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Model correlation for special types of rotor systems

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Abstract

The correlation between theoretical and experimental modal models can be quantified by the Modal Assurance Criterion, which has also been used for rotor systems. However, the conclusions drawn from the Modal Assurance Criterion may not be valid for non-symmetric systems since the underlying orthogonality relations are violated. Therefore, a Rotor Correlation Criterion is proposed, which incorporates right and left eigenvectors, eigenvalues, and mass and stiffness matrices. Applications of model correlation for test data assessment, sensor placement, mode pairing, and model updating are discussed in the context of rotors. For two numerical examples, the modified correlation approach is compared to the Modal Assurance Criterion. Considering the availability of experimentally determined left eigenvectors, these examples deal with a damped rotor under special support conditions and a purely gyroscopic rotor under more general conditions.

1 Introduction

The Modal Assurance Criterion (MAC) [2] is widely used to assess the correlation between theoretical and experimental modal models. If the mass or stiffness weighted MAC of a non-rotating structure is proportional to the identity matrix, mode shapes are perfectly correlated. For many structural models, the weighting matrix is not essential, and a similar MAC is obtained without weighting. The unweighted MAC only requires the mode shapes, which suggests that it could be suited for any type of system. However, the properties of the MAC depend on the underlying orthogonality relations, which are violated by non-symmetric models. Since the system matrices of rotors are in general non-symmetric, there is no guarantee that the usual conclusions from the MAC are still valid.

For a flexible rotor in journal bearings, the main MAC diagonal of theoretical and experimental modes has been calculated in [6]. Four out of ten values were below the recommended limit of 0.9 and could not be improved by model updating. Moreover, significant differences appeared between the MAC values of right and left eigenvectors. In this case, model correlation would have required a modified approach.

For the model correlation of rotor systems, one might consider the modal matching array from [1], which is defined for general second-order systems. It incorporates right and left eigenvectors as well as all system matrices. In a comparison between theoretical and experimental modal models, approximate mass and stiffness matrices will be available, while damping will hardly be known; this discourages the use of the modal matching array, which also involves rather complex calculations. The difficulty of obtaining left eigenvectors from experiments has been discussed in [3]. For purely gyroscopic rotors [7, 8] or special support conditions [9, 10], they can be calculated from right eigenvectors so that the rotor only has to be excited in one degree of freedom. In this paper, the results from [9, 10] are used to ensure the practical applicability of a modified correlation approach.

Besides the comparison between theoretical and experimental modal models, the MAC is used in other applications such as the validation of experimental modal models [2], optimal sensor placement [11], and the objective function for model updating algorithms [12]. Future success in the model updating of rotor systems will depend on an efficient treatment of these issues and requires an appropriate approach to model correlation.

This paper is based on a general orthogonality relation for non-symmetric systems. It proposes a correlation criterion that incorporates right and left eigenvectors, eigenvalues, and mass and stiffness matrices. Various applications of model correlation are discussed with an emphasis on their importance for rotors. The advantage of the modified correlation approach is demonstrated by numerical examples. These examples consider special types of rotor systems for which appropriate experiments are feasible. They investigate various applications of model correlation for a rotor in damped isotropic bearings and a rotor in undamped bearings with minor anisotropy.

2 Modified correlation criterion

To derive the required orthogonality relation, the equation of motion

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{C} + \mathbf{G})\dot{\mathbf{u}} + (\mathbf{K} + \mathbf{N})\mathbf{u} = \mathbf{0} \tag{1}$$

for an autonomous rotor system with the displacement vector $\mathbf{u}$, the symmetric mass matrix $\mathbf{M}$, the symmetric damping matrix $\mathbf{C}$, the skew-symmetric gyroscopic Matrix $\mathbf{G}$, the symmetric stiffness matrix $\mathbf{K}$, and the skew-symmetric circulatory matrix $\mathbf{N}$ is transformed into

$$\mathbf{A}\begin{bmatrix}\dot{\mathbf{u}}\\ \ddot{\mathbf{u}}\end{bmatrix} + \mathbf{B}\begin{bmatrix}\mathbf{u}\\ \dot{\mathbf{u}}\end{bmatrix} = \mathbf{0} \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix}\mathbf{C}+\mathbf{G} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix} \tag{3}$$

and

$$\mathbf{B} = \begin{bmatrix}\mathbf{K}+\mathbf{N} & 0\\ 0 & -\mathbf{M}\end{bmatrix}. \tag{4}$$

Equation (2) leads to the first order right eigenvalue problem

$$(\lambda_n\mathbf{A} + \mathbf{B})\boldsymbol{\theta}_{rn} = \mathbf{0}, \tag{5}$$

in which both eigenvalues λ_n and right eigenvectors $\boldsymbol{\theta}_{rn}$ appear as complex conjugate pairs. The first order left eigenvalue problem

$$\boldsymbol{\theta}_{ln}^T(\lambda_n\mathbf{A} + \mathbf{B}) = \mathbf{0}^T \tag{6}$$

yields the same eigenvalues λ_n and the left eigenvectors $\boldsymbol{\theta}_{ln}$, which also appear as complex conjugate pairs. In general, right and left eigenvectors do not coincide.

If Eq. (5) is premultiplied by the transpose of the left eigenvector $\boldsymbol{\theta}_{lm}$, one obtains

$$\boldsymbol{\theta}_{lm}^T(\lambda_n\mathbf{A} + \mathbf{B})\boldsymbol{\theta}_{rn} = 0. \tag{7}$$

Expressing Eq. (6) for the index m and postmultiplying by the right eigenvector $\boldsymbol{\theta}_{rn}$ results in

$$\boldsymbol{\theta}_{lm}^T(\lambda_m\mathbf{A} + \mathbf{B})\boldsymbol{\theta}_{rn} = 0. \tag{8}$$

For $m \neq n$, the combination of Eqs. (7) and (8) proves the orthogonality relation

$$\boldsymbol{\theta}_{lm}^T \mathbf{B} \boldsymbol{\theta}_{rn} = 0. \tag{9}$$

The right and left eigenvectors $\boldsymbol{\psi}_{rn}$ and $\boldsymbol{\psi}_{lm}$ of the system (1) satisfy the relations

$$\left(\lambda_n^2 \mathbf{M} + \lambda_n (\mathbf{C} + \mathbf{G}) + \mathbf{K} + \mathbf{N}\right) \boldsymbol{\psi}_{rn} = \mathbf{0} \tag{10}$$

and

$$\boldsymbol{\psi}_{lm}^T \left(\lambda_m^2 \mathbf{M} + \lambda_m (\mathbf{C} + \mathbf{G}) + \mathbf{K} + \mathbf{N}\right) = \mathbf{0}^T, \tag{11}$$

respectively. It follows from Eqs. (3-6), (10), and (11) that

$$\boldsymbol{\theta}_{rn} = \begin{bmatrix} \boldsymbol{\psi}_{rn} \\ \lambda_n \boldsymbol{\psi}_{rn} \end{bmatrix} \tag{12}$$

and

$$\boldsymbol{\theta}_{lm} = \begin{bmatrix} \boldsymbol{\psi}_{lm} \\ \lambda_m \boldsymbol{\psi}_{lm} \end{bmatrix}. \tag{13}$$

With Eqs. (4), (12), and (13), the orthogonality relation (9) becomes

$$\boldsymbol{\psi}_{lm}^T (\mathbf{K} + \mathbf{N}) \boldsymbol{\psi}_{rn} - \lambda_m \lambda_n \boldsymbol{\psi}_{lm}^T \mathbf{M} \boldsymbol{\psi}_{rn} = 0, \tag{14}$$

which constitutes the basis of the correlation criterion proposed in this paper.

In the various applications of model correlation, two sets A and B of both right and left eigenvectors are considered. They may be obtained from theory or by experiment and sometimes they coincide. Omitting the circulatory matrix for brevity, Eq. (14) suggests the abbreviation

$$S_{mAnB} = \boldsymbol{\psi}_{lmA}^T \mathbf{K} \boldsymbol{\psi}_{rnB} - \lambda_{mA} \lambda_{nB} \boldsymbol{\psi}_{lmA}^T \mathbf{M} \boldsymbol{\psi}_{rnB}; \tag{15}$$

using Eq. (15), the elements of the Rotor Correlation Criterion (RCC) matrix are defined as

$$\mathrm{RCC}_{mn} = \frac{|S_{mAnB}| \cdot |S_{nBmA}|}{|S_{mAmA}| \cdot |S_{nBnB}|}. \tag{16}$$

If the two sets of eigenvectors A and B coincide, an auto-RCC is obtained, which should be close to the identity matrix. In theory, it yields the identity matrix if the circulatory matrix remains in Eq. (15).

Section 4 uses the elements of the classical MAC matrix for comparison. They are given by

$$\mathrm{MAC}_{mn} = \frac{\left|\boldsymbol{\psi}_{rmA}^H \boldsymbol{\psi}_{rnB}\right| \cdot \left|\boldsymbol{\psi}_{rnB}^H \boldsymbol{\psi}_{rmA}\right|}{\left|\boldsymbol{\psi}_{rmA}^H \boldsymbol{\psi}_{rmA}\right| \cdot \left|\boldsymbol{\psi}_{rnB}^H \boldsymbol{\psi}_{rnB}\right|}, \tag{17}$$

which is based on the orthogonality relation

$$\boldsymbol{\psi}_{rm}^T \mathbf{M} \boldsymbol{\psi}_{rn} = 0 \quad (m \neq n) \tag{18}$$

for undamped symmetric systems [4].

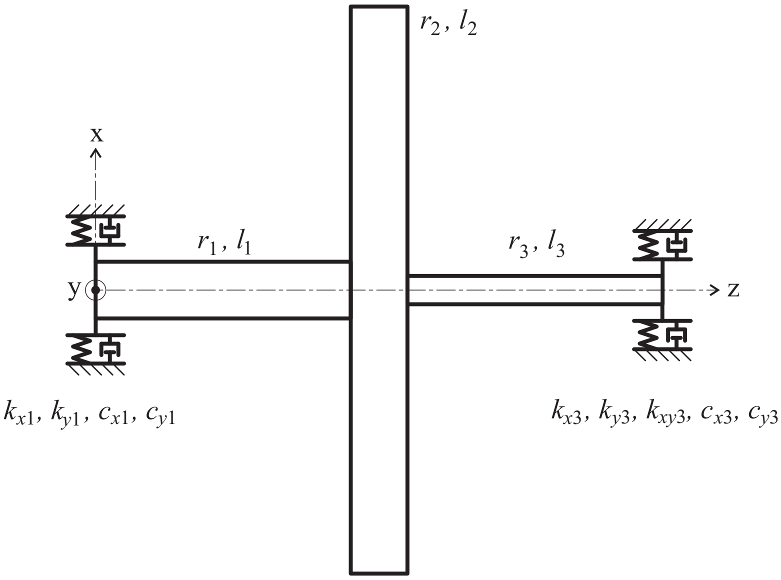

Figure 1: Schematic of a flexible rotor.

3 Applications of model correlation

In structural dynamics, the MAC has a wide range of applications; typical uses are listed in [2]. Some applications of model correlation will be essential for modal testing and model updating of rotor systems. In practical cases, access to a rotor is limited and vibration measurements are restricted to a small number of degrees of freedom. The preparation of measurement points may cause a considerable effort, which should be made for optimal positions to avoid repeated trials. This issue is addressed by optimal sensor placement. For non-rotating systems, an appropriate method has been described in [11]. It considers the auto-MAC matrix of Finite Element mode shapes that are sampled at the selected sensor positions. This matrix should come close to identity. For rotor systems, it can be replaced by the RCC matrix. In the RCC formula, right and left eigenvectors have to be sampled at the selected sensor positions, and mass and stiffness matrices must be reduced to the measured degrees of freedom.

In view of the practical difficulties that arise with excitation and measurement under rotation, an experimental modal rotor model should be validated to ensure the quality of measurements. The validation of experimental modal models was an original intention of the MAC [2]. For non-rotating systems, it requires an auto-MAC matrix of experimental mode shapes, which can be calculated in the absence of a theoretical model. If the RCC matrix is calculated for the validation of experimental mode shapes from a rotor system, the formula also requires experimental left eigenvectors as well as a first estimate of reduced mass and stiffness matrices. It seems easier to use the MAC, but this may lead to misjudgement. For an experimental modal rotor model, the auto-RCC matrix should be near identity.

The validation and, if possible, the successful correction of a physical model may be considered as the ultimate goal of modal testing. Once the quality of measurements has been ensured, the correlation between theoretical and experimental modal models can be used to validate the physical model. For non-rotating systems, the MAC matrix of theoretical and experimental mode shapes should have main diagonal values above 0.9 and off-diagonal values below 0.1 to ensure a good correlation [4]. For rotor systems, the RCC matrix can be used in a similar way. Reduced mass and stiffness matrices can be estimated from the theoretical model.

Model updating aims at the correction of physical model parameters. The objective function can be composed of theoretical and experimental eigenvalues, eigenvectors, or the correlation criterion itself [12]. For a reasonable definition of this function, it is essential to identify matching mode pairs, which are indicated by the correlation criterion. If the MAC is used for rotor mode pairing, the objective function may be misleading and model updating may fail. The RCC shoud be used instead. In an iterative approach, reduced mass and stiffness matrices could be recalculated from updated model parameters.

4 Numerical examples

The rotor under consideration is depicted in Fig. 1. The non-symmetric rotor system consists of a rigid disc attached to a flexible shaft, which is supported by a bearings at both ends. In the examples, two different configurations of these bearings will be considered. The coordinate system in Fig. 1 is aligned with the principal axes of the left bearing with stiffness constants in directions x and y denoted as k_{x1} and k_{y1}; respective damping constants c_{x1} and c_{y1} are used in the first example. The left shaft section with radius r_1 and length l_1 is described by its mass $m_1 = \rho l_1 r_1^2 \pi$ and transverse area moment of inertia $J_1 = r_1^4 \pi/4$; ρ denotes the mass density. The mass of the disc with radius r_2 and thickness l_2 is $m_2 = \rho l_2 r_2^2 \pi$. Its transverse and polar mass moments of inertia are $I_{t2} = \rho l_2 r_2^4 \pi/4$ and $I_{p2} = \rho l_2 r_2^4 \pi/2$, respectively. For the right shaft section with radius r_3 and length l_3, $m_3 = \rho l_3 r_3^2 \pi$ and $J_3 = r_3^4 \pi/4$. The stiffness constants of the right bearing are denoted as k_{x3} and k_{y3}. Respective damping constants c_{x3} and c_{y3} are used in the first example. Below, a cross-coupled stiffness k_{xy3} appears in the second example.

To keep the model simple, each of the two shaft sections is described by a beam element neglecting shear deflection [5]. The displacement vector is defined as

$$\mathbf{u} = \begin{bmatrix} x_1 & \beta_1 & x_2 & \beta_2 & x_3 & \beta_3 & y_1 & \alpha_1 & y_2 & \alpha_2 & y_3 & \alpha_3 \end{bmatrix}^T \tag{19}$$

with translations x_1, y_1, x_2, y_2, x_3, y_3 and rotations α_1, β_1, α_2, β_2, α_3, β_3 at the left bearing (index 1), the disc center (index 2), and the right bearing (index 3); α and β describe rotations about the x- and y-axes, respectively. The mass matrix associated with the displacement vector $\mathbf{u}$ reads

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_y \end{bmatrix} \tag{20}$$

with

$$\mathbf{M}_x = \mathbf{M}_2 + \frac{1}{420} \begin{bmatrix} 156 m_1 l_1 & 22 m_1 l_1^2 & 54 m_1 l_1 & -13 m_1 l_1^2 & 0 & 0 \\ 22 m_1 l_1^2 & 4 m_1 l_1^3 & 13 m_1 l_1^2 & -3 m_1 l_1^3 & 0 & 0 \\ 54 m_1 l_1 & 13 m_1 l_1^2 & 156(m_1 l_1 + m_3 l_3) & -22\left(m_1 l_1^2 - m_3 l_3^2\right) & 54 m_3 l_3 & -13 m_3 l_3^2 \\ -13 m_1 l_1^2 & -3 m_1 l_1^3 & -22\left(m_1 l_1^2 - m_3 l_3^2\right) & 4\left(m_1 l_1^3 + m_3 l_3^3\right) & 13 m_3 l_3^2 & -3 m_3 l_3^3 \\ 0 & 0 & 54 m_3 l_3 & 13 m_3 l_3^2 & 156 m_3 l_3 & -22 m_3 l_3^2 \\ 0 & 0 & -13 m_3 l_3^2 & -3 m_3 l_3^3 & -22 m_3 l_3^2 & 4 m_3 l_3^3 \end{bmatrix} \tag{21}$$

and

$$\mathbf{M}_y = \mathbf{M}_2 + \frac{1}{420} \begin{bmatrix} 156 m_1 l_1 & -22 m_1 l_1^2 & 54 m_1 l_1 & 13 m_1 l_1^2 & 0 & 0 \\ -22 m_1 l_1^2 & 4 m_1 l_1^3 & -13 m_1 l_1^2 & -3 m_1 l_1^3 & 0 & 0 \\ 54 m_1 l_1 & -13 m_1 l_1^2 & 156(m_1 l_1 + m_3 l_3) & 22\left(m_1 l_1^2 - m_3 l_3^2\right) & 54 m_3 l_3 & 13 m_3 l_3^2 \\ 13 m_1 l_1^2 & -3 m_1 l_1^3 & 22\left(m_1 l_1^2 - m_3 l_3^2\right) & 4\left(m_1 l_1^3 + m_3 l_3^3\right) & -13 m_3 l_3^2 & -3 m_3 l_3^3 \\ 0 & 0 & 54 m_3 l_3 & -13 m_3 l_3^2 & 156 m_3 l_3 & 22 m_3 l_3^2 \\ 0 & 0 & 13 m_3 l_3^2 & -3 m_3 l_3^3 & 22 m_3 l_3^2 & 4 m_3 l_3^3 \end{bmatrix}, \tag{22}$$

where

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{t2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

Note that signs are changed between Eqs. (21) and (22). The damping matrix is given by

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_y \end{bmatrix} \tag{24}$$

with

$$\mathbf{C}_x = \begin{bmatrix} c_{x1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{x3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_y = \begin{bmatrix} c_{y1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{y3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

The gyroscopic matrix reads

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{G}_0 \\ -\mathbf{G}_0^T & \mathbf{0} \end{bmatrix} \tag{26}$$

with

$$\mathbf{G}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{p2}\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

in which Ω denotes the rotational speed. The stiffness matrix is given by

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_x & \mathbf{K}_{xy} \\ \mathbf{K}_{xy}^T & \mathbf{K}_y \end{bmatrix} \tag{28}$$

with

$$\mathbf{K}_x = \begin{bmatrix} k_{x1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{x3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + E \begin{bmatrix} 12\frac{J_1}{l_1^3} & 6\frac{J_1}{l_1^2} & -12\frac{J_1}{l_1^3} & 6\frac{J_1}{l_1^2} & 0 & 0 \\ 6\frac{J_1}{l_1^2} & 4\frac{J_1}{l_1} & -6\frac{J_1}{l_1^2} & 2\frac{J_1}{l_1} & 0 & 0 \\ -12\frac{J_1}{l_1^3} & -6\frac{J_1}{l_1^2} & 12\left(\frac{J_1}{l_1^3}+\frac{J_3}{l_3^3}\right) & -6\left(\frac{J_1}{l_1^2}-\frac{J_3}{l_3^2}\right) & -12\frac{J_3}{l_3^3} & 6\frac{J_3}{l_3^2} \\ 6\frac{J_1}{l_1^2} & 2\frac{J_1}{l_1} & -6\left(\frac{J_1}{l_1^2}-\frac{J_3}{l_3^2}\right) & 4\left(\frac{J_1}{l_1}+\frac{J_3}{l_3}\right) & -6\frac{J_3}{l_3^2} & 2\frac{J_3}{l_3} \\ 0 & 0 & -12\frac{J_3}{l_3^3} & -6\frac{J_3}{l_3^2} & 12\frac{J_3}{l_3^3} & -6\frac{J_3}{l_3^2} \\ 0 & 0 & 6\frac{J_3}{l_3^2} & 2\frac{J_3}{l_3} & -6\frac{J_3}{l_3^2} & 4\frac{J_3}{l_3} \end{bmatrix}, \tag{29}$$

$$\mathbf{K}_y = \begin{bmatrix} k_{y1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{y3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + E \begin{bmatrix} 12\frac{J_1}{l_1^3} & -6\frac{J_1}{l_1^2} & -12\frac{J_1}{l_1^3} & -6\frac{J_1}{l_1^2} & 0 & 0 \\ -6\frac{J_1}{l_1^2} & 4\frac{J_1}{l_1} & 6\frac{J_1}{l_1^2} & 2\frac{J_1}{l_1} & 0 & 0 \\ -12\frac{J_1}{l_1^3} & 6\frac{J_1}{l_1^2} & 12\left(\frac{J_1}{l_1^3}+\frac{J_3}{l_3^3}\right) & 6\left(\frac{J_1}{l_1^2}-\frac{J_3}{l_3^2}\right) & -12\frac{J_3}{l_3^3} & -6\frac{J_3}{l_3^2} \\ -6\frac{J_1}{l_1^2} & 2\frac{J_1}{l_1} & 6\left(\frac{J_1}{l_1^2}-\frac{J_3}{l_3^2}\right) & 4\left(\frac{J_1}{l_1}+\frac{J_3}{l_3}\right) & 6\frac{J_3}{l_3^2} & 2\frac{J_3}{l_3} \\ 0 & 0 & -12\frac{J_3}{l_3^3} & 6\frac{J_3}{l_3^2} & 12\frac{J_3}{l_3^3} & 6\frac{J_3}{l_3^2} \\ 0 & 0 & -6\frac{J_3}{l_3^2} & 2\frac{J_3}{l_3} & 6\frac{J_3}{l_3^2} & 4\frac{J_3}{l_3} \end{bmatrix}, \tag{30}$$

and

$$\mathbf{K}_{xy} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{xy3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

where E denotes the modulus of elasticity. Again, signs are changed between Eqs. (29) and (30).

For a mixture of translations and rotations, the MAC cannot be calculated since this would require an addition of squared lengths and squared angles. The model is therefore reduced to the translational cooordinates

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \end{bmatrix}^T \tag{32}$$

by a static condensation [5]. The matrix $\mathbf{S}$ sorts the displacement vector $\mathbf{u}$ so that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{a} \end{bmatrix} = \mathbf{S}\mathbf{u} \tag{33}$$

with the remaining coordinates $\mathbf{x}$ and the condensed coordinates

$$\mathbf{a} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T . \tag{34}$$

The stiffness matrix associated with the sorted displacement vector is partitioned in 6×6 blocks according to

$$\left(\mathbf{S}^{-1}\right)^T \mathbf{K}\,\mathbf{S}^{-1} = \begin{bmatrix} \mathbf{K}_{xx} & \mathbf{K}_{xa} \\ \mathbf{K}_{ax} & \mathbf{K}_{aa} \end{bmatrix}, \tag{35}$$

and the static relationship can be expressed by

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{I}(6 \times 6) \\ -\mathbf{K}_{aa}^{-1}\mathbf{K}_{ax} \end{bmatrix} \mathbf{x}. \tag{36}$$

Using the reduction matrix

$$\mathbf{R} = \mathbf{S}^{-1} \begin{bmatrix} \mathbf{I}(6 \times 6) \\ -\mathbf{K}_{aa}^{-1}\mathbf{K}_{ax} \end{bmatrix}, \tag{37}$$

the condensed system matrices are given by $\mathbf{R}^T\mathbf{M}\,\mathbf{R}$, $\mathbf{R}^T\mathbf{C}\,\mathbf{R}$, $\mathbf{R}^T\mathbf{G}\,\mathbf{R}$, and $\mathbf{R}^T\mathbf{K}\,\mathbf{R}$.

4.1 Rotor in damped isotropic bearings

The first example considers a rotor system whose parameters are given by $r_1 = 10\,\mathrm{mm}$, $l_1 = 100\,\mathrm{mm}$, $r_2 = 100\,\mathrm{mm}$, $l_2 = 20\,\mathrm{mm}$, $r_3 = 5\,\mathrm{mm}$, $l_3 = 100\,\mathrm{mm}$, $\rho = 7860\,\mathrm{kg\,m^{-3}}$, $E = 2.1 \cdot 10^{11}\,\mathrm{N\,m^{-2}}$, $k_{x1} = k_{y1} = 30EJ_1/l_1^3$, $k_{x3} = k_{y3} = 0.1k_{x1}$, $k_{xy3} = 0$, $c_{x1} = c_{y1} = 0.01\sqrt{k_{x1}m_2/2}$, $c_{x3} = c_{y3} = 0.01\sqrt{k_{x3}m_2/2}$, and $\Omega = 20000\,\mathrm{rpm}$. Both the original and the reduced model comply with the conditions in [9, 10]. This means that the calculation of the RCC does not require additional measurements for the experimental determination of left eigenvectors.

To compare theoretical MAC and RCC matrices, the eigenvalue problem is solved for the reduced model. The auto-RCC of the reduced model yields the identity matrix. The respective auto-MAC matrix is given by

$$\text{MAC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & 0.2532 & 0.0000 & 0.0153 & 0.0000 \\ 0.0000 & \mathbf{1.0000} & 0.0000 & 0.2735 & 0.0000 & 0.0400 \\ 0.2532 & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.1249 & 0.0000 \\ 0.0000 & 0.2735 & 0.0000 & \mathbf{1.0000} & 0.0000 & \mathbf{0.4306} \\ 0.0153 & 0.0000 & 0.1249 & 0.0000 & \mathbf{1.0000} & 0.0000 \\ 0.0000 & 0.0400 & 0.0000 & \mathbf{0.4306} & 0.0000 & \mathbf{1.0000} \end{bmatrix}.$$

Since this matrix contains large off-diagonal values, it should not be used for the validation of an experimental modal rotor model. Otherwise, perfect measurements might be dismissed.

If the eigenvalue problem is solved for the original model and eigenvectors are sampled at the translational degrees of freedom, the question of sensor placement can be addressed. There is little alternative to measuring six translations, which results in the auto-RCC matrix

$$\text{RCC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & 0.1488 & 0.0000 & 0.0009 & 0.0000 \\ 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0848 & 0.0000 & 0.0511 \\ 0.1488 & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0488 & 0.0000 \\ 0.0000 & 0.0848 & 0.0000 & \mathbf{1.0000} & 0.0000 & \mathbf{0.6426} \\ 0.0009 & 0.0000 & 0.0488 & 0.0000 & \mathbf{1.0000} & 0.0000 \\ 0.0000 & 0.0511 & 0.0000 & \mathbf{0.6426} & 0.0000 & \mathbf{1.0000} \end{bmatrix};$$

relaxing the limit for off-diagonal values to 0.2, it can be concluded that the omission of rotations is acceptable for the lowest five modes. In contrast, the auto-MAC matrix becomes

$$\text{MAC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & \mathbf{0.9827} & 0.0000 & 0.0155 & 0.0000 \\ 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0007 & 0.0000 & 0.0040 \\ \mathbf{0.9827} & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0028 & 0.0000 \\ 0.0000 & 0.0007 & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.8001 \\ 0.0155 & 0.0000 & 0.0028 & 0.0000 & \mathbf{1.0000} & 0.0000 \\ 0.0000 & 0.0040 & 0.0000 & 0.8001 & 0.0000 & \mathbf{1.0000} \end{bmatrix};$$

this would mean that the omission of rotations cannot be accepted at all.

It is now assumed that experimental right eigenvectors are available for a rotor system whose actual model parameters differ from those of the theoretical model. The stiffness constants of the right bearing are $k_{x3} = k_{y3} = 0.06k_{x1}$ in this assumption. Experimental left eigenvectors can be obtained by changing the signs in the lower halves of the right eigenvectors [9, 10]. The RCC matrix of theoretical and experimental modal models reads

$$\text{RCC}: \quad \begin{bmatrix} \mathbf{0.9744} & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & \mathbf{0.9971} & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0005 & 0.0000 & \mathbf{0.9939} & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & \mathbf{1.0039} & 0.0000 & 0.0019 \\ 0.0127 & 0.0000 & 0.0035 & 0.0000 & \mathbf{0.9874} & 0.0000 \\ 0.0000 & 0.0016 & 0.0000 & 0.0045 & 0.0000 & \mathbf{0.9892} \end{bmatrix}.$$

Its main diagonal indicates a better correlation than the main diagonal of the MAC matrix

$$\text{MAC}: \quad \begin{bmatrix} \mathbf{0.9946} & 0.0000 & 0.1876 & 0.0000 & 0.0153 & 0.0000 \\ 0.0000 & \mathbf{0.9839} & 0.0000 & 0.1595 & 0.0000 & 0.0400 \\ 0.2191 & 0.0000 & \mathbf{0.9629} & 0.0000 & 0.1247 & 0.0000 \\ 0.0000 & 0.3471 & 0.0000 & \mathbf{0.8429} & 0.0000 & 0.4317 \\ 0.0384 & 0.0000 & 0.2776 & 0.0000 & \mathbf{1.0000} & 0.0000 \\ 0.0000 & 0.1042 & 0.0000 & 0.7483 & 0.0000 & \mathbf{1.0000} \end{bmatrix},$$

which underestimates the theoretical model. This demonstrates that the MAC may not be able to recognize the quality of a theoretical rotor system model.

4.2 Rotor in undamped bearings with minor anisotropy

In the second example, $c_{x1} = c_{y1} = c_{x3} = c_{y3} = 0$, and $k_{xy3} = 0.0005k_{x3}$. All the other parameters are the same as in the first example. Both the original and the reduced model comply with the conditions in [7]. Again, the calculation of the RCC does not require additional measurements for the experimental determination of left eigenvectors. While the auto-RCC of the reduced model yields the identity matrix, the auto-MAC matrix

$$\text{MAC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & 0.2532 & 0.0000 & 0.0126 & 0.0027 \\ 0.0000 & \mathbf{1.0000} & 0.0000 & 0.2699 & 0.0070 & 0.0328 \\ 0.2532 & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.1029 & 0.0220 \\ 0.0000 & 0.2699 & 0.0000 & \mathbf{1.0000} & 0.0845 & \mathbf{0.3931} \\ 0.0126 & 0.0070 & 0.1029 & 0.0845 & \mathbf{1.0000} & 0.0000 \\ 0.0027 & 0.0328 & 0.0220 & \mathbf{0.3931} & 0.0000 & \mathbf{1.0000} \end{bmatrix}$$

would not allow for a fair assessment of measurements. If the eigenvalue problem is solved for the original model, the auto-RCC matrix with eigenvectors sampled at the translational degrees of freedom reads

$$\text{RCC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & 0.1488 & 0.0000 & 0.0007 & 0.0003 \\ 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0843 & 0.0138 & 0.0361 \\ 0.1488 & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0347 & 0.0134 \\ 0.0000 & 0.0843 & 0.0000 & \mathbf{1.0000} & 0.1815 & \mathbf{0.4737} \\ 0.0007 & 0.0138 & 0.0347 & 0.1815 & \mathbf{1.0000} & 0.0000 \\ 0.0003 & 0.0361 & 0.0134 & \mathbf{0.4737} & 0.0000 & \mathbf{1.0000} \end{bmatrix}.$$

In contrast, the respective auto-MAC matrix

$$\text{MAC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & \mathbf{0.9827} & 0.0000 & 0.0112 & 0.0043 \\ 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0006 & 0.0011 & 0.0029 \\ \mathbf{0.9827} & 0.0000 & \mathbf{1.0000} & 0.0000 & 0.0020 & 0.0008 \\ 0.0000 & 0.0006 & 0.0000 & \mathbf{1.0000} & 0.2336 & 0.6058 \\ 0.0112 & 0.0011 & 0.0020 & 0.2336 & \mathbf{1.0000} & 0.0000 \\ 0.0043 & 0.0029 & 0.0008 & 0.6058 & 0.0000 & \mathbf{1.0000} \end{bmatrix}$$

would not recommend the omission of rotations.

Experimental right eigenvectors shall now be available for a reduced shaft stiffness according to $J_1 = 0.7r_1^4\pi/4$. Experimental left eigenvectors can be obtained as the complex conjugates of the right eigenvectors [8]. The RCC matrix of theoretical and experimental modal models then reads

$$\text{RCC}: \quad \begin{bmatrix} \mathbf{0.9999} & 0.0000 & 0.0000 & 0.0000 & 0.0003 & 0.0000 \\ 0.0000 & \mathbf{0.9973} & 0.0000 & 0.0000 & 0.0000 & 0.0003 \\ 0.0000 & 0.0000 & \mathbf{0.9963} & 0.0000 & 0.0074 & 0.0009 \\ 0.0000 & 0.0002 & 0.0000 & \mathbf{0.9996} & 0.0227 & 0.1876 \\ 0.0000 & 0.0002 & 0.0016 & 0.0001 & \mathbf{0.7813} & 0.0090 \\ 0.0000 & 0.0011 & 0.0003 & 0.0003 & 0.0075 & \mathbf{0.7811} \end{bmatrix}.$$

Its main diagonal indicates a worse correlation than the main diagonal of the MAC matrix

$$\text{MAC}: \quad \begin{bmatrix} \mathbf{1.0000} & 0.0000 & 0.2123 & 0.0000 & 0.0135 & 0.0016 \\ 0.0000 & \mathbf{0.9899} & 0.0000 & 0.2339 & 0.0036 & 0.0307 \\ 0.2514 & 0.0000 & \mathbf{0.9789} & 0.0000 & 0.0994 & 0.0118 \\ 0.0000 & 0.3102 & 0.0000 & \mathbf{0.9806} & 0.0479 & 0.4028 \\ 0.0133 & 0.0151 & 0.1938 & 0.1090 & \mathbf{0.9889} & 0.0102 \\ 0.0028 & 0.0706 & 0.0415 & 0.5071 & 0.0105 & \mathbf{0.9889} \end{bmatrix},$$

which overestimates the theoretical model. This demonstrates that the MAC may approve a theoretical rotor system model that does not comply with measurements.

5 Conclusions

It has been demonstrated by numerical examples that the Modal Assurance Criterion may not be suited for rotor systems. A Rotor Correlation Criterion has been defined as an alternative. Its advantage concerning the validation of experimental modal models, sensor placement, and the correlation between theoretical and experimental modal models has been demonstrated numerically. Since the Rotor Correlation Criterion requires left eigenvectors, the examples discuss rotors for which a relationship between right and left eigenvectors is known. Model correlation for other rotors would be facilitated if more general results on the determination of left eigenvectors were available.

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