VIENNA UNIVERSITY OF TECHNOLOGY

INSTITUTE OF MECHANICS AND MECHATRONICS
TECHNICAL ACOUSTICS

Advanced finite element formulation for viscothermal acoustics

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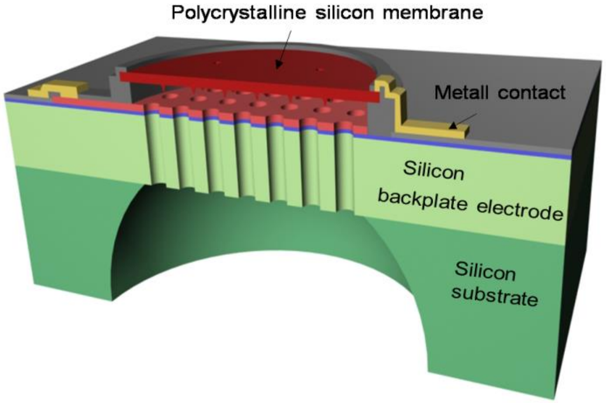
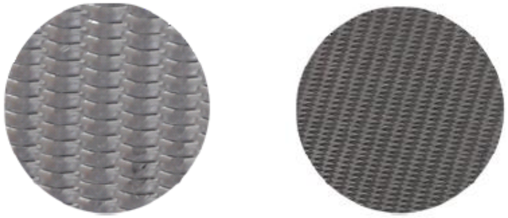
- MEMS



- Micro-Perforated-Panel Absorber



air, $f = 1\,\mathrm{kHz}$

$$\lambda = \frac{c}{f} \approx 0.34\,\mathrm{m}$$

$$\delta_{\mathrm{vis}} = \sqrt{\frac{\mu}{\pi f \rho}} \approx 68.9\,\mu\mathrm{m} \ \dots \ \text{viscous boundary layer}$$

$$\delta_{\mathrm{th}} = \sqrt{\frac{\gamma}{\pi f \rho C_{\mathrm{p}}}} \approx 63.6\,\mu\mathrm{m} \ \dots \ \text{thermal boundary layer}$$

μ	...	dynamic viscosity
γ	...	heat conductivity
ρ	...	density
c_{p}	...	specific heat at constant pressure
f	...	frequency

- Compressible flow equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

… conservation of mass

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot [\boldsymbol{\tau}]$$

… conservation of momentum

$$\rho T \frac{Ds}{Dt} = [\boldsymbol{\tau}] : \nabla \boldsymbol{v} - \nabla \cdot \boldsymbol{q}_{\mathrm{T}}$$

… conservation of energy

$$\lambda(\nabla \cdot \boldsymbol{v})\,[\boldsymbol{I}] + 2\mu[\boldsymbol{\epsilon}]$$

$$-\gamma \nabla T$$

$$[\boldsymbol{\epsilon}] = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla^t \boldsymbol{v}\right)$$

ρ	…	density	s	…	entropy
$\boldsymbol{v}$	…	velocity	T	…	temperature
p	…	pressure	$\boldsymbol{q}_{\mathrm{T}}$	…	heat flux
$[\boldsymbol{\tau}]$	…	viscous stress tensor	μ	…	dynamic viscosity
λ	…	bulk viscosity	γ	…	heat conductivity

- Relations for ideal gas

specific heat at const. pressure

$$\mathrm{d}h = T\,\mathrm{d}s + \frac{\mathrm{d}p}{\rho}\,;\;\; p = \rho R T\,; h = c_\mathrm{p} T\,;\;\; \frac{p}{p_0} = \frac{\rho}{\rho_0} + \frac{T}{T_0}\,;$$

enthalpy

- Perturbation ansatz

$$\boldsymbol{v} = \boldsymbol{v}'\,;\;\; p = p_0 + p'\,;\;\; \rho = \rho_0 + \rho'\,; T = T_0 + T'$$

$$\frac{1}{p_0}\frac{\partial p'}{\partial t} + \nabla\cdot\boldsymbol{v}' - \frac{1}{T_0}\frac{\partial T'}{\partial t} = 0$$

$$\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} - (\lambda + \mu)\nabla\nabla\cdot\boldsymbol{v}' - \mu\Delta\boldsymbol{v}' - \nabla p' = \boldsymbol{0}$$

$$\rho_0 c_\mathrm{p} \frac{\partial T'}{\partial t} - \nabla\cdot\gamma\nabla T' - \frac{\partial p'}{\partial t} = 0$$

- Fluid stress and thermal flux

$$\begin{aligned}[\boldsymbol{\sigma}'] &= -p'[\boldsymbol{I}] + [\boldsymbol{\tau}'] = -p'[\boldsymbol{I}] + \lambda\nabla\cdot\boldsymbol{v}'[\boldsymbol{I}] + \mu\left(\nabla\boldsymbol{v}' + \nabla^t\boldsymbol{v}'\right) \\ \boldsymbol{q}' &= -\gamma_{\mathrm{T}}\nabla T'\end{aligned}$$

Find $(\boldsymbol{v}', p', T') \in (\boldsymbol{X}, Y, Z)$ such that

$$\begin{aligned}\int_\Omega \cancel{\rho_0 \boldsymbol{w}\cdot\frac{\partial \boldsymbol{v}'}{\partial t}}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla\boldsymbol{w} : [\boldsymbol{\sigma}']\,\mathrm{d}\boldsymbol{x} - \int_\Gamma \boldsymbol{w}\cdot[\boldsymbol{\sigma}']\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} &= 0 \\ \int_\Omega \rho_0\nabla\cdot\boldsymbol{v}'\,\mathrm{d}\boldsymbol{x} + \cancel{\int_\Omega \frac{1}{\rho_0}\frac{\partial p'}{\partial t}\,\mathrm{d}\boldsymbol{x}} - \cancel{\int_\Omega \frac{1}{T_0}\frac{\partial T'}{\partial t}\,\mathrm{d}\boldsymbol{x}} &= 0 \\ \cancel{\int_\Omega \rho_0 c_{\mathrm{p}}\varphi\frac{\partial T'}{\partial t}\,\mathrm{d}\boldsymbol{x}} + \int_\Omega \gamma\nabla\varphi\cdot\nabla T'\,\mathrm{d}\boldsymbol{x} - \cancel{\int_\Omega \varphi\frac{\partial p'}{\partial t}\,\mathrm{d}x} - \int_\Gamma \varphi\boldsymbol{q}'\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} &= 0\end{aligned}$$

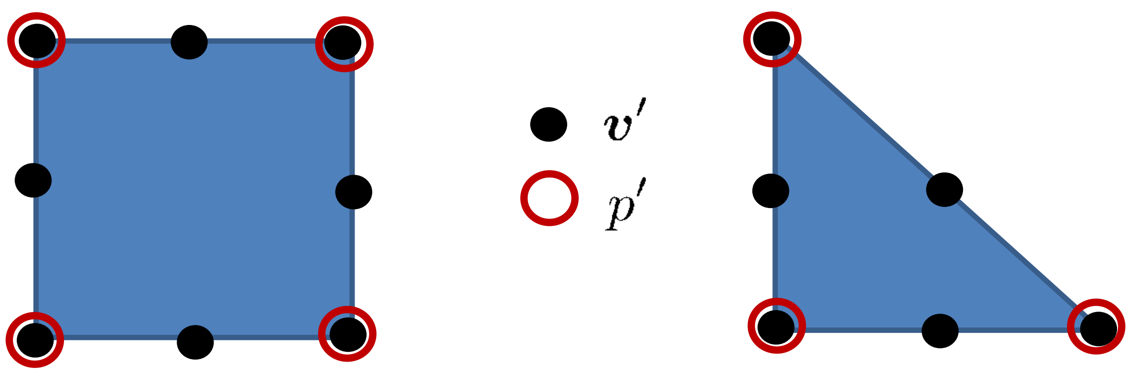
for all $(\boldsymbol{w}, \eta, \varphi) \in (\boldsymbol{X}', Y', Z')$.

Static case: Stokes problem and decoupling to heat PDE!

- Condition for well posedness of the weak formulation

  LBB (Ladyzhenskaya-Babuška-Breezi) condition (inf-sup condition)

  Use of Taylor – Hood elements: $P_{k+1}/P_k\ ;\ Q_{k+1}/Q_k$



- Discretization fluctuation temperature, T': $P_{k+1},\ Q_{k+1}$

Stokes boundary layer

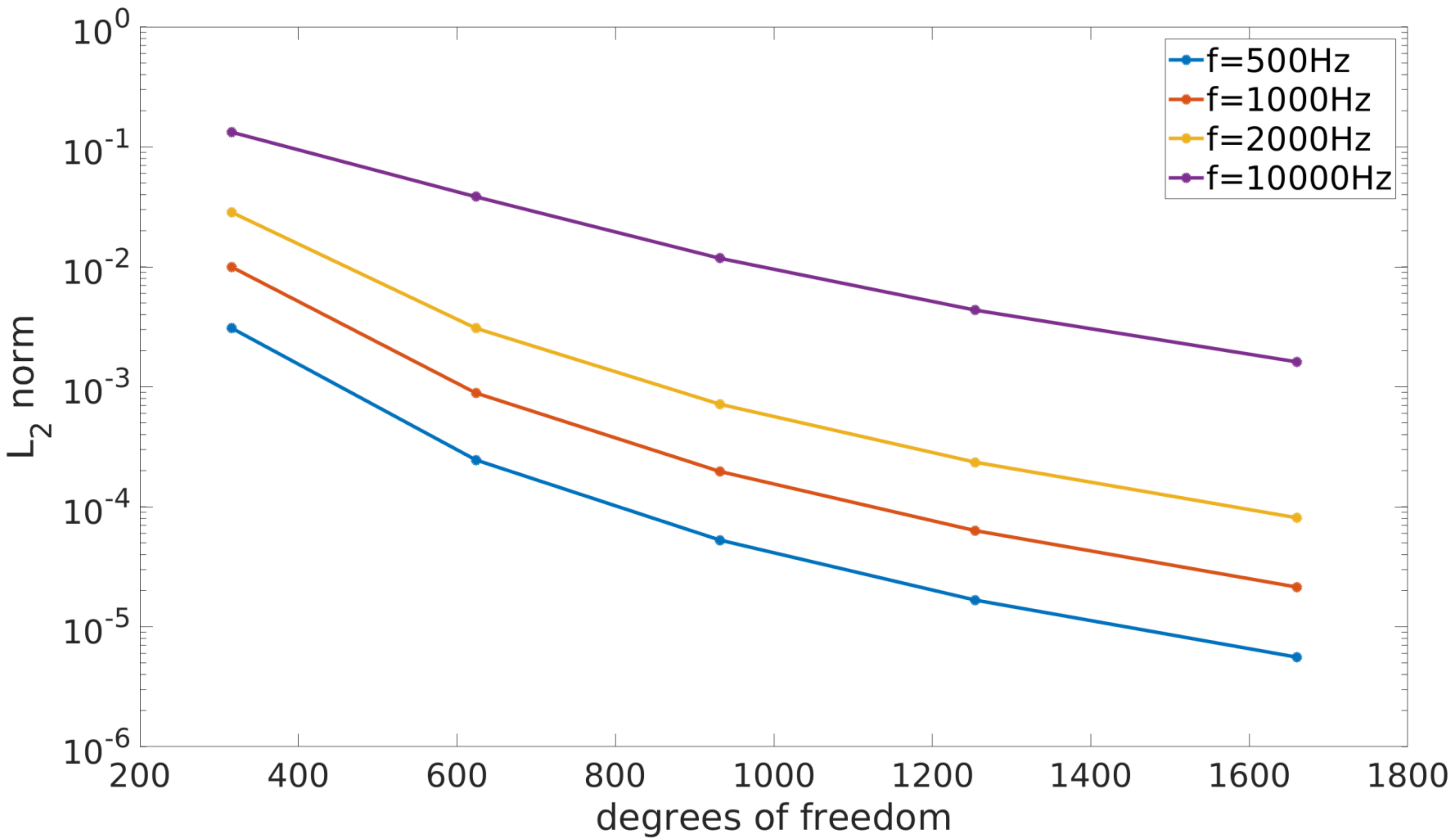
- Consider an infinitely long plate, which is oscillating with a velocity $U\cos(\omega t)$ in the x-direction, located at $y = 0$ in an infinite domain of fluid.

$$v_y = U\,\mathrm{Re}\left\{e^{i\omega t}\; e^{-\frac{1+i}{\sqrt{2}}\sqrt{\frac{\rho\omega}{\mu}}\,y}\right\}$$

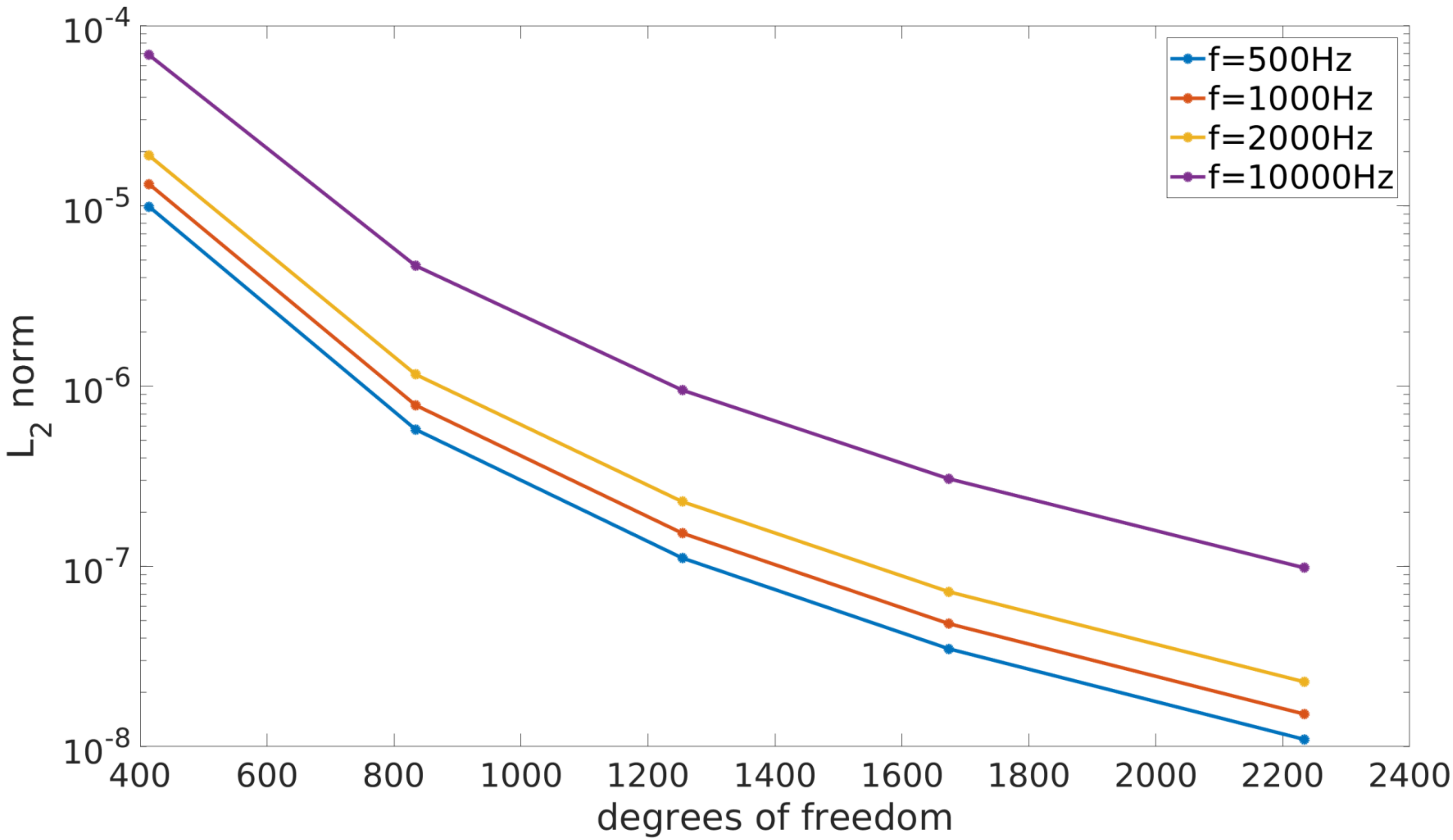
- Meshes
  - Uniform

  - Graded

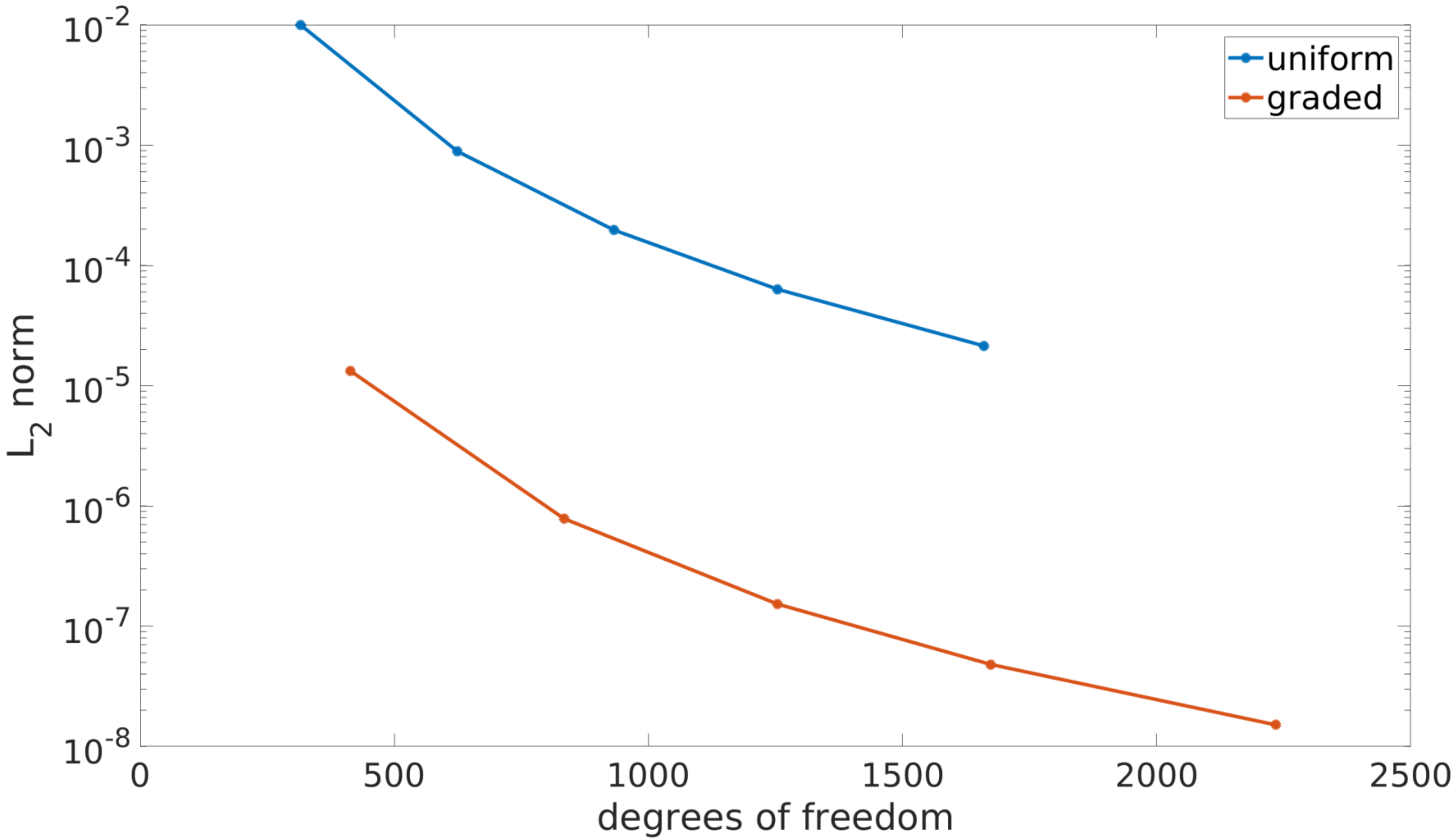
- Uniform mesh (h-refinement)



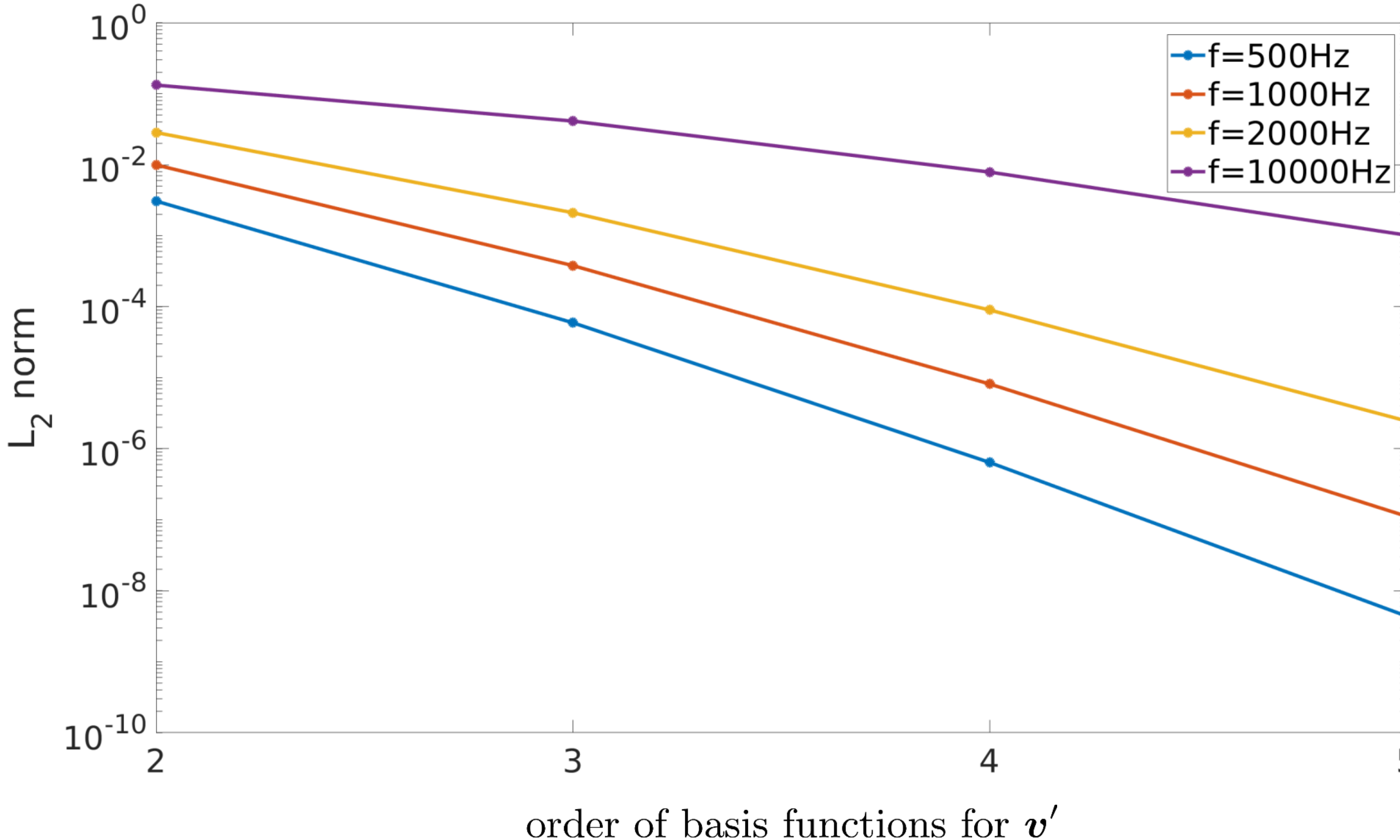
- Graded mesh (h-refinement)



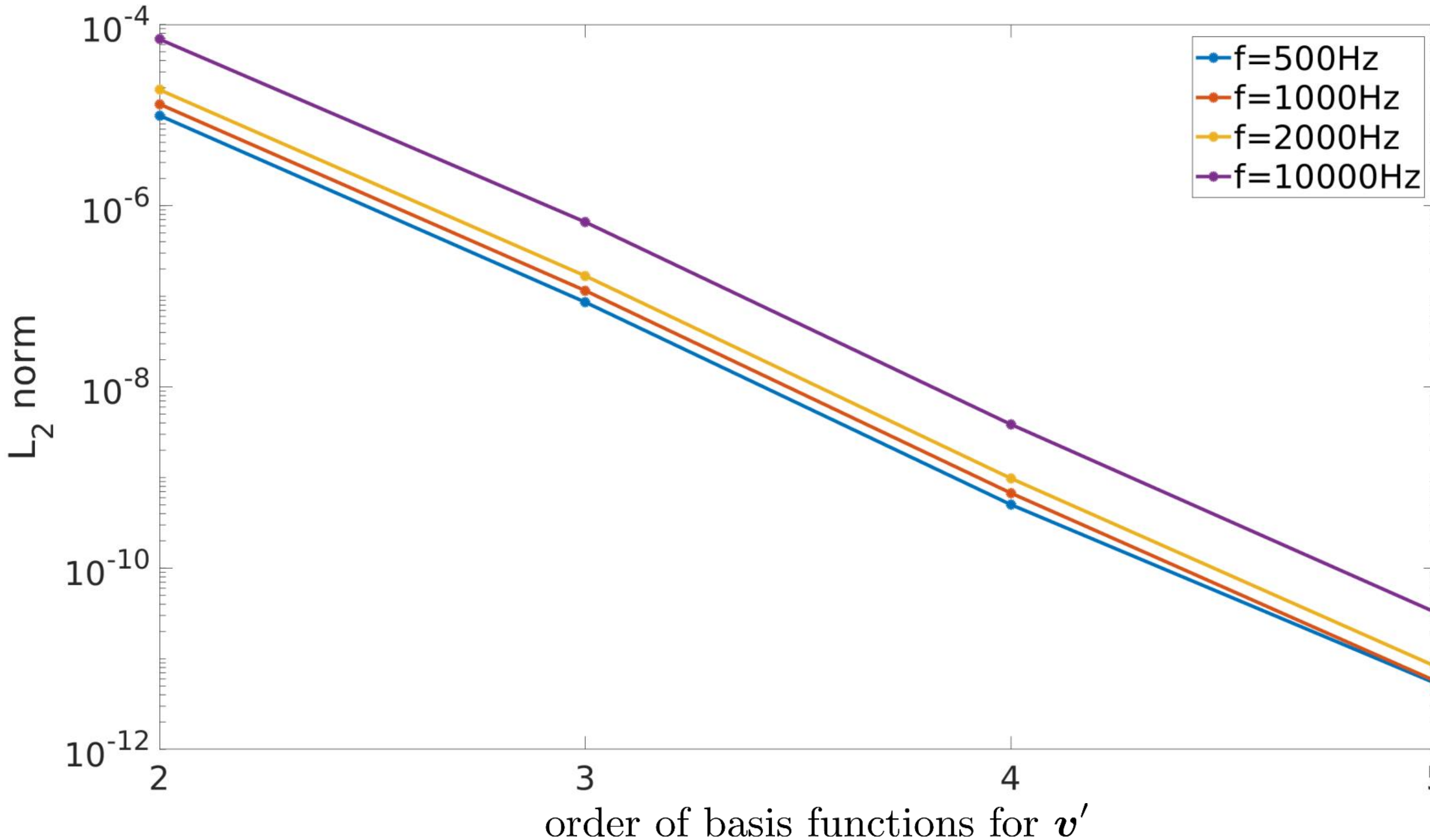
- Uniform and graded mesh refinement for *f*=1000Hz



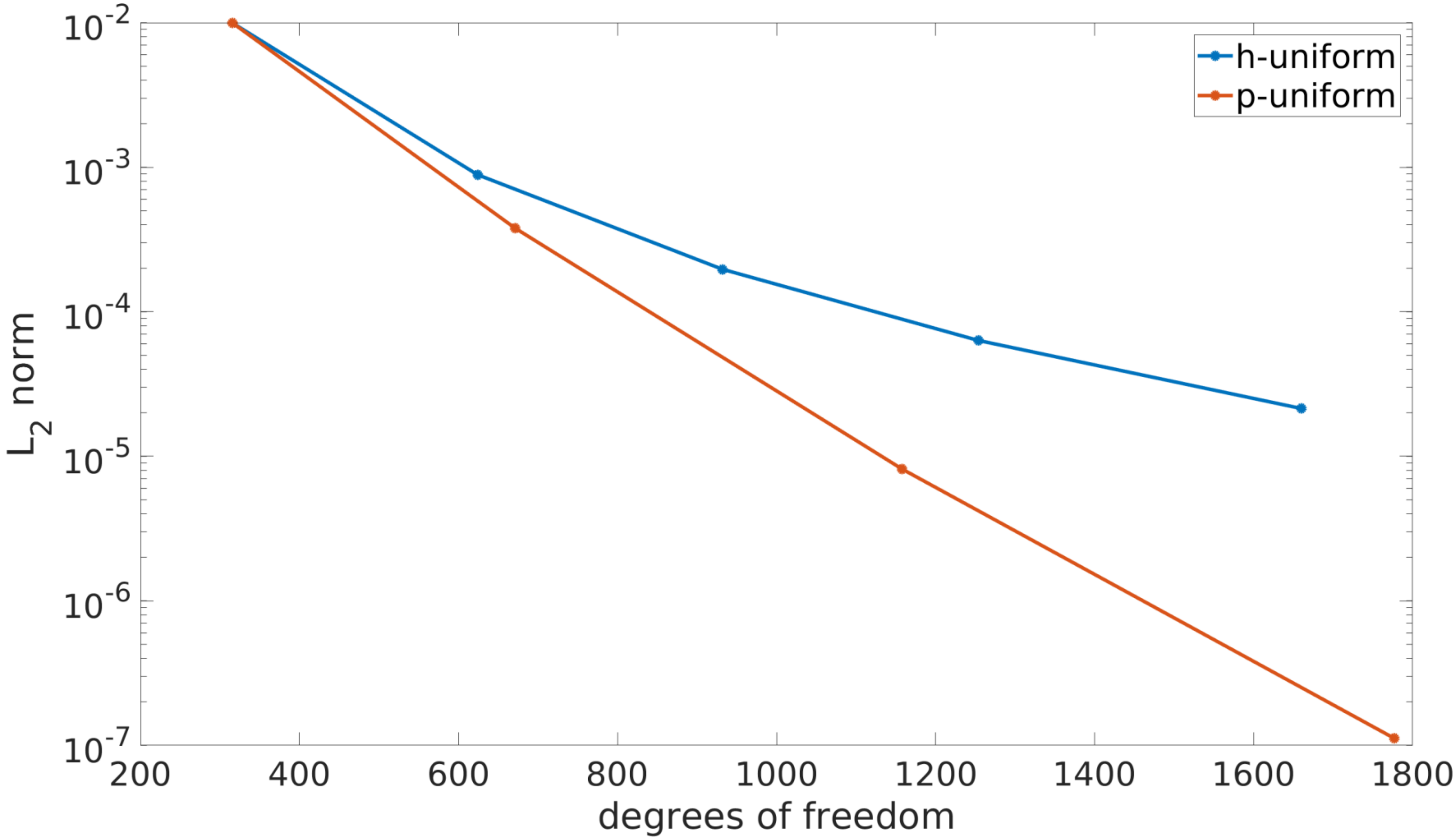
- pFEM at uniform mesh



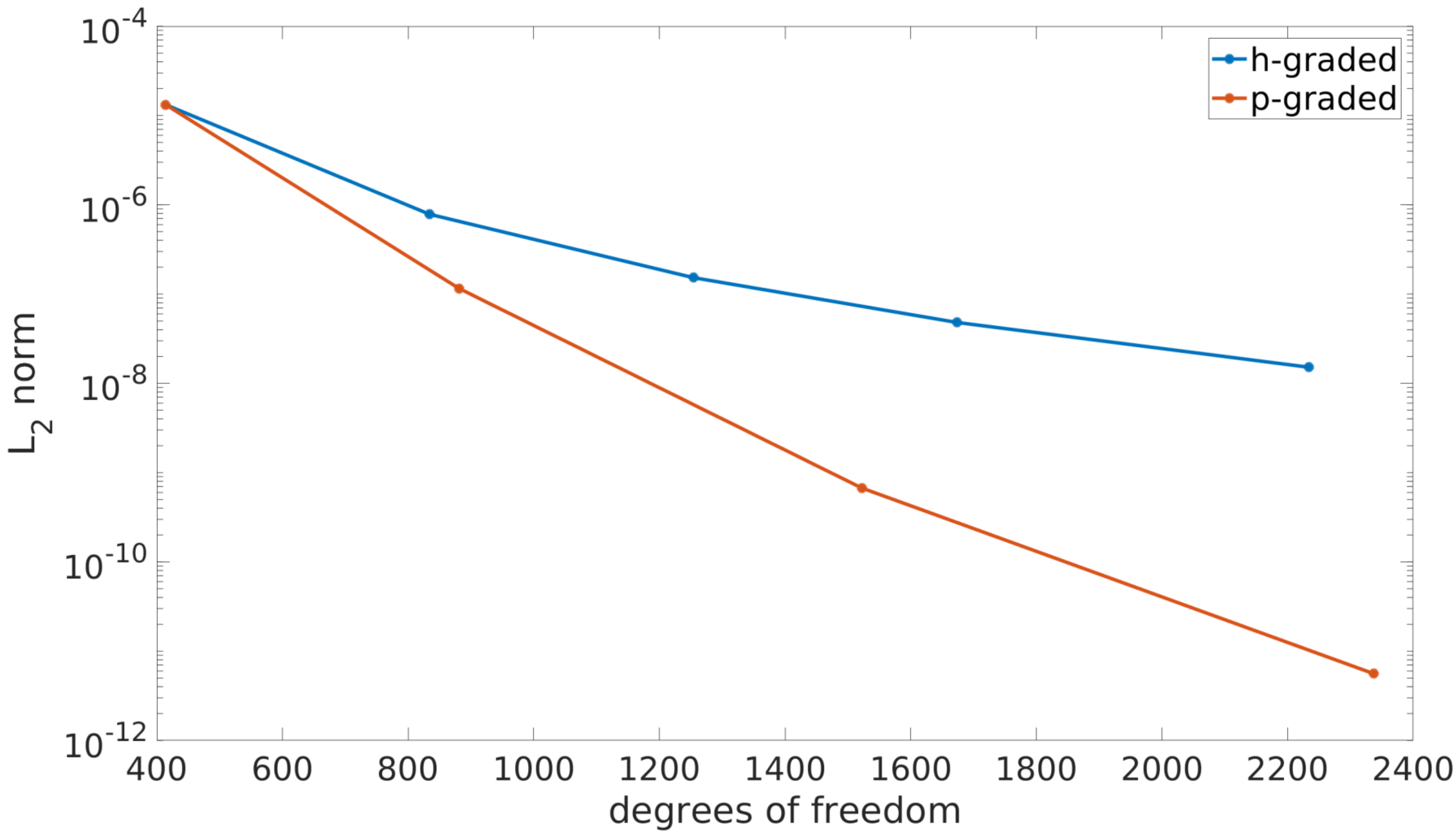
- pFEM at graded mesh



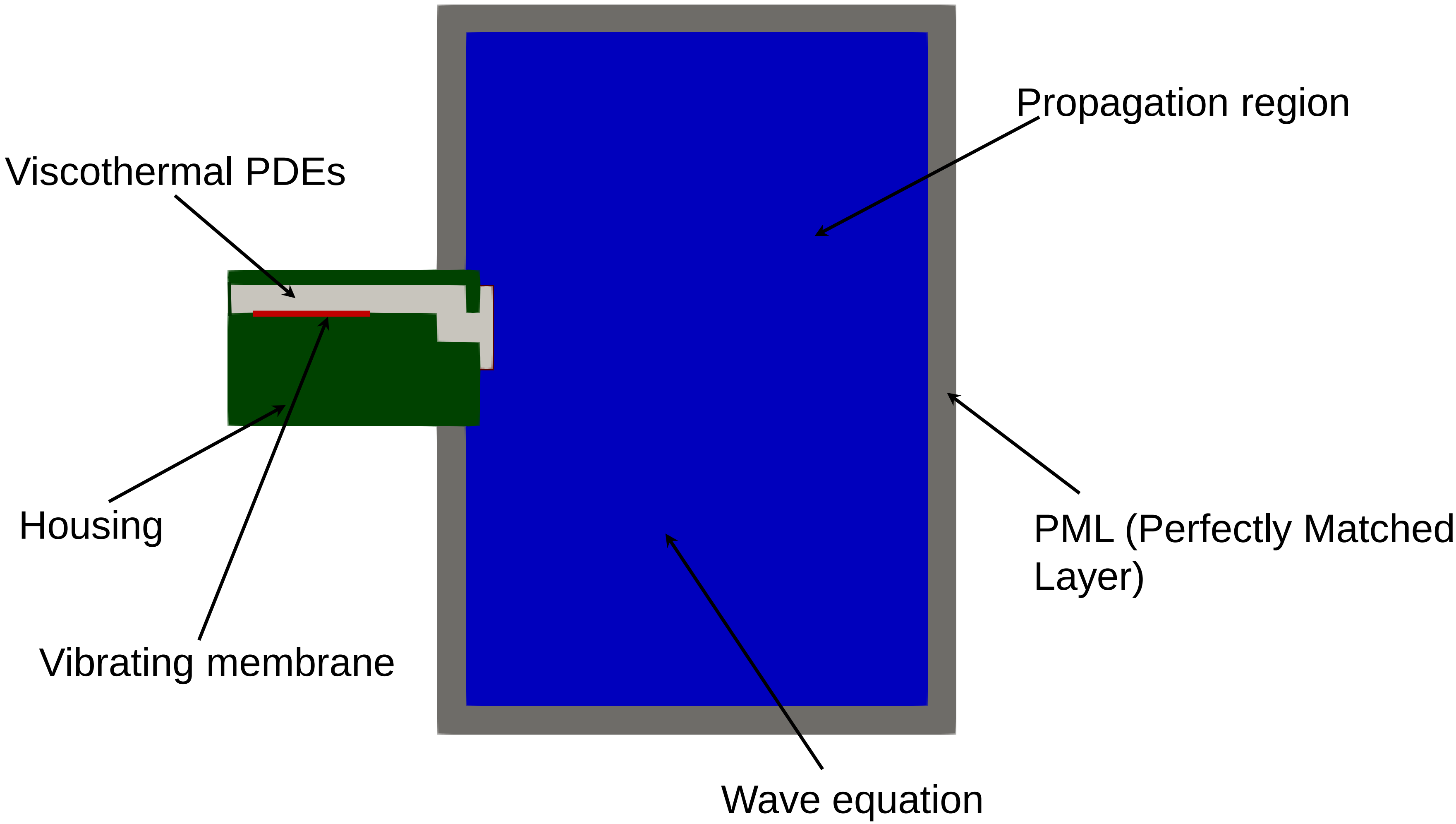
- hFEM / pFEM for uniform mesh for f=1000Hz



- hFEM / pFEM for graded mesh for f=1000Hz



- Computational setup



- Coupling between structural mechanics and viscothermal acoustics
  - Transmission condition I: continuity of surface traction

$$[\boldsymbol{\sigma}_{\mathrm{m}}] \cdot \boldsymbol{n} = [\boldsymbol{\sigma}'] \cdot \boldsymbol{n} = \boldsymbol{\lambda}$$

Cauchy stress tensor

Introduction of a Lagrange multiplier!

  - Transmission condition II: continuity of velocities

$$\boldsymbol{v}_{\mathrm{m}} = \boldsymbol{v}'$$

$$\int_{\Gamma_{\mathrm{I}}} \left(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v}' \right) \cdot \boldsymbol{\xi} \, \mathrm{d}\boldsymbol{s}$$

… fulfilled in a weak sense!

- Coupling between viscothermal acoustics and standard wave equation
  - Transmission condition I: continuity of surface traction

$$[\boldsymbol{\sigma}' \cdot \boldsymbol{n} = [\boldsymbol{\sigma}_\mathrm{a}] \cdot \boldsymbol{n}$$

$$= -p_\mathrm{a}[\boldsymbol{I}]$$

$$\left(-p'[\boldsymbol{I}] + \lambda \nabla \cdot \boldsymbol{v}'[\boldsymbol{I}] + \mu \left(\nabla \boldsymbol{v}' + \nabla^t \boldsymbol{v}' \right) \right)$$

Acoustic pressure enters weak form of viscothermal equation!

  - Transmission condition II: continuity of velocities in normal direction

$$\boldsymbol{v}' \cdot \boldsymbol{n} = \boldsymbol{v}_\mathrm{a} \cdot \boldsymbol{n}$$

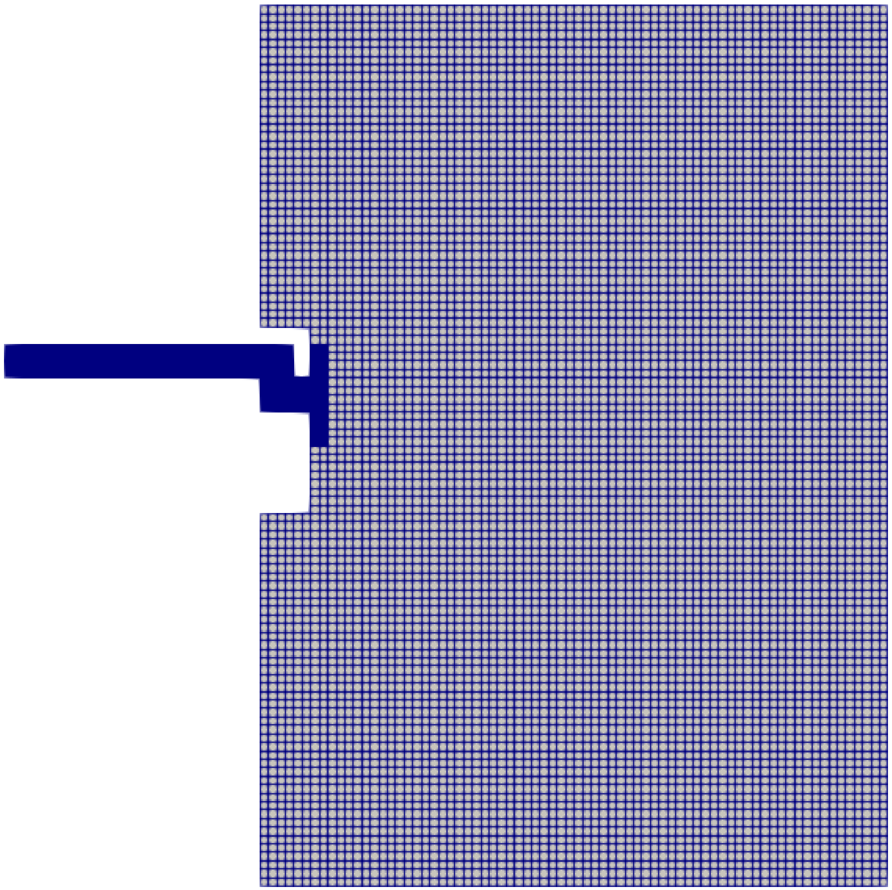
$$\nabla p_\mathrm{a} = -\rho_0 \frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t}$$

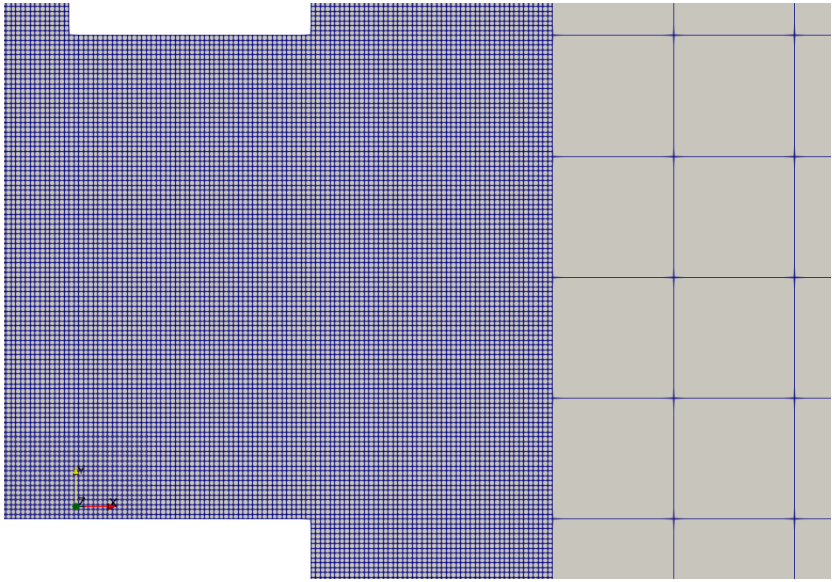
.. Lineariszed momentum conservation for acoustics

$$\frac{\partial p_\mathrm{a}}{\partial \boldsymbol{n}} = -\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} \cdot \boldsymbol{n}$$

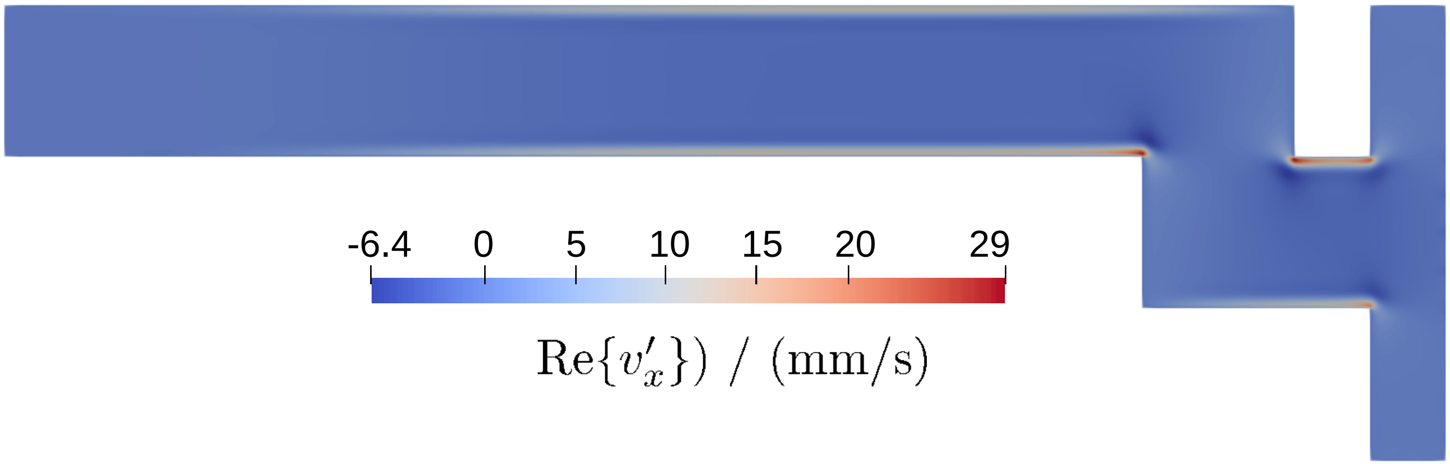
Viscothermal velocity entrers weak form of wave eqation!

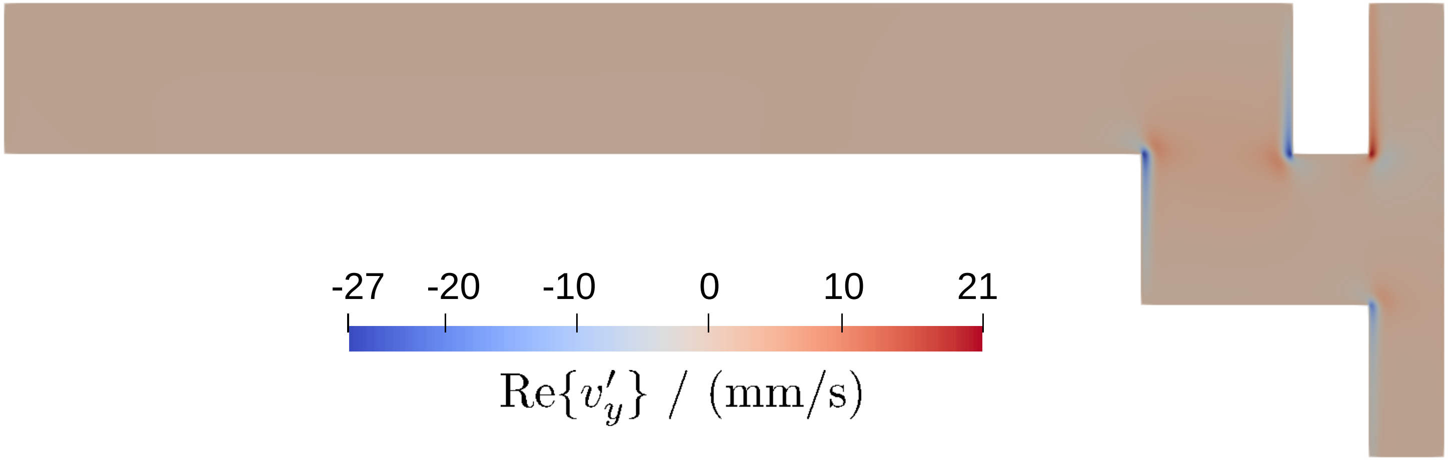
- Computational mesh: non-conforming interface



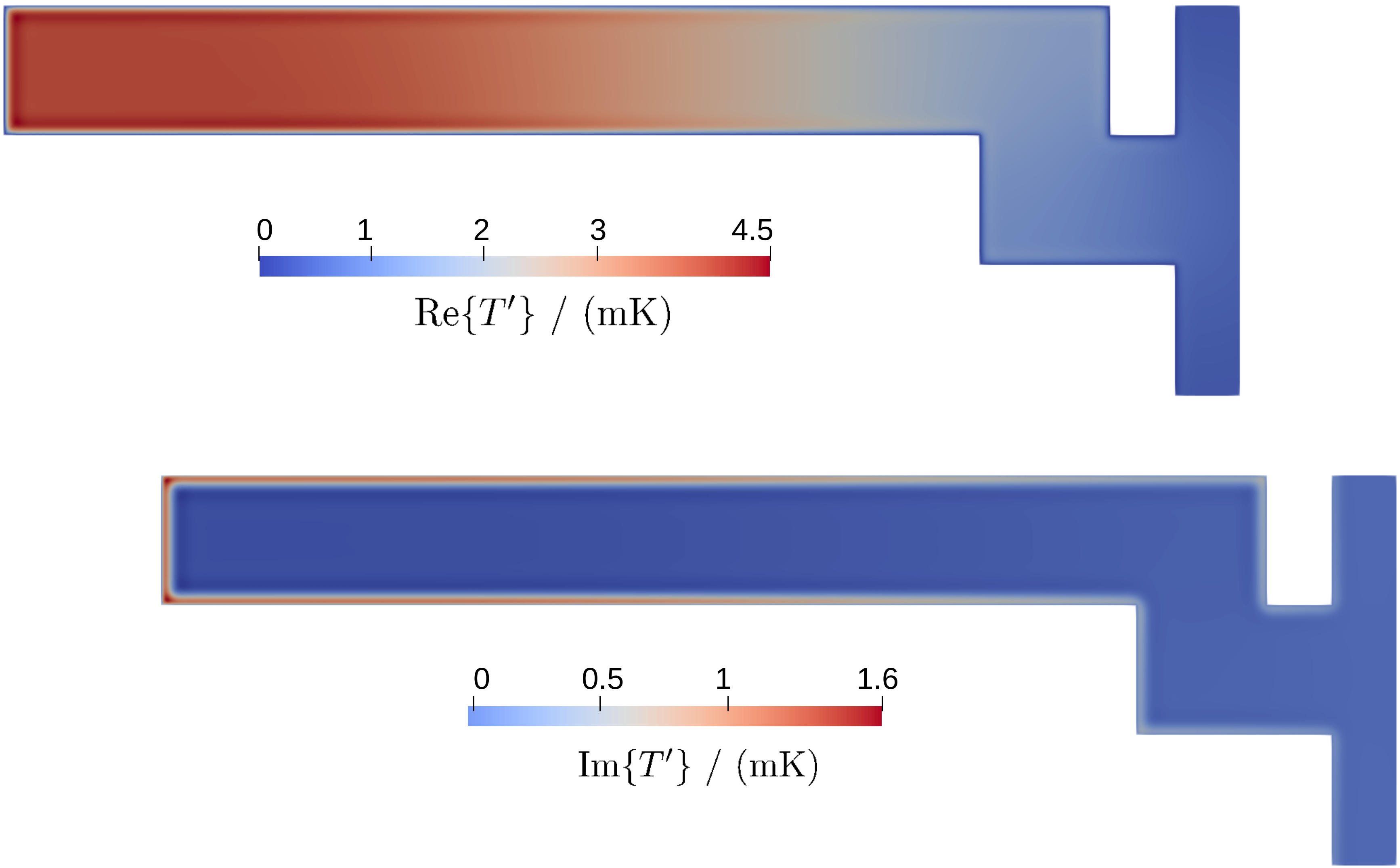


- Velocity at 1kHz

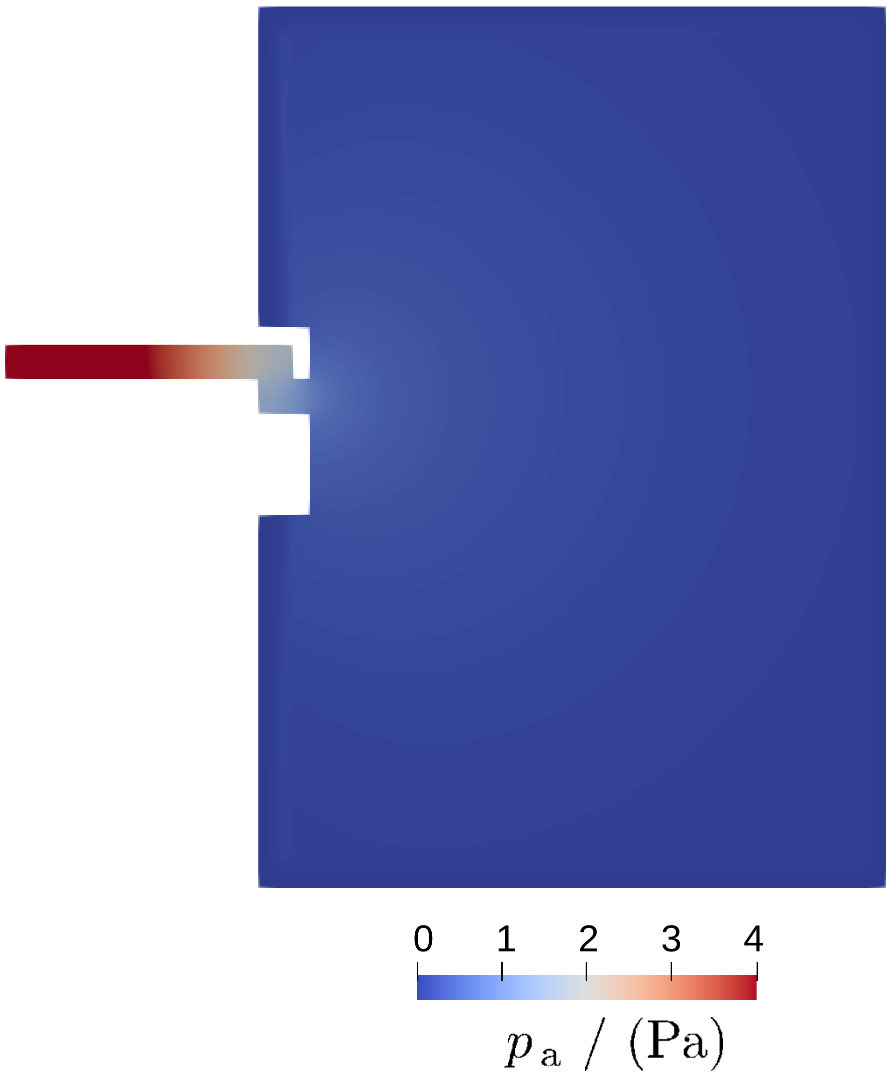




- Temperature rise at 1kHz



- Acoustic pressure at 1kHz



Conclusion / Outlook

- Viscothermal formulation using Taylor – Hood elements

- Coupling to (on non-conforming grids)
  - Structural mechanics
  - Acoustic wave equation

CFS++ Coupled Field Simulation

http://cfs-doc.mdmt.tuwien.ac.at

- Outlook
  - Development of a boundary layer condition for standard wave equation

WAVES 2019 Conference – Vienna, Austria: August 25 – 30, 2019

14th International Conference on Mathematical and Numerical Aspects of Wave Propagation