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Multiple scales analysis of the undular hydraulic jump over horizontal surfaces

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The basic equations for near-critical stationary turbulent open-channel flow over horizontal surfaces are solved by means of a multiple scales analysis. The multiple scales solution describing the first-order perturbation of the free-surface elevation is compared with the numerical solution of an extended steady-state version of the Korteweg–de Vries (KdV) equation, confirming the uniform validity of this non-linear third-order ODE. In comparison with experimental data the numerical solution of the extended KdV equation shows strong dependence on the initial curvature in terms of wavelength and amplitude.

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1 Introduction and problem description

The undular hydraulic jump is a peculiar but well known event occurring in slightly supercritical free-surface flows, characterized by a wavy (undular) surface shape with slowly changing amplitudes and lengths of the undulations, in hydraulics commonly called "wavelengths". The present work considers stationary turbulent open-channel flow over horizontal surfaces. This problem is governed by the continuity equation for incompressible flow and the Reynolds-averaged Navier–Stokes equations for the defect layer. Conventional kinematic and dynamic boundary conditions at the free surface are imposed. An asymptotic analysis of the governing equations in non-dimensional form is performed in the double limit of upstream Froude numbers $\mathrm{Fr} := \bar{u}_\mathrm{r}/\sqrt{g\bar{h}_\mathrm{r}}$ approaching the critical value 1, i.e., $(2/3)(\mathrm{Fr}-1) = \varepsilon \to 0$, and very small friction Froude numbers $\mathrm{Fr}_\tau := u_{\tau,\mathrm{r}}/\sqrt{g\bar{h}_\mathrm{r}} \to 0$, which is equivalent to very large Reynolds numbers. Here $\bar{u}_\mathrm{r}$, $u_{\tau,\mathrm{r}}$ and $\bar{h}_\mathrm{r}$ denote the velocity, friction velocity and free-surface elevation in the reference state, respectively, and g denotes the acceleration due to gravity.

In contrast to [1–3] where near-critical flow over an *inclined* bottom was considered, for the present case of a *horizontal* bottom the deviation of the reference state from fully developed flow cannot be assumed to be small. Incorporating this fact and aiming at an analysis that is free of turbulence-modeling leads to the coupling between the two small parameters ε and Fr_τ according to $\mathrm{Fr}_\tau^2 = B\varepsilon^3$ with $B = \mathrm{const} = O(1)$. This particular coupling represents an essential modification of the analysis in [2], where $\mathrm{Fr}_\tau^2 = O(\varepsilon^2)$ was assumed.

2 Multiple scales analysis and uniformly valid differential equation

A multiple scales analysis of the governing equations following [2], i.e., substituting the original contracted longitudinal coordinate X by a fast and a slow variable ξ and Ω, respectively, and introducing the spatially slowly changing "wave number" $\omega(\Omega) = \mathrm{d}\xi/\mathrm{d}X = \varepsilon^{-1/2}\,\mathrm{d}\Omega/\mathrm{d}X$ leads to a separation of the orders ε, $\varepsilon^{3/2}$, ε^2 and $\varepsilon^{5/2}$. As results of the orders ε and $\varepsilon^{3/2}$ the velocities in longitudinal and vertical direction and the pressure are expressed in terms of the yet unknown perturbations of the surface elevation $H_1(\xi,\Omega)$ and $H_{3/2}(\xi,\Omega)$. Investigation of the following orders, i.e., ε^2 and $\varepsilon^{5/2}$, leads to solvability conditions for H_1 and $H_{3/2}$. Combination of the solvability conditions gives the following results for the first-order perturbation of the free-surface elevation and the wave number:

$$H_1(\xi,\Omega) = h_2 + (h_3 - h_2)\mathrm{cn}^2\left[2K(m)(\xi-\xi_0)|m\right], \qquad \omega(\Omega) = \frac{\sqrt{h_3-h_1}}{4\sqrt{3}K(m)}, \tag{1}$$

where cn is the cnoidal Jacobian elliptic function, $K(m)$ denotes the complete elliptic integral of first kind with the parameter m defined by $m = (h_3-h_2)/(h_3-h_1)$. The constant of integration, ξ_0, is chosen such that $H_1(\xi=0,\Omega=0) = H_1(X=0) = 0$. The slowly varying auxiliary variables $h_1(\Omega)$, $h_2(\Omega)$ and $h_3(\Omega)$ are determined by the solution of:

$$\frac{\mathrm{d}h_1}{\mathrm{d}\Omega} = \frac{4B}{\sqrt{3}}\frac{I_1 - I_0 h_1}{(h_1-h_2)(h_1-h_3)}, \qquad \frac{\mathrm{d}h_2}{\mathrm{d}\Omega} = \frac{4B}{\sqrt{3}}\frac{I_1 - I_0 h_2}{(h_2-h_3)(h_2-h_1)}, \qquad \frac{\mathrm{d}h_3}{\mathrm{d}\Omega} = \frac{4B}{\sqrt{3}}\frac{I_1 - I_0 h_3}{(h_3-h_1)(h_3-h_2)}, \tag{2}$$

where the functions $I_0 = 2K(m)/\sqrt{h_3-h_1}$ and $I_1 = 2[h_1K(m) + (h_3-h_1)E(m)]/\sqrt{h_3-h_1}$ can be expressed in terms of $K(m)$ and $E(m)$, i.e., the complete elliptic integral of first and second kind, respectively, cf. [4].

Following [2], it is then shown that the extended steady-state version of the Korteweg–de Vries (KdV) equation,

$$H_{1XXX} + H_{1X}(H_1 - 1) = -\gamma, \tag{3}$$

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is *uniformly valid*, i.e., valid on both fast and slow scales. Subscripts X denote derivatives with respect to the original longitudinal coordinate. The right hand side of Eq. (3) represents the extension of the KdV equation by a constant damping term $\gamma = \mathrm{Fr}_\tau^2/(9\varepsilon^{5/2})$.

In Fig. 1 the multiple scales solution according to Eq. (1) is compared with the numerical solution of Eq. (3), subject to equal initial conditions and parameter values. Both solutions are in excellent agreement for quite a large distance. At $X \approx 21$ the auxiliary variables h_1 and h_2 coalesce. This point confines the region of possible multiple scales solutions. Shortly before, the numerical solution of Eq. (3) deviates from the multiple scales solution and experiences a breakdown, approaching the singular point X_s as $H_1 \sim -12/(X_s - X)^2$.

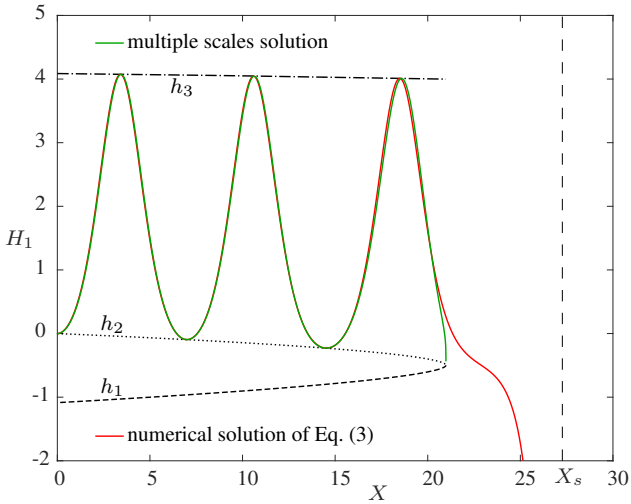

Fig. 1: Comparison of a numerical solution of Eq. (3) with the corresponding multiple scales solution, Eq. (1), for $\mathrm{Fr} = 1.015$ and $\mathrm{Fr}_\tau = 7.1 \cdot 10^{-4}$, i.e., $\varepsilon = 0.01$, $\gamma = 0.0056$. Initial conditions: $H_1(0) = 0$, $H_{1X}(0) = \gamma$, $H_{1XX}(0) = 0.74$.

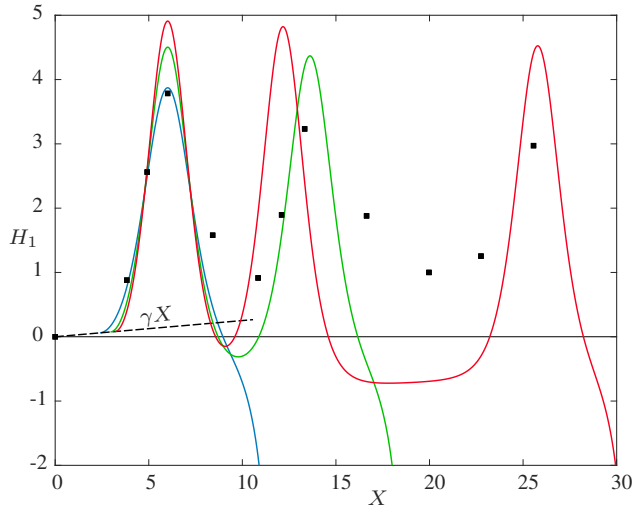

Fig. 2: Comparison of experimental data [5] (black squares) for $\mathrm{Fr} = 1.266$ and $\mathrm{Fr}_\tau = 0.055$ (i.e., $\varepsilon = 0.177$, $\gamma = 0.025$) with numerical solutions of Eq. (3). Initial conditions: $H_1(0) = 0$, $H_{1X}(0) = \gamma$; blue: $H_{1XX}(0) = 0.57$, green: $H_{1XX}(0) = 1.1$, red: $H_{1XX}(0) = 1.513$.

3 Comparison of experimental data with solutions of the extended KdV equation

In Fig. 2 numerical solutions of Eq. (3) are compared with surface elevation measurements of the undular hydraulic jump experiment CD1 described in [5]. Unfortunately, the Froude number of the experiment leads to a relatively large perturbation parameter, i.e., $\varepsilon = 0.177$. Thus, only modest agreement between measurements and theoretical predictions can be expected. Numerical solutions of Eq. (3) are obtained by keeping the initial values of the surface elevation and surface slope fixed, while altering the initial curvature of the surface, i.e., $H_{1XX}(0)$. The resulting curves are shifted along the straight line $H_1 = \gamma X$ such that the first wave crest corresponds to the experimental data. The solution for the lowest initial curvature, that is, the blue curve, shows good agreement in terms of the first wave amplitude but breaks down immediately afterwards. Increasing $H_{1XX}(0)$ leads to the development of a second crest (green curve). A solution with three crests, reasonably approximating the wavelengths between first and second as well as second and third crests, is found with $H_{1XX}(0) = 1.513$, i.e., the red curve. However, for the latter case the predicted amplitudes are too large.

The comparisons of Fig. 2 lead to the conclusion that the solution of Eq. (3) strongly depends on the initial curvature. Moreover, the theories for inclined bottom [1–3] show better agreement with experiments. Thus, an investigation of the limiting case of the bottom slope tending to zero, while Fr and Fr_τ remain fixed, seems necessary to gain better understanding of the undular hydraulic jump over horizontal surfaces.

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