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Accumulation and Obsolescence of Research Knowledge

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Abstract

In this paper we study how the scientific production of a representative researcher develops over his/her career. Using Pontryagin's maximum principle we derive the optimal capital accumulation over the life cycle of a scientist. In particular, we are able to show that the optimal production of human capital depends negatively on the obsolescence rate of knowledge as well as on the discount rate and that this reduction is largest at earlier ages. In a recent paper Feichtinger et al. (2020) also analyse scientific production over the life cycle with focus on greying of academia.

Keywords: Optimal control, scientific production over the life cycle, obsolescence of knowledge, optimal time allocation.

1 Introduction

In an interesting paper McDowell (1982) asked how does the durability of research knowledge affect the publication profiles over the career of scientists. For that he studied how the investment behavior of individuals in their human capital changes with the rate at which knowledge depreciates over time.

It is a fact that the durability of knowledge in natural sciences exceeds those in social sciences. The values for the various obsolescence rates are striking. According to McDowell (1982) a social scientist who interrupts her career, e.g. for childbearing and family duties, who returns to the job after some seven years will find the stock of effective knowledge reduced by half. For a physicist a similar skill reduction would require a career interruption of only four years but on he other hand almost three decades for an English professor.

To study the impact of obsolescence of research knowledge on career behavior McDowell uses human capital theory. In particular, he was able to show that the optimal production of human

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capital depends negatively on the obsolescence rate and that this reduction is largest at the early ages.

It is well-known that the production of scientific knowledge has shown a phenomenal upswing during the seven postwar decades. Remarkably, this rapid growth has been substantially accelerating in this period. An incredible flood of scientific books and papers published in an increasing number of journals illustrates this fact.

This rasant development has lead to a relatively new interesting field, namely to the science of science (compare Clauset et al., 2017). In the meantime, there are dozens or even hundreds of papers dealing with this meta-scientific matters.

Even most of the results obtained on this branch of knowledge are more or less recent, its origin is almost two hundred years old: the famous Belgian statistician A. Quetelet (1835) may be seen as the founder of the field. Another well-known contributor was A. Lotka (1926) who investigated the peculiar skew frequency of scientific productivity.[1]

While there are many kinds of such disparities in the context of scientific production (see, e.g. Feichtinger, 2019) age-dependency is one of the most remarkable ones. A more or less steep increase of the publication rate of a scientist followed by gradual decrease after reaching a maximum is the typical age pattern. Although this hump-shaped right-skew path of scientific productivity is well-kown since long time, recently additional life cycle patterns have been identified by real data; see Way et al. (2017). In this context and also as introduction to the intertemporal optimization model we are going on to analyze in the present paper we refer to Feichtinger et al, (2019). There it is shown that all the life patterns observed by Way et al. (2017) can be seen as the outcome of an optimal control model for certain parameter constellations.

Age is not only the key variable in demography and crucial in sociology, but it plays also an important role in economics. The development and interaction of socio-economic variables with the age of agents have been studied since the sixties (Becker, 1962). Other models dealing with efficient investment in human capital over the life cycle using optimal control techniques are Ben-Porath (1967) and Blinder and Weiss (1976). Later, Levine and Stephan (1991) considered a life cycle approach to explain age patterns of scientific production as result of age-specific research activity; see also Stephan (1996) for a survey.

The paper of McDowell (1982) which is along this line can be seen as the starting point of our analysis. Even he refers to the work of Ben-Porath (1967) where the efficiency of investment in the stock of research knowledge depends on this stock, McDowell does not analyze such a control-state interaction. Extending McDowell's approach by including capital stock in the production function of human capital we will prove a similar result. Note the relationship of McDowell's simplification with von Weizsäckers optimal training policies-model (see von Weizsäcker, 1967, and its extended treatment in Feichtinger and Hartl, 1986).

By using Pontryagin's maximum principle we are able to derive the optimal investment policy. Under rather general assumptions it turns out that one starts with full research followed by a

[1] He showed that the frequency of a scienist with n publications is proportional to n^{-2}.

phase in which only part of the time budget is used for research, while the rest of the working day is used for exploiting the human capital stock for earning reward in the form of reputation or income for publishing or presenting the research results.

Although we do not provide any analytical proof numerical simulations show that the optimal investment in human capital is monotonically decreasing over time whereas human capital follows a hump shaped pattern, at least for low initial values.

The paper is organized as follows. After this introduction the model will be presented in section 2. Section 3 states the necessary optimality conditions resulting from Pontryagin's maximum principle as well as results which can be obtained analytically. In section 4 the time-dependent solutions are presented which are obtained by numerical methods. To compute solutions of the canonical system in cases where a closed form solution could not be found we applied a Runge-Kutta algorithm (see, e.g., Ascher and Petzold, 1998). This section contains some insights gained by phase portrait analysis as well as time paths which are shown and discussed. Since our approach avoids to validate the model by concrete data, the results should be understood only qualitatively.

In section 5 a sensitivity analysis with respect to key parameters of the model is carried out. In particular, we could prove that the investment term shows the same behaviour with respect to the obsolescence rate as obtained by McDowell, although under a different dynamics. Additionally a similar dependence with respect to the discount rate can be observed.

Finally, our results are summarized in the concluding section 6 where also some ideas for extensions are presented.

2 The model

Consider a scientist and follow his/her career over the life cycle. Let time start at the begin of his career, say at the onset of his/her publishing activites. Assume that time t develops in a continuous manner and ends with the retirement of the scientist or his/her death at time T.

To keep the following analysis as simple as possible a deterministic approach is chosen, i.e. the end of the planning horizon is given, all variables and their dynamics are assumed to be known and develop in a deterministic way.

The model we are going on to introduce exhibits two key variables.

The first is the stock of knowledge an agent has at time t, denoted as $k(t)$. The dynamics of this individual human capital stock is influenced twofold. Firstly, in a positive way by learning. The accumulation of knowledge may take place in a quite heterogeneous manner. The scientist can sit at his/her desk trying to extend existing results or to think about new ideas, facts and methods, to trace out new models, to analyze them and to discuss the obtained innovations. Otherwise, and teamwork seems to be increasingly important in most of the sciences, the agent can discuss all this issues with colleagues, either in personal contact or via internet. Secondly, the stock of knowledge depreciates over time. As already mentioned in the introduction these values for obsolescence rates are striking.

While the depreciation of the human capital is beyond a scentist's radius of action, he/she can steer its accumulation. To simplify the analysis we assume that is only the time that is used for the agents investment activities (and not the intensity of research or the availability of resources).

Thus, in the tradition of early life cycle models (see Ben-Porath, 1967) we assume that a proportion of time, $s(t)$, is devoted to the investment in human capital, while the rest of the working day, i.e. $1 - s(t)$ time units are used for exploiting the stock of knowledge to produce scientific output as papers, books, patents. Note that it is this latter activity which generates the earnings of the agent. Note further that this bisection of the daily available time budget might be not quite realistic in practice. Firstly, the distinction between conceiving a paper and producing a final output is probably not that clear-cut as assumed in the present ansatz. And secondly, in reality there are additional forms of activities as, e.g., on-the job training.

The essential feature of intertemporal optimization is that the control $s(t)$ does not affect the state variable $k(t)$ directly but in an indirect way over its rate of change $\dot{k}(t)$. For the following it is crucial how $s(t)$ and $k(t)$ interact by influencing the dynamics of $k(t)$. Generally, this is described by the impact of a function $f(s(t), k(t))$ on the time derivative $\dot{k}(t)$ of the knowledge stock.

Thus, the dynamics of the system is described by the following ordinary differential equation

$$\dot{k}(t) = f(s(t), k(t)) - \delta k(t), \tag{1}$$

where the parameter $\delta > 0$ denotes the obsolecence rate of human capital. For the function f we assume as Ben-Porath (1967) does

$$f(s,k) = \beta_0 \left[s(t)k(t)\right]^{\beta} \tag{2}$$

where β_0 is a positive scaling parameter, and β is assumed to be within the interval $(0, 1)$. [2]

The objective of the agent is to maximize the discounted value of his/her scientific production

$$\max_{0 \leq s(t) \leq 1} \int_0^T e^{-rt} \left[k(t)(1 - s(t))\right] dt \tag{3}$$

where $r \geq 0$ denotes the non-negative discount rate.

As we assume that the scientist may even work as *professor emeritus* after retirement and could gain monetary benefits from his level of knowledge the salvage value is set to $S = 0$ because there are no pockets in the shroud.

Together with a given initial capital endowment [3]

$$k(0) = k_0 > 0 \tag{4}$$

[2]In case of the hairline case $\beta = 1$ we have a linear model. As a consequence bang-bang solutions or solutions with singular arcs may be optimal.

[3]Note that we have to start with a strictly positive value of initial knowledge ($k_0 > 0$) as otherwise no knowledge could be accumutaled due to the system dynamics in (1).

and the constraint for the control variable

$$0 \leq s(t) \leq 1 \tag{5}$$

we get an optimal control problem (1)- (5) with one state variable $k(t)$ and one control $s(t)$.

3 Optimality conditions

3.1 Canonical System

We apply Pontryagin's Maximum Principle (see, e.g., Grass et al., 2008) to find an optimal solution to the intertemporal optimization problem (1) - (5).

The Hamiltonian is given by

$$\mathcal{H} = k(t)(1 - s(t)) + \lambda(t) \left\{ \beta_0 \left[s(t)k(t) \right]^{\beta} - \delta k(t) \right\} \tag{6}$$

As the optimal control has to maximize the Hamiltonian, the first order condition leads to

$$\frac{\partial \mathcal{H}}{\partial s} = -k(t) + \lambda(t)\beta\beta_0 s(t)^{\beta-1} k(t)^{\beta} = 0 \quad \Rightarrow \quad s(t) = \frac{\left[\beta_0 \beta \lambda(t)\right]^{\frac{1}{1-\beta}}}{k(t)} \tag{7}$$

Due to the control constraint $0 \leq s(t) \leq 1$ the optimal control $s(t)$ is given by eq. (7) only as long as $k(t) \geq [\beta_0 \beta \lambda(t)]^{1/(1-\beta)}$; otherwise it is given by $s(t) = 1$.

The adjoint variable $\lambda(t)$ has to follow the differential equation

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial \mathcal{H}}{\partial k(t)} = \lambda(t) \left(r + \delta - \beta_0 \beta s(t)^{\beta} k(t)^{\beta-1} \right) - 1 + s(t), \quad \lambda(T) = 0. \tag{8}$$

Due to the control constraint $0 \leq s(t) \leq 1$ we distinguish two Regions in the following: .

1. **Region II:**

   In Region II, i.e. $k(t) > [\beta_0 \beta \lambda(t)]^{1/(1-\beta)}$, the canonical system is given by

$$\dot{k}(t) = \beta_0 \left[s(t)k(t) \right]^{\beta} - \delta k(t) \qquad k(0) = k_0 \geq 0 \tag{9}$$

$$\dot{\lambda}(t) = \lambda(t) \left(r + \delta - \beta_0 \beta s(t)^{\beta} k(t)^{\beta-1} \right) - 1 + s(t), \qquad \lambda(T) = 0 \tag{10}$$

   with optimal control given by

$$s(t) = \frac{\left[\beta_0 \beta \lambda(t)\right]^{\frac{1}{1-\beta}}}{k(t)} \tag{11}$$

Plugging the optimal control (11) into the dynamics (9)-(10) of state and adjoint variable leads to

$$\dot{k}(t) = \beta_0^{\frac{1}{1-\beta}} \beta^{\frac{\beta}{1-\beta}} \lambda^{\frac{\beta}{1-\beta}} - \delta k(t) \tag{12}$$

$$\dot{\lambda}(t) = \lambda(t)(r+\delta) - 1, \qquad \lambda(T) = 0. \tag{13}$$

By straight forward computation one can derive an explicit solution of the differential equation (13) resulting in

$$\lambda_{II}(t) = \frac{1}{r+\delta}\left[1 - e^{(r+\delta)(t-T)}\right] \tag{14}$$

2. **Region I:**

   In Region I, i.e. $k(t) < [\beta_0 \beta \lambda(t)]^{1/(1-\beta)}$, the optimal control is given by $s_I(t) = 1$ leading to the canonical system

$$\dot{k}(t) = \beta_0 \left[k(t)\right]^{\beta} - \delta k(t) \qquad k(0) = k_0 \geq 0 \tag{15}$$

$$\dot{\lambda}(t) = \lambda(t)\left(r + \delta - \beta_0 \beta k(t)^{\beta-1}\right) \tag{16}$$

   One can find an explicit solution of the differential equation (15) which is given by

$$k_I(t) = e^{-\delta t}\left[\frac{\beta_0}{\delta} e^{\delta(1-\beta)t} + c_1\right]^{\frac{1}{1-\beta}} \tag{17}$$

3. **Matching conditions:**

   Let us assume that the structure of the solution is the following (see Proposition 1 below):

   We start with a low initial value of knowledge $k(0)$. In a first phase (Region I) knowledge $k(t)$ is increased with $s(t) = 1$, then we switch to a second phase, i.e. Region II.

   To determine the constants of intergration c_0, c_1 and the switching time τ the following matching conditions have to hold:

   - Initial condition $k(0) = k_0$ implies

$$k_I(0) = \left[\frac{\beta_0}{\delta} + c_1\right]^{\frac{1}{1-\beta}} = k_0 \quad \Rightarrow \quad c_1 = {k_0}^{1-\beta} - \frac{\beta_0}{\delta} \tag{18}$$

- Continuity condition of the control implies $k_I(\tau) = [\beta_0 \beta \lambda_{II}(\tau)]^{\frac{1}{1-\beta}}$ leading to

$$k_I(\tau) = e^{-\delta\tau}\left[\frac{\beta_0}{\delta}e^{\delta(1-\beta)\tau} + c_1\right]^{\frac{1}{1-\beta}} = \left[\frac{\beta_0\beta}{r+\delta}\left[1 - e^{(r+\delta)(\tau-T)}\right]\right]^{\frac{1}{1-\beta}} \tag{19}$$

$$\Rightarrow \quad \frac{\beta_0}{\delta} + c_1 e^{-\delta(1-\beta)\tau} = \frac{\beta_0\beta}{r+\delta}\left[1 - e^{(r+\delta)(\tau-T)}\right] \tag{20}$$

  From (20) the switching time τ can only be determined numerically .

- Continuity condition of the state $k_I(\tau) = k_{II}(\tau)$ implies

$$k_I(\tau) = e^{-\delta\tau}\left[\frac{\beta_0}{\delta}e^{\delta(1-\beta)\tau} + c_1\right]^{\frac{1}{1-\beta}} = k_{II}(\tau) \tag{21}$$

  Unfortunately we do not have a closed form solution of $k_{II}(t)$.

3.2 Steady States and Isoclines:

To find steady states and carry out a phase portrait analysis we find the isoclines in a first step.

From (15) and (16) it follows that neither a $\dot{k} = 0-$isocline nor a $\dot{\lambda} = 0-$isocline exists in Region I, and thus there is also no steady state in this region.

From (13) it follows that in Region II the $\dot{\lambda} = 0-$isocline is given by a horizontal line at $\lambda = 1/(r+\delta)$ in the $(k, \lambda)-$plane. Below (above) this isocline $\lambda(t)$ is decreasing (increasing).

The $\dot{k} = 0-$ isocline can be determined from (12) and leads to the curve

$$\lambda(k) = \frac{(\delta k)^{\frac{1-\beta}{\beta}}}{\beta\beta_0^{1/\beta}} \tag{22}$$

The intersection of these two isoclines leads to a unique steady state

$$(k^\infty, \lambda^\infty) = \left(\frac{\beta_0^{\frac{1}{1-\beta}}}{\delta}\left[\frac{\beta}{r+\delta}\right]^{\frac{\beta}{1-\beta}}, \frac{1}{r+\delta}\right) \tag{23}$$

This steady state is saddle path stable and in case of an infinite time horizon the optimal solution would approach it asymptotically. For a finite time horizon, as in our model, the following result can be shown.

Proposition 1:

For solutions of the optimal control model (1) - (5) the following holds:

1. An optimal solution always ends in Region II.

2. A transition from Region II to Region I is not possible.

Proof:

1. As we start with $k(0) > 0$ this implies $k(T) > 0$; even without investing into knowledge (i.e. $s(t) = 0$) knowledge can decay exponentially at most, and thus never becomes 0. As $\lambda(T) = 0$ and due to continuity $k(t) > [\beta_0 \beta \lambda(t)]^{1/(1-\beta)}$ in a time interval $(T - \epsilon, T]$ and therefore every solution ends in Region II.

2. Let us consider the switching function

$$\sigma(t) = k(t) - [\beta_0 \beta \lambda(t)]^{\frac{1}{1-\beta}} \tag{24}$$

The optimal control is given by

$$s(t) = \begin{cases} 1 & \text{if} \quad \sigma(t) < 0 \qquad \text{Region I} \\ 1 & \text{if} \quad \sigma(t) = 0 \qquad \text{along the border of these 2 regionsI} \\ \frac{[\beta_0 \beta \lambda(t)]^{1/(1-\beta)}}{k(t)} & \text{if} \quad \sigma(t) > 0 \qquad \text{Region II} \end{cases}$$

A re-entry to Region I is not possible, iff $\dot{\sigma}(t)|_{\sigma(t)=0} > 0$.

Straight forward calculations lead to

$$\dot{\sigma}(t) = \dot{k} - \frac{[\beta_0 \beta \lambda(t)]^{\frac{1}{1-\beta}}}{1-\beta} \frac{\dot{\lambda}(t)}{\lambda(t)} \tag{25}$$

which reduces along the border $\sigma(t) = 0, s(t) = 1$ to

$$\dot{\sigma}(t)|_{\sigma(t)=0} = \dot{k} - \frac{k(t)}{1-\beta} \frac{\dot{\lambda}(t)}{\lambda(t)} \tag{26}$$

and after plugging in for the time dervaties of k and λ eventually to

$$\dot{\sigma}(t)|_{\sigma(t)=0} = \frac{k(t)^\beta}{1-\beta} \left[\beta_0 - k(t)^{1-\beta} \underbrace{(r + 2\delta - \beta\delta)}_{>0} \right] \tag{27}$$

As a consequence a transition from Region II into Region I only occurs at points with

$$k(t) > \tilde{k} := \left[\frac{\beta_0}{r + 2\delta - \beta\delta} \right]^{\frac{1}{1-\beta}} \tag{28}$$

Consider the point of intersection $(\hat{\lambda}, \hat{k})$ between the $\dot{\lambda} = 0-$isocline with the curve seperating Region I from Region II. This leads to

$$\hat{\lambda} = \frac{1}{r+\delta}, \qquad \hat{k} = \left(\frac{\beta_0 \beta}{r+\delta}\right)^{\frac{1}{1-\beta}} \tag{29}$$

Now it can be shown that $\hat{k} < \tilde{k}$ as

$$\hat{k}^{(1-\beta)} - \tilde{k}^{(1-\beta)} = \frac{\beta_0 \beta}{r+\delta} - \frac{\beta_0}{r+2\delta-\beta\delta} = \frac{-\beta_0(1-\beta)(r+\delta(1-\beta))}{(r+\delta)(r+2\delta-\beta\delta)} < 0$$

Therefore only orbits above the $\dot{\lambda} = 0-$ isocline may enter Region I from Region II. Due to the terminal condition $\lambda(T) = 0$ solutions of the control problem (1)- (5) are always below the $\dot{\lambda} = 0-$ isocline when they are in Region II and therefore cannot enter Region I anymore.

Remark: As a consequence there are only 2 types of solution patterns possible:

1. For low initial values of knowledge $k(0)$ we start with $s(t) = 1$ (i.e. in Region I), at time τ we switch to Region II, but we can never re-enter Region I and thus the optimal solution ends in Region II.

2. For sufficiently high values of initial knowledge $k(0)$ we immediately start in Region II, As a transition from Region II to Region I is not possible the whole solution stays in Region II.

4 Numerical analysis: phase portaits and time paths

To present phase portraits as well as time paths of optimal solutions we proceed with a numerical example. Without loss of generality we may assume $\beta_0 = 1$. As time horizon we choose $T = 40$ years and the parameter values $r = 0.025$, $\delta = 0.05$, $\beta = 0.5$.

The canonical system is

$$\begin{array}{lllll}
\dot{k} & = & \sqrt{sk} - 0.05k & = & \min\left\{\sqrt{k}, \frac{\lambda}{2}\right\} - 0.05k & k(0) = k_0 \\
\dot{\lambda} & = & \left(0.075 - \frac{1}{2}\sqrt{\frac{s}{k}}\right)\lambda - 1 + s & = & \left(0.075 - \min\left\{\frac{1}{2\sqrt{k}}, \frac{\lambda}{4k}\right\}\right)\lambda - 1 + \min\left\{1, \left(\frac{\lambda^2}{4k}\right)\right\} & \lambda(T) = 0 \\
s & = & \min\left\{1, \left(\frac{\lambda^2}{4k}\right)\right\} & & &
\end{array}$$

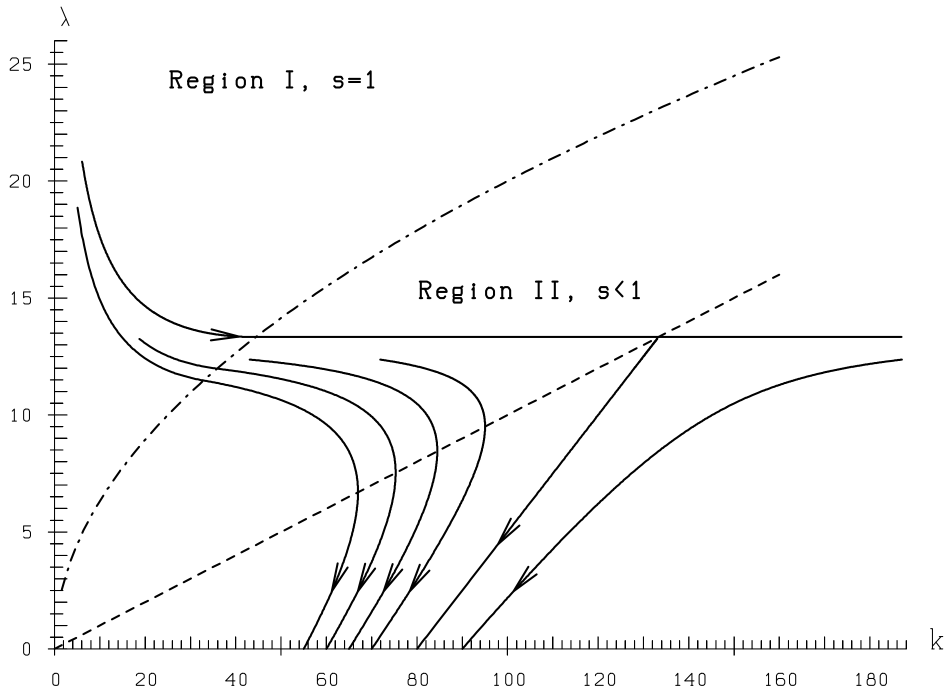

Fig. 1.: Phase portrait in the $(k, \lambda)-$plane.

Figure 1 shows a phase portrait in the $(k, \lambda)-$ plane. According to the general case we can distinguish two regions which are seperated by the curve $\lambda = 2\sqrt{k}$. This seperating curve can be seen as dashed-dotted line in Figure 1. Above this curve we are in Region I with the optimal control $s = 1$, below we are in Region II with the optimal control given by $s = \lambda^2/(4k)$.

In Region II the $\dot{\lambda} = 0-$ isocline is a horizontal straight line at $\lambda = 13.\dot{3}$, which coincides with the inset of the saddle path stable long run equilibrium. For $\beta = 0.5$ the $\dot{k} = 0-$ isocline is also a straight line, for the chosen parameter values it is $\lambda = k/10$ (the dashed line in Fig. 1). The intersection of these isoclines leads to the unique long run equilibrium at $(k^\infty, \lambda^\infty) = (133.\dot{3}, 13.\dot{3})$.

This figure also shows that for low initial values of human capital solution paths start in Region I (i.e. with $s = 1$) whereas for larger values solutions start already in Region II. Moreover left to the outset of the saddle human capital $k(t)$ evolves humped shaped, i.e. initially increasing and then decreasing, but right to this outset $k(t)$ is only decreasing. This can also be seen in the following Figure 2, where time paths of the state $k(t)$ are depicted.

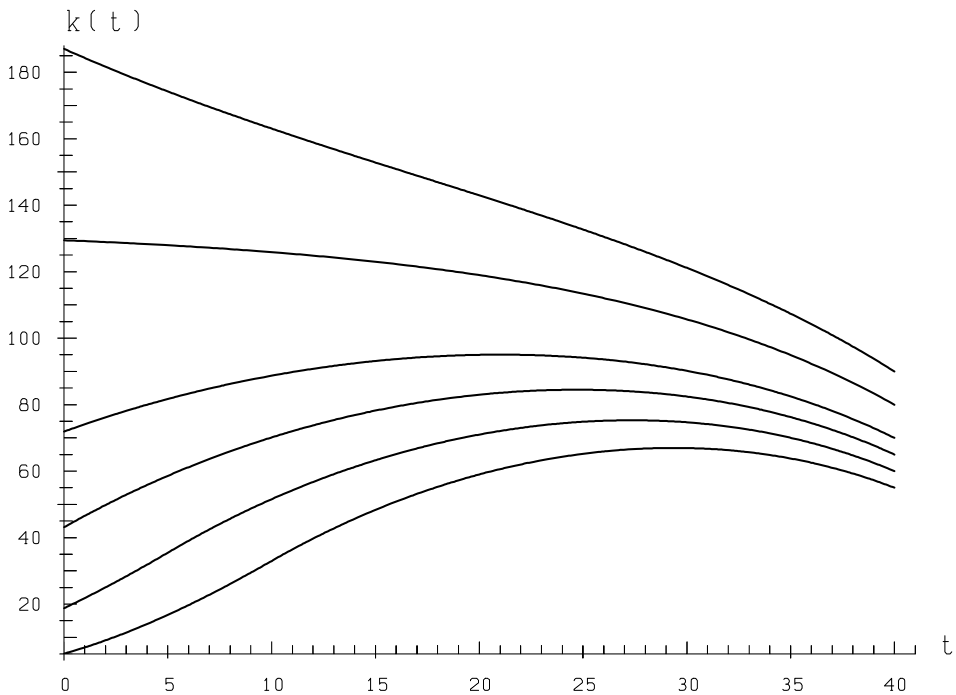

Fig. 2.: Time paths of the state $k(t)$.

The time paths in fig. 2 and fig. 4. correspond to the phase portraits in fig. 1 and fig. 3 concerning initial values of knowledge at the initial time $t = 0$.

The following figure 3 depicts orbits in the state/control space. For sufficiently small values of knowledge all available time is allocated to increase knowledge, afterwards the control is reduced.

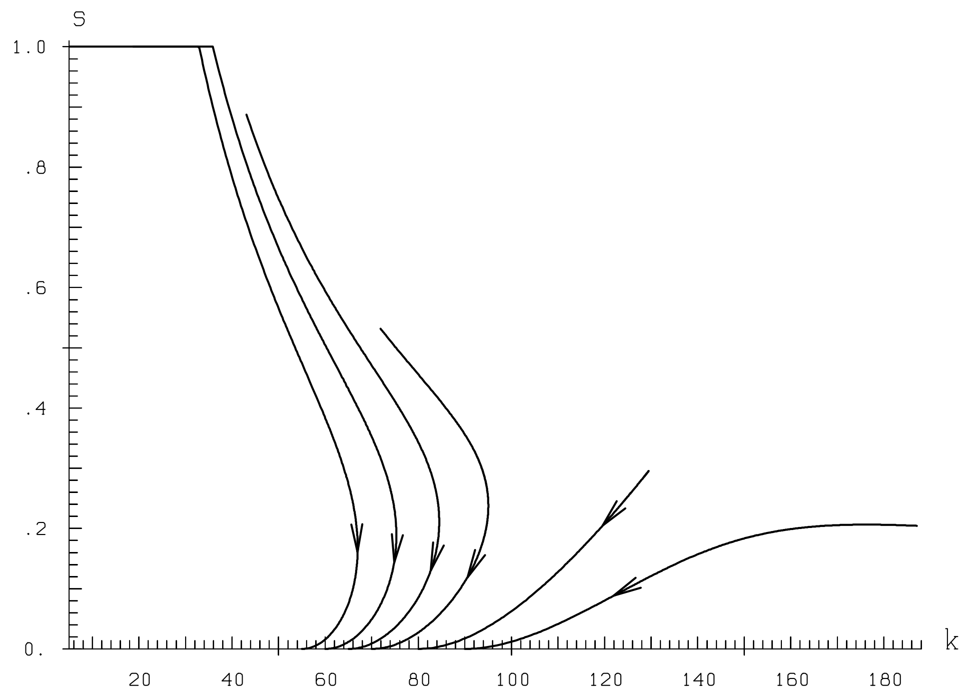

Fig. 3.: Phase portrait in the $(k, s)-$plane.

As can be seen from figure 4 the time paths of the control are in most cases monotonically decreasing, althought for very high values of knowledge also a slight increase of the control is possible.

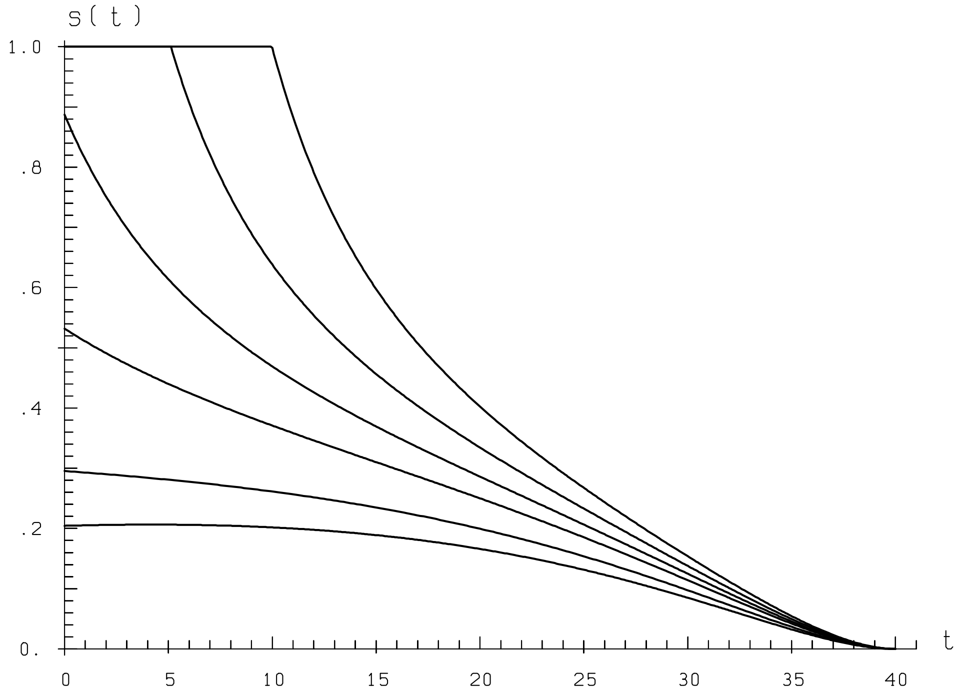

Fig. 4.: Time paths of the control $s(t)$.

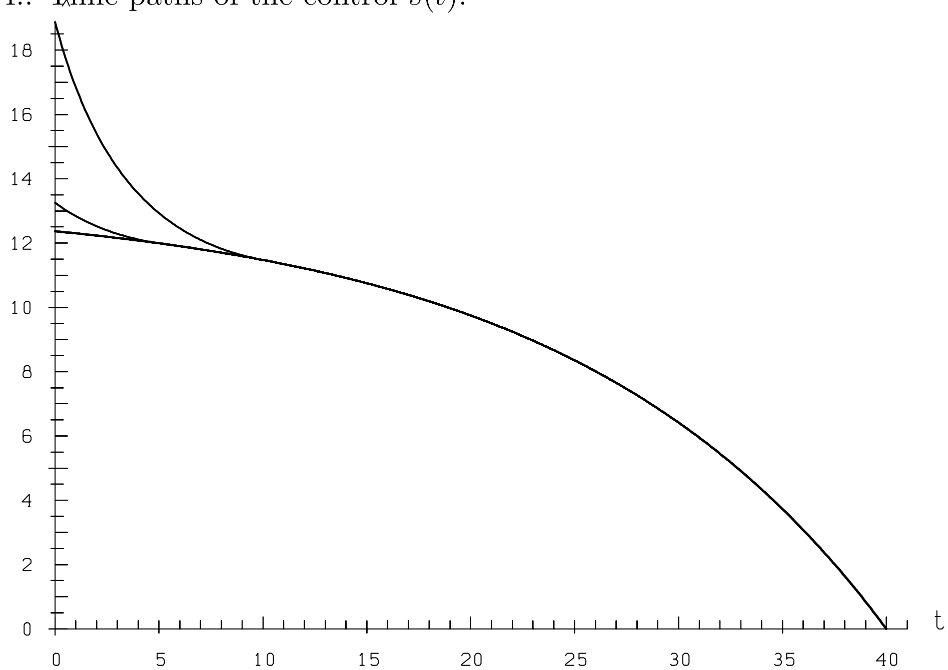

Fig. 5.: Time paths of adjoint variable $\lambda(t)$.

Finally the last figure shows the time paths of the adjoint variable. In Region II these time paths coincide.

5 Sensitivity analysis: dependency on delta in the style of McDowell

Following Mc Dowell (1982) we consider the sensitivity of the production of knowledge $L(t)$ with respect to various parameters, but in our case $L(t)$ is given by $L(t) = \beta_0[s(t)k(t)]^\beta$. Nevertheless we get similar results summarized in the following proposition.

Proposition 2:

In Region II it holds that

$$\frac{\partial L(t)}{\partial \delta} \leq 0, \qquad \text{and} \qquad \frac{\partial^2 L(t)}{\partial t \partial \delta} \geq 0 \qquad \forall t \in [0,T] \tag{30}$$

Proof:

1. According to (11) $L(t)$ is given by

$$L_{II}(t) = \beta_0 \left[s_{II}(t)k_{II}(t)\right]^\beta = \beta_0 \left\{ \frac{[\beta_0 \beta \lambda_{II}(t)]^{\frac{1}{1-\beta}}}{k_{II}(t)} k_{II}(t) \right\}^\beta = \beta_0[\beta_0 \beta \lambda_{II}(t)]^{\frac{\beta}{1-\beta}}$$

   As $\beta_0 > 0$ and $\beta \in (0,1)$ it is sufficient to show that the inequalities in (30) hold for $\lambda_{II}(t)$; i.e.

$$\frac{\partial \lambda_{II}(t)}{\partial \delta} \leq 0, \qquad \text{and} \qquad \frac{\partial^2 \lambda_{II}(t)}{\partial t \partial \delta} \geq 0 \qquad \forall t \in [0,T] \tag{31}$$

2. Taking the derivative of

$$\lambda_{II}(t) = \frac{1}{r+\delta}\left[1 - e^{(r+\delta)(t-T)}\right]$$

   with respect to δ leads to

$$\frac{\partial \lambda_{II}(t)}{\partial \delta} = \frac{e^{(r+\delta)(t-T)}\left[1 + (T-t)(r+\delta)\right] - 1}{(r+\delta)^2} \tag{32}$$

3. The time derivative of (32) is given by

$$\frac{\partial^2 \lambda_{II}(t)}{\partial t \partial \delta} = e^{(r+\delta)(t-T)}(T-t) \geq 0, \quad \forall t \in [0,T]. \tag{33}$$

4. At the end time point it follows from (32) that $\partial \lambda_{II}(T)/\partial \delta = 0$ and as $\partial^2 \lambda_{II}(t)/(\partial t \partial \delta) \geq 0$ this implies that $\partial \lambda_{II}(t)/\partial \delta \leq 0 \quad \forall t \in [0,T]$.

Remark:

In a similar way it can be shown that in Region II it holds that

$$\frac{\partial L(t)}{\partial r} \leq 0, \qquad \text{and} \qquad \frac{\partial^2 L(t)}{\partial t \partial r} \geq 0 \qquad \forall t \in [0,T] \tag{34}$$

6 Conclusion

The present paper may be seen in the tradition of life cycle models studied by Becker (1962), Ben-Porath (1967) and Blinder and Weiss (1976) (to mention a few important papers of pertinent earlier work). It is assumed that a representative scientist allocates each working day between investment in human capital and exploiting this stock by producing scientific output as writing papers, teaching etc. By using an intertemporal optimisation ansatz we have been able to characterize the optimal investment path in a qualitative way. Pontryagin's maximum principle delivers insights into the qualitative structure of optimal capital accumulation.

At early stages when endowment with human capital is low, all the time is allocated to increase knowledge until it reaches a certain threshold. Then it is optimal for the scientist to switch to a second stage, where part of time is also used to produce scientific output and earn money. The time being invested in still accumulating knowledge or at least reduce depreciation is monotonically decreasing tending towards zero, for reasonable parameter values.

The starting point of our resesrch was a paper by McDowell (1982) in which he analysed the impact of obsolescence of knowledge on research productivity over a career. For that he studied how the investment behaviour of individuals in their human capital changes with the rate at which the knowledge depreciates over time. In our contribution we corrected a mis-specification of the production function for human capital by using the original function proposed by Ben-Porath (1967).

One main result of our 'variant' of McDowell's life cycle model is that his results remain valid for a correct use of the human capital stock in Ben-Porath's style. In particular, we were able to show that the optimal production of human capital depends negatively on the obsolescence rate and that this reduction is largest at earlier ages.

Among the possible extensions we mention a larger space of state variables. Generally, it is not only the stock of knowledge a representative researcher has accumulated which is responsible for his/her scientific production, but also the reputation in the scientific world, the position in the networks of colleagues etc.; compare, e.g. Feichtinger et al. (2019). Another interesting and promising venue of further pertinent research in this field would be inclusion of the so-called Matthew effect (see Merton, 1968, Feichtinger et al., 2020).

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