpha)} > 0\,, \tag{1.2}$$

where Γ denotes the Gamma function.

Equation (1.1) with $\alpha = 1/3$ and either a quadratic flux, as above, or a cubic one, has been derived from one (quadratic flux) and two (cubic flux) layer shallow water flows, respectively, by performing formal asymptotic expansions associated to the triple-deck (boundary layer) theory used in fluid mechanics (see, e.g. [12] and [19]). In [19] numerical simulations indicate the existence of travelling waves that resemble the inner structure in a very

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particular limit of small amplitude shock waves for the original shallow water problem. In this manuscript we aim to study rigorously the existence and stability of these type of solutions for the quadratic flux.

In [1] travelling waves for (1.1) with $\tau = 0$ were analysed. In this case travelling waves are monotone, as it is the case for the classical (or local) Burgers equation. The existence prove relies on this fact. However, travelling waves are in general non-monotone if τ is larger that certain value $\tau_0 > 0$ in the (local) KdV-Burgers equation, see e.g. [3] (this can be inferred by linearisation of the critical points of the resulting travelling wave equation, an ODE in the local case). Numerical computations performed in [19] and in[12] suggest that we may expect a similar oscillatory behaviour of the travelling waves of (1.1). This has an immediate implication that the present existence proof (with $\tau > 0$) differs significantly from the existence proof in [1] as we shall see below. On the other hand, and in contrast to the classical KdV-Burgers equation, the presence of the non-local operator in (1.1) does not allow to approach the problem using phase-plane analysis of the travelling wave equation, since this becomes a (non-linear) integro-differential equation.

Let us first recall some basic properties of the fractional differential operator $\mathcal{D}^\alpha u$. Since it can be written as the convolution of u' with $\theta(x)x^{-\alpha}/\Gamma(1-\alpha)$ (where θ is the Heaviside function), $\mathcal{D}^\alpha$ is a pseudo-differential operator with symbol

$$\frac{ik\sqrt{2\pi}}{\Gamma(1-\alpha)}\mathcal{F}\left(\frac{\theta(x)}{x^\alpha}\right)(k) = (b_\alpha + ia_\alpha \operatorname{sgn}(k))\,|k|^\alpha\,, \tag{1.3}$$

i.e. $\mathcal{F}(\mathcal{D}^\alpha u)(k) = (b_\alpha + ia_\alpha\, sgn(k))\,|k|^\alpha \hat{u}(k)$ where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}\varphi(k) = \hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx}\varphi(x)dx\,,$$

and the coefficients a_α and b_α are given by

$$a_\alpha = \sin\left(\frac{\alpha\pi}{2}\right) > 0\,, \qquad b_\alpha = \cos\left(\frac{\alpha\pi}{2}\right) > 0\,, \tag{1.4}$$

(we refer to [2] for the details of the computation to obtain (1.3)). The operator on the right-hand side of (1.1) then is a pseudo-differential operator with symbol

$$\mathcal{F}(\partial_x\mathcal{D}^\alpha) = -\,(a_\alpha - ib_\alpha\, sgn(k))\,|k|^{\alpha+1}\,, \tag{1.5}$$

which is dissipative in the sense that the real part of (1.5) is negative.

For $s \in \mathbb{R}$ we shall adopt the following notation for the Sobolev of square integrable functions,

$$H^s := \{u : \|u\|_{H^s} < \infty\}\,, \qquad \|u\|_{H^s} := \|(1+|k|^2)^{s/2}\hat{u}\|_{L^2(\mathbb{R})}\,,$$

and the corresponding homogeneous norm

$$\|u\|_{\dot{H}^s} := \||k|^s\hat{u}\|_{L^2(\mathbb{R})}\,.$$

Using that $(a_\alpha^2 + b_\alpha^2) = 1$ it is easy to see that $\|\mathcal{D}^\alpha u\|_{\dot{H}^s} = \|u\|_{\dot{H}^{s+\alpha}}$, and this suggests that one can interpret $\mathcal{D}^\alpha$ as a differentiation operator of order α. We also observe that $\mathcal{D}^\alpha$ is a bounded linear operator from H^s to $H^{s-\alpha}$.

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We shall also let denote C_b^m with $m \geq 0$, the set of functions, whose derivatives up to order m are continuous and bounded. Then one can also infer that $\mathcal{D}^\alpha u$ is a bounded linear operator from $C_b^1(\mathbb{R})$ to $C_b(\mathbb{R})$. As explained in [1], this can be easily seen by splitting the domain of integration in (1.2) into $(-\infty, x-\delta]$ and $[x-\delta, x]$ for some positive $\delta > 0$. Then integration by parts in the first integral shows the boundedness of $\mathcal{D}^\alpha u$.

It is also known that $\mathcal{D}^\alpha$ can be inverted by multiplying it with $(z-\xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z. Applying this to (1.2) we obtain:

$$\mathcal{I}^\alpha \mathcal{D}^\alpha(u(x)) = u(x) - \lim_{x\to-\infty} u(x)\,, \tag{1.6}$$

with the integral operator

$$\mathcal{I}^\alpha u(x) = d_{1-\alpha}\int_{-\infty}^x \frac{u(y)}{(x-y)^{1-\alpha}}dy \qquad u \in C_b^1(\mathbb{R})\,. \tag{1.7}$$

We shall use this inversion of $\mathcal{D}^\alpha$ in Section 2.

In some instances we shall also need to split the integral operator (1.2) as follows

$$(\mathcal{D}^\alpha u)(x) = d_\alpha \int_{-\infty}^{x_0} \frac{u'(y)}{(x-y)^\alpha}dy + d_\alpha \int_{x_0}^x \frac{u'(y)}{(x-y)^\alpha}dy\,, \quad \text{for some} \quad x_0 < x\,, \tag{1.8}$$

and treat the first term as a known function, whereas the second one can be viewed as a left-sided Caputo derivative, see e.g. [11], and that we denote by $\mathcal{D}_{x_0}^\alpha$, indicating that the integration is from a finite value x_0, i.e. $u \in C_b^1([x_0,\infty))$ and $\alpha \in (0,1]$

$$\mathcal{D}_{x_0}^\alpha u(x) = \mathcal{I}_{x_0}^{1-\alpha} u'(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^x \frac{u'(y)}{(x-y)^\alpha}\,dy\,. \tag{1.9}$$

Notice that the first term in the right-hand side of (1.8), which is a function of x, is not equal to $(\mathcal{D}^\alpha u)(x_0)$, which is a number for fixed x_0.

2. Existence of Travelling Wave Solutions

We introduce the travelling wave variable $\xi = x - ct$ with wave speed c and look for solutions $u(x,t) = \phi(\xi)$ of (1.1) which connect two different far-field real values ϕ_- and ϕ_+. A straightforward calculation shows that if ϕ depends on x and t only through the travelling wave variable, then so does $\mathcal{D}^\alpha\phi$, and so the travelling wave problem becomes

$$-c\phi' + (\phi^2)' = (\mathcal{D}^\alpha\phi)' + \tau\phi'''\,, \tag{2.1}$$

subject to

$$\lim_{\xi\to-\infty}\phi(\xi) = \phi_-\,, \quad \lim_{\xi\to\infty}\phi(\xi) = \phi_+\,. \tag{2.2}$$

Here $'$ denotes differentiation with respect to ξ. We can then integrate (2.1) with respect to ξ and use (2.2) to arrive at the following travelling wave equation:

$$h(\phi) = \mathcal{D}^\alpha\phi + \tau\phi''\,, \quad \text{where} \quad h(\phi) := -c(\phi-\phi_-) + \phi^2 - \phi_-^2\,. \tag{2.3}$$

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If ϕ' decays to zero fast enough as $\xi \to \pm\infty$, then we obtain the Rankine-Hugoniot condition

$$c = \phi_+ + \phi_- \tag{2.4}$$

that we assume throughout. Since $h(\phi)$ is convex, the left hand side of (2.3) is negative between its only zeroes $\phi = \phi_-$ and $\phi = \phi_+$. In what follows we shall show the existence of solutions of (2.3) provided the entropy condition

$$\phi_- > \phi_+ \,, \tag{2.5}$$

is satisfied. We shall not make further assumptions on the far-field values (regarding the sign, for example), but just note (2.4) and (2.5) imply that

$$h'(\phi_-) = \phi_- - \phi_+ > 0 \quad \text{and} \quad h'(\phi_+) = \phi_+ - \phi_- < 0 \,. \tag{2.6}$$

We observe that (2.5) is a necessary condition for existence of the travelling wave if $\alpha = 1$. Their existence for $\tau = 0$ and $\alpha \in (0, 1)$, where this condition is crucial, is shown in [1].

As in [1], we shall start our analysis by proving a 'local' existence result on $(-\infty, \tilde{\xi}]$ with $\tilde{\xi} < 0$ and $|\xi|$ sufficiently large. Global existence will then follow by a continuation argument and global boundedness of solutions. The lack of monotonicity for $\tau > 0$ requires additional investigations in order to show that a travelling wave solution tends to ϕ_+ as $\xi \to \infty$. In order prove this we use that the functional $H(\phi) - H(\phi_-)$, where

$$H(\phi) = \int_0^\phi h(y)dy = -c\frac{\phi^2}{2} + \frac{\phi^3}{3} + A\phi\,, \quad \text{with} \quad A = c\phi_- - \phi_-^2\,, \tag{2.7}$$

is increasing with respect to ξ. This step allows to show that if a travelling wave tends to a constant value as ξ tends to ∞ then that constant must be ϕ_+. Then we show that indeed the solutions of (2.3) satisfying $\phi(-\infty) = \phi_-$ tend to a constant as ξ tends to ∞.

The local existence result is based on linearisation about $\xi = -\infty$ (or, equivalently, $\phi = \phi_-$). As it could be expected for ordinary differential equations, the linearisation about $\phi \equiv \phi_-$,

$$h'(\phi_-)v = \mathcal{D}^\alpha v + \tau v''\,, \tag{2.8}$$

has solutions of the form $v(\xi) = be^{\lambda\xi}$, $b \in \mathbb{R}$, where $\lambda > 0$ is a root of

$$P(z) = \tau z^2 + z^\alpha - h'(\phi_-)\,. \tag{2.9}$$

We observe that there is a unique positive real root of (2.9). Indeed, this follows from the fact that $P(z) \to \infty$ as $z \to \infty$ and

$$P(0) = -h'(\phi_-) < 0\,, \quad P'(z) = 2\tau z + \alpha z^{\alpha-1} \geq 0 \quad \text{for} \quad z \geq 0\,.$$

In Lemma B.1 of Appendix B we show, using Rouche's theorem, that (2.9) has exactly three roots, one positive real one and two complex conjugates with negative real part.

We assume for the moment that the only solutions of (2.8) that decay to 0 as $\xi \to -\infty$ are of the form $be^{\lambda\xi}$ for some constant b and λ being the real root of (2.9). We have not fully succeeded in proving this, however in Appendix A we do it in suitable weighted spaces (see Theorem A.2).

Henceforth, we assume that

$$\mathcal{N}\big(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id}\big) = \mathrm{span}\{e^{\lambda\xi}\} \quad \text{in} \quad H^4(\mathbb{R}) \tag{2.10}$$

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where Id denotes the identity operator.

The main result of this section is the following:

Theorem 2.1. *Let (2.5) and (2.10) hold. Then, there exists a solution $\phi \in C_b^3(\mathbb{R})$ of (2.1)-(2.2) that is unique (up to a shift in ξ) among all $\phi \in \phi_- + H^4((-\infty,0)) \cap C_b^3(\mathbb{R})$.*

We prove Theorem 2.1 in several steps that we write as lemmas. The first one below is a 'local' existence result that says that the nonlinear problem has, up to translations, only two nontrivial solutions, which can be approximated by $\phi_- \pm e^{\lambda\xi}$ for large negative ξ (observe that the shift in ξ gives a positive constant multiplying the exponential and that we have taken equal to 1 without loss of generality).

Lemma 2.1. *[Local existence] Let the assumptions of Theorem 2.1 hold. Then, for every small enough $\varepsilon > 0$, (2.3) has solutions ϕ_{up}, $\phi_{down} \in \phi_- + H^4(I_\varepsilon)$, where $I_\varepsilon = (-\infty, \xi_\varepsilon]$ and $\xi_\varepsilon = \log\varepsilon/\lambda$, such that*

$$\phi_{up}(\xi_\varepsilon) = \phi_- + \varepsilon\,, \qquad \phi_{down}(\xi_\varepsilon) = \phi_- - \varepsilon\,. \tag{2.11}$$

Moreover, these are unique among all functions ϕ satisfying $\|\phi-\phi_-\|_{H^4(I_\varepsilon)} \le \delta$, with δ small enough, but independent of ε. They satisfy, with an ε-independent constant C,

$$\|\phi_{up} - \phi_- - e^{\lambda\xi}\|_{H^4(I_\varepsilon)} \le C\varepsilon^2\,, \qquad \|\phi_{down} - \phi_- + e^{\lambda\xi}\|_{H^4(I_\varepsilon)} \le C\varepsilon^2\,.$$

Proof. We follow the proof of [1]. We only prove existence and uniqueness for ϕ_{down}, the proof of for ϕ_{up} is analogous and we do not do it here.

We start by writing (2.3) and the initial condition (2.11) in terms of the perturbation $\Phi(\xi) = \phi_{down}(\xi) - \phi_- + e^{\lambda\xi}$:

$$(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id})\Phi = h(\phi_- - e^{\lambda\xi} + \Phi) + h'(\phi_-)(e^{\lambda\xi} - \Phi)\,, \quad \Phi(\xi_\varepsilon) = 0\,. \tag{2.12}$$

We then define a fixed-point map by considering the right-hand side of (2.12) as given.

In order to use Fourier methods, we need a smooth enough extension of functions to $\xi \in \mathbb{R}$. Then, in general, for a $f \in H^4(I_\varepsilon)$ we let $\mathcal{E}(f) \in H^4(\mathbb{R})$ denote a smooth extension of f that satisfies

$$\mathcal{E}(f)\Big|_{I_\varepsilon} = f\,, \quad \|\mathcal{E}(f)\|_{H^4(\mathbb{R})} \le \gamma\|f\|_{H^4(I_\varepsilon)}\,.$$

And denote by Φ a bounded solution of

$$(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id})\Phi = \mathcal{E}(f) \quad \text{in} \quad \mathbb{R}\,,$$

then Φ and its derivatives with respect to ξ can be written as

$$\frac{d^m\Phi}{d\xi^m} = \mathcal{F}^{-1}\left[\left(-\tau k^2 + b_\alpha|k|^\alpha - h'(\phi_-) + ia_\alpha \mathrm{sgn}(k)|k|^\alpha\right)^{-1}\mathcal{F}\left(\frac{d^m\mathcal{E}(f)}{d\xi^m}\right)\right] \tag{2.13}$$

for $m = 0,1,2,3,4$. The Fourier symbol in (2.13) is uniformly bounded in k and this implies that there exist constants C_1, $C_2 > 0$ such that

$$\|\Phi|_{I_\varepsilon}\|_{H^4(I_\varepsilon)} \le \|\Phi\|_{H^4(\mathbb{R})} \le C_1\|\mathcal{E}(f)\|_{H^4(\mathbb{R})} \le C_2\|f\|_{H^4(I_\varepsilon)}\,.$$

By the assumption (2.10), the unique solution of

$$(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id})U = f \quad \text{in } I_\varepsilon\,, \quad U(\xi_\varepsilon) = 0\,,$$

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is $U[f](\xi) = \Phi(\xi) - \Phi(\xi_\varepsilon)e^{\lambda(\xi-\xi_\varepsilon)}$. This allows to write (2.12) as the fixed-point problem

$$\bar{\phi}(\xi) = U\left[h(\phi_- - e^{\lambda\xi} + \bar{\phi}(\xi)) + h'(\phi_-)(e^{\lambda\xi} - \bar{\phi}(\xi))\right].$$

The continuous embedding of $H^4(I_\varepsilon)$ in $C_b^3(I_\varepsilon)$ gives

$$\left\|h(\phi_- - e^{\lambda\xi} + \bar{\phi}) + h'(\phi_-)(e^{\lambda\xi} - \bar{\phi})\right\|_{H^4(I_\varepsilon)} \le L\left(\varepsilon^2 + \varepsilon\|\bar{\phi}\|_{H^4(I_\varepsilon)} + \|\bar{\phi}\|^2_{H^4(I_\varepsilon)}\right),$$

where L is a positive non-decreasing function. It is now easily seen that the fixed point map is a contraction in small enough balls (independent of ε) and that maps a ball with radius of $O(\varepsilon^2)$ into itself (see [1] for such similar details). ☐

Lemma 2.2. *[Continuation principle] Let $\phi \in C_b^3((-\infty,\xi_0])$ be a solution of (2.3) as constructed in Lemma 2.1. Then there exists a $\delta > 0$, such that ϕ can be extended uniquely to $C_b^3((-\infty,\xi_0+\delta))$.*

Proof. The idea is to write the integro-differential equation as a system of Caputo-differential equations. We use the definition of the Caputo derivative and the inversion formula for it(1.9). Since $\phi \in C^1([\xi_0,\infty))$ and $\alpha \in (0,1]$ then this allows to write down derivatives of entire order by using that $\mathcal{D}^\alpha_{\xi_0}\mathcal{I}^\alpha_{\xi_0} \equiv \mathrm{Id}$ (cf. [11]). Indeed, we can write

$$\phi'(\xi) = \mathcal{D}^\alpha_{\xi_0}\mathcal{D}^{1-\alpha}_{\xi_0}\,\phi(\xi) = \mathcal{D}^{1-\alpha}_{\xi_0}\mathcal{D}^\alpha_{\xi_0}\,\phi(\xi)\,,$$

hence, also

$$\phi''(x) = \mathcal{D}^{1-\alpha}_{\xi_0}\mathcal{D}^\alpha_{\xi_0}\mathcal{D}^{1-\alpha}_{\xi_0}\mathcal{D}^\alpha_{\xi_0}\,\phi(\xi)\,.$$

We can now express (2.3) as a system:

$$\mathcal{D}^\alpha_{\xi_0}\,\phi = \psi\,, \quad \mathcal{D}^{1-\alpha}_{\xi_0}\,\psi = \theta\,, \quad \mathcal{D}^\alpha_{\xi_0}\,\theta = \chi\,, \tag{2.14}$$

$$\tau\mathcal{D}^{1-\alpha}_{\xi_0}\chi = h(\phi) - \psi - \int_{-\infty}^{\xi_0}\frac{\phi'(y)}{(\xi-y)^\alpha}dy\,. \tag{2.15}$$

The system is locally Lipschitz continuous in $C_b^3(\xi_0,\xi_0+\delta)$. Local existence then follows by using a Picard-Lindelöf type of argument, taking as initial conditions the values of ϕ, $\mathcal{D}^\alpha_{\xi_0}\phi$, ϕ' and $\mathcal{D}^\alpha_{\xi_0}\phi'$ at $\xi = \xi_0$. The well-posedness of linear integro-differential systems of this form is given by Jafari and Daftardar-Gejji [9], so we do not give further details. ☐

It is now clear that boundedness of the solutions will guarantee global existence by applying repeatedly Lemma 2.2 as long as ϕ' remains integrable. First we show that a solution of (2.3) as constructed in lemmas 2.1 and 2.2 is uniformly bounded.

Lemma 2.3 (Uniform boundedness)**.** *Let $\phi \in C_b^3((-\infty,\xi_0])$ be a solution of (2.3) as constructed in Lemma 2.1. Then the solution is bounded for $\xi \in (-\infty,\xi_0)$ by*

$$\bar{\phi} < \phi(\xi) < \phi_-\,, \quad \textit{where} \quad \bar{\phi} = \frac{3\phi_+ - \phi_-}{2} < \phi_+ \tag{2.16}$$

is the second root of

$$\frac{H(\phi) - H(\phi_-)}{\phi - \phi_-} = 0\,.$$

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Proof. We first derive an energy type of estimate for (2.3). This is done, as in the local case, by multiplying the equation by ϕ' and integrating with respect to ξ:

$$H(\phi(\xi)) - H(\phi_-) = \frac{\tau}{2}\left(\phi'(\xi)\right)^2 + \int_{-\infty}^{\xi} \phi'(y)\,\mathcal{D}^\alpha \phi(y)dy\,. \tag{2.17}$$

The first term on the right-hand side of (2.17) is clearly non-negative.

Let us show that the second term is also non-negative.

We first observe that

$$\int_{-\infty}^{\xi} \phi'(y)\,\mathcal{D}^\alpha \phi(y)dy = \frac{1}{2}\int_{-\infty}^{\xi} \phi'(y) \int_{-\infty}^{\xi} \frac{\phi'(x)}{|x-y|^\alpha}\,dx\,dy \tag{2.18}$$

this is shown by noticing that

$$\int_{-\infty}^{\xi} \phi'(y) \int_{y}^{\xi} \frac{\phi'(x)}{(x-y)^\alpha}\,dx\,dy = \int_{-\infty}^{\xi} \phi'(x) \int_{-\infty}^{x} \frac{\phi'(y)}{(x-y)^\alpha}\,dy\,dx\,.$$

Then, we can consider an extension $\phi'_E \in L^2(\mathbb{R})$ of ϕ' to $\mathbb{R}$ so that $\phi'_E(y) = 0$ for $y > \xi$. Then, by applying Theorem 9.8[15] to (2.18) with this extension we obtain that

$$\int_{-\infty}^{\xi} \phi'(y)\,\mathcal{D}^\alpha \phi(y)dy = \frac{1}{2}\int_{\mathbb{R}} \phi'_E(x) \int_{\mathbb{R}} \frac{\phi'_E(y)}{|x-y|^\alpha}\,dy\,dx \geq 0\,. \tag{2.19}$$

Let us now prove the upper bound. Suppose that there exists a $\bar{\xi} < \infty$ such that $\phi(\bar{\xi}) = \phi_-$, then from (2.17) one gets that $\int_{-\infty}^{\bar{\xi}} \phi'(y)\,\mathcal{D}^\alpha \phi(y)dy = 0$, and (2.19) implies that $\phi'(\xi) = 0$ for all $\xi \in (-\infty, \bar{\xi}]$ (see [15]). Assume now that $\lim_{\xi\to\infty} \phi(\xi) = \phi_-$, then $\int_{-\infty}^{\infty} \phi'(y)\,\mathcal{D}^\alpha \phi(y)dy = 0$. But, we can write (2.18) with $\xi = \infty$ and without using an extension of ϕ';

$$\int_{-\infty}^{\xi} \phi'(y)\,\mathcal{D}^\alpha \phi(y)dy = \frac{1}{2}\int_{\mathbb{R}} \phi'(x) \int_{\mathbb{R}} \frac{\phi'(y)}{|x-y|^\alpha}\,dy\,dx = 0\,,$$

thus also $\phi'(\xi) = 0$ for all $\xi \in \mathbb{R}$. Then a non constant solution is always below ϕ_-.

In order to get the lower bound, we use that the right hand side of (2.17) is non-negative, thus

$$H(\phi) - H(\phi_-) = -\frac{c}{2}(\phi^2 - (\phi_-)^2) + \frac{1}{3}(\phi^3 - (\phi_-)^3) + A(\phi - \phi_-) \geq 0\,.$$

Since we have just shown that $\phi - \phi_- < 0$ in $(-\infty, \xi_0]$, we obtain the condition

$$\frac{H(\phi) - H(\phi_-)}{\phi - \phi_-} = -\frac{c}{2}(\phi + \phi_-) + \frac{1}{3}(\phi^2 + \phi\phi_- + (\phi_-)^2) + A < 0$$

and this implies (2.16). ☐

Lemma 2.4. *(Global uniqueness) Let $\phi \in \phi_- + H^4((-\infty, \xi_0))$ be a solution of (2.3). Then, up to a shift in ξ, ϕ is the continuation of either ϕ_{up} or ϕ_{down}, otherwise $\phi \equiv \phi_-$.*

Proof. For every $\delta > 0$ there exists a $\xi^* \leq \xi_0$, such that $\|\phi - \phi_-\|_{H^4((-\infty,\xi^*))} < \delta$ and, therefore, by Sobolev embedding, also $|\phi(\xi^*) - \phi_-| < \delta$. Choosing δ small enough, there are only two possibilities, either $\phi(\xi^*) = \phi_-$ (implying

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that $\phi \equiv \phi_-$) or $\phi(\xi^*) \neq \phi_-$. Whence, by local uniqueness, ϕ is, up to a shift, either equal to ϕ_{up} or ϕ_{down}, depending on the sign of $\phi(\xi^*) - \phi_-$. □

It remains to analyse the far-field behaviour.

Lemma 2.5. *Let $\phi \in \phi_- + H^4((-\infty, \xi_0))$ be a continuation of ϕ_{down} as in Lemma 2.4. Suppose that*

$$\lim_{\xi\to\infty} \phi = \phi_0 \in \mathbb{R}\,, \tag{2.20}$$

then $\phi_0 = \phi_+$.

Proof. We argue by contradiction. Assume that (2.20) holds with $\phi_0 \neq \phi_+$, then $h(\phi(\xi)) \to h(\phi_0) \neq 0$. Suppose first that for $\xi > \xi_0$, $h(\phi(\xi)) > C_+ > 0$, then applying the integral operator (1.7) to $h(\phi)$ we get

$$\begin{aligned} d_{1-\alpha}^{-1}\mathcal{I}^\alpha h(\phi(\xi)) &> \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + C_+\int_{\xi_0}^{\xi}\frac{dy}{(\xi-y)^{1-\alpha}} \\ &= \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + \frac{C_+}{\alpha}(\xi-\xi_0)^\alpha \to \infty \quad \text{as} \quad \xi\to\infty\,. \end{aligned}$$

This and (2.3) imply that $\mathcal{I}^\alpha\phi'' \to \infty$ as $\xi \to \infty$. Similarly, if for all $\xi > \xi_0$ we have $h(\phi(\xi)) < C_- < 0$, we obtain

$$\begin{aligned} d_{1-\alpha}^{-1}\mathcal{I}^\alpha h(\phi) &< \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + C_-\int_{\xi_0}^{\xi}\frac{dy}{(\xi-y)^{1-\alpha}} \\ &= \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + C_-(\xi-\xi_0)^\alpha \to -\infty \quad \text{as} \quad \xi\to\infty \end{aligned}$$

as before, this implies that $\mathcal{I}^\alpha\phi'' \to -\infty$ as $\xi \to \infty$. In both cases and using (2.3) we obtain that $|\mathcal{I}^\alpha\phi''|$ is unbounded as well. Let us see that this contradicts (2.20).

Since $\phi \in C_b^3(\mathbb{R})$ by Lemma 2.2 and (2.20) holds, we can take for any $\varepsilon > 0$ and ξ large enough, $\xi^* = \xi - \delta$ for a fixed $\delta > 0$ such that $|\phi''(\xi)| < \varepsilon$ for all $\xi > \xi^*$. Then

$$\left|\int_{\xi^*}^{\xi} \frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy\right| < \varepsilon(\xi-\xi^*)^\alpha = \varepsilon\delta^\alpha\,. \tag{2.21}$$

Now if we write

(2.22)

$$d_{1-\alpha}^{-1}\mathcal{I}^\alpha\phi'' = \int_{-\infty}^{\xi}\frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy = \int_{-\infty}^{\xi^*}\frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy + \int_{\xi^*}^{\xi}\frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy\,,$$

(2.21) implies that the second term of (2.22) converges. We integrate by parts the first term:

$$\begin{aligned} \int_{-\infty}^{\xi^*}\frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy &= \frac{\phi'(\xi^*)}{(\xi-\xi^*)^{1-\alpha}} - \lim_{y\to-\infty}\frac{\phi'(y)}{(\xi-y)^{1-\alpha}} \\ &\quad + (1-\alpha)\int_{-\infty}^{\xi^*}\frac{\phi'(y)}{(\xi-y)^{2-\alpha}}dy \\ &= \frac{\phi'(\xi^*)}{\delta^{1-\alpha}} + (1-\alpha)\int_{-\infty}^{\xi^*}\frac{\phi'(y)}{(\xi-y)^{2-\alpha}}dy\,. \end{aligned}$$

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The absolute value of the second term on the right hand side is also bounded by $C/\delta^{1-\alpha}$. Since δ was a fixed number, this contradicts the unboundedness of $\mathcal{I}^\alpha \phi''$. $\square$

Next we show that a solution as constructed in Lemma 2.1 approaches a constant value as $\xi \to \infty$. Once this is proved we can conclude the proof of Theorem 2.1 since this then implies that $\lim_{\xi\to\infty} \phi = \phi_+$ by Lemma 2.5.

Lemma 2.6. *Let ϕ be a solution of (2.3) as in Lemma 2.4. Then there exist a constant $\phi_0 \in \mathbb{R}$ such that $\lim_{\xi\to\infty} \phi(\xi) = \phi_0$.*

Proof. The solution ϕ can be extended to any interval of the form $(-\infty, \xi_0]$ by repeating the continuation result of Lemma 2.2 as necessary, since (2.16) is satisfied. Now, knowing that the smooth wave profile exists, we split the non-local differential operator and rewrite the travelling wave equation in the following form

$$\tau\phi'' + \mathcal{D}^\alpha_{\xi_0}\phi + \phi = q(\phi, \xi) \tag{2.23}$$

for $\xi \geq \xi_0$, where

$$q(\phi, \xi) = -d_\alpha \int_{-\infty}^{\xi_0} \frac{\phi'(y)}{(\xi - y)^\alpha} dy + h(\phi(\xi)) + \phi(\xi)\,.$$

We can know write down the solution to (2.23) implicitly. In order to do that one applies Laplace transform methods as in e.g. [8] to obtain a 'variations of constants' representation of the solution with initial conditions at $\xi = \xi_0$. One gets

$$\phi(\xi) = \phi(\xi_0)\, v(\xi) - \phi'(\xi_0)\, v'(\xi) - \int_{\xi_0}^{\xi} q(\phi(\xi - s), \xi - s) v'(s) ds$$

where the function v and its derivatives are uniformly bounded and satisfy (we give more details in Appendix C, see (C.12)-(C.14)):

$$\lim_{\xi\to\infty} (\xi - \xi_0)^\alpha v(\xi) = \frac{d_\alpha}{\tau}\,, \qquad \lim_{\xi\to\infty} (\xi - \xi_0)^{\alpha+1} v'(\xi) = \frac{d_{\alpha-1}}{\tau}\,.$$

Now using that ϕ is uniformly bounded in $\mathbb{R}$, we conclude that $q(\phi, \xi)$ is also uniformly bounded and it is easy to see the integrability of the term with the inhomogeneity q as well as the decay of ϕ towards a constant. $\square$

We end the section with the proof of the main theorem:

Proof of Theorem 2.1. The proof follows by applying the previous lemmas. First, Lemma 2.1 (local existence), then Lemma 2.2 (continuation principle) and then lemmas 2.3 and 2.4 imply the global existence and uniqueness up to translation in ξ of solutions of (2.3) satisfying $\phi(-\infty) = \phi_-$. Finally, Lemma 2.6 implies that such solution satisfies $\lim_{\xi\to\infty} |\phi(\xi)| < \infty$ and from Lemma 2.5 we conclude that in fact $\lim_{\xi\to\infty} \phi(\xi) = \phi_+$. $\square$

3. Analysis of the monotonicity of travelling waves

In this section we discuss the role of the parameter τ in the monotonicity of the travelling waves. To start with, we remark that one can show 'local' monotonicity for all $\tau > 0$ in the interval I_ε for ε small enough:

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Lemma 3.1 (Local monotonicity)**.** *Let the assumptions of Lemma 2.1 hold. Then, for ε small enough,*

$$\phi_{up} > \phi_- \,, \quad \phi'_{up} > 0 \,, \qquad \phi_{down} < \phi_- \,, \quad \phi'_{down} < 0 \,, \quad in \quad I_\varepsilon \,.$$

Proof. The proof follows as in [1]. □

Now, if $\tau = 0$ we know from [1] that travelling waves are monotone decreasing. Moreover, if $\tau \neq 0$ and $\alpha = 1$, thus in the classical KdV-Burgers case, it is easy to see that the waves are monotone if τ is smaller than some critical value (see [3]). In fact, travelling waves are heteroclinic connections of the corresponding ODE system. The critical points represent the far-field values ϕ_- and ϕ_+, linearisation about these points shows that the one associated to ϕ_- is a saddle point and the one associated to ϕ_+ is an attractor. It is important to notice that the attractor has the eigenvalues

$$\lambda_\pm = \frac{-1 \pm \sqrt{1 + 4\tau h'(\phi_+)}}{2\tau}$$

and that $h'(\phi_+) < 0$ (see (2.6)). It then becomes clear that heteroclinic connections give monotone travelling waves when $\tau \leq -1/(4h'(\phi_+))$.

We expect a similar behaviour for (2.3), although the decay of ϕ towards ϕ_+ is not exponential, as we have seen in the proof of Lemma 2.6.

Let us now prove that if τ is small enough then the solution of (2.2)-(2.3) that is a extension of ϕ_{down} is close to the solution with $\tau = 0$ (as constructed in [1]) on a large interval, thus implying monotonicity for small values of τ on such intervals. Before we give the result let us introduce the appropriate notation. Let us denote by ϕ_τ a travelling wave solution for a given τ and ϕ_0 a travelling wave of the problem with $\tau = 0$. Then:

Theorem 3.1 (Monotonicity)**.** *If τ is small enough, then there exist an interval $I_\tau = (-\infty, \xi_\tau]$ with $\xi_\tau = O(\tau^{-\frac{1}{2-\alpha}})$ as $\tau \to 0$, and a value $\xi = \xi_\tau^0 < \xi_\tau$ such that $\phi_\tau(\xi_\tau^0) = \phi_0(\xi_\tau^0)$, moreover, $|\phi_\tau(\xi) - \phi_0(\xi)| \leq \tau C$ and $|\phi'_\tau(\xi) - \phi'_0(\xi)| \leq \tau^{1/(2-\alpha)} C$ for all $\xi \in I_\tau$. Thus for τ small enough ϕ_τ is also monotone decreasing in I_τ.*

We prove this theorem in several lemmas. First we fix the shift in ξ:

Lemma 3.2. *For a given small τ there exists a $\xi_\tau^0 < \log\tau/(h'(\phi_-))^{1/\alpha}$ small and a travelling wave solution ϕ_τ such that, if ϕ_0 is the travelling wave solution of the problem with $\tau = 0$ such that $\phi_0(\log\tau/(h'(\phi_-))^{1/\alpha}) = \phi_- - \tau$, then ϕ_τ is monotone decreasing in $(-\infty, \xi_\tau^0]$ and*

(3.1)
$$\phi_\tau(\xi_\tau^0) = \phi_0(\xi_\tau^0) \,, \quad |\phi'_\tau(\xi) - \phi'_0(\xi)| \,, \; |\phi''_\tau(\xi) - \phi''_0(\xi)| \leq \tau C \quad for \quad \xi \in (-\infty, \xi_\tau^0]$$

with some order one constant $C > 0$.

Proof. We want to compare travelling wave solutions for a small $\tau > 0$ with solutions of the problem with $\tau = 0$. The later ones are monotone and are constructed 'locally' near $-\infty$ as in Lemma 2.1 in [1]. In particular, for a given small enough ε then $\phi_0(\xi_\varepsilon^0) = \phi_- - \varepsilon$ where $\xi_\varepsilon^0 = \log\varepsilon/(h'(\phi_-))^{1/\alpha}$. On the other hand, if λ_τ denotes the real root of (2.9) then $\phi_\tau(\log\varepsilon)/\lambda_\tau) = \phi_- - \varepsilon$. The asymptotic behaviour of λ_τ as $\tau \to 0$ (see (B.3) in Appendix B) and (2.6) imply that if τ is small enough then $\xi_\varepsilon^0 < \log\varepsilon/\lambda_\tau$, hence, by local

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monotonicity, $\phi_- - \varepsilon < \phi_\tau(\xi_\varepsilon^0) < \phi_-$ i.e. $\phi_\tau(\xi_\varepsilon^0) - \phi_0(\xi_\varepsilon^0) < \varepsilon$. Again by monotonicity, we can find a value $\xi_\varepsilon < \xi_\varepsilon^0$, by shifting ϕ_τ in ξ if necessary, such that $\phi_\tau(\xi_\varepsilon) = \phi_0(\xi_\varepsilon)$. Finally, by the construction of these waves, they are close to ϕ_- by an exponential difference in $H^3(-\infty, \xi_\tau^0)$, it holds that $|\phi_\tau'(\xi) - \phi_0'(\xi)|$, $|\phi_\tau''(\xi) - \phi_0''(\xi)| \leq \varepsilon C$ with an order 1 constant $C > 0$. Finally, we can do this same construction by taking $\varepsilon = \tau$. $\square$

Let $\psi_\tau := \phi_\tau - \phi_0$, then ψ_τ satisfies the following equation

$$\tau\psi_\tau'' + \mathcal{D}^\alpha_{\xi_\tau^0}\psi_\tau = \mathcal{R}(\phi_0, \phi_\tau, \xi) - \tau\phi_0''(\xi) \tag{3.2}$$

where

$$\mathcal{R}(\phi_0, \phi_\tau, \xi) = [-c + (\phi_\tau(\xi) + \phi_0(\xi))]\psi_\tau(\xi) - d_\alpha \int_{-\infty}^{\xi_\tau^0} \frac{\psi_\tau'(y)}{(\xi - y)^\alpha}\, dy\,, \tag{3.3}$$

(for simplicity, we do not write the dependency of $\mathcal{R}$ in ψ_τ which implicit in the dependency on ϕ_τ and ϕ_0) here we have used the expression of h in (2.3) to write $h(\phi_\tau) - h(\phi_0) = [-c + (\phi_\tau + \phi_0)]\psi_\tau$. That can be solved subject to the initial conditions (see (3.1))

$$\psi(\xi_\tau^0) = 0\,, \quad \psi'(\xi_\tau^0) = \phi_\tau'(\xi_\tau^0) - \phi_0'(\xi_\tau^0)\,. \tag{3.4}$$

Using Laplace transform we can write the solution to (3.2)-(3.4) taking ϕ_τ and ϕ_0 as given. In order to do that more conveniently we can first shift the independent variable so that $\eta = \xi - \xi_\tau^0$ and let $\bar\psi_\tau(\eta) = \psi_\tau(\eta + \xi_\tau^0)$, so (3.2) and (3.3) read

$$\tau\bar\psi_\tau'' + \mathcal{D}^\alpha_0\bar\psi_\tau + \bar\psi_\tau = \mathcal{Q}(\phi_0, \phi_\tau, \eta)\,, \quad {}' = \frac{d}{d\eta} \tag{3.5}$$

$$\mathcal{Q}(\phi_0, \phi_\tau, \eta) = \mathcal{R}(\phi_0, \phi_\tau, \eta + \xi_\tau^0) - \tau\phi_0''(\eta + \xi_\tau^0) + \bar\psi_\tau(\eta) \tag{3.6}$$

where we add an subtract the term $\bar\psi_\tau$ for technical reasons outlined below. Then, (3.5)-(3.6) must be solved with initial conditions (see (3.4))

$$\bar\psi_\tau(0^+) = 0\,, \quad \bar\psi_\tau'(0^+) = \psi'(\xi_\tau^0)\,. \tag{3.7}$$

Employing the computation performed in Appendix C, but here with $a = 1$, the solution of (3.5)-(3.7) is given implicitly by

$$\bar\psi(\eta) = -\tau\bar\psi'(0^+)v'(\eta) - \int_0^\eta v'(y)\mathcal{Q}(\phi_0, \phi_\tau, \eta - y)\, dy \tag{3.8}$$

where $v(\eta)$ reads (see (C.12) and (C.13))

$$v(\eta) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\eta r}K_{\tau,\alpha}(r)\, dr + 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\,, \tag{3.9}$$

and s_1 is the solution of

$$\tau z^2 + z^\alpha + 1 = 0 \tag{3.10}$$

with positive imaginary part, and $\beta = \arg(s_1) \in (\pi/2, \pi)$ (see Lemma B.1 and Appendix C), and where

$$K_{\tau,\alpha}(r) = r^{\alpha-1}\tilde K_{\tau,\alpha}(r) \tag{3.11}$$

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with $\tilde{K}_{\tau,\alpha}(r) = 1/\big((\tau r^2+1)^2 + 2(\tau r^2+1)r^\alpha \cos(\alpha\pi) + r^{2\alpha}\big)$. Then it is easy to see that

$$(3.12)\quad v'(\eta) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\eta r} r^\alpha \tilde{K}_{\tau,\alpha}(r)\,dr + 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1^2 + s_1^\alpha}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right).$$

The reason to introduce the term $\bar{\psi}_\tau$ is that this implies that the resulting algebraic function when applying the Laplace transform to the left-hand side of (3.5) has poles away from the negative real axis. Without this term, 0 would be a pole of such function, but is also a branch point, thus making the computation of the inverse Laplace transform a little cumbersome.

We also need to get pointwise estimates on $|\bar{\psi}_\tau(\eta)|$. We shall do this directly from the expression obtained differentiating (3.8):

(3.13)

$$\bar{\psi}_\tau'(\eta) = -\tau\bar{\psi}_\tau'(0^+)v''(\eta) - v'(\eta)\mathcal{Q}(\phi_0,\phi_\tau,0) - \int_0^\eta v'(y)\frac{d\mathcal{Q}}{d\eta}(\phi_0,\phi_\tau,\eta-y)\,dy\,.$$

The following estimates hold:

Lemma 3.3. *If $r < 1/\tau^{\frac{1}{2-\alpha}}$, for some τ small enough, then there exists a $C>0$ independent of τ such that*

$$\left|\tilde{K}_{\tau,\alpha}(r) - \frac{1}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}\right| \le C\,\tau\,.$$

If on the contrary $r \ge 1/\tau^{\frac{1}{2-\alpha}}$, for some τ small enough, there exists a $C>0$ independent of τ such that

$$|\tilde{K}_{\tau,\alpha}(r)| \le C\tau^{\frac{2}{2-\alpha}}\,.$$

Proof. We leave the proof to the reader. One can convince him or herself by inspecting the functions involved and a formal dominant balance analysis that can be made rigorous by performing the calculus. ☐

Regarding the second term on the right-hand side of (3.9) we have the following preliminary estimates that give exponential decay (observe that $\cos(\beta) < 0$:

Lemma 3.4. *If τ is small enough then, there exists a $0 < C(\tau) < 1$ such that $C(\tau) \sim 2/(2+\alpha)$ as $\tau \to 0$ and*

$$\left|Re\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\right| \le C(\tau)e^{(|s_1|\cos\beta)\,\eta}$$

where $\beta = arg(s_1) \in (\pi/2,\pi)$ with $\beta \to \pi/(2-\alpha)$ and $|s_1| = O(1/\tau^{1/(2-\alpha)})$ as $\tau \to 0$. Similarly, estimates on derivatives with respect to η of this term differ by a factor $|s_1|^n$ where n is the order of the derivative.

Proof. This is proved by using Lemma B.1 and (B.6) of Appendix B. Details are left to the reader. ☐

We next derive estimates that we apply to (3.8) and (3.13). Essentially, we get estimates on the uniform norms of v, v', $\mathcal{Q}$ and $d\mathcal{Q}/d\eta$, estimates on the integral of $|v'|$ over $[0,\eta]$ and on v''. We also need to get estimates on the integral terms of $\mathcal{Q}$ and $d\mathcal{Q}/d\eta$.

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Lemma 3.5. *The following estimates hold for all $\eta > 0$ and τ small enough:*

(i) *There exist constants $C(\|\phi_0''\|_{L^\infty}) > 0$ and $C > 0$ such that*

$$|\mathcal{Q}(\phi_0, \phi_\tau, \eta)| \leq (1 + 3|\phi_-| + |\phi_+|)\ |\bar{\psi}_\tau(\eta)| + \tau C(\|\phi_0''\|_{L^\infty}) \tag{3.14}$$

and

$$\left|\frac{d\mathcal{Q}}{d\eta}(\phi_0, \phi_\tau, \eta)\right| \leq C\left(|\bar{\psi}_\tau(\eta)| + |\bar{\psi}_\tau'(\eta)| + \tau\eta^{1-\alpha}\right) + \tau\,\|\phi_0'''\|_{L^\infty}\,. \tag{3.15}$$

(ii) *There exists a constant $C > 0$ such that*

$$|\mathcal{Q}(\phi_0, \phi_\tau, 0)| \leq \tau\,\left(C + |\phi_0''(\xi_\tau^0)|\right)\,.$$

(iii) *The functions v, v' and v'' are uniformly bounded on $[0, \infty)$. the first one by a constant independent of τ, whereas the other two by an constant that becomes unbounded as $\tau \to 0^+$. Moreover, for all $\eta > 0$ there exists a constant $C > 0$*

$$|v'(\eta)| \leq C\left(\tau^{\frac{\alpha}{2(2-\alpha)}}\eta^{\frac{\alpha-2}{2}} + \tau^{\frac{1-\alpha}{2-\alpha}}\eta^{-\alpha} + \tau^{\frac{3-\alpha}{2-\alpha}}\right) + 2C(\tau)e^{(|s_1|\cos\beta)\,\eta}|s_1| \tag{3.16}$$

s_1 being the zero of (3.10) with positive imaginary part and $\beta \in (\pi/2, \pi)$ its principal argument, and

$$|v''(\eta)| \leq C(\tau^{-1} + \tau^{-\frac{2}{2-\alpha}} + \tau^{-\frac{\alpha}{2-\alpha}} + \tau^{\frac{1-\alpha}{2-\alpha}})\,. \tag{3.17}$$

Proof. Statement *(i)* follows from the properties of ϕ_τ and ϕ_0. In order to estimate the integral term of $\mathcal{Q}$ we use the construction of the solutions ϕ_τ and ϕ_0 in the interval $(-\infty, \xi_\tau^0]$ and that $\lambda_\tau e^{\lambda_\tau y} - \lambda_0 e^{\lambda_0 y} = e^{\lambda_0 y}F(\tau, y)$ where $F(\tau, y)$ in uniformly bounded in $y < \xi_\tau^0$ $\tau > 0$ (see Lemma 3.2, (3.1)). One can show *(ii)* similarly, since

$$\mathcal{Q}(\phi_0, \phi_\tau, 0) = -d_\alpha\mathcal{D}^\alpha(\xi_\tau^0) - \phi_0''(\xi_\tau^0)\,.$$

The integral term of $dQ/d\eta$ reads, here $\eta > 0$,

$$I := -d_\alpha\int_{-\infty}^{\xi_\tau^0}\frac{\psi_\tau'(y)}{(\xi - y)^{\alpha+1}}dy = -d_\alpha\int_{-\infty}^{0}\frac{\bar{\psi}_\tau'(y)}{(\eta - y)^{\alpha+1}}dy\,.$$

Integration by parts gives

$$I = -\frac{d_\alpha}{\alpha}\frac{\bar{\psi}_\tau'(0)}{\eta^\alpha} + \frac{d_\alpha}{\alpha}\int_{-\infty}^{0}\frac{\bar{\psi}_\tau''(y)}{(\eta - y)^\alpha}dy$$

and using Lemma 3.2, (3.1) gives the estimate.

The statement in *(iii)* about v and v' follows from (C.5) and (C.14) of Appendix C.

Let us get the estimate on $|v'(\eta)|$. Using the expression of $v'(\eta)$ in (3.12) and Lemma 3.4 we obtain a first estimate

$$\begin{aligned}|v'(\eta)| &\leq \frac{1}{\pi}\int_0^\infty e^{-\eta r}r^\alpha\left|\tilde{K}_{\tau,\alpha}(r)\right|\,dr + 2\left|\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1^2 + s_1^\alpha}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\right| \\ &\leq \frac{1}{\pi}\int_0^\infty e^{-\eta r}r^\alpha\left|\tilde{K}_{\tau,\alpha}(r)\right|\,dr + Ce^{(|s_1|\cos\beta)\,\eta})|s_1|\,,\end{aligned}$$

for some positive constant independent of τ, where s_1 is the zero of (3.10) with positive imaginary part and β its principal argument. We estimate the first term on the right-hand side of the inequality above assuming that τ is

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small enough and so that we can apply Lemma 3.3. Thus we first split the integral over r and apply this lemma:

(3.18)
$$\begin{aligned}&\int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-\eta r} r^\alpha \left|\tilde{K}_{\tau,\alpha}(r)\right| dr \\ &\le \int_0^{\tau^{-\frac{1}{2-\alpha}}} \frac{e^{-\eta r} r^{\alpha-\gamma} r^\gamma}{1+2r^\alpha \cos(\alpha\pi) + r^{2\alpha}}\, dr + \tau^{1-\frac{1}{2-\alpha}} C \int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-\eta r} r^{\alpha-1}\, dr\,,\end{aligned}$$

where $\gamma \in (\alpha, 2\alpha)$ and

$$\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r} r^\alpha \left|\tilde{K}_{\tau,\alpha}(r)\right| dr \le \tau^{\frac{2}{2-\alpha}} C \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r} r^\alpha\, dr\,. \tag{3.19}$$

In order to estimate (3.18) we shall use that

$$\int_A^B e^{-\eta r} r^\sigma\, dr \le \int_0^\infty e^{-\eta r} r^\sigma\, dr = \frac{\Gamma(\sigma+1)}{\eta^{\sigma+1}} \tag{3.20}$$

with $\sigma > -1$, $0 \le A < B$, $\eta > 0$, and in each integral term we apply this as an estimate with different values of σ. For the first term on the right-hand side of (3.18) we first rescale $r = \tau^{-\frac{1}{2-\alpha}} \bar{r}$ and observe that for any $\gamma \in (\alpha, 2\alpha]$ the function $\bar{r}^\gamma / (\tau^{\frac{2\alpha}{2-\alpha}} a^2 + \tau^{\frac{\alpha}{2-\alpha}} 2\bar{r}^\alpha \cos(\alpha\pi) + \bar{r}^{2\alpha})$ is uniformly bounded for $\bar{r} \in [0,1]$ by a constant independent of τ. Then, after this change of variables, the estimate and applying (3.20) with $A = 0$, $B = 1$ and $\sigma = \alpha - \gamma$ one gets:

$$\begin{aligned}\int_0^{\tau^{-\frac{1}{2-\alpha}}} \frac{e^{-\eta r} r^\alpha}{1+2\cos(\alpha\pi) r^\alpha + r^{2\alpha}}\, dr &\le \tau^{\frac{\alpha-1}{2-\alpha}} C \int_0^1 e^{-\tau^{-\frac{1}{2-\alpha}} \eta \bar{r}} \bar{r}^{\alpha-\gamma}\, d\bar{r} \\ &\le \tau^{\frac{2\alpha-\gamma}{2-\alpha}} C(\alpha) \frac{1}{\eta^{\alpha-\gamma+1}}\,.\end{aligned}$$

We obtain

$$\int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-\eta r} r^\alpha \left|\tilde{K}_{\tau,\alpha}(r)\right| dr \le \tau^{\frac{2\alpha-\gamma}{2-\alpha}} C(\alpha) \eta^{\gamma-\alpha+1} + \tau^{\frac{1-\alpha}{2-\alpha}} C \Gamma(\alpha) \eta^{-\alpha}$$

and, taking $\gamma = 3\alpha/2$, this gives the first two terms on the right-hand side of (3.16). We further estimate (3.19) as follows:

$$\begin{aligned}&\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r} r^\alpha \left|\tilde{K}_{\tau,\alpha}(r)\right| dr \\ \le&\tau^{\frac{2}{2-\alpha}} C \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r} r^2 r^{\alpha-2}\, dr \le \tau^{\frac{2}{2-\alpha}} C \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty r^{\alpha-2}\, dr = \tau^{\frac{3-\alpha}{2-\alpha}} C\,,\end{aligned}$$

where we use Lemma 3.3 in the first step. Putting the estimates together we obtain (3.16).

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Let us get now the estimate on $|v''(\eta)|$. We differentiate (3.12) and estimate the last term as in Lemma 3.4, then

$$
\begin{aligned}
|v''(\eta)| &\leq \frac{|\sin(\alpha\pi)|}{\pi}\int_0^\infty e^{-r\eta}r^{\alpha+1}|\tilde{K}(r)|dr + 2\left|\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1^3 + s_1^{\alpha+1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\right| \\
&\leq \frac{|\sin(\alpha\pi)|}{\pi}\int_0^\infty e^{-r\eta}r^{\alpha+1}|\tilde{K}(r)|dr + C\,e^{(|s_1|\cos\beta)\eta}|s_1|^2\,. \qquad (3.21)
\end{aligned}
$$

We get estimates on the first term on the right-hand side by dividing the integral over the intervals $(0,\tau^{-1/(2-\alpha)})$ and $(\tau^{-1/(2-\alpha)},\infty)$ and apply the estimates on $\tilde{K}$ of Lemma 3.3, then

$$
\begin{aligned}
&\frac{|\sin(\alpha\pi)|}{\pi}\int_0^\infty e^{-r\eta}r^{\alpha+1}|\tilde{K}(r)|dr \\
\leq &C\left(\int_0^{\tau^{-\frac{1}{2-\alpha}}}\frac{e^{-r\eta}r^{2\alpha}r^{1-\alpha}}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}dr + \tau\int_0^{\tau^{-\frac{1}{2-\alpha}}}e^{-r\eta}r^{\alpha+1}dr\right. \\
&\qquad\left. + \tau^{\frac{2}{2-\alpha}}\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-r\eta}r^3r^{\alpha-2}dr\right) \\
\leq &C\left(\int_0^{\tau^{-\frac{1}{2-\alpha}}} r^{1-\alpha}dr + \tau^{-\frac{\alpha}{2-\alpha}} + \tau^{\frac{2}{2-\alpha}}\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty r^{\alpha-2}dr\right) \\
= &C\left(\frac{\tau^{-1}}{2-\alpha} + \tau^{-\frac{\alpha}{2-\alpha}} + \frac{\tau^{\frac{1-\alpha}{2-\alpha}}}{1-\alpha}\right).
\end{aligned}
$$

This together with (3.21) and the asymptotic behaviour of $|s_1|$ as $\tau \to 0^+$ (see Lemma 3.4) concludes the proof of (3.17). □

We can now prove Theorem 3.1

Proof of Theorem 3.1. We start estimating $|\bar{\psi}_\tau(\eta)|$ and $|\bar{\psi}'_\tau(\eta)|$ directly from (3.8) and (3.13) and using the estimates of Lemma 3.5:

$$
\begin{aligned}
|\bar{\psi}_\tau(\eta)| \leq &\tau C(\|\phi_0''\|_{L^\infty})\int_0^\eta |v'(y)|dy + \tau^2C|v'(\eta)| + \int_0^\eta |v'(y)|\,|\bar{\psi}_\tau(\eta-y)| \\
|\bar{\psi}'_\tau(\eta)| \leq &\tau C(|v'(\eta)| + \tau|v''(\eta)|) + \int_0^\eta |v'(y)|\,|\bar{\psi}'_\tau(\eta-y)|dy| \\
&+ \int_0^\eta |v'(y)|\left(|\bar{\psi}_\tau(\eta-y)| + \tau(\eta-y)^{1-\alpha} + \tau\|\phi_0'''\|_{L^\infty}\right)dy\,.
\end{aligned}
$$

The result follows by using Gronwall's Lemma, which implies that

$$
|\bar{\psi}_\tau(\eta)| \leq A_1(\eta) + \int_0^\eta A_1(x)B(x)e^{\int_x^\eta B(s)ds}dx \qquad (3.22)
$$

$$
|\bar{\psi}'_\tau(\eta)| \leq A_2(\eta) + \int_0^\eta A_2(x)B(x)e^{\int_x^\eta B(s)ds}dx \qquad (3.23)
$$

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where

$$\begin{aligned} A_1(\eta) =& \tau C(\|\phi_0''\|_{L^\infty}) \int_0^\eta |v'(y)| dy + \tau^2 C|v'(\eta)|\,, \quad B(\eta) = |v'(\eta - x)| \\ A_2(\eta) =& \tau C(|v'(\eta)| + \tau|v''(\eta)|) \\ &+ \int_0^\eta |v'(y)| \left(|\bar{\psi}_\tau(\eta - y)| + \tau(\eta - y)^{1-\alpha} + \tau\|\phi_0'''\|_{L^\infty}\right) dy\,. \end{aligned}$$

Observe that A_1, $\int_0^\eta B_1(y)dy$ and $\exp(\int_s^\eta B_1(s)ds)$ are uniformly bounded for $\eta < C\tau^{-1/(2-\alpha)}$ and, moreover, $|A_1(\eta)| \le \tau C$ for some $C > 0$. Using this in (3.22) implies $|\bar{\psi}_\tau(\eta)| \le \tau C$ for some $C > 0$ for all such η's. We can apply this last fact to (3.23) to conclude the proof, since $A_2 \le \tau C \eta^{1-\alpha}$, thus in this range of η's $A_2 \le \tau^{1/(2-\alpha)} C$. □

Finally, we discuss the fact that in the tail travelling waves are monotone as long as τ is small enough. This does not imply that the waves are decreasing in the whole of the domain, however. The following result holds:

Theorem 3.2. *Let ϕ be a solution of (2.1)-(2.2) as constructed in Theorem 2.1, then there exist a $\bar{\xi}$ large enough such that if τ is small enough ϕ is monotone decreasing in $(\bar{\xi}, \infty)$.*

Proof. We only sketch the proof. It can be done by a bootstrap argument based on the behaviour of the solutions in the tail for τ small enough. For every δ, let $\xi_\delta \in \mathbb{R}$ be such that $\xi_\delta = \inf\{\xi :\ \phi(\xi) - \phi_+ = \delta\}$. Let us write (2.1) as follows:

$$h(\phi) = \mathcal{D}^\alpha_{\xi_\delta}\phi + \int_{-\infty}^{\xi_\delta} \frac{\phi'(y)}{(\xi - y)^\alpha}\, dy + \tau\phi''\,. \tag{3.24}$$

Let $\psi(\xi) = \phi(\xi) - \phi_+$, then (3.24) reads

$$h(\phi) - h(\phi_+) - h'(\phi_+)\psi = \mathcal{D}^\alpha_{\xi_\delta}\psi + \int_{-\infty}^{\xi_\delta} \frac{\psi'(y)}{(\xi - y)^\alpha}\, dy + \tau\psi'' - h'(\phi_+)\psi\,,$$

where we use that $h(\phi_+) = 0$. It is convenient to shift the independent variable as follows $\zeta = \xi - \xi_\delta$ and let $\psi(\xi) = \Psi(\zeta)$. Then, (3.24) reads, rearranging terms,

$$\tau\Psi'' + \mathcal{D}^\alpha_0\Psi - h'(\phi_+)\Psi = h(\phi) - h(\phi_+) - h'(\phi_+)\Psi - \int_{-\infty}^0 \frac{\Psi'(y)}{(\zeta - y)^\alpha}\, dy\,, \tag{3.25}$$

Then, we express the solution implicitly as in Appendix C with $a = -h'(\phi_+) > 0$

$$\Psi(\zeta) = \bar{\psi}(0^+)v(\zeta) + \frac{\tau}{h'(\phi_+)}\bar{\psi}'(0^+)v'(\zeta) + \frac{1}{h'(\phi_+)}\int_0^\zeta v'(y)\mathcal{Q}(\zeta - y)\, dy\,, \tag{3.26}$$

$$Q(\zeta) := h(\phi(\zeta + \xi_\delta)) - h(\phi_+) - h'(\phi_+)\Psi(\zeta) - \int_{-\infty}^0 \frac{\Psi'(y)}{(\zeta - y)^\alpha}\, dy \tag{3.27}$$

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(cf. (C.6)) where $v(\zeta)$ has been computed in Appendix C and reads (see (C.12) and (C.13))

(3.28)
$$v(\zeta) = -h'(\phi_+)\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\zeta r}K_\alpha(r)\,dr + 2\mathrm{Re}\left(e^{s_1\zeta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right),$$

(cf. (3.9)) where, as before, s_1 is the solution of

$$\tau z^2 + z^\alpha - h'(\phi_+) = 0 \tag{3.29}$$

with positive imaginary part, and $\beta = \arg(s_1) \in (\pi/2, \pi)$ (see appendixes B and C), and where $K_\alpha(r) = r^{\alpha-1}\tilde{K}_\alpha(r)$ with

$$\tilde{K}_\alpha(r) = \frac{1}{(\tau r^2 - h'(\phi_+))^2 + 2(\tau r^2 - h'(\phi_+))r^\alpha\cos(\alpha\pi) + r^{2\alpha}}.$$

Observe that $v(\zeta)$ is the sum of a monotone (first term on the right-hand side of (3.28)) and a oscillatory term (second term on the right-hand side of (3.28)). On the other hand, the non-monotone contribution of $v'(\zeta)$ is given by the derivative of the exponential oscillatory term of $v(\zeta)$ if ζ is very large (thus if δ is very small), and the last term on the right hand side of (3.26) can be made arbitrarily small for δ small. Thus taking ζ large enough and τ small enough the small oscillations get damped by the algebraic decaying terms of the monotone part. Observe that $\mathrm{Re}\left(e^{s_1\zeta}\frac{\tau s_1+s_1^{\alpha-1}}{2\tau s_1+\alpha s_1^{\alpha-1}}\right)$ has infinitely many oscillations with frequency $\omega = \rho\sin\beta = \mathrm{Im}(s_1)$, but its amplitude decreases exponentially like $e^{(|s_1|\cos\beta)\zeta}$ as $\zeta\to\infty$ (recall that $|s_1| = O(\tau^{-1/(2-\alpha)})$ and $\cos(\beta) < 0$). □

4. Asymptotic stability of monotone travelling waves

In this section we assume that the travelling waves found in Theorem 2.1 are monotone (decreasing) and we prove their dynamic stability. Existence of such waves is guaranteed for small enough values of τ as the analysis of the previous section suggests. The stability analysis is done in a similar way as for the KdV-Burgers equation and the Burgers equation (see e.g. [17], and also [1] for the corresponding fractional diffusion Burgers equation). We next outline the key ideas of the proof.

It is convenient to first change variables to $x \to \xi = x - ct$ in (1.1), so it becomes

$$\partial_t u + \partial_\xi(u^2 - cu) = \partial_\xi\mathcal{D}^\alpha u + \tau\partial_\xi^3 u. \tag{4.1}$$

We then look for solutions of (4.1) which are a small perturbation of a travelling wave and that in particular share the same far-field values. Let $u_0(\xi)$ be an initial datum and $\phi(\xi)$ a monotone travelling wave as constructed in Theorem 2.1, with a shift in ξ chosen such that

$$\int_{\mathbb{R}}(u_0(\xi) - \phi(\xi))\,d\xi = 0. \tag{4.2}$$

Observe that conservation of mass, a property satisfied by (4.1), implies that

$$\int_{\mathbb{R}}(u(t,\xi) - \phi(\xi))d\xi = 0, \quad \text{for all} \quad t \geq 0.$$

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Now, the perturbation $U = u - \phi$ satisfies the equation

$$\partial_t U + \partial_\xi((2\phi - c)U) + \partial_\xi U^2 = \partial_\xi \mathcal{D}^\alpha U + \tau \partial_\xi^3 U\,. \tag{4.3}$$

The aim is to show that U tends to 0 in a suitable sense as t tends to ∞ for small enough $U_0 = u_0 - \phi$. We use integral estimates. For instance, testing (4.3) with U, we get

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \int_{\mathbb{R}} \phi' U^2 d\xi = -a_\alpha \|U\|^2_{\dot{H}^{\frac{1+\alpha}{2}}}\,, \tag{4.4}$$

where several integrations by parts have been carried out. Since we are assuming that $\phi' \leq 0$, the second term in (4.4) is non-positive. We next introduce the primitive of the perturbation and of the corresponding initial data

$$W(t,\xi) = \int_{-\infty}^{\xi} U(t,\eta) d\eta\,, \quad W_0(\xi) = \int_{-\infty}^{\xi} U_0(\eta) d\eta\,,$$

which satisfies the integrated version of (4.3),

$$\partial_t W + (2\phi - c)\partial_\xi W + (\partial_\xi W)^2 = \mathcal{D}^\alpha \partial_\xi W + \tau \partial_\xi^3 W\,, \tag{4.5}$$

and

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \int_{\mathbb{R}} \phi' W^2 d\,\xi + \int_{\mathbb{R}} (\partial_\xi W)^2 W\, d\xi = -a_\alpha \|W\|^2_{\dot{H}^{\frac{1+\alpha}{2}}}\,. \tag{4.6}$$

This integral identity has the crucial property that the term involving ϕ' is non-negative. In the cubic term (arising from the nonlinearity) we can estimate $|W|$ by the L^∞-norm, this factor can then be controlled by using the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

The right-hand side in (4.4) is obtained using Plancherel's theorem that, together with $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$, implies that

$$\int_{\mathbb{R}} \mathrm{sgn}(k)\, |k|^j \, |\hat{u}(k,t)|^2 \, dk = 0 \quad j \in \mathbb{N}\,.$$

We observe that (as one can easily check based on (1.3) and (1.5))

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha) = \mathcal{F}(\mathcal{D}^\alpha \partial_x) = -(a_\alpha - i b_\alpha \mathrm{sgn}(k))|k|^{\alpha+1}\,, \tag{4.7}$$

and from this we can obtain in the same way as for (4.4) the right-hand side of (4.6).

The well-posedness result below and the fact that (1.1), or (4.1), is a third order equation requires that we work with $U \in H^2$, in fact we shall require that at least $U_0 \in H^3(\mathbb{R})$ since we need integral estimates of higher order. We assume for the moment that the following theorem holds, and we prove it in Section 5:

Theorem 4.1. *For every $U_0 \in H^s(\mathbb{R})$, $s \geq 3$ and assuming that $\phi \in H^{s+1}(\mathbb{R})$, there is a $T > 0$ such that (4.3) with initial data $U(\xi,0) = U_0(\xi)$ has a unique solution $U \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R}))$ satisfying*

$$\|U\|_{H^s} \leq C\|U_0\|_{H^s}\,.$$

The same result applies to (4.5) with initial condition $W(\xi,0) = W_0(\xi)$.

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Then assuming this, we can locally perform further integral estimates of the consecutive differentiations of (4.5). Namely, from

$$\partial_t\partial_\xi^2 W + \partial_\xi^2((2\phi - c)\partial_\xi W) + \partial_\xi^2(\partial_\xi W)^2 = \partial_\xi^2\mathcal{D}^\alpha\partial_\xi W + \tau\partial_\xi^5 W\,. \tag{4.8}$$

we obtain, testing with $\partial_\xi^2 W$, the integral identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_\xi^2 W\|_{L^2}^2 - \int_{\mathbb{R}}\phi'''(\partial_\xi W)^2 d\xi + 3\int_{\mathbb{R}}\phi'(\partial_\xi^2 W)^2 d\xi + \int_{\mathbb{R}}(\partial_\xi^2 W)^3 d\xi\\ &= -\, a_\alpha\|\partial_\xi^2 W\|_{\dot H^{\frac{1+\alpha}{2}}}^2\,.\end{aligned} \tag{4.9}$$

Further, from the equation

$$\partial_t\partial_\xi^3 W + \partial_\xi^3((2\phi - c)\partial_\xi W) + \partial_\xi^3(\partial_\xi W)^2 = \partial_\xi^3\mathcal{D}^\alpha\partial_\xi W + \tau\partial_\xi^6 W\,, \tag{4.10}$$

we obtain testing now with $\partial_\xi^3 W$ the integral identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_\xi^3 W\|_{L^2}^2 + 2\int_{\mathbb{R}}\phi'''\partial_\xi W\,\partial_\xi^3 W d\xi - 3\int_{\mathbb{R}}\phi'''(\partial_\xi^2 W)^2 d\xi\\ &+ 5\int_{\mathbb{R}}\phi'(\partial_\xi^3 W)^2 d\xi + 5\int_{\mathbb{R}}\partial_\xi^2 W(\partial_\xi^3 W)^2 d\xi\\ &= -\, a_\alpha\|\partial_\xi^3 W\|_{\dot H^{\frac{1+\alpha}{2}}}^2\,.\end{aligned} \tag{4.11}$$

In order to justify the vanishing of the integral terms coming from the highest order term in each equation, we use Theorem 4.1 above, that allows to obtain these identities in $[0, T]$ provided the initial condition $W_0 \in H^{s+1}$ with $s \geq 3$. The proof of stability then uses a combination of the integral identities just obtained choosing the coefficients in such a way that the resulting functional is decreasing in time. The main point is that the terms with the wrong sign, coming in general from the nonlinear terms and the ones involving derivatives of ϕ, can be controlled by the dissipative ones via versions of the interpolation inequality

$$b^2\|g\|_{\dot H^1}^2 \leq b^{1+\alpha}\|g\|_{\dot H^{\frac{1+\alpha}{2}}}^2 + b^{3+\alpha}\|g\|_{\dot H^{\frac{3+\alpha}{2}}}^2\,, \quad b > 0 \tag{4.12}$$

that holds as a consequence of $(bk)^2 \leq |bk|^{1+\alpha} + |bk|^{3+\alpha}$, $k \in \mathbb{R}$ with $b > 0$. We shall also need the following one

$$\|g\|_{\dot H^1}^2 \leq \max\{1, 1/b\}\left(\|g\|_{\dot H^{\frac{1+\alpha}{2}}}^2 + b\|g\|_{\dot H^{\frac{3+\alpha}{2}}}^2\right)\,, \quad b > 0\,, \tag{4.13}$$

that follows from $(k)^2 \leq |k|^{1+\alpha} + |k|^{3+\alpha} \leq \min\{\tilde b|k|^{1+\alpha} + |k|^{3+\alpha}, |k|^{1+\alpha} + \tilde b|k|^{3+\alpha}\}$ for any $\tilde b > 1$.

After these preparations we can prove the following result.

Theorem 4.2. *Let ϕ be a travelling wave as in Theorem 2.1, and let $u_0(\xi)$ be an initial datum for (4.1), such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))d\eta$ satisfies $W_0 \in H^{s+1}$ with $s \geq 3$. Then if $\|W_0\|_{H^3}$ is small enough, the Cauchy problem for (4.1) with initial datum u_0 has a unique global solution with $u(t) \in H^{s-1}$ for all $t > 0$ converging to the travelling wave in the sense that*

$$\lim_{t\to\infty}\int_t^\infty \|u(\sigma, \cdot) - \phi\|_{H^2}^2 d\sigma = 0\,.$$

Note that (4.2), which can be translated into the condition $W_0(\pm\infty) = 0$, is part of the assumption $W_0 \in H^s$ in Theorem 4.2.

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Proof. As mentioned earlier the integral identities (4.4)-(4.11) are justified by Theorem 4.1 in an interval $[0,T]$. Then, (4.4) and (4.6) imply the estimates

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 \le -a_\alpha\|U\|_{\dot H^{\frac{1+\alpha}{2}}}^2 , \tag{4.14}$$

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \le -a_\alpha\|W\|_{\dot H^{\frac{1+\alpha}{2}}}^2 , \tag{4.15}$$

with $C_0 = \|\phi'\|_{L^\infty}$. Now, (4.9) and (4.11) imply the estimates

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_\xi^2 W\|_{L^2}^2 - C_1\|\partial_\xi W\|_{L^2}^2 - \left(\|\partial_\xi^2 W\|_{L^\infty} + 3C_0\right)\|\partial_\xi^2 W\|_{L^2}^2 \\ \le & - a_\alpha\|\partial_\xi^2 W\|_{\dot H^{\frac{1+\alpha}{2}}}^2 , \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_\xi^3 W\|_{L^2}^2 - C_1\|\partial_\xi W\|_{L^2}^2 - 3C_1\|\partial_\xi^2 W\|_{L^2}^2 \\ &- \left(5\|\partial_\xi^2 W\|_{L^\infty} + 5C_0 + C_1\right)\|\partial_\xi^3 W\|_{L^2}^2 \\ \le & - a_\alpha\|\partial_\xi^3 W\|_{\dot H^{\frac{1+\alpha}{2}}}^2 . \end{aligned} \tag{4.17}$$

with $C_1 = \|\phi'''\|_{L^\infty}$ and where we choose the constant $X > 0$ below.

Then we can combine the estimate by choosing three positive constants, say A, B and C to obtain the functional (that can be seen as a function of t)

$$J = \|W\|_{L^2}^2 + A\|\partial_\xi W\|_{L^2}^2 + B\|\partial_\xi^2 W\|_{L^2}^2 + C\|\partial_\xi^3 W\|_{L^2}^2$$

that clearly satisfies that there exist constants C_* and C^* such that

$$C_*\|W\|_{H^3}^2 \le J \le C^*\|W\|_{H^3}^2 . \tag{4.18}$$

Combining these estimates we obtain

$$\begin{aligned} &\frac{1}{2}\frac{dJ}{dt} - \left(\|W\|_{L^\infty} + AC_0 + BC_1 + CC_1\right)\|W\|_{\dot H^1}^2 \\ &-B\left(\|\partial_\xi^2 W\|_{L^\infty} + 3C_0\right)\|W\|_{\dot H^2}^2 - C\left(5\|\partial_\xi^2 W\|_{L^\infty} + 5C_0 + C_1\right)\|W\|_{\dot H^3}^2 \\ &+a_\alpha\left(\|W\|_{\dot H^{\frac{1+\alpha}{2}}}^2 + A\|W\|_{\dot H^{\frac{3+\alpha}{2}}}^2 + B\|W\|_{\dot H^{\frac{5+\alpha}{2}}}^2 + C\|W\|_{\dot H^{\frac{7+\alpha}{2}}}^2\right) \le 0 . \end{aligned}$$

Then we can estimate as follows

$$(AC_0 + BC_1 + CC_1)\|W\|_{\dot H^1}^2 \le \frac{a_\alpha}{2}\left(\|W\|_{\dot H^{\frac{1+\alpha}{2}}}^2 + \frac{A}{2}\|W\|_{\dot H^{\frac{3+\alpha}{2}}}^2\right) , \tag{4.19}$$

$$3(CC_1 + BC_0)\|W\|_{\dot H^2}^2 \le \frac{a_\alpha}{2}\left(\frac{A}{2}\|W\|_{\dot H^{\frac{3+\alpha}{2}}}^2 + \frac{B}{2}\|W\|_{\dot H^{\frac{5+\alpha}{2}}}^2\right) , \tag{4.20}$$

$$C\left(5C_0 + C_1\right)\|W\|_{\dot H^3}^2 \le \frac{a_\alpha}{2}\left(\frac{B}{2}\|W\|_{\dot H^{\frac{5+\alpha}{2}}}^2 + C\|W\|_{\dot H^{\frac{7+\alpha}{2}}}^2\right) . \tag{4.21}$$

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In order to obtain this we use (4.12), this implies, identifying coefficients, that the following must be satisfied:

$$\frac{2}{a_\alpha}(AC_0 + BC_1 + CC_1) = \left(\frac{A}{2}\right)^{\frac{1-\alpha}{2}},$$
$$\frac{12}{a_\alpha A}(CC_1 + BC_0) = \left(\frac{B}{A}\right)^{\frac{1-\alpha}{2}},$$
$$B = C\left(\frac{2^{\alpha+3}}{a_\alpha}(5C_0 + C_1)\right)^{\frac{2}{1+\alpha}}.$$

This can be solved using the third equation to eliminate C from the second one that can then be solve for B/A. Then one can eliminate C and B from the first equation to solve for A, and recovering B and C from the second and third equations, one gets

$$A = \frac{1}{2^{\frac{3-\alpha}{1+\alpha}}}\left(\frac{a_\alpha L_1 L_2}{C_0 L_1 L_2 + C_1(L_1+1)}\right)^{\frac{2}{1+\alpha}},\ B = \frac{A}{L_2},\ C = \frac{B}{L_1} \tag{4.22}$$

where

$$L_1 = \left(\frac{2^{\alpha+3}}{a_\alpha}(5C_0 + C_1)\right)^{\frac{2}{1+\alpha}},\ L_2 = \left(\frac{12}{a_\alpha}\left(C_0 + \frac{C_1}{L_1}\right)\right)^{\frac{2}{1+\alpha}}.$$

Finally, we can also estimate the terms that contain coefficients with L^∞ norms of W and/or its second derivative. Namely, the following hold easily from (4.13)

$$\|W\|^2_{\dot H^1} \le \max\{1, 2/A\}\left(\|W\|^2_{\dot H^{\frac{1+\alpha}{2}}} + \frac{A}{2}\|W\|^2_{\dot H^{\frac{3+\alpha}{2}}}\right) \tag{4.23}$$

$$B\|W\|^2_{\dot H^2} \le \frac{B}{A}\max\{1, 2A/B\}\left(\frac{A}{2}\|W\|^2_{\dot H^{\frac{3+\alpha}{2}}} + \frac{B}{2}\|W\|^2_{\dot H^{\frac{5+\alpha}{2}}}\right) \tag{4.24}$$

$$5C\|W\|^2_{\dot H^3} \le \frac{10C}{B}\max\{1, B/2C\}\left(\frac{B}{2}\|W\|^2_{\dot H^{\frac{5+\alpha}{2}}} + C\|W\|^2_{\dot H^{\frac{7+\alpha}{2}}}\right) \tag{4.25}$$

and hence the combined estimate reads:

$$\begin{aligned}&\frac{1}{2}\frac{dJ}{dt} + \left(\frac{a_\alpha}{2} - \max\{1, \frac{2}{A}\}\|W\|_{L^\infty}\right)\left(\|W\|^2_{\dot H^{\frac{1+\alpha}{2}}} + \frac{A}{2}\|W\|^2_{\dot H^{\frac{3+\alpha}{2}}}\right)\\ &+ \left(\frac{a_\alpha}{2} - \max\{\frac{B}{A}, 2\}\|\partial_\xi^2 W\|_{L^\infty}\right)\left(\frac{A}{2}\|W\|^2_{\dot H^{\frac{3+\alpha}{2}}} + \frac{B}{2}\|W\|^2_{\dot H^{\frac{5+\alpha}{2}}}\right)\\ &+ \left(\frac{a_\alpha}{2} - \max\{10\frac{C}{B}, 5\}\|\partial_\xi^2 W\|_{L^\infty}\right)\left(\frac{B}{2}\|W\|^2_{\dot H^{\frac{5+\alpha}{2}}} + C\|W\|^2_{\dot H^{\frac{7+\alpha}{2}}}\right)\\ &\le 0\,.\end{aligned} \tag{4.26}$$

By the Sobolev embedding and (4.18) we have

$$\|W\|_{L^\infty},\ \|\partial_\xi^2 W\|_{L^\infty} \le \|W\|_{H^3} \le \sqrt{\frac{1}{C_*}J}\,.$$

then letting the initial data be small enough such that

$$J(0) < C_* a_\alpha^2(\min\{1/5, B/10C, A/B, A/2\})^2/8\,;$$

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this and (4.26) imply the existence of a $\lambda > 0$ and a $\bar{\lambda} > 0$, such that

$$\frac{dJ}{dt} \leq -\lambda \left(\|W\|^2_{\dot{H}^{\frac{1+\alpha}{2}}} + A\|W\|^2_{\dot{H}^{\frac{3+\alpha}{2}}} + B\|W\|^2_{\dot{H}^{\frac{5+\alpha}{2}}} + C\|W\|^2_{\dot{H}^{\frac{7+\alpha}{2}}} \right) \leq -\bar{\lambda}\|U\|^2_{H^2}$$

for all $t > 0$. Integration with respect to time concludes the proof. □

5. The proof of Theorem 4.1

We now prove the well-posedness of the Cauchy problem for (4.5) for a given initial data $W(0,x) = W_0(x) \in H^s(\mathbb{R})$ with $s \geq 3$. In fact we show that the operators involved satisfy certain properties, collected in Lemma 5.1 below. Then, we can prove the existence of the Cauchy problem for (4.3) by applying Lemma 5.1 and then we can apply the lemma again (writing $(\partial_\xi W)^2 = U\partial_\xi W$ in (4.5)) to conclude local existence of the Cauchy problem associated to (4.5).

In the analysis we follow the semigroup approach for the Korteweg-de Vries equation by Pazy in [16, Section 8.5], which is a variant of Kato [10], Namely, one has to use [16, Theorem 6.4.3] and the fact that the conditions of the theorem can be relaxed for time independent and transport type operators, as is done in [16, Section 8.5] for the KdV equation. We can then use the following version of [16, Theorem 6.4.3] to conclude local existence:

Lemma 5.1. *Let X and Y be Banach spaces such that Y is densely and continuously embedded in X. For every $r > 0$, let $A(v)$ be a family of operators $A(v)$, $v \in B_r := \{v \in Y : \|v\|_Y \leq r\}$ that satisfies the conditions*

(i) *Each of the operators of family $A(v)$, with $v \in B_r$, generate a C_0 semigroup $T_v(t)$ in Y such that $\|T_v(t)\| \leq \exp(\beta t)$ where $\beta \geq c_0\|v\|_Y$ with c_0 independent of v.*

(ii) *There is an isomorphism S from Y onto X such that, for every $v \in B_r$, $SA(v)S^{-1} - A(v)$ is a bounded operator in X and*

$$\|SA(v)S^{-1} - A(v)\|_{X\to X} \leq C_1 \quad \text{for all } v \in B_r\,.$$

(iii) *For each $v \in B_r$, $D(A(v)) \subset Y$, $A(v)$ is a bounded linear operator from Y into X and*

$$\|A(v_1) - A(v_2)\|_{Y\to X} \leq C_2\|v_1 - v_2\|_X \quad \text{for all } v_1\,, v_2 \in B_r\,.$$

Then, there exists a $T > 0$ such that the quasilinear problem

$$\begin{cases} \partial_t u + A(u)u = 0 & \text{for } 0 \leq t \leq T\,, \\ u(0) = u_0 \in Y\,, \end{cases} \tag{5.1}$$

has a unique mild solution $u \in C([0,T], Y) \cap C^1([0,T], X)$.

In order to verify the conditions of the lemma, we shall split the homogeneous linear operators of (4.3) (and of (4.5)) into two operators. Namely, we take

$$A_0 : D(A_0) = H^3(\mathbb{R}) \mapsto L^2(\mathbb{R})\,,\, u \longmapsto \partial_\xi^3 u\,, \tag{5.2}$$

$$A_2 : D(A_2) = H^2(\mathbb{R}) \mapsto L^2(\mathbb{R})\,,\, u \longmapsto \partial_\xi \mathcal{D}^\alpha u\,. \tag{5.3}$$

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In addition we define the following family of transport operators for $v \in B_r$:

$$A_1(v) \,:\, D(A_1(v)) = H^1(\mathbb{R}) \mapsto L^2(\mathbb{R})\,,\; u \mapsto v\partial_\xi u\,. \tag{5.4}$$

In order to show that the conditions of Lemma 5.1 are satisfied, we first derive some properties of these operators.

Lemma 5.2. (i) *A_0 is the infinitesimal generator of a C_0 group of isometries on $L^2(\mathbb{R})$.*

(ii) *For every $v \in H^s(\mathbb{R})$ with $s \geq 3$, the operator $A_1(v)$ is well-defined with domain $D(A_1(v)) = H^1(\mathbb{R})$ (dense in $L^2(\mathbb{R})$). Moreover, the operator $-(A_1(v) + \beta I)$ is dissipative for all $\beta \geq \beta_0(v) = c_0\|v\|_{H^s}$ where c_0 is independent of v. Also, if $u \in H^3(\mathbb{R})$ and $\varepsilon > 0$, the estimate*

$$\|A_1(v)\|_{L^2} \leq \varepsilon\|\partial_x^3 u\|_{L^2} + C_1(\varepsilon, \|v\|_{H^s})\|u\|_{L^2} \tag{5.5}$$

holds for some positive constant C_1 depending on ε and on $\|v\|_{H^s}$.

(iii) *For every $0 < \alpha < 1$, the operator A_2 is well-defined with domain $D(A_2) = H^2(\mathbb{R})$ (dense in $L^2(\mathbb{R})$). Moreover, A_2 is dissipative with*

$$(A_2u, u) = -a_\alpha\|u\|^2_{\dot{H}^{\frac{\alpha+1}{2}}} \leq 0 \quad \text{for } u \in H^2(\mathbb{R})\,,$$

where $a_\alpha = \sin(\frac{\alpha\pi}{2}) > 0$. Finally, it satisfies for $u \in H^3(\mathbb{R})$ and $\varepsilon > 0$ the estimate

$$\|A_2u\|_{L^2} \leq \varepsilon\|\partial_x^3 u\|_{L^2} + C_2(\varepsilon)\|u\|_{L^2} \tag{5.6}$$

with $C_2(\varepsilon) = (\varepsilon p)^{\frac{1}{1-p}}(\frac{p}{p-1})^{-1}$ and $p = \frac{3}{\alpha+1} > 1$.

Proof. The proofs of *(i)* and *(ii)* can be found in [16, Lemma 8.5.2 and Lemma 8.5.3] respectively. We next prove *(iii)*.

That A_2 is dissipative follows from Plancherel's formula and the fact that $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$ imply

$$\int_{\mathbb{R}} \operatorname{sgn}(k)|k|^\gamma|\hat{u}(k,t)|^2\,dk = 0 \quad \text{for } \gamma > 0$$

(see [1]). Namely, for every $u \in H^2(\mathbb{R})$ and $0 < \alpha < 1$,

$$\begin{aligned}(A_2u, u) &= \int_{\mathbb{R}} \left(\partial_x \mathcal{D}^\alpha u\right) u dx = -\int_{\mathbb{R}} \left(a_\alpha - ib_\alpha \operatorname{sgn}(k)\right)|k|^{\alpha+1}|\mathcal{F}(u)(k)|^2 dk \\ &= -a_\alpha \int_{\mathbb{R}} |k|^{\alpha+1}|\mathcal{F}(u)(k)|^2 dk = -a_\alpha\|u\|^2_{\dot{H}^{\frac{\alpha+1}{2}}} \leq 0\,.\end{aligned}$$

It remains to prove (5.6). Indeed, for every $u \in H^3(\mathbb{R})$ and $0 < \alpha < 1$, we obtain the estimate

$$\begin{aligned}\|A_2u\|^2_{L^2} &= \int_{\mathbb{R}} \left||k|^{\alpha+1}\mathcal{F}(u)(k)\right|^2\,dk \\ &= \int_{\mathbb{R}} \left(|k|^{\alpha+1}|\mathcal{F}(u)(k)|^{\frac{\alpha+1}{3}}\right)^2 \left(|\mathcal{F}(u)(k)|^{1-\frac{\alpha+1}{3}}\right)^2\,dk \\ &\leq \left(\int_{\mathbb{R}} \left(|k|^3|\mathcal{F}(u)(k)|\right)^2\,dk\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |\mathcal{F}(u)(k)|^2\,dk\right)^{\frac{p-1}{p}} \\ &= \|\partial_x^3 u\|^{\frac{2}{p}}_{L^2}\,\|u\|^{\frac{2(p-1)}{p}}_{L^2}\,,\end{aligned}$$

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where we have used again Plancherel's formula, the fact that $(a_\alpha^2 + b_\alpha^2) = 1$, and Hölder's inequality with $p = 3/(\alpha+1) > 1$. Taking the square root of the last inequality and using Young's inequality [1] for some $\varepsilon > 0$, we infer that

$$\|A_2 u\|_{L^2} \le \varepsilon \|\partial_x^3 u\|_{L^2} + C_2(\varepsilon)\|u\|_{L^2}$$

with $C_2(\varepsilon) = (\varepsilon p)^{\frac{1}{1-p}}(\frac{p}{p-1})^{-1}$ and $p = \frac{3}{\alpha+1} > 1$. ☐

Lemma 5.3. (i) *For every $v \in H^s$, the operator $A_0 - A_1(v)$ is the infinitesimal generator of a C_0-semigroup $T_v(t)$ on L^2 satisfying $T_v(t) \le \exp(\beta t)$ for every $\beta \ge \beta_0(v) = c_0\|v\|_{H^s}$, where c_0 is a constant independent of v.*

(ii) *For every $0 < \alpha < 1$ and $v \in H^s(\mathbb{R})$ with $s \ge 3$, the operator $A_1(v) - A_2$ is well-defined from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Moreover, the operator $A(v) = A_0 + A_2 - A_1(v)$ is the infinitesimal generator of a C_0-semigroup $S_v(t)$ on L^2 satisfying*

$$\|S_v(t)\| \le \exp(\beta t) \tag{5.7}$$

for every $\beta \ge \beta_0(v) := c_0\|v\|_{H^s}$, where c_0 is a constant independent of v.

Proof. The proof of the statement *(i)* can be found in [16, Lemma 8.5.3]. We then prove *(ii)*.

Due to $v \in H^s(\mathbb{R})$ with $s \ge 3$, $\partial_x v \in H^{s-1}(\mathbb{R})$ and $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ such that $\|\partial_x v\|_{L^\infty} \le C\|\partial_x v\|_{H^{s-1}} \le C\|v\|_{H^s}$. For every $u \in H^2(\mathbb{R})$,

$$((A_1(v) - A_2)u, u) \ge -c_0\|v\|_{H^s}\|u\|_{L^2}^2 + a_\alpha \|u\|^2_{\dot H^{\frac{\alpha+1}{2}}} \ge -c_0\|v\|_{H^s}\|u\|_{L^2}^2\,,$$

since c_0 and a_α are positive constants. Therefore $-(A_1(v) - A_2 + \beta I)$ is dissipative for all $\beta \ge \beta_0(v) := c_0\|v\|_{H^s}$. A_0 is a skew-adjoint operator, whence $A_0 + A_2 - A_1(v) - \beta I$ is also dissipative for $\beta \ge \beta_0(v)$. Moreover, due to the estimates (5.5) and (5.6),

$$\begin{aligned}\|(A_1(v) - A_2 + \beta I)u\|_{L^2} &\le \|A_1(v)u\|_{L^2} + \|A_2 u\|_{L^2} + |\beta|\|u\|_{L^2}\\ &\le 2\varepsilon\|\partial_x^3 u\|_{L^2} + C_3(\beta,\varepsilon,\|v\|_{H^s})\|u\|_{L^2}\end{aligned}$$

holds with a positive constant $C_3(\beta,\varepsilon,\|v\|_{H^s}) := C_1(\varepsilon,\|v\|_{H^s}) + C_2(\varepsilon) + |\beta|$. Due to [16, Corollary 3.3.3] and the last estimate with $0 < \varepsilon < \frac{1}{2}$, we conclude that $A_0 + A_2 - A_1(v) - \beta I$ is the infinitesimal generator of a C_0-semigroup of contractions on X for every $\beta \ge \beta_0(v)$. Therefore $A_0 + A_2 - A_1(v)$ is the infinitesimal generator of a C_0-semigroup $T_v(t)$ satisfying (5.7). ☐

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We need only to check that the assumptions of Lemma 5.1 are satisfied for the operator, in the notation of this section

$$A(u)u = 2\phi' u + (2\phi - c + 2u)\partial_\xi u - \partial_x \mathcal{D}^\alpha u - \tau\partial_x^3 u\,.$$

[1]For positive real numbers a, b and ε, as well as $1 < p,\ q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the inequality

$$a\,b \;\le\; \varepsilon\, a^p \;+\; C(\varepsilon)\, b^q \quad \text{with } C(\varepsilon) = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$$

holds.

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We first observe that the second operator term can be seen as the sum of three operators of the form A_1 and the results of the previous lemmas apply. The first term has not been analysed in the previous lemmas, but since $\phi' \in H^{s+1}(\mathbb{R})$ the operator is bounded and adding it to the ones of the form A_1 preserves the properties shown above. Thus, Lemma 5.2 and 5.3 show that *(i)* holds with A_1 given by

$$A_1(v)u = (2\phi - c + 2w)\partial_x u \quad \text{with} \quad v := 2\phi - c + 2w\,,$$

we only observe that the constants that depend on $\|v\|_{H^s}$ in these lemmas now depend on $\|w\|_{H^s}$, c, $\|\phi\|_\infty$ and $\|\phi'\|_\infty$.

Let us show that *(ii)* holds. We proceed as in [16]. For $s \geq 3$ (for $s \geq 3/2$, in fact) the operator

$$f \to \Lambda^s f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(ix\cdot\xi)(1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi)\, d\xi\,,$$

is an isomorphism from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$. We notice that for u, $v \in H^s(\mathbb{R})$

$$(\Lambda^s A(v)\Lambda^{-s} - A(v))u = (\Lambda^s v - v\Lambda^s)\Lambda^{-s}\partial_x u + 2(\Lambda^s \phi' \Lambda^{-s} u - \phi' u)\,, \tag{5.8}$$

since for $u \in H^s$

$$\Lambda^s \partial_x^3 \Lambda^{-s} u = \partial_x^3 u \quad \text{and} \quad \Lambda^s \partial_x \mathcal{D}^\alpha \Lambda^{-s} u = \partial_x \mathcal{D}^\alpha u\,,$$

(see [16] for details). Therefore, for u, $v \in H^s$ and the multiplication operator $u \mapsto vu$, we deduce from (5.8) and [16, Lemma 8.5.4] that

$$\|(\Lambda^s v - v\Lambda^s)\Lambda^{1-s}\Lambda^{-1}\partial_x u\|_{L^2} \leq C\|v\|_{H^s}\|u\|_{L^2}\,.$$

It is easy to also show that

$$\|\Lambda^s \phi' \Lambda^{-s} u - \phi' u\|_{L^2} \leq (\|\phi'\|_{H^s} + \|\phi'\|_\infty)\|u\|_{L^2} \leq C\|u\|_{L^2}\,.$$

This estimate and $H^s(\mathbb{R})$ being dense in $\mathbb{R}$, implies that $\|SA(v)S^{-1} - A(v)\|_{L^2\to L^2} \leq C(\|w\|_{H^s} + c + \|\phi'\|_\infty + 1) \leq C$ for $w \in B_r \subset H^s$ and *(ii)* is satisfied with $S = \Lambda^s$.

It remains to show *(iii)*. Observe that for $s \geq 3$ and $0 < \alpha < 1$,

$$H^3(\mathbb{R}) = D(A(v)) \supset H^s(\mathbb{R}) \quad \text{for every} \quad v \in L^\infty(\mathbb{R})\,,$$

and also

$$\begin{aligned}\|A(v)u\|_{L^2} &\leq \|2\phi' u\|_{L^2} + \|v\partial_x u\|_{L^2} + \|\partial_x \mathcal{D}^\alpha u\|_{L^2} + \|\partial_x^3 u\|_{L^2} \\ &\leq 2\|\phi'\|_{L^\infty}\|u\|_{L^2} + \|v\|_{L^\infty}\|\partial_x u\|_{L^2} + \|u\|_{\dot{H}^{\alpha+1}} + \|\partial_x^3 u\|_{L^2} \\ &\leq C(1+\|v\|_\infty)\|u\|_{H^s}\,.\end{aligned}$$

Therefore, for $w \in B_r$, $A(v)$ is a bounded operator from $H^s(\mathbb{R})$ into $L^2(\mathbb{R})$. Moreover, if $v_1, v_2 \in B_r$ and $u \in H^s(\mathbb{R})$, then

$$\begin{aligned}\|(A(v_1) - A(v_2))u\|_{L^2} &= \|(w_1 - w_2)\partial_x u\|_{L^2} \\ &\leq \|w_1 - w_2\|_{L^2}\|\partial_x u\|_{L^\infty} \quad \leq C\|w_1 - w_2\|_{L^2}\|u\|_{H^s}\end{aligned}$$

and *(iii)* holds as well, since $v_1 - v_2 = w_1 - w_2$. □

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Appendix

Appendix A. The linear problem (2.8) on $(-\infty, \xi_0]$

In this appendix we show that the only solutions of the linear problem (2.8) are exponential functions in suitable weighted spaces. We shall assume without loss of generality that $\xi_0 = 0$ throughout this section. We use the approach introduced for *Wiener-Hopf* integral equations of the form

$$W(\xi) - \int_0^\infty K(\xi - y)W(y)dy = 0 \qquad \xi \geq 0\,, \tag{A.1}$$

which are related to the Fredholm property by conditions on its symbol, see [21]. We use the result by Krein [13, 14] that extends the method to equations with L^1-integrable kernels. Namely:

Theorem A.1 (Krein (1958&62))**.** *Let $K \in L^1(\mathbb{R})$. If the symbol $a(k) := 1 - \int_{\mathbb{R}} e^{-ixk}K(x)\,dx$ $(= 1 - \sqrt{2\pi}\mathcal{F}[K])$ is elliptic, i.e. $\inf_{s\in\mathbb{R}} |a(s)| > 0$, and the winding number of the curve $\{a(s) : s \in (-\infty, \infty)\}$ around the origin is a non-negative integer r, then (A.1) has exactly r linearly independent solutions in any of the Lebesgue spaces $L^p(\mathbb{R}_+)$, with $1 \leq p \leq \infty$.*

We observe that have adapted Theorem A.1 from the original result by Krein that is stated for $\sqrt{2\pi}\mathcal{F}(-k)$ instead of $\mathcal{F}(k)$.

It is not obvious that (2.8) can be transformed into a Wiener-Hopf equation, i.e. to the form (A.1). In particular, we will investigate the problem on weighted spaces, such that it is admissible to consider the integrated equation and compute its symbol.

For a generalisation of the Wiener-Hopf method to other spaces we refer to [5] and for generalisations to convolution kernels being distributions we refer to [7].

In order to write (2.8) as a Wiener-Hopf equation we first change variables so that it is posed in $\mathbb{R}_+$ rather than in $\mathbb{R}_-$:

Lemma A.1. *If $V \in H^3(\mathbb{R}_+)$ is a solution of the integral equation*

$$0 = \tau V(\xi) + \int_\xi^\infty \int_y^\infty \mathcal{D}^\alpha_-[V](z)\,dz\,dy - h'(\phi_-) \int_\xi^\infty \int_y^\infty V(z)\,dz\,dy \tag{A.2}$$

where $\mathcal{D}^\alpha_-[V](\xi) := -d_\alpha \int_\xi^\infty \frac{V'(y)}{(y-\xi)^\alpha}\,dy$, then $v(\xi) := V(-\xi)$ for $\xi \in \mathbb{R}_-$ is a solution of (2.8).

Moreover, if $v \in H^3(\mathbb{R}_-)$ is a solution of (2.8) whose primitives are integrable, then $V(\xi) := v(-\xi)$ for $\xi \in \mathbb{R}_+$ is a solution of (A.2).

Proof. Due to a Sobolev embedding $H^3(\mathbb{R}_-) \hookrightarrow C^2_b(\mathbb{R}_-)$, a solution $v \in H^3(\mathbb{R}_-)$ has a representative in $C^2_b(\mathbb{R}_-)$, such that equation (2.8) holds pointwise. We perform the change of variables in (2.8) $V(-\xi) = v(\xi)$, such that $V \in H^3(\mathbb{R}_+) \hookrightarrow C^2_b(\mathbb{R}_+)$, $\xi \to -\xi \in \mathbb{R}_+$ and $y \to -y$ inside the integral term, to get

$$0 = \tau V''(\xi) + \mathcal{D}^\alpha_-[V](\xi) - h'(\phi_-)V(\xi) \qquad \forall \xi \in \mathbb{R}_+\,. \tag{A.3}$$

Finally, $V \in H^3(\mathbb{R}_+) \hookrightarrow C^2_b(\mathbb{R}_+)$ implies that V has a representative in $C^2_b(\mathbb{R}_+)$, such that

$$\lim_{\xi\to+\infty} V(\xi) = 0\,, \quad \lim_{\xi\to+\infty} V'(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to+\infty} V''(\xi) = 0\,.$$

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Integrating (A.3) twice under the assumption that the primitives of V are integrable and reverting the change of variables yields (A.2). $\square$

Lemma A.2. *Suppose that $\mu > 0$ and let*

$$V \in H^3_\mu(\mathbb{R}_+) = \{f \in H^3(\mathbb{R}_+) :\ f(\xi) = e^{-\mu\xi} g(\xi) \quad \text{for some} \quad g \in H^3(\mathbb{R}_+)\}$$

be a solution of (A.2). Then, the corresponding equation for W where $V(\xi) = e^{-\mu\xi} W(\xi)$ and $W \in H^3(\mathbb{R}_+)$ can be written in the form (A.1) with the L^1-integrable kernel

$$K(z) := -(\theta(-z)e^{\mu z}(-z)^{\alpha-1}) + h'(\phi_-)\,(\theta(-z)e^{\mu z}) * (\theta(-z)e^{\mu z}) \tag{A.4}$$

having support on the negative real line $\mathbb{R}_-$, and has symbol

$$a_\mu(k) = \frac{\tau(\mu - ik)^2 + (\mu - ik)^\alpha - h'(\phi_-)}{(\mu - ik)^2}\,. \tag{A.5}$$

Proof. Let $V(\xi) = e^{-\mu\xi} W(\xi)$ with $W \in H^3(\mathbb{R}_+)$, then (A.2) becomes, after multiplying by $e^{\mu\xi}$,

(A.6)

$$0 = \tau W(\xi) + e^{\mu\xi} \int_\xi^\infty \int_y^\infty \mathcal{D}^\alpha_-[We^{-\mu\cdot}](z)\,dz\,dy - e^{\mu\xi} h'(\phi_-) \int_\xi^\infty \int_y^\infty W(z) e^{-\mu z}\,dz\,dy\,.$$

We have to extract alternative representations for the integral operators in (A.6). The first integral operator satisfies

$$\begin{aligned} e^{\mu\xi} \int_\xi^\infty \int_y^\infty \mathcal{D}^\alpha_-[We^{-\mu\cdot}](z)\,dz\,dy &= \int_\xi^\infty e^{\mu(\xi-y)} \int_y^\infty e^{\mu(y-z)} \mathcal{D}^\alpha_-[We^{\mu(z-\cdot)}](z)\,dz\,dy \\ &= (\theta(-\cdot)e^{\mu\cdot}) * (\theta(-\cdot)e^{\mu\cdot}) * \mathcal{D}^\alpha_-[We^{\mu(z-\cdot)}](z)\,. \end{aligned}$$

Observe that

$$\mathcal{D}^\alpha_-[We^{\mu(z-\cdot)}](z) = -d_\alpha \int_z^\infty \frac{\big(W(\sigma)e^{\mu(z-\sigma)}\big)'}{(\sigma - z)^\alpha}\,d\sigma = -d_\alpha \bigg([\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}] * W' - [\mu\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}] *$$

The convolution kernel $(\theta(-\cdot)e^{\mu\cdot})$ is L^1 integrable and its Fourier transform satisfies $\mathcal{F}[\theta(-\cdot)e^{\mu\cdot}](k) = (\mu - ik)^{-1}/\sqrt{2\pi}$. We use the identities $\mathcal{F}[f*g](k) = \sqrt{2\pi}\mathcal{F}[f](k)\,\mathcal{F}[g](k)$, $\mathcal{F}[fe^{\mu\cdot}](k) = \mathcal{F}[f](k + i\mu)$, and

$$\mathcal{F}\left[\frac{\theta(-\xi)}{(-\xi)^\alpha}\right](k) = \mathcal{F}\left[\frac{\theta(\xi)}{\xi^\alpha}\right](-k) = \frac{(-ik)^{\alpha-1}}{d_\alpha\sqrt{2\pi}}$$

for $k \in \mathbb{C}$, to compute

$$\begin{aligned} \mathcal{F}\bigg[e^{\mu\cdot}\mathcal{D}^\alpha[We^{-\mu\cdot}](.)\bigg](k) &= -d_\alpha \mathcal{F}\bigg[[\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}] * W' - [\mu\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}] * W\bigg] \\ &= -d_\alpha\sqrt{2\pi}\Big(\mathcal{F}[\theta(-\cdot)(-\cdot)^{-\alpha}](k + i\mu)\Big)\Big((ik - \mu)\mathcal{F}[W](k)\Big) \\ &= (\mu - ik)^\alpha \mathcal{F}[W](k)\,. \end{aligned}$$

Therefore, the first integral operator is a pseudo-differential operator with

$$\mathcal{F}\left[e^{\mu\xi} \int_\xi^\infty \int_y^\infty \mathcal{D}^\alpha_-[We^{-\mu\cdot}](z)\,dz\,dy\right](k) = (\mu - ik)^{\alpha-2}\,\mathcal{F}[W](k)\,. \tag{A.7}$$

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The second integral operator satisfies

$$-e^{\mu\xi}h'(\phi_-)\int_\xi^\infty\int_y^\infty W(z)e^{-\mu z}\,dz\,dy = -h'(\phi_-)\,(\theta(-.)e^{\mu\cdot})*(\theta(-.)e^{\mu\cdot})*W\,,$$

whence the integral operator is a pseudo-differential operator with

(A.8)

$$\mathcal{F}\Big[-e^{\mu\xi}h'(\phi_-)\int_\xi^\infty\int_y^\infty W(z)e^{-\mu z}\,dz\,dy\Big](k) = -h'(\phi_-)\,(\mu-ik)^{-2}\,\mathcal{F}[W](k)\,.$$

Thus the linear operator in (A.6) is a pseudo-differential operator with symbol (A.5).

It remains to justify that (A.6) is a Wiener-Hopf equation with some L^1 integrable kernel. Indeed, inverting the symbols (A.7) and (A.8) allows to write (A.6) as $\tau W(x) - K * W(x) = 0$ with K given by (A.4), which has support on the negative real line $\mathbb{R}_-$ and is L^1 integrable. □

Theorem A.2. *Suppose that* $0 < \mu < \min\{\lambda, h'(\phi_-)/(2-\alpha)\}$*, where* λ *is the unique positive real root of (2.9). Then, all solutions of (2.8) that are in the space*

$$L^\infty_w(\mathbb{R}_-) = \{f \in L^\infty(\mathbb{R}_+):\ f(\xi) = e^{\mu\xi}g(\xi) \quad \textit{for some} \quad g \in L^\infty(\mathbb{R}_-)\}$$

are given by the one-parameter family $\{be^{\lambda\xi}:\ b \in \mathbb{R}\}$.

Proof. Let us see that the conditions of Theorem A.1 are satisfied by the symbol (A.5). The symbol a_μ gives a closed curve $s \to a_\mu(s)$ for $s \in \mathbb{R}$, since $\lim_{s\to\pm\infty} a_\mu(s) = \tau$. The ellipticity follows from the fact that the numerator of (A.5) only vanishes identically at $s = 0$ and $\mu = \lambda$ (by assumption $0 < \mu < \lambda$) and the denominator of

$$|a_\mu(s)|^2 = \frac{|\tau(\mu-is)^2 + (\mu-is)^\alpha - h'(\phi_-)|^2}{(\mu^2-s^2)^2 + 4\mu^2 s^2}$$

does not vanish.

Moreover, the winding number of the closed curve is a well-defined integer. In order to compute the winding number around the origin we add the number of times that the curve crosses the negative real line in the anticlockwise direction and subtract the number of times it does it in the clockwise one.

There is a crossing at $s = 0$, since

$$a_\mu(0) = \frac{\tau\mu^2 + \mu^\alpha - h'(\phi_-)}{\mu^4} < 0 \quad \mu \in (0,\lambda)\,.$$

Let us see that this is the only one. In order to do that we compute

(A.9)

$$\mathrm{Re}(a_\mu(s)) = \tau + \frac{(\mu^2+s^2)^{\frac{\alpha}{2}}\big((\mu^2-s^2)\cos(\Theta_{s,\mu}\alpha) + 2s\mu\sin(\Theta_{s,\mu}\alpha)\big) - (\mu^2-s^2)h'(\phi_-)}{\mu^4+s^4+2\mu^2s^2}\,,$$

and

$$\mathrm{Im}(a_\mu(s)) = \frac{(\mu^2+s^2)^{\frac{\alpha}{2}}\big(2s\mu\cos(\Theta_{s,\mu}\alpha) - (\mu^2-s^2)\sin(\Theta_{s,\mu}\alpha)\big) - 2s\mu h'(\phi_-)}{\mu^4+s^4+2\mu^2s^2}\,.$$

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We observe that when the curve crosses the real line then $\mathrm{Im}(a\mu(s)) = 0$, imposing this condition and using (2.6) gives

$$h'(\phi_-) = (\mu^2 + s^2)^{\frac{\alpha+2}{2}} \frac{\sin(\Theta_{s,\mu}\alpha)}{2s\mu} > 0$$

and substituting this expression into (A.9) gives

$$\mathrm{Re} = \tau + (\mu^2 + s^2)^{\frac{\alpha}{2}} \frac{\sin(\Theta_{s,\mu}\alpha)}{2s\mu} > 0\,,$$

thus the curve crosses the negative real line only once. It remains to determine whether the crossing is in the clockwise or anticlockwise direction. We compute[2]

$$\frac{d}{ds}\mathrm{Re}(a_\mu(s))\big|_{s=0} = \frac{3h'(\phi_-)}{\mu^2} > 0$$

and, under the assumption on μ,

$$\frac{d}{ds}\mathrm{Im}(a_\mu(s))|_{s=0} = \frac{(2-\alpha)\mu^\alpha - h'(\phi_-)}{\mu} < 0\,.$$

Thus the curve $a_\mu(s)$ runs once around the origin in the anticlockwise sense, i.e. the winding number is 1. Applying Theorem A.1 and changing from W to V and then to the original variable imply the statement. □

Appendix B. The roots of (2.9), (3.10) and (3.29)

In this appendix we show that (2.9) has exactly one real positive root two complex conjugate roots with negative real part. We prove the result for the more general algebraic equation:

$$f(z) = z^2 + az^\alpha - b \quad \text{for} \quad a\,,b > 0\,, \quad \alpha \in (0,1)\,. \tag{B.1}$$

In order to prove this we use a version of Rouche's theorem as in [4], where it is shown that

$$g(z) = z^2 + az^\alpha + b \quad \text{for} \quad a\,,b > 0\,, \quad \alpha \in (0,1) \tag{B.2}$$

has exactly two complex conjugate roots with negative real part. We observe that (3.10) and (3.29) are of this form, so they have two complex conjugate roots with negative real part.

Lemma B.1. *For any positive values of a, b and any value $\alpha \in (0,1)$. Assume that z is the principal part of z^α $(-\pi < arg(z) < \pi)$, then (B.1) has exactly one real positive root and two complex conjugate roots with negative real part, and (B.2) has exactly two complex conjugate roots with negative real part on the principal branch.*

[2] We give the full expressions of the derivatives for completeness:

$$\frac{d}{ds}\mathrm{Re}(a_\mu(s)) = \frac{(2-\alpha)(\mu^2+s^2)^{\frac{\alpha}{2}}\Big((\mu^2-3s^2)\mu\sin(\Theta_{s,\mu}\alpha) - (3\mu^2-s^2)s\cos(\Theta_{s,\mu}\alpha)\Big) + (3\mu^2-s^2)h'(\phi_-)}{\mu^4+s^4+2\mu^2s^2}$$

and

$$\frac{d}{ds}\mathrm{Im}(a_\mu(s))|_{s=0} = -\frac{(\mu^2+s^2)^{\frac{\alpha}{2}}(2-\alpha)\left((3s^2-\mu^2)\mu\cos(\Theta_{s,\mu}\alpha) + (s^2-3\mu^2)s\sin(\Theta_{s,\mu}\alpha)\right) - 2\mu(\mu^2-3s^2)h'(\phi_-)}{\mu^4+s^4+2\mu^2s^2}\,.$$

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Proof. The statement about (B.2) has been shown in [4] (Theorem 13), we do not prove it here. In fact the proof for (B.1) can be done along the same lines, as follows.

First, it is easy to see that the unique positive real root of (B.1) is the only root with positive real part (see argument following (2.9)). Indeed, we argue by contradiction and assume that there exists a $z_0 \in \mathbb{C}$ that solves $f(z) = 0$ and that $\mathrm{Re}(z_0) > 0$. Since clearly $\bar{z}_0$ must also solve $f(z) = 0$ we can assume that $\arg(z_0) \in (0, \pi/2)$. Then, inspection of $f(z_0)$ shows that $\mathrm{Im}(f(z_0)) > 0$, which contradicts the assumption $f(z_0) = 0$.

It is also easy to show by simple inspection of $f(z)$ that there are neither purely imaginary roots of (B.1) nor negative real ones.

Since $f(z) = 0$ implies $f(\bar{z}) = 0$ we can restrict ourselves to the open sector

$$Q := \{z \in \mathbb{C} : \ \arg(z) \in (\pi/2, \pi)\} .$$

It then remains to show that there is only one $z \in Q$ such that $f(z) = 0$. In order to do that we use a version of Rouche's theorem by T.Estermann [6]. This theorem says that if f and l are regular functions on a simply connected region $\Omega \subset \mathbb{C}$ and if $|f - l| < |f| + |l|$ on $\partial\Omega$ then f and h have the same number of zeros in Ω counted with their multiplicity. We shall then apply this to f as in (B.1), which we shall compare to l given by

$$l(z) = z^2 + i .$$

Let $z \in Q \cap A_R$ where, for any given $R > 0$, A_R denotes the ring $\{z \in \mathbb{C} : 1/R < |z| < R\}$. One can check that if R is large enough so that for all $\theta \in [\pi/2, \pi]$ $z = Re^{\theta i}$ then

$$|f(z)| + |l(z)| > 2R^2 - aR^\alpha - b - 1 > aR^\alpha + b + 1 > |f(z) - l(z)| .$$

In order to prove the strict inequality on the rest of the boundary of $Q \cap A_R$, we argue by contradiction and assume that $|f - l| = |f| + |l|$ in this region. Thus, in particular, there exists a $L > 0$ such that $f = -L\, l$ there. Then for $z = e^{i\theta}/R$ with $\theta \in (\pi/2, \pi)$ we obtain $|\mathrm{Im}(l)| > |\mathrm{Im}(f)|$, but $|\mathrm{Re}(l)| < |\mathrm{Re}(f)|$ if R is sufficiently large, and this contradicts $f = -L\, l$. Finally, if $\theta \in \{\pi/2, \pi\}$ then $\mathrm{Im}(-l) = -c < 0$ and $\mathrm{Im}(f) = a|z| \sin(\alpha\theta) > 0$, and this also contradicts $f = -L\, lh$. Since we can take R as large as we want, this concludes the proof. □

Let us for completeness compute the two term expansion of the roots of (2.9) for very small values of τ (this can be made rigorous be applying the implicit function theorem): A regular expansion gives the real root, in this case it is easy to obtain by inserting the ansatz $\lambda = \lambda_0 + \tau\lambda_1 + O(\tau^2)$, and one gets that

$$\lambda = h'(\phi_-)^{\frac{1}{\alpha}} - \frac{\tau}{\alpha} h'(\phi_-)^{\frac{3-\alpha}{\alpha}} + O(\tau^2) . \tag{B.3}$$

The complex conjugated roots are obtained by first performing the scaling $\lambda = \tau^{-\frac{1}{2-\alpha}}\bar{\lambda}$, and inserting the ansatz $\bar{\lambda} = \bar{\lambda}_0 + \tau^{\frac{1}{2-\alpha}}\bar{\lambda}_1$ in the rescaled equation $\bar{\lambda}^2 + \lambda - \tau^{\frac{1}{2-\alpha}} = 0$. To leading order one gets three zeros, namely $\bar{\lambda}_0 = 0$, $e^{i\pi/(\alpha-2)}$ and $e^{-i\pi/(\alpha-2)}$. The first one corresponds to the real one

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already found, from the other two one then gets (in the original scaling):

$$\lambda = e^{\pm i\pi\frac{1}{\alpha-2}}\frac{1}{\tau^{\frac{1}{2-\alpha}}} + \frac{h'(\phi_-)}{2e^{\pm i\pi\frac{1}{\alpha-2}} + \alpha e^{\pm i\pi\frac{(\alpha-1)}{\alpha-2}}} + O(\tau^{\frac{1}{2-\alpha}})\,. \tag{B.4}$$

A similar approach can be used to compute the expansion of the zeros of

$$\tau z^2 + a z^\alpha + b = 0 \tag{B.5}$$

provided that a and b are of order 1 as $\tau \to 0$. In that case the zeros are approximated by

(B.6)

$$z = a^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}}\frac{1}{\tau^{\frac{1}{2-\alpha}}} - \frac{b}{2a^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}} + a^{\frac{1}{\alpha-2}+1}\alpha e^{\pm i\pi\frac{(\alpha-1)}{\alpha-2}}} + O(\tau^{\frac{1}{2-\alpha}}) \quad \text{as} \quad \tau \to 0^+\,.$$

Appendix C. Computation of the linear problems (2.23), (3.5) and (3.25)

In this appendix we give a way of solving implicitly equations of the type (3.5) and (3.25) for a given inhomogeneity and initial conditions on the unknown and its derivative. The method is by using the Laplace transform and the computations can be found in e.g. [4] and [8], we follow the latter.

Given the initial value problem

$$\tau\psi'' + \mathcal{D}_0^\alpha\psi + a\psi = Q(\eta)\,, \quad {}' = \frac{d}{d\eta} \tag{C.1}$$

subject to

$$\psi(0^+) = C_0\,, \quad \psi'(0^+) = C_1\,. \tag{C.2}$$

we apply the Laplace transform, $\mathcal{L}$ to get

$$\mathcal{L}(\psi)(s) = \frac{1}{\tau s^2 + s^\alpha + a}\left(\mathcal{L}(Q)(s) + (\tau s + s^{\alpha-1})\psi(0^+) + \tau\psi'(0^+)\right)\,, \tag{C.3}$$

we recall that $\mathcal{L}(f)(s) = \int_0^\infty e^{-s\eta} f(\eta)d\eta$. And using that $\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s)\,\mathcal{L}(g)(s)$ then:

$$\psi = \psi(0^+)\mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right) + \tau\bar{\psi}'(0^+)\mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right) + \mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right) * \mathcal{Q}\,.$$

For simplicity, we let

$$v(\eta) = \mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right) \quad \text{and} \quad \tilde{v}(s) = \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a} \tag{C.4}$$

and observe that, since

$$\frac{1}{\tau s^2 + s^\alpha + a} = \frac{1}{a}(1 - s\tilde{v}(s))$$

then

$$\mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right)(\eta) = -\frac{1}{a}v'(\eta)\,.$$

We also observe that:

$$\lim_{\eta\to 0^+} v(\eta) = \lim_{s\to\infty} s\tilde{v}(s) = 1 \quad \text{and} \quad \lim_{\eta\to 0^+} v'(\eta) = 0\,. \tag{C.5}$$

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We can write the expression of ψ in terms of v instead to get

$$\psi(\eta) = \psi(0^+)v(\eta) - \frac{\tau}{a}\bar{\psi}'(0^+)v'(\eta) - \frac{1}{a}\int_0^\eta v'(y)Q(\eta - y)\,dy\,. \tag{C.6}$$

For $a > 0$, let us sketch the computation of $v(\eta)$, we recall that since this is the inverse Laplace transform of $\tilde{v}(s)$, we have to compute:

$$v(\eta) = \frac{1}{2\pi i}\int_{Br} e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\,ds \tag{C.7}$$

where $Br \subset \mathbb{C}$ is a Bromwich contour:

$$Br := \{s : \ \mathrm{Re}(s) = \sigma \geq 1 \ \&\ \mathrm{Im}(s) \in (-\infty, \infty)\} \tag{C.8}$$

moreover, we restrict to the principal representation of s, namely, here $\arg(s) \in (-\pi, \pi]$. We follow the approach in [8], although they do it in some more detail for a different example and in the analogous case the estate the formulae. The results of [4] about the zeros of $\tau z^2 + z^\alpha + a$ apply here to the poles of the integrand in (C.7) for $a > 0$, thus, there exist two zeros that are complex conjugates and have negative real part, let them be denoted by s_1 and s_2. Then the contribution to the integral of these poles can be computed away from the Riemann surface cut (since $\alpha \in (0, 1)$) that is the negative part of the real line. One can then split the integral as follows:

(C.9)

$$v(\eta) = \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{Ha(\delta)} e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\,ds + \sum_{s=s_1,s_2}\mathrm{Res}\left(e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right)$$

where $Ha(\delta)$ is the Hankel path in $\mathbb{C}$

(C.10)

$$Ha(\delta) = \{s = -r + i\delta\,,\ r > 0\} \cup \{s = -r - i\delta\,,\ r > 0\} \cup \{s = \delta e^{i\beta}\,,\ \beta \in [-\pi/2, \pi/2]\}$$

It is easy to see by splitting the first integral term of (C.9) on these three contours that the one corresponding to the semicircle tends to 0 as δ tends to 0. The contribution of the other two is symmetric and gives:

(C.11)

$$v(\eta) = -\frac{1}{\pi}\int_0^\infty e^{-\eta r}\mathrm{Im}\left(\frac{\tau(re^{i\pi}) + (re^{i\pi})^{\alpha-1}}{\tau(re^{i\pi})^2 + (re^{i\pi})^\alpha + a}\right)dr + \sum_{s=s_1,s_2}\mathrm{Res}\left(e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right).$$

We compute the integrand and residues, and get (using trigonometry)

$$\mathrm{Im}\left(\frac{\tau(re^{i\pi}) + (re^{i\pi})^{\alpha-1}}{\tau(re^{i\pi})^2 + (re^{i\pi})^\alpha + a}\right) = -\frac{ar^{\alpha-1}\sin(\alpha\pi)}{(\tau r^2 + a)^2 + 2(\tau r^2 + a)r^\alpha\cos(\alpha\pi) + r^{2\alpha}}$$

and (using that s_1 and s_2 are complex conjugates)

$$\sum_{s=s_1,s_2}\mathrm{Res}\left(e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right) = 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right).$$

We then write $v(\eta)$ as

$$v(\eta) = \frac{a\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\eta r}K(r)\,dr + 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right), \tag{C.12}$$

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where

(C.13)
$$K(r) = r^{\alpha-1}\tilde{K}(r) \quad \text{with} \quad \tilde{K}(r) = \frac{1}{(\tau r^2 + a)^2 + 2(\tau r^2 + a) r^\alpha \cos(\alpha\pi) + r^{2\alpha}}.$$

That the integral term is bounded follows from application of Watson Lemma (see [20]), since $\tilde{K}$ is C^∞ near $r = 0$, $\tilde{K}(0) = 1 \neq 0$, $\alpha - 1 > -1$ and clearly there exist non-negative constants C and b such that $|K(r)| < Ce^{br}$. Then the integral is bounded and moreover if η is large enough the following approximation holds

$$\int_0^\infty e^{-\eta r} K(r) dr \sim \sum_{n=0}^{\infty} \frac{\tilde{K}^{n)}(0)\Gamma(\alpha+n)}{n!\eta^{\alpha+n}} \quad \text{as} \quad \eta \to \infty.$$

One can compute the derivatives of $\tilde{K}$ and show that the odd order ones are zero at $r = 0$ and the even order ones do not vanish there; a two-term expansion reads:

(C.14)
$$\int_0^\infty e^{-\eta r} K(r) dr \sim \frac{\Gamma(\alpha)}{a^2}\frac{1}{\eta^\alpha} - \frac{4\tau\Gamma(\alpha+2)}{a^3}\frac{1}{\eta^{\alpha+2}} + O\left(\frac{1}{\eta^{\alpha+4}}\right) \quad \text{as} \quad \eta \to \infty.$$

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ADDENDUM TO "TRAVELLING WAVES FOR A NON-LOCAL KORTEWEG-DE VRIES-BURGERS EQUATION" [J. DIFFERENTIAL EQUATIONS 257 (2014), NO. 3, 720–758]

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ABSTRACT. We add a theorem to [J. Differential Equations 257 (2014), no. 3, 720–758] by F. Achleitner, C.M. Cuesta and S. Hittmeir. In that paper we studied travelling wave solutions of a Korteweg-de Vries-Burgers type equation with a non-local diffusion term. In particular, the proof of existence and uniqueness of these waves relies on the assumption that the exponentially decaying functions are the only bounded solutions of the linearised equation. In this addendum we prove this assumption and thus close the existence and uniqueness proof of travelling wave solutions.

1. Introduction

In [1] we study the existence and stability of travelling waves of the following one-dimensional evolution equation:

$$\partial_t u + \partial_x u^2 = \partial_x \mathcal{D}^\alpha u + \tau \partial_x^3 u \,, \quad x \in \mathbb{R}\,,\ t \geq 0 \tag{1.1}$$

with $\tau > 0$, see also [2] for the case $\tau = 0$. The symbol $\mathcal{D}^\alpha$ denotes the non-local operator acting on x that, applied to a general function $f : \mathbb{R} \to \mathbb{R}$, reads

(1.2)
$$\mathcal{D}^\alpha f(x) = d_\alpha \int_{-\infty}^{x} \frac{f'(y)}{(x-y)^\alpha} dy \,, \quad \text{with} \quad 0 < \alpha < 1 \,, \quad d_\alpha := \frac{1}{\Gamma(1-\alpha)} > 0 \,,$$

here Γ denotes the Gamma function.This operator can be interpreted as a fractional derivative of order α in the Caputo sense, see e.g. [3], with integration taken from $-\infty$.

We recall that travelling wave solutions of (1.1) are solutions of the form $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ and $c \in \mathbb{R}$, that satisfy

$$h(\phi) = \mathcal{D}^\alpha \phi + \tau \phi'' \,, \quad \text{where} \quad h(\phi) := -c(\phi - \phi_-) + \phi^2 - \phi_-^2 \,. \tag{1.3}$$

and

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_- \,, \quad \lim_{\xi \to \infty} \phi(\xi) = \phi_+ \tag{1.4}$$

(see [1] for details) for some constant values ϕ_- and ϕ_+. Here $'$ denotes differentiation with respect to ξ. Further, it is assumed that $\phi_- > \phi_+$ (Lax

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entropy condition), which implies that $c = \phi_+ + \phi_-$ (Rankine-Hugoniot wave speed), and also that $h'(\phi_-) = \phi_- - \phi_+ > 0$.

The proof of existence and uniqueness of travelling wave solutions, for both $\tau > 0$ and $\tau = 0$, relies on the assumption that the functions $v(\xi) = Ce^{\lambda\xi}$, $C \in \mathbb{R}$, are the only solutions in the Sobolev spaces $H^s((-\infty, 0])$ with $s = 2$ if $\tau = 0$ (see [2]) and $s = 4$ if $\tau > 0$ (see [1]) of the linearised equation

$$h'(\phi_-)v = \mathcal{D}^\alpha v + \tau v'' , \tag{1.5}$$

where the exponent $\lambda > 0$ is the real and strictly positive zero of

$$P(z) = \tau z^2 + z^\alpha - h'(\phi_-) . \tag{1.6}$$

We recall that for $\tau > 0$ there is a unique positive real zero of (1.6), the other zeros being two complex conjugates with negative real part, see [1].

In [1] we do not give a complete proof of this assumption, however, we prove it in suitable weighted exponential spaces. We show this by writing the equation as a Wiener-Hopf equation ([6]) and applying the results by [4]. Namely, we show that if $0 < \mu < \min\{\lambda, h'(\phi_-)/(2-\alpha)\}$, then, all solutions of (1.5) that are in the space

$$L^\infty_w(-\infty, 0) = \{f \in L^\infty(-\infty, 0) : \ f(\xi) = e^{\mu\xi} g(\xi) \quad \text{for some } g \in L^\infty(-\infty, 0)\}$$

are given by the one-parameter family $\{Ce^{\lambda\xi} : \ C \in \mathbb{R}\}$. A similar result is given in [2] for the case $\tau = 0$, where it is also shown that bounded solutions decay to 0 as $\xi \to -\infty$ faster than algebraically.

The aim of the current addendum is thus to present a proof that removes the weight of the space. Namely, we show that:

Theorem 1.1. *All solutions of (1.5) with $\tau \geq 0$ that are in $H^s(-\infty, 0)$ with $s \geq 2$ are given by the one-parameter family $\{\xi \in (-\infty, 0) \to Ce^{\lambda\xi} : \ C \in \mathbb{R}\}$, where λ is the positive zero of (1.6).*

A crucial point in our proof will be the non-negativity of the integral

$$I[v] := \int_{-\infty}^{0} v'(\xi) \mathcal{D}^\alpha v(\xi) d\xi. \tag{1.7}$$

We give the proof of Theorem 1.1 in the next section. Let us now recall some notation and properties of (1.2). For $s \geq 0$ we shall adopt the following notation for the Sobolev space of square integrable functions,

$$H^s(\mathbb{R}) := \{u : \ \|u\|_{H^s(\mathbb{R})} < \infty\} , \qquad \|u\|_{H^s(\mathbb{R})} := \|(1 + |k|^2)^{s/2} \hat{u}\|_{L^2(\mathbb{R})} ,$$

and the corresponding homogeneous norm $\|u\|_{\dot{H}^s(\mathbb{R})} := \||k|^s \hat{u}\|_{L^2(\mathbb{R})}$, where $\hat{u}$ denotes the Fourier transform of u. It is easy to see that $\|\mathcal{D}^\alpha u\|_{\dot{H}^s(\mathbb{R})} = \|u\|_{\dot{H}^{s+\alpha}(\mathbb{R})}$, so that $\mathcal{D}^\alpha$ is a bounded linear operator from $H^s(\mathbb{R})$ to $H^{s-\alpha}(\mathbb{R})$.

We recall that the analysis in [1] starts out by proving a 'local' existence result on $(-\infty, \xi_0]$ with $\xi_0 < 0$ and $|\xi_0|$ sufficiently large. This proof is based on linearisation about $\xi = -\infty$ (or, equivalently, $\phi = \phi_-$), which is given by (1.5). It is then assumed that

$$\mathcal{N}\big(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id}\big) = \mathrm{span}\{e^{\lambda\xi}\} \quad \text{in} \quad H^s(-\infty, \xi_0) \tag{1.8}$$

where Id denotes the identity operator and $s = 4$ if $\tau > 0$ and $s = 2$ if $\tau = 0$. The assumption (1.8) follows if Theorem 1.1 is true. Notice that the

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problem is invariant under translation, so we can take $\xi_0 = 0$ without loss of generality in order to show (1.8).

2. Proof of Theorem 1.1

We work in the Hilbert space

$$H_0^2(-\infty,0) = \{v \in H^1(-\infty,0) : \ v(0) = 0\} \cap H^2(-\infty,0)\,.$$

We need two lemmas. First, we find a way of writing the potential in $\mathcal{D}^\alpha$ as an integral (see [5]):

Lemma 2.1. (i) *Let $h \in C_c^\infty(\mathbb{R})$ be an even function, then $H : \mathbb{R} \to \mathbb{R}$ defined by*

$$H(t) := \int_{\mathbb{R}} h(t-r)\,h(r)\,dr$$

is an even non-negative function with compact support. Moreover, h can be normalised such that $\int_0^\infty H(t)dt = 1$.

(ii) *Let $\beta > -1$ and H as in (i), then for all $x \in \mathbb{R}$*

$$|x|^{-(\beta+1)} = \int_0^\infty t^\beta H(tx)dt\,.$$

Moreover, for any ξ, $y \in \mathbb{R}$ and $a > 0$, we have

$$|\xi - y|^{-a} = \int_0^\infty t^a \int_{\mathbb{R}} h(t(z-\xi))\,h(t(z-y))\,dz\,dt. \tag{2.1}$$

We can now show the following key result:

Lemma 2.2. *Let $v \in H_0^2(-\infty,0)$, then the integral $I[v]$ in (1.7) is well-defined and is non-negative. Moreover, $I[v]$ is zero if and only $v \equiv 0$.*

Proof. We first show that $I[v]$ is well-defined. Indeed, using the Cauchy-Schwarz inequality and that $\alpha \in (0,1)$, it follows that

$$\left| \int_{-\infty}^0 v'(\xi)\mathcal{D}^\alpha v(\xi) d\xi \right| \le \|v'\|_{L^2(-\infty,0)} \|D^\alpha v\|_{L^2(-\infty,0)}\,. \tag{2.2}$$

We now use the reflection operator $\mathcal{E} : H_0^2(-\infty,0) \to H^2(\mathbb{R})$,

$$\mathcal{E}[u](x) := u^*(x) = \begin{cases} u(x) & \text{if } x \le 0, \\ -u(-x) & \text{if } x > 0, \end{cases}$$

so that $\|u^*\|^2_{L^2(\mathbb{R})} = 2\|u\|^2_{L^2(-\infty,0)}$. Then,

$$\|\mathcal{D}^\alpha v\|^2_{L^2(-\infty,0)} \le \|\mathcal{D}^\alpha v^*\|^2_{L^2(\mathbb{R})} = \|v^*\|^2_{\dot H^\alpha(\mathbb{R})} \le \|v^*\|^2_{H^1(\mathbb{R})} = 2\|v\|^2_{H_0^1(-\infty,0)} < \infty\,.$$

This and (2.2) imply that $I[v]$ is well-defined.

In order to determined the sign of $I[v]$, we first write (1.7) over integrals on $\mathbb{R}$:

$$I[v] = \int_{-\infty}^0 \int_y^0 \frac{v'(\xi)v'(y)}{(\xi-y)^\alpha}\,d\xi\,dy = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{[v'(\xi)\theta(-\xi)]\,[v'(y)\theta(-y)]}{|\xi-y|^\alpha}\,d\xi\,dy$$

where θ denotes the Heaviside function. In the first identity we have used the definition of $\mathcal{D}^\alpha$ and Fubini-Tonelli Theorem. Let us, for simplicity

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of notation, write $F(x) = v'(x)\theta(-x)$. Then, by expressing the potential according to Lemma 2.1 (2.1), we obtain

$$I[v] = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(\xi)F(y)\int_0^\infty t^\alpha \int_{\mathbb{R}} h(t(z-\xi))\, h(t(z-y))\, dz\, dt\, dy\, d\xi$$

and by Fubini-Tonelli Theorem we have that

$$I[v] \;=\; \frac{1}{2}\int_0^\infty t^\alpha \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(\xi)h(t(z-\xi))d\xi\right)^2 dz\, dt \geq 0\,.$$

Now, if $I[v] = 0$ then $F * h_t = 0$ almost everywhere, where $h_t(x) = h(tx)$. Since h has compact support, h_t acts as a mollifier as $t \to \infty$ and it is not hard to show that then $F \equiv 0$ ([5]). Recalling that $F(x) = v'(x)\theta(-x)$ with $v(0) = 0$ then $v \equiv 0$. □

Proof of Theorem 1.1. First let us prove the uniqueness of solutions of (1.5) in $H^2(-\infty,0)$ for a given data in $\xi = 0$. This is equivalent to proving that the only solution of (1.5) in $H_0^2(-\infty,0)$ is $v \equiv 0$; indeed, if v_1 and v_2 are two solutions of (1.5) with $v_1(0) = v_2(0)$, then $v = v_1 - v_2$ satisfies

$$\begin{cases} h'(\phi_-)v = \mathcal{D}^\alpha v + \tau v'' \\ v(0) = 0\,. \end{cases} \tag{2.3}$$

Testing (2.3) with $v' \in H^1(-\infty,0)$ and integrating with respect to ξ we obtain:

$$0 = \frac{h'(\phi_-)}{2}v(0)^2 = \int_{-\infty}^0 v'(\xi)\mathcal{D}^\alpha v(\xi)d\xi + \frac{\tau}{2}v'(0)^2$$

and Lemma 2.2 implies that $v \equiv 0$.

It is easy to see, just by a straight computation, that the exponential functions $Ce^{\mu\xi}$ with μ being a zero of (1.6) satisfy (1.5). If $\mu = \lambda$, then these exponential functions are the only solutions in $H^2(-\infty,0)$, by the uniqueness just established. On the other hand, since these functions are also solutions in $H^s(-\infty,0)$ with $s > 2$ and $H^s(-\infty,0) \subset H^2(-\infty,0)$, the result follows. □

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TWO CLASSES OF NONLOCAL EVOLUTION EQUATIONS RELATED BY A SHARED TRAVELING WAVE PROBLEM

FRANZ ACHLEITNER

ABSTRACT. We consider nonlocal reaction-diffusion equations and nonlocal Korteweg-de Vries-Burgers (KdVB) equations, i.e. scalar conservation laws with diffusive-dispersive regularization. We review the existence of traveling wave solutions for these two classes of evolution equations. For classical equations the traveling wave problem (TWP) for a local KdVB equation can be identified with the TWP for a reaction-diffusion equation. In this article we study this relationship for these two classes of evolution equations with nonlocal diffusion/dispersion. This connection is especially useful, if the TW equation is not studied directly, but the existence of a TWS is proven using one of the evolution equations instead. Finally, we present three models from fluid dynamics and discuss the TWP via its link to associated reaction-diffusion equations.

1. INTRODUCTION

We will consider two classes of (nonlocal) evolution equations and study the associated traveling wave problems in parallel: On the one hand, we consider scalar conservation laws with (nonlocal) diffusive-dispersive regularization

$$\partial_t u + \partial_x f(u) = \epsilon \mathcal{L}_1[u] + \delta \partial_x \mathcal{L}_2[u] \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{1}$$

for some nonlinear function $f : \mathbb{R} \to \mathbb{R}$, Lévy operators $\mathcal{L}_1$ and $\mathcal{L}_2$, as well as constants $\epsilon, \delta \in \mathbb{R}$. The Fourier multiplier operators $\mathcal{L}_1$ and $\partial_x \mathcal{L}_2$ model diffusion and dispersion, respectively. On the other hand, we consider scalar reaction-diffusion equations

$$\partial_t u = \sigma \mathcal{L}_3[u] + r(u) \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{2}$$

for some positive constant σ, as well as a nonlinear function $r : \mathbb{R} \to \mathbb{R}$ and a Lévy operator $\mathcal{L}_3$ modeling reaction and diffusion, respectively.

Definition 1.1. A traveling wave solution (TWS) of an evolution equation–such as (1) and (2)–is a solution $u(x,t) = \bar{u}(\xi)$ whose *profile* $\bar{u}$ depends on $\xi := x - ct$ for some *wave speed* c. Moreover, the profile $\bar{u} \in C^2(\mathbb{R})$ is assumed to approach distinct endstates $u_\pm$ such that

$$\lim_{\xi \to \pm\infty} \bar{u}(\xi) = u_\pm \ , \qquad \lim_{\xi \to \pm\infty} \bar{u}^{(n)}(\xi) = 0 \quad \text{with } n = 1, 2. \tag{3}$$

Such a TWS is also known as a *front* in the literature. A TWS $(\bar{u}, c)$ is called monotone, if its profile $\bar{u}$ is a monotone function.

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Definition 1.2. The traveling wave problem (TWP) associated to an evolution equation is to study for some distinct endstates $u_\pm$ the existence of a TWS $(\bar{u}, c)$ in the sense of Definition 1.1.

We want to identify classes of evolutions equations of type (1) and (2), which lead to the same TWP. This connection is especially useful, if the TWP is not studied directly, but the existence of a TWS is proven using one of the evolution equations instead. A classical example of (1) is a scalar conservation law with local diffusive-dispersive regularization

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x^2 u + \delta \partial_x^3 u \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{4}$$

for some nonlinear function $f : \mathbb{R} \to \mathbb{R}$ and some constants $\epsilon > 0$ and $\delta \in \mathbb{R}$. Equation (4) with Burgers flux $f(u) = u^2$ is known as Korteweg-de Vries-Burgers (KdVB) equation; hence we refer to Equation (4) with general f as *generalized* KdVB equation and Equation (1) as *nonlocal generalized* KdVB equation. A TWS $(\bar{u}, c)$ satisfies the traveling wave equation (TWE)

$$-c\bar{u}' + f'(\bar{u})\ \bar{u}' = \epsilon \bar{u}'' + \delta \bar{u}''' \ , \quad \xi \in \mathbb{R} \ , \tag{5}$$

or integrating on $(-\infty, \xi]$ and using (3),

$$h(\bar{u}) := f(\bar{u}) - c\bar{u} - (f(u_-) - c\ u_-) = \epsilon \bar{u}' + \delta \bar{u}'' \ , \quad \xi \in \mathbb{R} \ . \tag{6}$$

However, the TW ansatz $v(x, t) = \bar{u}(x - \epsilon t)$ for the scalar reaction-diffusion equation

$$\partial_t v = -h(v) + \delta \partial_x^2 v \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{7}$$

leads to the same TWE (6) except for a different interpretation of the parameters. The traveling wave speeds in the TWP of (4) and (7) are c and ϵ, respectively. For fixed parameters c, ϵ, and δ, the existence of a traveling wave profile $\bar{u}$ satisfying (3) and (6) reduces to the existence of a heteroclinic orbit for this ODE. This is an example, where the existence of TWS is studied directly via the TWE.

An example, where the TWE is not studied directly, is the TWP for a nonlocal KdVB equation (1) with $\mathcal{L}_1[u] = \partial_x^2 u$ and $\mathcal{L}_2[u] = \phi_\epsilon * u - u$ for some even non-negative function $\phi \in L^1(\mathbb{R})$ with compact support and unit mass, where $\phi_\epsilon(\cdot) := \phi(\cdot/\epsilon)/\epsilon$ with $\epsilon > 0$. It has been derived as a model for phase transitions with long range interactions close to the surface, which supports planar TWS associated to undercompressive shocks of (51), see [52]. In particular, the TWP for a cubic flux function $f(u) = u^3$ is related to the TWP for a reaction-diffusion equation (2) with $\mathcal{L}_3[u] = \mathcal{L}_2[u]$. The existence of TWS for this reaction-diffusion equation has been proven via a homotopy of (2) to a classical reaction-diffusion model (7), see [14].

Outline. In Section 2 we collect background material on Lévy operators $\mathcal{L}$, which will model diffusion in our nonlocal evolution equations. Special emphasize is given to convolution operators and Riesz-Feller operators. In Section 3 we review the classical results on the TWP for reaction-diffusion equations (7) and generalized Korteweg-de Vries-Burgers equation (4). We study their relationship in detail, especially the classification of function $h(u)$, which will be used again in Section 4. In Section 4, first we review the results on TWP for nonlocal reaction-diffusion equations (2) with operators $\mathcal{L}_3$ of convolution type and Riesz-Feller type, respectively. Finally, we study

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the example of nonlocal generalized Korteweg-de Vries-Burgers equation (1) with $\mathcal{L}_1[u] = \mathcal{D}_+^{1/3} u$ and $\mathcal{L}_2[u] = \partial_x^2 u$ modeling a shallow water flow [44], and Fowler's equation

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}_+^{1/3} u \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{8}$$

modeling dune formation [36], where $\mathcal{D}_+^\alpha$ is a Caputo derivative. In the Appendix, we collect background material on Caputo derivatives $\mathcal{D}_+^\alpha$ and the shock wave theory for scalar conservation laws, which will explain the importance of the TWP for KdVB equations.

Scalar conservation laws with fractional Laplacian are another example of equation (1) with $\mathcal{L}_1[u] = -(-\partial_x^2)^{\alpha/2}\, u$, $0 < \alpha < 2$, and $\mathcal{L}_2[u] \equiv 0$. However, its traveling wave problem can not be related to a nonlocal reaction-diffusion problem like our examples. Therefore, instead of discussing its traveling wave problem, we refer the interested reader to the literature [15, 31, 32, 7, 10, 43, 30, 23, 8, 33, 26] and references therein.

Notations. We use the conventions in probability theory, and define the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ for $g \in L^1(\mathbb{R})$ and $x, k \in \mathbb{R}$ as

$$\mathcal{F}[g](k) := \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}\, kx} g(x) \ \mathrm{d}x \ ; \qquad \mathcal{F}^{-1}[g](x) := \tfrac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\, kx} g(k) \ \mathrm{d}k \ .$$

In the following, $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote also their respective extensions to $L^2(\mathbb{R})$.

2. Lévy Operators

A Lévy process is a stochastic process with independent and stationary increments which is continuous in probability [12, 40, 53]. Therefore a Lévy process is characterized by its transition probabilities $p(t, x)$, which evolve according to an evolution equation

$$\partial_t p = \mathcal{L} p \tag{9}$$

for some operator $\mathcal{L}$, also called a *Lévy operator.* First, we define Lévy operators on the function spaces $C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) \mid \lim_{|x|\to\infty} f(x) = 0\}$ and $C_0^2(\mathbb{R}) := \{f, f', f'' \in C_0(\mathbb{R})\}$.

Definition 2.1. The family of Lévy operators in one spatial dimension consists of operators $\mathcal{L}$ defined for $f \in C_0^2(\mathbb{R})$ as

$$\mathcal{L}f(x) = \tfrac{1}{2} A f''(x) + \gamma\, f'(x) + \int_{\mathbb{R}} \big(f(x+y) - f(x) - y\, f'(x) 1_{(-1,1)}(y)\big)\, \nu(\,\mathrm{d}y) \tag{10}$$

for some constants $A \geq 0$ and $\gamma \in \mathbb{R}$, and a measure ν on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(1, |y|^2)\, \nu(\,\mathrm{d}y) \ < \infty \,.$$

Remark 1. The function $f(x+y) - f(x) - y\, f'(x) 1_{(-1,1)}(y)$ is integrable with respect to ν, because it is bounded outside of any neighborhood of 0 and

$$f(x+y) - f(x) - y\, f'(x) 1_{(-1,1)}(y) = O(|y|^2) \quad \text{as} \quad |y| \to 0$$

for fixed x. The indicator function $c(y) = 1_{(-1,1)}(y)$ is only one possible choice to obtain an integrable integrand. More generally, let $c(y)$ be a

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bounded measurable function from $\mathbb{R}$ to $\mathbb{R}$ satisfying $c(y) = 1 + o(|y|)$ as $|y| \to 0$, and $c(y) = O(1/|y|)$ as $|y| \to \infty$. Then (10) is rewritten as

(11)
$$\mathcal{L}f(x) = \tfrac{1}{2}Af''(x) + \gamma_c\, f'(x) + \int_{\mathbb{R}} \big(f(x+y) - f(x) - y\ f'(x)c(y)\big)\ \nu(\,\mathrm{d}y)\ ,$$

with $\gamma_c = \gamma + \int_{\mathbb{R}} y\ (c(y) - 1_{(-1,1)}(y))\ \nu(\,\mathrm{d}y)$.

Alternative choices for c:

(c 0) If a Lévy measure ν satisfies $\int_{|y|<1} |y|\ \nu(\,\mathrm{d}y) < \infty$ then $c \equiv 0$ is admissible.

(c 1) If a Lévy measure ν satisfies $\int_{|y|>1} |y|\ \nu(\,\mathrm{d}y) < \infty$ then $c \equiv 1$ is admissible.

We note that A and ν are invariant no matter what function c we choose.

Examples.

(a) The Lévy operators

$$\mathcal{L}f = \int_{\mathbb{R}} \big(f(x+y) - f(x)\big)\ \nu(\,\mathrm{d}y) \tag{12}$$

are infinitesimal generators associated to a compound Poisson process with finite Lévy measure ν satisfying (c 0). The special case of $\nu(\,\mathrm{d}y) = \phi(-y)\ \mathrm{d}y$ for some function $\phi \in L^1(\mathbb{R})$ yields

$$\mathcal{L}f(x) = \int_{\mathbb{R}} \big(f(x+y) - f(x)\big)\ \phi(-y)\ \mathrm{d}y = \Big(\phi * f - \int_{\mathbb{R}} \phi\ \ \mathrm{d}y\ \ f\Big)(x)\ . \tag{13}$$

(b) *Riesz-Feller operators.* The Riesz-Feller operators of order a and asymmetry θ are defined as Fourier multiplier operators

$$\mathcal{F}[D^a_\theta f](k) = \psi^a_\theta(k)\ \mathcal{F}[f](k)\ , \qquad k \in \mathbb{R}\ , \tag{14}$$

with symbol $\psi^a_\theta(k) = -|k|^a \exp[\mathrm{i}\ \mathrm{sgn}(k)\ \theta\pi/2]$ such that $(a, \theta) \in \mathfrak{D}_{a,\theta}$ and

$$\mathfrak{D}_{a,\theta} := \{\, (a,\theta) \in \mathbb{R}^2 \,|\, 0 < a \leq 2\ , \quad |\theta| \leq \min\{a, 2-a\}\,\}\ .$$

Special cases of Riesz-Feller operators are

- Fractional Laplacians $-(-\Delta)^{a/2}$ on $\mathbb{R}$ with Fourier symbol $-|k|^a$ for $0 < a \leq 2$. In particular, fractional Laplacians are the only symmetric Riesz-Feller operators with $-(-\Delta)^{a/2} = D^a_0$ and $\theta \equiv 0$.
- Caputo derivatives $-\mathcal{D}^\alpha_+$ with $0 < \alpha < 1$ are Riesz-Feller operators with $a = \alpha$ and $\theta = -\alpha$, such that $-\mathcal{D}^\alpha_+ = D^\alpha_{-\alpha}$, see also Section A.
- Derivatives of Caputo derivatives $\partial_x \mathcal{D}^\alpha_+$ with $0 < \alpha < 1$ are Riesz-Feller operators with $a = 1 + \alpha$ and $\theta = 1 - \alpha$, such that $\partial_x \mathcal{D}^\alpha_+ = D^{1+\alpha}_{1-\alpha}$.

Next we consider the Cauchy problem

$$\partial_t u(x,t) = D^a_\theta[u(\cdot,t)](x)\ , \quad u(x,0) = u_0(x)\ , \tag{15}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$ and initial datum u_0.

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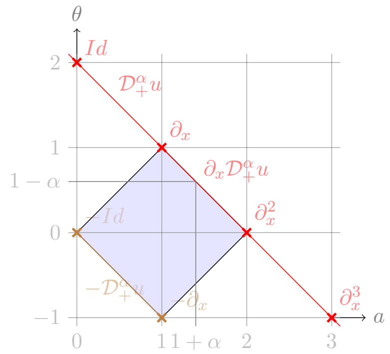

FIGURE 1. The family of Fourier multipliers $\psi_\theta^a(k) = -|k|^a \exp\left[\mathrm{i}\,\mathrm{sgn}(k)\theta\pi/2)\right]$ has two parameters a and θ. Some Fourier multiplier operators $\mathcal{F}[Tf](k) = \psi_\theta^a(k)\ \mathcal{F}[f](k)$ are inserted in the parameter space (a,θ): partial derivatives and Caputo derivatives $\mathcal{D}_+^\alpha$ with $0 < \alpha < 1$. The Riesz-Feller operators D_θ^a are those operators with parameters $(a,\theta) \in \mathfrak{D}_{a,\theta}$. The set $\mathfrak{D}_{a,\theta}$ is also called *Feller-Takayasu diamond* and depicted as a shaded region, see also [47].

Proposition 2.2. *For $(a,\theta) \in \mathfrak{D}_{a,\theta}$ with $\theta \neq \pm 1$ and $1 \leq p < \infty$, the Riesz-Feller operator D_θ^a generates a strongly continuous L^p-semigroup*

$$S_t : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \ , \quad u_0 \mapsto S_t u_0 = G_\theta^a(\cdot,t) * u_0 \ ,$$

with heat kernel $G_\theta^a(x,t) = \mathcal{F}^{-1}[\exp(t\ \psi_\theta^a(\cdot))](x)$. In particular, $G_\theta^a(x,t)$ is the probability measure of a Lévy strictly a-stable distribution.

The proof of this proposition for a subclass $1 < \alpha \leq 2$ in [6, Proposition 2.2] can be extended to cover all cases $(a,\theta) \in \mathfrak{D}_{a,\theta}$ with $\theta \neq \pm 1$. For $(a,\theta) \in \{(1,1),(1,-1)\}$, the probability measure G_θ^a is a delta distribution, e.g. $G_1^1(x,t) = \delta_{x+t}$ and $G_{-1}^1(x,t) = \delta_{x-t}$, and is called trivial [53, Definition 13.6]. However, we are interested in non-trivial probability measures G_θ^a for

$$(a,\theta) \in \mathfrak{D}_{a,\theta}^\diamond := \{\, (a,\theta) \in \mathfrak{D}_{a,\theta} \,|\, |\theta| < 1 \,\} \ ,$$

such that $\mathfrak{D}_{a,\theta} = \mathfrak{D}_{a,\theta}^\diamond \cup \{(1,1),\ (1,-1)\}$. Note, nonlocal Riesz-Feller D_θ^a operators are those with parameters

$$(a,\theta) \in \mathfrak{D}_{a,\theta}^\bullet := \{\, (a,\theta) \in \mathfrak{D}_{a,\theta} \,|\, 0 < a < 2 \ , \quad |\theta| < 1 \,\},$$

such that $\mathfrak{D}_{a,\theta}^\diamond = \mathfrak{D}_{a,\theta}^\bullet \cup \{(2,0)\}$.

Proposition 2.3 ([6, Lemma 2.1])**.** *For $(a,\theta) \in \mathfrak{D}_{a,\theta}^\diamond$ the probability measure G_θ^a is absolutely continuous with respect to the Lebesgue measure and possesses a probability density which will be denoted again by G_θ^a. For all $(x,t) \in \mathbb{R} \times (0,\infty)$ the following properties hold;*

(a) $G_\theta^a(x,t) \geq 0$. *If $\theta \neq \pm a$ then* $G_\theta^a(x,t) > 0$;
(b) $\|G_\theta^a(\cdot,t)\|_{L^1(\mathbb{R})} = 1$;

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(c) $G^a_\theta(x,t) = t^{-1/a} G^a_\theta(xt^{-1/a}, 1)$;
(d) $G^a_\theta(\cdot,s) * G^a_\theta(\cdot,t) = G^a_\theta(\cdot, s+t)$ *for all* $s,t \in (0,\infty)$;
(e) $G^a_\theta \in C^\infty_0(\mathbb{R} \times (0,\infty))$.

The Lévy measure ν of a Riesz-Feller operator D^a_θ with $(a,\theta) \in \mathfrak{D}^\bullet_{a,\theta}$ is absolutely continuous with respect to Lebesgue measure and satisfies

$$\nu(\,\mathrm{d}y) = \begin{cases} c_-(\theta) y^{-1-a} \;\mathrm{d}y & \text{on } (0,\infty)\;, \\ c_+(\theta)|y|^{-1-a} \;\mathrm{d}y & \text{on } (-\infty,0)\;, \end{cases} \tag{16}$$

with $c_\pm(\theta) = \Gamma(1+a)\sin((a\pm\theta)\pi/2)/\pi$, see [47, 54].

To study a TWP for evolution equations involving Riesz-Feller operators, it is necessary to extend the Riesz-Feller operators to $C^2_b(\mathbb{R})$. Their singular integral representations (10) may be used to accomplish this task.

Theorem 2.4 ([6])**.** *If* $(a,\theta) \in \mathfrak{D}^\bullet_{a,\theta}$ *with* $a \neq 1$*, then for all* $f \in \mathcal{S}(\mathbb{R})$ *and* $x \in \mathbb{R}$

$$\begin{aligned} D^a_\theta f(x) =& \frac{c_+(\theta)-c_-(\theta)}{1-a} f'(x) \\ &+ c_+(\theta) \int_0^\infty \frac{f(x+y)-f(x)-f'(x)\,y 1_{(-1,1)}(y)}{y^{1+a}} \;\mathrm{d}y \\ &+ c_-(\theta) \int_0^\infty \frac{f(x-y)-f(x)+f'(x)\,y 1_{(-1,1)}(y)}{y^{1+a}} \;\mathrm{d}y \end{aligned} \tag{17}$$

with $c_\pm(\theta) = \Gamma(1+a)\sin((a\pm\theta)\pi/2)/\pi$*. Alternative representations are*

- *If* $0<a<1$*, then*

$$D^a_\theta f(x) = c_+(\theta) \int_0^\infty \frac{f(x+y)-f(x)}{y^{1+a}} \;\mathrm{d}y + c_-(\theta) \int_0^\infty \frac{f(x-y)-f(x)}{y^{1+a}} \;\mathrm{d}y\;.$$

- *If* $1<a<2$*, then*

$$\begin{aligned} D^a_\theta f(x) = c_+(\theta) &\int_0^\infty \frac{f(x+y)-f(x)-f'(x)\,y}{y^{1+a}} \;\mathrm{d}y \\ &+ c_-(\theta) \int_0^\infty \frac{f(x-y)-f(x)+f'(x)\,y}{y^{1+a}} \;\mathrm{d}y\;. \end{aligned} \tag{18}$$

These representations allow to extend Riesz-Feller operators D^a_θ to $C^2_b(\mathbb{R})$ such that $D^a_\theta C^2_b(\mathbb{R}) \subset C_b(\mathbb{R})$. For example, one can show

Proposition 2.5 ([6, Proposition 2.4])**.** *For* $(a,\theta) \in \mathfrak{D}_{a,\theta}$ *with* $1<a<2$*, the integral representation* (18) *of* D^a_θ *is well-defined for functions* $f \in C^2_b(\mathbb{R})$ *with*

$$\sup_{x\in\mathbb{R}} |D^a_\theta f(x)| \leq \mathcal{K} \|f''\|_{C_b(\mathbb{R})} \frac{M^{2-a}}{2-a} + 4\mathcal{K}\|f'\|_{C_b(\mathbb{R})} \frac{M^{1-a}}{a-1} < \infty \tag{19}$$

for some positive constants M *and* $\mathcal{K} = \frac{\Gamma(1+a)}{\pi}|\sin((a+\theta)\frac{\pi}{2}) + \sin((a-\theta)\frac{\pi}{2})|$.

Estimate (19) is a key estimate, which is used to adapt Chen's approach [24] to the TWP for nonlocal reaction-diffusion equations with Riesz-Feller operators [6].

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3. TWP for classical evolution equations

In this section we review the importance of the TWP for reaction-diffusion equations and scalar conservation laws with higher-order regularizations, respectively.

3.1. **Reaction-diffusion equations.** A scalar reaction-diffusion equations is a partial differential equation

$$\partial_t u = \sigma \partial_x^2 u + r(u) \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{20}$$

for some positive constant $\sigma > 0$, as well as a nonlinear function $r : \mathbb{R} \to \mathbb{R}$ and second-order derivative $\partial_x^2 u$ modeling reaction and diffusion, respectively. The TWP for given endstates $u_\pm$ is to study the existence of a TWS $(\bar{u}, c)$ for (20) in the sense of Definition 1.1. If the profile $\bar{u} \in C^2(\mathbb{R})$ is bounded, then it satisfies $\lim_{\xi \to \pm\infty} \bar{u}^{(n)}(\xi) = 0$ for $n = 1, 2$. A TWS $(\bar{u}, c)$ satisfies the TWE

$$-c\bar{u}' = r(\bar{u}) + \sigma \bar{u}'' \ , \quad \xi \in \mathbb{R} \ . \tag{21}$$

Phase plane analysis. A traveling wave profile $\bar{u}$ is a heteroclinic orbit of the TWE (21) connecting the endstates $u_\pm$. To identify necessary conditions on the existence of TWS, TWE (21) is written as a system of first-order ODEs for u, $v := u'$:

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ (-r(u) - cv)/\sigma \end{pmatrix} =: F(u, v) \ , \quad \xi \in \mathbb{R} \ . \tag{22}$$

First, an endstate (u_s, v_s) of a heteroclinic orbit has to be a stationary state of F, i.e. $F(u_s, v_s) = 0$, which implies $v_s \equiv 0$ and $r(u_s) = 0$. Second, $(u_-, 0)$ has to be an unstable stationary state of (22) and $(u_+, 0)$ either a saddle or a stable node of (22). As long as a stationary state (u_s, v_s) is hyperbolic, i.e. the linearization of F at (u_s, v_s) has only eigenvalues λ with non-zero real part, the stability of (u_s, v_s) is determined by these eigenvalues. The linearization of F at (u_s, v_s) is

$$\mathrm{D}F(u_s, v_s) = \begin{pmatrix} 0 & 1 \\ -r'(u_s)/\sigma & -c/\sigma \end{pmatrix} \ . \tag{23}$$

Eigenvalues $\lambda_\pm$ of the Jacobian $\mathrm{D}F(u_s, v_s)$ satisfy the characteristic equation $\lambda^2 + \lambda c/\sigma + r'(u_s)/\sigma = 0$. Moreover, $\lambda_- + \lambda_+ = -c/\sigma$ and $\lambda_- \lambda_+ = r'(u_s)/\sigma$. The eigenvalues $\lambda_\pm$ of the Jacobian $\mathrm{D}F(u_s, v_s)$ are

$$\lambda_\pm = -\frac{c}{2\sigma} \pm \sqrt{\frac{c^2}{4\sigma^2} - \frac{r'(u_s)}{\sigma}} = \frac{-c \pm \sqrt{c^2 - 4\sigma r'(u_s)}}{2\sigma} \ . \tag{24}$$

Thus $r'(u_s) < 0$ ensures that $(u_s, 0)$ is a saddle point, i.e. with one positive and one negative eigenvalue.

Balance of potential. The potential R (of the reaction term r) is defined as $R(u) := \int_0^u r(v) \ \mathrm{d}v$. The potentials of the endstates $u_\pm$ are called *balanced* if $R(u_+) = R(u_-)$ and *unbalanced* otherwise. A formal computation reveals a connection between the sign of c and the balance of the potential $R(u)$: Multiplying TWE (21) with $\bar{u}'$, integrating on $\mathbb{R}$ and using (3), yields

$$-c\|\bar{u}'\|_{L^2}^2 = \int_{u_-}^{u_+} r(v) \ \mathrm{d}v = R(u_+) - R(u_-) \ , \tag{25}$$

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since $\int_{\mathbb{R}} \bar{u}''\bar{u}' \, \mathrm{d}\xi = 0$ due to (3). Thus $-\operatorname{sgn} c = \operatorname{sgn}(R(u_+) - R(u_-))$. In case of a balanced potential the wave speed c is zero, hence the TWS is stationary.

Definition 3.1. Assume $u_- > u_+$. A function $r \in C^1(\mathbb{R})$ with $r(u_\pm) = 0$ is

- *monostable* if $r'(u_-) < 0$, $r'(u_+) > 0$ and $r(u) > 0$ for $u \in (u_+, u_-)$.
- *bistable* if $r'(u_\pm) < 0$ and

$$\exists u_* \in (u_+, u_-) \ : \ r(u) \begin{cases} < 0 & \text{for } u \in (u_+, u_*) \ , \\ > 0 & \text{for } u \in (u_*, u_-) \ . \end{cases}$$

- *unstable* if $r'(u_\pm) > 0$.

We chose a very narrow definition compared to [56]. Moreover, in most applications of reaction-diffusion equations a quantity u models a density of a substance/population. In these situations only nonnegative states $u_\pm$ and functions u are of interest.

Proposition 3.2 ([56, §2.2])**.** *Assume $\sigma > 0$ and $u_- > u_+$.*

- *If r is monostable, then there exists a positive constant c_* such that for all $c \geq c_*$ there exists a monotone TWS $(\bar{u}, c)$ of* (20) *in the sense of Definition 1.1. For $c < c_*$ no such monotone TWS exists (however oscillatory TWS may exist).*
- *If r is bistable, then there exists an (up to translations) unique monotone TWS $(\bar{u}, c)$ of* (20) *in the sense of Definition 1.1.*
- *If r is unstable, then there does not exist a monotone TWS $(\bar{u}, c)$ of* (20).

If a TWS $(\bar{u}, c)$ exists, then a closer inspection of the eigenvalues (24) at $(u_+, 0)$ indicates the geometry of the profile $\bar{u}$ for large ξ:

$$c^2 - 4\sigma r'(u_+) \begin{cases} \geq 0 & \text{TWS with monotone decreasing profile } \bar{u} \text{ for large } \xi; \\ < 0 & \text{TWS with oscillating profile } \bar{u} \text{ for large } \xi. \end{cases}$$

3.2. **Korteweg-de Vries-Burgers equation (KdVB).** A generalized KdVB equation is a scalar partial differential equation

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x^2 u + \delta \partial_x^3 u, \quad x \in \mathbb{R}, \quad t > 0, \tag{26}$$

for some flux function $f : \mathbb{R} \to \mathbb{R}$ as well as constants $\epsilon > 0$ and $\delta \in \mathbb{R}$. The TWP for given endstates $u_\pm$ is to study the existence of a TWS $(\bar{u}, c)$ for (26) in the sense of Definition 1.1. The importance of the TWP for KdVB equations in the shock wave theory of (scalar) hyperbolic conservation laws is discussed in Section B. A TWS $(\bar{u}, c)$ satisfies the TWE

$$-c\bar{u}' + f'(\bar{u}) \, \bar{u}' = \epsilon \bar{u}'' + \delta \bar{u}''' \ , \quad \xi \in \mathbb{R} \ , \tag{27}$$

or integrating on $(-\infty, \xi]$ and using (3),

$$h(\bar{u}) := f(\bar{u}) - c\bar{u} - (f(u_-) - c \, u_-) = \epsilon \bar{u}' + \delta \bar{u}'' \ , \quad \xi \in \mathbb{R} \ . \tag{28}$$

Connection with reaction-diffusion equation. A TWS $u(x,t) = \bar{u}(x - ct)$ of a generalized Korteweg-de Vries-Burgers equation (26) satisfies TWE (28). Thus $v(x,t) = \bar{u}(x - \epsilon t)$ is a TWS $(\bar{u}, \epsilon)$ of the reaction-diffusion equation

$$\partial_t v = -h(v) + \delta \partial_x^2 v \ , \quad x \in \mathbb{R} \ , \quad t > 0 \ . \tag{29}$$

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Phase plane analysis. Following the analysis of TWE (21) for a reaction-diffusion equation (20) with $r(u) = -h(u)$ and $\sigma = \delta$, necessary conditions on the parameters can be identified. First, a TWE is rewritten as a system of first-order ODEs with vector field F. Then the condition on stationary states implies that endstates $u_\pm$ and wave speed c have to satisfy

$$f(u_+) - f(u_-) = c(u_+ - u_-) \ . \tag{30}$$

This condition is known in shock wave theory as Rankine-Hugoniot condition (54) on the shock triple $(u_-, u_+; c)$. The (nonlinear) stability of hyperbolic stationary states (u_s, v_s) of F is determined by the eigenvalues

$$\lambda_\pm = -\frac{1}{2}\frac{\epsilon}{\delta} \pm \frac{\sqrt{\epsilon^2 + 4\delta h'(u_s)}}{2|\delta|} \tag{31}$$

of the Jacobian $\mathrm{D}F(u_s, v_s)$. If $\epsilon, \delta > 0$, then $(u_+, 0)$ is always either a saddle or stable node, and $h'(u_-) = f'(u_-) - c > 0$ ensures that $(u_-, 0)$ is unstable. For example, Lax' entropy condition (55), i.e. $f'(u_+) < c < f'(u_-)$, implies the latter condition.

Convex flux functions.

Theorem 3.3. *Suppose $f \in C^2(\mathbb{R})$ is a strictly convex function. Let ϵ, δ be positive and let $(u_-, u_+; c)$ satisfy the Rankine-Hugoniot condition* (54) *and the entropy condition* (55)*, i.e. $u_- > u_+$. Then, there exists an (up to translations) unique TWS $(\bar{u}, c)$ of* (26) *in the sense of Definition 1.1.*

Proof. We consider the associated reaction-diffusion equation (29), i.e. $\partial_t u = r(u) + \delta\partial_x^2 u$ with $r(u) = -h(u)$. Due to (54) and (55), $r(u)$ is monostable in the sense of Definition 3.1. Moreover, function r is strictly concave, since $r''(u) = -f''(u)$ and $f \in C^2(\mathbb{R})$ is strictly convex. In fact, $(u_\pm, 0)$ are the only stationary points of system (22), where $(u_-, 0)$ is a saddle point and $(u_+, 0)$ is a stable node. Thus, for all wave speeds ϵ there exists a TWS $(\bar{u}, \epsilon)$ – with possibly oscillatory profile $\bar{u}$ – of reaction-diffusion equation (29). Moreover, $(\bar{u}, c)$ is a TWS of (26), due to (27)–(29). ☐

The TWP for KdVB equations (26) with Burgers' flux $f(u) = u^2$ has been investigated in [16]. The sign of δ in (26) is irrelevant, since it can be changed by a transformation $\tilde{x} = -x$ and $\tilde{u}(\tilde{x}, t) = -u(x, t)$, see also [41]. First, the results in Theorem 3.3 on the existence of TWS and geometry of its profiles are proven. More importantly, the authors investigate the convergence of profiles $\bar{u}(\xi; \epsilon, \delta)$ in the limits $\epsilon \to 0$, $\delta \to 0$, as well as ϵ and δ tending to zero simultaneously. Assuming that the ratio δ/ϵ^2 remains bounded, they show that the TWS converge to the classical Lax shocks for this vanishing diffusive-dispersive regularization [16].

Concave-convex flux functions.

Definition 3.4 ([45])**.** A function $f \in C^3(\mathbb{R})$ is called *concave-convex* if

$$uf''(u) > 0 \quad \forall u \neq 0 \ , \quad f'''(0) \neq 0 \ , \quad \lim_{u \to \pm\infty} f'(u) = +\infty \ . \tag{32}$$

Here the single inflection point is shifted without loss of generality to the origin. We consider a cubic flux function $f(u) = u^3$ as the prototypical concave-convex flux function with a single inflection point, see [39, 45].

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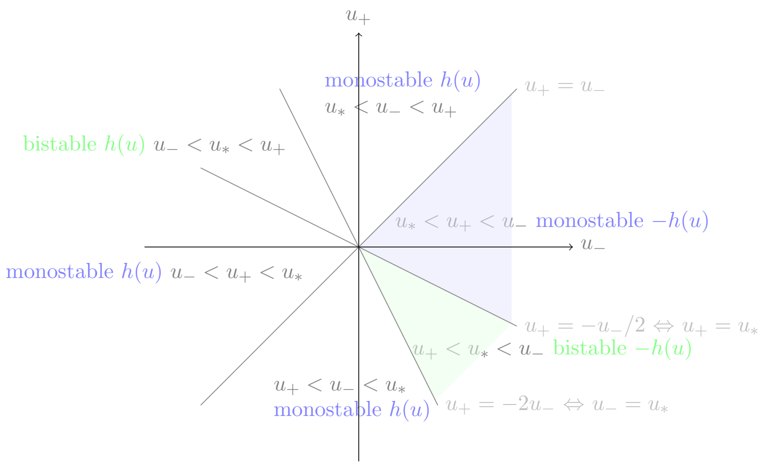

FIGURE 2. classification of the cubic reaction function $r(u) = -h(u)$ in (34) depending on its roots u_-, u_+ and $u_* = -u_- - u_+$ according to Definition 3.1.

Proposition 3.5 ([41, 38]). *Suppose $f(u) = u^3$ and $\epsilon > 0$.*

(a) If $\delta \leq 0$ then a TWS $(\bar{u}, c)$ of (26) *exists if and only if $(u_-, u_+; c)$ satisfy the Rankine-Hugoniot condition* (54) *and the entropy condition* (55).

(b) If $\delta > 0$ then a TWS $(\bar{u}, c)$ of (26) *exists for $u_- > 0$ if and only if $u_+ \in S(u_-)$ with*

$$S(u_-) = \begin{cases} [-\frac{u_-}{2}, u_-) & \text{if } u_- \leq 2\beta \,, \\ \{-u_- + \beta\} \cup [-\beta, u_-) & \text{if } u_- > 2\beta \,, \end{cases} \tag{33}$$

where the coefficient β is given by $\beta = \frac{\sqrt{2}}{3} \frac{\epsilon}{\sqrt{\delta}}$.

Proof. Following the discussion from (26)–(29), we consider the associated reaction-diffusion equation (29), i.e. $\partial_t u = r(u) + \delta \partial_x^2 u$ with $r(u) = -h(u)$. From this point of view, we need to classify the reaction term $r(u) = -h(u)$: Whereas $r(u_-) = 0$ by definition, $r(u_+) = 0$ if and only if $(u_-, u_+; c)$ satisfies the Rankine-Hugoniot condition (54). The Rankine-Hugoniot condition implies $c = u_+^2 + u_+ \, u_- + u_-^2$. Hence, the reaction term $r(u)$ has a factorization

$$r(u) = -(u^3 - u_-^3 - c(u - u_-)) = -(u - u_-)\,(u - u_+)\,(u + u_+ + u_-) \tag{34}$$

Thus, $r(u)$ is a cubic polynomial with three roots $u_1 \leq u_2 \leq u_3$, such that $r(u) = -(u - u_1)(u - u_2)(u - u_3)$. In case of distinct roots $u_1 < u_2 < u_3$ we deduce $r'(u_1) < 0$, $r'(u_2) > 0$ and $r'(u_3) < 0$. The ordering of the roots $u_\pm$ and $u_* = -u_- - u_+$ depending on $u_\pm$ is visualized in Figure 2. Next, we will discuss the results in Proposition 3.5(b) (for $u_- > 0$ and $\delta > 0$) via results on the existence of TWS for a reaction-diffusion equation (29).

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(1) For $u_+ < u_* < u_-$, function $r(u)$ is bistable, see also Figure 2. Due to Proposition 3.2, there exists an (up to translations) unique TWS $(\bar{u}, \epsilon)$ with possibly negative wave speed. Under our assumption that the wave speed ϵ is positive, relation (25) yields the restriction $-u_+ > u_-$. In fact, for $u_- > 2\beta$ and $u_+ = -u_- + \beta$ there exists a TWS $(\bar{u}, \epsilon)$ for reaction-diffusion equation (29), see [41, Theorem 3.4]. The function r is bistable with $u_* = -u_- - u_+ = -\beta$, hence $f'(u_\pm) > c$. This violates Lax' entropy condition (55) and is known in the shock wave theory as a slow undercompressive shock [45].
(2) For $u_* < u_+ < u_-$, function $r(u)$ is monostable, see Figure 2. Due to Proposition 3.2, there exists a critical wave speed c_*, such that monotone TWS $(\bar{u}, \epsilon)$ for (29) exist for all $\epsilon \geq c_*$. However, not all endstates (u_-, u_+) in the subset defined by $u_* < u_+ < u_-$ admit a TWS $(\bar{u}, c)$, see (33) and Figure 3B). The TWS $(\bar{u}, c)$ associated to non-classical shocks appear again, with reversed roles for the roots u_+ and u_*: For $u_- > 2\beta$ and $u_+ = -\beta$, there exists a TWS $(\bar{u}, \epsilon)$ for reaction-diffusion equation (29), see [41, Theorem 3.4]. These TWS form a horizontal halfline in Figure 3B) and divides the set defined by $u_* < u_+ < u_-$ into two subsets. In particular, TWS exist only for endstates (u_-, u_+) in the subset above this halfline.
(3) For $u_+ < u_- < u_*$, function $r(u) = -h(u)$ satisfies $r(u) < 0$ for all $u \in (u_+, u_-)$, see also Figure 2. Thus the necessary condition (25) can not be fulfilled for positive $c = \epsilon$, hence there exists no TWS $(\bar{u}, \epsilon)$ for the reaction-diffusion equation.
(4) For $u_* < u_- < u_+$, function $r(u)$ is monostable with reversed roles of the endstates $u_\pm$, see Figure 2. Due to Proposition 3.2, there exists a TWS $(\bar{u}, \epsilon)$ however satisfying $\lim_{\xi \to \mp\infty} \bar{u}(\xi) = u_\pm$.

If $\delta = 0$, then equation (26) is a viscous conservation law, and its TWE (28) is a simple ODE $-\epsilon \bar{u}' = r(\bar{u})$ with $r(u) = -h(u)$. Thus a heteroclinic orbit exists only for monostable $r(u)$, i.e. if the unstable node u_- and the stable node u_+ are not separated by any other root of r.

If $\delta < 0$, then we rewrite TWE (28) as $\epsilon \bar{u}' = h(u) + |\delta| \bar{u}''$. It is associated to a reaction-diffusion equation $\partial_t u = h(u) + |\delta| \partial_x^2 u$ via a TWS ansatz $u(x,t) = \bar{u}(x - (-\epsilon)t)$; note the change of sign for the wave speed. If $u_+ < u_* < u_-$ then $h(u)$ is an unstable reaction function. Thus there exists no TWS $(\bar{u}, -\epsilon)$ according to Proposition 3.2. If $u_* < u_+ < u_-$ then function $h(u) = -r(u)$ satisfies $h(u) < 0$ for all $u \in (u_+, u_-)$, see also Figure 2. The necessary condition (25) is still fine, since also the sign of the wave speed changed. In contrast to the case $\delta > 0$, there exists no TWS connecting u_- with u_*, which would indicate a bifurcation. Thus, the existence of TWS for all pairs (u_-, u_+) in the subset defined by $u_* < u_+ < u_-$ can be proven. The TWP for other pairs (u_-, u_+) is discussed similarly. □

4. TWP for nonlocal evolution equations

4.1. **Reaction-diffusion equations.** The first example of a reaction-diffusion equation with nonlocal diffusion is the integro-differential equation

$$\partial_t u = J * u - u + r(u) , \quad t > 0 , \quad x \in \mathbb{R} , \tag{35}$$

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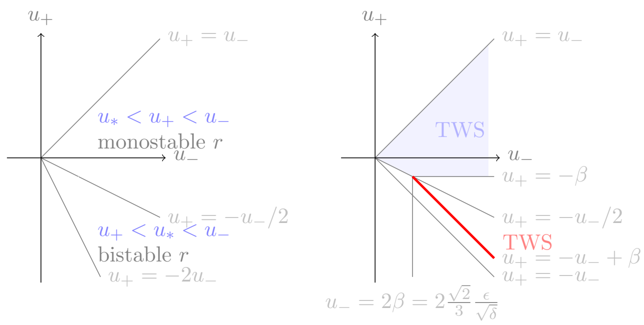

(A) (B)

FIGURE 3. A) classification of reaction function r depending on its roots u_-, u_+ and $u_* = -u_- - u_+$; B) Endstates $u_\pm$ in the shaded region and on the thick line can be connected by TWS of the cubic KdVB equation; TWS in the shaded region and on the thick line are associated to classical and non-classical shocks of $\partial_t u + \partial_x u^3 = 0$, respectively. For a classical shock the shock triple satisfies Lax' entropy condition $f'(u_-) > c > f'(u_+)$; i.e. characteristics in the Riemann problem meet at the shock. In contrast, the non-classical shocks are of slow undercompressive type, i.e. characteristics in the Riemann problem cross the shock.

for some even, non-negative function J with mass one, i.e. for all $x \in \mathbb{R}$

$$J \in C(\mathbb{R}) \;, \quad J \geq 0 \;, \quad J(x) = J(-x) \;, \quad \int_{\mathbb{R}} J(y) \;\, \mathrm{d}y \; = 1 \;, \tag{36}$$

and some function r. The operator $\mathcal{L}[u] = J * u - u$ is a Lévy operator, see (13), which models nonlocal diffusion. It is the infinitesimal generator of a compound Poisson stochastic process, which is a pure jump process.

The TWP for given endstates $u_\pm$ is to study the existence of a TWS $(\bar{u}, c)$ for (35) in the sense of Definition 1.1. Such a TWS $(\bar{u}, c)$ satisfies the TWE $-c\bar{u}' = J * \bar{u} - \bar{u} + r(\bar{u})$ for $\xi \in \mathbb{R}$. Next, we recall some results on the TWP for (35), which will depend crucially on the type of reaction function r and the tail behavior of a kernel function J. We will present the existence of TWS with monotone decreasing profiles $\bar{u}$, which will follow from the cited literature after a suitable transformation.

Proposition 4.1 ((monostable [27]), (bistable [14, 24]))**.** *Suppose $u_- > u_+$ and consider reaction functions r in the sense of Definition 3.1. Suppose $J \in W^{1,1}(\mathbb{R})$ and its continuous representative satisfies* (36)*.*

- *If r is monostable and there exists $\lambda > 0$ such that $\int_{\mathbb{R}} J(y) \; \exp(\lambda y) \;\, \mathrm{d}y < \infty$ then there exists a positive constant c_* such that for all $c \geq c_*$ there exists a monotone TWS $(\bar{u}, c)$ of* (35)*. For $c < c_*$ no such monotone TWS exists.*

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- *If r is bistable and $\int_{\mathbb{R}} |y|J(y) \, \mathrm{d}y < \infty$, then there exists an (up to translations) unique monotone TWS $(\bar{u}, c)$ of* (35).

For monostable reaction functions, the tail behavior of kernel function J is very important. There exist kernel functions J, such that TWS exist only for bistable – but not for monostable – reaction functions r, see [58]. The prime example are kernel functions J which decay more slowly than any exponentially decaying function as $|x| \to \infty$ in the sense that $J(x)\exp(\eta|x|) \to \infty$ as $|x| \to \infty$ for all $\eta > 0$.

For reaction-diffusion equations of bistable type, Chen established a unified approach [24] to prove the existence, uniqueness and asymptotic stability with exponential decay of traveling wave solutions. The results are established for a subclass of nonlinear nonlocal evolution equations

$$\partial_t u(x,t) = \mathcal{A}[u(\cdot,t)](x) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T] \ ,$$

where the nonlinear operator $\mathcal{A}$ is assumed to

(a) be independent of t;
(b) generate a L^∞ semigroup;
(c) be translational invariant, i.e. $\mathcal{A}$ satisfies for all $u \in \operatorname{dom} \mathcal{A}$ the identity

$$\mathcal{A}[u(\cdot + h)](x) = \mathcal{A}[u(\cdot)](x+h) \quad \forall x \ , h \in \mathbb{R} \, .$$

Consequently, there exists a function $r : \mathbb{R} \to \mathbb{R}$ which is defined by $\mathcal{A}[v\mathbf{1}] = r(v)\mathbf{1}$ for $v \in \mathbb{R}$ and the constant function $\mathbf{1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto 1$. This function r is assumed to be bistable in the sense of Definition 3.1;
(d) satisfy a comparison principle: If $\partial_t u \geq \mathcal{A}[u]$, $\partial_t v \leq \mathcal{A}[v]$ and $u(\cdot,0) \gneq v(\cdot,0)$, then $u(\cdot,t) > v(\cdot,t)$ for all $t > 0$.

Chen's approach relies on the comparison principle and the construction of sub- and supersolutions for any given traveling wave solution. Importantly, the method does not depend on the balance of the potential. More quantitative versions of the assumptions on $\mathcal{A}$ are needed in the proofs. Finally integro-differential evolution equations

$$\partial_t u = \epsilon \partial_x^2 u + G(u, J_1 * S^1(u), \dots, J_n * S^n(u)) \tag{37}$$

are considered for some diffusion constant $\epsilon \geq 0$, smooth functions G and S^k, and kernel functions $J_k \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ satisfying (36) where $k = 1, \dots, n$. Additional assumptions on the model parameters guarantee that an equation (37) can be interpreted as a reaction-diffusion equation with bistable reaction function including equations (20) and (35) as special cases.

Another example of reaction-diffusion equations with nonlocal diffusion are the integro-differential equations

$$\partial_t u = D^a_\theta u + r(u) \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{38}$$

for a (particle) density $u = u(x,t)$, some function $r = r(u)$, and a Riesz-Feller operator D^a_θ with $(\alpha,\theta) \in \mathfrak{D}_{a,\theta}$. The nonlocal Riesz-Feller operators are models for superdiffusion, where from a probabilistic view point a cloud of particle is assumed to spread faster than by following Brownian motion. Integro-differential equation (38) can be derived as a macroscopic equation for a particle density in the limit of modified Continuous Time Random

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Walk (CTRW), see [48]. In the applied sciences, equation (38) has found many applications, see [54, 57] for extensive reviews on modeling, formal analysis and numerical simulations.

The TWP for given endstates $u_\pm$ is to study the existence of a TWS $(\bar{u}, c)$ for (38) in the sense of Definition 1.1. Such a TWS $(\bar{u}, c)$ satisfies the TWE

$$-c\bar{u}' = D^a_\theta \bar{u} + r(\bar{u}) \ , \quad \xi \in \mathbb{R} \ . \tag{39}$$

First we collect mathematical rigorous results about the TWP associated to (38) in case of the fractional Laplacian $D^a_0 = -(-\Delta)^{a/2}$ for $a \in (0, 2)$, i.e. a Riesz-Feller operator D^a_θ with $\theta = 0$.

Proposition 4.2 ((monostable [17, 18, 34]), (bistable [21, 19, 20, 50, 25, 37])). *Suppose $u_- > u_+$. Consider the TWP for reaction-diffusion equation* (38) *with functions r in the sense of Definition 3.1 and fractional Laplacian D^a_0, i.e. symmetric Riesz-Feller operators D^a_θ with $0 < a < 2$ and $\theta = 0$.*

- *If r is monostable then there does not exist any TWS $(\bar{u}, c)$ of* (38).
- *If r is bistable then there exists an (up to translations) unique monotone TWS $(\bar{u}, c)$ of* (38).

For monostable reaction functions, Cabré and Roquejoffre prove that a front moves exponentially in time [17, 18]. They note that the genuine algebraic decay of the heat kernels G^a_0 associated to fractional Laplacians is essential to prove the result, which implies that no TWS with constant wave speed can exist. Engler [34] considered the TWP for (38) for a different class of monostable reaction functions r and non-extremal Riesz-Feller operators D^a_θ with $(a, \theta) \in \mathfrak{D}^+_{a,\theta}$ and $\mathfrak{D}^+_{a,\theta} := \{\, (a, \theta) \in \mathfrak{D}_{a,\theta} \mid |\theta| < \min\{a, 2 - a\} \,\}$. Again the associated heat kernels $G^a_\theta(x, t)$ with $(a, \theta) \in \mathfrak{D}^+_{a,\theta}$ decay algebraically in the limits $x \to \pm\infty$, see [47].

To our knowledge, we established the first result [6] on existence, uniqueness (up to translations) and stability of traveling wave solutions of (38) with Riesz-Feller operators D^a_θ for $(a, \theta) \in \mathfrak{D}_{a,\theta}$ with $1 < a < 2$ and bistable functions r. We present our results for monotone decreasing profiles, which can be inferred from our original result after a suitable transformation.

Theorem 4.3 ([6]). *Suppose $u_- > u_+$, $(a, \theta) \in \mathfrak{D}_{a,\theta}$ with $1 < a < 2$, and $r \in C^\infty(\mathbb{R})$ is a bistable reaction function. Then there exists an (up to translations) unique monotone decreasing TWS $(\bar{u}, c)$ of* (38) *in the sense of Definition 1.1.*

The technical details of the proof are contained in [6], whereas in [5] we give a concise overview of the proof strategy and visualize the results also numerically. In a forthcoming article [4], we extend the results to all non-trivial Riesz-Feller operators D^a_θ with $(a, \theta) \in \mathfrak{D}^\diamond_{a,\theta}$. The smoothness assumption on r is convenient, but not essential. To prove Theorem 4.3, we follow – up to some modifications – the approach of Chen [24]. It relies on a strict comparison principle and the construction of sub- and supersolutions for any given TWS. His quantitative assumptions on operator $\mathcal{A}$ are too strict, such that his results are not directly applicable. A modification allows to cover the TWP for (38) for all Riesz-Feller operators D^a_θ with $1 < a < 2$

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also for non-zero θ, and all bistable functions r regardless of the balance of the potential.

Next, we quickly review different methods to study the TWP of reaction-diffusion equations (38) with bistable function r and fractional Laplacian. In case of a classical reaction-diffusion equation (20), the existence of a TWS can be studied via phase-plane analysis [13, 35]. This method has no obvious generalization to our TWP for (38), since its traveling wave equation (39) is an integro-differential equation. The variational approach has been focused – so far – on symmetric diffusion operators such as fractional Laplacians and on balanced potentials, hence covering only stationary traveling waves [50]. Independently, the same results are achieved in [21, 19, 20] by relating the stationary TWE $(39)_{\theta=0,c=0}$ via [22] to a boundary value problem for a nonlinear partial differential equation. The homotopy to a simpler TWP has been used to prove the existence of TWS in case of (35), and $(38)_{\theta=0}$ with unbalanced potential [37].

Chmaj [25] also considers the TWP for $(38)_{\theta=0}$ with general bistable functions r. He approximates a given fractional Laplacian by a family of operators $J_\epsilon * u - (\int J_\epsilon)u$ such that $\lim_{\epsilon\to 0} J_\epsilon * u - (\int J_\epsilon)u = D_0^\alpha u$ in an appropriate sense. This allows him to obtain a TWS of $(38)_{\theta=0}$ with general bistable function r as the limit of the TWS u_ϵ of (35) associated to $(J_\epsilon)_{\epsilon\geq 0}$. It might be possible to modify Chmaj's approach to study reaction-diffusion equation (38) with asymmetric Riesz-Feller operators. This would give an alternative existence proof for TWS in Theorem 4.3. However, Chen's approach allows to establish uniqueness (up to translations) and stability of TWS as well.

4.2. **Nonlocal Korteweg-de Vries-Burgers equation.** First we consider the integro-differential equation in multi-dimensions $d \geq 1$

$$\partial_t u + \partial_x f(u) = \epsilon \Delta_x u + \gamma \epsilon^2 \sum_{j=1}^{d} \left(\phi_\epsilon * \partial_{x_j} u - \partial_{x_j} u\right) , \quad x \in \mathbb{R}^d , \quad t > 0 , \tag{40}$$

for parameters $\epsilon > 0$, $\gamma \in \mathbb{R}$, a smooth even non-negative function ϕ with compact support and unit mass, i.e. $\int_{\mathbb{R}^d} \phi(x) \ \mathrm{d}x = 1$, and the rescaled kernel function $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^d$. It has been derived as a model for phase transitions with long range interactions close to the surface, which supports planar TWS associated to undercompressive shocks of (51), see [52]. A planar TWS $(\bar u, c)$ is a solution $u(x,t) = \bar u(x - cte)$ for some fixed vector $e \in \mathbb{R}^d$, such that the profile is transported in direction e. The existence of planar TWS is proven by reducing the problem to a one-dimensional TWP for $(40)_{d=1}$, identifying the associated reaction-diffusion equation (35) and using results in Proposition 4.1. For cubic flux function u^3, the existence of planar TWS associated to undercompressive shocks of (51) is established. Moreover, the well-posedness of its Cauchy problem and the convergence of solutions u^ϵ as $\epsilon \searrow 0$ have been studied [52].

Another example is the **fractal Korteweg-de Vries-Burgers equation**

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x \mathcal{D}_+^\alpha u + \delta \partial_x^3 u, \quad x \in \mathbb{R}, \quad t > 0, \tag{41}$$

for some $\epsilon > 0$ and $\delta \in \mathbb{R}$.

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Equation (41) with $\alpha = 1/3$ has been derived as a model for shallow water flows, by performing formal asymptotic expansions associated to the triple-deck (boundary layer) theory in fluid mechanics, e.g. see [44, 55]. In particular, the situations of one-layer and two-layer shallow water flows have been considered, which yield a quadratic (one layer) and cubic flux function (two layer), respectively. In the monograph [49], similar models are considered and the well-posedness of the initial value problem and possible wave-breaking are studied.

The TWP for given endstates $u_\pm$ is to study the existence of a TWS $(\bar{u}, c)$ for (41) in the sense of Definition 1.1. Such a TWS $(\bar{u}, c)$ satisfies the TWE

$$h(\bar{u}) := f(\bar{u}) - f(u_-) - c(\bar{u} - u_-) = \epsilon \mathcal{D}^\alpha_+ \bar{u} + \delta \bar{u}'' \ . \tag{42}$$

We obtain a necessary condition for the existence of TWS – see also (25) – by multiplying the TWE with $\bar{u}'$ and integrating on $\mathbb{R}$,

$$\int_{u_-}^{u_+} h(u) \ \mathrm{d}u = \epsilon \int_{-\infty}^{\infty} \bar{u}' \ \mathcal{D}^\alpha_+ \bar{u}(\xi) \ \mathrm{d}\xi \ \geq 0 \ , \tag{43}$$

where the last inequality follows from (50).

Connection with reaction-diffusion equation. If a TWS $(\bar{u}, c)$ for (41) exists, then $u(x,t) = \bar{u}(x)$ is a stationary TWS $(\bar{u}, 0)$ of the evolution equation

$$\partial_t u = -\epsilon \mathcal{D}^\alpha_+ u - \delta \partial_x^2 u + h(u), \quad x \in \mathbb{R}, \quad t > 0. \tag{44}$$

To interpret equation (44) as a reaction-diffusion equation, we need to verify that $-\epsilon \mathcal{D}^\alpha_+ u - \delta \partial_x^2 u$ is a diffusion operator, e.g. that $-\epsilon \mathcal{D}^\alpha_+ u - \delta \partial_x^2 u$ generates a positivity preserving semigroup.

Lemma 4.4. *Suppose $0 < \alpha < 1$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. The operator $\gamma_1 \mathcal{D}^\alpha_+ u + \gamma_2 \partial_x^2 u$ is a Lévy operator if and only if $\gamma_1 \leq 0$ and $\gamma_2 \geq 0$. Moreover, the associated heat kernel is strictly positive if and only if $\gamma_2 > 0$.*

Proof. For $\alpha \in (0,1)$, the operator $-\mathcal{D}^\alpha_+$ is a Riesz-Feller operator $D^\alpha_{-\alpha}$ and generates a positivity preserving convolution semigroup with a Lévy stable probability distribution $G^\alpha_{-\alpha}$ as its kernel. The probability distribution is absolutely continuous with respect to Lebesgue measure and its density has support on a half-line [47]. For example the kernel associated to $-\mathcal{D}^{1/2}$ is the Lévy-Smirnov distribution. Thus, for $\gamma_1 \leq 0$ and $\gamma_2 \geq 0$, the operator $\gamma_1 \mathcal{D}^\alpha_+ u + \gamma_2 \partial_x^2 u$ is a Lévy operator, because it is a linear combination of Lévy operators. Using the notation for Fourier symbols of Riesz-Feller operators, the partial Fourier transform of equation

$$\partial_t u = -|\gamma_1| \mathcal{D}^\alpha[u] + \gamma_2 \partial_x^2 u$$

is given by $\partial_t \mathcal{F}[u](k) = (|\gamma_1| \psi^\alpha_{-\alpha}(k) - \gamma_2 k^2) \mathcal{F}[u](k)$. Therefore, the operator generates a convolution semigroup with heat kernel

$$\mathcal{F}^{-1}[\exp\{(|\gamma_1| \psi^\alpha_{-\alpha}(k) - \gamma_2 k^2) \ t\}](x) = G^\alpha_{-\alpha}(\cdot, |\gamma_1| t) * G^2_0(\cdot, \gamma_2 t) \ (x) \ ,$$

which is the convolution of two probability densities. The kernel is positive on $\mathbb{R}$ since probability densities are non-negative on $\mathbb{R}$ and the normal distribution G^2_0 is positive on $\mathbb{R}$ for positive $\gamma_2 t$.

The operator $\mathcal{D}^\alpha_+$ for $\alpha \in (0,1)$ is not a Riesz-Feller operator, see Figure 1, and it generates a semigroup which is not positivity preserving. Thus it and any linear combination with $\gamma_1 > 0$ is not a Lévy operator. □

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Convex flux functions.

Proposition 4.5. *Consider* (41) *with* $0 < \alpha < 1$, $\delta \in \mathbb{R}$ *and strictly convex flux function* $f \in C^3(\mathbb{R})$*. For a shock triple* $(u_-, u_+; c)$ *satisfying the Rankine-Hugoniot condition* (54)*, a non-constant TWS* $(\bar{u}, c)$ *can exist if and only if Lax' entropy condition* (55) *is fulfilled, i.e.* $u_- > u_+$.

Proof. The Rankine-Hugoniot condition (54) ensures that $h(u)$ in (42) has exactly two roots $u_\pm$. If Lax' entropy condition (55) is fulfilled, then $u_- > u_+$ and $-h(u)$ is monostable in the sense of Definition 3.1. Thus, the necessary condition (43) is satisfied. If $u_- = u_+$ then (43) implies that $\bar{u}$ is a constant function satisfying $\bar{u} \equiv u_\pm$. If $u_- < u_+$ then $-h(u)$ is monostable in the sense of Definition 3.1 with reversed roles of $u_\pm$. Thus, the necessary condition (43) is not satisfied. $\square$

Next, we recall some existence result which have been obtained by directly studying the TWE. In an Addendum [28], we removed an initial assumption on the solvability of the linearized TWE.

Theorem 4.6 ([3])**.** *Consider* (41) *with* $\delta = 0$ *and convex flux function* $f(u)$*. For a shock triple* $(u_-, u_+; c)$ *satisfying* (54) *and* (55)*, there exists a monotone TWS of* (41) *in the sense of Definition 1.1, whose profile* $\bar{u} \in C_b^1(\mathbb{R})$ *is unique (up to translations) among all functions* $u \in u_- + H^2(-\infty, 0) \cap C_b^1(\mathbb{R})$.

This positive existence result is consistent with the negative existence result in Proposition 4.2 and Engler [34] for (38) with non-extremal Riesz-Feller operators D_θ^α for $(a, \theta) \in \mathfrak{D}_{a,\theta}^+$. The reason is that $-\mathcal{D}_+^\alpha$ for $0 < \alpha < 1$ is the generator of a convolution semigroup with a one-sided strictly stable probability density function as its heat kernel; in contrast to heat kernels with genuine algebraic decay [17, 18, 34].

Theorem 4.7 ([2])**.** *Consider* (41) *with flux function* $f(u) = u^2/2$*. For a shock triple* $(u_-, u_+; c)$ *satisfying* (54) *and* (55)*, there exists a TWS of* (41) *in the sense of Definition 1.1, whose profile* $\bar{u}$ *is unique (up to translations) among all functions* $u \in u_- + H^4(-\infty, 0) \cap C_b^3(\mathbb{R})$.

If dispersion dominates diffusion then the profile of a TWS $(\bar{u}, c)$ will be oscillatory in the limit $\xi \to \infty$. For a classical KdVB equation this geometry of profiles depends on the ratio ϵ^2/δ and the threshold can be determined explicitly.

Concave-convex flux functions. We consider a cubic flux function $f(u) = u^3$ as the prototypical concave-convex flux function. Again the necessary condition (43) and the classification of function $h(u) = -r(u)$ in Figure 2 can be used to identify non-admissible shock triples $(u_-, u_+; c)$ for the TWP of (41).

We conjecture that a statement analogous to Proposition 3.5 holds true. Of special interest is again the occurrence of TWS $(\bar{u}, c)$ associated to non-classical shocks, which are only expected in case of (41) with $\epsilon > 0$ and $\delta > 0$.

Proposition 4.8. *Suppose* $f(u) = u^3$ *and* $\epsilon > 0$.

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(1) *If $\delta \leq 0$ then a TWS $(\bar{u}, c)$ of* (41) *exists if and only if $(u_-, u_+; c)$ satisfy the Rankine-Hugoniot condition* (54) *and the entropy condition* (55).
(2) *Conjecture: If $\delta > 0$ then a TWS $(\bar{u}, c)$ of* (41) *exists for $u_- > 0$ if and only if $u_+ \in S(u_-)$ for some set $S(u_-)$ similar to* (33).

Sketch of proof. If $\delta = 0$, then equation (41) is a viscous conservation law, and its TWE (42) is a fractional differential equation $\epsilon \mathcal{D}^\alpha_+ \bar{u} = h(\bar{u})$. Thus a heteroclinic orbit exists only for monostable $-h(u)$, i.e. if the unstable node u_- and the stable node u_+ are not separated by any other root of h. This follows from Theorem 4.6 and its proof in [3, 28].

If $\delta < 0$, then the TWE (42) is associated to a reaction-diffusion equation (44) via a stationary TWS ansatz $u(x, t) = \bar{u}(x)$. First we note that a stronger version of the necessary condition (43) is available

$$\int_{-\infty}^{\xi} h(\bar{u})\bar{u}'(y) \ \mathrm{d}y = \epsilon \int_{-\infty}^{\xi} \bar{u}' \ \mathcal{D}^\alpha_+ \bar{u}(y) \ \mathrm{d}y \geq 0 \ , \quad \forall \xi \in \mathbb{R} \ , \tag{45}$$

see [2]. If $u_+ < u_* < u_-$ then $h(u)$ is an unstable reaction function, see Figure 2. Thus there exists no TWS in the sense of Definition 1.1 satisfying the necessary condition (45). If $u_* < u_+ < u_-$ then function $-h(u)$ is monostable in the sense of Definition 3.1 and the necessary condition (43) can be satisfied. The existence of a TWS $(\bar{u}, c)$ can be proven by following the analysis in [2, 28]. The TWP for other pairs (u_-, u_+) is discussed similarly.

If $\delta > 0$ then the occurrence of TWS $(\bar{u}, c)$ associated to non-classical shocks is possible. Unlike in our previous examples, the associated evolution equation (44) is not a reaction-diffusion equation, since $-\epsilon \mathcal{D}^\alpha_+ \bar{u} - \delta \bar{u}''$ is not a Lévy operator. Especially, the results on existence of TWS for reaction-diffusion equations with bistable reaction function can not be used to prove the existence of TWS $(\bar{u}, c)$ associated to undercompressive shocks. Instead, we investigate the TWP directly [1], extending the analysis in [2, 28] for Burgers' flux to the cubic flux function $f(u) = u^3$. □

4.3. **Fowler's equation.** Fowler's equation (8) for dune formation is a special case of the evolution equation

$$\partial_t u + \partial_x f(u) = \delta \partial_x^2 u - \epsilon \partial_x \mathcal{D}^\alpha_+ u \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{46}$$

with $0 < \alpha < 1$, positive constant $\epsilon, \delta > 0$ and flux function f. Here the fractional derivative appears with the negative sign, but this instability is regularized by the second order derivative. The initial value problem for (8) is well-posed in L^2 [9]. However, it does not support a maximum principle, which is intuitive in the context of the application due to underlying erosions [9]. The existence of TWS of (8) – without assumptions (3) on the far-field behavior – has been proven [11].

For given endstates $u_\pm$, the TWP for (46) is to study the existence of a TWS $(\bar{u}, c)$ for (46) in the sense of Definition 1.1. Such a TWS $(\bar{u}, c)$ satisfies the TWE

$$h(\bar{u}) := f(\bar{u}) - f(u_-) - c(\bar{u} - u_-) = \delta \bar{u}' - \epsilon \mathcal{D}^\alpha_+ \bar{u} \ , \quad \xi \in \mathbb{R} \ . \tag{47}$$

For $\delta = 0$, the TWE reduces to a fractional differential equation $\epsilon \mathcal{D}^\alpha_+ \bar{u} = -h(\bar{u})$, which has been analyzed in [3, 28] for monostable functions $-h(u)$.

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Equation (47) is also the TWE for a TWS $(\bar{u}, \delta)$ of an evolution equation

$$\partial_t u = -\epsilon \mathcal{D}_+^\alpha u - h(u), \quad x \in \mathbb{R}, \quad t > 0. \tag{48}$$

For $\epsilon > 0$, the operator is $-\epsilon \mathcal{D}_+^\alpha \bar{u}$ is a Riesz-Feller operator $\epsilon D_{-\alpha}^\alpha$ whose heat kernel $G_{-\alpha}^\alpha$ has only support on a halfline. For a shock triple $(u_-, u_+; c)$ satisfying the Rankine-Hugoniot condition (54), at least $h(u_\pm) = 0$ holds. Under these assumptions, equation (48) is a reaction-diffusion equation with a Riesz-Feller operator modeling diffusion.

The abstract method in [11] does not provide any information on the far-field behavior. Thus, assume the existence of a TWS $(\bar{u}, c)$ in the sense of Definition 1.1, for some shock triple $(u_-, u_+; c)$ satisfying the Rankine-Hugoniot condition (54). Again, a necessary condition is obtained by multiplying TWE (47) with $\bar{u}'$ and integrating on $\mathbb{R}$; hence,

$$\int_{u_-}^{u_+} h(u) \ \mathrm{d}u = \int_{\mathbb{R}} (\bar{u}')^2 \ \mathrm{d}\xi - \int_{\mathbb{R}} \bar{u}' \mathcal{D}_+^\alpha \bar{u} \ \mathrm{d}\xi \ . \tag{49}$$

The left hand side is indefinite since each integral is non-negative, see also (50).

For a cubic flux function $f(u) = u^3$ and a shock triple $(u_-, u_+; c)$ satisfying the Rankine-Hugoniot condition (54), we deduce a bistable reaction function $r(u) = -h(u)$ as long as $u_+ < -u_+ - u_- < u_-$ see Figure 2. However, since the heat kernel has only support on a halfline, we can not obtain a strict comparison principle as needed in Chen's approach [24, 6, 4].

Appendix A. Caputo fractional derivative on $\mathbb{R}$

For $\alpha > 0$, the (Gerasimov-)Caputo derivatives are defined as, see [42, 54],

$$(\mathcal{D}_+^\alpha f)(x) = \begin{cases} f^{(n)}(x) & \text{if } \alpha = n \in \mathbb{N}_0 \ , \\ \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x \frac{f^{(n)}(y)}{(x-y)^{\alpha-n+1}} \ \mathrm{d}y & \text{if } n-1 < \alpha < n \text{ for some } n \in \mathbb{N}_0 \ . \end{cases}$$

$$(\mathcal{D}_-^\alpha f)(x) = \begin{cases} f^{(n)}(x) & \text{if } \alpha = n \in \mathbb{N}_0 \ , \\ \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^\infty \frac{f^{(n)}(y)}{(y-x)^{\alpha-n+1}} \ \mathrm{d}y & \text{if } n-1 < \alpha < n \text{ for some } n \in \mathbb{N}_0 \ . \end{cases}$$

Properties:

- For $\alpha > 0$ and $\lambda > 0$

$$(\mathcal{D}_+^\alpha \exp(\lambda \cdot))(x) = \lambda^\alpha \exp(\lambda x) \ , \quad (\mathcal{D}_-^\alpha \exp(-\lambda \cdot))(x) = \lambda^\alpha \exp(-\lambda x)$$

- For $\alpha > 0$ and $f \in \mathcal{S}(\mathbb{R})$, a Caputo derivative is a Fourier multiplier operator with $(\mathcal{F}\mathcal{D}_+^\alpha f)(k) = (\mathrm{i}\,k)^\alpha (\mathcal{F}f)(k)$ where $(\mathrm{i}\,k)^\alpha = \exp(\alpha\pi \,\mathrm{i} \operatorname{sgn}(k)/2)$.
- If $\bar{u}$ is the profile of a TWS $(\bar{u}, c)$ in the sense of Definition 1.1, then

$$\int_{-\infty}^\infty \bar{u}'(y) \, \mathcal{D}_+^\alpha \bar{u}(y) \ \mathrm{d}y = \tfrac{1}{2} \int_{\mathbb{R}} \bar{u}'(x) \int_{\mathbb{R}} \frac{\bar{u}'(y)}{|x-y|^\alpha} \ \mathrm{d}y \ \ \mathrm{d}x \geq 0 \ , \tag{50}$$

where the last inequality follows from [46, Theorem 9.8].

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Appendix B. Shock wave theory for scalar conservation laws

A standard reference on the theory of conservation laws is [29], whereas [45] covers the special topic of non-classical shock solutions. A scalar conservation law is a partial differential equation

$$\partial_t u + \partial_x f(u) = 0 \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{51}$$

for some flux function $f : \mathbb{R} \to \mathbb{R}$. For nonlinear functions f, it is well known that the initial value problem (IVP) for (51) with smooth initial data may not have a classical solution for all time $t > 0$ (due to shock formation). However, weak solutions may not be unique. The *Riemann problems* are a subclass of IVPs for (51), and especially important in some numerical algorithms: For given $u_-, u_+ \in \mathbb{R}$, find a weak solution $u(x,t)$ for the initial value problem of (51) with initial condition

$$u(x,0) = \begin{cases} u_- \ , & x < 0 \ , \\ u_+ \ , & x > 0 \, . \end{cases} \tag{52}$$

Weak solutions of a Riemann problem that are discontinuous for $t > 0$ may not be unique.

Example B.1. *A shock wave is a discontinuous solution of the Riemann problem,*

$$u(x,t) = \begin{cases} u_- \ , & x < ct \ , \\ u_+ \ , & x > ct \ , \end{cases} \tag{53}$$

if the shock triple $(u_-, u_+; c)$ *satisfies the Rankine-Hugoniot condition*

$$f(u_+) - f(u_-) = c(u_+ - u_-) \, . \tag{54}$$

The Rankine-Hugoniot condition (54) *is a necessary condition that* $u_\pm$ *are stationary states of an associated TWE* (28)*, see* (30)*.*

shock admissibility. Classical approaches to select a unique weak solution of the Riemann problem are

(a) *Lax' entropy condition:*

$$f'(u_+) < c < f'(u_-) \ . \tag{55}$$

It ensures that in the method of characteristics all characteristics enter the shock/discontinuity of a shock solution (53). For convex flux function f, condition (55) reduces to $u_- > u_+$. Shocks satisfying (55) are also called Lax or classical shocks. For non-convex flux functions f, also non-classical shocks can arise in experiments, called slow undercompressive shocks if $f'(u_\pm) > c$, and fast undercompressive shocks if $f'(u_\pm) < c$.

(b) *Oleinik's entropy condition.*

$$\frac{f(w) - f(u_-)}{w - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \quad \text{for all } w \text{ between } u_- \text{ and } u_+. \tag{56}$$

(c) *Entropy solutions* satisfying integral inequalities based on entropy-entropy flux pairs, such as Kruzkov's family of entropy-entropy flux pairs.

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(d) *Vanishing viscosity.* In the classical vanishing viscosity approach, instead of (51) one considers for $\epsilon > 0$ equation

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x^2 u \ , \quad t > 0 \ , \quad x \in \mathbb{R} \ , \tag{57}$$

where $\epsilon\partial_x^2 u$ models diffusive effects such as friction. Equation (57) is a parabolic equation, hence the Cauchy problem has global smooth solutions u^ϵ for positive times, especially for Riemann data (52). An admissible weak solution of the Riemann problem is identified by studying the limit of u^ϵ as $\epsilon \searrow 0$.
In other applications, different higher order effects may be important. For example, a nonlocal generalized KdVB equation (1) can be interpreted as a scalar conservation law (51) with higher-order effects $\mathcal{R}[u] := \epsilon\mathcal{L}_1[u] + \delta\partial_x\mathcal{L}_2[u]$.

Already for convex functions f, the convergence of solutions of the regularized equations (e.g. (1)) to solutions of (51) reveals a diverse solution structure. The solutions of viscous conservation laws (57) converge for $\epsilon \searrow 0$ to Kruzkov entropy solutions of (51). In contrast, in case of KdVB equation (4) the limit $\epsilon, \delta \to 0$ depends on the relative strength of diffusion and dispersion:

- **Weak dispersion:** $\delta = O(\epsilon^2)$ for $\epsilon \to 0$ e.g. $\delta = \beta\epsilon^2$ for some $\beta > 0$.
  TWS converge strongly to entropy solution of Burgers equation.
- **Moderate dispersion:** $\delta = o(\epsilon)$ for $\epsilon \to 0$ includes weak dispersion.
  TWS converge strongly to entropy solution of Burgers equation, see [51].
- **Strong dispersion:** weak limit of TWS for $\epsilon, \delta \to 0$ may not be a weak solution of Burgers equation.

For non-convex flux functions f, a TWS may converge to a weak solution of (51) which is not an Kruzkov entropy solution, but a non-classical shock.

A simplistic shock admissibility criterion based on the vanishing viscosity approach is the existence of TWS for a given shock triple:

Definition B.2 (compare with [41])**.** A solution u of the Riemann problem is called *admissible* (with respect to a fixed regularization $\mathcal{R}$), if there exists a TWS $(\bar{u}, c)$ in the sense of Definition 1.1 of the regularized equation (e.g. (1)) for every shock wave with shock triple $(u_-, u_+; c)$ in the solution u.

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CHAPTER 3

Reaction-diffusion equations

TRAVELING WAVES FOR A BISTABLE EQUATION WITH NONLOCAL-DIFFUSION

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Abstract. We consider a single component reaction-diffusion equation in one dimension with bistable nonlinearity and a nonlocal space-fractional diffusion operator of Riesz-Feller type. Our main result shows the existence, uniqueness (up to translations) and local asymptotic stability of a traveling wave solution connecting two stable homogeneous steady states. In particular, we provide an extension to classical results on traveling wave solutions involving local diffusion. This extension to evolution equations with Riesz-Feller operators requires several technical steps. These steps are based upon an integral representation for Riesz-Feller operators, a comparison principle, regularity theory for space-fractional diffusion equations, and control of the far-field behavior.

1. Introduction

We consider evolution equations

$$\partial_t u = \mathrm{D}_\theta^\alpha\, u + f(u)\,, \quad x \in \mathbb{R}\,, \quad t \in (0,\infty)\,, \tag{1}$$

where $f \in C^\infty(\mathbb{R})$ is a nonlinear function of bistable type, i.e., f has precisely three roots $u_- < a < u_+$ in the interval $[u_-, u_+]$ such that

$$f(u_-) = f(a) = f(u_+) = 0\,, \quad f'(u_-) < 0\,, \quad f'(u_+) < 0\,, \tag{2}$$

and D_θ^α is a Riesz-Feller operator for some fixed parameters $1 < \alpha \le 2$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$. A Riesz-Feller operator can be defined as a Fourier multiplier operator

$$\mathcal{F}[\mathrm{D}_\theta^\alpha\, u](\xi) = \psi_\theta^\alpha(\xi)\ \mathcal{F}[u](\xi), \qquad \xi \in \mathbb{R},$$

with $\mathcal{F}$ denoting the Fourier transform and symbol

$$\psi_\theta^\alpha(\xi) = -|\xi|^\alpha \exp\left[\mathrm{i}(\mathrm{sgn}(\xi))\theta\tfrac{\pi}{2}\right] \qquad \text{for } 0 < \alpha \le 2 \text{ and } |\theta| \le \min\{\alpha, 2-\alpha\}.$$

Special cases are the second order derivative $\mathrm{D}_0^2 = \partial_x^2$, the fractional Laplacians $\mathrm{D}_0^\alpha = -(-\partial_x^2)^{\alpha/2}$ for $0 < \alpha \le 2$ and $\theta = 0$, and Weyl fractional derivatives $\mathrm{D}_{2-\alpha}^\alpha$ for $0 < \alpha < 2$ and $\theta = 2-\alpha$; for details see Section 2.

The study of reaction-diffusion equations in the form (1) is motivated by the observation of ensembles of particles in experiments which do not spread according to normal diffusion modeled by $\partial_t u = \mathrm{D}_0^2\, u$. These diffusion processes are called anomalous and one distinguishes between subdiffusive and superdiffusive processes, given that the ensemble spreads slower, respectively, faster than normal diffusion [5, 29, 30]. In particular, diffusion

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equations with Riesz-Feller operator, $\partial_t u = \mathrm{D}^\alpha_\theta u$, are models which exhibit superdiffusive behavior. Consequently, reaction-diffusion equations of the form (1) have been used to model systems where reactions and superdiffusion occur simultaneously; examples include geophysical flows [13], the dynamics of fronts in magnetically confined plasmas [14], the spreading of epidemics due to complex mobility patterns of individuals [22], step-flow growth of a crystal surface [25], and experiments on the Belousov-Zhabotinsky reaction in a fluid forced by Faraday waves [30, 44].

A traveling wave solution of (1) is a solution of the form $u(t, x) = U(\xi)$, for some constant wave speed $c \in \mathbb{R}$, a traveling wave variable $\xi := x - ct$, and a function U connecting different endstates $\lim_{\xi\to\pm\infty} U(\xi) = u_\pm$. The profile U of a traveling wave solution has to satisfy the traveling wave equation

$$-cU'(\xi) = \mathrm{D}^\alpha_\theta U + f(U)$$

where D^α_θ has to be understood as its extension to C^2_b-functions.

The existence of traveling waves of (1) has been proved for $\mathrm{D}^2_0 = \partial^2_x$ [2, 18] and in case of fractional Laplacians $\mathrm{D}^\alpha_0 = -(-\partial^2_x)^{\alpha/2}$ modeling symmetric superdiffusion [7, 8, 21, 46, 35, 11]. However, some experiments indicate asymmetric superdiffusive behavior [13, 14, 22] and have been modeled by equation (1) with Riesz-Feller operator D^α_θ with non-zero asymmetry parameter θ.

Our aim is to prove existence, uniqueness (up to translations) and local asymptotic stability of traveling wave solutions $u(x, t) = U(x - ct)$ of (1) for $1 < \alpha < 2$ and $|\theta| \le \min\{\alpha, 2 - \alpha\}$. In the following we discuss only the case $u_- = 0$ and $u_+ = 1$ without loss of generality.

First, we briefly review previous results on traveling wave solutions of classical bistable reaction-diffusion equations in Subsection 1.1 and of bistable reaction-diffusion equations with fractional Laplacian in Subsection 1.2. Then we will present our main results in Subsection 1.3 and conclude with a discussion in Subsection 1.4.

1.1. **Classical Bistable Reaction-Diffusion equations.** Equation (1) with $D^2_0 = \partial^2_x$ and bistable nonlinear reaction term $f(u) = u(1 - u)(u - a)$ is known as Nagumo's equation to model propagation of signals [28, 31], as one-dimensional real Ginzburg-Landau equation to model long-wave amplitudes e.g. in case of convection in binary mixtures near the onset of instability [33, 39], as well as Allen-Cahn equation to model phase transitions in solids [1]. This equation has three homogeneous steady states (or equilibria) $0 = u_- < a < u_+ = 1$, where $u = u_\pm$ are locally asymptotically stable and $u = a$ is unstable. It is natural to search for monotone traveling wave solutions $u(x, t) = U(x - ct) = U(\xi)$ which connect two stable states

(3)
$$\lim_{\xi\to-\infty} U(\xi) = u_- \,, \qquad \lim_{\xi\to\infty} U(\xi) = u_+ \quad \text{and} \quad U'(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}\,.$$

The existence of—up to translation unique—traveling wave solutions $u(x, t) = U(x - ct)$ of reaction diffusion equations

(4)
$$\partial_t u = \partial^2_x u + f(u)\,, \quad x \in \mathbb{R}\,, \quad t > 0\,,$$

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with general bistable reaction function $f \in C^1(\mathbb{R})$ and their stability are well-known; see e.g. [2, 18, 42] and references therein.

It is important to highlight that phase plane methods may be used [19] to study the existence and uniqueness of traveling wave solutions of (4). In case of a partial integro-differential equation like (1) these classical geometric methods do not generalize immediately. A similar remark applies to the asymptotic stability of traveling wave solutions—with exponential rate of decay—which may be deduced from a special variational structure availabe in case of (4).

If a traveling wave solution with profile U exists then its wave speed c satisfies

$$c = -\frac{\int_{u_-}^{u_+} f(w) \, \mathrm{d}w}{\int_{\mathbb{R}} (U'(x))^2 \, \mathrm{d}x} \,. \tag{5}$$

Thus the potential $F(u) = F(u_-) + \int_{u_-}^{u} f(v) \, \mathrm{d}v$ indicates which stable state, either u_- or u_+, will replace the other one. In case of a balanced potential, $\int_{u_-}^{u_+} f(v) \, \mathrm{d}v = 0$, a stationary traveling wave will exist, i.e., both stable states will co-exist. In contrast, in case of an unbalanced potential, $\int_{u_-}^{u_+} f(v) \, \mathrm{d}v \neq 0$, the stable state with smaller potential value will replace the one with larger potential value, also called the metastable state. It is important to note that many of the following results are restricted to balanced bistable functions f.

In some applications a reaction-diffusion model with nonlocal diffusion may be more appropriate; Bates et al. [4] considered

$$\partial_t u = J * u - u + f(u) = \int_{\mathbb{R}} J(x-y) \, u(y,t) \, \mathrm{d}y - u(x,t) + f(u(x,t)) \,, \tag{6}$$

for $x \in \mathbb{R}$, $t > 0$, some suitable non-negative function $J \in C^1(\mathbb{R})$ and general bistable function $f \in C^2(\mathbb{R})$. This is an example of a nonlocal reaction-diffusion equation with diffusion operator of convolution type, $J*u-u$, which has—under suitable assumptions on J—similar properties as the Laplacian. They prove existence of traveling wave solutions via homotopy to a local reaction-diffusion model (4). However, local asymptotic stability of traveling wave solutions with exponential rate of decay is proven only for stationary traveling wave solutions, i.e., in case of a balanced bistable function f, since a variational structure used in (4) seems not to be available for (6).

Chen [10] established a unified approach to prove the existence, uniqueness and local asymptotic stability with exponential decay of traveling wave solutions for a class of nonlinear nonlocal evolution equations including (4) and (6) and many more examples from the literature. His approach is suitable for equations supporting a comparison principle and based on constructing suitable sub- and super-solutions. In Section 4.1 we recall his assumptions and results in more detail.

1.2. **Bistable Reaction-Diffusion equations with Fractional Laplacian.** We briefly review previous results on traveling wave solutions for equation (1) with fractional Laplacian D_0^α where $0 < \alpha < 2$.

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Zanette [45] considered equation (1) to study the effects of anomalous diffusion represented by a fractional Laplacian D_0^α in combination with a simple reaction function f introduced by McKean in his study [28] of Nagumo's equation [31]. Restricting to monotone traveling wave solutions, an explicit solution in integral form is derived, and the asymptotic behavior of front tails and front width are studied.

Volpert et al. [43] consider (1) for fractional Laplacians D_0^α with $\alpha \in [1,2]$ and general reaction function f. They notice that if a traveling wave solution $u(x,t) = U(x-ct)$ with monotone profile U exists, then its wave speed c has to satisfy again (5). Moreover, traveling wave profiles U are shown to approach the endstates at an algebraic rate $1/|\xi|^\alpha$ for $1 \leq \alpha < 2$, in contrast, to an exponential rate for $\alpha = 2$.

Nec et al. [32] consider (1) for fractional Laplacians D_0^α with $\alpha \in (1,2)$ and bistable reaction function $f(u) = u(1-u^2)$. They derive a variational formulation such that (1) is the associated Euler-Lagrange equation, and approximate traveling wave solutions. Similarly, they discuss examples with bistable reaction function f of unbalanced type.

A first rigorous proof for the existence and uniqueness of stationary traveling wave solutions of (1) for fractional Laplacian D_0^α and *balanced* bistable function f was given by Cabré and Sire [7, 8] as well as Palatucci, Savin and Valdinoci [35]. They consider equations in general space-dimensions, whereas we only discuss their results in one space-dimension.

Cabré and Sire [7, 8] consider the stationary problem

$$0 = \mathrm{D}_0^\alpha w + f(w) \quad \text{in } \mathbb{R} \tag{7}$$

for $\alpha \in (0,2)$ and a function $f \in C^{1,\gamma}(\mathbb{R})$ with $\gamma > \max(0, 1-\alpha)$. Due to a result by Caffarelli and Silvestre [9], they relate equation (7) to a boundary value problem for a nonlinear partial differential equation. Then they prove that a stationary traveling wave solution, $w(x) = U(x)$, of (7) exists if and only if f is a function of bistable type $f(u_-) = f(u_+) = 0$ with a balanced potential; if—in addition—$f'(u_\pm) > 0$ then they prove that a traveling wave solution is unique up to translations. Moreover, they derive the asymptotic behavior of front tails.

Palatucci, Savin and Valdinoci [35] investigate the existence, uniqueness and other geometric properties of the minimizers of the energy functional

$$\mathcal{E}(w,\Omega) := \mathcal{K}(w,\Omega) + \int_\Omega F(w(x))\ \mathrm{d}x \tag{8}$$

where $\mathcal{K}(w,\Omega)$ can be viewed as the contribution in Ω of the squared H^s semi-norm of w, and F is a double-well potential with $F(u_\pm) = 0$. First, they show that in one space-dimension stationary traveling wave solutions, $w(x) = U(x)$, of (7) are local minimizers of the functional $\mathcal{E}(w,\mathbb{R})$. For a bistable function $f \in C^1(\mathbb{R})$ with balanced potential, they prove the existence of a unique (up to translations) nontrivial global minimizer w of the energy $\mathcal{E}$ which is strictly increasing. This minimizer w solves (7) and is unique (up to translations) also in the class of monotone solutions of this equation. Moreover, they establish that w belongs to $C^2(\mathbb{R})$ and derive the asymptotic behavior of front tails.

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Chmaj [11] proved the existence of traveling wave solutions of (1) for fractional Laplacians D_0^α with $\alpha \in (0,2)$ and general bistable function f. A fractional Laplacian can be approximated by a family of convolution operators $J_\epsilon * U - U$ such that $J_\epsilon * U - U \longrightarrow \mathrm{D}_0^\alpha U$ for $\epsilon \to 0$ in a suitable sense. The associated (traveling wave) equations $-cU' = J_\epsilon * U - U + f(U)$ exhibit for all sufficiently small $\epsilon > 0$ a unique monotone solution (U_ϵ, c_ϵ) with $U_\epsilon' > 0$ see also [4, 10]. Finally the existence of a limit $U = \lim_{\epsilon\to 0} U_\epsilon$ is established and that U is the profile of a traveling wave solution of (1) with $\theta = 0$.

Gui and Zhao also consider (1) for fractional Laplacians D_0^α with $\alpha \in (0,2)$ and general bistable reaction function f. They prove existence of unique traveling wave solutions $u \in C^2(\mathbb{R})$ via homotopy to the balanced case. Moreover, they show qualitative properties of the traveling wave solutions such as the asymptotic behavior of front tails [21, 46].

1.3. **Bistable Reaction-Diffusion equations with Riesz-Feller operator.** The reaction-diffusion equation (1) with general Riesz-Feller operators has been considered in [3, 16]. Baeumer et al. [3] developed a numerical method for fractional reaction-diffusion equations based on operator splitting. Engler [16] considered the initial value problem for (1) with initial data having support on a half-line and studied how the spatial support of the solution spreads in time. Using comparison arguments he determines conditions on a given reaction function f to decide if the speed of the spread is bounded or unbounded. In case of (our) bistable reaction function f the spread of spatial support is proven to be bounded, supporting the existence of traveling wave solutions.

Remark. In comparison, the Fisher-KPP equation is a reaction-diffusion equation (1) with $D_0^2 = \partial_x^2$ and $f(u) = u(1-u)$ describing the competition of species. The stable state will invade the unstable state at a constant speed in case of $D_0^2 = \partial_x^2$ and at an exponential speed in case of a Riesz-Feller operator D_θ^α with $0 < \alpha < 2$ [14, 6, 16].

Main Results. Our main result is summarized in the following theorem.

Theorem 1.1. *Suppose $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ satisfies (2). Then equation (1) admits a traveling wave solution $u(x,t) = U(x-ct)$ satisfying (3). In addition, a traveling wave solution of (1) is unique up to translations. Furthermore, traveling wave solutions are locally asymptotically stable in the sense that there exists a positive constant κ such that if $u(x,t)$ is a solution of (1) with initial datum $u_0 \in C_b(\mathbb{R})$ satisfying $0 \le u_0 \le 1$ and*

$$\liminf_{x\to\infty} u_0(x) > a\,, \qquad \limsup_{x\to-\infty} u_0(x) < a\,,$$

then, for some constants $\xi \in \mathbb{R}$ and $K > 0$ depending on u_0,

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \le K e^{-\kappa t} \qquad \forall t \ge 0\,.$$

Our proof is structured as follows. In Section 2, first we consider the Riesz-Feller operators as Fourier multiplier operators on Schwartz functions, then we extend the Riesz-Feller operators in form of singular integrals to functions in $C_b^2(\mathbb{R})$.

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In Section 3, we investigate the Cauchy problem for (1) with initial datum $u_0 \in C_b(\mathbb{R})$ such that $0 \le u_0 \le 1$. We follow a standard approach, to consider the Cauchy problem in its mild formulation and to prove the existence of a mild solution. The Cauchy problem generates a nonlinear semigroup which allows us to prove uniform C_b^k estimates via a bootstrap argument and to conclude that mild solutions are also classical solutions. In Subsection 3.1 we establish a comparison principle for the partial integro-differential equation (1) and investigate the behavior of the spatial limits of solutions. The comparison principle is essential to prove our result on the existence, uniqueness and local asymptotic stability of traveling wave solutions and to allow for a larger class of admissible functions f in the result for the Cauchy problem. Moreover, in the existence proof we need to show that the (continuous) solution of the Cauchy problem with some prepared initial datum exhibits spatial limits at all times. Therefore, we prove Theorem 3.4 on the far-field behavior of solutions.

In Section 4, we consider the traveling wave problem for (1). First, we recall the results by Chen [10]. Then we study his necessary assumptions and notice that some estimates are not of the required form. However Chen's approach can be extended, which we prove in the Appendices A–C. Our main result in Theorem 1.1 will follow from the separate results on uniqueness in Theorem A.1, on local asymptotic stability in Theorem B.3 and on existence of a traveling wave solution in Theorem C.1. The details are given in Subsection 4.2.

1.4. **Discussion.** To our knowledge, we establish the first result on existence, uniqueness (up to translations) and local asymptotic stability of traveling wave solutions of (1) with Riesz-Feller operators D^α_θ for $1 < \alpha < 2$ with $|\theta| \le \min\{\alpha, 2-\alpha\}$ and general bistable function f.

The variational approach [35] is—at the moment—restricted to symmetric diffusion operators such as fractional Laplacians and *balanced* bistable functions. Whereas, Gui and Zhao deduce the existence of a traveling wave solution for an *unbalanced* bistable function via a homotopy argument from the case of a balanced one. It might be possible to modify Chmaj's approach to cover reaction-diffusion equation (1) with Riesz-Feller operators D^α_θ. However his approach is only concerned with the existence of traveling wave solutions. By following Chen's approach, we obtain uniqueness and local asymptotic stability of traveling wave solutions of (1) directly.

In contrast, the existence of traveling wave solutions of equation (1) with bistable function f and fractional Laplacian D^α_0 with $0 < \alpha \le 1$ has been established in case of balanced potentials [7, 8, 35] and in the unbalanced case by [11, 21]. However, to extend Chen's approach, if this is possible, to the general case of Riesz-Feller operators with $0 < \alpha \le 1$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$ remains an open problem.

2. Riesz-Feller Operators

We follow Mainardi, Luchko and Pagnini [27] in their definition of the Riesz-Feller fractional derivative as a Fourier multiplier operator. They use a definition of the Fourier transform which is customary in probability

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theory. For f in the Schwartz space

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^\beta \frac{\partial^\gamma f}{\partial x^\gamma}(x) \right| < \infty,\ \forall \beta, \gamma \in \mathbb{N}_0 \right\} \tag{9}$$

the Fourier transform is defined as

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}\xi x} f(x)\ \mathrm{d}x\ , \qquad \xi \in \mathbb{R}\,,$$

and the inverse Fourier transform as

$$\mathcal{F}^{-1}[f](x) := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi x} f(\xi)\ \mathrm{d}\xi\ , \qquad x \in \mathbb{R}\,.$$

In the following, $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote also their respective extensions to $L^2(\mathbb{R})$. Then, the Riesz-Feller space-fractional derivative of order α and skewness θ is the Fourier multiplier operator

$$\mathcal{F}[D^\alpha_\theta f](\xi) = \psi^\alpha_\theta(\xi)\ \mathcal{F}[f](\xi)\,, \qquad \xi \in \mathbb{R}\,, \tag{10}$$

with symbol

$$\psi^\alpha_\theta(\xi) = -|\xi|^\alpha \exp\left[\mathrm{i}(\mathrm{sgn}(\xi))\theta\tfrac{\pi}{2}\right]\ , \quad 0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,. \tag{11}$$

The symbol $\psi^\alpha_\theta(\xi)$ is the logarithm of the characteristic function of a Lévy strictly stable probability density with index of stability α and asymmetry parameter θ according to Feller's parameterization [17, 20]; see also [47, 36, 34].

2.1. **The Linear Space-Fractional Diffusion Equation.** To analyze the Cauchy problem for the reaction diffusion equation (1) we need to investigate the linear space-fractional diffusion equation

$$\partial_t u(x,t) = \mathrm{D}^\alpha_\theta\,[u(\cdot,t)](x)\,, \quad (x,t) \in \mathbb{R} \times (0,\infty)\,, \tag{12}$$

for $0 < \alpha \le 2$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$. A formal Fourier transform of the associated Cauchy problem yields

$$\partial_t \mathcal{F}[u](\xi,t) = \psi^\alpha_\theta(\xi)\ \mathcal{F}[u](\xi,t)\,, \qquad \mathcal{F}[u](\xi,0) = \mathcal{F}[u_0](\xi)\,,$$

which has a solution $\mathcal{F}[u](\xi,t) = \mathrm{e}^{t\psi^\alpha_\theta(\xi)}\mathcal{F}[u_0](\xi)$. Hence, a formal solution of the Cauchy problem is given by

$$u(x,t) = (G^\alpha_\theta(\cdot,t) * u_0)(x) = \int_{\mathbb{R}} G^\alpha_\theta(x-y,t)\ u_0(y)\ \mathrm{d}y \tag{13}$$

with kernel (or Green's function)

$$G^\alpha_\theta(x,t) := \mathcal{F}^{-1}\left[\mathrm{e}^{t\psi^\alpha_\theta(\cdot)}\right](x)\,. \tag{14}$$

To study the properties of the formal solution, first we investigate the kernel G^α_θ and then we verify that (13) defines a semigroup of solutions.

Lemma 2.1. *For $0 < \alpha \le 2$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$, $G^\alpha_\theta(x,t)$ is the probability measure of a Lévy strictly α-stable distribution.*

Moreover, for $|\theta| < 1$ the probability measure G^α_θ is absolutely continuous with respect to the Lebesgue measure and possesses a probability density which will be denoted again by G^α_θ. Furthermore, for all $(x,t) \in \mathbb{R} \times (0,\infty)$ the following properties hold;

(G1) $G^\alpha_\theta(x,t) \ge 0$,

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(G2) $G^\alpha_\theta(x,t) = t^{-1/\alpha} G^\alpha_\theta(xt^{-1/\alpha}, 1)$.
(G3) $\|G^\alpha_\theta(\cdot,t)\|_{L^1(\mathbb{R})} = 1$,
(G4) $G^\alpha_\theta(\cdot,s) * G^\alpha_\theta(\cdot,t) = G^\alpha_\theta(\cdot,s+t)$ *for all* $s,t \in (0,\infty)$,
(G5) $\|G^\alpha_\theta(\cdot,t)\|_{L^p(\mathbb{R})} \leq \|G^\alpha_\theta(\cdot,1)\|_{L^p(\mathbb{R})} t^{\frac{1-p}{\alpha p}}$ *for all* $1 \leq p < \infty$,
(G6) $G^\alpha_\theta \in C^\infty_0(\mathbb{R} \times (0,\infty))$.

Moreover, for $1 \leq \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ *and* $|\theta| < 1$,

(G7) For all $m \geq 0$ *there exists a constant* $B_m \in (0,\infty)$ *such that*

$$\left|\partial^m_x G^\alpha_\theta(x,t)\right| \leq t^{-(1+m)/\alpha} \frac{B_m}{1 + t^{-2/\alpha} x^2}\,, \qquad \forall (x,t) \in \mathbb{R} \times (0,\infty)\,. \tag{15}$$

(G8) For all $t > 0$*, there exists a* $\mathcal{K}$ *such that* $\|\partial_x G(\cdot,t)\|_{L^1(\mathbb{R})} = \mathcal{K} t^{-1/\alpha}$.
(G9) $G^\alpha_\theta(\cdot,s) * \partial_x G^\alpha_\theta(\cdot,t) = \partial_x G^\alpha_\theta(\cdot,s+t)$ *for all* $s,t \in (0,\infty)$.
(G10) For all $t > t_0 > 0$ *and* $u \in L^1(\mathbb{R})$ *we have* $(G^\alpha_\theta(\cdot,t) * u) \in C^\infty(\mathbb{R})$.

For $0 < \alpha \leq 2$ *and* $|\theta| \leq \min\{\alpha, 2-\alpha\}$ *and* $\alpha \neq \pm\theta$ *(i.e. excluding the so-called extremal pdfs)*

(G11) $G^\alpha_\theta(x,t) > 0$.

Proof. Due to Theorem [37, Theorem 14.19], the function $\mathrm{e}^{t\psi^\alpha_\theta(\xi)}$ is the characteristic function of a random variable with Lévy strictly α-stable distribution. Thus G^α_θ is the scaled probability measure of a Lévy strictly α-stable distribution. In case of $(\alpha,\theta) \in \{(0,0),(1,1),(1,-1)\}$, the probability measure G^α_θ is a delta distribution

$$G^0_0(x,t) = \delta_x\,, \qquad G^1_1(x,t) = \delta_{x+t}\,, \qquad G^1_{-1}(x,t) = \delta_{x-t}$$

and is called trivial [37, Definition 13.6]. In all other (non-trivial) cases, the probability measure G^α_θ is absolutely continuous with respect to the Lebesgue measure and has a continuous probability distribution density [37, Proposition 28.1], which we will denote again by G^α_θ. A non-trivial strictly α-stable probability density is pointwise non-negative (G1) and satisfies the scaling property (G2) due to [37, Remark 14.18]; hence the identity (G3) and the estimate (G5) follow. The (semigroup)-property is satisfied by the defining property of strictly α-stable probability density [37, Definition 13.1]. Moreover, a non-trivial strictly α-stable probability density is C^∞-smooth whose partial derivatives of all orders tend to 0 in the limits $x \to \pm\infty$ [37, Proposition 28.1; Example 28.2], hence (G6) holds. Subsequently, the properties (G7)–(G9) follow from direct calculations, see also [15] for the special case of fractional Laplacian D^α_0 and [12] for the general case $\alpha \in (1,\infty)\backslash\mathbb{N}$. To prove (G10), we consider the basic definition of the derivative as the limit of a finite difference. Moreover, for $t > t_0 > 0$, $G^\alpha_\theta(\cdot,t) \in C^\infty_b(\mathbb{R}) \cap W^{\infty,1}(\mathbb{R})$ due to the estimate (G7). Thus $\frac{1}{\epsilon}(G^\alpha_\theta(\cdot,t) - \tau_\epsilon G^\alpha_\theta(\cdot,t))$ converges uniformly to $\partial_x G^\alpha_\theta(\cdot,t)$, i.e. with respect to the norm $\|.\|_{L^\infty(\mathbb{R})}$. This fact and the Dominated Convergence Theorem imply that, $\frac{1}{\epsilon}(G^\alpha_\theta(\cdot,t) - \tau_\epsilon G^\alpha_\theta(\cdot,t)) * u$ converges uniformly to $\partial_x G^\alpha_\theta(\cdot,t) * u$, too. Finally, for $h(\cdot,t) := G^\alpha_\theta(\cdot,t) * u$, the identity $\frac{1}{\epsilon}(G^\alpha_\theta(\cdot,t) - \tau_\epsilon G^\alpha_\theta(\cdot,t)) * u = \frac{1}{\epsilon}(h(\cdot,t) - \tau_\epsilon h(\cdot,t))$ implies that the derivative $\partial_x h(\cdot,t)$ exists and is equal to $\partial_x G^\alpha_\theta(\cdot,t) * u$. A mathematical induction on the order of the derivative proves the general statement. Due to (G8) and a result by Sharpe [40], the support of $G^\alpha_\theta(\cdot,t)$ is either all of $\mathbb{R}$ or a half-line for each $t > 0$ [37, Remark 28.8]. Indeed only the strictly

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α-stable probability densities with $0 < \alpha < 1$ and $\theta = -\alpha$ or $\theta = \alpha$ have support on $(-\infty, 0]$ and $[0, \infty)$, respectively; all others have support $\mathbb{R}$ [36, Property 1.2.14]. □

Due to the properties of G_θ^α, it is easy to show that D_θ^α generates a semigroup.

Proposition 2.2. *For $1 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, the Riesz-Feller operator D_θ^α generates a strongly continuous, convolution semigroup*

$$S_t : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,, \quad u_0 \mapsto S_t u_0 = G_\theta^\alpha(\cdot, t) * u_0 \,,$$

for all $1 \leq p < \infty$.

Proof. Due to Lemma 2.1, the probability measure G_θ^α is absolutely continuous with respect to the Lebesgue measure and possesses a probability distribution density which will be denoted again by G_θ^α. Thus (G3) and Young's inequality for convolutions imply $\|S_t u\|_{L^p} \leq \|G_\theta^\alpha(\cdot, t)\|_{L^1} \|u\|_{L^p} = \|u\|_{L^p}$ for all $u \in L^p(\mathbb{R}^n)$. Therefore $S_t : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ are well-defined bounded linear operators for all $t \geq 0$. $(S_t)_{t\geq 0}$ is a semigroup, since $S_{t+s} = S_t S_s$ for all $s, t \geq 0$ holds due to (G4). Whereas the formal definition $S_0 = \mathrm{Id}$ is justified, since (G2) and a standard result about convolutions [26, p.64] yield strong continuity of $(S_t)_{t\geq 0}$. □

2.2. **Extensions to Bounded Continuous Functions.** We are interested in traveling wave solutions which will be $C_b^2(\mathbb{R})$ functions in space. Therefore, we are going to derive an extension for the nonlocal operators D_θ^α such that $\mathrm{D}_\theta^\alpha : C_b^2 \to C_b$ and it generates a semigroup on C_b.

2.2.1. *A Representation Formula.* To study the traveling wave problem, it is necessary to extend the nonlocal operator to $C_b^2(\mathbb{R})$. The following integral representations may be used to accomplish this task.

Proposition 2.3. *If $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, then for all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$*

(16)
$$\mathrm{D}_\theta^\alpha f(x) = c_1 \int_0^\infty \tfrac{f(x+\xi) - f(x) - f'(x)\,\xi}{\xi^{1+\alpha}} \, \mathrm{d}\xi + c_2 \int_0^\infty \tfrac{f(x-\xi) - f(x) + f'(x)\,\xi}{\xi^{1+\alpha}} \, \mathrm{d}\xi \,.$$

for some constants $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$.

Proof. This representation of the Riesz-Feller operator D_θ^α is well known in the literature on (generators of stable) stochastic processes [24, 12]. The representation is stated without proof [24, 12], therefore we show how to identify our chosen form (16) from a standard reference like [37].

Due to Lemma 2.1, G_θ^α is the scaled probability measure of a Lévy strictly α-stable distribution. Due to [37, Theorem 14.3], such a probability measure can be characterized by a *Lévy triplet* (A, ν, γ). In particular, for G_θ^α, the constants are determined as $A = 0$ and $\gamma \in \mathbb{R}$, and ν is an absolutely continuous (Lévy) measure

$$\nu(\,\mathrm{d}x) = \begin{cases} c_1 x^{-1-\alpha} & \text{on } (0, \infty) \,, \\ c_2 |x|^{-1-\alpha} & \text{on } (-\infty, 0) \,, \end{cases} \tag{17}$$

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for some constants $c_1, c_2 \geq 0$ such that $c_1 + c_2 > 0$, see also [37, page 80]. Moreover, there exists a Lévy process $(X_t)_{t\geq 0}$ (such that $P_{X_1} = G_\theta^\alpha$), which is unique up to identity in law [37, Corollary 11.6]. Due to [37, Theorem 31.5], the infinitesimal generator of the associated transition semigroup has a representation

$$\begin{aligned}(18)\quad \mathrm{D}_\theta^\alpha f(x) = \tfrac{c_1-c_2}{1-\alpha} f'(x) + c_1 \int_0^\infty \tfrac{f(x+\xi)-f(x)-f'(x)\,\xi 1_{(-1,1)}(\xi)}{\xi^{1+\alpha}}\ \mathrm{d}\xi \\ + c_2 \int_0^\infty \tfrac{f(x-\xi)-f(x)+f'(x)\,\xi 1_{(-1,1)}(\xi)}{\xi^{1+\alpha}}\ \mathrm{d}\xi\end{aligned}$$

for the given constants c_1 and c_2.

In [37, Remark 8.4], alternative representations are discussed. If $1 < \alpha < 2$, then the Lévy measure ν satisfies condition $\int_{|x|>1} |x|\,\nu(\,\mathrm{d}x) < \infty$, hence the characteristic function has a representation [37, Eq.(8.8)] with generating triplet $(A, \nu, \gamma_1)_1 = (0, \nu, \gamma_1)$. Due to [37, Theorem 14.7], a strictly α-stable distribution for $1 < \alpha < 2$ satisfies $\gamma_1 = 0$ which yields representation (16). □

This representation allows to extend the D_θ^α operator to $C_b^2(\mathbb{R})$ such that $\mathrm{D}_\theta^\alpha\, C_b^2(\mathbb{R}) \subset C_b(\mathbb{R})$.

Proposition 2.4. *The integral representation* (16) *of* D_θ^α *with* $1 < \alpha < 2$ *and* $|\theta| \leq \min\{\alpha, 2-\alpha\}$ *is well-defined for functions* $f \in C_b^2(\mathbb{R})$ *with*

$$(19)\quad \sup_{x\in\mathbb{R}} |\,\mathrm{D}_\theta^\alpha f(x)| \leq \tfrac{1}{2}(c_1+c_2)\|f''\|_{C_b(\mathbb{R})} \frac{M^{2-\alpha}}{2-\alpha} + 2(c_1+c_2)\|f'\|_{C_b(\mathbb{R})} \frac{M^{1-\alpha}}{\alpha-1} < \infty$$

for any positive constant $M > 0$ *and the positive constants* c_1 *and* c_2 *in Proposition 2.3.*

Proof. We consider the two summands in (16) separately, starting with the first integral for any $f \in C_b^2(\mathbb{R})$. The goal is to obtain an upper bound. Choose $M > 0$ and consider

$$\int_0^\infty \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi = \int_0^M \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi + \int_M^\infty \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi\ .$$

The first integral is written as

$$\begin{aligned}\int_0^M \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi &= \int_0^M \tfrac{1}{\xi^{1+\alpha}} \left[\int_0^1 f'(x+\theta\xi)\,\xi\ \mathrm{d}\theta - f'(x)\xi\right] \mathrm{d}\xi \\ &= \int_0^M \tfrac{\xi}{\xi^{1+\alpha}} \left[\int_0^1 \int_0^1 f''(x+s\theta\xi)\,\theta\xi\ \mathrm{d}s\ \ \mathrm{d}\theta\right] \mathrm{d}\xi \\ &= \int_0^M \tfrac{\xi^2}{\xi^{1+\alpha}} \underbrace{\left[\int_0^1 \int_0^1 f''(x+s\theta\xi)\,\theta\ \mathrm{d}s\ \ \mathrm{d}\theta\right]}_{\text{bounded by } \|f''\|_{C_b}} \mathrm{d}\xi\end{aligned}$$

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where we use the shorthand notation $\|\cdot\|_{C_b} = \|\cdot\|_{C_b(\mathbb{R})}$. Thus

$$\left|\int_0^M \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi\right| \leq \tfrac{1}{2}\|f''\|_{C_b}\int_0^M \xi^{1-\alpha}\ \mathrm{d}\xi\ = \tfrac{1}{2}\|f''\|_{C_b}\tfrac{M^{2-\alpha}}{2-\alpha}\,.$$

The second integral is written as

$$\begin{aligned}\int_M^\infty \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi &= \int_M^\infty \tfrac{1}{\xi^{1+\alpha}}\left[\int_0^1 f'(x+\theta\xi)\,\xi\ \mathrm{d}\theta\ - f'(x)\xi\right]\ \mathrm{d}\xi \\ &= \int_M^\infty \tfrac{\xi}{\xi^{1+\alpha}}\underbrace{\left[\int_0^1 f'(x+\theta\xi)-f'(x)\ \mathrm{d}\theta\right]}_{\text{bounded by } 2\|f'\|_{C_b}}\ \mathrm{d}\xi\end{aligned}$$

Thus

$$\left|\int_M^\infty \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi\right| \leq 2\|f'\|_{C_b}\int_M^\infty \xi^{-\alpha}\ \mathrm{d}\xi\ = 2\|f'\|_{C_b}\tfrac{M^{1-\alpha}}{\alpha-1}\,.$$

Summarizing we estimate

$$\left|\int_0^\infty \tfrac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi\right| \leq \tfrac{1}{2}\|f''\|_{C_b}\tfrac{M^{2-\alpha}}{2-\alpha} + 2\|f'\|_{C_b}\tfrac{M^{1-\alpha}}{\alpha-1} < \infty$$

and similarly

$$\left|\int_0^\infty \tfrac{f(x-\xi)-f(x)+f'(x)\xi}{\xi^{1+\alpha}}\ \mathrm{d}\xi\right| \leq \tfrac{1}{2}\|f''\|_{C_b}\tfrac{M^{2-\alpha}}{2-\alpha} + 2\|f'\|_{C_b}\tfrac{M^{1-\alpha}}{\alpha-1} < \infty$$

for any $M > 0$. Consequently, the integral representation (16) of D^α_θ satisfies estimate (19). □

The estimate (19) shows that for $1 < \alpha < 2$ there exists a bound for D^α_θ involving first and second derivatives. This is one key estimate we are going to use to adapt the assumptions (B3) and (C3) discussed in Section 4.1.

For a self-contained derivation of the representation of fractional Laplacians D^α_0, $0 < \alpha < 2$, see the work of Droniou and Imbert [15, Theorem 1]. Their results on continuity [15, Proposition 1] and on sequences [15, Theorem 2] generalize to Riesz-Feller operators with obvious modifications in their proofs.

Proposition 2.5. *Let $1 < \alpha < 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $f \in C^2_b(\mathbb{R})$. If $(f_n)_{n\geq 1} \in C^2_b(\mathbb{R})$ is bounded in $L^\infty(\mathbb{R})$ and $\mathrm{D}^2_0 f_n \to \mathrm{D}^2_0 f$ locally uniformly on $\mathbb{R}$, then $\mathrm{D}^\alpha_\theta f_n \to \mathrm{D}^\alpha_\theta f$ locally uniformly on $\mathbb{R}$.*

Theorem 2.6. *Let $1 < \alpha < 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $f \in C^2_b(\mathbb{R})$. If $(x_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}^n$ such that $f(x_k) \to \sup_{\mathbb{R}^n} f$ as $k \to \infty$, then $\lim_{k\to\infty} \nabla f(x_k) = 0$ and $\liminf_{k\to\infty} \mathrm{D}^\alpha_\theta\,[f](x_k) \geq 0$.*

2.2.2. *Semigroup Properties.* A non-degenerate Riesz-Feller operator generates a strongly continuous convolution semigroup on $C_0(\mathbb{R})$, which can be extended to a convolution semigroup on $L^\infty(\mathbb{R})$.

Theorem 2.7. *For $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, the Riesz-Feller operator D^α_θ generates a convolution semigroup $S_t : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $u_0 \mapsto S_t u_0 = G^\alpha_\theta(\cdot,t) * u_0$, with kernel $G^\alpha_\theta(x,t)$. Moreover, the convolution semigroup with $u(x,t) := S_t u_0$ satisfies*

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(1) $u \in C^\infty(\mathbb{R} \times (t_0, \infty))$ *for all* $t_0 > 0$*;*
(2) $\partial_t u = \mathrm{D}^\alpha_\theta\, u$ *for all* $(x,t) \in \mathbb{R} \times (t_0, \infty)$ *and any* $t_0 > 0$*;*
(3) $u(\cdot,t) \overset{*}{\rightharpoonup} u_0$ *for* $t \searrow 0$ *in the weak-$*$ topology of* $L^\infty(\mathbb{R})$*;*
(4) If $u_0 \in C_b(\mathbb{R})$ *then* $\lim_{\mathbb{R}\times(0,\infty)\ni(x,t)\to(x_0,0)} u(x,t) = u_0(x_0)$ *for each* $x_0 \in \mathbb{R}$*.*

Proof. Due to the assumptions and Lemma 2.1, the kernel is a smooth probability density function with $G^\alpha_\theta(\cdot,t) \in L^1(\mathbb{R})$. This observation and Young's inequality for convolutions show that $S_t : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $t > 0$, are well-defined bounded linear operators. We define $S_0 = \mathrm{Id}$ and the semigroup property follows from property (G4) in Lemma 2.1. The semigroup $(S_t)_{t\geq 0}$ of bounded linear operators on $L^\infty(\mathbb{R})$ is not necessarily strongly continuous, see also [23, page 427 ff.]. However $S_t u_0$ converges for $t \searrow 0$ in the weak-$*$ topology of $L^\infty(\mathbb{R})$, see also [41].

The function u is smooth, since u is a convolution of $u_0 \in L^\infty(\mathbb{R})$ with an integrable smooth function G^α_θ having bounded integrable derivatives (G6)–(G8). Furthermore, u is a solution of (12), since G^α_θ is a solution of (12) for positive times. Finally, G^α_θ is an approximate unit with respect to t due to (G1)–(G3) which is sufficient for the stated convergence to the initial datum u_0. □

In the analysis of the traveling wave problem, we are mostly interested in the evolution of initial data in C_b. Therefore, it is important to notice the following corollary.

Corollary 2.8. *For* $1 < \alpha < 2$ *and* $|\theta| \leq \min\{\alpha, 2-\alpha\}$*, the Riesz-Feller operator* D^α_θ *generates a convolution semigroup* $S_t : C_b(\mathbb{R}) \to C_b(\mathbb{R})$*,* $u_0 \mapsto S_t u_0 = G^\alpha_\theta(\cdot,t) * u_0$*, with kernel* $G^\alpha_\theta(x,t)$*. Moreover, the convolution semigroup with* $u(x,t) := S_t u_0$ *satisfies*
(1) $u \in C^\infty(\mathbb{R} \times (t_0, \infty))$ *for all* $t_0 > 0$*;*
(2) $\partial_t u = \mathrm{D}^\alpha_\theta\, u$ *for all* $(x,t) \in \mathbb{R} \times (t_0, \infty)$ *and any* $t_0 > 0$*;*
(3) If $u_0 \in C_b(\mathbb{R})$ *then* $u \in C_b(\mathbb{R} \times [0,T])$ *for any* $T > 0$*.*

Since $S_t : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is not a strongly continuous semigroup, the relation between the C_b-extension of the strongly continuous semigroup $(S_t)_{t\geq 0}$ on $C_0(\mathbb{R})$ and the C^2_b-extension of the Fourier multiplier operators D^α_θ is not obvious. This issue is discussed in [38], see also [23, Section 4.8].

3. Cauchy Problem and Comparison Principle

We consider the Cauchy problem

$$
\begin{cases} \partial_t u = \mathrm{D}^\alpha_\theta\, u + f(u) & \text{for } (x,t) \in \mathbb{R} \times (0,\infty)\,, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}\,, \end{cases} \tag{20}
$$

for $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ satisfying (2). We follow a standard approach, and consider the Cauchy problem in its mild formulation to prove the existence of a mild solution. The Cauchy problem generates a nonlinear semigroup which allows to prove uniform C^k_b estimates via a bootstrap argument and to conclude that mild solutions are also classical solutions.

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Droniou and Imbert [15] studied partial integro-differential equations

$$\partial_t u(x,t) = \mathrm{D}_0^\alpha\,[u(\cdot,t)](x) + F(t,x,u(x,t),\nabla u(x,t)) \quad \text{for } x\in\mathbb{R}^n\,, t>0\,,$$

involving the fractional Laplacian D_0^α for $0<\alpha<2$. First, they introduce the fractional Laplacian D_0^α as a Fourier multiplier operator on the Schwartz class $\mathcal{S}(\mathbb{R})$, and then they extend it to $C_b^2(\mathbb{R})$ functions in [15, Lemma 2]. In particular, they consider the Cauchy problem for $\alpha\in(1,2)$

$$\begin{cases} \partial_t u(x,t) = \mathrm{D}_0^\alpha\,[u(\cdot,t)](x) + F(t,x,u,\nabla u) & \text{for } x\in\mathbb{R}^n\,, t>0\,,\\ u(x,0)=u_0(x) & \text{for } x\in\mathbb{R}^n\,, \end{cases} \tag{21}$$

where $u_0\in W^{1,\infty}(\mathbb{R}^n)$ and $F\in C^\infty([0,\infty)\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n)$.

In the one-dimensional case ($n=1$) of a smooth function $F=F(u)$ depending only on the unknown u and $u_0\in L^\infty(\mathbb{R})$, their assumptions on F simplify to

(DI1) $\forall R>0$, $\forall k\in\mathbb{N}$, $\exists\mathcal{K}_{R,k}>0$ such that $\forall v\in[-R,R]$ and $\forall\beta\in\mathbb{N}$ satisfying $\beta\le k$,

$$|\partial_v^\beta F(v)|\le\mathcal{K}_{R,k}\,.$$

(DI2) $\exists\Lambda:[0,\infty)\mapsto(0,\infty)$ continuous and non-decreasing such that $\int_0^\infty \frac{1}{\Lambda(a)}\,\mathrm{d}a=\infty$ and $\forall v\in\mathbb{R}$

$$\operatorname{sgn}(v)F(v)\le\Lambda(|v|)\,.$$

(DI3) $\forall R>0$, $\exists\Gamma_R:[0,\infty)\mapsto(0,\infty)$ continuous and non-decreasing such that $\int_0^\infty\frac{1}{\Gamma_R(a)}\,\mathrm{d}a=\infty$ and $\forall v\in[-R,R]$

$$|\nu|\partial_v F(v)\le\Gamma_R(|\nu|)\,.$$

A smooth function $F=F(u)$ that depends only on u and satisfies (DI1) also satisfies (DI3), since

$$|\nu|\partial_v F(t,x,v,\nu)=|\nu|F'(v)\le|\nu|\max_{v\in[-R,R]}|F'(v)|\le\mathcal{K}_{R,1}|\nu|=:\Gamma_R(|\nu|)$$

implies

$$\int_0^\infty\frac{1}{\Gamma_R(u)}\,\mathrm{d}u=\int_0^\infty\frac{1}{\mathcal{K}_{R,1}\,u}\,\mathrm{d}u=\tfrac{1}{\mathcal{K}_{R,1}}\lim_{\epsilon\to0}\ln(u)\Big|_{u=\epsilon}^{u=\frac{1}{\epsilon}}=\infty\,.$$

In this case a simplified proof of [15, Theorem 3] allows to show the existence of a solution for the initial value problem (IVP) with $u_0\in L^\infty(\mathbb{R})$.

Theorem 3.1. *Let $\alpha\in(1,2)$, $u_0\in L^\infty(\mathbb{R})$ and $F=F(u)$ satisfy (DI1) and (DI2). There exists a unique solution of (21) in the following sense: for all $T>0$*

(DI4) $u\in C_b(\mathbb{R}\times(0,T))$ and for all $a\in(0,T)$ $u\in C_b^\infty(\mathbb{R}\times(a,T))$;
(DI5) u satisfies the partial integro-differential equation (21) on $\mathbb{R}\times(0,T)$,
(DI6) If $u_0\in C_b(\mathbb{R})$ then $u(\cdot,t)\to u_0$ uniformly on $\mathbb{R}$ as $t\to0$.

There are also the following estimates on the solution: for all $0<t<T<\infty$,

(DI7) $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}\le\mathcal{L}^{-1}\big(t+\mathcal{L}(\|u_0\|_{L^\infty(\mathbb{R})})\big)$,

where $\mathcal{L}:[0,\infty)\to[0,\infty)$, $a\mapsto\mathcal{L}(a)=\int_0^a\frac{1}{\Lambda(b)}\,\mathrm{d}b$.

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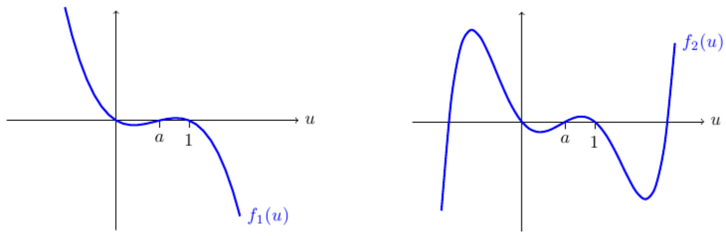

FIGURE 1. The functions $f_1(u) = u\,(1-u)\,(u-a)$ and $f_2(u) = (u+1)\,u\,(u-a)\,(u-1)\,(u-2)$ for any $a \in (0,1)$ are bistable in the sense of (2) and depicted in the left and the right figure, respectively. Whereas f_1 satisfies the assumptions (DI1)–(DI3), function f_2 does not satisfy (DI2).

Remark. The function $\mathcal{L} : [0,\infty) \to [0,\infty)$, $a \mapsto \mathcal{L}(a) = \int_0^a \frac{1}{\Lambda(b)}\, \mathrm{d}b$, is a non-decreasing C^1-diffeomorphism from $[0,\infty)$ to $[0,\infty)$, due to the assumptions on Λ.

For our purposes we need to extend the result of Theorem 3.1 to the case of all Riesz-Feller operators D^α_θ in (20) and to adapt the result to admissible functions f which do not satisfy the growth condition (DI2) see also Figure 1.

First, Droniou and Imbert note in [15, Remark 5] that their proof of [15, Theorem 3] still applies if D^α_0 is replaced by more general operators which satisfy [15, Theorem 2] and whose associated kernel $K_\alpha(x,t)$ has the properties [15, (30)]

(P1) $K_\alpha \in C^\infty(\mathbb{R} \times (0,\infty))$ and $(K_\alpha(\cdot,t))_{t\to 0}$ is an approximate unit (in particular, $K_\alpha \geq 0$ and, for all $t > 0$, $\|K_\alpha(\cdot,t)\|_{L^1(\mathbb{R})} = 1$),

(P2) $\forall t > 0$, $\forall t' > 0$, $K_\alpha(\cdot, t+t') = K_\alpha(\cdot,t) * K_\alpha(\cdot,t')$,

(P3) $\exists \mathcal{K} > 0$, $\forall t > 0$, $\|\nabla K_\alpha(\cdot,t)\|_{L^1(\mathbb{R})} \leq \mathcal{K} t^{-1/\alpha}$,

and [15, (59)]

(P4) $(0,\infty) \ni t \mapsto K_\alpha(\cdot,t) \in L^1(\mathbb{R})$ is continuous.

The Riesz-Feller operators D^α_θ for $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$ satisfy the properties (P1)–(P3), due to Theorem 2.6 and Lemma 2.1, and (P4) follows from the regularity of G^α_θ and the scaling property (G2). Therefore the result of Theorem 3.1 still holds if the operator D^α_0 in (21) is replaced by a Riesz-Feller operator D^α_θ for $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$.

Second, the prototype of a function f satisfying assumption (2) is a cubic polynomial of the form

$$f_1(u) := u\,(1-u)\,(u-a) \quad \text{for some } a \in (0,1)\,,$$

which satisfies $(\operatorname{sgn} v) f_1(v) \leq \max_{u\in[0,1]} f_1(u)$ for all $v \in \mathbb{R}$ and hence assumption (DI2) with a constant Λ. In contrast, other admissible function such as

$$f_2(u) := (u+1)\,u\,(u-a)\,(u-1)\,(u-2) \quad \text{for some } a \in (0,1)$$

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do not satisfy assumption (DI2). The estimate

$$(\operatorname{sgn} v) f_2(v) \le \Lambda(|v|) = c(|v|+2)^5$$

for some $c > 0$, implies that

$$\int_0^\infty \frac{1}{\Lambda(u)} \,\mathrm{d}u = \int_0^\infty \frac{1}{c(u+2)^5} \,\mathrm{d}u = - \lim_{R\to\infty} \frac{1}{4c(u+2)^4}\bigg|_{u=0}^{u=R} = \frac{1}{c2^6} < \infty \,.$$

However, we are interested in solutions taking values in $[0,1]$ and the partial integro-differential equation exhibits a comparison principle see also Lemma 3.3. Thus we will modify the function f_2 outside of $[0,1]$, such that it satisfies the assumptions (DI1)–(DI2), see also Figure 1. Consequently (a generalization of) Theorem 3.1 applies to the associated Cauchy problem and the solution—taking values in $[0,1]$—will be a solution of the original Cauchy problem (20).

Theorem 3.2. *Suppose $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ satisfies (2). The Cauchy problem (20) with initial condition $u(\cdot,0) = u_0 \in C_b(\mathbb{R})$ and $0 \le u_0 \le 1$ has a solution $u(x,t)$ in the sense of Theorem 3.1 satisfying $0 \le u(x,t) \le 1$ for all $(x,t) \in \mathbb{R}\times(0,\infty)$. Moreover, for all $k \in \mathbb{N}$ and $t_0 > 0$ there exists a $\mathcal{K} > 0$ such that $\|u(\cdot,t)\|_{C_b^k(\mathbb{R})} \le \mathcal{K}$ for all $0 < t_0 < t$.*

Proof. The first assumption (DI1) is satisfied, since f is a smooth function, hence all derivatives are continuous and bounded on any compact interval $[-R,R]$. We are interested in solutions taking values in $[0,1]$. Moreover, the partial integro-differential equation exhibits a comparison principle, such that classical solutions $u(x,t)$ of our Cauchy problem will satisfy $0 \le u \le 1$. Therefore, we can modify f in such a way that its modification $\widetilde{f}$ satisfies assumption (DI2) but does not change the dynamics as long as u takes values in $[0,1]$. First, we define $f_{min} := \min_{u\in[0,1]} f(u)$, $f_{max} := \max_{u\in[0,1]} f(u)$, and a bounded function $\overline{f}(u) := \max\{f_{min}, \min\{f(u), f_{max}\}\}$. Finally, we consider a smooth function $\widetilde{f} \in C^\infty(\mathbb{R})$, such that $\widetilde{f}(u) = \overline{f}(u) = f(u)$ for all $u \in [0,1]$ and $|\widetilde{f}(u)| \le |\overline{f}(u)|$ for all $u \in \mathbb{R}$. Then, assumption (DI2) holds for $\widetilde{f}$ with $(\operatorname{sgn} v)\widetilde{f}(v) \le \Lambda(|v|) := \|\widetilde{f}\|_\infty < \infty$. Assumption (DI1) continues to hold. Thus, due to (a generalization of) Theorem 3.1, there exists a unique solution to the Cauchy problem

$$\begin{cases} \partial_t u = \mathrm{D}^\alpha_\theta\, u + \widetilde{f}(u) & \text{for } (x,t) \in \mathbb{R}\times(0,T]\,, \\ u(\cdot,0) = u_0 & \text{for } x \in \mathbb{R}\,. \end{cases} \tag{22}$$

Due to the assumptions on the initial datum $0 \le u_0 \le 1$ and a comparison principle—formulated in Lemma 3.3—$0 \le u(x,t) \le 1$ for all $(x,t) \in \mathbb{R}\times[0,T]$. Thus the solution $u(x,t)$ is a solution of the original Cauchy problem, whose uniqueness has to be verified. Suppose two solutions of (20) with the stated properties exist, then they are solutions of the modified Cauchy problem (22) as well. However, the modified Cauchy problem has a unique solution, hence the two solutions are identical.

Due to (a generalization of) Theorem 3.1 a solution u exists for all $T > 0$ on a time interval $(0,T)$. However the comparison principle proves that $0 \le u(\cdot,t) \le 1$ for all $t \ge 0$, such that $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ satisfies not only (DI7)

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but also $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq 1$ for all $t \geq 0$. The solution u is also a mild solution and satisfies

$$u(x,t) = (G^\alpha_\theta(\cdot,t) * u_0)(x) + \int_0^t [G^\alpha_\theta(\cdot,t-\tau) * f(u(\cdot,\tau))](x)\ \mathrm{d}\tau$$

for $t \geq 0$. For $t \geq t_0 > 0$ the solution is differentiable and satisfies the mild formulation

$$\partial_x u(x,t) = \Big(\partial_x G^\alpha_\theta(\cdot,t) * u_0\Big)(x) + \int_0^t \Big[\partial_x G^\alpha_\theta(\cdot,t-\tau) * f(u(\cdot,\tau))\Big](x)\ \mathrm{d}\tau$$

and hence the estimate

$$\sup_{x\in\mathbb{R}} |\partial_x u(x,t)| \leq \mathcal{K} t^{-\frac{1}{\alpha}} \underbrace{\|u_0\|_{L^\infty}}_{\leq 1} + \max_{u\in[0,1]} |f(u)|\, \mathcal{K} \frac{t^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}} \tag{23}$$

due to $0 \leq u(\cdot,t) \leq 1$ for all $t \geq 0$ and Lemma 2.1. In particular, assumption $1 < \alpha \leq 2$ implies

$$t^{-\frac{1}{\alpha}} \leq t_0^{-\frac{1}{\alpha}} \quad \text{and} \quad t^{1-\frac{1}{\alpha}} \leq (2t_0)^{1-\frac{1}{\alpha}}$$

for all $t \in [t_0, 2t_0]$. Thus, for $t \in [t_0, 2t_0]$, estimate (23) yields

$$\sup_{x\in\mathbb{R}} |\partial_x u(x,t)| \leq \mathcal{K} t_0^{-\frac{1}{\alpha}} + \max_{u\in[0,1]} |f(u)|\, \mathcal{K} \frac{(2t_0)^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}}.$$

This gives an estimate on bounded intervals, but not a global estimate on $[t_0,\infty)$. However, the IVP generates a nonlinear semigroup; the solution u of the IVP with initial condition $u(\cdot,0) = u_0(\cdot)$ is equal to the solution v of the IVP with initial condition $v(\cdot,t_0) = u(\cdot,t_0)$ on the time interval $[t_0,\infty)$. Hence, u and its derivative $\partial_x u(x,t)$ satisfy

$$u(x,t) = (G^\alpha_\theta(\cdot,t-t_0) * u(\cdot,t_0))(x) + \int_{t_0}^t [G^\alpha_\theta(\cdot,t-\tau) * f(u(\cdot,\tau))](x)\ \mathrm{d}\tau$$

and

$$\partial_x u(x,t) = \Big(\partial_x G^\alpha_\theta(\cdot,t-t_0) * u(\cdot,t_0)\Big)(x) + \int_{t_0}^t \Big[\partial_x G^\alpha_\theta(\cdot,t-\tau) * f(u(\cdot,\tau))\Big](x)\ \mathrm{d}\tau$$

for $t \geq t_0 > 0$. The estimate now reads

$$\sup_{x\in\mathbb{R}} |\partial_x u(x,t)| \leq \mathcal{K}(t-t_0)^{-\frac{1}{\alpha}} \|u(\cdot,t_0)\|_{L^\infty} + \max_{u\in[0,1]} |f(u)|\, \mathcal{K} \frac{(t-t_0)^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}}$$

for $t \geq t_0 > 0$ and $t \in [2t_0, 3t_0]$ we obtain again

$$\sup_{x\in\mathbb{R}} |\partial_x u(x,t)| \leq \mathcal{K} t_0^{-\frac{1}{\alpha}} + \max_{u\in[0,1]} |f(u)|\, \mathcal{K} \frac{(2t_0)^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}}$$

due to $1 < \alpha \leq 2$ and the uniform estimate on u. By induction we obtain the uniform estimate of $\partial_x u(x,t)$ on $(x,t) \in \mathbb{R} \times [t_0,\infty)$, and in a similar way the uniform estimates for all other derivatives of u. $\square$

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3.1. **Comparison principles and far-field behavior.**

Lemma 3.3. *Assume* $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$, $T > 0$ *and* $u, v \in C_b(\mathbb{R} \times [0,T]) \cap C_b^2(\mathbb{R} \times (t_0, T])$ *for all* $t_0 \in (0,T)$ *such that*

$$\partial_t u \le \mathrm{D}_\theta^\alpha\, u + f(u) \qquad \textit{and} \qquad \partial_t v \ge \mathrm{D}_\theta^\alpha\, v + f(v) \qquad \textit{in } \mathbb{R} \times (0,T]\,.$$

(i) If $v(\cdot,0) \ge u(\cdot,0)$ *then* $v(x,t) \ge u(x,t)$ *for all* $(x,t) \in \mathbb{R} \times (0,T]$.
(ii) If $v(\cdot,0) \gneqq u(\cdot,0)$ *then* $v(x,t) > u(x,t)$ *for all* $(x,t) \in \mathbb{R} \times (0,T]$.
(iii) Moreover, there exists a positive continuous function

$$\eta : [0,\infty) \times (0,\infty) \to (0,\infty)\,, \qquad (m,t) \mapsto \eta(m,t)\,,$$

such that if $v(\cdot,0) \ge u(\cdot,0)$ *then for all* $(x,t) \in \mathbb{R} \times (0,T)$

$$v(x,t) - u(x,t) \ge \eta(|x|,t) \int_0^1 v(y,0) - u(y,0)\ \mathrm{d}y\ .$$

Proof. (i) The function $w := v - u$ satisfies $w \in C_b(\mathbb{R}\times[0,T]) \cap C_b^2(\mathbb{R}\times(t_0,T])$ for all $t_0 \in (0,T)$, $w(\cdot,0) \ge 0$ in $\mathbb{R}$ and

$$\begin{aligned}
\partial_t w = \partial_t(v-u) &\ge \mathrm{D}_\theta^\alpha(v-u) + f(v) - f(u) \\
&= \mathrm{D}_\theta^\alpha(v-u) + \int_0^1 f'(\theta v + (1-\theta)u)\,(v-u)\ \mathrm{d}\theta \\
&= \mathrm{D}_\theta^\alpha\, w + \underbrace{\left(\int_0^1 f'(\theta v + (1-\theta)u)\ \mathrm{d}\theta\right)}_{=:k(x,t)} w\,.
\end{aligned}$$

In particular, $k : \mathbb{R} \times [0,T] \to \mathbb{R}$, $(x,t) \mapsto k(x,t)$, is a bounded continuous function, due to the properties of u and v. To prove $w \ge 0$ in $\mathbb{R} \times (0,T]$, we will derive a contradiction following [10, page 153]. Assume w takes negative values in $\mathbb{R} \times [0,T]$. Due to $w \in C_b(\mathbb{R} \times [0,T])$ and $w(\cdot,0) \ge 0$, for any $\kappa > 0$ there exist $\epsilon > 0$ and $T \ge \tilde{T} > 0$ such that

$$w(x,t) > -\epsilon\ \mathrm{e}^{2\kappa t} \quad \text{ in } \mathbb{R} \times [0,\tilde{T}) \quad \text{ and } \inf_{x\in\mathbb{R}} w(x,\tilde{T}) = -\epsilon\ \mathrm{e}^{2\kappa\tilde{T}}\,.$$

In the following we use again T instead of $\tilde{T}$ and assume without loss of generality $w(0,T) < -\frac{7}{8}\epsilon\ \mathrm{e}^{2\kappa T}$. Consider $\omega(x,t) := -\epsilon(\frac{3}{4} + \sigma z(x))\mathrm{e}^{2\kappa t}$ where $\sigma > 0$ and $z \in C^\infty(\mathbb{R})$, $z(0) = 1$, $\lim_{x\to\pm\infty} z(x) = 3$, as well as $3 \ge z \ge 1$, $|z'| \le 1$ and $|z''| \le 1$ in $\mathbb{R}$. The function ω satisfies for $\sigma \ge 0$

$$\omega(x,t) = -\epsilon\left(\tfrac{3}{4} + \sigma z(x)\right)\mathrm{e}^{2\kappa t} \le -\epsilon\left(\tfrac{3}{4} + \sigma\right)\mathrm{e}^{2\kappa t}$$

where the upper bound is monotone decreasing with respect to σ. Thus there exists a $\sigma^* \in (\frac{1}{8}, \frac{1}{4}]$ such that $w \ge \omega$ in $\mathbb{R} \times [0,T]$, where $\frac{1}{8} < \sigma^*$ due to the restrictions at $x = 0$. Moreover

$$\begin{aligned}
\lim_{x\to\pm\infty} -\epsilon(\tfrac{3}{4} + \sigma^* z(x))\mathrm{e}^{2\kappa t} &= -\epsilon(\tfrac{3}{4} + 3\sigma^*)\mathrm{e}^{2\kappa t} < -\epsilon\tfrac{9}{8}\mathrm{e}^{2\kappa t} < -\epsilon\ \mathrm{e}^{2\kappa t} \\
&\le \liminf_{x\to\pm\infty} w(x,t)\,.
\end{aligned}$$

In summary, there exists $\sigma^* \in (\frac{1}{8}, \frac{1}{4}]$ and $(x_0,t_0) \in \mathbb{R} \times (0,T]$ such that $w \ge \omega$ in $\mathbb{R} \times [0,T]$ and $w(x_0,t_0) = \omega(x_0,t_0)$. Thus $w - \omega \in C_b(\mathbb{R} \times [0,T]) \cap C_b^2(\mathbb{R} \times (t_0,T])$ is a non-negative function which attains its minimum at $(x_0,t_0) \in \mathbb{R} \times (0,T]$, hence $\partial_t w(x_0,t_0) \le \partial_t \omega(x_0,t_0)$, $\partial_x w(x_0,t_0) = \partial_x \omega(x_0,t_0)$, $\partial_x^2 w(x_0,t_0) \ge \partial_x^2 \omega(x_0,t_0)$.

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First we deduce from the integral representation of D^α_θ in Proposition 2.3 the estimate $\mathrm{D}^\alpha_\theta[w(\cdot,t_0)](x_0) \geq \mathrm{D}^\alpha_\theta[\omega(\cdot,t_0)](x_0)$. Second we deduce the estimate

$$\begin{aligned}
-\tfrac{7}{4}\epsilon\kappa e^{2\kappa t_0} &\geq \partial_t\omega(x_0,t_0) \geq \partial_t w(x_0,t_0) \\
&\geq \mathrm{D}^\alpha_\theta[w(\cdot,t_0)](x_0) + k(x_0,t_0)w(x_0,t_0) \\
&\geq \mathrm{D}^\alpha_\theta[\omega(\cdot,t_0)](x_0) - \sup|k|\,|\omega(x_0,t_0)| \\
&\geq -\mathcal{K}\|\omega''\|_{C_b(\mathbb{R})}\tfrac{M^{2-\alpha}}{2-\alpha} - 4\mathcal{K}\|\omega'\|_{C_b(\mathbb{R})}\tfrac{M^{1-\alpha}}{\alpha-1} - \sup|k|\,\epsilon(\tfrac{3}{4}+\sigma^* z(x_0))e^{2\kappa t_0} \\
&\geq -\mathcal{K}\tfrac{M^{2-\alpha}}{2-\alpha}\epsilon\sigma^* e^{2\kappa t_0} - 4\mathcal{K}\tfrac{M^{1-\alpha}}{\alpha-1}\epsilon\sigma^* e^{2\kappa t_0} - \sup|k|\,\epsilon\tfrac{6}{4}e^{2\kappa t_0}\,,
\end{aligned}$$

where we use Proposition 2.4 with some positive constants M and $\mathcal{K}$. Thus if we choose $\kappa > 0$ such that

$$-\tfrac{7}{4}\kappa < -\tfrac{\mathcal{K}}{4}\tfrac{M^{2-\alpha}}{2-\alpha} - \mathcal{K}\tfrac{M^{1-\alpha}}{\alpha-1} - \tfrac{6}{4}\sup|k|$$

then we obtain a contradiction. Therefore $w \geq 0$ in $\mathbb{R}\times(0,T]$.

(ii) For another constant $K_2 \in \mathbb{R}$, the function $w_2 := e^{K_2 t}w$ satisfies $w_2 \in C_b(\mathbb{R}\times[0,T]) \cap C^2_b(\mathbb{R}\times(t_0,T])$ for all $t_0 \in (0,T)$, $w_2 \geq 0$ in $\mathbb{R}\times(0,T]$, and

$$\partial_t w_2 \geq \mathrm{D}^\alpha_\theta\, w_2 - c_2(x,t)w_2$$

with $c_2(x,t) := -(K_2 + k(x,t))$. Choosing $K_2 \in \mathbb{R}$ such that $c_2(x,t) = -(K_2+k(x,t)) \leq 0$ and using $w_2 \geq 0$ in $\mathbb{R}\times(0,T]$, yields

$$\partial_t w_2 \geq \mathrm{D}^\alpha_\theta\, w_2 - c_2(x,t)w_2 \geq \mathrm{D}^\alpha_\theta\, w_2\,.$$

Due to the first part,

$$w_2(x,t) \geq [G^\alpha_\theta(\cdot,t) * w_2(\cdot,0)](x) = e^{K_2 t}[G^\alpha_\theta(\cdot,t) * w(\cdot,0)](x)\,.$$

The assumption $v(\cdot,0) \gneqq u(\cdot,0)$ implies that there exists $x_0 \in \mathbb{R}$ and $\epsilon > 0$ such that $w(x,0) > 0$ for all $x \in (x_0-\epsilon, x_0+\epsilon)$ due to continuity of w. Moreover, the nonlocal diffusion equation $\partial_t w = \mathrm{D}^\alpha_\theta\, w$ generates a convolution semigroup with a positive convolution kernel $G^\alpha_\theta(x,t)$, i.e. $G^\alpha_\theta(x,t) > 0$ in $\mathbb{R}\times(0,T]$, see Lemma 2.1. Therefore,

$$w_2(x,t) \geq \int_{U_\epsilon(x_0)} G^\alpha_\theta(x-y,t)\, w_2(y,0)\; \mathrm{d}y > 0 \quad \text{for all } (x,t)\in\mathbb{R}\times(0,T]\,,$$

which implies $w(x,t) > 0$ for all $(x,t) \in \mathbb{R}\times(0,T]$.

(iii) If $v(\cdot,0) \geq u(\cdot,0)$ then as before

$$\begin{aligned}
w_2(x,t) \geq [G^\alpha_\theta(\cdot,t) * w_2(\cdot,0)](x) &\geq \int_0^1 [G^\alpha_\theta(x-y,t)w_2(y,0)]\;\mathrm{d}y \\
&\geq \min_{y\in[0,1]} G^\alpha_\theta(x-y,t)\int_0^1 w_2(y,0)\;\mathrm{d}y\,,
\end{aligned}$$

since G^α_θ is an integrable positive smooth function, and $w_2(\cdot,0) \geq 0$ in $\mathbb{R}$. Thus

$$\begin{aligned}
e^{K_2 t}\big(v(x,t)-u(x,t)\big) &\geq \min_{z\in[-|x|-1,|x|]} G^\alpha_\theta(z,t)\int_0^1 w(y,0)\;\mathrm{d}y \\
&= \tilde{\eta}(|x|,t)\int_0^1 \big(v(y,0)-u(y,0)\big)\;\mathrm{d}y\,,
\end{aligned}$$

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where $\tilde{\eta}(m,t) = \min_{z\in[-m-1,m]} G_\theta^\alpha(z,t)$, is a positive continuous function, since $G_\theta^\alpha(\cdot,t)$ for $t>0$ is a positive smooth function. Consequently the function $\eta : [0,\infty)\times(0,\infty)\to(0,\infty)$, $(m,t)\mapsto \mathrm{e}^{-K_2 t}\tilde{\eta}(|x|,t)$, is a positive continuous function, and the statement follows. □

We need to investigate the behavior of solutions in the limits $x\to\pm\infty$, see also [42, Theorem 5.2] for the case of a system of reaction-diffusion equations with local derivatives. We consider the Cauchy problem

$$\begin{cases} \partial_t u = \mathrm{D}_\theta^\alpha\, u + F(u) & \text{for } (x,t)\in\mathbb{R}\times(0,\infty)\,, \\ u(x,0) = u_0(x) & \text{for } x\in\mathbb{R}\,, \end{cases} \tag{24}$$

for some unknown function $u:\mathbb{R}\times(0,\infty)\to\mathbb{R}$ and a given bounded continuous function $F:\mathbb{R}\to\mathbb{R}$, $u\mapsto F(u)$, satisfying a Lipschitz condition in u.

Theorem 3.4. *Let $1<\alpha\le 2$ and $|\theta|\le\min\{\alpha,2-\alpha\}$. Suppose $u_0\in C_b(\mathbb{R})$ and that the limits*

$$\lim_{x\to\pm\infty} u_0(x) = u_{0,\pm}$$

exist. If $u\in C_b(\mathbb{R}\times[0,T])\cap C_b^2(\mathbb{R}\times(t_0,T])$ for all $t_0\in(0,T)$ is a solution of the Cauchy problem (24) then the limits $\lim_{x\to\pm\infty} u(x,t) = u_\pm(t)$ exist and satisfy

$$\tfrac{\mathrm{d}\,u_\pm}{\mathrm{d}\,t} = F(u_\pm) \quad \text{for } t\in[0,T]\,, \qquad u_\pm(0) = u_{0,\pm}\,. \tag{25}$$

Proof of Theorem 3.4. The result is a variation of [42, Theorem 5.2] where the case $D_0^2 = \partial_x^2$ is considered. Again, for $1<\alpha\le 2$ and $|\theta|\le\min\{\alpha,2-\alpha\}$ the fundamental solution G_θ^α of

$$\partial_t u = \mathrm{D}_\theta^\alpha\, u \quad \text{for } (x,t)\in\mathbb{R}\times(0,T]$$

is for all $t>0$ an integrable positive smooth function $G_\theta^\alpha(\cdot,t)\in L^1(\mathbb{R})$ with finite mean, see Lemma 2.1. Like in the proof of [42, Theorem 5.1], we obtain the unique mild solution as the limit of an iterated sequence

$$u^0(x,t) = \int_{-\infty}^{+\infty} G_\theta^\alpha(x-y,t)u_0(y)\ \mathrm{d}y$$
$$u^{k+1}(x,t) = u^0(x,t) + \int_0^t \int_{-\infty}^{+\infty} G_\theta^\alpha(x-y,t-\tau)F(u^k(y,\tau))\ \mathrm{d}y\ \ \mathrm{d}\tau$$

for $k\in\mathbb{N}$. The functions u^k are bounded and continuous, hence measurable. To study the limits of a solution u, we consider the limits of the functions u^k. The dominated convergence theorem yields

$$u_\pm^0(t) := \lim_{x\to\pm\infty} u^0(x,t) = \lim_{x\to\pm\infty}\int_{-\infty}^{+\infty} G_\theta^\alpha(y,t)u_0(x-y)\ \mathrm{d}y$$
$$= \int_{-\infty}^{+\infty} G_\theta^\alpha(y,t) \lim_{x\to\pm\infty} u_0(x-y)\ \mathrm{d}y = \int_{-\infty}^{+\infty} G_\theta^\alpha(y,t)\, u_{0,\pm}\ \mathrm{d}y = u_{0,\pm}\,.$$

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A mathematical induction on $k \in \mathbb{N}$ proves that the limits of u^k satisfy

$$\begin{aligned} u_{\pm}^{k+1}(t) :&= \lim_{x\to\pm\infty} u^{k+1}(x,t) \\ &= u_{0,\pm} + \lim_{x\to\pm\infty} \int_0^t \int_{-\infty}^{+\infty} G_\theta^\alpha(x-y,t-\tau) F(u^k(y,\tau))\, \mathrm{d}y\ \mathrm{d}\tau \\ &= u_{0,\pm} + \int_0^t \int_{-\infty}^{+\infty} G_\theta^\alpha(y,t-\tau) \lim_{x\to\pm\infty} F(u^k(x-y,\tau))\, \mathrm{d}y\ \mathrm{d}\tau \\ &= u_{0,\pm} + \int_0^t F(u_\pm^k(\tau))\ \mathrm{d}\tau\ . \end{aligned}$$

The sequence of functions $u_\pm^k(t)$ converges uniformly for $0 < t \le T$ to some function $u_\pm(t)$, by virtue of the uniform convergence of the sequence of functions $u^k(x,t)$, $k \in \mathbb{N}$. Passing to the limit, we obtain

$$u_\pm(t) = u_{0,\pm} + \int_0^t F(u_\pm(\tau))\ \mathrm{d}\tau\ ,$$

which is equivalent to the stated differential equation. □

4. Traveling Wave Problem

We consider the traveling wave problem for the local reaction-nonlocal diffusion equation

$$\partial_t u = \mathrm{D}_\theta^\alpha\, u + f(u)\,, \quad x \in \mathbb{R}\,, \quad t \in (0,\infty)\,, \tag{26}$$

with $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ being a bistable function in the sense of (2). The profile U—of a traveling wave solution $u(t,x) = U(x-ct)$—has to satisfy the traveling wave equation

$$-cU'(\xi) = \mathrm{D}_\theta^\alpha\, U + f(U)$$

where D_θ^α has to be understood in the sense of the singular integral in Proposition 2.3 which is well-defined for $C_b^2(\mathbb{R})$ functions due to Proposition 2.4.

4.1. Chen's Approach and Results.

In this section we briefly review the results from [10] as they provide the basis for this work. Consider the evolution equation

$$\partial_t u(x,t) = \mathcal{A}[u(\cdot,t)](x), \qquad (x,t) \in \mathbb{R}\times[0,\infty)\,, \tag{27}$$

where $\mathcal{A}$ is a nonlinear operator. We shall also need the Fréchet derivative of $\mathcal{A}$ defined by

$$\mathcal{A}'[u](v) := \lim_{\epsilon\to 0} \frac{\mathcal{A}[u+\epsilon v] - \mathcal{A}[u]}{\epsilon}.$$

The basic assumptions on the operator $\mathcal{A}$ are:

(1) (**semigroup**) $\mathcal{A}$ generates a semigroup on $L^\infty(\mathbb{R})$,
(2) (**translation invariance**) $\mathcal{A}[u(\cdot+h)](x) = \mathcal{A}[u(\cdot)](x+h)$ for all $x, h \in \mathbb{R}$,
(3) (**bistability**) there exists a function $f(\cdot)$ such that $\mathcal{A}[\alpha 1] = f(\alpha)1$ for all $\alpha \in \mathbb{R}$ with

$$f \in C^1(\mathbb{R})\,, \qquad f(0) = 0 = f(1)\,, \qquad f'(0) < 0\,, \qquad f'(1) < 0\,, \tag{28}$$

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(4) (**comparison principle**)

$$\text{if } \partial_t u \geq \mathcal{A}[u],\ \partial_t v \leq \mathcal{A}[v],\ u(\cdot,0) \geq v(\cdot,0),\ u(\cdot,0) \not\equiv v(\cdot,0) \quad \text{then } u(\cdot,t) > v(\cdot,t)\ \forall t > 0. \tag{29}$$

Chen [10] studies the existence, uniqueness and local asymptotic stability of traveling fronts $u(x,t) = U(x-ct)$ for (27) connecting the two homogeneous stable states i.e. in a moving coordinate frame $\xi = x - ct$ one demands

$$\lim_{\xi\to-\infty} U(\xi) = 0\,, \qquad \lim_{\xi\to\infty} U(\xi) = 1 \qquad \text{and} \qquad \lim_{|\xi|\to\infty} U'(\xi) = 0\,. \tag{30}$$

We state the three main results from [10] which will follow from the semigroup property, several variants of the other three properties and additional estimates for $\mathcal{A}$.

Theorem 4.1. *(uniqueness, [10, Thm. 2.1]) Suppose the following assumptions hold:*

(A1) $\mathcal{A}$ is translation invariant and f is bistable in the sense of (28).
(A2) $\mathcal{A}$ satisfies the comparison principle (29).
(A3) There exists constants $K_1 > 0$ and $K_2 > 0$ and a probability measure ν such that for any functions u, v with $-1 \leq u, v \leq 2$ and every $x \in \mathbb{R}$

$$\begin{aligned} &\left|\mathcal{A}'[u+v](1)(x) - \mathcal{A}'[u](1)(x)\right| \\ \leq &K_1 \int_{\mathbb{R}} |v(x-y)|\nu(\,\mathrm{d}y) + K_2 \|v(x+\cdot)\|_{C^0([-1,1])}. \end{aligned} \tag{31}$$

Then monotonic traveling waves are unique up to translation. More precisely, suppose (27) has a traveling wave $U \in C^1(\mathbb{R})$ with speed c satisfying (30) and $U'(\xi) > 0\ \forall \xi \in \mathbb{R}$, then any other traveling wave solution $(\tilde{U}, \tilde{c})$ with $\tilde{U} \in C^0(\mathbb{R})$ and $0 \leq \tilde{U} \leq 1$ on $\mathbb{R}$ satisfies

$$c = \tilde{c} \qquad \text{and} \qquad \tilde{U}(\cdot) = U(\cdot + \xi_0) \text{ for some fixed } \xi_0 \in \mathbb{R}$$

i.e. $\tilde{U}$ is a translate of the original wave U.

To obtain local asymptotic stability of the traveling wave one has to extend the assumptions (A1)–(A3).

Theorem 4.2. *(local asymptotic stability, [10, Thm. 3.1]) Suppose (A1)–(A3) hold and, in addition, we have:*

(B1) There exist constants a^- and a^+ with $0 < a^- \leq a^+ < 1$ such that f satisfies $f > 0$ in $(-1,0) \cup (a^+,1)$ and $f < 0$ in $(0,a^-) \cup (1,2)$.
(B2) There exists a positive non-increasing function $\eta(m)$ defined on $[1,\infty)$ such that for any functions $u(x,t), v(x,t)$ satisfying $-1 \leq u, v \leq 2$, $\partial_t u \geq \mathcal{A}[u]$, $\partial_t v \leq \mathcal{A}[v]$ and $u(\cdot,0) \geq v(\cdot,0)$, there holds

$$\min_{x\in[-m,m]} [u(x,1) - v(x,1)] \geq \eta(m) \int_0^1 [u(y,0) - v(y,0)] \ \,\mathrm{d}y \quad \forall m \geq 1. \tag{32}$$

(B3) With K_1, K_2, ν, u and v as in (A3), there holds, for every $x \in \mathbb{R}$,

$$|\mathcal{A}[u+v](x) - \mathcal{A}[u](x)| \leq K_1 \int_{\mathbb{R}} |v(x-y)| \ \nu(\,\mathrm{d}y) + K_2 \left\|v''\right\|_{C^0(\mathbb{R})}. \tag{33}$$

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Then monotonic traveling waves are exponentially stable. More precisely, suppose (27) *has a traveling wave* $U \in C^1(\mathbb{R})$ *with speed* c *satisfying* (30) *and* $U'(\xi) > 0$ $\forall \xi \in \mathbb{R}$. *Then there exists a constant* κ *such that for any* $u_0 \in L^\infty(\mathbb{R})$ *satisfying* $0 \le u_0 \le 1$ *and*

$$\liminf_{x\to\infty} u_0(x) > a^+, \qquad \limsup_{x\to-\infty} u_0(x) < a^-,$$

the solution $u(x,t)$ *of* (27) *with initial value* $u(\cdot,0) = u_0(\cdot)$ *satisfies the exponential stability estimate*

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \le K\mathrm{e}^{-\kappa t} \qquad \text{for all } t \le 0,$$

where ξ *and* K *are constants depending on* u_0.

The strongest set of assumptions is required to show the existence of a traveling wave.

Theorem 4.3. *(existence,* [10, Thm. 4.1]*) Suppose the following assumptions are satisfied:*

- *(C1)* $\mathcal{A}$ *is translation invariant and the function* f *satisfies for some* $a \in (0,1)$,

$$f(u) \begin{cases} > 0 & \text{for } u \in (-1,0) \cup (a,1), \\ < 0 & \text{for } u \in (0,a) \cup (1,2), \end{cases}$$

where $f'(0) < 0$, $f'(1) < 0$, $f'(a) > 0$.

- *(C2) There exists a positive continuous function* $\eta(x,t)$ *defined on* $[0,\infty)\times(0,\infty)$ *such that if* $u(x,t), v(x,t)$ *satisfy* $-1 \le u,v \le 2$, $\partial_t u \ge \mathcal{A}[u]$, $\partial_t v \le \mathcal{A}[v]$ *and* $u(\cdot,0) \ge v(\cdot,0)$, *then for all* $(x,t) \in \mathbb{R}\times(0,\infty)$

$$u(x,t) - v(x,t) \ge \eta(|x|,t) \int_0^1 [u(y,0) - v(y,0)] \;\mathrm{d}y\,. \tag{34}$$

- *(C3) There exist positive constants* K_1, K_2, K_3, *and a probability measure* ν *such that for any* $u, v \in L^\infty(\mathbb{R})$ *with* $-1 \le u,v \le 2$, *and* $x \in \mathbb{R}$ *we have*

$$\begin{aligned} &|\mathcal{A}[u+v](x) - \mathcal{A}[u](x)| \\ &\le K_1 \int_{\mathbb{R}} |v(x-y)|\;\nu(\,\mathrm{d}y) + K_2\left\|v''\right\|_{C^0([x-1,x+1])}\,, \end{aligned} \tag{35}$$

$$\left|\mathcal{A}[u+v] - \mathcal{A}[u] - \mathcal{A}'[u](v)\right| \le K_3\|v\|^2_{C^0(\mathbb{R})}, \tag{36}$$

$$\begin{aligned} &\left|\mathcal{A}'[u+v](1)(x) - \mathcal{A}'[u](1)(x)\right| \\ &\le K_1 \int_{\mathbb{R}} |v(x-y)|\;\nu(\,\mathrm{d}y) + K_2\left\|v\right\|_{C^0([x-1,x+1])}\,. \end{aligned} \tag{37}$$

- *(C4) For any function* $u_0(\cdot)$ *satisfying* $0 \le u_0 \le 1$ *and* $\|u_0\|_{C^3(\mathbb{R})} < \infty$, *the solution* $u(x,t)$ *of* (27) *with initial condition* $u(\cdot,0) = u_0(\cdot)$ *satisfies*

$$\sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{C^2(\mathbb{R})} \;<\; \infty\,.$$

Then there exists a traveling wave $U \in C^1(\mathbb{R})$ *with speed* c *satisfying* (30) *and* $U'(\xi) > 0$ $\forall\xi \in \mathbb{R}$.

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Observe that the assumption (Ci) for $i = 1, 2, 3$ implies (Ai) as well as (Bi). Furthermore, the first assumption for each theorem prescribes the nonlinear bistability behavior, the second one is a comparison principle and the third assumption yields estimates on the nonlinear operator $\mathcal{A}$ as well as on its linearization $\mathcal{A}'$.

Chen proved Theorems 4.1–4.3 with a view towards a general class of integro-differential evolution equations of the form

$$\partial_t u = \delta \partial_x^2 u + G(u, J_1 * S^1(u), \dots, J_n * S^n(u)) \tag{38}$$

for some diffusion constant $\delta \geq 0$, smooth functions G and S^k, and non-negative functions $J_k \in C^1(\mathbb{R})$ of unit mass $\int_{\mathbb{R}} J_k(y)\ \mathrm{d}y = 1$ and bounded total variation $\int_{\mathbb{R}} |J_k'(y)|\ \mathrm{d}y < \infty$ where $k = 1, \dots, n$. In [10, Section 5] further assumptions are specified such that the conditions (C1)–(C4) hold, which imply the existence, uniqueness and local exponential stability of traveling wave solutions for these equations. It turns out that the approach does not apply directly when we replace the Laplacian in (38) by a more general Riesz-Feller operator D_θ^α.

4.2. **The Bistable Case with Nonlocal Diffusion.** The analysis of equation (26) in the Sections 2 and 3 show that we only need a relatively mild generalization of Chen's results [10] which we reviewed in Section 4.1.

First we identify the operator $\mathcal{A}$ as $\mathcal{A}[u] := \mathrm{D}_\theta^\alpha\, u + f(u)$ and take a look at the assumptions (C1)–(C4).

(C1) The Riesz-Feller operators D_θ^α are translational invariant with respect to the spatial variable, which follows from their integral representation in Proposition 2.3. The nonlinearity $u \mapsto f(u)$ depends on the spatial variable only through the function u itself, hence the operator is again translational invariant. Consequently, the operator $\mathcal{A}$ is translational invariant, since it is the sum of translational invariant operators.

Due to translational invariance, the operator $\mathcal{A}$ maps a constant function to a constant function. In particular, $\mathcal{A}[c\mathbf{1}] = \mathrm{D}_\theta^\alpha\,[c\mathbf{1}] + f(c)\mathbf{1} = f(c)\mathbf{1}$ for all $c \in \mathbb{R}$, where $\mathbf{1}$ denotes the constant function $x \mapsto 1$. The additional assumptions on f identify the admissible nonlinear functions.

(C2) The property follows Lemma 3.3.

(C3') In the following, we consider $u, v \in L^\infty(\mathbb{R})$ with $-1 \leq u, v \leq 2$, see assumption (C3). The quantity in (35) is estimated as

$$\begin{aligned} |\mathcal{A}[u+v](x) - \mathcal{A}[u](x)| &= |\,\mathrm{D}_\theta^\alpha\,[u+v] + f(u+v) - \mathrm{D}_\theta^\alpha\, u - f(u)| \\ &\leq |\,\mathrm{D}_\theta^\alpha\, v|(x) + |f(u+v) - f(u)|(x) \\ &\leq K_2 \|v''\|_{C_b(\mathbb{R})} + K_4 \|v'\|_{C_b(\mathbb{R})} + K_1 |v(x)| \end{aligned}$$

for some positive constants K_1, K_2 and K_4, due to Proposition 2.4 and

$$\begin{aligned} |f(u+v) - f(u)|(x) &= \left| \int_0^1 f'(u+tv)\ \mathrm{d}t\ \ v(x) \right| \\ &\leq \|f'\|_{C([-2,4])} |v(x)|\,. \end{aligned}$$

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Note that the estimate involves $\|v''\|_{C(\mathbb{R})}$ instead of $\|v''\|_{C([x-1,x+1])}$ due to the estimate of the Riesz-Feller operator in Proposition 2.4. The Fréchet derivative $\mathcal{A}'[u](v)$ of $\mathcal{A}$ is $\mathcal{A}'[u](v) = \mathrm{D}^{\alpha}_{\theta}\, v + f'(u)v$. The second estimate (36) follows from

$$|\mathcal{A}[u+v] - \mathcal{A}[u] - \mathcal{A}'[u](v)| = |f(u+v) - f(u) - f'(u)v|$$
$$= \left| \int_0^1 \int_0^t f''(u+sv)\ \mathrm{d}s\ \ \mathrm{d}t\ v^2 \right| \leq \|f''\|_{C([-2,4])} |v(x)|^2 \,.$$

The third estimate (37) follows from

$$|\mathcal{A}'[u+v](1)(x) - \mathcal{A}'[u](1)(x)| = |f'(u+v) - f'(u)|$$
$$= \left| \int_0^1 f''(u+tv)\ \mathrm{d}t\ v(x) \right| \leq \|f''\|_{C([-2,4])} |v(x)| \,.$$

(C4) Due to Theorem 3.2, the Cauchy problem with initial datum $u_0 \in C^3(\mathbb{R})$ and $0 \leq u_0 \leq 1$ has a solution $u(x,t)$ which satisfies the properties (DI4)–(DI7), $0 \leq u \leq 1$ and the uniform estimates $\sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{C^2(\mathbb{R})} < \infty$. We observe that a solution u of the IVP with initial datum $u_0 \in L^\infty(\mathbb{R})$ and $0 \leq u_0 \leq 1$ almost everywhere becomes smooth for positive times and its $C^k_b(\mathbb{R})$-norm for any $k \in \mathbb{N}$ can be uniformly bounded.

The modifications in the estimates in (C3') are due to our replacement of a second-order derivative with a Riesz-Feller operator, which demand a global estimate see Proposition 2.4. Furthermore, we prefer to work in a C_b setting instead of a L^∞ setting.

Theorem 4.4. *Theorems 4.1-4.3 still hold if each term $K_2\|v''\|_{C^0(\mathbb{R})}$ is replaced by*

$$\tilde{K}_2\|v'\|_{C_b(\mathbb{R})} + K_2\|v''\|_{C_b(\mathbb{R})}$$

occurring in the inequalities (33) and (35).

Proof. Precise statements and details are given in Appendix A for uniqueness, Appendix B for stability and Appendix C for existence. □

Finally we prove the main result stated in Theorem 1.1.

Proof of Theorem 1.1. Under the assumption of this theorem, we studied at the beginning of this subsection Chen's original conditions (C1)–(C4). We noticed that only in condition (C3) one estimate has to be modified. This implies that the same estimate has to be changed also in condition (B3). However, in the Appendices we verify that his approach can be modified to obtain the stated results on existence in Theorem C.1, uniqueness in Theorem A.1 and stability in Theorem B.3 of traveling wave solutions of (1). □

Appendix A. Proof - Uniqueness

The problem (1) under consideration fulfills the assumptions (A2) and (A3) due to the discussion in Section 4. For nonlinear functions satisfying the assumptions (A1), the uniqueness result in Theorem [10, Theorem 2.1] is applicable. In the following, we modify the proof to incorporate Riesz-Feller operators.

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A traveling wave solution of (1) is a solution of the form $u(t,x) = U(\xi)$, for some constant wave speed $c \in \mathbb{R}$, a traveling wave variable $\xi := x - ct$, and a function U connecting different endstates $\lim_{\xi\to\pm\infty} U(\xi) = u_\pm$. The profile U of a traveling wave solution has to satisfy the traveling wave equation $-cU'(\xi) = \mathrm{D}^\alpha_\theta U + f(U)$ where D^α_θ has to be understood as its extension to C^2_b-functions.

Theorem A.1. *Suppose (A1) holds and (U,c) is a traveling wave solution of (1) satisfying*

$$
\begin{aligned}
U \in C^1(\mathbb{R}) \,, \quad \lim_{\xi\to-\infty} U(\xi) = 0 =: u_- \,, \quad \lim_{\xi\to+\infty} U(\xi) = 1 =: u_+ \,, \\
U'(\xi) > 0 \quad \text{on } \mathbb{R} \,, \quad \lim_{|\xi|\to\infty} U'(\xi) = 0 \,.
\end{aligned} \tag{39}
$$

Then, for any traveling wave solution $(\tilde{U},\tilde{c})$ of (1) with

$$
\tilde{U} \in C(\mathbb{R}) \,, \quad \lim_{\xi\to\pm\infty} \tilde{U}(\xi) = u_\pm \quad \text{and} \quad u_- \le \tilde{U} \le u_+ \quad \text{on } \mathbb{R} \,,
$$

we have $\tilde{c} = c$ and $\tilde{U}(\cdot) = U(\cdot + \xi_0)$ for some $\xi_0 \in \mathbb{R}$.

First, we need to construct sub- and supersolutions.

Lemma A.2. *Suppose (U,c) is a traveling wave solution of (1) satisfying (39). Then, there exists a small positive constant δ_* (which is independent of U) and a large positive constant σ^* (which depends on U) such that for any $\delta \in (0,\delta_*]$ and every $\xi_0 \in \mathbb{R}$, the functions w^+ and w^- defined by*

$$
w^\pm(x,t) := U\big(x - ct + \xi_0 \pm \sigma^*\delta[1 - \mathrm{e}^{-\beta t}]\big) \pm \delta \mathrm{e}^{-\beta t} \tag{40}
$$

with $\beta := \frac{1}{2}\min\{-f'(0), -f'(1)\}$ are a supersolution and a subsolution of (1), respectively.

Proof. Let $y := x - ct + \xi_0 + \sigma^*\delta[1 - \mathrm{e}^{-\beta t}]$ and note that the function $w^+(x,t)$ satisfies

$$
\begin{aligned}
&\partial_t w^+ - \mathrm{D}^\alpha_\theta w^+ - f(w^+) \\
=&U'(y)\big(-c + \sigma^*\delta\beta \mathrm{e}^{-\beta t}\big) - \delta\beta \mathrm{e}^{-\beta t} - \mathrm{D}^\alpha_\theta w^+ - f(w^+) \,;
\end{aligned}
$$

a traveling wave satisfies $-cU' = \mathrm{D}^\alpha_\theta U + f(U)$ as well as $\mathrm{D}^\alpha_\theta U(y) = \mathrm{D}^\alpha_\theta w^+(x,t)$, hence

$$
\begin{aligned}
&= \mathrm{D}^\alpha_\theta U(y) + f(U) - \mathrm{D}^\alpha_\theta w^+(x,t) - f(w^+) + U'(y)\sigma^*\delta\beta \mathrm{e}^{-\beta t} - \delta\beta \mathrm{e}^{-\beta t} \\
&= f(U) - f\big(U + \delta \mathrm{e}^{-\beta t}\big) + \delta\beta \mathrm{e}^{-\beta t}\big(U'(y)\sigma^* - 1\big) \,.
\end{aligned}
$$

Due to the properties (39) of U, there exists for any $\delta_* \in (0, \frac{1}{2})$ a positive constant $M = M(U)$ such that

$$
U(\xi) > 1 - \delta_* \quad \text{for all } \xi \ge M \,, \quad U(\xi) < \delta_* \quad \text{for all } \xi \le -M \,. \tag{41}
$$

We consider three cases $|y| \le M$, $y < -M$ and $y > M$.

In case $|y| \le M$, the estimate

$$
\begin{aligned}
f(U) - f\big(U + \delta \mathrm{e}^{-\beta t}\big) &= -\delta \mathrm{e}^{-\beta t} \int_0^1 f'\big(U + \theta\delta \mathrm{e}^{-\beta t}\big) \,\mathrm{d}\theta \\
&\ge -\|f'\|_{C([-1,2])}\, \delta \mathrm{e}^{-\beta t}
\end{aligned}
$$

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yields

$$\begin{aligned}&f(U) - f\big(U + \delta \mathrm{e}^{-\beta t}\big) + \delta\beta \mathrm{e}^{-\beta t}\big(U'(y)\sigma^* - 1\big)\\ \geq&\delta \mathrm{e}^{-\beta t}\big(- \|f'\|_{C([-1,2])} + \beta\big(U'(y)\sigma^* - 1\big)\big)\,.\end{aligned}$$

The last expression is non-negative, if σ^* is chosen according to

$$\sigma^* \geq \sup_{|y|\leq M} \frac{\|f'\|_{C([-1,2])} + \beta}{\beta U'(y)} = \frac{\|f'\|_{C([-1,2])} + \beta}{\beta \inf_{|y|\leq M} U'(y)}\,, \tag{42}$$

where $\inf_{|y|\leq M} U'(y)$ is positive, since U' is a continuous positive function and $|y| \leq M$ is a compact subset. For σ^* in (42), $\partial_t w^+ - \mathrm{D}^\alpha_\theta\, w^+ - f(w^+) \geq 0$ for all $|y| \leq M$.

In case $y \geq M$, we have

$$\begin{aligned}&f(U) - f\big(U + \delta \mathrm{e}^{-\beta t}\big) + \delta\beta \mathrm{e}^{-\beta t}\big(U'(y)\sigma^* - 1\big)\\ &\qquad= \delta \mathrm{e}^{-\beta t}\left(\int_0^1 -f'\big(U(y) + \theta\delta \mathrm{e}^{-\beta t}\big) - \beta \;\mathrm{d}\theta \,+ \beta\sigma^* U'(y)\right).\end{aligned}$$

The last expression is non-negative, if $\delta \in (0, \delta_*]$ and δ_* is chosen sufficiently small according to

$$\min_{u\in[1-\delta_*,1+\delta_*]} -f'(u) \geq \beta = \tfrac{1}{2} \min\{-f'(0), -f'(1)\}\,, \tag{43}$$

since $\beta\sigma^* U'(y)$ is non-negative anyway.

In case $y \leq -M$, we have

$$\begin{aligned}&f(U) - f\big(U + \delta \mathrm{e}^{-\beta t}\big) + \delta\beta \mathrm{e}^{-\beta t}\big(U'(y)\sigma^* - 1\big)\\ &\qquad= \delta \mathrm{e}^{-\beta t}\left(\int_0^1 -f'\big(U(y) + \theta\delta \mathrm{e}^{-\beta t}\big) - \beta \;\mathrm{d}\theta \,+ \beta\sigma^* U'(y)\right).\end{aligned}$$

The last expression is non-negative, if $\delta \in (0, \delta_*]$ and δ_* is chosen sufficiently small according to

$$\min_{u\in[0,2\delta_*]} -f'(u) \geq \beta = \tfrac{1}{2} \min\{-f'(0), -f'(1)\}\,, \tag{44}$$

since $\beta\sigma^* U'(y)$ is non-negative anyway.

Choosing δ_* sufficiently small such that (43) and (44), then M sufficiently large such that (41) and finally σ^* sufficiently large such that (42) are satisfied, respectively, we deduce that

$$\partial_t w^+ - \mathrm{D}^\alpha_\theta\, w^+ - f(w^+) \geq 0\,.$$

In contrast, to prove that w^- is a subsolution, i.e.

$$\partial_t w^- - \mathrm{D}^\alpha_\theta\, w^- - f(w^-) \leq 0\,,$$

we have to choose δ_* sufficiently small such that

$$\min_{u\in[-\delta_*,\delta_*]\cup[1-2\delta_*,1]} -f'(u) \geq \beta = \tfrac{1}{2} \min\{-f'(0), -f'(1)\}.$$

then M sufficiently large such that (41) and finally σ^* sufficiently large such that (42) are satisfied, respectively.

Together, the result follows if we choose δ_* sufficiently small such that

$$\min_{u\in[-\delta_*,2\delta_*]\cup[1-2\delta_*,1+\delta_*]} -f'(u) \geq \beta = \tfrac{1}{2} \min\{-f'(0), -f'(1)\}.$$

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then M sufficiently large such that (41) and finally σ^* sufficiently large such that (42) are satisfied, respectively. $\square$

Proof of Theorem A.1. The proof is based upon [10, Proof of Theorem 2.1], and we highlight the necessary modifications. The problem (1) under consideration fulfills the assumptions (A2) and (A3) due to the discussion in Section 4.

Step 1. Since $U(\xi)$ and $\tilde{U}(\xi)$ have the same limits as $\xi \to \pm\infty$, there exist $\xi_1 \in \mathbb{R}$ and $h \gg 1$ such that

$$U(\cdot + \xi_1) - \delta_* < \tilde{U}(\cdot) < U(\cdot + \xi_1 + h) + \delta_* \quad \text{on } \mathbb{R}\,, \tag{45}$$

where δ_* is taken from Lemma A.2. Considering the translated profile $U(\cdot + \xi_1)$ instead of U, we can set $\xi_1 = 0$ without loss of generality. Comparing $\tilde{U}(x - \tilde{c}t)$ with $w^\pm$ in (40) (with $\xi_0 = 0$ for w^- and $\xi_0 = h$ for w^+), we obtain from inequality (45), Lemma A.2 and Lemma 3.3

$$\begin{aligned} U\big(x - ct - \sigma^*\delta_*[1 - \mathrm{e}^{-\beta t}]\big) - \delta_*\mathrm{e}^{-\beta t} \\ < \tilde{U}(x - \tilde{c}t) < U\big(x - ct + h + \sigma^*\delta_*[1 - \mathrm{e}^{-\beta t}]\big) + \delta_*\mathrm{e}^{-\beta t} \end{aligned} \tag{46}$$

for all $x \in \mathbb{R}$ and $t > 0$. Keeping $\xi := x - \tilde{c}t$ fixed, sending $t \to \infty$, and using $\lim_{\xi\to\pm\infty} U(\xi) = \lim_{\xi\to\pm\infty} \tilde{U}(\xi) = u_\pm$, we obtain from the first inequality in (46) that $c \geq \tilde{c}$ and from the second inequality in (46) that $c \leq \tilde{c}$, so that $c = \tilde{c}$. In addition,

$$U(\xi - \sigma^*\delta_*) < \tilde{U}(\xi) < U(\xi + h + \sigma^*\delta_*) \qquad \forall \xi \in \mathbb{R}\,. \tag{47}$$

Step 2. Due to (47), the shifts

$$\xi^* := \inf\{\, \xi \in \mathbb{R} \,|\, \tilde{U}(\cdot) \leq U(\cdot + \xi)\,\} \geq -\sigma^*\delta_*$$

and

$$\xi_* := \sup\{\, \xi \in \mathbb{R} \,|\, \tilde{U}(\cdot) \geq U(\cdot + \xi)\,\} \leq h + \sigma^*\delta_*$$

are well-defined and satisfy $\xi_* \leq \xi^*$. To finish the proof, it suffices to show that $\xi_* = \xi^*$. To do this, we use a contradiction argument. Hence, we assume that $\xi_* < \xi^*$ and $\tilde{U}(\cdot) \not\equiv U(\cdot + \xi^*)$.

Since we assume $\lim_{|\xi|\to\infty} U'(\xi) = 0$, there exists a large positive constant $M_2 = M_2(U)$ such that

$$2\sigma^* U'(\xi) \leq 1 \quad \text{if } |\xi| \geq M_2\,. \tag{48}$$

The definition of ξ^* implies $\tilde{U}(\cdot) \leq U(\cdot + \xi^*)$. The functions $\tilde{U}(\cdot)$ and $U(\cdot + \xi^*)$ are stationary solutions of (1) and $\tilde{U}(\cdot) - U(\cdot + \xi^*) \in C([0,T]; C_0(\mathbb{R}))$. Thus the comparison result in Lemma 3.3 implies $\tilde{U}(\cdot) < U(\cdot + \xi^*)$ on $\mathbb{R}$. Consequently, by the continuity of U and $\tilde{U}$, there exists a small constant $\hat{h} \in (0, \frac{1}{2\sigma^*}]$ such that

$$\tilde{U}(\xi) < U(\xi + \xi^* - 2\sigma^*\hat{h}) \qquad \forall \xi \text{ with } |\xi + \xi^*| \leq M_2 + 1\,. \tag{49}$$

When $|\xi + \xi^*| \geq M_2 + 1$, then for some $\theta \in [0,1]$

$$\begin{aligned} U(\xi + \xi^* - 2\sigma^*\hat{h}) - \tilde{U}(\xi) &> U(\xi + \xi^* - 2\sigma^*\hat{h}) - U(\xi + \xi^*) \\ &= -2\sigma^*\hat{h}U'(\xi + \xi^* - 2\theta\sigma^*\hat{h}) > -\hat{h} \end{aligned}$$

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by the definition of M_2. Hence, in conjunction with (49), $U(\xi+\xi^*-2\sigma^*\hat{h})+\hat{h} > \tilde{U}(\xi)$ on $\mathbb{R}$. Due to Lemma A.2 and Lemma 3.3, for all $x \in \mathbb{R}$ and $t > 0$,

$$U\big(x-ct+\xi^*-2\sigma^*\hat{h}+\sigma^*\hat{h}[1-\mathrm{e}^{-\beta t}]\big)+\hat{h}\mathrm{e}^{-\beta t} > \tilde{U}(x-ct). \tag{50}$$

Keeping $\xi := x-ct$ fixed and sending $t \to \infty$, we obtain $U\big(\xi+\xi^*-\sigma^*\hat{h}\big) \geq \tilde{U}(\xi)$ for all $\xi \in \mathbb{R}$. But this contradicts the definition of ξ^*. Hence, $\xi_* = \xi^*$, which completes the proof of the theorem.

Appendix B. Proof - Stability

We follow the proof of Chen in [10, Section 3]. In Section 4 we studied the properties (C1)–(C4) in case of $\mathcal{A}[u] := \mathrm{D}^{\alpha}_{\theta}\, u + f(u)$. Indeed the properties (C1), (C2) and (C4) are satisfied, whereas one estimate in (C3) has to be modified. This implies that the properties (A1)–(A3) and (B1)–(B2) hold, whereas the estimate in property (B3) has to be modified.

First, we construct sub- and supersolutions of (1). In the sequel, $\zeta \in C^\infty(\mathbb{R})$ is a fixed function having the following properties:

$$\begin{cases} \zeta(s) = 0 & \text{if } s \leq 0\,, \\ \zeta(s) = 1 & \text{if } s \geq 4\,, \\ 0 < \zeta'(s) < 1 \ \text{ and } \ |\zeta''(s)| \leq 1 & \text{if } s \in (0,4)\,. \end{cases} \tag{51}$$

Lemma B.1. *Assume that (B1) holds. Then, for every $\delta \in (0\,,\min\{a^-/2\,,(1-a^+)/2\}]$, there exists a small positive constant $\epsilon = \epsilon(\delta)$ and a large positive constant $\mathcal{K} = \mathcal{K}(\delta)$ such that, for every $\xi \in \mathbb{R}$, the function $w^+(x,t)$ and $w^-(x,t)$ defined by*

$$w^+(x,t) := (1+\delta) - [1-(a^- - 2\delta)\mathrm{e}^{-\epsilon t}]\zeta(-\epsilon(x-\xi+\mathcal{K}t))\,,$$
$$w^-(x,t) := -\delta + [1-(1-a^+ - 2\delta)\mathrm{e}^{-\epsilon t}]\zeta(\epsilon(x-\xi-\mathcal{K}t))\,,$$

are respectively a supersolution and a subsolution of (1) in $\mathbb{R}\times(0,\infty)$.

Proof. We only prove the assertion of the lemma for w^-. The proof for w^+ is analogous and is omitted. By translational invariance, we need only consider the case $\xi = 0$. We want to estimate the right-hand side of

$$\partial_t w^-(x,t) - \mathcal{A}[w^-(\cdot,t)](x) = \partial_t w^-(x,t) - \mathrm{D}^{\alpha}_{\theta}\,[w^-(\cdot,t)](x) - f(w^-(x,t))\,.$$

On the one hand

$$\partial_t w^-(x,t) = -\mathcal{K}\epsilon[1-(1-a^+-2\delta)\mathrm{e}^{-\epsilon t}]\zeta' + \epsilon(1-a^+-2\delta)\mathrm{e}^{-\epsilon t}\zeta \leq -\mathcal{K}\epsilon a^+\zeta' + \epsilon\,,$$

due to the assumptions on ζ and δ. On the other hand,

$$|\,\mathrm{D}^{\alpha}_{\theta}\,[w^-(\cdot,t)](x)| \leq \mathcal{K}\Big[\|\partial_x^2 w^-\|_{C_b(\mathbb{R}\times[0,T])} + \|\partial_x w^-\|_{C_b(\mathbb{R}\times[0,T])}\Big] \leq \mathcal{K}\epsilon\,,$$

due to Proposition 2.4, the assumptions on ζ and

$$\partial_x w^- = \epsilon[1-(1-a^+-2\delta)\mathrm{e}^{-\epsilon t}]\zeta'(\epsilon(x-\xi+\mathcal{K}t))$$

as well as $\partial_x^2 w^- = \epsilon^2[1-(1-a^+-2\delta)\mathrm{e}^{-\epsilon t}]\zeta''(\epsilon(x-\xi+\mathcal{K}t))$. Consequently, the estimate

$$\partial_t w^-(x,t) - \mathcal{A}[w^-(\cdot,t)](x) \leq -\mathcal{K}_1\epsilon a^+\zeta' - f(w^-) + \mathcal{K}_2\epsilon \tag{52}$$

for some positive constants $\mathcal{K}_1$ and $\mathcal{K}_2$ follows. To show that w^- is a subsolution, we have to find ϵ and $\mathcal{K}_1$ such that the right-hand side of (52) is

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negative. This is possible by the same arguments as in [10, Proof of Lemma 3.2]. □

Second, we investigate estimates on a solution of (1) as time evolves.

Lemma B.2. *Assume that the hypothesis of Theorem B.3 hold and the constants δ_* and σ^* are taken from Lemma A.2. Then, there exist a small positive constant ϵ^* (independent of u_0) such that if, for some $\tau \geq 0$, $\xi \in \mathbb{R}$, $\delta \in (0, \min\{1, 1/\sigma^*\}\delta_*/2]$, and $h > 0$, there holds*

$$U(x - c\tau + \xi) - \delta \leq u(x, \tau) \leq U(x - c\tau + \xi + h) + \delta \qquad \forall x \in \mathbb{R}, \tag{53}$$

then for every $t > \tau + 1$, there exist $\hat{\xi}(t)$, $\hat{\delta}(t)$, and $\hat{h}(t)$ satisfying

$$\begin{aligned} &\hat{\xi}(t) \in [\xi - \sigma^*\delta\,, \xi + h + \sigma^*\delta]\,, \\ &\hat{\delta}(t) \leq \mathrm{e}^{-\beta(t-\tau-1)}[\delta + \epsilon^* \min\{h\,, 1\}]\,, \\ &\hat{h}(t) \leq [h - \sigma^*\epsilon^* \min\{h\,, 1\}] + 2\sigma^*\delta\,, \end{aligned}$$

such that (53) holds with (τ, ξ, δ, h) replaced by $(t, \hat{\xi}(t), \hat{\delta}(t), \hat{h}(t))$.

Proof. Equation (1) is invariant with respect to spatial translations and time shifts. Thus we set $\xi = 0$ and $\tau = 0$ without loss of generality and obtain

$$U(x) - \delta \leq u(x, 0) \leq U(x + h) + \delta \qquad \forall x \in \mathbb{R}.$$

We want to deduce

$$U\big(x-ct-\sigma^*\delta[1-\mathrm{e}^{-\beta t}]\big)-\delta\mathrm{e}^{-\beta t} \leq u(x,t) \leq U\big(x-ct+h+\sigma^*\delta[1-\mathrm{e}^{-\beta t}]\big)+\delta\mathrm{e}^{-\beta t}, \tag{54}$$

with the help of Lemma 3.3. First, the functions

$$w^-(x,t) := U\big(x - ct - \sigma^*\delta[1 - \mathrm{e}^{-\beta t}]\big) - \delta\mathrm{e}^{-\beta t} \quad \text{ with } \ w^-(x,0) = U(x) - \delta\,, \tag{55}$$

and

$$w^+(x,t) := U\big(x-ct+h+\sigma^*\delta[1-\mathrm{e}^{-\beta t}]\big)+\delta\mathrm{e}^{-\beta t} \quad \text{ with } \ w^+(x,0) = U(x+h)+\delta\,, \tag{56}$$

are a subsolution and a supersolution of (1), respectively, due to Lemma A.2. Second, a solution $u(x,t)$ of (1) with initial datum $u(\cdot, 0) = u_0(\cdot) \in C_b(\mathbb{R})$ and $0 \leq u_0 \leq 1$ satisfies $0 \leq u(x,t) \leq 1$ for all $(x,t) \in \mathbb{R} \times (0,T]$, due to Theorem 3.2. Finally, inequality (54) follows from the comparison principle in Lemma 3.3.

Define $\overline{h} := \min\{h, 1\}$ and $\epsilon_1 := \frac{1}{2}\min_{\xi\in[0,2]} U'(\xi)$, then $\int_0^1 U(y + \overline{h}) - U(y) \ \mathrm{d}y \geq 2\epsilon_1\overline{h}$. Due to (54), at least one of the estimates

$$\int_0^1 u(y,0) - U(y) + \delta \ \mathrm{d}y \geq \epsilon_1\overline{h} + \delta$$
$$\text{or} \quad \int_0^1 U(y + \overline{h}) + \delta - u(y,0) \ \mathrm{d}y \geq \epsilon_1\overline{h} + \delta$$

is true. Here the first case is considered, whereas the second case is similar and omitted. Comparing u with w^- in (55) and using property (B2)—see

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also Lemma 3.3—yields

$$\min_{x\in[-M_2-2-|c|,M_2+2+|c|]} u(x,1) - \left[U\big(x - c - \sigma^*\delta[1-\mathrm{e}^{-\beta}]\big) - \delta\mathrm{e}^{-\beta}\right] \tag{57}$$
$$\geq \eta(M_2+2+|c|)\int_0^1 u(y,0) - (U(y)-\delta)\ \mathrm{d}y \geq \eta\epsilon_1\overline{h} + \eta\delta$$

with $\eta := \eta(M_2+2+|c|)$. Defining $\xi_1 := c + \sigma^*\delta[1-\mathrm{e}^{-\beta}]$, which satisfies

$$-|c| \leq \xi_1 = c + \sigma^*\delta[1-\mathrm{e}^{-\beta}] \leq |c| + \sigma^* \min\{1, 1/\sigma^*\}\delta_*/2 \leq |c| + 1\,,$$

and

$$\epsilon^* := \min\left\{\frac{\delta_*}{2}\,,\frac{1}{2\sigma^*}\,,\min_{x\in[-M_2-2-2|c|,M_2+2+2|c|]}\frac{\eta\epsilon_1+\eta\delta/\overline{h}}{2\sigma^*U'(x)}\right\} \tag{58}$$

yields

$$U(x-\xi_1+2\sigma^*\epsilon^*\overline{h}) - U(x-\xi_1) = U'(\theta)2\sigma^*\epsilon^*\overline{h} \leq \eta\epsilon_1\overline{h} + \eta\delta$$

for all $x \in [-M_2-1-|c|\,, M_2+1+|c|]$ and some $\theta \in [x-\xi_1\,, x-\xi_1+2\sigma^*\epsilon^*\overline{h}]$. Consequently, together with (57),

$$u(x,1) \geq U(x-\xi_1+2\sigma^*\epsilon^*\overline{h}) - \delta\mathrm{e}^{-\beta} \qquad \forall x \in [-M_2-1-|c|\,, M_2+1+|c|]\,.$$

In contrast, for $|x| \geq M_2+1+|c|$, the definition of M_2 in (48) yields $U(x-\xi_1) \geq U(x-\xi_1+2\sigma^*\epsilon^*\overline{h}) - \epsilon^*\overline{h}$. Together with (54) for $t=1$ and the previous estimate, we obtain

$$u(x,1) \geq U(x-\xi_1) - \delta\mathrm{e}^{-\beta} \geq U(x-\xi_1+2\sigma^*\epsilon^*\overline{h}) - \big(\delta\mathrm{e}^{-\beta} + \epsilon^*\overline{h}\big) \qquad \forall x \in \mathbb{R}\,.$$

Next, we want to show

$$u(x,1+\tau) \geq U(x-c\tau-\xi_1+2\sigma^*\epsilon^*\overline{h} - \sigma^*q(1-\mathrm{e}^{-\beta\tau})) - q\mathrm{e}^{-\beta\tau} =: w_2^-(x,\tau)$$

for $q := \delta\mathrm{e}^{-\beta} + \epsilon^*\overline{h}$ and all $\tau \geq 0$. The estimate $q = \delta\mathrm{e}^{-\beta} + \epsilon^*\overline{h} \leq \delta_*$ and Lemma A.2 imply that

$$w_2^-(x,\tau) \quad \text{with} \quad w_2^-(x,0) := U(x-\xi_1+2\sigma^*\epsilon^*\overline{h}) - q$$

is a subsolution of (1). Thus we deduce from Lemma 3.3, $u(x,1+\tau) \geq w_2^-(x,\tau)$ for all $\tau \geq 0$. Furthermore, we conclude

$$\begin{aligned} u(x,1+\tau) \geq w_2^-(x,\tau) &= U(x-c\tau-\xi_1+2\sigma^*\epsilon^*\overline{h} - \sigma^*q(1-\mathrm{e}^{-\beta\tau})) - q\mathrm{e}^{-\beta\tau} \\ &\geq U(x-c\tau-c+\sigma^*\epsilon^*\overline{h} - \sigma^*\delta) - \mathrm{e}^{-\beta\tau}\big(\delta+\epsilon^*\overline{h}\big)\,, \end{aligned}$$

using the definitions of ξ_1 and q, and the monotonicity of U. Hence, setting $t = 1+\tau$, $\hat{\xi}(t) := \sigma^*\epsilon^*\overline{h} - \sigma^*\delta$, and $\hat{\delta}(t) = \mathrm{e}^{-\beta(t-1)}(\delta+\epsilon^*\overline{h})$, we obtain from the last inequality the lower bound. Whereas, estimate (54) with $\hat{h}(t) := [h+\sigma^*\delta(1-\mathrm{e}^{-\beta t})] - \hat{\xi}(t) = h - \sigma^*\epsilon^*\overline{h} + \sigma^*\delta[2-\mathrm{e}^{-\beta\tau}]$, implies the upper bound. ☐

Finally, we show local asymptotic stability of traveling wave solutions of (1)

Theorem B.3. *Assume that (A1)–(A3), (B1)–(B2) and (C3') hold. Also assume that* (1) *has a traveling wave solution* (U,c) *satisfying* (39)*, and*

$$0 < \delta \leq \min\left\{\min\{1,1/\sigma^*\}\frac{\delta_*}{2}\,,\frac{a^-}{2}\,,\frac{1-a^+}{2}\right\}, \tag{59}$$

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where σ^* and δ_* are the positive constants in Lemma A.2. Then, there exists a positive constant κ such that for any $u_0 \in C_b(\mathbb{R})$ satisfying $0 \le u_0 \le 1$ and

$$\liminf_{x\to+\infty} u_0(x) > 1-\delta > a^+ , \qquad \limsup_{x\to-\infty} u_0(x) < \delta < a^- , \tag{60}$$

the solution $u(x,t)$ of (1) with initial condition $u(\cdot,0) = u_0(\cdot)$ has the property that

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \le K\mathrm{e}^{-\kappa t} \quad \forall t \ge 0 ,$$

where $\xi \in \mathbb{R}$ and $K > 0$ are constants depending on u_0.

Proof. We follow the four step procedure in [10, Proof of Theorem 3.1].

Step 1. We prove that for any admissible $\delta > 0$, there exist large positive constants T and H such that

$$U(x - cT - H/2) - \delta \le u(x,T) \le U(x - cT + H/2) + \delta \qquad \forall x \in \mathbb{R} . \tag{61}$$

First, auxiliary smooth functions $w^\pm(x,t)$ are introduced as in Lemma B.1 which are constant except on a bounded interval. The functions

$$w^+(x,t) = (1+\delta) - [1 - (a^- - 2\delta)\mathrm{e}^{-\epsilon t}]\zeta(-\epsilon(x - \xi^+ + \mathcal{K}t))$$
$$w^-(x,t) = -\delta + [1 - (1 - a^+ - 2\delta)\mathrm{e}^{-\epsilon t}]\zeta(\epsilon(x - \xi^- - \mathcal{K}t))$$

are a supersolution and a subsolution of (1), respectively, for any

$$0 < \delta \le \min\{\tfrac{a^-}{2}, \tfrac{1-a^+}{2}\} ,$$

$\xi^\pm \in \mathbb{R}$, and appropriate constants $\epsilon = \epsilon(\delta)$ and $\mathcal{K} = \mathcal{K}(\delta)$. If, for a suitable choice of the parameters $\xi^\pm$, the inequality

$$w^-(x,0) \le u(x,0) = u_0(x) \le w^+(x,0) \qquad \forall x \in \mathbb{R} \tag{62}$$

holds, then

$$w^-(x,T) \le u(x,T) \le w^+(x,T) \qquad \forall x \in \mathbb{R} \tag{63}$$

will follow from Lemma 3.3. Thus, we determine suitable parameters $\xi^\pm$ to satisfy (62). Due to assumption (60), the biggest $x^* \in \mathbb{R}$ such that $u_0(x^*) = 1 - \delta > a^+$ is a finite number. Moreover, $w^-(x,0) \le a^+ + \delta$ for all $x \in \mathbb{R}$ where assumption (59) implies $a^+ + \delta \le 1 - \delta$. Thus the choice $\xi^- = x^*$ implies the estimate

$$w^-(x,0) \le \begin{cases} -\delta & \text{for all } x \le x^* , \\ a^+ + \delta & \text{for all } x \ge x^* , \end{cases}$$

hence $w^-(x,0) \le u_0(x)$ for all $x \in \mathbb{R}$.

Again, due to assumption (60), the smallest $x_* \in \mathbb{R}$ such that $u_0(x_*) = \delta$ is a finite number. Moreover, $w^+(x,0) \ge a^- - \delta$ for all $x \in \mathbb{R}$ where assumption (59) implies $\delta \le a^- - \delta$. Thus the choice $\xi^+ = x_*$ implies the estimate

$$w^+(x,0) \ge \begin{cases} a^- - \delta & \text{for all } x \le x_* , \\ 1 + \delta & \text{for all } x \ge x_* , \end{cases}$$

hence $u_0(x) \le w^+(x,0)$ for all $x \in \mathbb{R}$. Consequently, for our choice of parameters $\xi^\pm$, Lemma 3.3 implies estimate (63) for all $T > 0$.

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Additionally, for $\delta_U > 0$ satisfying (59), we determine $T > 0$ and $H > 0$ such that

$$U(x-cT-H/2)-\delta_U \le w^-(x,T) \quad \text{and} \quad w^+(x,T) \le U(x+cT+H/2)+\delta_U$$

for all $x \in \mathbb{R}$. The functions $U(\cdot)$ and $w^\pm$ are continuous differentiable and monotone increasing. Moreover, $w^-(x,t) = -\delta+[1-(1-a^+-2\delta)\mathrm{e}^{-\epsilon t}]\zeta(\epsilon(x-\xi^- - \mathcal{K}t))$ satisfies

$$w^-(x,T) \begin{cases} = 1-\delta-(1-a^+-2\delta)\mathrm{e}^{-\epsilon T} \ge a^+ + \delta & \text{for all } x \ge \xi^- + \mathcal{K}T + \frac{4}{\epsilon}\,, \\ \ge -\delta & \text{everywhere.} \end{cases}$$

Given an admissible δ_U, we choose $0 < \delta < \delta_U$ and $H^- > 0$ such that

$$U(x-cT-H^-/2)-\delta_U = -\delta \quad \text{for } x = \xi^- + \mathcal{K}T + \frac{4}{\epsilon}\,.$$

Then, $U(x-cT-H^-/2)-\delta_U \le w^-(x,T)$ for all $x \in \mathbb{R}$, if

(64)
$$\lim_{x\to-\infty} U(x-cT-H^-/2)-\delta_U = -\delta_U < -\delta\,,$$

(65)
$$\lim_{x\to+\infty} U(x-cT-H^-/2)-\delta_U = 1-\delta_U \le 1-\delta-(1-a^+-2\delta)\mathrm{e}^{-\epsilon T}\,.$$

Similarly, we find that $w^+(x,t) = (1+\delta)-[1-(a^- -2\delta)\mathrm{e}^{-\epsilon t}]\zeta(-\epsilon(x-\xi^+ +\mathcal{K}t))$ satisfies

$$w^+(x,T) \begin{cases} = \delta+(a^- -2\delta)\mathrm{e}^{-\epsilon T} & \text{for all } x \le \xi^+ - \mathcal{K}T - \frac{4}{\epsilon}\,, \\ \le 1+\delta & \text{everywhere.} \end{cases}$$

Given $0 < \delta < \delta_U$, we choose $H^+ > 0$ such that

$$U(x-cT+H^+/2)+\delta_U = 1+\delta \quad \text{for } x = \xi^+ - \mathcal{K}T - \frac{4}{\epsilon}\,.$$

Then, $U(x-cT+H^+/2)+\delta_U \ge w^+(x,T)$ for all $x \in \mathbb{R}$, if

$$\lim_{x\to-\infty} U(x-cT+H^+/2)+\delta_U = \delta_U \ge \delta+(a^- -2\delta)\mathrm{e}^{-\epsilon T}\,, \tag{66}$$

$$\lim_{x\to+\infty} U(x-cT+H^+/2)+\delta_U = 1+\delta_U > 1+\delta\,. \tag{67}$$

The Conditions (64) and (67) are equivalent to $\delta_U > \delta$. We consider $d := \delta_U - \delta > 0$. To fulfill the Conditions (65) and (66), we need $d \ge \max\{a^- - 2\delta\,, 1-a^+-2\delta\}\mathrm{e}^{-\epsilon T} > 0$. In particular, we choose $d = \max\{a^- - 2\delta\,, 1-a^+-2\delta\}\mathrm{e}^{-\epsilon T}$ for some $T > 0$ sufficiently large, such that $\delta = \delta_U - d$ and $\delta \in (0\,, \min\{a^-/2\,, (1-a^+)/2\}]$, which is an assumption in Lemma B.1. Finally, due to the monotonicity of the functions $U(\cdot)$ and $w^\pm$, inequality (61) will hold for the choice $H = \max\{H^-\,, H^+\}$.

Step 2. First, define

$$\delta^* := \min\left\{\tfrac{\delta_*}{2}, \tfrac{\epsilon^*}{4}\right\} \quad \text{and} \quad \kappa^* := \sigma^*\epsilon^* - 2\sigma^*\delta^* \ge \tfrac{\sigma^*\epsilon^*}{2} > 0\,,$$

then fix $t^* \ge 2$ such that

$$0 \le \mathrm{e}^{-\beta(t^*-1)}\left(1+\tfrac{\epsilon^*}{\delta^*}\right) \le 1-\kappa^* < 1\,.$$

Due to Step 1, for $\delta = \delta^*$, there exist positive constants T and H such that (61) holds. In particular, we can assume without loss of generality that

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$0 < H < 1$. Otherwise (61) and Lemma B.2 imply that (53) holds with $\tau = T + t^*$, some $\xi \in \big[-\frac{H}{2} - \sigma^*\delta^*, \frac{H}{2} + \sigma^*\delta^*\big]$, $\delta = \delta^*$ and $h = H - \kappa^*$, since

$$\hat{\delta}(T + t^*) \le \mathrm{e}^{-\beta(t^*-1)}[\delta^* + \epsilon^*] \le \delta^*,$$

and

$$\hat{h}(T + t^*) \le H - \sigma^*\epsilon^* + 2\sigma^*\delta^* \le H - \kappa^*,$$

due to the definition of t^* and κ^*. Repeating the procedure shows that (53) holds for $\tau = T + Nt^*$, $\delta = \delta^*$ and $h = H - N\kappa^*$ for all $N \in \mathbb{N}_0$ such that $H - (N-1)\kappa^* \ge 1$. Hence there exists a finite time $T_1 \ge T$ such that (53) holds for $\tau = T_1$, $\delta = \delta^*$ and $h = 1$ and some $\xi \in \mathbb{R}$ (which will be denoted by ξ^0).

Step 3. A mathematical induction on $k \in \mathbb{N}_0$ shows that (53) holds for some $\xi = \xi^k \in \mathbb{R}$, and

$$\tau = T^k := T_1 + kt^*, \quad \delta = \delta^k := (1-\kappa^*)^k\delta^* \quad \text{and } h = h^k := (1-\kappa^*)^k.$$

Induction start: For $k = 0$, the assertion holds due to Step 2. Induction step: Assuming the assertion for $k = l \in \mathbb{N}_0$ (induction hypothesis), we prove the assertion for $k = l + 1$. The induction hypothesis for $k = l \in \mathbb{N}_0$ is equivalent to (53) for $(\tau, \xi, \delta, h) = (T^l, \xi^l, \delta^l, h^l)$. Lemma B.2 implies that estimate (53) holds for $(\tau, \xi, \delta, h) = (T^{l+1}, \hat{\xi}, \hat{\delta}, \hat{h})$ satisfying

$$\begin{aligned}
&\hat{\xi} \in [\xi^l - \sigma^*\delta^l, \xi^l + h^l + \sigma^*\delta^l],\\
&\hat{\delta} \le \mathrm{e}^{-\beta(t^*-1)}(\delta^l + \epsilon^* h^l) = (1-\kappa^*)^l\delta^*\mathrm{e}^{-\beta(t^*-1)}\big(1 + \tfrac{\epsilon^*}{\delta^*}\big) \le (1-\kappa^*)^{l+1}\delta^*,\\
&\hat{h} \le h^l - \sigma^*\epsilon^* h^l + 2\sigma^*\delta^l = (1-\kappa^*)^l(1 - \sigma^*\epsilon^* + 2\sigma^*\delta^*) = (1-\kappa^*)^{l+1},
\end{aligned}$$

due to the definition of δ^*, κ^* and t^*. Consequently, (53) holds for $\tau = T^{l+1}$, some $\xi = \xi^{l+1} \in [\xi^l - \sigma^*\delta^l, \xi^l + h^l + \sigma^*\delta^l]$, $\delta = (1-\kappa^*)^{l+1}\delta^*$ and $h = (1-\kappa^*)^{l+1}$. This finishes the proof of the induction step, and—hence—the proof of the mathematical induction.

Step 4. Step 3 shows that (53) holds for $(\tau, \xi, \delta, h) = (T^k, \xi^k, \delta^k, h^k)$ for all $k \in \mathbb{N}_0$, i.e. at discrete times $\tau = T^k$. Like in Step 1, it follows that (53) holds for all $\tau \ge T^k$, $\delta = \delta^k$, $h = h^k + 2\sigma^*\delta^k$, $\xi = \xi^k - \sigma^*\delta^k$ and $k \in \mathbb{N}_0$. To deduce the (best) estimate for $t \ge T_1$, we define $\delta(t) = \delta^k$, $\xi(t) = \xi^k - \sigma^*\delta^k$, $h(t) = h^k + 2\sigma^*\delta^k$ on each interval $t \in [T^k, T^{k+1})$ for all $k \in \mathbb{N}_0$. Then,

$$U(x - ct + \xi(t)) - \delta(t) \le u(x,t) \le U(x - ct + \xi(t) + h(t)) + \delta(t)$$

for all $t \ge T_1$, $x \in \mathbb{R}$. Recalling $T^k = T_1 + kt^*$ for $k \in \mathbb{N}_0$, we conclude that $T^k \le t < T^{k+1}$ is equivalent to $k \le \frac{t-T_1}{t^*} < k+1$. Thus using the definitions of κ^*, $\delta(t)$ and $h(t)$, we deduce $\ln(1-\kappa^*) < 0$,

$$\begin{aligned}
\delta(t) &\le \delta^* \exp\left\{\left(\tfrac{t-T_1}{t^*} - 1\right)\ln(1-\kappa^*)\right\},\\
h(t) &\le (1 + 2\sigma^*\delta^*)\exp\left\{\left(\tfrac{t-T_1}{t^*} - 1\right)\ln(1-\kappa^*)\right\}
\end{aligned}$$

for all $t \ge T_1$. Consequently, $\xi(t) \in [\xi(\tau) - \sigma^*\delta(\tau), \xi(\tau) + h(\tau) + \sigma^*\delta(\tau)]$ for any $t \ge \tau \ge T_1$ implies that

$$|\xi(t) - \xi(\tau)| \le h(\tau) + 2\sigma^*\delta(\tau) \le (1 + 4\sigma^*\delta^*)\exp\left\{\left(\tfrac{\tau-T_1}{t^*} - 1\right)\ln(1-\kappa^*)\right\}.$$

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Therefore, the limit $\xi_\infty = \lim_{t\to\infty} \xi(t)$ exists and

$$|\xi_\infty - \xi(\tau)| \le h(\tau) + 2\sigma^*\delta(\tau) \le (1+4\sigma^*\delta^*)\exp\left\{\left(\tfrac{\tau-T_1}{t^*} - 1\right)\ln(1-\kappa^*)\right\}$$

for all $\tau \ge T_1$. The estimates imply the statement of Theorem B.3 for the choice $\kappa = -\frac{1}{t^*}\ln(1-\kappa^*)$. □

Following the proof of [10, Theorem 3.5], we determine bounds on the traveling wave speed c.

Theorem B.4. *Assume that (B1), (A2) and (C3') hold. Then, for any traveling wave solution (U, c) of (1), the wave speed c satisfies*

$$|c| \le \bar{C} := \frac{\|f\|_{C([0,1])}}{\bar{\epsilon}} \frac{3+\bar{a}}{\bar{a}} \tag{68}$$

where $\bar{a} := \min\{a^-, 1-a^+\}$ and $\bar{\epsilon}$ is a positive constant defined implicitly by

$$\rho(\bar{\epsilon}) := \overline{\mathcal{K}}\left[\tfrac{2\sqrt{3}}{9}\,\bar{\epsilon}^2 + \tfrac{1}{2}\,\bar{\epsilon}\right] = \min\{\,|f(s)|\mid s\in[\tfrac{\bar a}{3},\tfrac{2\bar a}{3}]\cup[1-\tfrac{2\bar a}{3}, 1-\tfrac{\bar a}{3}]\,\}$$

with the constant $\overline{\mathcal{K}}$ determined in Proposition 2.4.

Proof. Estimate (68) will be proven with the help of explicit sub- and supersolutions in traveling wave form. Due to assumption (B1), $0 < \bar{a} \le \frac{1}{2}$ and $\min\{\,|f(s)|\mid s\in[\frac{\bar a}{3},\frac{2\bar a}{3}]\cup[1-\frac{2\bar a}{3}, 1-\frac{\bar a}{3}]\,\} > 0$. The traveling wave U takes only values in $[0, 1]$, hence we can modify f without loss of generality such that $\|f\|_{C([0,1])} = \|f\|_{C([1,2])}$ as well as $f(u) = -f(\frac{\bar a}{3}) > 0$ for $u \in [-1, -\frac{\bar a}{3}]$ and $f(u) = -f(1-\frac{\bar a}{3}) < 0$ for $u \in [1+\frac{\bar a}{3}, 2]$.

To prove the upper bound $c \le \bar{C}$, we will use a subsolution $w^-(x,t)$. We recall the definition of $\bar{\epsilon}$ and $\bar{C}$ in the statement of Theorem B.4 and define $\zeta(s) := \frac{1}{2}[1+\tanh(s)]$, $\delta = \frac{\bar a}{3}$, $w^-(x,t) := -2\delta + (1+\delta)\zeta(\bar{\epsilon}(x-\bar{C}t))$. A direct calculation like in the proof of Lemma B.1 yields

$$\begin{aligned}\partial_t w^-(x,t) - \mathcal{A}[w^-(\cdot,t)](x) &= \partial_t w^-(x,t) - \mathrm{D}^\alpha_\theta\,[w^-(\cdot,t)](x) - f(w^-(x,t)) \\ &\le -\bar{\epsilon}\bar{C}(1+\delta)\zeta'(\bar{\epsilon}(x-\bar{C}t)) + \rho(\bar{\epsilon}) - f(w^-(x,t))\end{aligned} \tag{69}$$

where

$$\begin{aligned}|\,\mathrm{D}^\alpha_\theta\,[w^-(\cdot,t)](x)| &\le \overline{\mathcal{K}}\Big[\|\partial_x^2 w^-\|_{C_b(\mathbb{R}\times[0,T])} + \|\partial_x w^-\|_{C_b(\mathbb{R}\times[0,T])}\Big] \\ &\le \overline{\mathcal{K}}\Big[\bar{\epsilon}^2\|\zeta''\|_{C_b(\mathbb{R})} + \bar{\epsilon}\|\zeta'\|_{C_b(\mathbb{R})}\Big] =: \rho(\bar{\epsilon})\,,\end{aligned}$$

due to Proposition 2.4. To show that $w^-(x,t)$ is a subsolution, i.e. $\partial_t w^-(x,t) - \mathcal{A}[w^-(\cdot,t)](x) \le 0$ for all $(x,t) \in \mathbb{R}\times(0,\infty)$, we consider three subcases $\zeta \in (0, \frac{\delta}{1+\delta}]$, $\zeta \in (\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ and $\zeta \in [\frac{1}{1+\delta}, 1)$. First, $\zeta \in (0, \frac{\delta}{1+\delta}]$ implies that $w^-(x,t) = -2\delta + (1+\delta)\zeta(\bar{\epsilon}(x-\bar{C}t)) \in (-2\delta, -\delta]$ hence $f(w^-(x,t)) = -f(\frac{\bar a}{3}) > 0$ and $f(w^-(x,t)) \ge \rho(\bar{\epsilon})$. In this case the right hand side of (69) is nonpositive. Second, $\zeta \in (\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ implies that $w^-(x,t) = -2\delta + (1+\delta)\zeta(\bar{\epsilon}(x-\bar{C}t)) \in (-\delta, 1-2\delta)$ hence $f(w^-(x,t))$ has no definite sign. However, we have

$$\min\left\{\zeta'(s)\,\middle|\,\zeta(s)\in\left(\tfrac{\delta}{1+\delta},\tfrac{1}{1+\delta}\right)\right\} = \frac{2\delta}{(1+\delta)^2}$$

appeared as: **F. Achleitner and C. Kuehn. "Traveling waves for a bistable equation with nonlocal diffusion". In: *Adv. Differential Equations* 20.9-10 (2015), pp. 887–936** due to $\zeta'(s) = \frac{1}{2}(1-\tanh^2(s)) = \frac{1}{2}(1-(2\zeta(s)-1)^2) = -2\zeta(s)\,(\zeta(s)-1)$. Thus, for our choice of $\bar{\epsilon}$ and $\bar{C}$, the right hand side of (69) satisfies

$$-\bar{\epsilon}\bar{C}(1+\delta)\zeta'(\bar{\epsilon}(x-\bar{C}t))+\rho(\bar{\epsilon})-f(w^-(x,t)) \leq -\bar{\epsilon}\bar{C}(1+\delta)\tfrac{2\delta}{(1+\delta)^2}+2\|f\|_{C([0,1])} \leq 0$$

Third, $\zeta \in [\frac{1}{1+\delta}, 1)$ implies $w^-(x,t) = -2\delta+(1+\delta)\zeta(\bar{\epsilon}(x-\bar{C}t)) \in [1-2\delta, 1-\delta)$ hence $f(w^-(x,t)) > 0$ and $f(w^-(x,t)) \geq \rho(\bar{\epsilon})$. In this case the right hand side of (69) is nonpositive. Therefore $\partial_t w^-(x,t) - \mathcal{A}[w^-(\cdot,t)](x) \leq 0$ for all $(x,t) \in \mathbb{R}\times(0,\infty)$, hence $w^-(x,t)$ is a subsolution.

Like in the first step of the proof of Theorem B.3, we can find $X \gg 1$ such that $U(\cdot) \geq w^-(\cdot - X, 0)$ and deduce $U(x-ct) \geq w^-(x-X,t) = w^-(x-\bar{C}t-X,0)$ for all $(x,t) \in \mathbb{R}\times(0,\infty)$ from Lemma 3.3. Setting $\xi = x-ct$ yields $U(\xi) \geq w^-(\xi+(c-\bar{C})t-X,0)$ for all $(\xi,t) \in \mathbb{R}\times(0,\infty)$. In case of $c \geq \bar{C}$ taking the limit $t\to\infty$ would lead to a contradiction with $U(\cdot) \geq w^-(\cdot - X,0)$, hence the estimate $c \leq \bar{C}$ follows.

To prove the lower bound $-\bar{C} \leq c$, we use a supersolution $w^+(x,t) := \delta + (1+\delta)\zeta(\bar{\epsilon}(x+\bar{C}t))$. □

Appendix C. Proof - Existence

Theorem C.1. *Assume that the assumptions (C1), (C2), (C3') and (C4) hold. There exists a traveling wave solution (U,c) of (1) that satisfies (39).*

Proof. Consider the IVP

$$\begin{cases} \partial_t v = \mathcal{A}[v] & \text{in } \mathbb{R}\times(0,\infty)\,, \\ v(\cdot,0) = \zeta(\cdot) & \text{in } \mathbb{R}\,, \end{cases} \tag{70}$$

where the function ζ is defined in (51). The idea is to show that, for some diverging sequence $(t_j)_{j\in\mathbb{N}}$ with $t_j \to \infty$, the sequence $(v(\cdot+\xi(t_j),t_j))_{j\in\mathbb{N}}$—where $v(\xi(t),t) = a$ for all $t \geq 0$—has a pointwise limit $U(\cdot)$ which is the profile of a traveling wave solution of (1).

The IVP (70) has a unique solution $v \in C_b^\infty(\mathbb{R}\times(t_0,\infty))$ for any $t_0 > 0$ due to Theorem 3.2, which satisfies

$$0 \leq v(\cdot,t) \leq 1\,, \quad \lim_{x\to-\infty} v(x,t) = 0 \quad \text{and} \quad \lim_{x\to+\infty} v(x,t) = 1 \quad \text{for all } t \geq 0 \tag{71}$$

due to Theorem 3.4. The function v is monotone increasing in x, since $v(x,0) = \zeta(x) \leq \zeta(x+h) = v(x+h,0)$ and the comparison principle (C2). The function v is smooth for positive times, hence $v_x(x,t) > 0$ for all $(x,t) \in \mathbb{R}\times(0,\infty)$; actually $v_x(x,t) \geq \eta(|x|,t)\zeta(1) > 0$ for all $(x,t) \in \mathbb{R}\times(0,\infty)$ follows from studying the difference quotients $\frac{v(x+h,t)-v(x,t)}{h}$ with the help of (C2). Then the implicit function theorem implies the existence of a smooth function $z : (0,1)\times(0,\infty) \to \mathbb{R}$, such that $v(z(\tilde{a},t),t) = \tilde{a}$ for all $(\tilde{a},t) \in (0,1)\times(0,\infty)$. The following three lemmas can be proved in the same way as in Step 2 of the proof of [10, Theorem 4.1].

Lemma C.2. *Under the assumptions of Theorem C.1, there exist a small positive constant δ_1 and a function $m_1 : (0,\delta_1/2] \to (0,\infty)$ such that for all $\delta \in (0,\delta_1/2]$ the function $z : (0,1)\times(0,\infty) \to \mathbb{R}$ satisfies*

$$z(1-\delta,t) - z(\delta,t) \leq m_1(\delta) \qquad \forall t \geq 0\,. \tag{72}$$

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Lemma C.3. *Under the assumptions of Theorem C.1, for every $M > 0$ there exists a constant $\hat\eta(M) > 0$ such that*

$$\partial_x v(x + z(a,t), t) \geq \hat\eta(M) \qquad \forall t \geq 1\,, \quad x \in [-M, M]\,. \tag{73}$$

Similar to Lemma A.2 sub- and supersolutions of (1) are constructed.

Lemma C.4. *Under the assumptions of Theorem C.1, there exists a small positive constant δ_0 and a large positive constant σ_2 such that for any $\delta \in (0, \delta_0]$ and every $\xi \in \mathbb{R}$, the functions W^+ and W^- defined by*

$$W^{\pm}(x,t) := v\big(x + \xi \pm \sigma_2 \delta [1 - \mathrm{e}^{-\beta t}]\big) \pm \delta \mathrm{e}^{-\beta t} \tag{74}$$

with $\beta := \frac{1}{2}\min\{-f'(0), -f'(1)\}$ are a supersolution and a subsolution of (1), respectively.

Lemma C.5. *Under the assumptions of Theorem C.1, there exists a sequence $(t_j)_{j\in\mathbb{N}}$ and a non-decreasing function $U : \mathbb{R} \to (0,1)$, such that $(t_j)_{j\in\mathbb{N}}$ diverges to $+\infty$ as $j \to +\infty$ and*

$$\lim_{j\to\infty} v(\xi + z(a, t_j), t_j) = U(\xi) \quad \text{for all } \xi \in \mathbb{R}\,.$$

Moreover, U satisfies $\lim_{\xi\to-\infty} U(\xi) = 0$ and $\lim_{\xi\to+\infty} U(\xi) = 1$.

Proof of Lemma C.5. The sequence $\{f_k(\cdot) := v(\cdot + z(a,k), k)\}_{k\in\mathbb{N}}$ of real-valued functions on $\mathbb{R}$ consists of bounded functions which are uniformly equicontinuous. Due to the Arzelà-Ascoli theorem, there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ and a bounded continuous function $U \in C_b(\mathbb{R})$ such that $f_{k_j}(\cdot) = v(\cdot + z(a,k_j), k_j) \to U(\cdot)$ for $j \to \infty$ uniformly on compact subsets of $\mathbb{R}$. Obviously, the function U inherits from the function v the properties $U(0) = a$, $0 \leq U \leq 1$, and to be non-decreasing in x. For sufficiently small positive δ estimate (72) implies $U(-m_1(\delta)) \leq \delta$ and $U(m_1(\delta)) \geq 1-\delta$ consequently $\lim_{\xi\to-\infty} U(\xi) = 0$ and $\lim_{\xi\to\infty} U(\xi) = 1$.

First, we show $U(\xi) \leq \delta$ for all $\xi \leq -m_1(\delta)$: Estimate (73) implies $U(\xi+h) - U(\xi) \geq \hat\eta(|\xi|+1)h \geq 0$ for all $h \in [0,1]$ and all $\xi \in \mathbb{R}$. Therefore, we only need to show $U(-m_1(\delta)) \leq \delta$, where $v(-m_1(\delta) + z(a,k_j), k_j) \to U(-m_1(\delta))$ for $j \to \infty$. The function $v(x,t)$ is monotone increasing in the first argument, hence the function $z(\tilde a, t)$ is monotone increasing in its first argument as well. Due to Lemma C.2 for $\delta \in (0, \delta_1/2]$ with $\delta < a < 1-\delta$, we deduce $z(\delta,t) < z(a,t) < z(1-\delta,t)$,

$$-m_1(\delta) + z(a,t) \leq -z(1-\delta,t) + z(\delta,t) + z(a,t) < z(\delta,t)\,,$$

$v(-m_1(\delta) + z(a,k_j), k_j) < v(z(\delta,t),t) = \delta$ and finally

$$v(-m_1(\delta) + z(a,k_j), k_j) \to U(-m_1(\delta)) \leq \delta \quad \text{ for } j \to \infty\,.$$

In a similar way, we show $v(\xi + z(a,t)) > 1-\delta$ for all $\xi > m_1(\delta)$ and deduce $U(\xi) \geq 1-\delta$ for all $\xi > m_1(\delta)$.

Moreover, the convergence $\lim_{j\to\infty} v(\xi + z(a,t_j), t_j) = U(\xi)$ is uniform on $\mathbb{R}$: For sufficiently small $\delta > 0$ we deduce for all $j \in \mathbb{N}$ that

$$|U(\xi) - v(\xi + z(a,t_j), t_j)| \leq |U(\xi)| + |v(\xi + z(a,t_j), t_j)| \leq \delta \qquad \forall \xi \leq -m_1(\delta/2)$$

and $|U(\xi) - v(\xi + z(a,t_j), t_j)| \leq |1 - U(\xi)| + |1 - v(\xi + z(a,t_j), t_j)| \leq \delta$ for all $\xi \geq m_1(\delta/2)$. Due to the uniform convergence on compact intervals, we

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can choose $J(\delta)$ sufficiently large such that

$$|U(\xi)-v(\xi+z(a,t_j),t_j)| \le \delta \qquad \forall \xi \in [-m_1(\delta/2), m_1(\delta/2)] \quad \text{and} \quad \forall j \ge J(\delta)$$

hence—using the short hand notation $w(\xi,t_j) := U(\xi)-v(\xi+z(a,t_j),t_j)$—it follows that

$$\sup_{\xi\in\mathbb{R}} |w(\xi,t_j)| = \max\{\sup_{\xi\in I_1} |w(\xi,t_j)|, \sup_{\xi\in I_2} |w(\xi,t_j)|, \sup_{\xi\in I_3} |w(\xi,t_j)|\} \le \delta$$

for $I_1 := (-\infty, -m_1(\frac{\delta}{2}))$, $I_2 := [-m_1(\frac{\delta}{2}), m_1(\frac{\delta}{2})]$, $I_3 := (m_1(\frac{\delta}{2}),\infty)$, and all $j \ge J(\delta)$.

More precisely, the solution v is a smooth function for positive times and has uniformly bounded derivatives due to Theorem 3.2. Therefore, an iterated application of the Arzelà-Ascoli Theorem implies that $U \in C_b^m(\mathbb{R})$ of any order $m \in \mathbb{N}$ and the existence of a diverging sequence $\{k_j\}_{j\in\mathbb{N}}$ such that $f_{k_j}(\cdot) = v(\cdot + z(a,k_j),k_j) \to U(\cdot)$ for $j \to \infty$ uniformly w.r.t. the C^m norm on compact subsets of $\mathbb{R}$. Moreover, the function v converges to constant endstates, and its spatial derivative of any order converge to zero in the limits $x \to \pm\infty$. These properties are passed on to the function U and —as before with the help of Lemma C.2—the convergence $f_{k_j}(\cdot) = v(\cdot + z(a,k_j),k_j) \to U(\cdot)$ for $j \to \infty$ turns out to be uniform on $\mathbb{R}$. □

Finally, we prove that the function U is the profile of a traveling wave solution of (1) and satisfies (39). The IVP

$$\begin{cases} \partial_t \tilde{U} = \mathcal{A}[\tilde{U}] & \text{in } \mathbb{R}\times(0,\infty)\,, \\ \tilde{U}(\cdot,0) = U(\cdot) & \text{in } \mathbb{R}\,, \end{cases} \tag{75}$$

has a unique solution $\tilde{U} \in C_b^\infty(\mathbb{R}\times(t_0,\infty))$ for any $t_0 > 0$ due to Theorem 3.2. First, we need to establish

$$\lim_{j\to\infty} v(\xi + z(a,t_j), t_j + t) = \tilde{U}(\xi,t) \qquad \text{for all } (\xi,t) \in \mathbb{R}\times(0,\infty)\,. \tag{76}$$

For any $\hat{\epsilon} > 0$ there exists $J(\hat{\epsilon})$ such that if $j > J(\hat{\epsilon})$ then

$$v(\cdot - \hat{\epsilon} + z(a,t_j),t_j) - \hat{\epsilon} < U(\cdot) < v(\cdot + \hat{\epsilon} + z(a,t_j),t_j) + \hat{\epsilon}\,.$$

Considering these functions as the initial data of the IVP (75), we obtain from Lemma C.4 the estimate

$$\begin{aligned} v(\cdot - \hat{\epsilon} + z(a,t_j) - \sigma_2\hat{\epsilon}[1-\mathrm{e}^{-\beta t}], t_j + t) - \hat{\epsilon}\mathrm{e}^{-\beta t} &\le \tilde{U}(\cdot,t) \\ &\le v(\cdot - \hat{\epsilon} + z(a,t_j) + \sigma_2\hat{\epsilon}[1-\mathrm{e}^{-\beta t}], t_j + t) + \hat{\epsilon}\mathrm{e}^{-\beta t}\,. \end{aligned}$$

Noticing that $\tilde{U}$ is smooth and taking the limit $\hat{\epsilon} \to 0$ and then $j \to \infty$ yields statement (76). More precisely, the first estimate is rewritten as

$$v(\cdot + z(a,t_j), t_j + t) - \hat{\epsilon}\mathrm{e}^{-\beta t} \le \tilde{U}(\cdot + \hat{\epsilon} + \sigma_2\hat{\epsilon}[1-\mathrm{e}^{-\beta t}], t)$$

taking the limits yields

$$\limsup_{j\to\infty} v(\cdot + z(a,t_j), t_j + t) \le \tilde{U}(\cdot,t)\,.$$

Using the second estimate yields $\liminf_{j\to\infty} v(\cdot + z(a,t_j),t_j+t) \ge \tilde{U}(\cdot,t)$. Taken together

$$\tilde{U}(\cdot,t) \le \liminf_{j\to\infty} v(\cdot + z(a,t_j),t_j+t) \le \limsup_{j\to\infty} v(\cdot + z(a,t_j),t_j+t) \le \tilde{U}(\cdot,t)\,,$$

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we deduce statement (76). The monotonicity of v w.r.t. to x and its limiting behavior allow to find a large positive constant m_0 such that $v(\cdot - m_0, 1) - \delta_0 \le v(\cdot, 0) \le v(\cdot + m_0, 1) + \delta_0$. Again a comparison principle and Lemma C.4 imply

$$\begin{aligned} & v(\cdot - m_0 - \sigma_2\delta_0(1-\mathrm{e}^{-\beta t}), t+1) - \delta_0 \mathrm{e}^{-\beta t} \\ \le & v(\cdot, t) \\ \le & v(\cdot + m_0 + \sigma_2\delta_0(1-\mathrm{e}^{-\beta t}), t+1) + \delta_0 \mathrm{e}^{-\beta t} \end{aligned}$$

consequently evaluating at $\xi + z(a,t)$, setting $t = t_j$ and taking the limit $j \to \infty$ yields

$$\tilde{U}(\xi - m_0 - \sigma_2\delta_0, 1) \le U(\xi) \le \tilde{U}(\xi + m_0 + \sigma_2\delta_0, 1) \quad \text{for all } \xi \in \mathbb{R}\,. \tag{77}$$

To prove that the function U is the profile of a traveling wave solution of (1), we show that $\tilde{U}(\cdot, t) = U(\cdot - ct)$ for some $c \in \mathbb{R}$ and all t. Due to estimate (77) the numbers

$$\begin{aligned} \xi_* &:= \sup\{\, \xi \in \mathbb{R} \,|\, \tilde{U}(\cdot + \xi, 1) \le U(\cdot) \,\}\,, \\ \xi^* &:= \inf\{\, \xi \in \mathbb{R} \,|\, U(\cdot) \le \tilde{U}(\cdot + \xi, 1) \,\}\,, \end{aligned}$$

are well-defined and satisfy $-m_0 - \sigma_2\delta_0 \le \xi_* \le \xi^* \le m_0 + \sigma_2\delta_0$. However $\xi_* = \xi^*$ arguing as in the proof of Theorem A.1. In particular we noted that $U \in C_b^\infty(\mathbb{R})$ and for some diverging sequence $\{k_j\}_{j\in\mathbb{N}}$ the convergence $v(\cdot + z(a,k_j), k_j) \to U(\cdot)$ for $j \to \infty$ is uniform w.r.t. the $C_b^m(\mathbb{R})$-norm for any order $m \in \mathbb{N}$. In a similar way we can establish that $\lim_{x\to\pm\infty} \tilde{U}_x(x,t) = 0$ for all $t \ge 0$ and the uniform convergence $v(\cdot + z(a,t_j), t_j + t) \to \tilde{U}(\cdot, t)$ w.r.t. the $C_b^1(\mathbb{R})$-norm for all $t > 0$.

Comparing $\tilde{U}(\cdot, t)$ with $U(\cdot)$ for $t \in (1,2]$ in the same way one obtains the existence of a function $c : [1,2] \to \mathbb{R}$ with $c(1) = \xi_* = \xi^*$ such that $\tilde{U}(\cdot, t) = U(\cdot - c(t))$. The function c is differentiable and equation $\partial_t \tilde{U} = \mathcal{A}[\tilde{U}]$ implies $-c'(t)U'(\xi) = \mathcal{A}[U](\xi)$. The right-hand side of the identity does not depend on t explicitly (only through ξ), hence $c'(t)$ is constant for all t and (U, c') is a traveling wave solution of (1). To establish the properties of U' in (39), we notice that $\tilde{U}$ and hence U are bounded smooth functions approaching constant endstates in the limits $\xi \to \pm\infty$. □

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ANALYSIS AND NUMERICS OF TRAVELING WAVES FOR ASYMMETRIC FRACTIONAL REACTION-DIFFUSION EQUATIONS

FRANZ ACHLEITNER AND CHRISTIAN KUEHN

ABSTRACT. We consider a scalar reaction-diffusion equation in one spatial dimension with bistable nonlinearity and a nonlocal space-fractional diffusion operator of Riesz-Feller type. We present our analytical results on the existence, uniqueness (up to translations) and stability of a traveling wave solution connecting two stable homogeneous steady states. Moreover, we review numerical methods for the case of reaction-diffusion equations with fractional Laplacian and discuss possible extensions to our reaction-diffusion equations with Riesz-Feller operators. In particular, we present a direct method using integral operator discretization in combination with projection boundary conditions to visualize our analytical results about traveling waves.

1. INTRODUCTION.

A scalar reaction-diffusion equation is a partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,, \tag{1}$$

where the spatial derivative models diffusion and (a nonlinear) function f models reaction of some quantity $u = u(x,t)$ over time. The application and analysis of reaction-diffusion equations has a long history [6, 58, 61].

In the following, we consider equation (1) with a *bistable* nonlinear function $f \in C^1(\mathbb{R})$ such that

$$\exists u_- < a < u_+ \text{ in } \mathbb{R}: \quad f(u) \begin{cases} = 0 & \text{for } u \in \{u_-\,, a\,, u_+\}\,, \\ < 0 & \text{for } u \in (u_-, a)\,, \\ > 0 & \text{for } u \in (a, u_+)\,, \end{cases} \tag{2}$$

$$f'(u_-) < 0\,, \quad f'(u_+) < 0\,.$$

This kind of reaction-diffusion equation is known as Nagumo's equation to model propagation of signals [43, 46], as one-dimensional real Ginzburg-Landau equation (RGLE) to model long-wave amplitudes e.g. in case of convection in binary mixtures near the onset of instability [48, 55], as well as Allen-Cahn equation to model phase transitions in solids [5].

Following Allen and Cahn, a stable stationary state - such as u_- and u_+ - represents a phase of the system, whereas a traveling wave solution $u(x,t) =$

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$U(x-ct)$ with $\lim_{\xi\to\pm\infty} U(\xi) = u_\pm$ represents a phase transition. Each stationary state u_* has an associated potential $F(u_*) = F(u_-) + \int_{u_-}^{u_*} f(v)\ \mathrm{d}v$. One distinguishes between the balanced case, i.e. the stable states u_- and u_+ have the same potential $F(u_-) = F(u_+)$, and the unbalanced case, where the stable state with lesser potential value $F(u)$ is called the metastable state. Then a traveling wave solution $u(x,t) = U(x-ct)$ connecting the stable states u_- and u_+ will be stationary ($c = 0$) in the balanced case and moving in the direction of the metastable state in the unbalanced case.

In some applications it is important to include nonlocal effects. For example, Bates et al. [8] proposed a non-local model

$$\frac{\partial u}{\partial t} = J * u - u + f(u) \quad \text{for } (x,t) \in \mathbb{R}\times(0,T]\,, \tag{3}$$

for even, non-negative functions $J \in C^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} J(y)\ \mathrm{d}y = 1\,, \quad \int_{\mathbb{R}} |y| J(y)\ \mathrm{d}y < \infty\,, \quad J' \in L^1(\mathbb{R})\,,$$

and bistable functions f. The assumptions on J ensure that the problem exhibits a maximum principle and a variational formulation. The existence of traveling wave solutions $u(x,t) = U(x-ct)$ is concluded from a homotopy of (3) to a classical reaction-diffusion model (1). Moreover the traveling wave again will move depending on the balance of the potential values of the stable states. In contrast, the asymptotic stability is established only for stationary traveling wave solutions, i.e. in the balanced case, where an additional variational structure is available.

Chen established a unified approach [15] to prove the existence, uniqueness and asymptotic stability with exponential decay of traveling wave solutions for the previous reaction-diffusion equations and many more examples from the literature. He considers general nonlinear nonlocal evolution equations in the form

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{A}[u(\cdot,t)](x) \quad \text{for } (x,t) \in \mathbb{R}\times(0,T]\,,$$

where the nonlinear operator $\mathcal{A}$ is assumed to

(1) be independent of t;
(2) generate a L^∞ semigroup;
(3) be translational invariant, i.e. $\mathcal{A}$ satisfies for all $u \in \operatorname{dom}\mathcal{A}$ the identity
$$\mathcal{A}[u(\cdot+h)](x) = \mathcal{A}[u(\cdot)](x+h) \quad \forall x\,, h \in \mathbb{R}\,.$$
Consequently, there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is defined by $\mathcal{A}[\alpha\mathbf{1}] = f(\alpha)\mathbf{1}$ for $\alpha \in \mathbb{R}$ and the constant function $\mathbf{1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto 1$. This function f is assumed to be bistable (2);
(4) satisfy a comparison principle
If $\frac{\partial u}{\partial t} \geq \mathcal{A}[u]$, $\frac{\partial v}{\partial t} \leq \mathcal{A}[v]$ and $u(\cdot,0) \gneq v(\cdot,0)$, then $u(\cdot,t) > v(\cdot,t)$ for all $t > 0$.

Chen's approach relies on the comparison principle and the construction of sub- and supersolutions for any given traveling wave solution. Importantly, the method does not depend on the balance of the potential.

At the same time, Zanette [67] proposed a model

$$\frac{\partial u}{\partial t} = D_0^\alpha u + f(u) \quad \text{for } (x,t) \in \mathbb{R}\times(0,T]\,, \tag{4}$$

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with a fractional Laplacian D_0^α for some $\alpha \in (0,2)$ and an explicit bistable function f. This model exhibits monotone traveling wave solutions having an explicit integral representation, hence the asymptotic behavior of front tails and the front width can be studied directly. Subsequently, the reaction-diffusion equation (4) with fractional Laplacian and general bistable function f has been studied in the literature [67, 47, 62, 11, 12, 49, 32, 17].

Engler [20] was one of the first to consider the scalar partial integro-differential equations

$$\frac{\partial u}{\partial t} = D_\theta^\alpha u + f(u) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,, \tag{5}$$

where $u = u(x,t)$, $f \in C^1(\mathbb{R})$ is a (bistable) nonlinear function, and D_θ^α is a Riesz-Feller operator with $1 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. A Riesz-Feller operator D_θ^α of order α and skewness θ can be defined as a Fourier multiplier operator, see also the exposition of Mainardi, Luchko and Pagnini [41]. Starting from the fundamental solution of $\frac{\partial u}{\partial t} = D_\theta^\alpha u$, Engler constructs traveling wave solutions for some appropriate bistable function f. Assuming the existence of traveling wave solutions for general functions f, Engler studies the finiteness of the wave speed. The existence, uniqueness (up to translations), and stability of traveling wave solutions for general bistable functions is left open.

1.1. **Main analytical result.** Our main result is summarized in the following theorem.

Theorem 1 ([1]). *Suppose $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ satisfies (2). Then equation (5) admits a traveling wave solution $u(x,t) = U(x-ct)$ satisfying*

$$\lim_{\xi\to\pm\infty} U(\xi) = u_\pm \quad \textit{and } U'(\xi) > 0 \quad \textit{for all } \xi \in \mathbb{R}\,. \tag{6}$$

In addition, a traveling wave solution of (5) is unique up to translations. Furthermore, traveling wave solutions are globally asymptotically stable in the sense that there exists a positive constant κ such that if $u(x,t)$ is a solution of (5) with initial datum $u_0 \in C_b(\mathbb{R})$ satisfying $0 \leq u_0 \leq 1$ and

$$\liminf_{x\to\infty} u_0(x) > a\,, \qquad \limsup_{x\to-\infty} u_0(x) < a\,, \tag{7}$$

then, for some constants ξ and K depending on u_0,

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \leq K e^{-\kappa t} \qquad \forall t \geq 0\,.$$

1.2. **Discussion.** To our knowledge, we established the first result [1] on existence, uniqueness (up to translations) and stability of traveling wave solutions of (5) with Riesz-Feller operators D_θ^α for $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and bistable functions f satisfying (2). The technical details of the proof are contained in [1], whereas in this paper we give a concise overview of the proof strategy and visualize the results also numerically.

To prove Theorem 1, we follow - up to some modifications - the approach of Chen [15]. His approach relies on the comparison principle and the construction of sub- and supersolutions for any given traveling wave solution. It allows to cover all bistable functions f satisfying (2) regardless of the balance of the potential and all Riesz-Feller operators D_θ^α for $1 < \alpha < 2$ regardless of $|\theta| \leq \min\{\alpha, 2-\alpha\}$.

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Next, we quickly review different methods to study the traveling wave problem of a reaction-diffusion equation. In case of a classical reaction-local diffusion equation (1), the existence of traveling wave solutions can be studied via phase-plane analysis [6, 26]. This method has no obvious generalization to our traveling wave problem for (5), since its traveling wave equation is an integro-differential equation.

The variational approach has been focused - so far - on symmetric diffusion operators such as fractional Laplacians and on balanced potentials, hence covering only stationary traveling waves [13, 11, 12, 49]. The homotopy to a simpler traveling wave problem has been used to prove the existence of traveling wave solutions in case of (3), and (4) with unbalanced potential [32].

Chmaj [17] also considers the traveling wave problem for (4) with general bistable functions f. He approximates a given fractional Laplacian by a family of operators $J_\epsilon * u - (\int J_\epsilon)u$ such that $\lim_{\epsilon\to 0} J_\epsilon * u - (\int J_\epsilon)u = D_0^\alpha u$ in an appropriate sense. This allows him to obtain a traveling wave solution of (4) with general bistable function f as the limit of the traveling wave solutions u_ϵ of (3) associated to $(J_\epsilon)_{\epsilon\geq 0}$. It might be possible to modify Chmaj's approach to study also our reaction-diffusion equation (5) with Riesz-Feller operators. This would give an alternative existence proof of a traveling wave solutions.

However, Chen's approach allows to establish uniqueness (up to translations) and stability of traveling wave solutions as well. It remains an open problem to extend Chen's approach, if this is possible, to the general case of Riesz-Feller operators with $0 < \alpha \leq 1$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$.

1.3. **Outline.** Our article is structured as follows. In Section 2, we give a non-technical review of our analytical results in a companion article [1]. We introduce the Riesz-Feller operators as Fourier multiplier operators on Schwartz functions, and extend the Riesz-Feller operators in form of singular integrals to functions in $C_b^2(\mathbb{R})$. The Riesz-Feller operators D_θ^α generate a convolution semigroup which we deduce from the theory of Lévy processes.

Then we present the analysis of the Cauchy problem for (5) with initial datum $u_0 \in C_b(\mathbb{R})$ such that $0 \leq u_0 \leq 1$. The proof follows a standard approach, to consider the Cauchy problem in its mild formulation and to prove the existence of a mild solution. The Cauchy problem generates a nonlinear semigroup which allows to prove uniform C_b^k estimates via a bootstrap argument and to conclude that mild solutions are also classical solutions.

A comparison principle is essential to prove our result on the existence, uniqueness and stability of traveling wave solutions and to allow for a larger class of admissible functions f in the result for the Cauchy problem.

Finally, we consider the traveling wave problem for (5). In [1] we consider a general approach by Chen [15]. There we study his necessary assumptions and notice that some estimates are not of the required form. However Chen's approach can be extended, which we prove in [1, Appendices A–C]. We sketch the proof of Theorem 1 in Section 2, and refer to [1, Subsection 4.2] for more details.

In Section 3, we review numerical methods for reaction-diffusion equations with fractional Laplacian and discuss the (im-)possibility of extensions to our

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reaction-diffusion equations with Riesz-Feller operators. Then we present a direct method using integral operator discretization based on quadrature in combination with projection boundary conditions. Furthermore, we visualize the analytical results from Section 2 and outline several challenges for the numerical analysis of asymmetric Riesz-Feller operators.

2. Traveling wave solutions.

A Riesz-Feller operator of order α and skewness θ can be defined as a Fourier multiplier operator,

$$\mathcal{F}[D^\alpha_\theta f](\xi) = \psi^\alpha_\theta(\xi)\mathcal{F}[f](\xi)\,, \qquad \xi \in \mathbb{R}\,, \tag{8}$$

with symbol

$$\psi^\alpha_\theta(\xi) = -|\xi|^\alpha \exp\left[i(\mathrm{sgn}(\xi))\theta\tfrac{\pi}{2}\right]\,, \tag{9}$$

for some $0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. The symbol $\psi^\alpha_\theta(\xi)$ is the logarithm of the characteristic function of a Lévy strictly stable probability density with index of stability α and asymmetry parameter θ according to Feller's parameterization [25, 29].

Remark 1. *We follow the convention in probability theory and define the Fourier transform of f in the Schwartz space $\mathcal{S}(\mathbb{R})$ as*

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}} e^{+i\xi x} f(x)\ \mathrm{d}x\,, \qquad \xi \in \mathbb{R}\,,$$

and the inverse Fourier transform as

$$\mathcal{F}^{-1}[f](x) := \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} f(\xi)\ \mathrm{d}\xi\,, \qquad x \in \mathbb{R}\,.$$

Moreover, $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote also their respective extensions to $L^2(\mathbb{R})$.

To analyze the Cauchy problem for the reaction diffusion equation (5) we need to investigate the linear space-fractional diffusion equation

$$\tfrac{\partial u}{\partial t}(x,t) = D^\alpha_\theta[u(\cdot,t)](x) \quad \text{for } (x,t) \in \mathbb{R}\times(0,\infty)\,, \tag{10}$$

$0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. A formal Fourier transform of the associated Cauchy problem yields

$$\tfrac{\partial}{\partial t}\mathcal{F}[u](\xi,t) = \psi^\alpha_\theta(\xi)\mathcal{F}[u](\xi,t)\,, \qquad \mathcal{F}[u](\xi,0) = \mathcal{F}[u_0](\xi)\,,$$

which has a solution $\mathcal{F}[u](\xi,t) = e^{t\psi^\alpha_\theta(\xi)}\mathcal{F}[u_0](\xi)$. Hence, a formal solution of the Cauchy problem is given by

$$u(x,t) = (G^\alpha_\theta(\cdot,t) * u_0)(x) \tag{11}$$

with kernel (or Green's function) $G^\alpha_\theta(x,t) := \mathcal{F}^{-1}\left[\exp(t\psi^\alpha_\theta(\cdot))\right](x)$.

Due to Theorem [51, Theorem 14.19], the function $e^{t\psi^\alpha_\theta(\xi)}$ is the characteristic function of a random variable with Lévy strictly α-stable distribution. Thus G^α_θ is the scaled probability measure of a Lévy strictly α-stable distribution. In case of $(\alpha,\theta) \in \{(0,0),(1,1),(1,-1)\}$, the probability measure G^α_θ is a delta distribution

$$G^0_0(x,t) = \delta_x\,, \qquad G^1_1(x,t) = \delta_{x+t}\,, \qquad G^1_{-1}(x,t) = \delta_{x-t}$$

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and called trivial [51, Definition 13.6]. In all other (non-trivial) cases, the probability measure G^α_θ is absolutely continuous with respect to the Lebesgue measure and has a continuous probability density [51, Proposition 28.1], which we will denote again by G^α_θ. For every infinitely divisible distribution μ on $\mathbb{R}^d$, such as G^α_θ, there exists an associated Lévy process $(X_t)_{t\geq 0}$. In particular, every Lévy process exhibits an associated strongly continuous semigroup on $C_0(\mathbb{R}^d)$, see also [51, Theorem 31.5].

The infinitesimal generator of our Lévy process has the following representation, which allows to extend the Riesz-Feller operator to $C^2_b(\mathbb{R})$-functions.

Theorem 2. *If $0<\alpha<1$ or $1<\alpha<2$ and $|\theta|\leq\min\{\alpha,2-\alpha\}$, then for all $f\in\mathcal{S}(\mathbb{R})$ and $x\in\mathbb{R}$*

$$D^\alpha_\theta f(x) = \tfrac{c_1-c_2}{1-\alpha} f'(x) + c_1\int_0^\infty \frac{f(x+\xi)-f(x)-f'(x)\,\xi\mathbf{1}_{(-1,1)}(\xi)}{\xi^{1+\alpha}}\,\mathrm{d}\xi + c_2\int_0^\infty \frac{f(x-\xi)-f(x)+f'(x)\,\xi\mathbf{1}_{(-1,1)}(\xi)}{\xi^{1+\alpha}}\,\mathrm{d}\xi \tag{12}$$

where $\mathbf{1}_{(-1,1)}(\cdot)$ is an indicator function and some constants $c_1,c_2\geq 0$ with $c_1+c_2>0$.

Proof. The result follows from [51, Theorem 31.7] see also [1, Theorem 2.4]. □

In the analysis of the traveling wave problem, we are mostly interested in the evolution of initial data in C_b. Therefore, it is important to notice the following proposition.

Proposition 3 ([1, Corollary 2.10])**.** *For $1<\alpha<2$ and $|\theta|\leq\min\{\alpha,2-\alpha\}$, the Riesz-Feller operator D^α_θ generates a convolution semigroup $S_t: C_b(\mathbb{R})\to C_b(\mathbb{R})$, $u_0\mapsto S_tu_0 = G^\alpha_\theta(\cdot,t)*u_0$, with kernel $G^\alpha_\theta(x,t)$. Moreover, the convolution semigroup with $u(x,t):=S_tu_0$ satisfies*

(1) *$u\in C^\infty(\mathbb{R}\times(t_0,\infty))$ for all $t_0>0$;*
(2) *$\frac{\partial u}{\partial t} = D^\alpha_\theta u$ for all $(x,t)\in\mathbb{R}\times(t_0,\infty)$ and any $t_0>0$;*
(3) *If $u_0\in C_b(\mathbb{R})$ then $u\in C_b(\mathbb{R}\times[0,T])$ for any $T>0$.*

This result states that Riesz-Feller operators D^α_θ for $0<\alpha\leq 2$ and $|\theta|\leq\min\{\alpha,2-\alpha\}$ generate conservative C_b-Feller semigroups. This can be deduced from a criterion on the symbol of Fourier multiplier operators in [54].

It is important to notice that $S_t: C_b(\mathbb{R})\to C_b(\mathbb{R})$ is not a strongly continuous semigroup. Thus the C^2_b-extension of D^α_θ are not the infinitesimal generators of the C_b-extension of the strongly continuous semigroup $(S_t)_{t\geq 0}$ on $C_0(\mathbb{R})$ in the usual sense.

2.1. **Cauchy problem.** We consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = D^\alpha_\theta u + f(u) & \text{for } (x,t)\in\mathbb{R}\times(0,\infty)\,, \\ u(x,0) = u_0(x) & \text{for } x\in\mathbb{R}\,, \end{cases} \tag{13}$$

for $1<\alpha\leq 2$, $|\theta|\leq\min\{\alpha,2-\alpha\}$ and $f\in C^\infty(\mathbb{R})$ satisfying (2). We follow a standard approach, and consider the Cauchy problem in its mild formulation to prove the existence of a mild solution. The Cauchy problem generates

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a nonlinear semigroup which allows to prove uniform C_b^k estimates via a bootstrap argument and to conclude that mild solutions are also classical solutions.

Theorem 4 ([1, Theorem 3.3])**.** *Suppose* $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ *and* $f \in C^\infty(\mathbb{R})$ *satisfies* (2)*. The Cauchy problem* (5) *with initial condition* $u(\cdot,0) = u_0 \in C_b(\mathbb{R})$ *and* $0 \leq u_0 \leq 1$ *has a solution* $u(x,t)$ *in the following sense: for all* $T > 0$

(1) $u \in C_b(\mathbb{R} \times (0,T))$ *and* $u \in C_b^\infty(\mathbb{R} \times (t_0,T))$ *for all* $t_0 \in (0,T)$*;*
(2) u *satisfies* (5) *on* $\mathbb{R} \times (0,T)$*;*
(3) *If* $u_0 \in C_b(\mathbb{R})$ *then* $u(\cdot,t) \to u_0$ *uniformly as* $t \to 0$*;*
(4) $0 \leq u(x,t) \leq 1$ *for all* $(x,t) \in \mathbb{R} \times (0,\infty)$*;*
(5) $\forall k \in \mathbb{N}\ \forall t_0 > 0\ \exists C > 0$ *such that* $\|u(\cdot,t)\|_{C_b^k(\mathbb{R})} \leq C\ \forall 0 < t_0 < t$*.*

The following comparison principle is essential to prove our result on the existence, uniqueness and stability of traveling wave solutions and to allow for a larger class of admissible functions f in the result for the Cauchy problem.

Lemma 5 ([1, Lemma 3.4])**.** *Assume* $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $T > 0$ *and* $u, v \in C_b(\mathbb{R} \times [0,T]) \cap C_b^2(\mathbb{R} \times (t_0,T])$ *for all* $t_0 \in (0,T)$ *such that*

$$\tfrac{\partial u}{\partial t} \leq D_\theta^\alpha u + f(u) \quad \textit{and} \quad \tfrac{\partial v}{\partial t} \geq D_\theta^\alpha v + f(v) \quad \textit{in } \mathbb{R} \times (0,T]\,.$$

(1) *If* $v(\cdot,0) \geq u(\cdot,0)$ *then* $v(x,t) \geq u(x,t)$ *for all* $(x,t) \in \mathbb{R} \times (0,T]$.
(2) *If* $v(\cdot,0) \gneqq u(\cdot,0)$ *then* $v(x,t) > u(x,t)$ *for all* $(x,t) \in \mathbb{R} \times (0,T]$.
(3) *Moreover, there exists a positive continuous function*

$$\eta : [0,\infty) \times (0,\infty) \to (0,\infty)\,, \quad (m,t) \mapsto \eta(m,t)\,,$$

such that if $v(\cdot,0) \geq u(\cdot,0)$ *then for all* $(x,t) \in \mathbb{R} \times (0,T)$

$$v(x,t) - u(x,t) \geq \eta(|x|,t) \int_0^1 v(y,0) - u(y,0)\ \mathrm{d}y\ .$$

Sketch of the proof of Theorem 1. We present here a sketch of the proof of Theorem 1 and refer to our article [1] for more details. To prove existence of traveling wave solutions satisfying (6), we consider the Cauchy problem for (5) with some smooth initial datum $u_0 \in C_b(\mathbb{R})$ satisfying (6). Due to Theorem 4 there exists a classical solution $u(x,t)$. We consider a diverging sequence $\{t_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} t_j = \infty$ and the associated sequence $\{u(\cdot,t_j)\}_{j\in\mathbb{N}}$ in $C_b(\mathbb{R})$. Then, due to Arzela-Ascoli Theorem, there exists a subsequence and a limiting function $\tilde{u}$ such that $\lim_{k\to\infty} u(\cdot,t_{j_k}) = \tilde{u}(\cdot)$. The final and most important step is to verify that $\tilde{u}$ is a traveling wave solution of (5) satisfying (6).

To prove uniqueness (up to translations) of a traveling wave solution, sub- and super-solutions of (5) are constructed from any given traveling wave solution. Assuming the existence of two traveling wave solutions, one traveling wave solution is bounded from below and from above by suitable sub- and super-solutions associated to the other traveling wave solution, respectively. The comparison principle in Lemma 5 allows to show that one traveling wave solution is a translated version of the other traveling wave solution.

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To prove stability of a traveling wave solution, considering the Cauchy problem for (5) with initial datum u_0 satisfying (7), then the associated solution v can be bounded from below and from above by suitable sub- and super-solutions associated to the traveling wave solution, respectively. The comparison principle and the evolution of sub- and super-solutions show that these bounds on the solution v get tighter and allow to prove the exponential convergence to (a translated version of) the traveling wave solution.

For more details see the proof of [1, Theorem 4.6]. □

3. Numerical methods.

In this section, we illustrate our results from Theorem 1 and discuss numerical methods for (5). The case $\theta = 0$ yields the fractional Laplacian $D_0^\alpha = -(-\Delta)^{\alpha/2}$ which has been discussed frequently from a numerical perspective in the literature. Hence, there is a notational convention to write (5) for $\theta = 0$ as

$$\frac{\partial u}{\partial t} + (-\Delta)^{\alpha/2} = f(u) \qquad \text{or} \qquad \frac{\partial u}{\partial t} = -(-\Delta)^{\alpha/2} + f(u).$$

However, we shall adhere to the convention D_0^α as introduced previously. First, we review some of the available numerical schemes for this case. We restrict the computational domain from $x \in \mathbb{R}$ to $x \in [-b, b] =: \Omega$ for some (sufficiently large) $b > 0$ and with Neumann or Dirichlet boundary conditions. A numerical comparison of various methods for the case D_0^α has already been carried out in [59, 63] so we shall focus our small survey in Sections 3.1-3.4 on the difficulties in the numerical generalization from $\theta = 0$ to $\theta \neq 0$ for space-fractional equations. Furthermore, we only cover spatial grid bases schemes and do not discuss stochastic particle methods.

The main novel results are our direct method using integral operator discretization in combination with projection boundary conditions in Section 3.5 and the numerical results in Section 3.6 for (5).

3.1. **Spectral methods.** One idea is to generalize *spectral methods* to the fractional Laplacian case [9]. Let λ_j denote the Laplacian eigenvalues and ϕ_j the corresponding eigenfunctions for $D_0^2\phi_l = \lambda_l\phi_l$ with $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider $L^2(\Omega)$ then we may write $v \in L^2(\Omega)$ as a series expansion

$$v = \sum_{l=0}^{\infty} \hat{v}_l \phi_l, \qquad \hat{v}_l := \langle v, \phi_l \rangle \tag{14}$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product. Fix some α with $1 < \alpha \leq 2$ and consider

$$H^{\alpha/2}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{l=0}^{\infty} |\hat{v}_l|^2 |\lambda_l|^{\alpha/2} < \infty \right\}. \tag{15}$$

The spectral decomposition of the fractional Laplacian implies [7] that $-(-\lambda_l)^{\alpha/2}$ are eigenvalues with eigenfunctions ϕ_l for D_0^α and for any $v \in H^{\alpha/2}(\Omega)$ we have

$$D_0^\alpha v = -\sum_{l=0}^{\infty} (-\lambda_l)^{\alpha/2} \hat{v}_l \phi_l. \tag{16}$$

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As a remark, we note that all the minus signs on the right-hand side in (16) disappear if we would write $(-\Delta)^{\alpha/2}u$ on the left-hand side instead and would let λ_l denote the eigenvalues of the negative Laplacian. It is suggested in [9] to apply a backward Euler-type time discretization on a mesh

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < T \tag{17}$$

for (5) where we set $t_{m+1} - t_m =: (\delta t)_m$. Denote by $u^m := u(x, t_m)$ the solution at time t_m. For the time step t_m to t_{m+1} one may consider the semi-implicit backward Euler scheme

$$\frac{u^{m+1} - u^m}{(\delta t)_m} = D_0^\alpha u^{m+1} + f(u^m). \tag{18}$$

Making the Fourier spectral ansatz

$$u(x,t) = \sum_{l=0}^{\infty} \hat{u}_l(t)\phi_l(x) \approx \sum_{l=0}^{L} \hat{u}_l(t)\phi_l(x)$$

in (18), using the orthogonality of the basis functions ϕ_l and employing (16) leads to the numerical method

$$\hat{u}_l^{m+1} = \frac{1}{1 + (-\lambda_l)^{\alpha/2}(\delta t)_m} \left(\hat{u}_l^m + (\delta t)_m \hat{f}_l(u^m)\right) \tag{19}$$

where $\hat{f}_l$ is the l-th Fourier coefficient of f. In particular, the $L+1$ Fourier modes in (19) are decoupled and relatively easy to solve for. Further implementation details of (19) can be found in [9, Code 4, p.10]. However, the generalization of (19) from the fractional Laplacian case D_0^α to the asymmetric case D_θ^α with $\theta \neq 0$ is not straightforward. In fact, in the asymmetric case one generically obtains complex eigenvalues and a continuous spectrum [3]. This means that (16) is no longer valid for $\theta \neq 0$. For another approach using transform/Fourier-type techniques we refer to [52].

3.2. **Finite difference methods.** A second possible approach to solve (5) is to use a *finite difference method* (FDM) [45] combined with the Grünwald-Letnikov representation of the space fractional derivative. Let us consider a spatial discretization of $\Omega = [-b, b]$ as follows

$$-b = x_1 < x_2 < \cdots < x_N = b. \tag{20}$$

We still use the temporal discretization (17). For D_0^α the Grünwald-Letnikov representation of D_0^α is given by

$$\begin{aligned} (D_0^\alpha u)(x,t) &= \lim_{N\to\infty} \frac{1}{h_+^\alpha} \sum_{r=0}^{N} \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)\Gamma(r+1)} u(x - rh_+, t) \\ &\quad + \lim_{N\to\infty} \frac{1}{h_-^\alpha} \sum_{r=0}^{N} \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)\Gamma(r+1)} u(x + rh_+, t), \end{aligned} \tag{21}$$

where $h_+ = (x+b)/N$ and $h_- = (b-x)/N$. Let us assume for simplicity that the spatial grid is equidistant and let $h := 2b/N$. Furthermore, we let

$$u_n^m := u(x_n, t_m).$$

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Then one possible finite-difference discretization of (5) is given by [45, 59]

$$\frac{u_n^{m+1}-u_n^m}{(\delta t)_m} = \frac{1}{h^\alpha}\left[\sum_{r=0}^{n+1} g_r u_{n-r+1}^{m+1} + \sum_{r=0}^{N-n+1} g_r u_{n+r-1}^{m+1}\right] + f(u_n^{m+1}),$$

with $g_r := \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)\Gamma(r+1)}$. For a similar approach using the Grünwald-Letnikov representation to obtain finite-difference schemes we also refer to [14, 28, 39, 40, 44, 53, 60, 68]. For even more details on finite-difference methods for space-fractional diffusion equations consider [38, 57, 66]. In some sense, our scheme in Section 3.5 has an analogous starting point. However, instead of the Grünwald-Letnikov representation we use the integral representation formula which we also employed in the existence-uniqueness-stability proof of Theorem 1; see also Section 3.5.

3.3. **Finite element methods.** Another quite natural possibility is to follow the classical *finite element method* (FEM) variational approach. We follow [22, 36] in our exposition for the case $\theta = 0$. Let $X := H_0^{\alpha/2}(\Omega)$ denote the usual fractional Sobolev space obtained as a closure of $C_0^\infty(\Omega)$ in $H^{\alpha/2}(\Omega)$ and define

$$A(v,w) := -\langle D_0^{\alpha/2} v, D_0^{\alpha/2} w\rangle\,, \tag{22}$$

where representation (16) is used. Then one may check that A is coercive and continuous. Consider the space X_h of piecewise linear continuous functions in X with compact support given by

$$X_h := \{v \in C_0(\Omega) : v \text{ is linear over } [x_n, x_{n+1}],\ n = 1, 2, \ldots, N-1\}.$$

Then we may define a discrete operator $A_h : X_h \to X_h$ associated to A via

$$\langle A_h v_h, w_h\rangle = A(v_h, w_h) \qquad \forall v_h, w_h \in X_h.$$

A semi-discrete Galerkin FEM scheme for (5) is to find $u_h = u_h(t) \in X_h$ such that

$$\left\langle \frac{\partial u_h}{\partial t}(t), v_h \right\rangle = \langle A_h u_h(t), v_h\rangle + \langle f(u_h(t)), v_h\rangle \qquad \forall v_h \in X\,, \tag{23}$$

and projected initial condition $\langle u_h(0), v_h\rangle = \langle u(0), v_h\rangle$. Choosing a basis $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ of X_h we may write

$$u_h(x,t) = \sum_{n=1}^N u_n(t)\varphi_n(x).$$

One defines the usual mass matrix $M \in \mathbb{R}^{N\times N}$ and stiffness matrix $A \in \mathbb{R}^{N\times N}$ with entries

$$M_{nm} = \langle \varphi_n, \varphi_m\rangle, \qquad A_{nm} = A(\varphi_n, \varphi_m), \qquad m, n \in \{1, 2, \ldots, N\}. \tag{24}$$

This converts (23) into the ODEs

$$M\frac{dU}{dt} = AU + f(U) \tag{25}$$

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where $U = (u_1, \dots, u_N)^T$ and $f(U) = (\langle f(u_h), \varphi_1\rangle, \dots, \langle f(u_h), \varphi_N\rangle)^T$. Then one may use a time-stepping scheme directly. For example, a backward Euler semi-implicit discretization yields for $U^m := U(t_m)$ the method

$$(\mathrm{Id} - (\delta t)_m M^{-1} A) U^{m+1} = U^m + (\delta t)_m M^{-1} f(U^m). \tag{26}$$

These considerations show that we can, at least formally, just follow the classical FEM theory to derive numerical methods for equations involving D_0^α. However, for the fractional Laplacian D_0^α the matrix entries for A defined in (24) are not as easy to compute as for D_0^2. FEM techniques also seem to generalize formally to the asymmetric case as coercivity and continuity hold for classes of fractional operators more general than D_0^α [22, p.574-575]. However, we are again faced with the practical problem of computing (an approximation of) $A(\varphi_n, \varphi_m)$. This observation is one reason which motivates the method presented in the next section. For more details on FEM for space fractional equations we refer to [23, 24, 27, 50].

3.4. **Matrix-transfer techniques.** The symmetry of D_0^2 and the view of fractional powers D_0^α can be employed in conjunction with FEM or FDM discretizations for (5). Again, we consider the case $\theta = 0$ following [34, 35, 10]. Let $A_\Delta \in \mathbb{R}^{N\times N}$ be the usual FEM stiffness matrix and M_Δ be the FEM mass matrix for D_0^2. One natural idea is to use a fractional power of the matrix $B_\Delta := M_\Delta^{-1} A_\Delta$ in a numerical scheme to represent the fractional Laplacian. Suppose we can compute $(B_\Delta)^\alpha$ then a backward semi-implicit Euler-type time discretization, similar to (26), leads to

$$(\mathrm{Id} - (\delta t)_m (B_\Delta)^\alpha) U^{m+1} = U^m + M_\Delta^{-1} f(U^m). \tag{27}$$

To solve (27) one has to also compute the function

$$q(z) = \frac{1}{1 - (\delta t)_m z^\alpha} \tag{28}$$

efficiently for matrices, which has been discussed in [10]. However, we still have to define B_Δ^α. Standard theory implies that A_Δ, M_Δ are real, symmetric matrices [21]. Furthermore, A_Δ is non-negative definite and M_Δ is positive definite. A direct calculation shows that

$$(M_\Delta)^{1/2} B_\Delta (M_\Delta)^{-1/2} = (M_\Delta)^{-1/2} A_\Delta (M_\Delta)^{-1/2}.$$

Therefore, B_Δ is similar to a real, symmetric matrix with well-defined point spectrum $\sigma(B_\Delta) \subset \mathbb{R}$ and eigenvalues $\xi_1 \leq \xi_2 \leq \cdots \leq \cdots \xi_N$. Then it is very natural to define a matrix function $q(Z)$, including (28) as a special case, by

$$q(Z) = Q q(\Xi) Q^{-1},$$

where Ξ is a diagonal matrix with $\Xi_{nn} = \xi_n$, Q consists of the eigenvectors associated to the eigenvalues ξ_n and $[q(\Xi)]_{nn} = q(\xi_n)$. This yields a well-defined fractional power $(B_\Delta)^\alpha$ when applied to $q(z) = z^\alpha$ and can then also be applied to define (28). Unfortunately, the matrix transfer technique does not generalize immediately to the case $\theta \neq 0$ as the spectrum $\sigma(D_\theta^\alpha)$ for $\theta \neq 0$ is generically continuous with complex eigenvalues as already discussed in Section 3.1. For more on the matrix transfer technique we refer to [64, 65].

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3.5. **Integral representation, quadrature and projection boundary conditions.** Sections 3.1-3.4 explain why, to the best of our knowledge, there seem to be very few (if any) detailed numerical studies of the asymmetric case $\theta \neq 0$ for the nonlinear Allen-Cahn/Nagumo-type Riesz-Feller reaction-diffusion equation (5).

Here we present an easy-to-implement method to study (5) numerically with a focus on the dynamics of traveling waves. Our approach is to use the integral representation of Riesz-Feller operators to view (5) as an integro-differential equation. For $\alpha \in (1,2)$ the representation formula is given by [1]

$$
\begin{aligned}
(D^\alpha_\theta u)(x,t) &= c_1 \int_0^\infty \frac{u(x+\xi,t)-u(x,t)-\xi\frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}}\,\mathrm{d}\xi \\
&\quad + c_2 \int_0^\infty \frac{u(x-\xi,t)-u(x,t)+\xi\frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}}\,\mathrm{d}\xi
\end{aligned} \tag{29}
$$

where the constants $c_{1,2}$ are given in [41] as

$$
c_1 = \frac{\Gamma(1+\alpha)\sin\left((\alpha+\theta)\frac{\pi}{2}\right)}{\pi} \quad \text{and} \quad c_2 = \frac{\Gamma(1+\alpha)\sin\left((\alpha-\theta)\frac{\pi}{2}\right)}{\pi}.
$$

Note that there is also an integral representation for $\alpha \in (0,1)$ [1] for $x \in \mathbb{R}$. Furthermore, there is an analogous integral representation formula for fractional Laplacians in $\mathbb{R}^d$ in [19]. Hence, starting from a representation like (29) is not really a restriction, even for higher-dimensional cases. Furthermore, a similar strategy has also been applied successfully in a similar to context to other nonlocal operator equations [4] involving traveling waves. If we write

$$
\begin{aligned}
g_1(\xi,x,t) &:= \frac{u(x+\xi,t)-u(x,t)-\xi\frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}}, \\
g_2(\xi,x,t) &:= \frac{u(x-\xi,t)-u(x,t)+\xi\frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}},
\end{aligned}
$$

then we can simply re-write (5) as an integro-differential equation

$$
\frac{\partial u}{\partial t}(x,t) = c_1 \int_0^\infty g_1(\xi,x,t)\,\mathrm{d}\xi + c_2 \int_0^\infty g_2(\xi,x,t)\,\mathrm{d}\xi + f(u(x,t)). \tag{30}
$$

For simplicity, we shall just introduce our method for a uniform spatial mesh (20), i.e. we have

$$
-b = x_1 < x_2 < \cdots < x_N = b \qquad \text{with} \quad x_{n+1} - x_n = \frac{2b}{N-1} =: h, \tag{31}
$$

where we assume that $N \geq 3$ is odd so that $x_{(N+1)/2} = 0$. Furthermore, we use another spatial mesh to approximate the integral operators (29) over a finite domain obtained as a sub-mesh from (31) as follows

$$
\xi_1 = x_{\frac{N+1}{2}+1}, \quad \xi_2 = x_{\frac{N+1}{2}+2}, \quad \ldots, \quad \xi_{M+1} = x_N, \tag{32}
$$

which has M subintervals $[\xi_m, \xi_{m+1}]$. We may easily relate M to the number of points N in our original mesh by $M = (N-1)/2$. For b, N sufficiently large we may just use a quadrature rule to approximate (29); we remark that the possibility to use quadrature techniques for time-fractional

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Caputo-derivative fractional equations has already been noticed in [2, 37]. Very recently (in fact, during the preparation of this work), Huang and Oberman [33] proposed a quadrature-scheme based upon a singular integral presentation of the symmetric case D_0^α. We use the regularized, fully asymmetric representation (29) and obtain for the trapezoidal rule, with $\rho \in \{1, 2\}$, that

$$\begin{aligned} c_\rho \int_0^\infty g_\rho(\xi, x, t)\ \mathrm{d}\xi &\approx c_\rho \int_h^b g_\rho(\xi, x, t)\ \mathrm{d}\xi \\ &\approx \frac{c_\rho(b-h)}{2M}\left[g_\rho(\xi_1, x, t) + g_\rho(\xi_{M+1}, x, t) + 2\sum_{m=2}^{M} g_\rho(\xi_m, x, t)\right]. \end{aligned}$$

Using this approximation in (30) yields a system of (formal) ODEs for $u_n(t) = u(x_n, t)$, which can be written as

$$\begin{aligned} \frac{\mathrm{d}u_n}{\mathrm{d}t} \quad = \quad \frac{b-h}{2M}\Bigg[& c_1\left((g_1(u))_{n,1} + (g_1(u))_{n,M+1}\right) + 2c_1\sum_{m=2}^{M}(g_1(u))_{n,m} \\ & + c_2\left((g_2(u))_{n,1} + (g_2(u))_{n,M+1}\right) + 2c_2\sum_{m=2}^{M}(g_2(u))_{n,m}\Bigg] + f(u_n) \end{aligned} \tag{33}$$

for $n \in \{1, 2, \ldots, N\}$, where the terms involving $(g_\rho(u))_{n,m}$ are given by

$$(g_1(u))_{n,m} = \tfrac{u_{n+m} - u_n}{\xi_m^{1+\alpha}} - \tfrac{u_{n+1} - u_n}{\xi_m^\alpha h}, \quad (g_2(u))_{n,m} = \tfrac{u_{n-m} - u_n}{\xi_m^{1+\alpha}} + \tfrac{u_{n+1} - u_n}{\xi_m^\alpha h}.$$

Of course, the system (33) is, as yet, only a formal representation as it involves spatial mesh indices for u which lie outside the range i.e. $u_n = u(x_n, t)$ for $n \in \{1, 2, \ldots, N\}$. There is a choice of boundary conditions. However, instead of classical Neumann or Dirichlet conditions, we want to compute traveling waves which satisfy

$$\lim_{x \to -\infty} u(x, t) = u_-, \qquad \lim_{x \to +\infty} u(x, t) = u_+$$

for constants $u_\pm$. Hence, we adopt the following projection-type boundary conditions for the numerical method

$$u_n = \begin{cases} u_N & \text{if } n \geq N, \\ u_1 & \text{if } n \leq 1, \end{cases} \tag{34}$$

Using (34), we get a well-defined ODE system (33) which can be solved using forward integration, i.e. we adopt a method-of-lines approach; for more details on using projection boundary conditions to compute traveling waves in the classical FitzHugh-Nagumo equation we refer e.g. to [18, 30, 31].

Regarding our algorithm (33)-(34) for waves of the Riesz-Feller bistable equation, we emphasize that our approach is clearly non-optimal from a numerical perspective. For example, there are straightforward generalizations to non-uniform meshes and higher-order schemes by using non-uniform-mesh higher-order quadrature methods. We leave these generalizations as future challenges. Here, we are primarily interested in developing a simple scheme for (5) and to visualize some of the results from Theorem 1.

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3.6. **Numerical results.** In this section, we briefly discuss some numerical simulations of (5) with $f(u) = u(1-u)(u-a)$ for some $a \in (0,1)$ using our algorithm from Section 3.5. Unless stated otherwise, we fix $\Omega = [-b, b] = [-30, 30]$, $N = 181$, spatial mesh points and always employ a standard stiff ODE solver to solve (33)-(34) (more precisely, `ode15s` from MatLab [56]) for $t \in [0, T]$. Figure 1(a) shows the initial condition

$$u_0(x) = u(x,0) = \begin{cases} 0 & \text{if } x \in [-30,-2), \\ \frac{1}{4}x + \frac{1}{2} & \text{if } x \in [-2,2], \\ 1 & \text{if } x \in (2,30]. \end{cases} \tag{35}$$

The initial condition (35) is important as it has been used in the existence part of the proof of Theorem 1 as discussed in [1, 16]. In particular, u_0 is shown to converge to a traveling wave.

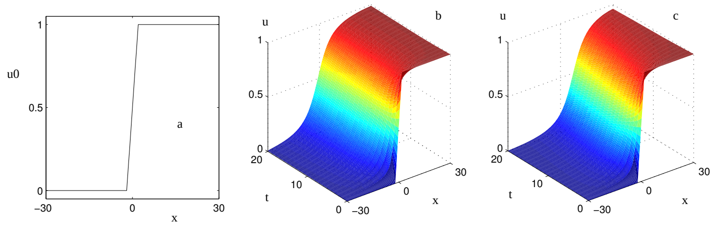

FIGURE 1. Fixed parameter values are $\theta = 0.1$, $\alpha = 1.8$, $T = 20$. (a) Initial condition $u_0 = u(x, 0)$ given by (35). (b) Simulation with $a = 0.5$. (c) Simulation with $a = 0.6$, the wave travels to the right.

Figure 1(b)-(c) show the fully asymmetric fractional case with D^α_θ for $\alpha = 1.8$ and $\theta = 0.1$. In both cases we observe a rapid smoothing effect of the solution as predicted by the smoothing result in Theorem 1. Furthermore, in both cases, convergence to a traveling wave profile is observed, where moving the parameter a changes the wave speed. Again, this is expected since the supremum-norm of the nonlinearity $f(u) = u(1-u)(u-a)$ does influence the wave speed.

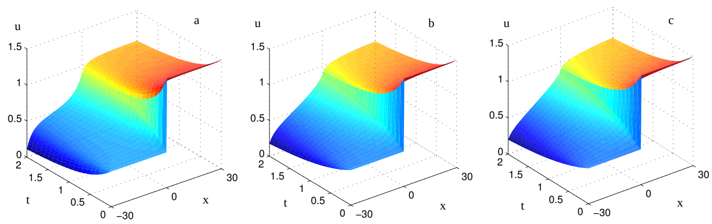

FIGURE 2. Fixed parameter values are $\theta = 0.1$, $a = 0.5$, $T = 2$ with initial condition (36). (a) $\alpha = 1.8$. (b) $\alpha = 1.2$. (c) $\alpha = 1.01$.

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As a second interesting part we are interested in discontinuous initial conditions bounded away from the traveling wave, and even violating one of the stability assumptions ($0 \leq u_0 \leq 1$) from Theorem 1. One example is

$$u_0(x) = u(x,0) = \begin{cases} 0.49 & \text{if } x \in [-30,0], \\ 1.51 & \text{if } x \in (0,30]. \end{cases} \tag{36}$$

Furthermore, we vary the fractional exponent α. Figure 2 shows the results. Although the initial condition is not within the framework of the theoretical analysis, we still observe extremely rapid convergence to a wave profile where the end-states move to $u(-b,t) = 0$ and $u(b,t) = 1$. Note however, that the convergence, as well as the regularization effect, seems to be slower for smaller exponents α.

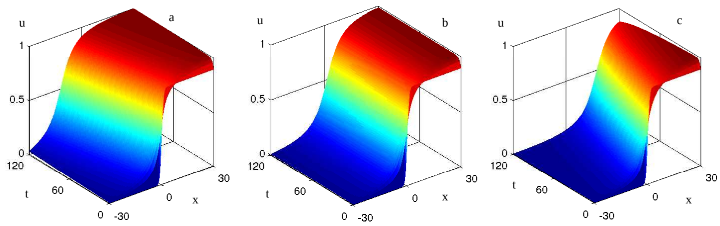

FIGURE 3. Fixed parameter values are $\alpha = 1.5$, $a = 0.5$, $T = 120$ with initial condition (35). (a) $\theta = 0.2$; wave moves to the left. (b) $\theta = 0.0$; standing wave. (c) $\theta = -0.2$; wave moves to the right.

Another question is the effect of the asymmetry parameter θ. Figure 3 shows three different cases for $\theta = 0.2, 0, -0.2$. It is clearly visible that the wave speed is directly affected. Within the time $t \in [0,T]$, the wave in Figure 2(b) barely moves while there is a drift to the right in Figure 2(c) and to the left in Figure 2(a). Hence, we may conclude that the asymmetry parameter definitely has an effect on quantitative properties of traveling waves. Based on the relation to microscopic super-diffusion processes and previous studies for other nonlinearities [42], a quantitative change is expected.

As a last issue, we briefly discuss the influence of the asymmetry parameter on numerical stability. Figure 4 shows simulations for the same parameter values $\alpha = 1.5$, $\theta = 0.4$ where θ is chosen closer to the critical line $2 - \alpha$ (see Section 2) than before. The absolute error tolerance for the numerical time step is different in Figures 4(a)-(b). Whereas we observe numerically induced oscillations in Figure 4(a) for a relatively low tolerance, the oscillations are suppressed for the more accurate computation in Figure 4(b). We checked that the numerical solution poses no problem for the lower error tolerance when θ is lower as well, for example, $\theta = 0.1$. This gives a strong indication that the ODE problem may be stiff, respectively that the region of A-stability shrinks when θ is changed. In particular, this leads to the conjecture that the asymmetric case is not only more complicated with respect to the design and implementation of numerical algorithms but also with respect to numerical stability.

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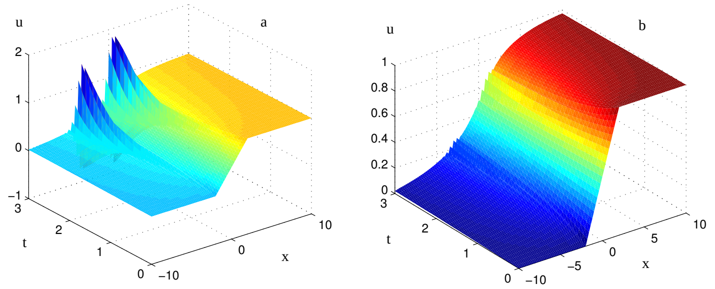

Figure 4. Fixed parameter values are $\alpha = 1.5$, $a = 0.5$, $T = 3$, $\theta = 0.4$ with $b = 10$ and $L = 101$ i.e. on a coarser grid than in the previous figures. (a) Absolute tolerance for the ODE time stepper is 10^{-6}. (b) Absolute tolerance for the ODE time stepper is 10^{-9}.

3.7. **Numerical analysis: some challenges.** In this section, we would like to highlight some numerical challenges/conjectures which are relevant for future work:

(1) Provide a generalization of our scheme to higher-order quadrature rules and non-uniform meshes, including convergence and error analysis.
(2) Generalize the scheme to 2- and 3-dimensional cases. What about the computation of coherent/localized structures for this case?
(3) Investigate the numerical stability properties of algorithms for asymmetric fractional evolution equations regarding the (α, θ)-dependence.
(4) Compare various approaches to truncate the domain $\mathbb{R}$. What is the influence of boundary conditions for space-fractional equations?
(5) What about adaptive algorithms to resolve wave profiles? What is the influence of α and θ on the adaptive mesh selection?
(6) Provide robust methods, including error estimates, to calculate the wave-speed and far-field/tail behavior.
(7) Which methods for fractional diffusion equations, derived by different approaches such as FDM, FEM or quadrature, are equivalent?

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