A Sequent-Type Calculus for Three-Valued Default Logic, Or: Tweety Meets Quartum Non Datur

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Abstract. Sequent-type proof systems constitute an important and widely-used class of calculi well-suited for analysing proof search. In this paper, we introduce a sequent-type calculus for a variant of default logic employing Łukasiewicz's three-valued logic as the underlying base logic. This version of default logic has been introduced by Radzikowska addressing some representational shortcomings of standard default logic. More specifically, our calculus axiomatises brave reasoning for this version of default logic, following the sequent method first introduced in the context of nonmonotonic reasoning by Bonatti, which employs a *complementary calculus* for axiomatising invalid formulas, taking care of expressing the consistency condition of defaults.

1 Introduction

Sequent-type proof systems, first introduced in the 1930s by Gerhard Gentzen [15] for classical and intuitionistic logic, are among the basic calculi used in automated deduction for analysing proof search. In the area of nonmonotonic reasoning, Bonatti [8] introduced in the early 1990s sequent-style systems for default logic [31] and autoepistemic logic [25], and a few years later together with Olivetti [9] also for circumscription [24]. A distinguishing feature of these calculi is the usage of a *complementary calculus* for axiomatising invalid formulas, i.e., of non-theorems, taking care of formalising consistency conditions, which makes these calculi arguably particularly elegant and suitable for proof-complexity elaborations as, e.g., recently undertaken by Beyersdorff et al. [5]. In a complementary calculus, the inference rules formalise the propagation of refutability instead of validity and establish invalidity by deduction and thus in a purely syntactic manner. Complementary calculi are also referred to as *refutation calculi* or *rejection calculi* and the first axiomatic treatment of rejection was done by Łukasiewicz in his formalisation of Aristotle's syllogistic [18].

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In this paper, we introduce a sequent-type calculus for brave reasoning in the style of Bonatti [8] for a variant of default logic due to Radzikowska [30]. This version of default logic uses Lukasiewicz's three-value logic [17] as underlying logical apparatus for addressing certain shortcomings of standard default logic [31]. In particular, three-valued default logic allows a more fine-grained distinction between formulas obtained by applying defaults and formulas which are known for certain, in order to avoid counter-intuitive conclusions by successive applications of defaults.

Our calculus, called B_3, consists of three parts, similar to Bonatti's calculus for standard default logic [31], viz. a sequent calculus for Łukasiewiz's three-valued logic, a complementary *anti-sequent* calculus for three-valued logic, and specific default inference rules. For many-valued logics, different kinds of sequent-style systems exist in the literature, like systems [4,6] based on (two-sided) sequents in the style of Gentzen [15] employing additional non-standard rules, or using *hypersequents* [2], which are tuples of Gentzen-style sequents. In our sequent and anti-sequent calculi for Łukasiewicz's three-valued logic, we adopt the approach of Rousseau [33], which is a natural generalisation for many-valued logics of the classical two-sided sequent formulation of Gentzen. The respective calculi are obtained from a systematic construction for many-valued logics as described by Zach [37] and Bogojeski [7].

2 Background

We first recapitulate the basic elements of the three-valued logic $\mathbf{Ł_3}$ of Łukasiewicz for the propositional case and afterwards we discuss a propositional version of the three-valued default logic $\mathbf{DL_3}$ of Radzikowska [30] which is based on $\mathbf{Ł_3}$. Our exposition of both formalisms follows Radzikowska [30].

Łukasiewicz's Three-Valued Logic $\mathbf{Ł_3}$. The alphabet of $\mathbf{Ł_3}$ consists of (i) a denumerable set $\mathcal{P}$ of propositional constants, (ii) the primitive logical connectives $\neg$ ("negation") and $\supset$ ("implication"), (iii) the truth constants $\top$ ("truth"), $\bot$ ("falsehood"), and $\sqcup$ ("undetermined"), and (iv) the punctuation symbols "(" and ")". The class of *formulas* is built from the elements of the alphabet of $\mathbf{Ł_3}$ in the usual inductive fashion, whereby the propositional constants and the truth constants constitute the *atomic formulas*.

The additional connectives $\vee$ ("disjunction"), $\wedge$ ("conjunction"), and $\equiv$ ("equivalence") are defined in the following way: $(A \vee B) := ((A \supset B) \supset B)$, $(A \wedge B) := \neg(\neg A \vee \neg B)$, and $(A \equiv B) := ((A \supset B) \wedge (B \supset A))$. Furthermore, we make use of the unary operators L ("certainty operator") and M ("possibility operator"), defined by $\mathrm{L}A := \neg(A \supset \neg A)$ and $\mathrm{M}A := (\neg A \supset A)$, which, according to Łukasiewicz [17], were first formalised in 1921 by Tarski. Intuitively, $\mathrm{L}A$ expresses that A is certain, whilst $\mathrm{M}A$ means that A is possible. These operators will be used below to distinguish between *certain knowledge* and *defeasible conclusions*. Given L and M, we also define $\mathrm{I}A := (\mathrm{M}A \wedge \neg\mathrm{L}A)$, expressing that A is *contingent* or *modally indifferent*.

$\neg$	
t	**f**
u	**u**
f	**t**

$\supset$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**t**	**t**	**u**
f	**t**	**t**	**t**

$\vee$	**t**	**u**	**f**
t	**t**	**t**	**t**
u	**t**	**u**	**u**
f	**t**	**u**	**f**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**f**
f	**f**	**f**	**f**

$\equiv$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**t**	**u**
f	**f**	**u**	**t**

L	
t	**t**
u	**f**
f	**f**

M	
t	**t**
u	**t**
f	**f**

I	
t	**f**
u	**t**
f	**f**

Fig. 1. Truth tables for the connectives of $\mathbf{Ł_3}$.

A (*three-valued*) *interpretation* is a mapping m assigning to each propositional constant from $\mathcal{P}$ an element from $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$. We refer to each of the symbols $\mathbf{t}$, $\mathbf{f}$, and $\mathbf{u}$ as a *truth value*, and of $m(p)$ as the *truth value of* p *under* m. We assume a total order $\leq$ over the truth values such that $\mathbf{f} \leq \mathbf{u} \leq \mathbf{t}$ holds.

The truth value, $\mathrm{V}^m(A)$, of an arbitrary formula A under an interpretation m is given subject to the following conditions: (i) if $A = \top$, then $\mathrm{V}^m(A) = \mathbf{t}$; (ii) if $A = \sqcup$, then $\mathrm{V}^m(A) = \mathbf{u}$; (iii) if $A = \bot$, then $\mathrm{V}^m(A) = \mathbf{f}$; (iv) if A is an atomic formula, then $\mathrm{V}^m(A) = m(A)$; and (v) if $A = \neg B$, for some formula B, or $A = (C \supset D)$, for some formulas C and D, then $\mathrm{V}^m(A)$ is determined according to the truth tables in Fig. 1 (there, the corresponding truth conditions for the defined connectives are also given).

If $\mathrm{V}^m(A) = \mathbf{t}$, then A is *true under* m, if $\mathrm{V}^m(A) = \mathbf{u}$, then A is *undetermined under* m, and if $\mathrm{V}^m(A) = \mathbf{f}$, then A is *false under* m. If A is true under m, then m is a *model* of A. If A is true in every interpretation, then A is *valid* (*in* $\boldsymbol{Ł_3}$), written $\models_{\mathbf{Ł_3}} A$.

Clearly, the classically valid principle of *tertium non datur*, i.e., the law of excluded middle, $A \vee \neg A$, as well as the corresponding law of non-contradiction, $\neg(A \wedge \neg A)$, are not valid in $\mathbf{Ł_3}$. However, their three-valued pendants, viz., the principle of *quartum non datur*, i.e., the law of excluded fourth, $A \vee \mathrm{I}A \vee \neg A$, and the corresponding extended non-contradiction principle, $\neg(A \wedge \neg\mathrm{I}A \wedge \neg A)$, are valid in $\mathbf{Ł_3}$.

As usual, by a *theory* we understand a set of formulas. An interpretation is a model of a theory T iff it is a model of each element of T. A theory is *satisfiable* iff it has a model; otherwise, it is *unsatisfiable*. Theories T and T' are *equivalent* iff they have the same models. A theory T is said to *entail* a formula A, or A is a *consequence* of T, symbolically $T \models_{\mathbf{Ł_3}} A$, iff every model of T is also a model of A.

Sound and complete Hilbert-style axiomatisations of $\mathbf{Ł_3}$ can be readily found in the literature [22,32]; the first one was introduced by Wajsberg in 1931 [36]. We write $T \vdash_{\mathbf{Ł_3}} A$ if A has a derivation (in some fixed Hilbert-style calculus) from T in $\mathbf{Ł_3}$. As well, the *deductive closure operator* of $\mathbf{Ł_3}$ is given by $\mathrm{Th}_{\mathbf{Ł_3}}(T) := \{A \mid T \vdash_{\mathbf{Ł_3}} A\}$, where T is a theory. A theory T is *deductively closed* iff $T = \mathrm{Th}_{\mathbf{Ł_3}}(T)$. We say that a theory T is *consistent* iff there is a formula A such that $T \nvdash_{\mathbf{Ł_3}} A$. Clearly, T is consistent iff it is satisfiable. Moreover, a formula A is *consistent with* T iff $T \nvdash_{\mathbf{Ł_3}} \neg A$. Note that the consistency of a formula A with a theory T implies the consistency of the theory $T \cup \{\mathrm{M}A\}$, but not necessarily of the theory $T \cup \{A\}$. For instance, $\neg p$ is consistent with $\{\mathrm{M}p\}$, so $\{\mathrm{M}p, \mathrm{M}\neg p\}$ is consistent, but $\{\mathrm{M}p, \neg p\}$ is not.

Three-Valued Default Logic. Radzikowska's three-valued default logic $\mathbf{DL_3}$ [30] differs from Reiter's standard default logic [31] in two aspects: not only is in $\mathbf{DL_3}$ the deductive machinery of classical logic replaced with $\mathbf{Ł_3}$, but there is also a modified consistency check for default rules employed in which the consequent of a default is taken into account as well. The latter feature is somewhat reminiscent to the consistency checks used in *justified default logic* [19] and *constrained default logic* [12,34], where a default may only be applied if it does not lead to a contradiction *a posteriori*.

Formally, a *default rule*, or simply a *default*, d, is an expression of form

$$\frac{A : B_1, \dots, B_n}{C} \ ,$$

where A is the *prerequisite*, $B_1, \dots, B_n$ are the *justifications*, and C is the *consequent* of d. The intuitive meaning of such a default is: if A is believed, and $B_1, \dots, B_n$ and $\mathrm{L}C$ are consistent with what is believed, then $\mathrm{M}C$ is asserted. Note that under this reading, by applying a default of the above form, it is assumed that C cannot be false, but it is not assumed that C is true in all situations. It is only assumed that C must be true in at least one such situation. This reflects the intuition that accepting a default conclusion, we are prepared to rule out all situations where it is false, but we can imagine at least one such situation in which it is true. As a consequence, we cannot conclude both $\mathrm{M}C$ and $\mathrm{M}\neg C$ simultaneously.

In what follows, formulas of the form $\mathrm{M}C$ obtained by applying defaults will be referred to as *default assumptions*. For simplicity, defaults will also be written in the form $(A : B_1, \dots, B_n/C)$.

A *default theory*, T, is a pair $\langle W, D\rangle$, where W is a set of formulas (i.e., a theory in $\mathbf{Ł_3}$) and D is a set of defaults. An *extension* of a default theory $T = \langle W, D\rangle$ in the three-valued default logic $\mathbf{DL_3}$ is defined thus: For a set S of formulas, let $\Gamma_T(S)$ be the smallest set K of formulas obeying the following conditions:

(i) $K = \mathrm{Th}_{\mathbf{Ł_3}}(K)$;
(ii) $W \subseteq K$;
(iii) if $(A : B_1, \dots, B_n/C) \in D$, $A \in K$, $\neg B_1 \notin S, \dots, \neg B_n \notin S$, and $\neg \mathrm{L}C \notin S$, then $\mathrm{M}C \in K$.

Then, E is an extension of T iff $\Gamma_T(E) = E$.

Note that the criterion of the applicability of a default in $\mathbf{DL_3}$ makes the two defaults $d = (A : B_1, \dots, B_n/C)$ and $d' = (A : \mathrm{M}B_1, \dots, \mathrm{M}B_n/C)$ equivalent in the sense that the application of d implies the application of d' and vice versa. Thus, in a default theory $T = \langle W, D\rangle$, we can replace all $d \in D$ with their corresponding version d' without changing extensions.

There are two basic reasoning tasks in the context of default logic, viz., *brave reasoning* and *skeptical reasoning*. The former task is the problem of checking whether a closed formula A belongs to at least one extension of a given closed default theory T, whilst the latter task examines whether A belongs to all extensions of T. Our aim is to give a sequent-type axiomatisation of brave default reasoning, following the approach of Bonatti [8] for standard default logic.

To conclude our review of three-valued default logic, we give two examples, as discussed by Radzikowska [30], showing the representational advantages of $\mathbf{DL_3}$.

Example 1 ([20]). Consider $T = \langle W, D \rangle$, where $W = \{Summer, \neg Sun\_Shining\}$ and $D = \{(Summer : \neg Rain / Sun\_Shining)\}$. The only default of this theory is inapplicable since $W \vdash_{\mathbf{Ł_3}} \neg \mathrm{L} Sun\_Shining$. Hence, T has one extension, $E = \mathrm{Th}_{\mathbf{Ł_3}}(W)$. Note that T has no extension in Reiter's default logic due to the weaker consistency check which yields to a vicious circle where the application of the default violates its justification for applying it. □

Example 2 ([29]). Consider the default rules $d_1 = (P : Q/Q)$ and $d_2 = (Q : R/R)$, where P, Q, and R are atomic formulas respectively standing for "Joe recites passages from Shakespeare", "Joe can read and write", and "Joe is over seven years old". Obviously, common sense suggests that, given P, there are perfect reasons to apply both defaults to infer that Joe is over seven years old. Suppose now that we add the default rule $d_3 = (S : Q/Q)$, where S stands for "Joe is a child prodigy". Given S, it is reasonable to infer that Joe can read and write, but the inference that Joe is over seven years old seems to be unjustified. In standard default logic, a common way of suppressing R in the second case of this example would be to employ a default rule with exceptions of the form $d'_2 = (Q : R \wedge \neg S)/R)$. However, this remedy is somewhat unsatisfactory as it requires every default with a possibly large number of conceivable exceptions which, each time a new default is added, the previous ones must be revised, which is arguably ad hoc. In $\mathbf{DL_3}$, however, this can easily be accommodated by using the defaults $(P : \mathrm{L}Q/\mathrm{L}Q)$, $(Q : \mathrm{L}R/\mathrm{L}R)$, and $(\mathrm{M}S : Q/Q)$ instead of d_1, d_2, and d_3, respectively. □

3 Preparatory Characterisations: Residues and Extensions

We now discuss some properties of extensions concerning adding defaults to default theories which lays the groundwork on which our subsequent calculus is built. In doing so, we first introduce an alternative formulation of $\mathbf{DL_3}$ extensions, adapting a proof-theoretical characterisation as described by Marek and Truszczyński [23] for standard default logic, and afterwards we provide results concerning so-called *residues*, which are inference rules resulting from defaults satisfying their consistency condition. The latter endeavour generalises the approach of Bonatti [8] to the three-valued case and follows the exposition given by Tompits [35] for standard default logic.

Definition 1. *Let E be a set of formulas. A default $(A : B_1, \ldots, B_n/C)$ is* active *in E iff $E \vdash_{\mathbf{Ł_3}} A$ and $\{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset$.*

Definition 2. *Let D be a set of defaults and E a set of formulas. The* reduct of D with respect to E*, denoted by D_E, is the set consisting of the following inference rules:*

$$D_E := \left\{ \frac{A}{\mathrm{M}C} \;\middle|\; \frac{A : B_1, \ldots, B_n}{C} \in D \, and \, \{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset \right\}.$$

An inference rule $A/\mathrm{M}C$ is called residue *of a default $(A : B_1, \ldots, B_n/C)$.*

For a set R of inference rules, let $\vdash^R_{\mathbf{Ł_3}}$ be the inference relation obtained from $\vdash_{\mathbf{Ł_3}}$ by augmenting the postulates of the Hilbert-type calculus underlying $\vdash_{\mathbf{Ł_3}}$ with the inference rules from R. The corresponding deductive closure operator for $\vdash^R_{\mathbf{Ł_3}}$ is given by $\mathrm{Th}^R_{\mathbf{Ł_3}}(W) := \{A \mid W \vdash^R_{\mathbf{Ł_3}} A\}$. Clearly, $\mathrm{Th}^{\emptyset}_{\mathbf{Ł_3}}(W) = \mathrm{Th}_{\mathbf{Ł_3}}(W)$.

We then obtain the following characterisation of the operator Γ_T, mirroring the analogous property for standard default logic as discussed by Marek and Truszczyński [23]:

Theorem 1. *Let $T = \langle W, D \rangle$ be a default theory, E a set of formulas, and D_E the reduct of D with respect to E. Then, $\Gamma_T(E) = \mathrm{Th}^{D_E}_{\mathbf{Ł_3}}(W)$.*

Corollary 1. *Let $T = \langle W, D \rangle$ be a default theory. A set E of formulas is an extension of T iff $\mathrm{Th}^{D_E}_{\mathbf{Ł_3}}(W) = E$.*

Now we show some properties of extensions with respect to active and non-active defaults which are relevant for our sequent calculus. We start with two obvious results whose proofs are straightforward.

Lemma 1. *Let R, R' be sets of inference rules, and W, W' sets of formulas. Then:*

(i) $W \subseteq \mathrm{Th}^R_{\mathbf{Ł_3}}(W)$;
(ii) $\mathrm{Th}^R_{\mathbf{Ł_3}}(W) = \mathrm{Th}^R_{\mathbf{Ł_3}}(\mathrm{Th}^R_{\mathbf{Ł_3}}(W))$;
(iii) *if* $R \subseteq R'$, *then* $\mathrm{Th}^R_{\mathbf{Ł_3}}(W) \subseteq \mathrm{Th}^{R'}_{\mathbf{Ł_3}}(W)$; *and*
(iv) *if* $W \subseteq W'$, *then* $\mathrm{Th}^R_{\mathbf{Ł_3}}(W) \subseteq \mathrm{Th}^R_{\mathbf{Ł_3}}(W')$.

Lemma 2. *Let A and B be formulas, W and E sets of formulas, and R a set of inference rules. Then:*

(i) *if* $A \notin \mathrm{Th}^R_{\mathbf{Ł_3}}(W)$, *then* $\mathrm{Th}^R_{\mathbf{Ł_3}}(W) = \mathrm{Th}^{R \cup \{A/B\}}_{\mathbf{Ł_3}}(W)$;
(ii) *if* $A \in \mathrm{Th}^{R \cup \{A/B\}}_{\mathbf{Ł_3}}(W)$, *then* $\mathrm{Th}^{R \cup \{A/B\}}_{\mathbf{Ł_3}}(W) = \mathrm{Th}^R_{\mathbf{Ł_3}}(W \cup \{B\})$.

For convenience, we employ the following notation in what follows: for a default $d = (A : B_1, \ldots, B_n/C)$, we write $\mathsf{p}(d) := A$, $\mathsf{j}(d) := \{B_1, \ldots, B_n, \mathrm{L}C\}$, and $\mathsf{c}(d) := \mathrm{M}C$. Furthermore, for a set S of formulas, $\neg S := \{\neg A \mid A \in S\}$.

Theorem 2. *Let $T = \langle W, D \rangle$ be a default theory, E a set of formulas, and d a default not active in E. Then, E is an extension of $\langle W, D \rangle$ iff E is an extension of $\langle W, D \cup \{d\} \rangle$.*

Proof. If $\neg\mathsf{j}(d) \cap E \neq \emptyset$, then $(D \cup \{d\})_E = D_E$. So, $\mathrm{Th}_{\mathbf{Ł_3}}^{(D\cup\{d\})_E}(W) = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W)$ and the statement of the lemma holds quite trivially by Corollary 1.

For the rest of the proof, assume thus $\neg\mathsf{j}(d) \cap E = \emptyset$. Since d is not active in E, $E \nvdash_{\mathbf{Ł_3}} \mathsf{p}(d)$ must then hold. Furthermore, $(D \cup \{d\})_E = D_E \cup \{\mathsf{p}(d)/\mathsf{c}(d)\}$ holds.

Suppose E is an extension of $T = \langle W, D\rangle$, i.e., $E = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W)$. Since $E \nvdash_{\mathbf{Ł_3}} \mathsf{p}(d)$ and E is deductively closed, we obtain $\mathsf{p}(d) \notin E$, and so $\mathsf{p}(d) \notin \mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W)$. By Lemma 2(i), $\mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W) = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E\cup\{\mathsf{p}(d)/\mathsf{c}(d)\}}(W)$. But $D_E \cup \{\mathsf{p}(d)/\mathsf{c}(d)\} = (D \cup \{d\})_E$, hence $\mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W) = \mathrm{Th}_{\mathbf{Ł_3}}^{(D\cup\{d\})_E}(W)$. Since $E = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W)$, we obtain $E = \mathrm{Th}_{\mathbf{Ł_3}}^{(D\cup\{d\})_E}(W)$ and E is extension of $\langle W, D \cup \{d\}\rangle$.

This proves the "only if" direction; the "if" direction follows by essentially the same arguments, but employing additionally Lemma 1(iii). ☐

Theorem 3. *Let E be a set of formulas and d a default.*

(i) If E is an extension of $\langle W, D \cup \{d\}\rangle$ and d is active in E, then E is an extension of $\langle W \cup \{\mathsf{c}(d)\}, D\rangle$.
(ii) If E is an extension of the default theory $\langle W \cup \{\mathsf{c}(d)\}, D\rangle$, $W \vdash_{\mathbf{Ł_3}} \mathsf{p}(d)$, and $\neg\mathsf{j}(d) \cap E = \emptyset$, then E is an extension of $\langle W, D \cup \{d\}\rangle$.

Proof. We only show item (ii); the proof of (i) is similarly. Assume that the preconditions of (ii) hold. Since E is an extension of $\langle W \cup \{\mathsf{c}(d)\}, D\rangle$, $E = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W \cup \{\mathsf{c}(d)\})$. Furthermore, by the hypothesis $W \vdash_{\mathbf{Ł_3}} \mathsf{p}(d)$, we have $\mathsf{p}(d) \in \mathrm{Th}_{\mathbf{Ł_3}}^{D_E\cup\{\mathsf{p}(d)/\mathsf{c}(d)\}}(W)$. We thus get $\mathrm{Th}_{\mathbf{Ł_3}}^{D_E\cup\{\mathsf{p}(d)/\mathsf{c}(d)\}}(W) = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E}(W \cup \{\mathsf{c}(d)\})$ in view of Lemma 2(ii), and therefore $E = \mathrm{Th}_{\mathbf{Ł_3}}^{D_E\cup\{\mathsf{p}(d)/\mathsf{c}(d)\}}(W)$. By observing that the assumption $\neg\mathsf{j}(d) \cap E = \emptyset$ implies $D_E \cup \{\mathsf{p}(d)/\mathsf{c}(d)\} = (D \cup \{d\})_E$, the result follows. ☐

4 A Sequent Calculus for $\mathbf{DL_3}$

We now introduce our sequent calculus for brave reasoning in $\mathbf{DL_3}$. Our calculus adapts the approach of Bonatti [8], defined for standard default logic, for the three-valued case.

Analogous to Bonatti's system, our calculus, which we denote by B_3, comprises three kinds of sequents: assertional sequents for axiomatising validity in $\mathbf{Ł_3}$, *anti-sequents* for axiomatising *invalidity*, i.e., non-theorems of $\mathbf{Ł_3}$, taking care of the consistency check of defaults, and proper default sequents. Although it would be possible to use just one kind of sequents, this would be at the expense of losing clarity of the sequents' structure. As well, the current separation of types of sequents also reflects the interactions between the underlying monotonic proof machinery the nonmonotonic inferences in a much clearer manner.

As far as sequent-type calculi for three-valued logics are concerned—or, more generally, many-valued logics—, different techniques have been discussed in the literature [2,3,6,11,16,37]. Here, we use an approach due to Rousseau [33],

$$\frac{\Gamma \mid \Delta \mid \Pi, A \qquad \Gamma, B \mid \Delta \mid \Pi}{\Gamma, A \supset B \mid \Delta \mid \Pi} \ (\supset : \mathbf{f})$$

$$\frac{\Gamma \mid \Delta, A, B \mid \Pi \qquad \Gamma, B \mid \Delta \mid \Pi, A}{\Gamma \mid \Delta, A \supset B \mid \Pi} \ (\supset : \mathbf{u})$$

$$\frac{\Gamma, A \mid \Delta, A \mid \Pi, B \qquad \Gamma, A \mid \Delta, B \mid \Pi, B}{\Gamma \mid \Delta \mid \Pi, A \supset B} \ (\supset : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta \mid \Pi, A}{\Gamma, \neg A \mid \Delta \mid \Pi} \ (\neg : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta, A \mid \Pi}{\Gamma \mid \Delta, \neg A \mid \Pi} \ (\neg : \mathbf{u}) \qquad \frac{\Gamma, A \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, \neg A} \ (\neg : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta \mid \Pi}{\Gamma, A \mid \Delta \mid \Pi} \ (w : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta \mid \Pi}{\Gamma \mid \Delta, A \mid \Pi} \ (w : \mathbf{u}) \qquad \frac{\Gamma \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, A} \ (w : \mathbf{t})$$

Fig. 2. Rules of the sequent calculus $\mathsf{SŁ}_3$.

which is a natural generalisation for many-valued logics of the classical two-sided sequent formulation as pioneered by Gentzen [15]. In Rousseau's approach, a sequent for a three-valued logic is a triple of sets of formulas where each component of the sequent represents one of the three truth values.

Formally, we introduce sequents for **Ł3** as follows:

Definition 3. *A* (three-valued) sequent *is a triple of form* $\Gamma_1 \mid \Gamma_2 \mid \Gamma_3$*, where each* Γ_i ($i \in \{1, 2, 3\}$) *is a finite set of formulas, called* component *of the sequent.*

For an interpretation m*, a sequent* $\Gamma_1 \mid \Gamma_2 \mid \Gamma_3$ *is* true under m *if, for at least one* $i \in \{1, 2, 3\}$*,* Γ_i *contains some formula* A *such that* $\mathrm{V}^m(A) = \mathsf{v}_i$*, where* $\mathsf{v}_1 = \mathbf{f}$*,* $\mathsf{v}_2 = \mathbf{u}$*, and* $\mathsf{v}_3 = \mathbf{t}$*. Furthermore, a sequent is* valid *if it is true under each interpretation.*

Obviously, a standard classical sequent $\Gamma \vdash \Delta$ corresponds to a pair $\Gamma \mid \Delta$ with the usual two-valued semantics. As customary for sequents, we write sequent components comprised of a singleton set $\{A\}$ simply as "A" and similarly $\Gamma \cup \{A\}$ as "Γ, A".

For obtaining the postulates of a many-valued logic in Rousseau's approach, the conditions of the logical connectives of the given logic are encoded in two-valued logic by means of a so-called *partial normal form* [32] and expressed by suitable inference rules.

The calculus we introduce for **Ł3**, which we denote by $\mathsf{SŁ}_3$, is obtained from a systematic construction of sequent-style calculi for many-valued logics due to Zach [37] and by applying some optimisations of the corresponding partial normal form [7].[1] The axioms of $\mathsf{SŁ}_3$ consist of sequents of the form $A \mid A \mid A$, where A is a formula, and the inference rules depicted in Fig. 2. Note that from

[1] $\mathsf{SŁ}_3$ has optimised rules for $\supset$ compared to those of the calculus for **Ł3** given by Malinowski [22]; also note that $\mathsf{SŁ}_3$ does not require a cut rule.

$$\frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma, A \supset B \nmid \Delta \nmid \Pi}\ (\supset : \mathbf{f}^1)^r \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi}{\Gamma, A \supset B \nmid \Delta \nmid \Pi}\ (\supset : \mathbf{f}^2)^r$$

$$\frac{\Gamma \nmid \Delta, A, B \nmid \Pi}{\Gamma \nmid \Delta, A \supset B \nmid \Pi}\ (\supset : \mathbf{u}^1)^r \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi, A}{\Gamma \nmid \Delta, A \supset B \nmid \Pi}\ (\supset : \mathbf{u}^2)^r$$

$$\frac{\Gamma, A \nmid \Delta, A \nmid \Pi, B}{\Gamma \nmid \Delta \nmid \Pi, A \supset B}\ (\supset : \mathbf{t}^1)^r \qquad \frac{\Gamma, A \nmid \Delta, B \nmid \Pi, B}{\Gamma \nmid \Delta \nmid \Pi, A \supset B}\ (\supset : \mathbf{t}^2)^r$$

$$\frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma, \neg A \nmid \Delta \nmid \Pi}\ (\neg : \mathbf{f})^r \qquad \frac{\Gamma \nmid \Delta, A \nmid \Pi}{\Gamma \nmid \Delta, \neg A \nmid \Pi}\ (\neg : \mathbf{u})^r \qquad \frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi, \neg A}\ (\neg : \mathbf{t})^r$$

$$\frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi}\ (w : \mathbf{f})^r \qquad \frac{\Gamma \nmid \Delta, A \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi}\ (w : \mathbf{u})^r \qquad \frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma \nmid \Delta \nmid \Pi}\ (w : \mathbf{t})^r$$

Fig. 3. Rules of the anti-sequent calculus $\mathsf{R}Ł_3$.

the inference rules of $\mathsf{S}Ł_3$ we can easily obtain derived rules for the defined connectives of **Ł₃**. Furthermore, the last three rules in Fig. 2 are also referred to as *weakening rules*.

Soundness and completeness of $\mathsf{S}Ł_3$ follows directly from the method described by Zach [37].

Theorem 4. *A sequent* $\Gamma \mid \Delta \mid \Pi$ *is valid iff it is provable in* $\mathsf{S}Ł_3$.

Note that sequents in the style of Rousseau are *truth functional* rather than formalising entailment directly, but the latter can be expressed simply as follows:

Theorem 5. *For a theory* T *and a formula* A, $T \vdash_{\mathbf{Ł_3}} A$ *iff the sequent* $T \mid T \mid A$ *is provable in* $\mathsf{S}Ł_3$.

As for axiomatising non-theorems of **Ł₃**, Bogojeski [7] describes a systematic construction of refutation calculi for many-valued logics, which is obtained by adapting the approach of Zach [37]. The refutation calculus we introduce now for axiomatising invalid sequents in **Ł₃**, denoted by $\mathsf{R}Ł_3$, is obtained from the method of Bogojeski [7].

Definition 4. *A* (three-valued) anti-sequent *is a triple of form* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$, *where each* Γ_i $(i \in \{1,2,3\})$ *is a finite set of formulas, called* component *of the anti-sequent.*

For an interpretation m, *an anti-sequent* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$ *is* refuted by m, *or* m refutes $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$, *if, for every* $i \in \{1,2,3\}$ *and every formula* $A \in \Gamma_i$, $\mathrm{V}^m(A) \neq \mathsf{v}_i$, *where* v_i *is defined as in Definition 3.*

If m *refutes* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$, *then* m *is also said to be a* counter-model *of* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$. *An anti-sequent* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$ *is* refutable, *if there is at least one interpretation that refutes* $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$.

$$\frac{\Gamma \mid \Gamma \mid A}{\Gamma; \emptyset \Rightarrow A; \emptyset}\ l_1 \qquad \frac{\Gamma \nmid \Gamma \nmid A}{\Gamma; \emptyset \Rightarrow \emptyset; A}\ l_2 \qquad \frac{\Gamma; \emptyset \Rightarrow \Sigma_1; \Theta_1 \quad \Gamma; \emptyset \Rightarrow \Sigma_2; \Theta_2}{\Gamma; \emptyset \Rightarrow \Sigma_1, \Sigma_2; \Theta_1, \Theta_2}\ mu$$

$$\frac{\Gamma; \Delta \Rightarrow \Sigma; \Theta, A}{\Gamma; \Delta, (A : B_1, \ldots, B_n / C) \Rightarrow \Sigma; \Theta}\ d_1 \qquad \frac{\Gamma; \Delta \Rightarrow \Sigma, \neg B; \Theta}{\Gamma; \Delta, (A : \ldots, B, \ldots / C) \Rightarrow \Sigma; \Theta}\ d_2$$

$$\frac{\Gamma; \Delta \Rightarrow \Sigma, \neg \mathrm{L}C; \Theta}{\Gamma; \Delta, (A : B_1, \ldots, B_n / C) \Rightarrow \Sigma; \Theta}\ d_3$$

$$\frac{\Gamma; \emptyset \Rightarrow A; \emptyset \qquad \Gamma, \mathrm{M}C; \Delta \Rightarrow \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C}{\Gamma; \Delta, (A : B_1, \ldots, B_n / C) \Rightarrow \Sigma; \Theta}\ d_4$$

Fig. 4. Additional rules of the calculus B_3.

Clearly, an anti-sequent $\Gamma_1 \nmid \Gamma_2 \nmid \Gamma_3$ is refutable iff the corresponding sequent $\Gamma_1 \mid \Gamma_2 \mid \Gamma_3$ is not valid.

The postulates of $\mathsf{RŁ}_3$ are as follows: the axioms of $\mathsf{RŁ}_3$ are anti-sequents whose components are sets of propositional constants such that no constant appears in all components. Furthermore, the inference rules of $\mathsf{RŁ}_3$ are given in Fig. 3.

Note that, in contrast to $\mathsf{SŁ}_3$, the inference rules of $\mathsf{RŁ}_3$ have only single premisses. Indeed, this is a general pattern in sequent-style rejection calculi: if an inference rule for standard (assertional) sequents for a connective have n premisses, then there are usually n corresponding unary inference rules in the associated rejection calculus. Intuitively, what is exhaustive search in a standard sequent calculus becomes nondeterminism in a rejection calculus.

Again, soundness and completeness of $\mathsf{RŁ}_3$ follows from the systematic construction as described by Bogojeski [7]. Likewise, non-entailment in **Ł₃** is expressed similarly as for $\mathsf{SŁ}_3$.

Theorem 6. *An anti-sequent $\Gamma \nmid \Delta \nmid \Pi$ is refutable iff it is provable in $\mathsf{RŁ}_3$. Moreover, for a theory T and a formula A, $T \not\models_{\mathbf{Ł_3}} A$ iff $T \nmid T \nmid A$ is provable in $\mathsf{RŁ}_3$.*

We are now in a position to specify our calculus B_3 for brave reasoning in $\mathbf{DL_3}$.

Definition 5. *By a* (brave) default sequent *we understand an ordered quadruple of the form $\Gamma; \Delta \Rightarrow \Sigma; \Theta$, where Γ, Σ, and Θ are finite sets of formulas and Δ is a finite set of defaults.*

A default sequent $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ is true *iff there is an extension E of the default theory $\langle \Gamma, \Delta \rangle$ such that $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$; E is called a* witness *of $\Gamma; \Delta \Rightarrow \Sigma; \Theta$.*

The default sequent calculus B_3 consists of three-valued sequents, anti-sequents, and default sequents. It incorporates the systems $\mathsf{SŁ}_3$ for three-valued sequents and $\mathsf{RŁ}_3$ for anti-sequents. Additionally, it has axioms of the form

$\Gamma; \emptyset \Rightarrow \emptyset; \emptyset$, where Γ is a finite set of formulas, and the inference rules as depicted in Fig. 4.

The informal meaning of the nonmonotonic inference rules is the following. First of all, rules l_1 and l_2 combine three-valued sequents and anti-sequents with default sequents, respectively. Rule *mu* is the rule of "monotonic union"; it allows the joining of information in case that no default is present. Rules d_1–d_4 are the default introduction rules: rules d_1, d_2, and d_3 take care of introducing non-active defaults, whilst rule d_4 allows to introduce an active default.

Theorem 7 (Soundness). *If $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ is provable in B_3, then it is true.*

Proof. We show that all axioms are true, and that the conclusions of all inference rules are true whenever its premisses are true (resp., valid or refutable in case of l_1 and l_2).

First of all, an axiom $\Gamma; \emptyset \Rightarrow \emptyset; \emptyset$ is trivially true, because $\mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$ is the unique extension of the default theory $\langle \Gamma, \emptyset \rangle$ and hence the unique witness of $\Gamma; \emptyset \Rightarrow \emptyset; \emptyset$.

Suppose $\Gamma \mid \Gamma \mid A$ is the premiss of the rule l_1 and assume it is valid. Hence, $\Gamma \vdash_{\mathbf{Ł_3}} A$ and therefore $A \in \mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$. But $\mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$ is the unique extension of $\langle \Gamma, \emptyset \rangle$, so $\mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$ is the unique witness of $\Gamma; \emptyset \Rightarrow A; \emptyset$. Likewise, if the premiss $\Gamma \nmid \Gamma \nmid A$ of the rule l_2 is refutable, then $A \notin \mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$, and therefore $\mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$ is the (unique) witness of $\Gamma; \emptyset \Rightarrow \emptyset; A$.

If the two premises $\Gamma; \emptyset \Rightarrow \Sigma_1; \Theta_1$ and $\Gamma; \emptyset \Rightarrow \Sigma_2; \Theta_2$ of the rule *mu* are true, then they must have the same witness $E = \mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$. Hence, E is also the (unique) witness of $\Gamma; \emptyset \Rightarrow \Sigma_1, \Sigma_2; \Theta_1, \Theta_2$.

As for the soundness of the rules d_1, d_2, and d_3, we only show the case for d_3; the other two are similar. Let E be a witness of $\Gamma; \Delta \Rightarrow \Sigma, \neg \mathrm{L}C; \Theta$. Then, E is an extension of $\langle \Gamma, \Delta \rangle$, $\Sigma \cup \{\neg \mathrm{L}C\} \subseteq E$, and $\Theta \cap E = \emptyset$. So, $\neg \mathrm{L}C \in E$ and thus the default $(A : B_1, \dots, B_n / C)$ is not active in E. By Theorem 2, it follows that E is an extension of $\langle \Gamma, \Delta \cup \{(A : B_1, \dots, B_n/C)\} \rangle$. Moreover, since $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$, E is a witness of $\Gamma; \Delta, (A : B_1, \dots, B_n/C) \Rightarrow \Sigma; \Theta$.

Finally, assume that the premisses of rule d_4 are true. Let E_1 be a witness of $\Gamma; \emptyset \Rightarrow A; \emptyset$ and E_2 a witness of $\Gamma, \mathrm{M}C; \Delta \Rightarrow \Sigma; \Theta, \neg B_1, \dots, \neg B_n, \neg \mathrm{L}C$. Thus, E_2 is an extension of $\langle \Gamma \cup \{\mathrm{M}C\}, \Delta \rangle$ and $\{\neg B_1, \dots, \neg B_n, \neg \mathrm{L}C\} \cap E_2 = \emptyset$. So, E_1 is an extension of $\langle \Gamma, \emptyset \rangle$ with $A \in E_1$, and therefore $\Gamma \vdash_{\mathbf{Ł_3}} A$. Hence, by Theorem 3(ii), E_2 is an extension of $\langle \Gamma, \Delta \cup \{(A : B_1, \dots, B_n/C)\} \rangle$. Clearly, $\Sigma \subseteq E_2$ and $\Theta \cap E_2 = \emptyset$, so E_2 is a witness of $\Gamma; \Delta, (A : B_1, \dots, B_n/C) \Rightarrow \Sigma; \Theta$. □

Theorem 8 (Completeness). *If $\Gamma; \Delta \Rightarrow \Sigma; \Theta$ is true, then it is provable in B_3.*

Proof. Suppose $S = \Gamma; \Delta \Rightarrow \Sigma; \Theta$ is true, with E as its witness. The proof proceeds by induction on the cardinality $|\Delta|$ of Δ.

Induction Base. Assume $|\Delta| = 0$. If $\Sigma = \Theta = \emptyset$, then S is an axiom and hence provable in B_3. So suppose that either $\Sigma \neq \emptyset$ or $\Theta \neq \emptyset$. Since $\mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$ is the unique extension of $\langle \Gamma, \emptyset \rangle$, we have $E = \mathrm{Th}_{\mathbf{Ł_3}}(\Gamma)$. Furthermore, $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$. It follows that for any $A \in \Sigma$, $\Gamma \mid \Gamma \mid A$ is provable in $\mathsf{SŁ}_3$, and for any $B \in \Theta$, $\Gamma \nmid \Gamma \nmid B$ is provable in $\mathsf{RŁ}_3$. Repeated applications of rules l_1, l_2, and *mu* yield a proof of S in B_3.

Induction Step. Assume $|\Delta| > 0$, and let the statement hold for all default sequents $\Gamma'; \Delta' \Rightarrow \Sigma'; \Theta'$ such that $|\Delta'| < |\Delta|$. We distinguish two cases: (i) there is some default in Δ which is active in E, or (ii) none of the defaults in Δ is active in E.

If (i) holds, then there must be some default $d = (A : B_1, \ldots, B_n / C)$ in Δ such that d is active in E and $\Gamma \vdash_{\mathbf{Ł_3}} A$. Consider $\Delta_0 := \Delta \setminus \{d\}$. Then, $|\Delta_0| = |\Delta| - 1$ and $\Delta_0 \cup \{d\} = \Delta$. By Theorem 3(i), E is an extension of $\langle \Gamma \cup \{\mathrm{M}C\}, \Delta_0 \rangle$. Since d is active in E, $\{\neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C\} \cap E = \emptyset$; and since E is a witness of $S = \Gamma; \Delta \Rightarrow \Sigma; \Theta$, $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$. So, E is a witness of $S' = \Gamma, \mathrm{M}C; \Delta_0 \Rightarrow \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C$. Since $|\Delta_0| < |\Delta|$, by induction hypothesis there is some proof α in B_3 of S'. Furthermore, $\Gamma \vdash_{\mathbf{Ł_3}} A$, so there is some proof β of $\Gamma \mid \Gamma \mid A$ in $\mathsf{SŁ}_3$. The following figure is a proof of S in B_3:

$$\dfrac{\dfrac{\overset{\beta}{\Gamma \mid \Gamma \mid A}}{\Gamma; \emptyset \Rightarrow A; \emptyset}\, l_1 \qquad \overset{\alpha}{\Gamma, \mathrm{M}C; \Delta_0 \Rightarrow \Sigma; \Theta, \neg B_1, \ldots, \neg B_n, \neg \mathrm{L}C}}{\Gamma; \Delta \Rightarrow \Sigma; \Theta}\, d_4$$

Now assume that (ii) holds, i.e., no default in Δ is active in E. Since $|\Delta| > 0$, there is some default $d = (A : B_1, \ldots, B_n / C)$ in Δ such that $\Delta = \Delta_0 \cup \{d\}$ with $\Delta_0 := \Delta \setminus \{d\}$. Since d is not active in E, according to Theorem 2, E is an extension of $\langle \Gamma, \Delta_0 \rangle$. Furthermore, either (i) $E \nvdash_{\mathbf{Ł_3}} A$, (ii) there is some $B_{i_0} \in \{B_1, \ldots, B_n\}$ such that $\neg B_{i_0} \in E$, or (iii) $\neg \mathrm{L}C \in E$. Consequently, E is either a witness of (i) $\Gamma; \Delta_0 \Rightarrow \Sigma; \Theta, A$, (ii) $\Gamma; \Delta_0 \Rightarrow \Sigma, \neg B_{i_0}; \Theta$, or (iii) $\Gamma; \Delta_0 \Rightarrow \Sigma, \neg \mathrm{L}C; \Theta$. Since $|\Delta_0| < |\Delta|$, by induction hypothesis there is either (i) a proof α in B_3 of $\Gamma; \Delta_0 \Rightarrow \Sigma; \Theta, A$, (ii) a proof β in B_3 of $\Gamma; \Delta_0 \Rightarrow \Sigma, \neg B_{i_0}; \Theta$, or (iii) a proof γ in B_3 of $\Gamma; \Delta_0 \Rightarrow \Sigma, \neg \mathrm{L}C; \Theta$. Therefore, one of the three figures below constitutes a proof of S :

$$\dfrac{\overset{\alpha}{\Gamma; \Delta_0 \Rightarrow \Sigma; \Theta, A}}{\Gamma; \Delta \Rightarrow \Sigma; \Theta}\, d_1 \qquad \dfrac{\overset{\beta}{\Gamma; \Delta_0 \Rightarrow \Sigma, \neg B_{i_0}; \Theta}}{\Gamma; \Delta \Rightarrow \Sigma; \Theta}\, d_2 \qquad \dfrac{\overset{\gamma}{\Gamma; \Delta_0 \Rightarrow \Sigma, \neg \mathrm{L}C; \Theta}}{\Gamma; \Delta \Rightarrow \Sigma; \Theta}\, d_3$$

□

5 Conclusion

In this paper, we introduced a sequent-type calculus for brave reasoning for a three-valued version of default logic [30] following the method of Bonatti [8]. This form of axiomatisation yields a particular elegant formulation mainly due to their usage of anti-sequents. Also, the approach is flexible and can be applied to formalise different versions of nonmonotonic reasoning. Indeed, other variants of default logic besides the three-valued version studied in our paper, including justified default logic [19] and constrained default logic [12,34], have also been axiomatised by this sequent method [13,21].

Related to the sequent approach discussed here are also works employing tableau methods. In particular, Niemelä [26] introduces a tableau calculus for inference under circumscription. Other tableau approaches, however, do not encode inference directly, rather they characterise models (resp., extensions) associated with a particular nonmonotonic reasoning formalism [1,10,14,28].

A variation of our calculus can be obtained by using different calculi for the underlying three-valued logic. We opted here for the style of calculi as discussed by Rousseau [33] and Zach [37] because they naturally model the underlying semantic conditions of the considered logic. Alternatively, we could also use two-sided sequent and anti-sequent calculi like the ones described by Avron [2] and Oetsch and Tompits [27], respectively. By employing such two-sided sequents, however, one then deals with calculi having also "non-standard" inference rules introducing two connectives simultaneously. Another prominent proof method for many-valued logics are *hypersequent calculi* [3] which are basically disjunctions of two-sided sequents. However, no rejection calculus based on hypersequents exist as far as we know; establishing such a system in particular for $\mathbf{Ł_3}$ would be worthwhile.

Another topic for future work is to develop a calculus for skeptical reasoning in $\mathbf{DL_3}$ and other variants of default logic, similar to the system for skeptical reasoning in standard default logic as introduced by Bonatti and Olivetti [9]. In that work, they introduced also a different version of a calculus for brave default reasoning—extending this calculus to $\mathbf{DL_3}$ would provide an alternative to B_3.

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