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On the parameterized complexity of (k, s)-SAT

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ABSTRACT

Let (k, s)-SAT be the k-SAT problem restricted to formulas in which each variable occurs in at most s clauses. It is well known that $(3, 3)$-SAT is trivial and $(3, 4)$-SAT is NP-complete. Answering a question posed by Iwama and Takaki (DMTCS 1997), Berman, Karpinski and Scott (DAM 2007) gave, for every fixed $t \geq 0$, a polynomial-time algorithm for $(3, 4)$-SAT restricted to formulas in which the number of variables that occur in four clauses is t. Parameterized by t, their algorithm runs in XP time. We extend their result by giving, for every $k \geq 3$ and $s \geq k$, an FPT algorithm for (k, s)-SAT when parameterized by the number t of variables occurring in more than k clauses.

1. Introduction

In this note we consider some special variant of the Satisfiability problem from a parameterized point of view. In order to define it we first give the necessary terminology. A *literal* is a (propositional) variable x or a negated variable $\overline{x}$. A set S of literals is *tautological* if $S \cap \overline{S} \neq \emptyset$, where we write $\overline{S} = \{\overline{x} \mid x \in S\}$. A *clause* is a finite non-tautological set of literals. A *(CNF) formula* is a finite set of clauses. For $k \geq 1$, a *k-CNF formula* is a formula in which each clause contains exactly k different literals. A variable x *occurs* in a clause C if $x \in C$ or $\overline{x} \in C$. For $k, s \geq 1$, a *(k, s)-formula* is a k-CNF formula in which each variable occurs in at most s clauses. A variable is *k-exceeding* if it occurs in more than k clauses. A *truth assignment* for a set X of variables is a mapping $\tau : X \to \{0, 1\}$. In order to define τ on literals we set $\tau(\overline{x}) = 1 - \tau(x)$. A truth assignment τ *satisfies* a clause C if C contains at least one literal x with $\tau(x) = 1$, and τ *satisfies* a formula F if it satisfies every clause of F. In the latter case we call F *satisfiable*.

The Satisfiability problem (SAT) is to decide whether a given formula is satisfiable. For $k \geq 3$, the k-SAT problem is the restriction of SAT to k-CNF formulas. It is well known and readily seen that 2-SAT is polynomial-time solvable, whereas 3-SAT is NP-complete [10]. This led to numerous studies on further restrictions and variants of SAT. We focus on the (k, s)-SAT problem, which is the restriction of k-SAT to (k, s)-formulas. We say that (k, s)-SAT is *satisfiable* if every (k, s)-formula is satisfiable. Tovey proved the following.

Theorem 1 *([14]).* $(3, 3)$-SAT *is satisfiable and* $(3, 4)$-SAT *is* NP-*complete.*

Dubois [4] extended Theorem 1 by proving that if (k, s)-SAT is satisfiable, then (k', s')-SAT is satisfiable for every $k' = k + \ell$ and $s' \leq s + \ell \cdot [\frac{s}{k}]$ (where $[x]$ denotes the integral part of a number x). This result, combined with Theorem 1, implies that (k, k)-SAT is satisfiable for every $k \geq 1$. Kratochvíl, Savický and Tuza [11] extended Theorem 1 by proving that there exists a natural function f

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(that grows exponentially) such that (k,s)-SAT is satisfiable if $s \le f(k)$ and NP-complete if $s \ge f(k)+1$. Exact values of $f(k)$ are only known for very small values of k [1,7], but the asymptotic behaviour has been settled as $f(k) = \Theta(\frac{2^k}{k})$ by Gebauer [6]. Iwama and Takaki [9] proved that every $(3,4)$-formula with at most three 3-exceeding variables is satisfiable, and they also gave an unsatisfiable $(3,4)$-formula with nine 3-exceeding variables. Answering a question of Iwama and Takaki [9], Berman, Karpinski and Scott proved the following result.

Theorem 2 *([2]). $(3,4)$-SAT can be solved in $2^{\frac{t}{3}} n^{\frac{t}{3}}\,\mathrm{poly}(n)$ time on $(3,4)$-formulas with n variables, t of which are 3-exceeding.*

In the terminology of *Parameterized Complexity* [3], Theorem 2 implies that $(3,4)$-SAT, when parameterized by the number of 3-exceeding variables, is in the complexity class XP. Problems in this class are polynomial-time solvable if the parameter is a fixed constant. However, the order of the polynomial may depend on the parameter. In the viewpoint of Parameterized Complexity, the main question is now whether one can remove this dependency and show *fixed-parameter tractability* (FPT), which refers to running times of the form $g(t)n^{O(1)}$, where g is a computable (and possibly exponential) function of the parameter t.

In Section 2 we extend Theorem 2 by proving that for every $k \ge 3$ and $s \ge k$, (k,s)-SAT is fixed-parameter tractable when parameterized by the number t of k-exceeding variables.

Theorem 3. *For $k \ge 3$ and $s \ge k$, (k,s)-SAT can be solved in $O(2^{\frac{t(s-k)}{k}} n^3)$ time on (k,s)-formulas with n variables, t of which are k-exceeding.*

Recall that, when $s \le f(k)$ for the function f defined by Kratochvíl, Savický and Tuza [11], (k,s)-SAT is not only FPT but even polynomial-time solvable.

Berman, Karpinski and Scott [2] also proved that 3-SAT is NP-complete even if exactly one variable is 3-exceeding. This result shows that Theorems 2 and 3 cannot be extended to k-SAT.

2. Fixed-parameter tractability

To prove Theorem 3 we need to introduce some additional terminology. Let α be a truth assignment defined on a set X of variables, and let F be a formula. Then α is *autark* for F if each variable in X occurs in at least one clause of F and α satisfies all the clauses of F in which the variables of X occur. The formula obtained from F by deleting all clauses satisfied by α is denoted by $F[\alpha]$. We make the following observation.

Observation 4. *Let F be a k-CNF formula for some $k \ge 1$, and let α be an autark truth assignment for F. Then $F[\alpha]$ is also a k-CNF formula.*

We also need the following lemma due to Monien and Speckenmeyer.

Lemma 1 *([12]). Let α be an autark truth assignment for F. Then F is satisfiable if and only if $F[\alpha]$ is satisfiable.*

Let F be a formula. The *length* of F is $\sum_{C \in F} |C|$. The *incidence graph* of F is the bipartite graph $I(F)$ whose partition classes are the set of clauses of F and the set of variables occurring in these clauses, such that there is an edge between a variable x and a clause C if and only if x occurs in C.

A matching M in a graph G *covers* a vertex u of G if u incident with an edge of M. We need the following known results.

Theorem 5 *([8]). A maximum matching of a bipartite graph $G = (V,E)$ can be computed in $O(\sqrt{|V|}|E|)$ time.*

Lemma 2 *([5,13]). Let F be a formula of length ℓ and M be a maximum matching of $I(F)$. It is possible to find in $O(\ell)$ time an autark truth assignment α for F such that the edges of M in $I(F[\alpha])$ form a maximum matching of $I(F[\alpha])$ covering every variable of $F[\alpha]$.*

We say that the truth assignment α from Lemma 2 is an *M-truth assignment* of the formula F.

Now let F be a formula with m clauses and n variables. The *deficiency* of F is $\delta(F) = m - n$. The *maximum deficiency* of F is $\delta^*(F) = \max_{F' \subseteq F} \delta(F')$. The following result shows that SAT is FPT when parameterized by the maximum deficiency.

Theorem 6 *([13]). Let F be a formula with n variables. It is possible to decide in $O(2^{\delta^*(F)} n^3)$ time whether F is satisfiable.*

We also need the following lemma.

Lemma 3. *For $k \ge 3$ and $s \ge k$, let F be a (k,s)-formula with t k-exceeding variables. Let α be an M-truth assignment for some maximum matching M of $I(F)$. Then $\delta^*(F[\alpha]) \le \frac{t(s-k)}{k}$.*

Proof. By Lemma 2, the edges of M in $I(F[\alpha])$ form a maximum matching M' of $I(F[\alpha])$ that covers every variable of $I(F[\alpha])$. Let S be the set of all clauses of $I(F[\alpha])$ that are not covered by M'. We observe that $\delta^*(F[\alpha]) \le |S|$. Hence, it suffices to show that $|S| \le \frac{t(s-k)}{k}$.

As F is a (k,s)-formula and thus a k-CNF formula, $F[\alpha]$ is a k-CNF formula as well due to Observation 4. So, every clause of $F[\alpha]$ contains k literals. Hence, the sum of the vertex degrees of the clauses in $I(F[\alpha])$ is $(|S|+|M'|)k$. Recall that M' covers every variable of $I(F[\alpha])$. Hence, the sum of the vertex degrees of the variables in $I(F[\alpha])$ is at most $ts + (|M'|-t)k$. This means that $(|S|+|M'|)k \le ts + (|M'|-t)k$, or equivalently, $|S| \le \frac{t(s-k)}{k}$, as desired. □

We are now ready to prove Theorem 3, which we restate below.

Theorem 3. *For $k \geq 3$ and $s \geq k$, (k,s)-SAT can be solved in $O(2^{\frac{t(s-k)}{k}} n^3)$ time on (k,s)-formulas with n variables, t of which are k-exceeding.*

Proof. Let F be a (k,s)-formula with m clauses, n variables, t of which are k-exceeding, and let ℓ be the length of F. We have $\ell \leq ts + (n-t)k = t(s-k) + nk \leq sn$, as well as $\ell = mk$, and hence, $m = \ell/k \leq \frac{s}{k}n$. We first compute a maximum matching M of $I(F)$. As $I(F)$ has $m + n = O(\frac{s}{k}n)$ vertices and $\ell = O(sn)$ edges, this takes $O(\frac{s}{k}\sqrt{s}n^{\frac{3}{2}})$ time by Theorem 5. We now apply Lemma 2. This takes $O(sn)$ time and gives us an M-truth assignment α. By Lemma 1 it suffices to decide whether $F[\alpha]$ is satisfiable. As $\delta^*(F[\alpha]) \leq \frac{t(s-k)}{k}$ due to Lemma 3, the latter takes $O(2^{\frac{t(s-k)}{k}} n^3)$ time by Theorem 6. Hence the total running time is $O(2^{\frac{t(s-k)}{k}} n^3 + \frac{s}{k}\sqrt{s}n^{\frac{3}{2}} + sn)$. According to the statement of the theorem, s and k are constants. Hence, the running time for deciding whether $F[\alpha]$ is satisfiable dominates the time needed for computing M and α, respectively. $\square$

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