Kolmogorov-type conditional probabilities among distinct context

http://tph.tuwien.ac.at/~svozil/publ/2019-Svozil-Prague-pres.pdf
based on https://arxiv.org/abs/1903.10424

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Prague, Czech Republic, May 18th, 2019

Quantum bistochasticity

In what follows any “largest” domain of mutually commuting observables will be termed *context*. For quantum mechanics grounded in Hilbert space, a context can be equivalently represented by (i) an orthonormal basis, (ii) the respective one-dimensional orthogonal projection operators associated with the basis elements, or (iii) a single maximal operator whose spectral sum is non-degenerated.
An essential assumption entering Gleason's derivation of the Born rule for quantum probabilities is the validity of classical probability theory whenever the respective observables are co-measurable. Formally, this amounts to the validity of Kolmogorov probability theory for mutually commuting observables; and in particular, to the assumption of Kolmogorov's axioms within contexts.

Quantum bistochasticity cntd.

Consider two orthonormal bases aka two contexts. Their respective conditional probabilities can be arranged into a matrix form: The entry in the i-th row j-th column element corresponds to the conditional probability associated with the probability of occurrence of the j-th element (observable) of the second context, given the i-th element (observable) of the first context.

By Gleason's assumption of the validity of Kolmogorov's axioms within contexts resulting in a conditional quantum probability of the Born rule form, as well as by utilizing the dual role of projection operators in quantum mechanics as elementary two-valued observables as well as of pure states, and by taking into account that cyclically interchanging factors inside a trace does not change its value, this matrix needs to be doubly stochastic (bistochastic) [Auffeves-Grangier-2017 DOI:10.1038/srep43365, Auffeves-Grangier-2018; DOI:10.1098/rsta.2017.0311]; that is, the sum taken within every single row and every single column adds up to one.

Generalization of Kolmogorov axioms for multi-context environments

In order to generalize the quantum case, we suggest to postulate that the quantum case is just one instance satisfying a very general axiom: That, given two arbitrary contexts $\mathcal{C}_1 = \{\mathbf{e}_1, \ldots \mathbf{e}_m\}$ and $\mathcal{C}_2 = \{\mathbf{f}_1, \ldots \mathbf{f}_n\}$, the associated $(n \times m)$-matrix whose entries are the conditional probabilities $P(\mathbf{f}_j|\mathbf{e}_i)$ of "$\mathbf{f}_j$ given $\mathbf{e}_i$" must be such that the sum taken within every single row adds up to one.

We shall be mostly concerned with cases for which $n = m$; that is, the associated matrix is a row (aka right) stochastic (square) matrix. Formally, such a matrix $\mathbf{A}$ has nonnegative entries $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$ whose row sums add up to one: $\sum_{j=1}^{n} a_{ij} = 1$ for $i = 1, \ldots, n$.

The above criterium is a generalization of Kolmogorov's axioms, as it allows cases in which both contexts do not coincide. For coinciding contexts this rule just reduces to Kolmogorov's axioms.

Quasi-classical partition logics I: Two non-intertwining two-atomic contexts

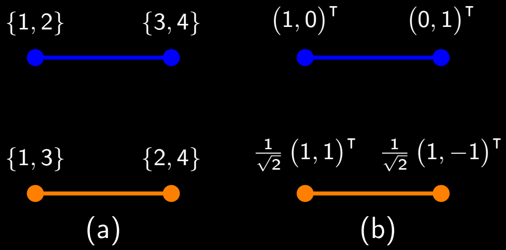


Figure: Greechie orthogonality diagram of a logic consisting of two nonintertwining contexts. (a) The associated (quasi)classical partition logic representations obtained through in inverse construction using all two-valued measures thereon (Svozil; DOI: 10.1007/s10773-005-7052-0); (b) a faithful orthogonal representation (Lovasz, DOI: 10.1109/TIT.1979.1055985) rendering a quantum *double*.

Quasi-classical partition logics I: Two non-intertwining two-atomic contexts cntd.

This logic labels the atoms (aka elementary propositions) obtained by an "inverse construction" using all two-valued measures thereon. With the identifications $\mathbf{e}_1 \equiv \{1,2\}$, $\mathbf{e}_2 \equiv \{3,4\}$, $\mathbf{f}_1 \equiv \{1,3\}$, and $\mathbf{f}_2 \equiv \{2,4\}$ we obtain all classical probabilities by identifying $i \rightarrow \lambda_i > 0$. The respective conditional probabilities are

$$
\begin{aligned}
&[P(\mathcal{C}_2|\mathcal{C}_1)] = [P(\{\mathbf{f}_1,\mathbf{f}_2\}|\{\mathbf{e}_1,\mathbf{e}_2)] \\
&\equiv \begin{pmatrix} P(\mathbf{f}_1|\mathbf{e}_1) & P(\mathbf{f}_2|\mathbf{e}_1) \\ P(\mathbf{f}_1|\mathbf{e}_2) & P(\mathbf{f}_2|\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} \frac{P(\mathbf{f}_1\cap\mathbf{e}_1)}{P(\mathbf{e}_1)} & \frac{P(\mathbf{f}_2\cap\mathbf{e}_1)}{P(\mathbf{e}_1)} \\ \frac{P(\mathbf{f}_1\cap\mathbf{e}_2)}{P(\mathbf{e}_2)} & \frac{P(\mathbf{f}_2\cap\mathbf{e}_2)}{P(\mathbf{e}_2)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{P(\{1,3\}\cap\{1,2\})}{P(\{1,2\})} & \frac{P(\{2,4\}\cap\{1,2\})}{P(\{1,2\})} \\ \frac{P(\{1,3\}\cap\{3,4\})}{P(\{3,4\})} & \frac{P(\{2,4\}\cap\{3,4\})}{P(\{3,4\})} \end{pmatrix} \\
&= \begin{pmatrix} \frac{P(\{1\})}{P(\{1,2\})} & \frac{P(\{2\})}{P(\{1,2\})} \\ \frac{P(\{3\})}{P(\{3,4\})} & \frac{P(\{4\})}{P(\{3,4\})} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} \\ \frac{\lambda_3}{\lambda_3+\lambda_4} & \frac{\lambda_4}{\lambda_3+\lambda_4} \end{pmatrix},
\end{aligned}
\tag{1}
$$

as well as

$$
\begin{aligned}
&[P(\mathcal{C}_1|\mathcal{C}_2)] = [P(\{\mathbf{e}_1,\mathbf{e}_2\}|\{\mathbf{f}_1,\mathbf{f}_2\})] \\
&\equiv \begin{pmatrix} \frac{P(\{1\})}{P(\{1,3\})} & \frac{P(\{3\})}{P(\{1,3\})} \\ \frac{P(\{2\})}{P(\{2,4\})} & \frac{P(\{4\})}{P(\{2,4\})} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_3} & \frac{\lambda_3}{\lambda_1+\lambda_3} \\ \frac{\lambda_2}{\lambda_2+\lambda_4} & \frac{\lambda_4}{\lambda_2+\lambda_4} \end{pmatrix}.
\end{aligned}
\tag{2}
$$

Quasi-classical partition logics II: Two intertwining three-atomic contexts

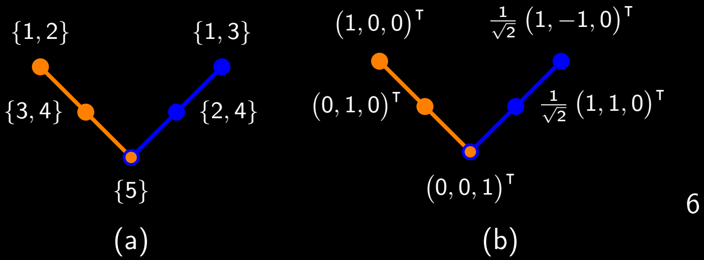


Figure: Greechie orthogonality diagram of the L_{12} "firefly" logic. (a) The associated (quasi)classical partition logic representation obtained through in inverse construction using all two-valued measures thereon (Svozil; DOI: 10.1007/s10773-005-7052-0); (b) a faithful orthogonal representation (Lovasz, DOI: 10.1109/TIT.1979.1055985) rendering a quantum *double*.

Quasi-classical partition logics II: Two intertwining three-atomic contexts cntd.

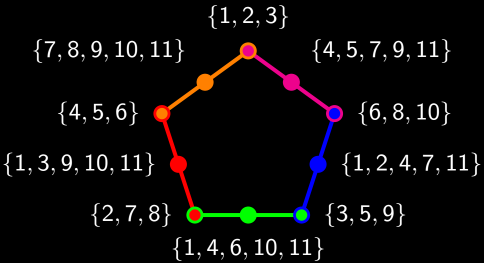
This L_{12} "firefly" logic labels the atoms (aka elementary propositions) obtained by an "inverse construction" using all two-valued measures thereon. By design, it will be very similar to the earlier logic with four atoms. With the identifications $\mathbf{e}_1 \equiv \{1,2\}$, $\mathbf{e}_2 \equiv \{3,4\}$, $\mathbf{e}_3 = \mathbf{f}_3 \equiv \{5\}$, $\mathbf{f}_1 \equiv \{1,3\}$, and $\mathbf{f}_2 \equiv \{2,4\}$ we obtain all classical probabilities by identifying $i \rightarrow \lambda_i > 0$. The respective conditional probabilities are

$$[P(\mathcal{C}_2|\mathcal{C}_1)] \equiv \begin{pmatrix} \frac{P(\{1\})}{P(\{1,2\})} & \frac{P(\{2\})}{P(\{1,2\})} & \frac{P(\emptyset)}{P(\{1,2\})} \\ \frac{P(\{3\})}{P(\{3,4\})} & \frac{P(\{4\})}{P(\{3,4\})} & \frac{P(\emptyset)}{P(\{3,4\})} \\ \frac{P(\emptyset)}{P(\{5\})} & \frac{P(\emptyset)}{P(\{5\})} & \frac{P(\{5\})}{P(\{5\})} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} & 0 \\ \frac{\lambda_3}{\lambda_3+\lambda_4} & \frac{\lambda_4}{\lambda_3+\lambda_4} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$[P(\mathcal{C}_1|\mathcal{C}_2)] \equiv \begin{pmatrix} \frac{P(\{1\})}{P(\{1,3\})} & \frac{P(\{3\})}{P(\{1,3\})} & \frac{P(\emptyset)}{P(\{1,3\})} \\ \frac{P(\{2\})}{P(\{2,4\})} & \frac{P(\{4\})}{P(\{2,4\})} & \frac{P(\emptyset)}{P(\{2,4\})} \\ \frac{P(\emptyset)}{P(\{5\})} & \frac{P(\emptyset)}{P(\{5\})} & \frac{P(\{5\})}{P(\{5\})} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_3} & \frac{\lambda_3}{\lambda_1+\lambda_3} & 0 \\ \frac{\lambda_2}{\lambda_2+\lambda_4} & \frac{\lambda_4}{\lambda_2+\lambda_4} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Quasi-classical partition logics II: Pentagon/pentagram/house logic with five cyclically intertwining three-atomic contexts

By now it should be clear how classical conditional probabilities work on partition logics. Consider the pentagon/pentagram/(orthomodular) house logic in Fig. 3 labels the atoms (aka elementary propositions) obtained by an "inverse construction" using all 11 two-valued measures thereon. take, for example, one of the two contexts $\mathcal{C}_4 = \{\{2,7,8\},\{1,3,9,10,11\},\{4,5,6\}\}$ "opposite" to the context $\mathcal{C}_1 = \{\{1,2,3\},\{4,5,7,9,11\},\{6,8,10\}\}$.



Quasi-classical partition logics II: Pentagon/pentagram/house logic with five cyclically intertwining three-atomic contexts cntd.

With the identifications $\mathbf{e}_1 \equiv \{1,2,3\}$, $\mathbf{e}_2 \equiv \{4,5,7,9,11\}$, $\mathbf{e}_3 \equiv \{6,8,10\}$, $\mathbf{f}_1 \equiv \{2,7,8\}$, $\mathbf{f}_2 \equiv \{1,3,9,10,11\}$, and $\mathbf{f}_3 \equiv \{4,5,6\}$. The respective conditional probabilities are

$$
\begin{aligned}
[P(\mathcal{C}_2|\mathcal{C}_1)] &= [P(\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}|\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\})] \\
\equiv & \begin{pmatrix}
\frac{P(\{2,7,8\}\cap\{1,2,3\})}{P(\{1,2,3\})} & \frac{P(\{1,3,9,10,11\}\cap\{1,2,3\})}{P(\{1,2,3\})} & \frac{P(\{4,5,6\}\cap\{1,2,3\})}{P(\{1,2,3\})} \\
\frac{P(\{2,7,8\}\cap\{4,5,7,9,11\})}{P(\{4,5,7,9,11\})} & \frac{P(\{1,3,9,10,11\}\cap\{4,5,7,9,11\})}{P(\{4,5,7,9,11\})} & \frac{P(\{4,5,6\}\cap\{4,5,7,9,11\})}{P(\{4,5,7,9,11\})} \\
\frac{P(\{2,7,8\}\cap\{6,8,10\})}{P(\{6,8,10\})} & \frac{P(\{1,3,9,10,11\}\cap\{6,8,10\})}{P(\{6,8,10\})} & \frac{P(\{4,5,6\}\cap\{6,8,10\})}{P(\{6,8,10\})}
\end{pmatrix} \\
= & \begin{pmatrix}
\frac{P(\{2\})}{P(\{1,2,3\})} & \frac{P(\{1,3\})}{P(\{1,2,3\})} & \frac{P(\emptyset)}{P(\{1,2,3\})} \\
\frac{P(\{7\})}{P(\{4,5,7,9,11\})} & \frac{P(\{11\})}{P(\{4,5,7,9,11\})} & \frac{P(\{4,5\})}{P(\{4,5,7,9,11\})} \\
\frac{P(\{8\})}{P(\{6,8,10\})} & \frac{P(\{10\})}{P(\{6,8,10\})} & \frac{P(\{6\})}{P(\{6,8,10\})}
\end{pmatrix} \\
= & \begin{pmatrix}
\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} & \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+\lambda_3} & 0 \\
\frac{\lambda_7}{\lambda_4+\lambda_5+\lambda_7+\lambda_9+\lambda_{11}} & \frac{\lambda_9+\lambda_{11}}{\lambda_4+\lambda_5+\lambda_7+\lambda_9+\lambda_{11}} & \frac{\lambda_4+\lambda_5}{\lambda_4+\lambda_5+\lambda_7+\lambda_9+\lambda_{11}} \\
\frac{\lambda_8}{\lambda_6+\lambda_8+\lambda_{10}} & \frac{\lambda_{10}}{\lambda_6+\lambda_8+\lambda_{10}} & \frac{\lambda_6}{\lambda_6+\lambda_8+\lambda_{10}}
\end{pmatrix}.
\end{aligned}
$$

Extrema of conditional probabilities in row and doubly stochastic matrices

The row stochastic matrices representing conditional probabilities form a polytope in $\mathbb{R}^{n^2}$ whose vertices are the n^n matrices $\mathbf{T}_i$, $i = 1, \ldots, n^n$, with exactly one entry 1 in each row. Therefore, a row stochastic matrix can be represented as the convex sum $\sum_{i=1}^{n^n} \lambda_i \mathbf{T}_i$, with nonnegative $\lambda_i \geq 0$ and $\sum_{i=1}^{n^n} \lambda_i = 1$.

For conditional probabilities yielding doubly stochastic matrices, such as, for instance, the quantum case, the Birkhoff theorem yields more restricted linear bounds: it states that any doubly stochastic $(n \times n)$–matrix is the convex hull of $m \leq (n-1)^2 + 1 \leq n!$ permutation matrices. That is, if $\mathbf{A} \equiv a_{ij}$ is a doubly stochastic matrix such that $a_{ij} \geq 0$ and $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ji} = 1$ for $1 \leq i, j \leq n$, then there exists a convex sum decomposition $\mathbf{A} = \sum_{k=1}^{m \leq (n-1)^2+1 \leq n!} \lambda_k \mathbf{P}_k$ in terms of $m \leq (n-1)^2 + 1$ linear independent permutation matrices $\mathbf{P}_k$ such that $\lambda_k \geq 0$ and $\sum_{k=1}^{m \leq (n-1)^2+1 \leq n!} \lambda_k = 1$.

Thank you for your attention!