Functional a-posteriori error estimates for BEM

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Introduction. We consider the Poisson model problem

(1) $\Delta u = 0$ in Ω subject to Dirichlet boundary conditions $u = g$ on $\Gamma := \partial\Omega$,

where $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain with compact boundary, and (1) is supplemented by a natural radiation condition if Ω is unbounded.

For the ease of presenting the idea of [16], we consider an indirect BEM ansatz

(2) $$u = \widetilde{V}\phi \text{ in } \Omega \quad \text{with unknown density } \phi \in H^{-1/2}(\Gamma),$$

where $\widetilde{V}\colon H^{-1/2}(\Gamma) \to H^1_{\ell oc}(\mathbb{R}^d)$ is the single-layer potential operator defined by

(3) $$\widetilde{V}\phi(x) := \int_\Gamma G(x-y)\phi(y)\,\mathrm{d}\Gamma(y) \quad \text{with} \quad G(z) = \begin{cases} -\frac{1}{2\pi}\log|z|, & \text{for } d=2, \\ +\frac{1}{4\pi}|z|^{-1}, & \text{for } d=3. \end{cases}$$

Taking the trace of (2), we obtain the weakly-singular integral equation

(4) $$g = V\phi \text{ on } \Gamma \text{ with unknown density } \phi \in H^{-1/2}(\Gamma),$$

where the integral representation of the weakly-singular boundary integral operator $V = (\widetilde{V}\cdot)|_\Gamma\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ formally coincides with that of the single-layer potential (3). Since V is continuous and elliptic, the Lax–Milgram lemma proves that (4) admits a unique solution $\phi \in H^{-1/2}(\Gamma)$.

While the exact solution ϕ (as well as the corresponding potential $u = \widetilde{V}\phi$) can hardly be computed in practice, we suppose that $\phi_h \in H^{-1/2}(\Gamma)$ is a computed (but essentially arbitrary) approximation $\phi_h \approx \phi$. Owing to the mapping properties of the single-layer potential, it then holds that

(5) $$\Delta u_h = 0 \text{ in } \Omega \text{ for the induced approximate potential } u_h := \widetilde{V}\phi_h \in H^1(\Omega)$$

We stress that, for the indirect BEM ansatz, the potential u has a physical meaning, while the integral density ϕ has not. In our recent work [16], we derive computable a-posteriori error estimators μ_h and η_h for the potential error

(6) $$\mu_h \le \|\nabla(u-u_h)\|_{L^2(\Omega)} \le \eta_h \quad (+\text{ data oscillations})$$

with a particular emphasis on constant-free estimates. Clearly, the computation of u_h in Ω is computationally expensive and thus shall be avoided, until a prescribed accuracy $\eta_h \le \mathsf{tol}$ can be guaranteed. As a matter of fact, the advertised approach applies to *any* approximation $\phi_h \approx \phi$ and hence covers Galerkin BEM and collocation BEM as well as inexact computations based on iterative solvers.

State of the art. Earlier works provide a-posteriori estimates for the density error $\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)}$ (mainly exploiting that ϕ_h is a Galerkin approximation of ϕ by piecewise polynomials). See, e.g., [6, 7, 2, 3] for weighted-residual error estimators, [17, 18, 15, 14] for two-level estimators, [20] for estimators exploiting the Calderón projector, [9, 10, 4] for estimators based on the localization of the

$H^{1/2}$-residual norm, and [5, 13, 8] for averaging and $(h-h/2)$-type error estimators. Moreover, we refer to the recent review [11]. Clearly, a-posteriori error control of the density also provides a bound for the potential error: From boundedness of the single-layer potential operator $\widetilde{V}$, it follows that

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \le C\,\|\phi-\phi_h\|_{H^{-1/2}(\Gamma)}, \tag{7}$$

where, however, the constant $C>0$ is generic, depends on Ω, and is hardly accessible. Finally, one particular advantage of BEM is its potentially high convergence order for point errors of the potential, i.e., $|u(x)-u_h(x)|$ for some fixed $x\in\Omega$, where we refer to [12, 1] for a-posteriori error control and adaptive algorithms.

Functional a-posteriori error estimate. We employ the so-called hypercircle method (see, e.g., [19]), which even simplifies because of $\Delta u = 0 = \Delta u_h$.

Theorem 1 (Functional error identity)**.** *Exploiting* (1) *and* (5)*, it holds that*

$$\max_{\substack{\boldsymbol{\tau}\in L^2(\Omega)\\ \nabla\cdot\boldsymbol{\tau}=0}} \underline{\mathfrak{M}}(\boldsymbol{\tau}; g-u_h|_\Gamma) = \|\nabla(u-u_h)\|^2_{L^2(\Omega)} = \min_{\substack{w\in H^1(\Omega)\\ w|_\Gamma = g-u_h|_\Gamma}} \overline{\mathfrak{M}}(\nabla w), \tag{8}$$

where $\underline{\mathfrak{M}}(\boldsymbol{\tau}; f) = 2\,\langle f\,,\,\boldsymbol{\tau}|_\Gamma\cdot\boldsymbol{n}\rangle_{L^2(\Gamma)} - \|\boldsymbol{\tau}\|^2_{L^2(\Omega)}$ *and* $\overline{\mathfrak{M}}(\nabla w) = \|\nabla w\|^2_{L^2(\Omega)}$. *Moreover, the unique maximizer (resp. minimizer) is* $w = u-u_h$ *(resp.* $\boldsymbol{\tau} = \nabla(u-u_h)$*).*

With Theorem 1, each $\boldsymbol{\tau}\in L^2(\Omega)$ with $\nabla\cdot\boldsymbol{\tau}=0$ provides a lower bound for the potential error $\|\nabla(u-u_h)\|_{L^2(\Omega)}$, while each $w\in H^1(\Omega)$ with $w|_\Gamma = g-u_h|_\Gamma$ provides an upper bound. To compute such functions, we employ the FEM. To lower the computational costs, we fix a *bounded* boundary layer domain $\omega\subseteq\Omega$ with $\Gamma\subset\partial\omega$ and $\mathrm{dist}(\Gamma,\partial\omega\backslash\Gamma)$ and let $\mathcal{T}_h$ be a conforming triangulation of ω.

To motivate the following corollaries, consider for $\omega=\Omega$ the problem

$$\Delta(u-u_h) = 0 \text{ in } \Omega \quad \text{subject to} \quad (u-u_h)|_\Gamma = g-u_h|_\Gamma, \tag{9}$$

Then, (10) is the mixed FEM formulation of (9) with $\boldsymbol{\tau}_h \approx \boldsymbol{\tau} = \nabla(u-u_h)$, while (13) is the primal FEM formulation of (9) with $w_h\approx u-u_h$. In either case, the zero boundary conditions on $\partial\omega\backslash\Gamma$ are motivated by the fact that, first, they are clearly exact if $g-u_h|_\Gamma = 0$ and, second, they allow for a trivial extension of the computed functions from ω to Ω (which is needed to apply Theorem 1).

Corollary 2 (Computable lower bound)**.** *Let* $P^q(\mathcal{T}_h)$ *be the space of* $\mathcal{T}_h$*-piecewise polynomials of degree* q*. Let* $RT^q(\mathcal{T}_h)\subset P^{q+1}(\mathcal{T}_h)$ *be the corresponding Raviart–Thomas space on* ω *and* $RT^q_0(\mathcal{T}_h) := \{\boldsymbol{\tau}_h\in RT^q(\mathcal{T}_h) : \boldsymbol{\tau}_h|_{\partial\omega\backslash\Gamma}\cdot\boldsymbol{n} = 0\}$*. Then, there exists a unique pair of* $\boldsymbol{\tau}_h\in RT^q_0(\mathcal{T}_h)$ *and* $p_h\in P^q(\mathcal{T}_h)$ *such that*

$$\left\{\begin{array}{rcrcl} \langle\boldsymbol{\tau}_h\,,\,\boldsymbol{\sigma}_h\rangle_{L^2(\omega)} & + & \langle\nabla\cdot\boldsymbol{\sigma}_h\,,\,p_h\rangle_{L^2(\omega)} & = & \langle g-u_h|_\Gamma\,,\,\boldsymbol{\sigma}_h|_\Gamma\cdot\boldsymbol{n}\rangle_{L^2(\Gamma)},\\ & & \langle\nabla\cdot\boldsymbol{\tau}_h\,,\,q_h\rangle_{L^2(\omega)} & = & 0, \end{array}\right. \tag{10}$$

for all $\boldsymbol{\sigma}_h\in RT^q_0(\mathcal{T}_h)$ *and all* $q_h\in P^q(\mathcal{T}_h)$*. Extending* $\boldsymbol{\tau}_h$ *by zero to* $\Omega\backslash\omega$*, it holds that* $\boldsymbol{\tau}_h\in L^2(\Omega)$ *with* $\nabla\cdot\boldsymbol{\tau}_h = 0$*. In particular, we obtain that*

$$2\,\langle g-u_h|_\Gamma\,,\,\boldsymbol{\tau}_h|_\Gamma\cdot\boldsymbol{n}\rangle_{L^2(\Gamma)} - \|\boldsymbol{\tau}_h\|^2_{L^2(\omega)} \le \|\nabla(u-u_h)\|^2_{L^2(\Omega)}. \tag{11}$$

For the upper bound, we proceed analogously. However, since discrete FEM functions cannot satisfy continuous Dirichlet conditions, we must discretize the Dirichlet conditions of (9). Therefore, the upper bounds additionally involves certain data oscillation terms.

Corollary 3 (Computable upper bound)**.** *Let $S^q(\mathcal{T}_h) := P^q(\mathcal{T}_h) \cap H^1(\omega)$ be the standard Courant FEM space and $S^q_0(\mathcal{T}_h) := \{v_h \in S^q(\mathcal{T}_h) \,:\, v_h|_{\partial\omega\setminus\Gamma} = 0\}$. Suppose that $J_h : H^{1/2}(\Gamma) \to \{v_h|_\Gamma \,:\, v_h \in S^q(\mathcal{T}_h)\}$ is, e.g., the $L^2(\Gamma)$-orthogonal projection. Then, there exists a unique $w_h \in S^q_0(\mathcal{T}_h)$ such that*

$$w_h|_\Gamma = J_h(g - u_h|_\Gamma) \quad \text{and} \quad \langle \nabla w_h \,,\, \nabla v_h \rangle_{L^2(\omega)} = 0 \quad \text{for all} \;\; v_h \in S^q_0(\mathcal{T}_h). \tag{12}$$

Extend w_h by zero to $\Omega\setminus\omega$ to get $w_h \in H^1(\Omega)$, and the triangle inequality leads to

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le \|\nabla w_h\|_{L^2(\omega)} + C\,\|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}, \tag{13}$$

where $C > 0$ depends only on the definition of the $H^{1/2}$-norm and $C = 1$ if the latter is defined by harmonic extension.

Adaptive algorithm. The discrete majorant $\|\nabla w_h\|_{L^2(\omega)}$ from Corollary 3 can be used to drive an adaptive mesh-refining algorithm.

In the initial step, let $\mathcal{T}^\Gamma_h$ be a (boundary) triangulation of Γ, which is extended to some (volume) triangulation $\mathcal{T}_h$ of some boundary layer $\omega \subseteq \Omega$.

In the successive steps of the adaptive loop, we proceed as follows: Given the triangulation $\mathcal{T}_h$ of ω, we extract the triangulation $\mathcal{T}^\Gamma_h$, compute the BEM solution ϕ_h and the FEM majorant w_h, use $\|\nabla w_h\|_{L^2(T)}$ to mark elements $T \in \mathcal{T}_h$, and refine the marked elements. By this procedure, we obtain a refined triangulation $\mathcal{T}_h$ of ω. We then use the latter to shrink the FEM domain ω (e.g., as a fixed-order patch of the new boundary mesh $\mathcal{T}^\Gamma_h$), restrict $\mathcal{T}_h$ to the shrunken ω, and continue with the next step of the adaptive loop.

Note that by shrinking ω, we practically ensure that the dimension of the BEM space stays proportional to the dimension of the auxiliary FEM space. Of course, it is important that Corollary 2–3 are independent of ω.

Conclusions. The proofs of Theorem 1 and Corollary 2–3 are independent of the approximation $\phi_h \approx \phi$ and avoid, in particular, the use of any Galerkin-type orthogonality. This makes the proposed approach interesting in engineering, where primarily collocation BEM is used. Moreover, the analysis applies to indirect BEM (as presented here) as well as direct BEM for both interior and exterior problems. Finally, the mathematical concepts go beyond the Poisson model problem.

In first experiments with lowest-order BEM for the 2D Poisson problem (1), our empirical observations are as follows [16]: For lowest-order BEM, first-order FEM elements are sufficient (i.e., $q = 1$ in Corollary 2–3). After a few adaptive steps, the oscillation term $\|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$ becomes negligible and there even holds $\|\nabla(u - u_h)\|_{L^2(\Omega)} \le \|\nabla w_h\|_{L^2(\omega)}$ with $\|\nabla w_h\|_{L^2(\omega)}/\|\nabla(u - u_h)\|_{L^2(\Omega)} \approx 1.2$. Mathematically, it remains to prove that upper and lower bound in (6) are equivalent (at least up to oscillations terms) and that the proposed adaptive strategy leads to optimal convergence. Both is observed experimentally. We are currently working on a 3D implementation, where the overall goal will be to deal with 3D Maxwell.

An obvious drawback of the approach is that it requires a FEM mesh on a boundary layer $\omega \subseteq \Omega$ along Γ. However, we stress that the generation of such a FEM mesh is a standard problem for FEM computations, where the geometry is usually given in terms of a CAD representation.

References

[1] M. Bakry: *A goal-oriented a posteriori error estimate for the oscillating single layer integral equation*, Appl. Math. Lett. 69 (2017), 133–137.

[2] C. Carstensen: *An a-posteriori error estimate for a first-kind integral equation*, Math. Comp. **66** (1997), 139–155.

[3] C. Carstensen, M. Maischak, E.P. Stephan: *A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation*, Numer. Math. **90** (2001), 197–213.

[4] C. Carstensen, D. Praetorius: *A posteriori error control in adaptive qualocation boundary element analysis for a logarithmic-kernel integral equation of the first kind*, SIAM J. Sci. Comp. **25** (2003), 259–283.

[5] C. Carstensen, D. Praetorius: *Averaging techniques for the effective numerical solution of Symm's integral equation of the first kind*, SIAM J. Sci. Comp. **27** (2006), 1226–1260.

[6] C. Carstensen, E.P. Stephan: *A posteriori error estimates for boundary element methods*, Math. Comp. **64** (1995), 483–500.

[7] C. Carstensen, E.P. Stephan: *Adaptive boundary element methods for some first kind integral equations*, SIAM J. Numer. Anal. **33** (1996), 2166–2183.

[8] C. Erath, S. Ferraz-Leite, S.A. Funken, D. Praetorius: *Energy norm based a posteriori error estimation for boundary element methods in two dimensions*, Appl. Numer. Math. **59** (2009), 2713–2734.

[9] B. Faermann: *Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. Part I: The two-dimensional case*, IMA J. Numer. Anal. **20** (2000), 203–234.

[10] B. Faermann: *Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. Part II: The three-dimensional case*, Numer. Math. **92** (2002), 467–499.

[11] M. Feischl, T. Führer, N. Heuer, M. Karkulik, D. Praetorius: *Adaptive boundary element methods*, Arch. Comput. Methods Eng. **22** (2015), 309–389.

[12] M. Feischl, T. Führer, G. Gantner, A. Haberl, D. Praetorius: *Adaptive boundary element methods for optimal convergence of point errors*, Numer. Math. 132 (2016), 541–567.

[13] S. Ferraz-Leite, D. Praetorius: *Simple a posteriori error estimates for the h-version of the boundary element method*, Computing **83** (2008), 135–162.

[14] N. Heuer, V.J. Ervin: *An adaptive boundary element method for the exterior Stokes problem in three dimensions*, IMA J. Numer. Anal. **26** (2006), 297–325.

[15] N. Heuer, M.E. Mellado, E.P. Stephan: *hp-adaptive two-level methods for boundary integral equations on curves*, Computing **67** (2001), 305–334.

[16] S. Kurz, D. Pauly, D. Praetorius, S. Repin, and D. Sebastian: *Functional a posteriori error estimates for boundary element methods*, Preprint `arXiv:1912.05789`, 2019.

[17] M. Maischak, P. Mund, E.P. Stephan: *Adaptive multilevel BEM for acoustic scattering*, Comput. Methods Appl. Mech. Engrg. **150** (1997), 351–367.

[18] P. Mund, E.P. Stephan, J. Weiße: *Two-level methods for the single layer potential in* $\mathbb{R}^3$, Computing **60** (1998), 243–266.

[19] S. Repin: *A posteriori estimates for partial differential equations*, de Gruyter, Berlin 2008.

[20] H. Schulz, O. Steinbach: *A new a posteriori error estimator in adaptive direct boundary element methods: the Dirichlet problem*, Calcolo **37** (2000), 79–96.