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Polynomial Chaos Expansion Approach for Stochastic Partial Differential Equations with Applications

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#### Abstract

In this thesis we study some types of stochastic partial differential equations (SPDEs) in the framework of white noise analysis and thier particular applications in optimal control. The thesis is divided in two parts: theoretical results and applications. In the first part we developed the theoretical framework for studying different classes of SPDEs with singular data. Particularly, we developed generalized Malliavin calculus on spaces of generalized stochastic functions based on the chaos expansions. We solved different classes of stochastic evolution equations using the chaos expansion method and generalized some of these results to the related optimal control problem.

The second part of the thesis is devoted to applications. We study infinite dimensional stochastic linear quadratic optimal control problems related to evolution equations discussed in the previous chapter. We proved an optimal feedback synthesis along with well-posedness of the Riccati equation in a general setting. We provided a novel numerical framework for solving this type of control problems using the method of chaos expansions. We also presented an approximation framework for computing the solution of the stochastic linear quadratic control problem on Hilbert spaces. For the finite horizon case, we proved convergence results of the finite-dimensional problem to the infinite-dimensional one. In addition, we developed a stochastic treatment of unbounded control action problems arising in a general class of dynamical systems which exhibit singular estimates, but are not necessarily analytic. Moreover, in the same setting we present a regularization scheme for operator differential algebraic equations with noise disturbances. Finally, we combined a polynomial chaos expansion method with splitting methods for solving particular classes of SPDEs.


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## Preface

This thesis is devoted to stochastic partial differential equations, their theoretical treatment and their applications in the framework of white noise analysis. A major contribution of this thesis is the development of generalized Malliavin calculus in the framework of white noise analysis, based on chaos expansion representation of stochastic processes and its application for solving several classes of stochastic differential equations with singular data. Especially, stochastic equations with singular coefficients and singular initial conditions involving the main operators of Malliavin calculus are considered. The polynomial chaos expansion method is also combined with the operator semigroup theory in order to prove existence and uniqueness of solutions of nonlinear stochastic evolution equations with Wick-polynomial nonlinearities, random force and random initial condition. These equations include the stochastic Fujita equation, the stochastic Fisher-KPP equation, the stochastic FitzHugh-Nagumo equation and the stochastic ChaffeeInfante equation. These equations arise in ecology, medicine, engineering and physics. Additionaly, we proved existence and uniqueness of solutions of a large class of parabolic stochastic partial differential equations with multiplicative noise. Special cases include the heat equation with random potential, the Langevin equation, the Schrödinger equation, the transport equation driven by white noise. Moreover, a novel approach for numerical treatment of stochastic evolution equations which combines the polynomial chaos approach with splitting methods is also included in the thesis. Significant contributions are made in applications of the polynomial chaos expansion approach to stochastic control problems. Particularly, in the stochastic linear quadratic optimal control problem as well as in the regularization of stochastic operator differential algebraic equations.

Most of the results of this thesis are summarized in:
Book T. Levajković, H. Mena, Equations involving Malliavin calculus operators: Applications and numerical approximation. SpringerBriefs in Mathematics. Cham, Springer International Publishing Switzerland, 2017. ISBN: 978-3-319-65677-9.

The thesis is divided into two chapters. Chapter 1 deals with the theoretical framework of white noise analysis based on chaos expansion represen-
tation and solutions of particular equations. The individual sections correspond to the author's contributions concerning the chaos expansion method in Malliavin calculus (Section 1.1 and Section 1.2), the study of fundamental equations with higher order Malliavin operators (Section 1.3), a theoretical framework for solving stochastic evolution equations with multiplicative noise (Section 1.4), the solution of Malliavin-type differential equations (Section 1.5 and Section 1.6) and a theoretical framework for the study of stochastic evolution equations with Wick-polynomial nonlinearities (Section 1.7). Chapter 1 is based on the following publications:
1.1 T. Levajković, D. Seleši, Malliavin calculus for generalized and test processes. Filomat 31(13), 4231-4256, 2017.
1.2 T. Levajković, S. Pilipović, D. Seleši, Chaos expansion methods in Malliavin calculus: A survey of recent results. Novi Sad J. Math. 45(1), 45-103, 2015.
1.3 T. Levajković, S. Pilipović, D. Seleši, Fundamental equations with higher order Malliavin operators. Stochastic 88(1), 106-127, 2016.
1.4 T. Levajković, S. Pilipović, D. Seleši, M. Žigić, Stochastic evolution equations with multiplicative noise. Electron. J. Prob. 20(19), 1-23, 2015.
1.5 T. Levajković, H. Mena, Equations involving Malliavin derivative: A chaos expansion approach, in S. Pilipović, J. Toft (Eds.) PseudoDifferential Operators and Generalized Functions, Operator Theory: Advances and Applications, Vol. 245, 197-214, Springer International Publishing, 2015.
1.6 T. Levajković, D. Seleši, Nonhomogeneous first order linear Malliavin type differential equation, in S. Molahajloo, S. Pilipović, J. Toft, M. W. Wong (Eds.), Pseudo-Differential Operators: Generalized Functions and Asymptotic, 353-369, Springer, 2013.
1.7 T. Levajković, S. Pilipović, D. Seleši, M. Žigić, Stochastic evolution equations with Wick-polynomial nonlinearities. Electron. J. Probab. 23(116), 1-25, 2018.

Chapter 2 addresses applications of the theoretical results, extensions and generalizations to different classes of stochastic differential equations. Particularly, applications in optimal control problems are shown. Namely, a novel theoretical framework for solving generalized linear quadratic optimal control problems (Section 2.1 and Section 2.2), a feedback synthesis of the stochastic linear quadratic optimal control problem with singular estimates on the finite time interval (Section 2.3), a convergence analysis
in Hilbert spaces (Section 2.4), a numerical treatment of a stochastic linear quadratic regulator problem for the infinite horizon case (Section 2.5), a splitting/polynomial chaos expansion approach for stochastic evolution equations (Section 2.6) and a regularization approach for operator differential algebraic equations with noise (Section 2.7). Chapter 2 is based on the following publications:
2.1 T. Levajković, H. Mena, A. Tuffaha, The stochastic linear quadratic control problem: A chaos expansion approach. Evol. Equ. Control Theory 5(1), 105-134, 2016.
2.2 T. Levajković, H. Mena, L.-M. Pfurtscheller, Solving stochastic LQR problems by polynomial chaos. IEEE Control Systems Letters 2(4), 641-646, 2018.
2.3 T. Levajković, H. Mena, A. Tuffaha, The stochastic $L Q R$ optimal control with fractional Brownian motion, in M. Oberguggenberger, J. Toft, J. Vindas, P. Wahlberg (Eds.) Advanced in Partial Differential Equations, Generalized Functions and Fourier Analysis, Dedicated to Stevan Pilipović on the Occasion of his 65th Birthday, 115-151, Birkhäuser, 2017.
2.4 C. Hafizoglu, I. Lasiecka, T. Levajković, H. Mena, A. Tuffaha, The stochastic linear quadratic problem with singular estimates. SIAM J. Control Optim. 55(2), 595-626, 2017.
2.5 T. Levajković, H. Mena, A. Tuffaha, A numerical approximation framework for the stochastic linear quadratic regulator problem on Hilbert spaces. Appl. Math. Optim. 75(3), 499-523, 2017.
2.6 A. Kofler, T. Levajković, H. Mena, A. Ostermann, A splitting/polynomial chaos expansion approach for stochastic evolution equations, Submitted to: Stoch. PDE: Anal. Comp., 2019, arXiv.1903.10786.
2.7 R. Altmann, T. Levajković, H. Mena, Operator differential algebraic equations with noise arising in fluid dynamics. Monatsh. Math. 182(4), 741-780, 2017.

All of the above publications were written after the completion of the author's Ph.D. degree in April 2012. An effort was made to use a consistent notation in the introductory paragraphs which link to the individual papers.

In addition, the following publications were completed in the same period of time:

Publications:

1. D. Babić, B. Begović, T. Levajković, Analysis of the impact of passenger' preferences on the airline network structure: A probabilistic approach. Submitted to: Transportation Research Part B, 2018.
2. T. Levajković, D. Babić, M. Kalić, Airline revenue management for complex networks. Proceedings of the XLIV International Symposium on Operational Research, 758-764, 2017.
3. T. Levajković, H. Mena, M. Zarfl, Lévy processes, subordinators and crime modelling. Novi Sad J. Math. 46 (2), 65-86, 2016.
4. T. Levajković, H. Mena, On deterministic and stochastic linear quadratic control problems, in V. Mityushev, M. Ruzhansky (Eds.), Current Trends in Analysis and Its Applications, Trends in Mathematics, Research Perspectives, pp. 315-322, Springer International Publishing Switzerland, 2015.
5. T. Levajković, D. Seleši, Chaos expansion methods of stochastic processes for Malliavin-type equations. Electronic Notes in Discrete Mathematics 43, Elsevier, 289-298, 2013.

Textbooks for bachelor studies:
7. T. Levajković, K. Kukić, M. Borisavljavić, A. Jelović, N. Ćirić, D. Ilić, A. Perović, Mathematics 1: Book of exercises with solutions (in Serbian). Faculty of Traffic and Transport Engineering, University of Belgrade, 2015, ISBN: 978-86-7395-333-5.
8. M. Borisavljavić, N. Ćirić, S. Miloradović, T. Levajković, D. Ilić, K. Kukić, Mathematics book of exercises with solutions: Preparatory book for higher education entrance examination (in Serbian). Faculty of Traffic and Transport Engineering, University of Belgrade, Seventh edition, 2015, ISBN: 978-86-7395-302-1.

Acknowledgments. All the publications that are part of this thesis were written during my appointment at University of Innsbruck. I would like to express my gratitude to Professor Alexander Ostermann and Professor Michael Oberguggenberger for giving me the opportunity to work in great environments. I would also like to thank the support of the research grant for Austrian graduates (support for scientific purposes) for the project Nonlinear stochastic evolution equations: Theory and applications, financed by the Office of the Vice Rector for Research at University of Innsbruck. Additionally, I would like to thank the ÖAD bilateral project Austria-Serbia Solutions of stochastic equations involving differential and pseudodifferential operators in algebras of generalized stochastic processes funded by the Austrian agency for international mobility and cooperation in education, science and research. Special thanks go to all my co-authors and colleagues, who contributed greatly in making the recent years enjoyable and successful. I would also like to thank the staff of the Applied Statistics group at the Institute of Statistics and Mathematical Methods in Economics, Faculty of Mathematics and Geoinformation, Vienna University of Technology, where this Habilitation thesis was finished. Last but not least, I would like to thank my family for their love and support.

## Author's contribution

This Habilitation thesis is based on fourteen papers and one book. I contributed to all publications in every stage of the work. In most cases the results presented here are product of extended discussions between coauthors. Thus, it is extremely difficult to quantify the percentage of contribution of each author. However, in the following I provide an approximate percentage of my personal contribution of each publication that is part of this thesis.

Book
T. Levajković, H. Mena, Equations involving Malliavin calculus operators: Applications and numerical approximation. SpringerBriefs in Mathematics. Cham, Springer International Publishing Switzerland, 2017. ISBN: 978-3-319-65677-9, 2017.

I contributed in all stages of the work, investigating the properties of the generalized operators of the Malliavin calculus, providing proofs of lemmas and theorems and obtaining the convergence results. Percentage of personal contribution $65 \%$.
1.1 T. Levajković, D. Seleši, Malliavin calculus for generalized and test processes. Filomat 31(13), 4231-4256, 2017.

I contributed in all stages of the work, developing the Malliavin calculus for stochastic processes, studying the properties of the operators, proving lemmas and theorems. Percentage of personal contribution $60 \%$.
1.2 T. Levajković, S. Pilipović, D. Seleši, Chaos expansion methods in Malliavin calculus: A survey of recent results. Novi Sad J. Math. 45(1),45-103, 2015.

I contributed in all stages of the work, extending the classical results to the generalized setting, proving the main properties and their generalizations. Percentage of personal contribution $50 \%$.
1.3 T. Levajković, S. Pilipović, D. Seleši, Fundamental equations with higher order Malliavin operators. Stochastic 88(1), 106-127, 2016.

I contributed in all stages of the work, developing the theoretical framework, solving equations and proving theorems. Percentage of personal contribution $35 \%$.
1.4 T. Levajković, S. Pilipović, D. Seleši, M. Žigić, Stochastic evolution equations with multiplicative noise. Electron. J. Prob. 20(19), 1-23, 2015.

I contributed in all stages of the work, developing the theoretical framework based on chaos expansions and the operator semigroup theory, proving lemmas and theorems and providing illustrative examples. Percentage of personal contribution $25 \%$.
1.5 T. Levajković, H. Mena, Equations involving Malliavin derivative: A chaos expansion approach, in S. Pilipović, J. Toft (Eds.) Pseudo-Differential Operators and Generalized Functions, Operator Theory: Advances and Applications, Vol. 245, 197-214, Springer International Publishing, 2015.

I contributed in all stages of the work, developing the theoretical framework suitable for solving equations with the Malliavin derivative operator and proving existence and uniqueness of generalized solutions of these equations. Percentage of personal contribution $60 \%$.
1.6 T. Levajković, D. Seleši, Nonhomogeneous first order linear Malliavin type differential equation, in S. Molahajloo, S. Pilipović, J. Toft, M. W. Wong (Eds.), Pseudo-Differential Operators: Generalized Functions and Asymptotic, 353-369, Springer, 2013.

I contributed in all stages of the work, solving nonhomogeneous linear Malliavin differential equations and proving theorems. Percentage of personal contribution $75 \%$.
1.7 T. Levajković, S. Pilipović, D. Seleši, M. Žigić, Stochastic evolution equations with Wick-polynomial nonlinearities. Electron. J. Probab. 23(116), 1-25, 2018.

I contributed in all stages of the work, proving lemmas and theorems and applications. The results extend the developments of the chaos approach combined with the operator semigroup theory presented in Section 1.4 for solving nonlinear evolution SPDEs. Percentage of personal contribution $30 \%$.
2.1 T. Levajković, H. Mena, A. Tuffaha, The stochastic linear quadratic control problem: A chaos expansion approach. Evol. Equ. Control Theory 5(1), 105-134, 2016.

I contributed in all stages of the work, developing the theoretical framework for solving stochastic linear quadratic optimal control problems and proving the two main theorems. The results extend the ideas of the polynomial chaos approach for solving SPDEs to optimal control problems. Percentage of personal contribution $40 \%$.
2.2 T. Levajković, H. Mena, L.-M. Pfurtscheller, Solving stochastic LQR problems by polynomial chaos. IEEE Control Systems Letters 2(4), 641-646, 2018.

I contributed in all stages of the work, providing theoretical analysis, performing the convergence analysis, proving the optimality result and developing the algorithm. Percentage of personal contribution $30 \%$.
2.3 T. Levajković, H. Mena, A. Tuffaha, The stochastic LQR optimal control with fractional Brownian motion, in M. Oberguggenberger, J. Toft, J. Vindas, P. Wahlberg (Eds.) Advanced in Partial Differential Equations, Generalized Functions and Fourier Analysis, Dedicated to Stevan Pilipović on the Occasion of his 65th Birthday, 115-151, Birkhäuser, 2017.

I contributed in all stages of the work, developing the theoretical framework and proving the optimality result. Percentage of personal contribution $45 \%$.
2.4 C. Hafizoglu, I. Lasiecka, T. Levajković, H. Mena, A. Tuffaha, The stochastic linear quadratic problem with singular estimates. SIAM J. Control Optim. 55(2), 595-626, 2017.

I contributed in all stages of the work, proving lemmas and theorems. Particularly, the theorem related with the characterization of the optimal control. Percentage of personal contribution $20 \%$.
2.5 T. Levajković, H. Mena, A. Tuffaha, A numerical approximation framework for the stochastic linear quadratic regulator problem on Hilbert spaces. Appl. Math. Optim. 75(3), 499-523, 2017.

I contributed in all stages of the work, developing the numerical framework, obtaining the theoretical results and proving of theorems. Percentage of personal contribution $35 \%$.
2.6 A. Kofler, T. Levajković, H. Mena, A. Ostermann, A splitting/ polynomial chaos expansion approach for stochastic evolution equations. Submitted to: Stoch. PDE: Anal. Comp., 2019, arXiv.1903.10786.

I contributed to this work performing the convergence analysis and proving the related theorems. Percentage of personal contribution $25 \%$.
2.7 R. Altmann, T. Levajković, H. Mena, Operator differential algebraic equations with noise arising in fluid dynamics. Monatsh. Math. 182(4), 741-780, 2017.

I contributed in all stages of the work, developing the theoretical framework for solving operator differential algebraic equations with noise, proving theorems and deriving a connection between specific classes of problems involving operators of Malliavin calculus and stochastic operator differential algebraic equations. Percentage of personal contribution $40 \%$.

## Chapter 1

## Theoretical Results

In this chapter we present the theoretical contributions of this thesis. Namely, the development of generalized Malliavin calculus in the framework of white noise analysis based on chaos expansion representation of stochastic processes and its application for solving several classes of stochastic differential equations with singular coefficients and singular initial conditions. Generalized operators of Malliavin calculus, the Malliavin derivative operator $\mathbb{D}$, the Skorokhod integral $\delta$ and the OrnsteinUhlenbeck operator $\mathcal{R}$ are introduced in terms of chaos expansions. The main properties of the operators, which are known in the literature for stochastic processes with finite second moments, are proven using the chaos expansion approach and extended for generalized stochastic processes. Moreover, fractional versions of these operators are also discussed and the connection with the corresponding operators of Mallaivin calculus through an isometry mapping is proven. Also, several classes of equations involving Malliavin calculus operators are solved with this technique.

The Malliavin derivative $\mathbb{D}$, the Skorokhod integral $\delta$ and the OrnsteinUhlenbeck operator $\mathcal{R}$ play a crucial role in the stochastic calculus of variations. They are part of the infinite-dimensional differential calculus on white noise spaces and is also called the Malliavin calculus [16, 24, 83, 87, 93, 98. In stochastic analysis, the Malliavin derivative characterizes densities of distributions, the Skorokhod integral is an extension of the Itô integral to non-adapted processes, and the Ornstein-Uhlenbeck operator plays the role of the stochastic Laplacian. Additionally, the Malliavin derivative appears as the adjoint operator of the Skorokhod integral, while their composition, the Ornstein-Uhlenbeck operator, is a linear, unbounded and self-adjoint operator. These operators are interpreted in quantum theory respectively as the annihilation, the creation and the number operators.

Since the pioneer work of Itô [47] that characterized stochastic integrals in terms of Hermite polynomials, another important keystone was the development of white noise analysis proposed by Hida [42] who set up
an appropriate functional analytical framework using nuclear operators to characterize Gaussian processes. Second quantization operator techniques are used to obtain weighted spaces of generalized stochastic processes such as the Hida and Kondratiev spaces. For infinite-dimensional analysis with a probabilistical approach we refer the reader to [24, 43, 80].

Originally, the Malliavin derivative was introduced by Paul Malliavin in order to provide a probabilistic proof of Hörmander's sum of squares theorem for hypoelliptic operators and to study the existence and regularity of a density for the solution of stochastic differential equations [82]. Nowadays, besides applications concerning the existence and smoothness of a density for the probability law of random variables, it has found significant applications in stochastic control and mathematical finance, particularly in option pricing and computing the Greeks (the Greeks measure the stability of the option price under variations of the parameters) via the Clark-Ocone formula [23, 83, 96]. Recently, in [89] a novel connection between the Malliavin calculus and the Stein method was discovered, which can be used to estimate the distance of a random variable from Gaussian variables. In Section 1.2 [74] this relationship was reviewed using the chaos expansion method.

In the classical setting [24, 79, 87], the domain of these operators is a strict subset of the set of processes with finite second moments leading to Sobolev type normed spaces. We recall these classical results and denote the corresponding domains with a "zero" in order to retain a symmetry between test and generalized processes. A more general characterization of the domain of these operators in Kondratiev generalized function spaces has been derived in [69, 72, 73]. Surjectivity of the operators for generalized processes for $\rho=1$ has been developed in [74, 75], while a setting for the domains of these operators for $\rho \in[0,1]$ and for test processes was developed in 62, 71]. We summarize these recent results, construct the domain of the operators and prove that they are linear and bounded within the corresponding spaces. We adopt the notation from [71, 74, 75] and denote the domains of all the operators in the Kondratiev space of distributions by a "minus" sign to reflect the fact that they correspond to generalized processes and the domains for test processes denote by a "plus" sign.

The Malliavin derivative of generalized stochastic processes has first been considered in [15] using the $\mathcal{S}$-transform of stochastic exponentials and chaos expansions with $n$-fold Itô integrals with some vague notion of the Itô integral of a generalized function. Our approach is different, it relies on chaos expansions via Hermite polynomials (in the Gaussian case) and provides more precise results. A fine gradation of generalized and test functions is followed where each level has a Hilbert structure and consequently each level of singularity has its own domain, range, set of multipliers, etc. We developed the calculus including the integration by parts formula, product rules, the chain rule, using the interplay of generalized processes with their test processes and different types of dual pairings. We apply the chaos expan-
sion method to illustrate several known results in Malliavin calculus and thus provide a comprehensive insight into its capabilities. For example, we proved some well-known classical results, such as the commutator relationship between $\mathbb{D}$ and $\delta$ and the relation between Itô integration and Riemann integration. These results are included in the first part of this chapter and associated to Section 1.1 [71], Section 1.2 [74] and Section 1.3 [75].

In second part of this chapter we apply the chaos expansion method for solving stochastic partial differential equations (SPDEs) with singular data. The focus is on the study of different classes of equations that involve the operators of Malliavin calculus, the study of stochastic evolution equations with multiplicative noise and stochastic evolution equations with Wick-polynomial nonlinearities and random force and random initial condition. These equations include the heat equation with random potential, the Langevin equation, the Schrödinger equation, the transport equation driven by white noise, the stochastic Fujita equation, the stochastic Fisher-KPP equation and the stochastic FitzHugh-Nagumo equation.

The main difficulty that may arise when solving equations with singular data (both linear and nonlinear) is the problem of multiplication of generalized functions. In this thesis we overcome this difficulty by interpreting the product as the Wick product (stochastic convolution) within the white noise analysis. Also, the Wick product is known for representing the highest order stochastic approximation of the ordinary product [86]. Alternative approaches have been developed in the theory of regularity structures by Martin Hairer 40] and in rough path theory and paracontrolled distributions by Massimiliano Gubinelli, Peter Imkeller and Nicolas Perkowski 38. Another possibility is to consider the equations in Colombeau algebras of generalized functions and after regularization interpret the product as a classical product [21, 90].

The method of chaos expansions has been applied successfully to several classes of SPDEs [50, [72, 73, 80, 85] to obtain an explicit form of the solution. The basic idea is to construct the solution of the considered SPDE as a Fourier series in terms of a Hilbert space basis of orthogonal stochastic polynomials, resulting in an infinite triangular system system of deterministic equations for the coefficients, which can be solved by induction. Summing up all coefficients of the expansion, i.e., the solutions of the deterministic system, and proving its convergence in an appropriate space of stochastic processes, one obtains the solution of the initial equation.

Besides the fact that the chaos expansion method is easy to apply (since it uses orthogonal bases and series expansions), the advantage of the method is that it provides an explicit form of the solution. We avoid using the Hermite transform 43] or the $\mathcal{S}$-transform [42], since these methods depend on the feasibility to apply their inverse transforms. The chaos expansion method requires only to find an appropriate weight factor to make the resulting series convergent. It is also known for being an efficient method in
numerical approximations, the so-called stochastic Galerkin method. Moreover, for non-Gaussian processes, convergence can be easily improved by changing the Hermite basis to another family of the Askey-scheme of hypergeometric orthogonal polynomials (Charlier, Laguerre, Meixner, etc.) [102]. The results of the second part of this chapter are related to Section 1.3 [75], Section 1.4 [76], Section 1.5 [62], Section 1.6 [70] and Section 1.7 [77].

## Preliminaries

We consider the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel sigma-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to the characteristic function

$$
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \phi \in S(\mathbb{R})
$$

given by the Bochner-Minlos theorem [43].
The Hilbert space of random variables with finite second moments is denoted by $L^{2}(\mu)$. The set of multi-indices $\mathcal{I}$ is $\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$, i.e. the set of sequences of non-negative integers which have only finitely many nonzero components. Particularly, we denote by $\mathbf{0}=(0,0,0, \ldots)$ the zero multi-index with all entries equal to zero and the $k$ th unit vector $\varepsilon^{(k)}=(0, \cdots, 0,1,0, \cdots)$, $k \in \mathbb{N}$, i.e., the sequence of zeros with the number 1 as the $k$ th component. The length of a multi-index is $|\alpha|=\sum_{i=1}^{\infty} \alpha_{i}$ for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots\right) \in \mathcal{I}$ and $\alpha!=\prod_{i=1}^{\infty} \alpha_{i}!$. We will use the convention that $\alpha-\beta$ is defined if $\alpha_{n}-\beta_{n} \geq 0$ for all $n \in \mathbb{N}$, i.e., if $\alpha-\beta \geq \mathbf{0}$. Let $(2 \mathbb{N})^{\alpha}=\prod_{k=1}^{\infty}(2 k)^{\alpha_{k}}$. Note that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ for $p>1$, see 43].

The Wiener-Itô theorem (sometimes also referred as the Cameron-Martin theorem) states that one can define an orthogonal basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ of $L^{2}(\mu)$, where $H_{\alpha}$ are constructed by means of Hermite orthogonal polynomials $h_{n}$ and Hermite functions $\xi_{n}$,

$$
H_{\alpha}(\omega)=\prod_{n=1}^{\infty} h_{\alpha_{n}}\left(\left\langle\omega, \xi_{n}\right\rangle\right), \quad \alpha \in \mathcal{I}, \quad \omega \in \Omega
$$

Then, every $F \in L^{2}(\Omega, \mu)$ has a unique chaos expansion representation

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega), \quad \omega \in S^{\prime}(\mathbb{R})
$$

such that

$$
\sum_{\alpha \in \mathcal{I}}\left|f_{\alpha}\right|^{2} \alpha!<\infty, \quad f_{\alpha} \in \mathbb{R}, \quad \alpha \in \mathcal{I}
$$

We denote by $\mathcal{H}_{1}$ the subspace of $L^{2}(\mu)$, spanned by the polynomials $H_{\varepsilon_{k}}(\cdot), k \in \mathbb{N}$. The subspace $\mathcal{H}_{1}$ contains Gaussian stochastic processes, e.g. Brownian motion is given by $B_{t}(\omega)=\sum_{k=1}^{\infty} \int_{0}^{t} \xi_{k}(s) d s H_{\varepsilon_{k}}(\omega)$.

Similarly, we denote by $\mathcal{H}_{m}$ the $m$ th order chaos space, i.e. the closure in $L^{2}(\Omega, \mu)$ of the linear subspace spanned by the orthogonal polynomials $H_{\alpha}(\cdot)$ with $|\alpha|=m, m \in \mathbb{N}_{0}$. Then, the Wiener-Itô chaos expansion states that $L^{2}(\Omega, \mu)=\bigoplus_{m=0}^{\infty} \mathcal{H}_{m}$, where $\mathcal{H}_{0}$ is the set of constants in $L^{2}(\Omega, \mu)$.

Changing the topology on $L^{2}(\mu)$ to a weaker one, Hida [42] defined spaces of generalized random variables containing the white noise as a weak derivative of the Brownian motion. Using the same technique as in 43] one can define Banach spaces $(S)_{\rho, p}$ of test functions and their topological duals $(S)_{-\rho,-p}$ of stochastic distributions for all $\rho \in[0,1]$ and $p \geq 0$.

Definition 1 Let $\rho \in[0,1]$, the stochastic test function spaces are defined by

$$
(S)_{\rho, p}=\left\{F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in L^{2}(\mu):\|F\|_{(S)_{\rho, p}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1+\rho}\left|f_{\alpha}\right|^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}
$$

for all $p \geq 0$. Their topological duals, the stochastic distribution spaces, are given by formal sums

$$
(S)_{-\rho,-p}=\left\{F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}:\|F\|_{(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho}\left|f_{\alpha}\right|^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
$$

for all $p \geq 0$. The Kondratiev space of test random variables is $(S)_{\rho}=$ $\bigcap_{p \geq 0}(S)_{\rho, p}$, endowed with the projective topology. Its dual, the space of Kondratiev generalized random variables is $(S)_{-\rho}=\bigcup_{p \geq 0}(S)_{-\rho,-p}$, endowed with the inductive topology.

The action of $F=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha} \in(S)_{-\rho}$ onto $f=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha} \in(S)_{\rho}$ is given by $\langle F, f\rangle=\sum_{\alpha \in \mathcal{I}}\left(b_{\alpha}, c_{\alpha}\right) \alpha$ !, where $\left(b_{\alpha}, c_{\alpha}\right)$ stands for the inner product in $\mathbb{R}$. The following Gel'fand triple is obtained

$$
(S)_{\rho} \subseteq L^{2}(\mu) \subseteq(S)_{-\rho}
$$

The spaces $(S)_{\rho, p}$ and $(S)_{-\rho,-p}$ are separable Hilbert spaces. Moreover, $(S)_{\rho}$ and $(S)_{-\rho}$ are nuclear spaces. For $\rho=0$ we obtain the space of Hida stochastic distributions $(S)_{-0}$ and for $\rho=1$ the Kondratiev space of generalized random variables $(S)_{-1}$. It holds that

$$
(S)_{1} \hookrightarrow(S)_{0} \hookrightarrow L^{2}(\mu) \hookrightarrow(S)_{-0} \hookrightarrow(S)_{-1}
$$

where $\hookrightarrow$ denotes dense inclusions.
The time derivative of the Brownian motion exists in a generalized sense and for each fixed $t$ it belongs to the Kondratiev space $(S)_{-1,-p}$ for $p \geq \frac{5}{12}$. We refer it as the white noise and its formal expansion is given by $W(t, \omega)=$ $\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon_{k}}(\omega)$.

The definition of stochastic processes can be extended to processes with the chaos expansion form $U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}$, where the coefficients $u_{\alpha}$ are elements of some Banach space of functions $X$. We say that $U$ is an $X$-valued generalized stochastic process, i.e. $U \in X \otimes(S)_{-\rho}$ if there exists $p \geq 0$ such that

$$
\|U\|_{X \otimes(S)_{-\rho,-p}^{2}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

For example, let $X=C^{k}([0, T]), k \in \mathbb{N}$. We have proved in [76] that the differentiation of a stochastic process can be carried out componentwise in the chaos expansion, i.e. due to the fact that $(S)_{-\rho}$ is a nuclear space it holds that $C^{k}\left([0, T],(S)_{-\rho}\right)=C^{k}[0, T] \hat{\otimes}(S)_{-\rho}$ where $\hat{\otimes}$ denotes the completion of the tensor product which is the same for the $\varepsilon$-completion and $\pi$-completion. In the following, we will use the notation $\otimes$ instead of $\hat{\otimes}$. Hence $C^{k}([0, T]) \otimes(S)_{-\rho,-p}$ and $C^{k}([0, T]) \otimes(S)_{\rho, p}$ denote subspaces of the corresponding completions. We keep the same notation when $C^{k}([0, T])$ is replaced by another Banach space. This means that a stochastic process $U(t, \omega)$ is $k$ times continuously differentiable if and only if all of its coefficients $u_{\alpha}(t), \alpha \in \mathcal{I}$ are in $C^{k}([0, T])$.

The same holds for Banach space valued stochastic processes, i.e. elements of $C^{k}([0, T], X) \otimes(S)_{-\rho}$, where $X$ is an arbitrary Banach space. It holds that
$C^{k}\left([0, T], X \otimes(S)_{-\rho}\right)=C^{k}([0, T], X) \otimes(S)_{-\rho}=\bigcup_{p \geq 0} C^{k}([0, T], X) \otimes(S)_{-\rho,-p}$.
In addition, if $X$ is a Banach algebra, then the Wick product of the stochastic processes $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$ and $G=\sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta} \in X \otimes(S)_{-\rho,-p}$ is given by

$$
F \diamond G=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} H_{\gamma}=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} f_{\beta} g_{\alpha-\beta} H_{\alpha}
$$

and $F \diamond G \in X \otimes(S)_{-\rho,-(p+k)}$ for all $k>1$, see [43]. The $n$th Wick power is defined by $F^{\diamond n}=F^{\diamond(n-1)} \diamond F, F^{\diamond 0}=1$. Note that $H_{n \varepsilon^{(k)}}=H_{\varepsilon^{(k)}}^{\diamond n}$ for $n \in \mathbb{N}_{0}, k \in \mathbb{N}$. Through the thesis we will mostly assume that $X$ is a Banach algebra.

We also consider processes which are elements of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$. They are represented in chaos expansion of the form

$$
F=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} h_{k} \otimes \xi_{k}
$$

where $g_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), h_{k}=\sum_{\alpha \in \mathcal{I}} f_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$ and $f_{\alpha, k} \in X$. Thus, for some $p, l \in \mathbb{N}_{0}$,

$$
\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}(\alpha!)^{1-\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty
$$

The generalized expectation is given by $\mathbb{E} F=\sum_{k \in \mathbb{N}} f_{(0,0, \ldots), k} \otimes \xi_{k}=g_{(0,0, \ldots)}$.

## Operators of generalized Malliavin calculus

Some of the most important operators of stochastic calculus are the operators of the Malliavin calculus. We recall their definitions in the generalized $S^{\prime}(\mathbb{R})$ setting as they appear in Section 1.1 [71], Section 1.2 [74] and Section 1.3 [75]. These definitions are used through this chapter.

## The Malliavin derivative

We define the Malliavin derivative operator $\mathbb{D}$ on spaces of generalized stochastic processes, test stochastic processes and classical stochastic processes. We also describe the domains in terms of chaos expansion representations.

Definition $2([71])$ Let $\rho \in[0,1]$ and let $u \in X \otimes(S)_{-\rho}$ be a generalized stochastic process given in the chaos expansion form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$, $u_{\alpha} \in X, \alpha \in \mathcal{I}$. Then, $u$ belongs to $\operatorname{Dom}_{-\rho,-p}(\mathbb{D})$ if there exists $p \in \mathbb{N}_{0}$ such that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{1.1}
\end{equation*}
$$

and its Malliavin derivative is defined by

$$
\begin{equation*}
\mathbb{D} u=\sum_{|\alpha|>0} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}} \tag{1.2}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$, i.e., for a multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k}, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$ if $\alpha_{k} \geq 1$ we have $H_{\alpha-\varepsilon^{(k)}}=$ $H_{\left(\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right)}$

Thus, the domain of the Malliavin derivative in $X \otimes(S)_{-\rho}$ is given by

$$
\begin{align*}
\operatorname{Dom}_{-\rho}(\mathbb{D}) & =\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-\rho,-p}(\mathbb{D})  \tag{1.3}\\
& =\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{-\rho}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
\end{align*}
$$

All processes that belong to $\operatorname{Dom}_{-\rho}(\mathbb{D})$ are called Malliavin differentiable. The operator $\mathbb{D}$ is also called the stochastic gradient.
The range of the Malliavin derivative operator is characterized in the following theorem. Particularly, for $\rho=1$ this characterization was proven in [73] and for $\rho=0$ it was considered in [74].
Theorem $3\left([\mathbf{6 2},[71])\right.$ The Malliavin derivative of a process $u \in X \otimes(S)_{-\rho}$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{-\rho,-p}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}
$$

for $l>p+1$ and $p \in \mathbb{N}_{0}$.

Definition 4 Let $\rho \in[0,1]$ and let $v \in X \otimes(S)_{\rho}$ be given in the form $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \otimes H_{\alpha}, v_{\alpha} \in X, \alpha \in \mathcal{I}$. We say that $u$ belongs to $D^{\prime} m_{\rho, p}(\mathbb{D})$ if

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1-\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty, \quad \text { for all } \quad p \in \mathbb{N}_{0}
$$

Thus, the domain of the Malliavin derivative operator in $X \otimes(S)_{\rho}$ is the projective limit of the spaces $\operatorname{Dom}_{\rho, p}(\mathbb{D})$, i.e.,

$$
\begin{align*}
\operatorname{Dom}_{\rho}(\mathbb{D}) & =\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{\rho, p}(\mathbb{D})  \tag{1.4}\\
& =\bigcap_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{\rho, p}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{1-\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}
\end{align*}
$$

Theorem 5 ([71]) The Malliavin derivative of a test stochastic process $v \in$ $X \otimes(S)_{\rho}$ is a linear and continuous mapping

$$
\mathbb{D}: \quad \operatorname{Dom}_{\rho, p}(\mathbb{D}) \rightarrow X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}, \quad \text { for } \quad p>l+1
$$

Definition 6 The domain of $\mathbb{D}$ of a stochastic process $u \in X \otimes L^{2}(\mu)$ is given by

$$
\begin{equation*}
\operatorname{Dom}_{0}(\mathbb{D})=\left\{u \in X \otimes L^{2}(\mu): \sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\} \tag{1.5}
\end{equation*}
$$

Theorem 7 ([71]) The Malliavin derivative of a process $u \in \operatorname{Dom}_{0}(\mathbb{D})$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{0}(\mathbb{D}) \rightarrow X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)
$$

For $\rho \in[0,1]$ and $p \in \mathbb{N}$ we obtained $\operatorname{Dom}_{\rho, p}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$, and therefore $\operatorname{Dom}_{\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-\rho}(\mathbb{D})$.

## The Skorokhod integral

The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $L^{2}(\mu)$-sense. In [73] the definition of the Skorokhod integral from Hilbert space valued processes to the class of $S^{\prime}$-valued generalized processes was extended. Further development in this direction was proposed in 62, 71, [73, 74]. In the following we summarize these results.

Definition 8 Let $\rho \in[0,1]$. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ such that $f_{\alpha} \in X \otimes S^{\prime}(\mathbb{R})$ is given by $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k}, f_{\alpha, k} \in X$. Then, $F$ belongs to $\operatorname{Dom}_{-\rho,-l,-p}(\delta)$ if it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1-\rho} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{1.6}
\end{equation*}
$$

Thus, the chaos expansion of its Skorokhod integral is given by

$$
\begin{equation*}
\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}=\sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}} f_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} \tag{1.7}
\end{equation*}
$$

The domain of the Skorokhod integral operator for generalized stochastic processes in $\mathcal{X}=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ is denoted by $\operatorname{Dom}_{-\rho}(\delta)$ and is given as the inductive limit of the spaces $\operatorname{Dom}_{-\rho,-l,-p}(\delta), l, p \in \mathbb{N}_{0}$, i.e.,

$$
\operatorname{Dom}_{-\rho}(\delta)=\bigcup_{p>l+1} \operatorname{Dom}_{-\rho,-l,-p}(\delta)=\bigcup_{p>l+1}\left\{F \in \mathcal{X}:\|F\|_{D o m_{-\rho,-l,-p}}^{2}<\infty\right\}
$$

where $\|F\|_{D o m_{-\rho,-l,-p}}^{2}$ is given by (1.6). Each stochastic process $F \in \operatorname{Dom}_{-\rho}(\delta)$ is called integrable in the Skorokhod sense.

Theorem 9 ([64]) Let $\rho \in[0,1]$. The Skorokhod integral $\delta$ is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{-\rho,-l,-p}(\delta) \rightarrow X \otimes(S)_{-\rho,-p}, \quad p>l+1
$$

Particularly, the domain $\operatorname{Dom}_{-1}(\delta)$ was characterized in [73, 75].
In the following, we characterize the domains $\operatorname{Dom}_{\rho}(\delta)$ and $\operatorname{Dom}_{0}(\delta)$ of the Skorokhod integral operator for test processes from $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ and processes from $X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$, as modifications of those presented in [71, 74].

Definition 10 ([64]) Let $\rho \in[0,1]$. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes S(\mathbb{R}) \otimes$ $(S)_{\rho}$ and let $f_{\alpha} \in X \otimes S(\mathbb{R})$ be given by the expansion $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k}$, $f_{\alpha, k} \in X$. We say that the process $F$ belongs to $\operatorname{Dom}_{\rho, l, p}(\delta)$ if

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}(2 \mathbb{N})^{p \alpha}<\infty \tag{1.8}
\end{equation*}
$$

Then, the chaos expansion form of the Skorokhod integral of $F$ is given by the expression 1.7).

The domain of the Skorokhod integral for test stochastic processes in $X \otimes$ $S(\mathbb{R}) \otimes(S)_{\rho}$ is denoted by $\operatorname{Dom}_{\rho}(\delta)$ and is given as the projective limit of the spaces $\operatorname{Dom}_{\rho, l, p}(\delta), l, p \in \mathbb{N}_{0}$, i.e.,

$$
\begin{aligned}
\operatorname{Dom}_{\rho}(\delta) & =\bigcap_{l>p+1} \operatorname{Dom}_{\rho, l, p}(\delta) \\
& =\bigcap_{l>p+1}\left\{F \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}:\|F\|_{\left.D_{o m_{\rho, l, p}(\delta)}^{2}<\infty\right\}}\right.
\end{aligned}
$$

where $\|F\|_{\operatorname{Dom}_{\rho, l, p}(\delta)}^{2}$ is defined by (1.8). All test processes $F$ that belong to $\operatorname{Dom}_{\rho}(\delta)$ are called Skorokhod integrable.

Theorem 11 ([64]) The Skorokhod integral $\delta$ of a $S_{l}(\mathbb{R})$-valued stochastic test process is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{\rho, l, p}(\delta) \rightarrow X \otimes(S)_{\rho, p}, \quad l>p+1, \quad p \in \mathbb{N}
$$

Definition $12([64])$ Let $F \in X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$ be represented in the form $F=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}, f_{\alpha, k} \in X$. The process $F$ is Skorokhod integrable if it belongs to the space $\operatorname{Dom}_{0}(\delta)$, i.e., if it holds

$$
\begin{equation*}
\operatorname{Dom}_{0}(\delta)=\left\{F \in X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu): \sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|f_{\alpha}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2}<\infty\right\} \tag{1.9}
\end{equation*}
$$

Theorem 13 ([64]) The Skorokhod integral $\delta$ is a linear and continuous mapping

$$
\delta: \quad \operatorname{Dom}_{0}(\delta) \rightarrow X \otimes L^{2}(\mu)
$$

## The Ornstein-Uhlenbeck operator

The third main operator of the Malliavin calculus is the Ornstein-Uhlenbeck operator. We describe the domain and the range of the Ornstain-Uhlenbeck operator for different classes of stochastic processes [62, 71, 74, 75].

Definition 14 The operator $\mathcal{R}=\delta \circ \mathbb{D}$ defined as the composition of the Malliavin derivative and the Skorokhod integral is denoted by and is called the Ornstein-Uhlenbeck operator.

Since the estimate $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ holds for all $\alpha \in \mathcal{I}$, the image of the Malliavin derivative is included in the domain of the Skorokhod integral and thus we can define their composition. For example, for $v \in \operatorname{Dom}_{-\rho,-l,-p}(\delta)$ and $q+1-\rho \leq p$ we obtain

$$
\begin{aligned}
\|v\|_{D o m_{-\rho,-l,-p}(\delta)}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha|^{1-\rho} \alpha!^{1-\rho}\left\|v_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}} \alpha!^{1-\rho}\left\|v_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-q \alpha}=\|v\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}}^{2},
\end{aligned}
$$

i.e., $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q} \subseteq \operatorname{Dom}_{-\rho,-l,-p}(\mathbb{D})$ for $q+1-\rho \leq p$. From Theorem 3 and Theorem 9 we obtain additional conditions $l>q+1$ and $p>l+1$ and thus for $p>q+2$ the operator $\mathcal{R}$ is well defined in $X \otimes(S)_{-\rho}$.

Theorem 15 ([71]) For a Malliavin differentiable stochastic process $u$ that is represented in the form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$, the Ornstein-Uhlenbeck operator is given by

$$
\begin{equation*}
\mathcal{R}(u)=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} \tag{1.10}
\end{equation*}
$$

For a special choice of $u=u_{\alpha} \otimes H_{\alpha}, \alpha \in \mathcal{I}$ we obtain that the Fourier-Hermite polynomials are eigenfunctions of $\mathcal{R}$ and the corresponding eigenvalues are $|\alpha|, \alpha \in \mathcal{I}$, i.e.,

$$
\begin{equation*}
\mathcal{R}\left(u_{\alpha} \otimes H_{\alpha}\right)=|\alpha| u_{\alpha} \otimes H_{\alpha} \tag{1.11}
\end{equation*}
$$

Moreover, Gaussian processes with zero expectation are the only fixed points of the Ornstein-Uhlenbeck operator [74].

The domain of the Ornstein-Uhlenbeck operator in $X \otimes(S)_{-\rho}$ is given as the inductive limit $\operatorname{Dom}_{-\rho}(\mathcal{R})=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-\rho,-p}(\mathcal{R})$ of the spaces

$$
\begin{equation*}
\operatorname{Dom}_{-\rho,-p}(\mathcal{R})=\left\{u \in X \otimes(S)_{-\rho,-p}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{2} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\} \tag{1.12}
\end{equation*}
$$

Theorem 16 ([71]) The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \operatorname{Dom}_{-\rho,-p}(\mathcal{R}) \rightarrow X \otimes(S)_{-\rho,-p}, \quad p \in \mathbb{N}_{0}
$$

Moreover, $\operatorname{Dom}_{-\rho}(\mathcal{R}) \subseteq \operatorname{Dom}_{-\rho}(\mathbb{D})$, while for $\rho=1$ they coincide.
The domain of the Ornstein-Uhlenbeck operator in the space $X \otimes(S)_{\rho}$ is defined as the projective limit $\operatorname{Dom}_{\rho}(\mathcal{R})=\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{\rho, p}(\mathcal{R})$ of the spaces

$$
\begin{equation*}
\operatorname{Dom}_{\rho, p}(\mathcal{R})=\left\{v \in X \otimes(S)_{\rho, p}: \sum_{\alpha \in \mathcal{I}} \alpha!^{1+\rho}|\alpha|^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} \tag{1.13}
\end{equation*}
$$

Theorem $17([71,[75])$ The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \quad \operatorname{Dom}_{\rho, p}(\mathcal{R}) \rightarrow X \otimes(S)_{\rho, p}, \quad p \in \mathbb{N}_{0}
$$

Moreover, it holds $\operatorname{Dom}_{\rho}(\mathbb{D}) \supsetneq \operatorname{Dom}_{\rho}(\mathcal{R})$.
The definition of the domain of the Ornstein-Uhlenbeck operator in the space $X \otimes L^{2}(\mu)$ corresponds to the classical definition. Denote by

$$
\begin{equation*}
\operatorname{Dom}_{0}(\mathcal{R})=\left\{u \in X \otimes L^{2}(\mu): \sum_{\alpha \in \mathcal{I}} \alpha!|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\} \tag{1.14}
\end{equation*}
$$

Theorem $18([71,[75])$ The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \quad \operatorname{Dom}_{0}(\mathcal{R}) \rightarrow X \otimes L^{2}(\mu)
$$

Moreover, it holds $\operatorname{Dom}_{0}(\mathbb{D}) \supsetneq \operatorname{Dom}_{0}(\mathcal{R})$.
The characterization of the domain, the range of the operator $\mathcal{R}$ and its properties on $X \otimes(S)_{1}$ and $X \otimes L^{2}(\mu)$ were discussed in [69, 74]. Moreover, for this particular cases the surjectivity of the mappings was proven in [71, 74, 75].

## Main results of Section 1.1 and Section 1.2

The main results of Section 1.1 and Section 1.2 are twofold. The first group is related to the proofs of several properties and relations between the operators of generalized Malliavin calculus based on chaos expansions. The second group of results includes some applications of the Malliavin calculus.

## Properties of the operators $\mathbb{D}, \delta$ and $\mathcal{R}$

Based on the definitions of the operators of generalized Malliavin calculus we proved the integration by parts formula, i.e., the duality relation between $\mathbb{D}$ and $\delta$, product rules for $\mathbb{D}$ and $\mathcal{R}$, the Leibniz formula and the chain rule.

In the classical $L^{2}$ setting it is known that the Skorokhod integral is the adjoint of the Malliavin derivative [87]. We extend this result in the following theorem and prove their duality by pairing a generalized process with a test process (the classical result is revisited in part $3^{\circ}$ ).

Theorem 19 ([71, 74]) (Duality) Assume that either of the following hold:

$$
\begin{aligned}
& 1^{\circ} F \in \operatorname{Dom}_{-\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{\rho}(\delta) \\
& 2^{\circ} F \in \operatorname{Dom}_{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{-\rho}(\delta) \\
& 3^{\circ} \quad F \in \operatorname{Dom}_{0}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{0}(\delta) .
\end{aligned}
$$

Then, the following duality relationship between the operators $\mathbb{D}$ and $\delta$ holds

$$
\begin{equation*}
\mathbb{E}(F \cdot \delta(u))=\mathbb{E}(\langle\mathbb{D} F, u\rangle) \tag{1.15}
\end{equation*}
$$

where 1.15 denotes the equality of the generalized expectations of two objects in $X \otimes(S)_{-\rho}$ and $\langle\cdot, \cdot\rangle$ denotes the dual paring of $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.

Theorem 19 is a special case of a more general identity, i.e. under suitable assumptions that make all the products well defined, the following holds

$$
\begin{equation*}
F \delta(u)=\delta(F u)+\langle\mathbb{D}(F), u\rangle \tag{1.16}
\end{equation*}
$$

By taking the expectation in 1.16 and using the fact that $\mathbb{E}(\delta(F u))=0$, we obtain the duality relation (1.15).

The higher order duality formula, which connects the $k$ th order iterated Skorokhod integral and the Malliavin derivative operator of $k$ th order, $k \in \mathbb{N}$ is proven in [74]. A weaker type of duality than (1.15), which holds in Hida spaces was proven in [71]. Here we formulate the weak duality and omit its proof. A similar result is obtained in [74] for Kondratiev spaces when $\rho=1$.

Theorem 20 ([71, [74]) (Weak duality) Let $\rho=0 . F \in \operatorname{Dom}_{-0,-p}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-0,-q}(\mathbb{D})$, for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R})$, $n<q-1$, it holds that

$$
\ll\langle\mathbb{D} F, \varphi\rangle_{-r}, u>_{-r}=\ll F, \delta(\varphi u) \gg_{-r}
$$

for $r>\max \{q, p+1\}$.
The following theorem states that the Malliavin derivative indicates the rate of change in time between the ordinary product and the Wick product.

Theorem $21([75])$ Let $h \in X \otimes(S)_{-\rho}$ and let $w_{t}$ denote white noise. Then,

$$
\begin{equation*}
h \cdot w_{t}-h \diamond w_{t}=\mathbb{D}(h) \tag{1.17}
\end{equation*}
$$

The relation 1.17 gave us the motivation to study the fundamental equations involving $k$ th order operators of Malliavin calculus.
The Malliavin derivative $\mathbb{D}$ is not the inverse operator of the Skorokhod integral $\delta$ and also they do not commute. However, the relation (1.18) holds.

Theorem $22\left([71,[74])\right.$ If $u \in \operatorname{Dom}_{-\rho}(\delta)$ then $\mathbb{D} u \in \operatorname{Dom}_{-\rho}(\delta)$ and it holds

$$
\begin{equation*}
\mathbb{D}(\delta u)=u+\delta(\mathbb{D} u) \tag{1.18}
\end{equation*}
$$

The commutation relation 1.18 holds for processes $u \in \operatorname{Dom}_{\rho}(\delta)$ and also for $u \in \operatorname{Dom}_{0}(\delta)$.

The following theorem states the product rule for the Ornstein-Uhlenbeck operator. Its special case for $F, G \in \operatorname{Dom}_{0}(\mathcal{R})$ states that $F \cdot G$ is also in $\operatorname{Dom}_{0}(\mathcal{R})$ and 1.19 holds. The proof can be found for example in [48].

Theorem 23 ([74]) (Product rule for $\mathcal{R}$ )
$1^{\circ}$ Let $F \in \operatorname{Dom}_{\rho}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-\rho}(\mathcal{R})$. Then $F \cdot G \in \operatorname{Dom}_{-\rho}(\mathcal{R})$ and

$$
\begin{equation*}
\mathcal{R}(F \cdot G)=F \cdot \mathcal{R}(G)+G \cdot \mathcal{R}(F)-2 \cdot\langle\mathbb{D} F, \mathbb{D} G\rangle \tag{1.19}
\end{equation*}
$$

holds, where $\langle\cdot, \cdot\rangle$ is the dual paring between $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.
$2^{\circ}$ Let $F, G \in \operatorname{Dom}_{-\rho}(\mathcal{R})$. Then $F \diamond G \in \operatorname{Dom}_{-\rho}(\mathcal{R})$ and

$$
\begin{equation*}
\mathcal{R}(F \diamond G)=F \diamond \mathcal{R}(G)+\mathcal{R}(F) \diamond G \tag{1.20}
\end{equation*}
$$

$3^{\circ}$ Let $F \in \operatorname{Dom}_{\rho}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-\rho}(\mathcal{R})$ or vice versa (including also the possibility $\left.F, G \in \operatorname{Dom}_{0}(\mathcal{R})\right)$. Then,

$$
\begin{equation*}
\mathbb{E}(F \cdot \mathcal{R}(G))=\mathbb{E}(\langle\mathbb{D} F, \mathbb{D} G\rangle) \tag{1.21}
\end{equation*}
$$

The property (1.21) holds also for $F, G \in \operatorname{Dom}_{0}(\mathcal{R})$.

In the classical literature, e.g. [83, 87, it is proven that the Malliavin derivative satisfies the product rule (with respect to ordinary multiplication), i.e., if $F, G \in \operatorname{Dom}_{0}(\mathbb{D})$, then $F \cdot G \in \operatorname{Dom}_{0}(\mathbb{D})$ and 1.22 holds. The following theorem recapitulates this result and extends it for generalized and test processes, and also for the Wick multiplication [15, [74].

Theorem $24([71,[74])$ (Product rule for $\mathbb{D})$
$1^{\circ}$ Let $F \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and $G \in \operatorname{Dom}_{\rho}(\mathbb{D})$. Then $F \cdot G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and it holds

$$
\begin{equation*}
\mathbb{D}(F \cdot G)=F \cdot \mathbb{D} G+\mathbb{D} F \cdot G \tag{1.22}
\end{equation*}
$$

$2^{\circ}$ Let $F, G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$. Then $F \diamond G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and

$$
\mathbb{D}(F \diamond G)=F \diamond \mathbb{D} G+\mathbb{D} F \diamond G
$$

Theorem 25 ([71, [74]) Assume that either of the following hold:

$$
\begin{aligned}
& 1^{\circ} F \in \operatorname{Dom}_{-\rho}(\mathbb{D}), G \in \operatorname{Dom}_{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{\rho}(\delta), \\
& 2^{\circ} F, G \in \operatorname{Dom}_{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{-\rho}(\delta), \\
& 3^{\circ} F, G \in \operatorname{Dom}_{0}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{0}(\delta) .
\end{aligned}
$$

Then, the second integration by parts formula holds

$$
\begin{equation*}
\mathbb{E}(F\langle\mathbb{D} G, u\rangle)+\mathbb{E}(G\langle\mathbb{D} F, u\rangle)=\mathbb{E}(F G \delta(u)) \tag{1.23}
\end{equation*}
$$

A generalization of Theorem 24 for higher order derivatives, i.e., the Leibnitz formula is given [71, 74]. The chain rule for the Malliavin derivative for processes with finite second moments has been known in the literature as a direct consequence of the definition of Malliavin derivatives as Fréchet derivatives [15]. An alternative proof suited for chaos expansions setting was presented in [71, 74].

Theorem 26 ([71, [74]) (The chain rule) Let $\phi$ be a twice continuously differentiable function with bounded derivatives.
$1^{\circ}$ If $F \in \operatorname{Dom}_{\rho}(\mathbb{D})$ (or $F \in \operatorname{Dom}_{0}(\mathbb{D})$ ) then $\phi(F) \in \operatorname{Dom}_{\rho}(\mathbb{D})$ (respectively $\phi(F) \in \operatorname{Dom}_{0}(\mathbb{D})$ ) and the chain rule holds

$$
\begin{equation*}
\mathbb{D}(\phi(F))=\phi^{\prime}(F) \cdot \mathbb{D}(F) \tag{1.24}
\end{equation*}
$$

$2^{\circ}$ If $F \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and $\phi$ is analytic then $\phi^{\diamond}(F) \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and

$$
\begin{equation*}
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\phi^{\prime \diamond}(F) \diamond \mathbb{D}(F) \tag{1.25}
\end{equation*}
$$

Additionally, several illustrative examples are provided in [74].

## Applications of the Malliavin calculus

One of the first and most important applications of the Malliavin calculus concerns the existence and smoothness of a density for the probability law of random variables. More recent applications in finance have been developed for option pricing and computing Greeks (Greeks measure the stability of the option price under variations of the parameters) via the Clark-Ocone formula [16, 83, 88]. A few years ago it was also discovered that Malliavin calculus is in a close relationship with the Stein method and can be used for estimating the distance of a random variable from Gaussian variables [89]. In this section we assume that $X=\mathbb{R}$. The following results appeared in [74.

## Measurability and densities

Let $A \in \mathcal{B}$ be a Borel set in $S^{\prime}(\mathbb{R})$. Denote by $\kappa_{A}$ its indicator function, i.e. the random variable $\kappa_{A}(\omega)=1$ for $\omega \in A$ and $\kappa_{A}(\omega)=0$ for $\omega \in A^{c}$. Then, $\kappa_{A}=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}$, where $a_{\alpha}=E\left(\kappa_{A} \cdot H_{\alpha}\right), \alpha \in \mathcal{I}$. Especially, $a_{0}=E\left(\kappa_{A}\right)=P(A)$.

Proposition 27 ([87]) The indicator function $\kappa_{A} \in \operatorname{Dom}_{0}(\mathbb{D})$ if and only if $P(A)=0$ or $P(A)=1$.

If $P(A) \in(0,1)$, then $\kappa_{A} \notin \operatorname{Dom}_{0}(\mathbb{D})$. For example, $f(\omega)=\kappa_{\left\{B_{t}(\omega)>0\right\}} \notin$ $\operatorname{Dom}_{0}(\mathbb{D})$ since $P\left\{B_{t}>0\right\} \in(0,1)$. On the other hand, $\kappa_{A} \in \operatorname{Dom}_{-}(\mathbb{D})$ regardless of the value of $P(A)$.

For a closed subspace $A$ of $S^{\prime}(\mathbb{R})$, we denote by $\sigma[A]$ the sub- $\sigma$-algebra of $\mathcal{B}$ generated by $A$. A random variable $f$ is measurable with respect to $\sigma[A]$ if and only if $\mathbb{D}(f)=0$ a.e. on $A^{c}$. In particular, it was proven $([17,53,87])$, that if a stochastic process $f_{t}$ is adapted to the Brownian filtration $A_{t}=$ $\sigma\left[B_{s}: s \leq t\right]$, then $\operatorname{supp} \mathbb{D}\left(f_{t}\right)=[0, t]$, i.e. $\mathbb{D} f_{t}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha}(t) \otimes$ $\xi_{k}(s) \otimes H_{\alpha-\varepsilon^{(k)}}=0$ for $s>t$.

Theorem 28 (Clark-Ocone formula) Let $F \in \operatorname{Dom}_{0}(\mathbb{D})$ be adapted to the Brownian filtration. Then,

$$
F(s)=E(F)+\int_{0}^{s} E\left(\mathbb{D} F(s) \mid A_{t}\right) d B_{t}
$$

In [74] we also showed that absolutely continuous distributions can be characterized via the Malliavin derivative. Moreover, there exists an explicit formula for the density of the distribution. We point out that $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}=$ $\langle\mathbb{D} F, \mathbb{D} F\rangle_{L^{2}(\mathbb{R})}$ is an element in $L^{2}(\mu)$. If $F$ is of the form $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$, then $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}=\sum_{k \in \mathbb{N}}\left(\sum_{\alpha \in \mathcal{I}} f_{\alpha+\varepsilon^{(k)}}\left(\alpha_{k}+1\right) H_{\alpha}\right)^{2}$.

Theorem $29([53])$ Let $F \in \operatorname{Dom}_{0}(\mathbb{D})$ be such that $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})} \neq 0$ a.e. and $\frac{\mathbb{D} F}{\|\mathbb{D} F\|^{2}} \in \operatorname{Dom}_{0}(\delta)$. Then for every $\phi \in C_{0}^{2}(\mathbb{R})$,

$$
\begin{equation*}
E\left(\phi^{\prime}(F)\right)=E\left(\phi(F) \cdot \delta\left(\frac{\mathbb{D} F}{\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}}\right)\right) \tag{1.26}
\end{equation*}
$$

Moreover, $F$ is an absolutely continuous random variable and its density $\varphi$ is given by

$$
\begin{equation*}
\varphi(t)=E\left(\kappa_{\{F>t\}} \cdot \delta\left(\frac{\mathbb{D} F}{\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}}\right)\right) \tag{1.27}
\end{equation*}
$$

## Gaussian approximations

In [74] we proved some results which combine the Malliavin calculus with the Stein method [89. The properties were proven by using the method of chaos expansions. It is well-known that a random variable $N$ has $\mathcal{N}(0,1)$ distribution if and only if $E\left(N \cdot F(N)-F^{\prime}(N)\right)=0$, for every smooth function $F$. Thus, according to the Stein lemma, one can measure the distance to $N \sim \mathcal{N}(0,1)$, for an arbitrary random variable $Z$ by measuring the expectation of $Z \cdot F(Z)-F^{\prime}(Z)$. By using Malliavin calculus we showed that

$$
E(Z \cdot F(Z))=E\left(F^{\prime}(Z)\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right)
$$

holds for every $F \in C^{2}(\mathbb{R})$. Thus, in order to measure the distance to $N \sim \mathcal{N}(0,1)$, one needs to estimate

$$
\begin{equation*}
E\left|1-\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right| \tag{1.28}
\end{equation*}
$$

where $E\left|1-\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right|=0$ if and only if $Z \sim \mathcal{N}(0,1)$.
Theorem $30([74])$ Let $f \in \operatorname{Dom}_{+}(\mathbb{D})$ or $f \in \operatorname{Dom}_{0}(\mathbb{D})$ such that $E(f)=$ 0 and let $F \in C^{2}(\mathbb{R})$. Then,

$$
E(f \cdot F(f))=E\left(F^{\prime}(f) \cdot\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle\right)
$$

Thus, if $f \in \operatorname{Dom}_{+}(\mathbb{D})$ or $f \in \operatorname{Dom}_{0}(\mathbb{D})$ such that $E(f)=0$, then $f \sim \mathcal{N}(0,1)$ if and only if $\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle=1$.

Theorem 31 ([74]) A random variable $f$ has $\mathcal{N}(0,1)$ distribution if and only if $f \in L^{2}(\mu) \cap \mathcal{H}_{1}$ and $\|f\|_{L^{2}(\mu)}^{2}=1$, i.e. if it is of the form $f=$ $\sum_{j=1}^{\infty} f_{j} H_{\varepsilon^{(j)}}$ and $\sum_{j=1}^{\infty}\left|f_{j}\right|^{2}=1$ holds.

The previous theorem was also extended for generalized random variables, e.g. the white noise process at a fixed time point. These processes have an infinite variance and can be regarded as elements of the Kondratiev spaces. Recall that $\langle\cdot, \cdot\rangle_{-p}$ denotes the scalar product in the Schwartz space $S_{-p}(\mathbb{R})$.

Theorem $32([74])$ Let $f \in \operatorname{Dom}_{-p}(\mathbb{D})$ and $E(f)=0$. The following statements are equivalent:

- $f$ has a generalized Gaussian distribution,
- $f \in \mathcal{H}_{1}$,
- $\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle_{-p}=\|f\|_{(S)_{-1,-p}}^{2}<\infty$.

Theorem 31 and Theorem 32 provide a complete characterization of Gaussian processes (classical and generalized processes). All Gaussian processes belong to $\mathcal{H}_{1}$ and $\mathcal{H}_{1}$ contains nothing else apart from Gaussian processes.

Theorem $33([89])$ Let $Z \in \operatorname{Dom}_{+}(\mathbb{D})$ or $Z \in \operatorname{Dom}_{0}(\mathbb{D})$ be such that $E(Z)=0$ and $\operatorname{Var}(Z)=1$. Then the expectation (1.28) satisfies

$$
E\left(\left|1-\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right|\right) \leq \sqrt{\operatorname{Var}\left(\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right)}
$$

In order to measure how close is $Z$ to being normally distributed, one has to estimate how close is $\operatorname{Var}\left(\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right)$ to zero. This quantity is larger than the Kolmogorov distance, but nevertheless still a good approximation.

## Equations involving Mallivin calculus operators

This section is devoted to the study of several classes of stochastic equations involving generalized operators of the Malliavin calculus. In particular, equations that were discussed in Section 1.2 [74], Section 1.3 [75], Section 1.4 [76], Section 1.5 [62], Section 1.6 [70] and Section 1.7 [77]. We also consider equations involving the Malliavin derivative operator and the Wick product with a Gaussian process. Additionally, we study stochastic evolution equations with multiplicative noise and stochastic evolution equations with Wick-power nonlinearities. Applying the chaos expansion method in white noise spaces, we solve these equations and obtain explicit forms of the solutions in appropriate spaces of stochastic processes.

## Fundamental equations

It is of great importance to solve explicitly stochastic differential equations involving operators of Malliavin calculus, since explicit expansions of solutions can be used in numerical simulations [28, 84, 101]. Particularly, we consider the following fundamental equations with the $k$ th order operators of the Malliavin calculus

$$
\begin{equation*}
\mathcal{R}^{(k)} u=g \quad \mathbb{D}^{(k)} u=h, \quad \delta^{(k)} u=f \tag{1.29}
\end{equation*}
$$

as well as

$$
\begin{equation*}
P_{m}(\mathcal{R}) u=g \tag{1.30}
\end{equation*}
$$

where $P_{m}$ is a polynomial of order $m$. We also consider Wick-type equations involving Malliavin derivative and a nonhomogeneous linear equation with $\mathbb{D}$, i.e., respectively

$$
\begin{equation*}
\mathbb{D} u=\mathbf{G} \diamond(\mathbf{A} u)+h, \quad \text { and } \quad \mathbb{D} u=c \otimes u+h, \tag{1.31}
\end{equation*}
$$

satisfying the initial condition $\mathbb{E} u=\widetilde{u}_{0}$. Here, $\mathbf{G}$ is a Gaussian process, $\mathbf{A}$ a coordinatewise operator, $c \in S^{\prime}(\mathbb{R})$ and $h$ is a Schwartz space valued generalized stochastic process. The three equations in 1.29 have been solved in Section 1.3 [75]. Particularly, for $k=1$ they provide a full characterization of the range of all three operators, and were considered in [74, 75]. The study of the Wick-type equation in (1.31) was motivated by [75]. There it was shown that Malliavin derivative indicates the rate of change in time between ordinary product and the Wick product (1.17). Moreover, the Wick product and the Malliavin derivative play an important role in the analysis of nonlinear problems. For instance, in 100 the authors proved that in random fields, random polynomial nonlinearity can be expanded in a Taylor series involving Wick products and Malliavin derivatives, the so-called Wick-Malliavin series expansion.

## Equations with the Ornstein-Uhlenbeck operator

We consider stochastic equations involving polynomials of the OrnsteinUhlenbeck operator and generalize results from [71, 74, 75].

Theorem 34 ([64]) Let $\rho \in[0,1]$ and let $P_{m}(t)=\sum_{k=0}^{m} p_{k} t^{k}, t \in \mathbb{R}$ be a polynomial of degree $m$ with real coefficients.
a) If $P_{m}(k) \neq 0$, for $k \in \mathbb{N}_{0}$, then the equation $P_{m}(\mathcal{R}) u=g$ has a unique solution represented in the form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} \frac{g_{\alpha}}{P_{m}(|\alpha|)} \otimes H_{\alpha} \tag{1.32}
\end{equation*}
$$

b) If $P_{m}(k)=0$ for $k \in M$, where $M$ is a finite subset of $\mathbb{N}_{0}$ and $g_{\alpha}=0$ for $|\alpha|=i \in M$ then the equation $P_{m}(\mathcal{R}) u=g$ with the conditions $u_{\alpha}=c_{i}$ for $|\alpha|=i \in M$ has a unique solution given by

$$
\begin{equation*}
u=\sum_{|\alpha| \notin M} \frac{g_{\alpha}}{P_{m}(|\alpha|)} \otimes H_{\alpha}+\sum_{|\alpha|=i \in M} c_{i} \otimes H_{\alpha} \tag{1.33}
\end{equation*}
$$

Moreover, the following hold:

$$
\begin{aligned}
& 1^{\circ} \text { If } g \in X \otimes(S)_{-\rho,-p}, p \in \mathbb{N} \text { then } u \in \operatorname{Dom}_{-\rho,-p}\left(\mathcal{R}^{m}\right) \\
& 2^{\circ} \text { If } g \in X \otimes(S)_{\rho, p}, p \in \mathbb{N} \text { then } u \in \operatorname{Dom}_{\rho, p}\left(\mathcal{R}^{m}\right)
\end{aligned}
$$

$3^{\circ}$ If $g \in X \otimes L^{2}(\mu)$ then $u \in \operatorname{Dom}_{0}\left(\mathcal{R}^{m}\right)$.
Remark 35 For $P_{m}(t)=t^{m}, t \in \mathbb{R}$ the equation $P_{m}(\mathcal{R}) u=g$ reduces to

$$
\begin{equation*}
\mathcal{R}^{m} u=g, \quad \mathbb{E} u=\tilde{u}_{0} \in X \tag{1.34}
\end{equation*}
$$

This case was considered in [75]. Assuming that $g$ has zero generalized expectation, from Theorem 34 it follows that the equation (1.34) has a unique solution of the form

$$
u=\tilde{u}_{0}+\sum_{|\alpha|>0} \frac{g_{\alpha}}{|\alpha|^{m}} \otimes H_{\alpha}
$$

We also note that each stochastic process $g$ can be represented as $g=$ $\mathbb{E} g+\mathcal{R}(u)$, for some $u \in \operatorname{Dom}(\mathcal{R})$, where $\operatorname{Dom}(\mathcal{R})$ denotes the domain of $\mathcal{R}$ in one of the spaces $X \otimes(S)_{\rho}, X \otimes(S)_{-\rho}$ or $X \otimes L^{2}(\mu)$.

## First order equation with the Malliavin derivative operator

A first order equation involving the Malliavin derivative operator is studied in Section 1.2 [74]. It also appears as a special case in Section 1.3 [75]. The following result characterizes the family of stochastic processes that can be written as the Malliavin derivative of some stochastic process. The results from [71, 74, 75] are generalized here.

Theorem $36([\mathbf{6 4},[74])$ Let $\rho \in[0,1]$. Let a process $h$ be given in the chaos expansion representation form $h=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$ such that the coefficients $h_{\alpha, k}$ satisfy the condition

$$
\begin{equation*}
\frac{1}{\alpha_{k}} h_{\alpha-\varepsilon^{(k)}, k}=\frac{1}{\beta_{j}} h_{\beta-\varepsilon^{(j)}, j} \tag{1.35}
\end{equation*}
$$

for all $\alpha+\varepsilon^{(k)}=\beta+\varepsilon^{(j)}$. Then, for each $\widetilde{u}_{0} \in X$ the equation

$$
\begin{equation*}
\mathbb{D} u=h, \quad \mathbb{E} u=\widetilde{u}_{0} \tag{1.36}
\end{equation*}
$$

has a unique solution $u$ represented in the form

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} \tag{1.37}
\end{equation*}
$$

Moreover, the following holds:

$$
\begin{aligned}
& 1^{\circ} \text { If } h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q>p+1 \text { then } u \in \operatorname{Dom}_{-\rho,-q}(\mathbb{D}) . \\
& 2^{\circ} \text { If } h \in X \otimes S_{p}(\mathbb{R}) \otimes(S)_{\rho, q}, p>q+1 \text {, then } u \in \operatorname{Dom}_{\rho, q}(\mathbb{D}) . \\
& 3^{\circ} \text { If } h \in \operatorname{Dom}_{0}(\delta) \text { then } u \in \operatorname{Dom}_{0}(\mathbb{D}) .
\end{aligned}
$$

In 73 ] for $\rho=1$ we provided another way for solving equation 1.36). Applying the chaos expansion method directly, we transformed equation (1.36) into a system of infinitely many equations of the form

$$
\begin{equation*}
u_{\alpha+\varepsilon^{(k)}}=\frac{1}{\alpha_{k}+1} h_{\alpha, k}, \quad \text { for all } \quad \alpha \in \mathcal{I}, k \in \mathbb{N} \tag{1.38}
\end{equation*}
$$

from which we calculated $u_{\alpha}$, by induction on the length of $\alpha$.
Denote by $r=r(\alpha)=\min \left\{k \in \mathbb{N}: \alpha_{k} \neq 0\right\}$, for a nonzero multiindex $\alpha \in \mathcal{I}$, i.e., let $r$ be the position of the first nonzero component of $\alpha$. Then, the first nonzero component of $\alpha$ is the $r$ th component $\alpha_{r}$, i.e., $\alpha=\left(0, \ldots, 0, \alpha_{r}, \ldots, \alpha_{m}, 0, \ldots\right)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of $\alpha$, except the $r$ th, which is $\alpha_{r}-1$. With the given notation we call $\alpha_{\varepsilon^{(r)}}$ the representative of $\alpha$ and write $\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}$. For $\alpha \in \mathcal{I},|\alpha|>0$ the set

$$
\mathcal{K}_{\alpha}=\left\{\beta \in \mathcal{I}: \alpha=\beta+\varepsilon^{(j)}, \text { for those } j \in \mathbb{N} \text { such that } \alpha_{j}>0\right\}
$$

is a nonempty set, because it contains at least the representative of $\alpha$, i.e., $\alpha_{\varepsilon^{(r)}} \in \mathcal{K}_{\alpha}$. Note that, if $\alpha=n \varepsilon^{(r)}, n \in \mathbb{N}$ then $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)=1$ and in all other cases $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)>1$. Further, for $|\alpha|>0, \mathcal{K}_{\alpha}$ is a finite set because $\alpha$ has finitely many nonzero components and $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)$ is equal to the number of nonzero components of $\alpha$. In [73] the coefficients $u_{\alpha}$ of the solution of (1.38) are obtained as functions of the representative $\alpha_{\varepsilon^{(r)}}$ of a nonzero multi-index $\alpha \in \mathcal{I}$ in the form

$$
u_{\alpha}=\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r)}, r, \quad \text { for }|\alpha| \neq 0, \alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}
$$

Theorem 37 ([73]) Let $h=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in X \otimes S_{-p}(\mathbb{R}) \otimes$ $(S)_{-\rho,-p}$, for some $p \in \mathbb{N}_{0}$ with $h_{\alpha, k} \in X$ such that

$$
\begin{equation*}
\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}=\frac{1}{\alpha_{j}} h_{\beta, j} \tag{1.39}
\end{equation*}
$$

for the representative $\alpha_{\varepsilon^{(r)}}$ of $\alpha \in \mathcal{I},|\alpha|>0$ and all $\beta \in \mathcal{K}_{\alpha}$, such that $\alpha=\beta+\varepsilon^{(j)}$, for $j \geq r, r \in \mathbb{N}$. Then, 1.36 has a unique solution in $X \otimes(S)_{-\rho,-p}$ given in the chaos expansion form

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha=\alpha_{\varepsilon}(r)+\varepsilon^{(r)} \in \mathcal{I}} \frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r} \otimes H_{\alpha} \tag{1.40}
\end{equation*}
$$

Corollary 38 It holds that $\mathbb{D}(u)=0$ if and only if $u=\mathbb{E} u$.
In other words, the kernel of the operator $\mathbb{D}$ is $\mathcal{H}_{0}$.
If the input function $h$ is a constant random variable, i.e. an element of $\mathcal{H}_{0}$, then the solution $u$ of 1.36 is a Gaussian process. Additionally, for every Skorokhod integrable process $h$ there exists a unique $u \in \operatorname{Dom}(\mathbb{D})$ such that $\mathbb{E} u=0$ and $h=\mathbb{D}(u)$ holds.

## Nonhomogeneous equation with the Malliavin derivative operator

In Section 1.6 [70] we solved the nonhomogeneous linear Malliavin differential equation

$$
\begin{equation*}
\mathbb{D} u=c \otimes u+h, \quad \mathbb{E} u=\widetilde{u}_{0} \tag{1.41}
\end{equation*}
$$

where $c \in S^{\prime}(\mathbb{R}), h$ is a Schwartz space valued generalized stochastic process and $\widetilde{u}_{0} \in X$. Especially, for $h=0$ the equation (1.41) reduces to the corresponding homogeneous equation $\mathbb{D} u=c \otimes u$ satisfying $\mathbb{E} u=\widetilde{u}_{0}$, i.e., the generalized eigenvalue problem for the Malliavin derivative operator that was solved in [73]. Moreover, it was proved that in a special case, the obtained solution coincide with the stochastic exponential. Additionally, setting $c=0$, the initial equation (1.41) transforms to (1.36).

Let $\alpha_{\varepsilon(r)}$ be the representative of a nonzero multi-index $\alpha$, i.e., $\alpha=$ $\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)},\left|\alpha_{\varepsilon(r)}\right|=|\alpha|-1$ and let $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)>1$. Then, we denote by $r_{1}$ the first nonzero component of $\alpha_{\varepsilon^{(r)}}$ and by $\alpha_{\varepsilon^{\left(r_{1}\right)}}$ its representative, i.e., $\alpha_{\varepsilon(r)}=\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon\left(r_{1}\right)}$ and $\left|\alpha_{\varepsilon}\left(r_{1}\right)\right|=|\alpha|-2$. If $\operatorname{Card}\left(\mathcal{K}_{\alpha_{\varepsilon}\left(r_{1}\right)}\right)>1$, we denote by $r_{2}$ the first nonzero component of $\alpha_{\varepsilon^{\left(r_{1}\right)}}$ and with $\alpha_{\varepsilon^{\left(r_{2}\right)}}$ its representative, i.e., $\alpha_{\varepsilon^{\left(r_{1}\right)}}=\varepsilon^{\left(r_{2}\right)}+\alpha_{\varepsilon^{\left(r_{2}\right)}}$ and so on. With such a procedure we decompose $\alpha \in \mathcal{I}$ recursively by new representatives of the previous representatives and we obtain a sequence of $\mathcal{K}$-sets. Thus, for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$, $|\alpha|=s+1$ there exists an increasing family of integers $1 \leq r \leq r_{1} \leq r_{2} \leq$ $\ldots \leq r_{s} \leq m, s \in \mathbb{N}$ such that $\alpha_{\varepsilon\left(r_{s}\right)}=\mathbf{0}$ and every $\alpha$ is decomposed by the recurrent sum

$$
\begin{align*}
\alpha=\varepsilon^{(r)}+\alpha_{\varepsilon^{(r)}} & =\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}}=\ldots \\
& =\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\ldots+\varepsilon^{\left(r_{s}\right)}+\alpha_{\varepsilon\left(r_{s}\right)} . \tag{1.42}
\end{align*}
$$

Theorem 39 ( 70$]$ ) Let $\rho \in[0,1]$. Let $c=\sum_{k=1}^{\infty} c_{k} \xi_{k} \in S^{\prime}(\mathbb{R})$ and let $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ with coefficients $h_{\alpha, k} \in X$ such that the following conditions ( $C$ )

$$
\begin{aligned}
\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r} & =\frac{1}{\beta_{k}} h_{\beta, k}, & & \beta \in \mathcal{K}_{\alpha},|\alpha|=1 \\
\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}} & =\frac{1}{\beta_{k} \beta_{k_{1}}} c_{k} h_{\beta_{1}, k_{1}}, & & \beta \in \mathcal{K}_{\alpha}, \beta_{1} \in \mathcal{K}_{\alpha_{\varepsilon}(r)},|\alpha|=2
\end{aligned}
$$

hold for all possible decompositions of $\alpha$ of the form (1.42). If $c_{k} \geq 2 k$ for
all $k \in \mathbb{N}$, then (1.41) has a unique solution in $X \otimes(S c)_{-\rho}$ given by

$$
\begin{align*}
& u=u^{h o m}+u^{n h o m}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{h o m} \otimes H_{\alpha}+\sum_{|\alpha|>0} u_{\alpha}^{\text {nhom }} \otimes H_{\alpha} \\
& =\widetilde{u}_{0} \otimes \sum_{\alpha \in \mathcal{I}} \frac{c^{\alpha}}{\alpha!} H_{\alpha}+\sum_{|\alpha|>0}\left(\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}+\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon\left(r_{1}\right)}, r_{1}}\right. \\
& \left.+\frac{1}{\alpha_{r} \alpha_{r_{1}} \alpha_{r_{2}}} c_{r} c_{r_{1}} h_{\alpha_{\varepsilon\left(r_{2}\right)}, r_{2}}+\ldots+\frac{1}{\alpha!} c_{r} c_{r_{1} \ldots} c_{r_{s-1}} h_{0, r_{s}}\right) \otimes H_{\alpha}, \tag{1.43}
\end{align*}
$$

where $u^{\text {hom }}$ is the solution of the corresponding homogeneous equation $\mathbb{D} u=$ $c \otimes u$. The nonhomogeneous part $u^{\text {nhom }}$ of the solution $u$ is given by the the second sum in (1.43), which runs through nonzero $\alpha$ represented in the recursive form 1.42).

The proof for $\rho=1$ was given in [70]. Note that the first subcondition in $(C)$ corresponds to 1.35 and equals 1.39 .

## Wick-type equations involving the Malliavin derivative

We consider a nonhomogeneous first order equation involving the Malliavin derivative operator

$$
\begin{equation*}
\mathbb{D} u=G \diamond u+h, \quad \mathbb{E} u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X \tag{1.44}
\end{equation*}
$$

This type of problems was considered in Section 1.5 62]. It is assumed that $h$ is a $S^{\prime}(\mathbb{R})$-valued generalized stochastic process and $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}$ is a Gaussian process represented in the form

$$
\begin{equation*}
G=\sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes H_{\varepsilon^{(k)}} \tag{1.45}
\end{equation*}
$$

Moreover, for some $l, q>0$ the condition

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} g_{k}^{2}(2 k)^{-q-l}<\infty \tag{1.46}
\end{equation*}
$$

holds. First we solve the homogeneous version of 1.44 .
Theorem 40 ([62]) Let $\rho \in[0,1]$ and let $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q, l>0$ be a Gaussian process of the form (1.45) whose coefficients $g_{k}, k \in \mathbb{N}$ satisfy the condition 1.46 . If $g_{k} \geq 2 k$ for all $k \in \mathbb{N}$ then the initial value problem

$$
\begin{equation*}
\mathbb{D} u=G \diamond u, \quad \mathbb{E} u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X \tag{1.47}
\end{equation*}
$$

has a unique solution in $\operatorname{Dom}(\mathbb{D} g)_{-\rho,-p}$ represented in the form

$$
\begin{equation*}
u=\widetilde{u}_{0} \otimes \sum_{\alpha=2 \beta \in \mathcal{I}} \frac{C_{\alpha}}{|\alpha|!!}\left(\prod_{k=1}^{\infty} g_{k}^{\beta_{k}}\right) H_{\alpha}=\widetilde{u}_{0} \otimes \sum_{2 \beta \in \mathcal{I}} C_{2 \beta} \frac{g^{\beta}}{|2 \beta|!!} H_{2 \beta} \tag{1.48}
\end{equation*}
$$

where $C_{\alpha}$ represents the number of all possible decomposition chains connecting multi-indices $\alpha$ and $\tilde{\alpha}$, such that $\tilde{\alpha}$ is the first successor of $\alpha$ having only one nonzero component that is obtained by the subtractions $\alpha-2 \varepsilon^{\left(p_{1}\right)}-$ $\ldots-2 \varepsilon^{\left(p_{s}\right)}=\tilde{\alpha}$, for $p_{1}, \ldots, p_{s} \in \mathbb{N}, s \geq 0$.

Theorem 41 ([62]) Let $\rho \in[0,1]$ and let $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q, l>0$ be a Gaussian process of the form (1.45) whose coefficients $g_{k}, k \in \mathbb{N}$ satisfy (1.46). If $g_{k} \geq 2 k$ for all $k \in \mathbb{N}$ and if the coefficients of $h \in X \otimes S_{-l} \otimes$ $(S)_{-\rho,-p}, l, p>0$ satisfy $(C)$ for all possible decompositions of $\alpha$ of the form (1.42), then the nonhomogeneous equation

$$
\begin{equation*}
\mathbb{D} u=G \diamond u+h, \quad \mathbb{E} u=\widetilde{u}_{0} \tag{1.49}
\end{equation*}
$$

for each $\widetilde{u}_{0} \in X$ has a unique solution in $\operatorname{Dom}(\mathbb{D} g)_{-\rho,-p}$ represented in the form $u=u^{\text {hom }}+u^{\text {nhom }}$, where $u^{\text {hom }}$ is the solution of the corresponding homogeneous equation (1.47) and is of the form (1.48) and $u^{n h o m}$ is the nonhomogeneous part.

The study of more general types of equations is also included in Section 1.6 62].

## Integral equation

We consider an integral type equation involving the Skorokhod integral operator. In the following theorem we generalize results from [74, 75] for processes in $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ and generalized processes from $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$, $\rho \in[0,1]$.

Theorem 42 Let $\rho \in[0,1]$. Let $f$ be a stochastic process with zero expectation and chaos expansion representation of the form $f=\sum_{|\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}$, $f_{\alpha} \in X$. Then the integral equation

$$
\begin{equation*}
\delta(u)=f \tag{1.50}
\end{equation*}
$$

has a unique solution $u$ given by

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} \tag{1.51}
\end{equation*}
$$

Moreover, the following hold:
$1^{\circ}$ If $f \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D}), p \in \mathbb{N}$ then $u \in \operatorname{Dom}_{-\rho,-l,-p}(\delta)$ for $l>p+1$.
$2^{\circ}$ If $f \in \operatorname{Dom}_{\rho, p}(\mathbb{D}), p \in \mathbb{N}$ then $u \in \operatorname{Dom}_{\rho, l, p}(\delta)$ for $l<p-1$.
$3^{\circ}$ If $f \in \operatorname{Dom}_{0}(\mathbb{D})$, then $u \in \operatorname{Dom}_{0}(\delta)$.
As a consequence, we conclude that each stochastic process $f$ can be represented as $f=\mathbb{E} f+\delta(u)$ for some Schwartz valued process $u$. In classical setting, this result is known as the Itô representation theorem.

Higher order integral equations were solved in Section 1.3 [75].

## Stochastic evolution equations

We consider stochastic evolution equations with multiplicative noise and stochastic evolution equations with Wick-polynomial nonlinearities. These results are related to Section 1.4 [76] and Section 1.7 [77], respectively.

## Operators

We consider two classes of operators defined on sets of stochastic processes, coordinatewise operators and convolution type operators. These classes include the generalized operators of Malliavin calculus. We follow the classification given in [68, 76]. Let $X$ be a Banach algebra and let $\rho \in[0,1]$.

Definition 43 We say that an operator $\mathbf{A}$ defined on $X \otimes(S)_{-\rho}$ is:
$1^{\circ} a$ coordinatewise operator if there exists a family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, $A_{\alpha}: X \rightarrow X, \alpha \in \mathcal{I}$, such that

$$
\begin{equation*}
\mathbf{A} u=\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha} \otimes H_{\alpha}, \tag{1.52}
\end{equation*}
$$

for all $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$.
$2^{\circ} a$ simple coordinatewise operator if $A_{\alpha}=A$ for all $\alpha \in \mathcal{I}$, i.e., if it holds that

$$
\mathbf{A} u=\sum_{\alpha \in \mathcal{I}} A\left(u_{\alpha}\right) \otimes H_{\alpha}=A\left(u_{\mathbf{0}}\right)+\sum_{|\alpha|>0} A\left(u_{\alpha}\right) \otimes H_{\alpha} .
$$

Definition 43 can be modified for the operators acting on the spaces $X \otimes L^{2}(\mu)$ and $X \otimes(S)_{\rho}$.

Lemma 44 ([76, 64]) Let A be a coordinatewise operator for which all $A_{\alpha}, \alpha \in \mathcal{I}$, are polynomially bounded, i.e., $\left\|A_{\alpha}\right\|_{L(X)} \leq R(2 \mathbb{N})^{r \alpha}$ for some $r, R>0$. Then, A is a bounded operator:
$1^{\circ} \mathbf{A}: X \otimes(S)_{-\rho,-p} \rightarrow X \otimes(S)_{-\rho,-q}$ for $q \geq p+2 r$, and
$2^{\circ} \mathbf{A}: X \otimes(S)_{\rho, p} \rightarrow X \otimes(S)_{\rho, q} \quad$ for $\quad q+2 r \leq p$.
The condition stating that the deterministic operators $A_{\alpha}, \alpha \in \mathcal{I}$ are polynomially bounded can be formulated as $\sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-r \alpha}<\infty$ for some $r>0$.

Definition 45 The Wick convolution type operator $\mathbf{B} \diamond$ is defined by

$$
\begin{equation*}
\mathbf{B} \diamond(y)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} B_{\beta}\left(y_{\alpha-\beta}\right) H_{\alpha}=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} B_{\alpha}\left(y_{\beta}\right) H_{\gamma} \tag{1.53}
\end{equation*}
$$

for $y=\sum_{\alpha \in \mathcal{I}} y_{\alpha} H_{\alpha}$.
If the operators $B_{\alpha}, \alpha \in \mathcal{I}$ are polynomially bounded and linear on $X$, then $\mathbf{B} \diamond$ is well-defined operator on $X \otimes(S)_{-\rho}$ and, similarly, also on $X \otimes(S)_{\rho}$.
Lemma 46 ([76]) If the operators $B_{\alpha}, \alpha \in \mathcal{I}$, for some $p>0$ satisfy the condition $\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$ then $\mathbf{B} \diamond$ is well-defined as a mapping $\mathbf{B} \diamond: X \otimes(S)_{-\rho,-p} \rightarrow X \otimes(S)_{-\rho,-q}$, for $q \geq p+r+1$.

The operator of differentiation and the Fourier transform are simple coordinatewise operators, while, for example $\mathbf{A}(u)=u^{\diamond 2}$ cannot be written in this form. The Ornstein-Uhlenbeck operator, defined by (1.10), is a coordinatewise operator, but it is not a simple coordinatewise operator. In [76] we proved that the Skorokhod integral, defined by (1.7), can be represented in the form of a convolution type operator. There exists an operator $\mathbf{M}$ such that $\delta(\mathbf{M} u)=\mathbf{B} \diamond u$.

## Stochastic evolution equations with multiplicative noise

We consider a stochastic Cauchy problem of the form

$$
\begin{align*}
\frac{d}{d t} U(t, x, \omega) & =\mathbf{A} U(t, x, \omega)+\mathbf{B} \diamond U(t, x, \omega)+F(t, x, \omega)  \tag{1.54}\\
U(0, x, \omega) & =U^{0}(x, \omega)
\end{align*}
$$

where $t \in(0, T], \omega \in \Omega$, and $U(t, \cdot, \omega)$ belongs to $X$. The operator $\mathbf{A}$ is densely defined, generating a $C_{0}-$ semigroup and $\mathbf{B}$ is a linear bounded operator which combined with the Wick product $\diamond$ introduces convolutiontype perturbations into the equation. All stochastic processes are considered in the setting of Wiener-Itô chaos expansions.

This study was inspired by [81, where the authors provide a comprehensive analysis of equations of the form

$$
\begin{aligned}
\frac{d}{d t} u(t, x, \omega) & =\mathbf{A} u(t, x, \omega)+\delta(\mathbf{M} u(t, x, \omega)) \\
& =\mathbf{A} u(t, x, \omega)+\int \mathbf{M} u(t, x, \omega) \diamond W(x, \omega) d x
\end{aligned}
$$

where $\delta$ denotes the Skorokhod integral and $W$ denotes the spatial white noise process. In [76] we proved that for every operator $\mathbf{M}$ there exists a corresponding operator $\mathbf{B}$ such that $\mathbf{B} \triangleleft u=\delta(\mathbf{M} u)$. On the other hand, the class of operators $\mathbf{B}$ is much larger.

We have studied elliptic SPDEs in [72, 92], particularly the stochastic Dirichlet problem of the form $\mathbf{L} \diamond u+f=0$. Equations (1.54) also include as a special case equations of the form $\frac{d}{d t} u=\mathbf{L} u+f$ and $\frac{d}{d t} u=\mathbf{L} \diamond u+$ $f$, where $L$ is a strictly elliptic second order partial differential operator. These equations describe the heat conduction in random media, where the properties of the material are modeled by a positively definite stochastic matrix. Other special cases of (1.54) include the heat equation with random potential $\frac{d}{d t} u=\Delta u+\mathbf{B} \diamond u$, the Schrödinger equation $(i \hbar) \frac{d}{d t} u=\Delta u+\mathbf{B} \diamond u+$ $f$, the transport equation $\frac{d}{d t} u=\frac{d^{2}}{d x^{2}} u+W \diamond \frac{d}{d x} u$ driven by white noise, the generalized Langevin equation $\frac{d}{d t} u=\mathbf{J} u+\mathbf{C}\left(Y^{\prime}\right)$, where $Y$ is a Lévy process, $\mathbf{J}$ the infinitesimal generator of a $C_{0}$-semigroup and $\mathbf{C}$ a bounded operator, which was studied in [4], as well as the equation $\frac{d}{d t} u=\mathbf{L} u+W \diamond u$, where $\mathbf{L}$ is a strictly elliptic partial differential operator as studied in [19] and [44]. Equations of the form $\frac{d}{d t} u=\mathbf{A} u+\mathbf{B} W$ were also studied in [85], where $\mathbf{A}$ is not necessarily generating a $C_{0}$-semigroup, but an $r$-integrated or a convolution semigroup.

We prove existence and uniqueness of solution of 1.54 by combining the chaos expansion method with the operator semigroup theory.

Definition 47 ([76]) It is said that $U$ is a solution of the equation (1.54) if $U \in C([0, T], X) \otimes(S)_{-1} \cap C^{1}((0, T], X) \otimes(S)_{-1}$ and $U$ satisfies 1.54).

Theorem 48 ([76]) Let the following assumptions hold:
(A1) Let $\mathbf{A}$ be a coordinatewise operator of the form (1.52), acting on processes $U \in \operatorname{Dom}(\mathbf{A}) \subseteq D \otimes(S)_{-1}$, where
$\operatorname{Dom}(\mathbf{A})=\left\{U \in D \otimes(S)_{-1}: \exists p_{U}>0, \sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\left(u_{\alpha}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p_{U} \alpha}<\infty\right\}$.
The operators $A_{\alpha}, \alpha \in \mathcal{I}$, defined on the same domain $D$ dense in $X$, are infinitesimal generators of $C_{0}$-semigroups $\left(T_{t}\right)_{\alpha}, t \geq 0, \alpha \in \mathcal{I}$, uniformly bounded by

$$
\begin{equation*}
\left\|\left(T_{t}\right)_{\alpha}\right\|_{L(X)} \leq M e^{w t}, t \geq 0, \quad \text { for some } M, w>0 \tag{1.55}
\end{equation*}
$$

(A2) Let $\mathbf{B} \diamond$ be of the form (1.53), where $B_{\alpha}, \alpha \in \mathcal{I}$, are bounded linear operators on $X$ so that there exists $p>0$ such that

$$
\begin{equation*}
K:=\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|(2 \mathbb{N})^{-p \frac{\alpha}{2}}<\infty \tag{1.56}
\end{equation*}
$$

(A3) Let the initial value $U^{0} \in X \otimes(S)_{-1}$ be such that $U^{0} \in \operatorname{Dom}(\mathbf{A})$, i.e.,

$$
\begin{gather*}
U^{0}(\omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha}(\omega) \in X \otimes(S)_{-1,-p}, \text { satisfies }  \tag{1.57}\\
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{gather*}
$$

and

$$
\begin{gather*}
\mathbf{A} U^{0}(\omega)=\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha}^{0} H_{\alpha}(\omega) \in X \otimes(S)_{-1,-p}, \text { satisfies } \\
\sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha} u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{1.58}
\end{gather*}
$$

(A4) Let

$$
F(t, \omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega) \in C^{1}([0, T], X) \otimes(S)_{-1},
$$

where $t \mapsto f_{\alpha}(t) \in C^{1}([0, T], X), \alpha \in \mathcal{I}$ so that

$$
\begin{align*}
& \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \quad=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}+\sup _{t \in[0, T]}\left\|f_{\alpha}^{\prime}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{1.59}
\end{align*}
$$

Then, the stochastic Cauchy problem (1.54) has a unique solution $U$ in $C^{1}([0, T], X) \otimes(S)_{-1,-p}$.

## Stationary equations

We consider stationary equations of the form

$$
\begin{equation*}
\mathbf{A} y+\mathbf{T} \diamond y+f=0 \tag{1.60}
\end{equation*}
$$

where $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}, \rho \in[0,1]$ and $\mathbf{T} \diamond: X \otimes(S)_{-\rho} \rightarrow$ $X \otimes(S)_{-\rho}$ are the operators of the forms (1.52) and 1.53), respectively. We assume that $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{T_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ are bounded operators such that $A_{\alpha}=\widetilde{A}_{\alpha}+C_{\alpha}, \alpha \in \mathcal{I}$. We also assume that $T_{0}$ and $\widetilde{A}_{\alpha}, \alpha \in \mathcal{I}$ are compact operators and $C_{\alpha}$ are self adjoint for all $\alpha \in \mathcal{I}$ such that $C_{\alpha}\left(H_{\alpha}\right)=r_{\alpha} H_{\alpha}$, $\alpha \in \mathcal{I}$. The chaos expansion method is combined with classical results of elliptic PDEs and the Fredholm alternative [34] in order to prove existence and uniqueness of the solution of 1.60 .

Theorem 49 ([76]) Let $\rho \in[0,1]$. Let $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ and $\mathbf{T} \diamond: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ be the operators, for which the following is satisfied:
(a1) $\mathbf{A}$ is of the form $\mathbf{A}=\mathbf{B}+\mathbf{C}$, where $\mathbf{B} y=\sum_{\alpha \in \mathcal{I}} B_{\alpha} y_{\alpha} \otimes H_{\alpha}$ and $B_{\alpha}$ : $X \rightarrow X$ are compact operators for all $\alpha \in \mathcal{I}, \mathbf{C} y=\sum_{\alpha \in \mathcal{I}} r_{\alpha} y_{\alpha} \otimes H_{\alpha}$, $r_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, and $\mathbf{T}$ is of the form (1.53), where $T_{\mathbf{0}}: X \rightarrow X$ is a compact operator. Assume that there exists $K>0$ such that for all $\alpha \in \mathcal{I}$

$$
-r_{\alpha}-\left\|B_{\alpha}\right\|-\left\|T_{\mathbf{0}}\right\| \geq 0 \quad \text { and } \quad \sup _{\alpha \in \mathcal{I}}\left(\frac{1}{-r_{\alpha}-\left\|B_{\alpha}\right\|-\left\|T_{0}\right\|}\right)<K .
$$

(a2) $\mathbf{T}$ is of the form 1.53), where $T_{\beta}: X \rightarrow X, \beta>\mathbf{0}$ are bounded operators and there exists $p>0$ such that

$$
K \sqrt{2} \sum_{\beta>0}\left\|T_{\beta}\right\|(2 \mathbb{N})^{\frac{-p \beta}{2}}<1 .
$$

(a3) For every $\alpha \in \mathcal{I}$

$$
\operatorname{Ker}\left(B_{\alpha}+\left(1+r_{\alpha}\right) \operatorname{Id}+T_{0}\right)=\{0\} .
$$

Then, for every $f \in X \otimes(S)_{-\rho,-p}$ there exists a unique solution $y \in X \otimes$ $(S)_{-\rho,-p}$ of the equation (1.60).

Remark 50 Some special cases of the equation (1.60):

1. If $A_{\alpha}=0$ for all $\alpha \in \mathcal{I}$ and $T_{\alpha}, \alpha \in \mathcal{I}$ are second order strictly elliptic partial differential operators in divergent form

$$
\begin{equation*}
T_{\alpha}=\sum_{i=1}^{n} D_{i}\left(\sum_{j=1}^{n} a_{\alpha}^{i j}(x) D_{j}+b_{\alpha}^{i}(x)\right)+\sum_{i=1}^{n} c_{\alpha}^{i}(x) D_{i}+d_{\alpha}(x) \tag{1.61}
\end{equation*}
$$

with essentially bounded coefficients, then equation (1.60) reduces to the elliptic equation

$$
\mathbf{T} \diamond U=F,
$$

which was solved in 92 .
2. Let $\widetilde{A}_{\alpha}=0$ for all $\alpha \in \mathcal{I}$ and let $T_{\alpha}, \alpha \in \mathcal{I}$, be second order strictly elliptic partial differential operators in divergent form 1.61. Let $\mathbf{C}=c P(\mathcal{R})$, for some $c \in \mathbb{R}$, where $\mathcal{R}$ is the Ornstein-Uhlenbeck operator, $P$ a polynomial of degree $m$ with real coefficients and $P(\mathcal{R})$ the differential operator $P(\mathcal{R})=p_{m} \mathcal{R}^{m}+p_{m-1} \mathcal{R}^{m-1}+\ldots+p_{1} \mathcal{R}+p_{0} I d$. Then, the corresponding eigenvalues are $r_{\alpha}=c P(|\alpha|), \alpha \in \mathcal{I}$. Hence, equation (1.60) transforms to the elliptic equation with a perturbation term driven by the polynomial of the Ornstein-Uhlenbeck operator

$$
\mathbf{T} \diamond U+c P(\mathcal{R}) U=F,
$$

that was solved in [72.

## Stochastic evolution equations with Wick-polynomial nonlinearities

We study stochastic nonlinear evolution equations of the form

$$
\begin{align*}
& u_{t}(t, \omega)=\mathbf{A} u(t, \omega)+\sum_{k=0}^{n} a_{k} u^{\diamond k}(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{1.62}\\
& u(0, \omega)=u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

where $u(t, \omega)$ is an $X$-valued generalized stochastic process, $\mathbf{A}$ corresponds to a densely defined infinitesimal generator of a $C_{0}-$ semigroup and $a_{k}, 1 \leq$ $k \leq n$ are constants and $a_{n} \neq 0$. The nonlinear part is the Wick-power product $u^{\diamond n}=u^{\diamond n-1} \diamond u=u \diamond \ldots \diamond u, n \in \mathbb{N}$. The Wick product is involved due to the fact that we allow random terms to be present both in the initial condition $u_{0}$ and the driving force $f$. This leads to singular solutions that do not allow to use ordinary multiplication, but require a renormalization of the multiplication, which is done by introducing the Wick product into the equation.

Some special examples of (1.62) are the stochastic versions of Fujita-type equations $u_{t}=\mathbf{A} u+u^{\diamond n}+f$, the stochastic FitzHugh-Nagumo equations $u_{t}=\mathbf{A} u+u^{\diamond 2}-u^{\diamond 3}+f$, the stochastic Fisher-KPP equations $u_{t}=\mathbf{A} u+$ $u-u^{\diamond 2}+f$ and the stochastic Chaffee-Infante equations $u_{t}=\mathbf{A} u+u^{\diamond 3}-$ $u+f$. These equations arise in ecology, medicine, engineering and physics. For example, the FitzHugh-Nagumo equation is used to study electrical activity of neurons in neurophysiology by modeling the conduction of electric impulses down a nerve axon. The Fisher-KPP equation provides a model for the spread of an epidemic in a population or for the distribution of an advantageous gene within a population. Other applications in medicine involve the modeling of cellular reactions to the introduction of toxins, and the process of epidermal wound healing. In plasma physics it has been used to study neutron flux in nuclear reactors, while in ecology it models flame propagation of fire outbreaks. Thus, the study of their stochastic versions that arise, e.g. when some of the input factors are disturbed by an external noise, is very important.

We combined the chaos expansion method with operator semigroup theory in order to prove the existence and the uniqueness of a solution for (1.62). To solve the propagator system, we exploit the intrinsic relationship between the Wick product and the Catalan numbers that was discovered in [49]. We build upon these ideas in order to solve a general class of stochastic nonlinear equations $(1.62)$.

We first solved the equation 1.62 for $a_{0}=\cdots=a_{n-1}=0$ and $a_{n}=1$.
Definition 51 An $X$-valued generalized stochastic process

$$
\begin{equation*}
u(t)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha} \in X \otimes(S)_{-1}, \quad t \in[0, T] \tag{1.63}
\end{equation*}
$$

is a coordinatewise classical solution of

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+u^{\diamond n}(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{1.64}\\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

if $u_{\mathbf{0}}$ is a classical solution of

$$
\begin{equation*}
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+u_{\mathbf{0}}^{n}(t)+f_{\mathbf{0}}(t), \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0} \tag{1.65}
\end{equation*}
$$

and for every $\alpha \in \mathcal{I} \backslash\{\mathbf{0}\}$, the coefficient $u_{\alpha}$ is a classical solution of

$$
\begin{equation*}
\frac{d}{d t} u_{\alpha}(t)=B_{\alpha, n}(t) u_{\alpha}(t)+g_{\alpha, n}(t), \quad t \in(0, T], \quad u_{\alpha}(0)=u_{\alpha}^{0} \tag{1.66}
\end{equation*}
$$

where $B_{\alpha, n}(t)=A_{\alpha}+n u_{0}^{n-1}(t) I d$ and $g_{\alpha, n}(t)=r_{\alpha, n}(t)+f_{\alpha}(t), t \in[0, T]$ for all $\alpha>\mathbf{0}$, and the functions $r_{\alpha, n}, n>1$ contain only coordinate functions $u_{\beta}, \beta<\alpha$. The coordinatewise solution $u(t) \in X \otimes(S)_{-1}, t \in[0, T]$ is an almost classical solution of (1.64) if $u \in C([0, T], X) \otimes(S)_{-1}$, an almost classical solution is a classical solution if $u \in C([0, T], X) \otimes(S)_{-1} \cap$ $C^{1}((0, T], X) \otimes(S)_{-1}$.

We assume that the following conditions hold:
(B1) The operators $A_{\alpha}, \alpha \in \mathcal{I}$, are infinitesimal generators of $C_{0}$-semigroups $\left\{T_{\alpha}(s)\right\}_{s \geq 0}$ with a common domain $D_{\alpha}=D, \alpha \in \mathcal{I}$, dense in $X$. We assume that there exist constants $m \geq 1$ and $w \in \mathbb{R}$ such that

$$
\left\|T_{\alpha}(s)\right\| \leq m e^{w s}, s \geq 0 \quad \text { for all } \quad \alpha \in \mathcal{I}
$$

The action of $\mathbf{A}$ is given by 1.52 for $u \in \mathbb{D} \subseteq D \otimes(S)_{-1}$ of the form (1.63), where

$$
\mathbb{D}=\left\{u \in D \otimes(S)_{-1}: \exists p_{0} \geq 0, \sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\left(u_{\alpha}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p_{0} \alpha}<\infty\right\}
$$

(B2) The initial value $u^{0}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha} \in \mathbb{D}$, i.e. $u_{\alpha}^{0} \in D$ for every $\alpha \in \mathcal{I}$ and there exists $p \geq 0$ such that

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \\
\sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\left(u_{\alpha}^{0}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{gathered}
$$

(B3) The inhomogeneous part $f(t, \omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega), t \in[0, T], \omega \in$ $\Omega$ belongs to $C^{1}([0, T], X) \otimes(S)_{-1}$; hence $t \mapsto f_{\alpha}(t) \in C^{1}([0, T], X), \alpha \in$ $\mathcal{I}$ and there exists $p \geq 0$ such that

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \quad=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}+\sup _{t \in[0, T]}\left\|f_{\alpha}^{\prime}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{aligned}
$$

(B4-n) The Cauchy problem

$$
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+u_{\mathbf{0}}^{n}(t)+f_{\mathbf{0}}(t), \quad t \in(0, T] ; \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0}
$$

has a classical solution $u_{0} \in C^{1}([0, T], X)$.
Particularly, if $A_{\mathbf{0}}=\Delta$ is the Laplace operator and $f_{\mathbf{0}} \equiv 0$, then 1.65 belongs to the class of Fujita equations

$$
\begin{equation*}
u_{t}=\Delta u+u^{p}, \quad u(0)=u_{0} \tag{1.67}
\end{equation*}
$$

studied by Fujita, Chen and Watanabe [32, 33]. The authors proved that for a nonnegative initial condition $u^{0} \in C\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$, equation 1.67) has a unique classical solution on some $\left[0, T_{1}\right)$. Moreover, if $p>1+\frac{2}{N}$ then there exist a positive bounded solution. For $\alpha=\mathbf{0}$ equation (1.65) can also be solved by the Fixed point theorem [104].

Theorem 52 ([77]) Let the assumptions $(B 1)-(B 4-n)$ be fulfilled. Then, there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ of the stochastic nonlinear evolution equation (1.64).

## The linear nonautonomous case

Our analysis gives a simple observation for the linear nonautonomous equation

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A}(t) u(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{1.68}\\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

We assume the following:
(b1) The operator $\mathbf{A}(t): \mathbb{D}^{\prime} \subset X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}, t \in[0, T]$ is a coordinatewise operator depending on $t$ that corresponds to a family of deterministic operators $A_{\alpha}(t): D\left(A_{\alpha}\right) \subset X \rightarrow X, \alpha \in \mathcal{I}$. For every $\alpha \in \mathcal{I}$ the operator family $\left\{A_{\alpha}(t)\right\}_{t \in[0, T]}$ is a stable family of infinitesimal generators of $C_{0}$-semigroups on $X$ with stability constants $m>1$ and $w \in \mathbb{R}$ not depending on $\alpha$, therefore the corresponding evolution systems $S_{\alpha}(t, s)$ satisfy

$$
\left\|S_{\alpha}(t, s)\right\| \leq m e^{w(t-s)} \leq m e^{w T}, \quad 0 \leq s<t \leq T, \quad \alpha \in \mathcal{I}
$$

The domain $D\left(A_{\alpha}(t)\right)=D$ is independent of $t \in[0, T]$ and $\alpha \in$ $\mathcal{I}$. For every $x \in D$ the function $A_{\alpha}(t) x, t \in[0, T]$ is continuously differentiable in $X$ for each $\alpha \in \mathcal{I}$.
The action of $\mathbf{A}(t), t \in[0, T]$ is given by

$$
\mathbf{A}(t)(u)=\sum_{\alpha \in \mathcal{I}} A_{\alpha}(t)\left(u_{\alpha}\right) H_{\alpha}
$$

for $u \in \mathbb{D}^{\prime} \subseteq D \otimes(S)_{-1}$ of the form 1.63 , where
$\mathbb{D}^{\prime}=\left\{u \in D \otimes(S)_{-1}: \exists p_{0} \geq 0, \sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|A_{\alpha}(t)\left(u_{\alpha}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p_{0} \alpha}<\infty\right\}$.
(b2) The initial value $u^{0}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha} \in \mathbb{D}^{\prime}$, i.e. $u_{\alpha}^{0} \in D$ for every $\alpha \in \mathcal{I}$ and there exists $p \geq 0$ such that

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \\
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|A_{\alpha}(t) u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{gathered}
$$

For the inhomogeneous part $f(t, \omega), \omega \in \Omega, t \in[0, T]$ we assume (B3).

Theorem 53 ([77]) Let the assumptions (b1), (b2) and (B3) be fulfilled. Then, there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ of the linear nonautonomous equation 1.68 .

## Extensions to nonlinear equations

The results of Theorem 52 are extended to a more general case of stochastic evolution equation of the form 1.62 . In order to apply Theorem 52 we replace $(B 4-n)$ with the following assumption:
(C4-n) The Cauchy problem

$$
\begin{equation*}
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+\sum_{k=0}^{n} a_{k} u_{\mathbf{0}}^{k}(t)+f_{\mathbf{0}}(t), t \in(0, T] ; \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0} \tag{1.69}
\end{equation*}
$$

has a classical solution $u_{0} \in C^{1}([0, T], X)$.

For the sake of simplicity in [77] we presented only a procedure for solving (1.62) for $n=3$. The general case can be done in the same way. From the form of the process (1.63) and its Wick-powers, we obtain the expansion of
the Wick-polynomial nonlinearity

$$
\begin{aligned}
p_{3}^{\diamond}(u) & =a_{0}+a_{1} u+a_{2} u^{\diamond 2}+a_{3} u^{\diamond 3} \\
& =\left(a_{0}+a_{1} u_{\mathbf{0}}+a_{2} u_{\mathbf{0}}^{2}+a_{3} u_{\mathbf{0}}^{3}\right) H_{\mathbf{0}} \\
& +\sum_{\alpha>\mathbf{0}}\left(\left(3 a_{3} u_{\mathbf{0}}^{2}+2 a_{2} u_{\mathbf{0}}+a_{1}\right) u_{\alpha}+\left(3 a_{3} u_{\mathbf{0}}+a_{2}\right) \sum_{0<\beta<\alpha} u_{\alpha-\beta} u_{\beta}\right. \\
& \left.+a_{3} \sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta} u_{\beta-\gamma} u_{\gamma}\right) H_{\alpha} \\
& =p_{3}\left(u_{\mathbf{0}}\right)+\sum_{\alpha>\mathbf{0}}\left(p_{3}^{\prime}\left(u_{\mathbf{0}}\right) u_{\alpha}+\frac{1}{2!} \cdot p_{3}^{\prime \prime}\left(u_{\mathbf{0}}\right) \sum_{0<\beta<\alpha} u_{\alpha-\beta} u_{\beta}\right. \\
& \left.+\frac{1}{3!} \cdot p_{3}^{\prime \prime \prime}\left(u_{\mathbf{0}}\right) \sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta} u_{\beta-\gamma} u_{\gamma}\right) H_{\alpha}
\end{aligned}
$$

where $p_{3}^{\prime}, p_{3}^{\prime \prime}$ and $p_{3}^{\prime \prime \prime}$ denote respectively the first, the second and the third derivative of the polynomial $p_{3}$. Thus, by applying the chaos expansion method to the nonlinear stochastic problem 1.62 we obtained the system of infinitely many deterministic Cauchy problems that have the forms 1.69 and 1.66 .

Theorem 54 ([77]) Let the assumptions $(B 1)-(B 3)$ and $(C 4-3)$ be fulfilled. Then, there exists a unique almost classical solution $u \in C([0, T], X) \otimes$ $(S)_{-1}$ of the stochastic nonlinear equations 1.62 .

## Fractional operators of the Malliavin calculus

In 64, 68, 69 we defined fractional operators of generalized Malliavin calculus. They are connected with the corresponding classical operators through an isometry mapping denoted by $\mathcal{M}$, see [64]. The equations with fractional operators can be considered in an analogue way as the ones presented in this thesis.

We denote by $\mathbb{D}$ the Malliavin derivative and $\mathbb{D}^{(H)}$ the fractional Malliavin derivative on $X \otimes(S)_{-\rho}$ (respectively on $X \otimes(S)_{\rho}$ and $\left.X \otimes L^{2}(\mu)\right)$. We say that a process $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}, f_{\alpha} \in X$ is differentiable in Malliavin sense if its coefficients satisfy (1.3) (respectively (1.4) and 1.5). Then, the chaos expansion form of its Malliavin derivative is given by $(1.2)$, while the chaos expansion form of its fractional Malliavin derivative is given by

$$
\begin{equation*}
\mathbb{D}^{(H)} F=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes e_{k}^{(H)} \otimes H_{\alpha-\varepsilon^{(k)}} \tag{1.70}
\end{equation*}
$$

where $e_{k}^{(H)}=M^{(1-H)} \xi_{k}, k \in \mathbb{N}$. Denote by $\widetilde{\mathbb{D}}$ the Malliavin derivative and by $\widetilde{\mathbb{D}}^{(H)}$ the fractional Malliavin derivative on $X \otimes(S)_{-\rho}^{(H)}$ (respectively on
$X \otimes(S)_{\rho}^{(H)}$ and $\left.X \otimes L^{2}\left(\mu_{H}\right)\right)$. If the coefficients of $\widetilde{F}=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes \widetilde{H}_{\alpha}, f_{\alpha} \in$ $X, \alpha \in \mathcal{I}$ satisfy (1.3) (respectively (1.4) and (1.5), then chaos expansion forms of these operators are

$$
\begin{align*}
\widetilde{\mathbb{D}} F & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes e_{k}^{(H)} \otimes \widetilde{H}_{\alpha-\varepsilon^{(k)}}  \tag{1.71}\\
\widetilde{\mathbb{D}}^{(H)} F & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes M^{(1-H)} e_{k}^{(H)} \otimes \widetilde{H}_{\alpha-\varepsilon^{(k)}},
\end{align*}
$$

Note that both $\operatorname{Dom}(\mathbb{D})=\operatorname{Dom}\left(\mathbb{D}^{(H)}\right)$ and $\operatorname{Dom}(\widetilde{\mathbb{D}})=\operatorname{Dom}(\widetilde{\mathbb{D}}(H))$ are determined by the condition (1.3) (respectively by (1.4) and (1.5)). The connection between $\mathbb{D}^{(H)}$ and $\mathbb{D}$ on a classical space and also between $\widetilde{\mathbb{D}}^{(H)}$ and $\widetilde{\mathbb{D}}$ on a fractional space is given through the mapping $\mathbf{M}=M^{(H)} \otimes I d$, see 64. In particular, let $\mathbb{D}^{(H)}: X \otimes(S)_{-\rho} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ and $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}\left(\mathbb{D}^{(H)}\right)$. Then,

$$
\begin{equation*}
\mathbb{D}^{(H)} F=\mathbf{M}^{-1}\left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon(k)}\right)=\mathbf{M}^{-1} \circ \mathbb{D} F . \tag{1.72}
\end{equation*}
$$

Similarly, $\widetilde{\mathbb{D}}^{(H)}: X \otimes(S)_{-\rho}^{(H)} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}^{(H)}$ and for $\widetilde{F} \in \operatorname{Dom}\left(\widetilde{\mathbb{D}}^{(H)}\right)$ it holds $\widetilde{\mathbb{D}}^{(H)} \widetilde{F}=\mathrm{M}^{-1} \circ \widetilde{\mathbb{D}} \widetilde{F}$.

Theorem 55 ([69]) For $F \in \operatorname{Dom}(\mathbb{D})$ it holds

$$
\begin{equation*}
\mathbb{D}^{(H)} F=\mathbf{M}^{-1} \circ \mathbb{D} F=\mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1} F . \tag{1.73}
\end{equation*}
$$

We denote by $\delta^{(H)}$ the fractional Skorokhod integral on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ (respectively on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ and $\left.X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)\right)$ and by $\widetilde{\delta}$ the Skorokhod integral on the corresponding fractional space $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}^{(H)}$ (respectively on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{\rho}^{(H)}$ and $\left.X \otimes L^{2}(\mathbb{R}) \otimes L^{2}\left(\mu_{H}\right)\right)$. In particular, $u \in \operatorname{Dom}(\delta)$ if its coefficients satisfy (1.6) (respectively (1.8) and (1.9)) and the fractional Skorokhod integral is defined by

$$
\begin{equation*}
\delta^{(H)}(u)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha, k}^{H} \otimes H_{\alpha+\varepsilon^{(k)}}, \tag{1.74}
\end{equation*}
$$

where $u_{\alpha, k}^{H}=\left(u_{\alpha}, e_{k}^{(H)}\right), \alpha \in \mathcal{I}$ and $k \in \mathbb{N}$. Let $\widetilde{u}=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes$ $S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}^{(H)}$ (respectively on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{\rho}^{(H)}$ and $\left.X \otimes L^{2}(\mathbb{R}) \otimes L^{2}\left(\mu_{H}\right)\right)$, such that the coefficients $u_{\alpha}=\sum_{k \in \mathbb{N}} u_{\alpha, k} \otimes \xi_{k}$ with $u_{\alpha, k} \in X$ satisfy (1.6) (respectively (1.8) and (1.9)). Then, the Skorokhod integral $\tilde{\delta}$ is of the form

$$
\begin{equation*}
\widetilde{\delta}(\widetilde{u})=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha, k} \otimes \widetilde{H}_{\alpha+\varepsilon^{(k)}} . \tag{1.75}
\end{equation*}
$$

Theorem $56([68])$ For $\widetilde{u} \in \operatorname{Dom}_{0}(\widetilde{\delta})$ it holds $\mathcal{M}(\widetilde{\delta}(\widetilde{u}))=\delta(\mathcal{M}(\widetilde{u}))$.

The fractional Ornstein-Uhlenbeck operator $\mathcal{R}^{(H)}$ on the classical space is defined as the composition $\mathcal{R}^{(H)}=\delta^{(H)} \circ \mathbb{D}^{(H)}$ and can be represented in the form

$$
\mathcal{R}^{(H)} u=\mathcal{R}^{(H)}\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\mathcal{R} u
$$

Similarly, the Ornstein-Uhlenbeck operator $\widetilde{\mathcal{R}}=\widetilde{\delta} \circ \widetilde{\mathbb{D}}$ and the fractional Ornstein-Uhlenbeck operators $\widetilde{\mathcal{R}}^{(H)}=\widetilde{\delta}^{(H)} \circ \widetilde{\mathbb{D}}^{(H)}$ in fractional spaces are also equal

$$
\widetilde{\mathcal{R}}^{(H)} \widetilde{u}=\widetilde{\mathcal{R}}^{(H)}\left(\sum_{\alpha \in \mathcal{I}} \widetilde{u}_{\alpha} \otimes \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}}|\alpha| \widetilde{u}_{\alpha} \otimes \widetilde{H}_{\alpha}=\widetilde{\mathcal{R}} \widetilde{u}
$$

The corresponding domains remain the same and, depending on a set of processes, are determined by 1.12 , 1.13 or 1.14 . More details can be found in 64.

# Malliavin Calculus for Generalized and Test Stochastic Processes 

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#### Abstract

We extend the Malliavin calculus from the classical finite variance setting to generalized processes with infinite variance and their test processes. The domain and the range of the basic Malliavin operators is characterized in terms of test processes and generalized processes. Various properties are proved such as the duality of the integral and the derivative in strong and in weak sense, the product rule with respect to ordinary and Wick multiplication and the chain rule in classical and in Wick sense.


## 1. Introduction

Stochastic processes with infinite variance (e.g. the white noise process) appear in many cases as solutions to stochastic differential equations. The Hida spaces and the Kondratiev spaces (see e.g. [3, 4]) have been introduced as the stochastic analogues of the Schwartz spaces of tempered distributions in order to provide a strict theoretical meaning for these kind of processes. The spaces of the test processes contain highly regular processes which are needed as windows through which one can detect the action of generalized processes.

The Malliavin derivative, the Skorokhod integral and the Ornstein-Uhlenbeck operator are fundamental for the stochastic calculus of variations. Each of them has a meaning also in quantum theory: they represent the annihilation, the creation and the number operator respectively. In stochastic analysis, the Malliavin derivative charachterizes densities of distributions, the Skorokhod integral is an extension of the Itô integral to non-adapted processes, and the Ornstein-Uhlenbeck operator plays the role of the stochastic Laplacian.

In the classical setting followed by $[2,13,15]$, the domain of these operators is a strict subset of the set of processes with finite second moments $(L)^{2}$, leading to Sobolev type normed spaces. A more general characterization of the domain of these operators in Kondratiev generalized function spaces has been derived in $[5,6,9,10]$. The range of the operators for generalized processes for $\rho=1$ has been studied in [8]. As a conclusion to this series of papers, in the current paper we provide a setting for the domains of these operators for $\rho \in[0,1]$ and a similar setting for test processes: first we construct a subset of the Kondratiev space which will be the domain of the operators, then we prove that the operators are linear, bounded, non-injective within the corresponding spaces and develop a representation of their range. In

[^0]the second part of the paper we fully develop the calculus including the integration by parts, Leibnitz rule and chain rule etc. using the interplay of generalized processes with their test processes and different types of dual pairings.

The Malliavin derivative of generalized stochastic processes has first been considered in [1] using the $S$-transform of stochastic exponentials and chaos expansions with $n$-fold Itô integrals with some vague notion of the Ito integral of a generalized function. Our approach is different, it relies on chaos expansions via Hermite polynomials and it provides more precise results: a fine gradation of generalized and test functions is followed where each level has a Hilbert structure and consequently each level of singularity has its own domain, range, set of multipliers etc.

The organisation of the paper is the following: After a short preview of the basic setting and notions of chaos expansions (Subsection 2.1), spaces of generalized stochastic processes and test stochastic processes (Subsection 2.2-2.3), we turn to the question of their multiplication in Subsection 2.4. In Section 3 we provide the characterisation of the domains of the basic operators of Malliavin calculus and prove their linearity and boundedness. In Section 4 we provide explicit solutions to the equations $\mathcal{R} u=g$, $\mathbb{D} u=h, \delta u=f$. In Section 5 we prove some rules of the Malliavin calculus for generalized and test processes, such as the duality between the derivative $\mathbb{D}$ and the integral operator $\delta$ (integration by parts formula), the product rule for $\mathbb{D}$ and $\mathcal{R}$ both for ordinary multiplication and Wick multiplication, and eventually we prove the chain rule. Some accompanying examples, applications and supplementary material to our results are provided in [11].

## 2. Preliminaries

Consider the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel $\sigma$-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to the characteristic function $\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \phi \in S(\mathbb{R})$, given by the Bochner-Minlos theorem.

Denote by $h_{n}(x)=(-1)^{n} e^{\frac{x^{2}}{2}} \frac{d^{n}}{d x^{n}}\left(e^{-\frac{x^{2}}{2}}\right), n \in \mathbb{N}_{0}, \mathbb{N}_{0}=\mathbb{N} \cup\{0\}$, the family of Hermite polynomials and $\xi_{n}(x)=$ $\frac{1}{\sqrt[4]{\pi} \sqrt{(n-1)!}} e^{-\frac{x^{2}}{2}} h_{n-1}(\sqrt{2} x), n \in \mathbb{N}$, the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^{2}(\mathbb{R})$. We follow the characterization of the Schwartz spaces in terms of the Hermite basis: The space of rapidly decreasing functions as a projective limit space $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$ and the space of tempered distributions as an inductive limit space $S^{\prime}(\mathbb{R})=\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$ where

$$
S_{l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k}:\|f\|_{l}^{2}=\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{Z}, \mathbb{Z}=-\mathbb{N} \cup \mathbb{N}_{0}
$$

Note that $S_{l}(\mathbb{R})$ is a Hilbert space endowed with the scalar product $\langle\cdot, \cdot\rangle_{l}$ given by

$$
\left\langle\xi_{k}, \xi_{i}\right\rangle_{l}=\left\{\begin{array}{rl}
0, & k \neq i \\
\left\|\xi_{k}\right\|_{l}^{2}=(2 k)^{l}, & k=i .
\end{array}, \quad l \in \mathbb{Z} .\right.
$$

### 2.1. The Wiener chaos spaces

Let $I=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ denote the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right), \alpha_{i} \in \mathbb{N}_{0}, i=1,2, \ldots, m, m \in \mathbb{N}$. The $k$ th unit vector $\varepsilon^{(k)}=(0, \cdots, 0,1,0, \cdots), k \in \mathbb{N}$ is the sequence of zeros with the only entry 1 as its $k$ th component. The multi-index $0=(0,0,0,0, \ldots)$ has all zero entries. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$.

Operations with multi-indices are carried out componentwise e.g. $\alpha+\beta=\left(\alpha_{1}+\beta_{1}, \alpha_{2}+\beta_{2}, \ldots\right)$, $\alpha!=\prod_{k=1}^{\infty} \alpha_{k}!$ and $\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$. Note that $\alpha>0$ (equivalently $|\alpha|>0$ ) if there is at least one component $\alpha_{k}>0$. We adopt the convention that $\alpha-\beta$ exists only if $\alpha-\beta>0$ and otherwise it is not defined.

Let $(2 \mathbb{N})^{\alpha}=\prod_{k=1}^{\infty}(2 k)^{\alpha_{k}}$. Note that $\sum_{\alpha \in I}(2 \mathbb{N})^{-p \alpha}<\infty$ for $p>1$ (see e.g. [4]).

Lemma 2.1. The following estimates hold:

$$
\begin{aligned}
& 1^{\circ}\binom{\alpha}{\beta} \leq 2^{|\alpha|} \leq(2 \mathbb{N})^{\alpha}, \quad \alpha \in \mathcal{I}, \\
& 2^{\circ}(\theta+\beta)!\leq \theta!\beta!(2 \mathbb{N})^{\theta+\beta}, \quad \theta, \beta \in \mathcal{I} .
\end{aligned}
$$

Proof. $1^{\circ}$ Since $\binom{n}{k} \leq 2^{n}$, for all $n \in \mathbb{N}_{0}$ and $0 \leq k \leq n$, it follows that

$$
\binom{\alpha}{\beta}=\prod_{i \in \mathbb{N}}\binom{\alpha_{i}}{\beta_{i}} \leq \prod_{i \in \mathbb{N}} 2^{\alpha_{i}}=2^{|\alpha|} \leq \prod_{i \in \mathbb{N}}(2 i)^{\alpha_{i}}=(2 \mathbb{N})^{\alpha}
$$

for all $\alpha \in \mathcal{I}$ and $\mathbf{0} \leq \beta \leq \alpha$.
$2^{\circ}$ From $\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$ and (i) it follows that $\alpha!\leq \beta!(\alpha-\beta)!(2 \mathbb{N})^{\alpha}$. By substituting $\theta=\alpha-\beta$, we obtain $(\theta+\beta) \leq \theta!\beta!(2 \mathbb{N})^{\theta+\beta}$, for all $\theta, \beta \in \mathcal{I}$.

Let $(L)^{2}=L^{2}\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$ be the Hilbert space of random variables with finite second moments. Then

$$
\begin{equation*}
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in I \tag{1}
\end{equation*}
$$

forms the Fourier-Hermite orthogonal basis of $(L)^{2}$ such that $\left\|H_{\alpha}\right\|_{(L)^{2}}^{2}=\alpha!$. In particular, $H_{0}=1$ and for the $k$ th unit vector $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$. The prominent Wiener-Itô chaos expansion theorem states that each element $F \in(L)^{2}$ has a unique representation of the form

$$
F(\omega)=\sum_{\alpha \in I} c_{\alpha} H_{\alpha}(\omega), \quad \omega \in S^{\prime}(\mathbb{R}), c_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}
$$

such that $\|F\|_{(L)^{2}}^{2}=\sum_{\alpha \in I} c_{\alpha}^{2} \alpha!<\infty$.

### 2.2. Kondratiev spaces and Hida spaces

The stochastic analogue of Schwartz spaces as generalized function spaces are the Kondratiev spaces of generalized random variables. Let $\rho \in[0,1]$.

Definition 2.2. The space of the Kondratiev test random variables $(S)_{\rho}$ consists of elements $f=\sum_{\alpha \in I} c_{\alpha} H_{\alpha} \in(L)^{2}$, $c_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|f\|_{\rho, p}^{2}=\sum_{\alpha \in I} c_{\alpha}^{2}(\alpha!)^{1+\rho}(2 \mathbb{N})^{p \alpha}<\infty, \quad \text { for all } p \in \mathbb{N}_{0}
$$

The space of the Kondratiev generalized random variables $(S)_{-\rho}$ consists of formal expansions of the form $F=\sum_{\alpha \in I} b_{\alpha} H_{\alpha}, b_{\alpha} \in \mathbb{R}, \alpha \in I$, such that

$$
\|F\|_{-\rho,-p}^{2}=\sum_{\alpha \in I} b_{\alpha}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha}<\infty, \quad \text { for some } p \in \mathbb{N}_{0}
$$

This provides a sequence of spaces $(S)_{\rho, p}=\left\{f \in(L)^{2}:\|f\|_{\rho, p}<\infty\right\}, \rho \in[0,1]$, such that

$$
\begin{aligned}
& (S)_{1, p} \subseteq(S)_{\rho, p} \subseteq(S)_{0, p} \subseteq(L)^{2} \subseteq(S)_{0,-p} \subseteq(S)_{-\rho,-p} \subseteq(S)_{-1,-p} \\
& (S)_{\rho, p} \subseteq(S)_{\rho, q} \subseteq(L)^{2} \subseteq(S)_{-\rho,-q} \subseteq(S)_{-\rho,-p}
\end{aligned}
$$

for all $p \geq q \geq 0$, the inclusions denote continuous embeddings and $(S)_{0,0}=(L)^{2}$. Thus, $(S)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{\rho, p}$, can be equipped with the projective topology, while $(S)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-\rho,-p}$ as its dual with the inductive topology. Note that $(S)_{\rho}$ is nuclear and the following Gel'fand triple

$$
(S)_{\rho} \subseteq(L)^{2} \subseteq(S)_{-\rho}
$$

is obtained. Especially, the case $\rho=0$ corresponds to the Hida spaces.
We will denote by $<\cdots>_{\rho}$ the dual pairing between $(S)_{-\rho}$ and $(S)_{\rho}$. Its action is given by $\ll A, B \gg_{\rho}=\ll \sum_{\alpha \in I} a_{\alpha} H_{\alpha}, \sum_{\alpha \in I} b_{\alpha} H_{\alpha} \gg_{\rho}=\sum_{\alpha \in I} \alpha!a_{\alpha} b_{\alpha}$. In case of random variables with finite variance it reduces to the scalar product $\ll A, B>_{(L)^{2}}=E(A B)$. Especially, the Hida case will be of importance, thus note that for any fixed $p \in \mathbb{Z},(S)_{0, p}, p \in \mathbb{Z}$, is a Hilbert space (we identify the case $p=0$ with $\left.(L)^{2}\right)$ endowed with the scalar product

$$
\ll H_{\alpha}, H_{\beta}>_{0, p}=\left\{\begin{aligned}
0, & \alpha \neq \beta, \\
\alpha!(2 \mathbb{N})^{p \alpha}, & \alpha=\beta,
\end{aligned} \quad \text { for } p \in \mathbb{Z},\right.
$$

extended by linearity and continuity to

$$
\ll A, B \gg_{0, p}=\sum_{\alpha \in I} \alpha!a_{\alpha} b_{\alpha}(2 \mathbb{N})^{p \alpha}, \quad p \in \mathbb{Z}
$$

In the framework of white noise analysis, the problem of pointwise multiplication of generalized functions is overcome by introducing the Wick product. It is well defined in the Kondratiev spaces of test and generalized stochastic functions $(S)_{\rho}$ and $(S)_{-\rho}$; see for example $[3,4]$.
Definition 2.3. Let $F, G \in(S)_{-\rho}$ be given by their chaos expansions $F(\omega)=\sum_{\alpha \in I} f_{\alpha} H_{\alpha}(\omega)$ and $G(\omega)=\sum_{\beta \in I} g_{\beta} H_{\beta}(\omega)$, for unique $f_{\alpha}, g_{\beta} \in \mathbb{R}$. The Wick product of $F$ and $G$ is the element denoted by $F \diamond G$ and defined by

$$
F \diamond G(\omega)=\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} H_{\alpha+\beta}(\omega)=\sum_{\gamma \in I}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) H_{\gamma}(\omega) .
$$

The same definition is provided for the Wick product of test random variables belonging to $(S)_{\rho}$.
For the Fourier-Hermite polynomials (1), for all $\alpha, \beta \in \mathcal{I}$ it holds $H_{\alpha} \diamond H_{\beta}=H_{\alpha+\beta}$.
The $n$th Wick power is defined by $F^{\diamond n}=F^{\diamond(n-1)} \diamond F, F^{\diamond 0}=1$. Note that $H_{n \varepsilon^{(k)}}=H_{\varepsilon(k)}^{\diamond n}$ for $n \in \mathbb{N}_{0}, k \in \mathbb{N}$.
Note that the Kondratiev spaces $(S)_{\rho}$ and $(S)_{-\rho}$ are closed under the Wick multiplication [4], while the space $(L)^{2}$ is not closed under it. The most important property of the Wick multiplication is its relation to the Itô-Skorokhod integration [3, 4], since it reproduces the fundamental theorem of calculus. It also represents a renormalization of the ordinary product and the highest order stochastic approximation of the ordinary product [14].

In the sequel we will need the notion of Wick-versions of analytic functions.
Definition 2.4. If $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a real analytic function at the origin represented by the power series

$$
\varphi(x)=\sum_{n=0}^{\infty} a_{n} x^{n}, \quad x \in \mathbb{R}
$$

then its Wick version $\varphi^{\diamond}:(S)_{-\rho} \rightarrow(S)_{-\rho}$, for $\rho \in[0,1]$, is given by

$$
\varphi^{\diamond}(F)=\sum_{n=0}^{\infty} a_{n} F^{\diamond n}, \quad F \in(S)_{-\rho} .
$$

### 2.3. Generalized stochastic processes

Let $\tilde{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\tilde{X}}$ and let $\tilde{X}^{\prime}$ denote its dual space. In this section we describe $\tilde{X}$-valued random variables. Most notably, if $\tilde{X}$ is a space of functions on $\mathbb{R}$, e.g. $\tilde{X}=C^{k}([a, b])$, $-\infty<a<b<\infty$ or $\tilde{X}=L^{2}(\mathbb{R})$, we obtain the notion of a stochastic process. We will also define processes where $\tilde{X}$ is not a normed space, but a nuclear space topologized by a family of seminorms, e.g. $\tilde{X}=S(\mathbb{R})$ (see e.g. [16]).

Definition 2.5. Let $f$ have the formal expansion

$$
\begin{equation*}
f=\sum_{\alpha \in I} f_{\alpha} \otimes H_{\alpha}, \quad \text { where } f_{\alpha} \in X, \alpha \in I \tag{2}
\end{equation*}
$$

Let $\rho \in[0,1]$. Define the following spaces:

$$
\begin{aligned}
X \otimes(S)_{\rho, p} & =\left\{f:\|f\|_{X \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha \in I} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}, \\
X \otimes(S)_{-\rho,-p} & =\left\{f:\|f\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in I} \alpha 1^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\},
\end{aligned}
$$

where $X$ denotes an arbitrary Banach space (allowing both possibilities $X=\tilde{X}, X=\tilde{X}^{\prime}$ ). Especially, for $\rho=0$ and $p=0, X \otimes(S)_{0,0}$ will be denoted by

$$
X \otimes(L)^{2}=\left\{f:\|f\|_{X \otimes(L)^{2}}^{2}=\sum_{\alpha \in I} \alpha!\left\|f_{\alpha}\right\|_{X}^{2}<\infty\right\} .
$$

We will denote by $E(F)=f_{0}$ the generalized expectation of the process $F$.
Definition 2.6. Generalized stochastic processes and test stochastic processes in Kondratiev sense are elements of the spaces

$$
X \otimes(S)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}} X \otimes(S)_{-\rho,-p}, \quad X \otimes(S)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}} X \otimes(S)_{\rho, p}, \quad \rho \in[0,1]
$$

respectively.
Remark 2.7. The symbol $\otimes$ denotes the projective tensor product of two spaces, i.e. $\tilde{X}^{\prime} \otimes(S)_{-\rho}$ is the completion of the tensor product with respect to the $\pi$-topology.

The Kondratiev space $(S)_{\rho}$ is nuclear and thus $\left(\tilde{X} \otimes(S)_{\rho}\right)^{\prime} \cong \tilde{X}^{\prime} \otimes(S)_{-\rho}$. Note that $\tilde{X}^{\prime} \otimes(S)_{-\rho}$ is isomorphic to the space of linear bounded mappings $\tilde{X} \rightarrow(S)_{-\rho}$, and it is also isomporphic to the space of linear bounded mappings $(S)_{\rho} \rightarrow \tilde{X}^{\prime}$.

In [19] and [20] a general setting of $S^{\prime}$-valued generalized stochastic process is provided: $S^{\prime}(\mathbb{R})$-valued generalized stochastic processes are elements of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ and they are given by chaos expansions of the form

$$
\begin{equation*}
f=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in I} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k} \tag{3}
\end{equation*}
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in I} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$ and $a_{\alpha, k} \in X$. Thus,
$X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}=\left\{f=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}:\|f\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha!^{1-\rho}\left\|a_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty\right\}$
and

$$
X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}=\bigcup_{p, l \in \mathbb{N}_{0}} X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}
$$

The generalized expectation of an $S^{\prime}$-valued stochastic process $f$ is given by $E(f)=\sum_{k \in \mathbb{N}} a_{(0,0, \ldots), k} \otimes \xi_{k}=b_{0}$.
In an analogue way, we define $S$-valued test processes as elements of $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$, which are given by chaos expansions of the form (3), where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S(\mathbb{R}), c_{k}=\sum_{\alpha \in I} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{\rho}$ and $a_{\alpha, k} \in X$. Thus,

$$
X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}=\left\{f=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}:\|f\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha!^{1+\rho}\left\|a_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha}<\infty\right\}
$$

and

$$
X \otimes S(\mathbb{R}) \otimes(S)_{\rho}=\bigcap_{p, l \in \mathbb{N}_{0}} X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}
$$

The Hida spaces are obtained for $\rho=0$. Especially, for $p=l=0$, one obtains the space of processes with finite second moments and square integrable trajectories $X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$. It is isomporphic to $X \otimes L^{2}(\mathbb{R} \times \Omega)$ and if $X$ is a separable Hilbert space, then it is also isomorphic to $L^{2}(\mathbb{R} \times \Omega ; X)$.

### 2.4. Multiplication of stochastic processes

We generalize the definition of the Wick product of random variables to the set of generalized stochastic processes in the way as it is done in [7,17] and [18]. For this purpose we will assume that $X$ is closed under multiplication, i.e. that $x \cdot y \in X$, for all $x, y \in X$.
Definition 2.8. Let $F, G \in X \otimes(S)_{ \pm \rho}, \rho \in[0,1]$, be generalized (resp. test) stochastic processes given in chaos expansions of the form (2). Then the Wick product $F \diamond G$ is defined by

$$
\begin{equation*}
F \diamond G=\sum_{\gamma \in I}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) \otimes H_{\gamma} . \tag{4}
\end{equation*}
$$

Theorem 2.9. Let $\rho \in[0,1]$ and let the stochastic processes $F$ and $G$ be given in their chaos expansion forms $F=\sum_{\alpha \in I} f_{\alpha} \otimes H_{\alpha}$ and $G=\sum_{\alpha \in I} g_{\alpha} \otimes H_{\alpha}$.
$1^{\circ}$ If $F \in X \otimes(S)_{-\rho,-p_{1}}$ and $G \in X \otimes(S)_{-\rho_{,}-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{-\rho,-q}$, for $q \geq p_{1}+p_{2}+4$.
$2^{\circ}$ If $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{\rho, p_{2}}$ for $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{\rho, q}$, for $q \leq \min \left\{p_{1}, p_{2}\right\}-4$.
Proof. $1^{\circ}$ By the Cauchy-Schwartz inequality, the following holds

$$
\begin{aligned}
\|F \diamond G\|_{X \otimes(S)_{-\rho,-q}}^{2} & =\sum_{\gamma \in I}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(\gamma!)^{1-\rho}(2 \mathbb{N})^{-q \gamma} \leq \sum_{\gamma \in I}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(\gamma!)^{1-\rho}(2 \mathbb{N})^{-\left(p_{1}+p_{2}+4\right) \gamma} \\
& \left.=\sum_{\gamma \in I} \| \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}(\alpha+\beta)\right)^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{1}+1}{2} \gamma}(2 \mathbb{N})^{-\frac{p_{2}+1}{2} \gamma} \|_{X}^{2}(2 \mathbb{N})^{-2 \gamma} \\
& \leq \sum_{\gamma \in I}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\left(\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}\right)^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{1}+1}{2} \alpha}(2 \mathbb{N})^{-\frac{p_{2}+1}{2} \beta}\right\|_{X}^{2}(2 \mathbb{N})^{-2 \gamma} \\
& \leq \sum_{\gamma \in I}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} \alpha!^{\frac{1-\rho}{2}} \beta!^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{1}+\rho}{2} \alpha}(2 \mathbb{N})^{-\frac{p_{2}+\rho}{2} \beta}\right\|_{X}^{2}(2 \mathbb{N})^{-2 \gamma} \\
& \leq \sum_{\gamma \in I}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} \alpha!^{\frac{1-\rho}{2}} \beta!^{1-\rho}(2 \mathbb{N})^{-\frac{p_{1} \alpha}{2}}(2 \mathbb{N})^{-\frac{p_{2} \beta}{2}}\right\|_{X}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \leq \sum_{\gamma \in I}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma}\left\|f_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p_{1} \alpha}\right)\left(\sum_{\alpha+\beta=\gamma}\left\|g_{\beta}\right\|_{X}^{2}(\beta!)^{1-\rho}(2 \mathbb{N})^{-p_{2} \beta}\right) \\
& \leq \sum_{\gamma \in I}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha \in I}\left\|f_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p_{1} \alpha}\right)\left(\sum_{\beta \in I}\left\|g_{\beta}\right\|_{X}^{2}(\beta!)^{1-\rho}(2 \mathbb{N})^{-p_{2} \beta}\right) \\
& =M \cdot\|F\|_{X \otimes(S)_{-\rho_{,}-p_{1}}^{2}}^{2} \cdot\|G\|_{X \otimes(S)_{-\rho_{1}-p_{2}}^{2}}^{2}<\infty,
\end{aligned}
$$

since $M=\sum_{\gamma \in I}(2 \mathbb{N})^{-2 \gamma}<\infty$. We also applied Lemma 2.1 part $1^{\circ}$, inequalities $(2 \mathbb{N})^{-\frac{p_{1}+1}{2} \gamma} \leq(2 \mathbb{N})^{-\frac{p_{1}+1}{2} \alpha}$ and $(2 \mathbb{N})^{-\frac{p_{2}+1}{2} \gamma} \leq(2 \mathbb{N})^{-\frac{p_{2}+1}{2} \beta}$ since $\gamma \geq \alpha, \gamma \geq \beta$, as well as $(2 \mathbb{N})^{-\frac{p_{1}+\rho}{2} \alpha} \leq(2 \mathbb{N})^{-\frac{p_{1} \alpha}{2}}$ because $\rho \in[0,1]$.
$2^{\circ}$ Let now $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{\rho, p_{2}}$ for all $p_{1}, p_{2} \in \mathbb{N}_{0}$. Then the chaos expansion form of $F \diamond G$ is given by (4) and

$$
\begin{aligned}
\|F \diamond G\|_{X \otimes(S)_{p, q}}^{2} & =\sum_{\gamma \in I} \gamma!^{1+\rho}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{q \gamma}=\sum_{\gamma \in I}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} \gamma!^{\frac{1+\rho}{2}} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2} \gamma}\right\|_{X}^{2} \\
& \leq \sum_{\gamma \in I}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} \alpha!^{\frac{1+\rho}{2}} \beta!^{\frac{1+\rho}{2}}(2 \mathbb{N})^{\frac{1+\rho}{2}(\alpha+\beta)} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \leq M\left(\sum_{\alpha+\beta=\gamma} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p_{1} \alpha}\right)\left(\sum_{\alpha+\beta=\gamma} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p_{2} \beta}\right) \\
& \leq M\left(\sum_{\alpha \in I} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{(q+4)) \alpha}\right)\left(\sum_{\beta \in I} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{(q+4) \beta}\right) \\
& =M \cdot\|F\|_{X \otimes(S)_{\rho, p_{1}}^{2}}^{2} \cdot\|G\|_{X \otimes(S)_{\rho, p_{2}}}^{2}<\infty,
\end{aligned}
$$

if $q \leq p_{1}-4$ and $q \leq p_{2}-4$. We used the Cauchy-Schwartz inequality along with the estimate $(\alpha+\beta)$ ! $\leq$ $\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}$, from Lemma 2.1.

Remark 2.10. A test stochastic process $u \in X \otimes(S)_{p, p}, p \geq 0$ can be considered as a generalized stochastic process from $X \otimes(S)_{-\rho,-q}, q \geq 0$ since $\|u\|_{X \otimes(S)_{-\rho,-q}}^{2} \leq\|u\|_{X \otimes(S)_{\rho, \phi}}^{2}$. Therefore, if $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{-\rho_{,}-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{-p,-q}$, for $q \geq p_{2}+4$. This follows from Theorem 2.9 part $1^{\circ}$ by letting $p_{1}=0$.

Applying the well-known formula for the Fourier-Hermite polynomials (see [4])

$$
\begin{equation*}
H_{\alpha} \cdot H_{\beta}=\sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \tag{5}
\end{equation*}
$$

one can define the ordinary product $F \cdot G$ of two stochastic processes $F$ and $G$. Thus, by applying formally (5) we obtain

$$
\begin{aligned}
F \cdot G & =\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \otimes H_{\alpha} \cdot H_{\beta}=\sum_{\alpha \in I} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \otimes \sum_{0 \leq \gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \\
& =F \diamond G+\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \otimes \sum_{0<\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}
\end{aligned}
$$

After a change of variables $\delta=\alpha-\gamma, \theta=\beta-\gamma$, we obtain $H_{\alpha} \cdot H_{\beta}=\sum_{\substack{\gamma, \delta, \theta \\ \gamma+\theta=\beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!\theta!} H_{\delta+\theta}$.

$$
H_{\alpha} \cdot H_{\beta}=\sum_{\substack{0 \leq \tau \delta+\beta \\ \gamma+\tau=\delta+\beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau}=\sum_{\substack{0 \leq \tau<+\beta \\ \alpha+\tau \tau \beta+2 \delta}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau}, \quad \alpha, \beta \in \mathcal{I}
$$

After another change of variables $\tau=\delta+\theta$ we finally obtain the chaos expansion of $H_{\alpha} \cdot H_{\beta}$ in $(L)^{2}$ :

$$
H_{\alpha} \cdot H_{\beta}=\sum_{\tau \in I} \sum_{\substack{\gamma \in I, \delta \leq \tau \\ \gamma+\tau-\delta=\beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau}=H_{\alpha+\beta}+\sum_{\tau \in I} \sum_{\substack{\gamma \geq 0, \delta \leq \tau \\ \gamma+\tau-\delta=\beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau} .
$$

Similarly, we can rearrange the sums for $F \cdot G$ to obtain

$$
\begin{equation*}
F \cdot G=F \diamond G+\sum_{\tau \in I} \sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\substack{\gamma>0,0 \leq \tau \\ \gamma+\tau-\delta \beta, \gamma+\delta \delta \alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau}=\sum_{\tau \in I} \sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} a_{\alpha, \beta, \tau} H_{\tau}, \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{\alpha, \beta, \tau}=\sum_{\substack{\gamma \in \in, \overline{2} \tau \tau, \gamma+\tau \tau \delta \beta, \gamma+\delta \delta \alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} \tag{7}
\end{equation*}
$$

Note the following facts: for each $\alpha, \beta, \tau \in I$ fixed there exists a unique pair of multi-indices $\gamma, \delta \in I$ such that $\delta \leq \tau$ and $\gamma+\tau-\delta=\beta, \gamma+\delta=\alpha$. Moreover, both $\alpha+\beta$ and $|\alpha-\beta|$ are odd (resp. even) if and only if $\tau$ is odd (resp. even). Also, $\alpha+\beta \geq \tau \geq|\alpha-\beta|$. Thus,

$$
a_{\alpha, \beta, \tau}=\frac{\alpha!\beta!}{\left(\frac{\alpha+\beta-\tau}{2}\right)!\left(\frac{\alpha-\beta+\tau}{2}\right)!\left(\frac{\beta-\alpha+\tau}{2}\right)!} .
$$

For example, if $\tau=(2,0,0,0, \ldots)$, then the coefficient next to $H_{\tau}$ in (6) is $f_{(0,0,0, \ldots)} g_{(2,0,0, \ldots)}+f_{(1,0,0, \ldots)} g_{(1,0,0, \ldots)}+$ $f_{(2,0,0, \ldots)} g_{(0,0,0, \ldots)}+3 f_{(1,0,0, \ldots)} g_{(3,0,0, \ldots)}+4 f_{(2,0,0, \ldots)} g_{(2,0,0, \ldots)}+3 f_{(3,0,0, \ldots)} g_{(1,0,0, \ldots)}+18 f_{(3,0,0, \ldots)} g_{(3,0,0, \ldots)}+\cdots$.

Lemma 2.11. Let $\alpha, \beta, \tau \in I$ and $a_{\alpha, \beta, \tau}$ be defined as in (7). Then

$$
a_{\alpha, \beta, \tau} \leq(2 \mathbb{N})^{\alpha+\beta}
$$

Proof. From the estimate $\alpha!=\frac{(2 \alpha)!}{2^{(\alpha)}} \geq \frac{(2 \alpha)!}{(2 \mathbb{N})^{a}}$, which follows from Lemma 2.1 part $1^{\circ}$, we obtain

$$
a_{\alpha, \beta, \tau}=\frac{\alpha!\beta!}{\left(\frac{\alpha+\beta-\tau}{2}\right)!\left(\frac{\alpha-\beta+\tau}{2}\right)!\left(\frac{\beta-\alpha+\tau}{2}\right)!} \leq \frac{\alpha!\beta!}{(\alpha+\beta-\tau)!(\alpha-\beta+\tau)!(\beta-\alpha+\tau)!(2 \mathbb{N})^{-(\alpha+\beta-\tau)}} .
$$

Without loss of generality we may assume that $\alpha \leq \beta$. The case $\beta \leq \alpha$ can be considered similarly.
First case, if $\alpha \leq \beta \leq \tau$. Then, $\beta \leq \tau$ implies that $\frac{\alpha!}{(\alpha-\beta+\tau)!} \leq 1$, while $\alpha \leq \tau$ implies that $\frac{\beta!}{(\beta-\alpha+\tau)!} \leq 1$. Thus

$$
a_{\alpha, \beta, \tau} \leq \frac{(2 \mathbb{N})^{\alpha+\beta-\tau}}{(\alpha+\beta-\tau)!} \leq(2 \mathbb{N})^{\alpha+\beta}
$$

Second case, if $\alpha \leq \tau \leq \beta$. Then, $\alpha \leq \tau$ implies again $\frac{\beta!}{(\beta-\alpha+\tau)!} \leq 1$, while $\tau \leq \beta$ now implies that $\frac{\alpha!}{(\alpha+\beta-\tau)!} \leq 1$. Thus,

$$
a_{\alpha, \beta, \tau} \leq \frac{(2 \mathbb{N})^{\alpha+\beta-\tau}}{(\alpha-\beta+\tau)!} \leq(2 \mathbb{N})^{\alpha+\beta}
$$

Third case, if $\tau \leq \alpha \leq \beta$. Then $\beta-\alpha+\tau \leq \beta$ and $\alpha-\beta+\tau \leq \tau$. Thus, we obtain

$$
\begin{aligned}
a_{\alpha, \beta, \tau} & \leq \prod_{i \in \mathbb{N}} \frac{\alpha_{i}!\beta_{i}!}{\left(\alpha_{i}+\beta_{i}-\tau_{i}\right)!\left(\alpha_{i}-\beta_{i}+\tau_{i}\right)!\left(\beta_{i}-\alpha_{i}+\tau_{i}\right)!(2 i)^{-\left(\alpha_{i}+\beta_{i}-\tau_{i}\right)}} \\
& =\prod_{i \in \mathbb{N}} \frac{\left(\alpha_{i}-\beta_{i}+\tau_{i}\right)!\cdot\left(\alpha_{i}-\beta_{i}+\tau_{i}+1\right) \ldots\left(\alpha_{i}-1\right) \cdot \alpha_{i}\left(\beta_{i}-\alpha_{i}+\tau_{i}\right)!\cdot\left(\beta_{i}-\alpha_{i}+\tau_{i}+1\right) \ldots\left(\beta_{i}-1\right) \cdot \beta_{i}}{\left(\alpha_{i}-\beta_{i}+\tau_{i}\right)!\left(\beta_{i}-\alpha_{i}+\tau_{i}\right)!\left(\alpha_{i}+\beta_{i}-\tau_{i}\right)!(2 i)^{-\left(\alpha_{i}+\beta_{i}-\tau_{i}\right)}} \\
& \leq 1 \cdot(2 \mathbb{N})^{\alpha+\beta-\tau} \leq(2 \mathbb{N})^{\alpha+\beta}
\end{aligned}
$$

## Theorem 2.12. The following holds:

$1^{\circ}$ If $F \in X \otimes(S)_{\rho, r_{1}}$ and $G \in X \otimes(S)_{\rho, r_{2}}$, for some $r_{1}, r_{2} \in \mathbb{N}_{0}$, then the ordinary product $F \cdot G$ is a well defined element in $X \otimes(S)_{\rho, q}$ for $q \leq \min \left\{r_{1}, r_{2}\right\}-8$.
$2^{\circ}$ If $F \in X \otimes(S)_{\rho, r_{1}}$ and $G \in X \otimes(S)_{-\rho,-r_{2}}$, for $r_{1}-r_{2}>8$, then their ordinary product $F \cdot G$ is well defined and belongs to $X \otimes(S)_{-p,-q}$ for $r_{2} \leq q \leq r_{1}-8$.

Proof. $1^{\circ}$ Let $q=p-8$, where $p \leq \min \left\{p_{1}, p_{2}\right\}-8$. By Lemma 2.11, Lemma 2.1 and the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
& \|F \cdot G\|_{X \otimes(S)_{p q}}^{2}=\sum_{\tau \in I} \tau!^{1+\rho}\left\|\sum_{\alpha, \beta \in I} f_{\alpha} g_{\beta} a_{\alpha, \beta, \tau}\right\|_{X}^{2}(2 \mathbb{N})^{q \tau} \\
& \leq \sum_{\tau \in I} \tau \tau^{1+\rho}\left\|\sum_{\substack{a \in \beta=I \\
\tau \leq \alpha+\beta}} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\alpha+\beta}\right\|_{X}^{2}(2 \mathbb{N})^{(p-8) \tau} \\
& =\sum_{\tau \in I}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\substack{\alpha, \beta \in I \\
\tau \leq \alpha+\beta}} f_{\alpha} g_{\beta} \tau!^{\frac{1+\rho}{2}}(2 \mathbb{N})^{\alpha+\beta}(2 \mathbb{N})^{\frac{p-6}{2} \tau}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in I}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\substack{\alpha, \beta \in+\\
\tau \alpha \alpha \beta}} f_{\alpha} g_{\beta} \alpha!^{\frac{1+p}{2}} \beta!^{\frac{1+p}{2}}(2 \mathbb{N})^{\frac{1+p}{2}}(2 \mathbb{N})^{\alpha+\beta}(2 \mathbb{N})^{\frac{p-6}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in I}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\alpha, \beta \in I} \alpha!^{\frac{1+\rho}{2}} f_{\alpha}(2 \mathbb{N})^{\frac{p \alpha}{2}}(2 \mathbb{N})^{-\beta} \beta!^{\frac{1+\rho}{2}} g_{\beta}(2 \mathbb{N})^{\frac{p \beta}{2}}(2 \mathbb{N})^{-\alpha}\right\|_{X}^{2} \\
& =\sum_{\tau \in I}(2 \mathbb{N})^{-2 \tau}\left(\sum_{\alpha, \beta \in I} \alpha 1^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}(2 \mathbb{N})^{-2 \beta} \sum_{\alpha, \beta \in I} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta}(2 \mathbb{N})^{-2 \alpha}\right) \\
& \leq \sum_{\tau \in I}(2 \mathbb{N})^{-2 \tau}\left(\sum_{\beta \in I}(2 \mathbb{N})^{-2 \beta} \sum_{\alpha \in I} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}\right)\left(\sum_{\alpha \in I}(2 \mathbb{N})^{-2 \alpha} \sum_{\beta \in I} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta}\right) \\
& \leq M C_{1} C_{2} \sum_{\alpha \in I} \alpha!!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \sum_{\beta \in I} \alpha!!^{2}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta} \\
& =M C_{1} C_{2}\|F\|_{X \otimes(S)_{p, p}}^{2}\|G\|_{X \otimes(S)_{p, p}}^{2}<\infty \text {, }
\end{aligned}
$$

where $M=\sum_{\tau \in \mathcal{I}}(2 \mathbb{N})^{-2 \tau}<\infty, C_{1}=\sum_{\beta \in \mathcal{I}}(2 \mathbb{N})^{-2 \beta}<\infty$ and $C_{2}=\sum_{\alpha \in I}(2 \mathbb{N})^{-2 \alpha}<\infty$.
$2^{\circ}$ Let $\varphi \in(S)_{\rho, q}$ and $F \in X \otimes(S)_{\rho, r_{1}}$. Then by Theorem 2.12 part $1^{\circ}, F \cdot \varphi \in(S)_{\rho, s}$ for $s \leq \min \left\{r_{1}, q\right\}-8=r_{1}-8$. Also, $G \in(S)_{-\rho, r_{2}}$ implies that $G \in(S)_{-\rho,-c}$ for $c \geq r_{2}$. Thus for any $c$ such that $r_{2} \leq c \leq s \leq r_{1}-8$ we have $F \cdot \varphi \in(S)_{\rho, c}$ and $G \in(S)_{-\rho,-c}$. Now,

$$
\begin{aligned}
\|F \cdot G\|_{-\rho,-q}^{2} & =\sup _{\|\varphi\|_{q} \leq 1}|\ll F \cdot G, \varphi \gg \rho|=\sup _{\|\varphi\|_{q} \leq 1}|\ll G, F \cdot \varphi \gg \rho| \\
& \leq \sup _{\|\varphi\|_{q} \leq 1}\|G\|_{-\rho,-c} \cdot\|F \cdot \varphi\|_{\rho, c} \leq \sup _{\|\varphi\|_{q} \leq 1}\|G\|_{-\rho,-c} \cdot\|F\|_{\rho, r_{1}} \cdot\|\varphi\|_{\rho, q} .
\end{aligned}
$$

This implies

$$
\|F \cdot G\|_{-\rho,-q}^{2} \leq M \cdot\|G\|_{-\rho,-r_{2}} \cdot\|F\|_{\rho, r_{1},}
$$

for some $M>0$.

Remark 2.13. Note, for $F, G \in X \otimes(L)^{2}$ the ordinary product $F \cdot G$ will not necessarily belong to $X \otimes(L)^{2}(f o r ~ a ~$ counterexample see [11]), but due to the Hölder inequality it will belong to $X \otimes(L)^{1}$.

## 3. Operators of the Malliavin Calculus

In the classical literature $[2,12,13,15]$ the Malliavin derivative and the Skorokhod integral are defined on a subspace of $(L)^{2}$ so that the resulting process after application of these operators necessarily remains in $(L)^{2}$. We will recall of these classical results and denote the corresponding domains with a "zero" in order to retain a nice symmetry between test and generalized processes. In $[6,7,9,10]$ we allowed values in the Kondratiev space $(S)_{-1}$ and thus obtained larger domains for all operators. These domains will be denoted by a "minus" sign to reflect the fact that they correspond to generalized processes. In this paper we introduce also domains for test processes. These domains will be denoted by a "plus" sign.

Definition 3.1. Let a generalized stochastic process $u \in X \otimes(S)_{-\rho}$ be of the form $u=\sum_{\alpha \in I} u_{\alpha} \otimes H_{\alpha}$. If there exists $p \in \mathbb{N}_{0}$ such that

$$
\begin{equation*}
\sum_{\alpha \in I}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty, \tag{8}
\end{equation*}
$$

then the Malliavin derivative of $u$ is defined by

$$
\begin{equation*}
\mathbb{D} u=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha} \tag{9}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$, i.e. $H_{\alpha-\varepsilon^{(k)}}=\left\{\begin{array}{cc}0, & \alpha_{k}=0 \\ H_{\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right)}, & \alpha_{k} \geq 1\end{array}\right.$,for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in I$.

For two processes $u=\sum_{\alpha \in I} u_{\alpha} \otimes H_{\alpha}, v=\sum_{\alpha \in I} v_{\alpha} \otimes H_{\alpha}$ and constants $a, b$ the linearity property holds, i.e. $\mathbb{D}(a u+b v)=a \mathbb{D}(u)+b \mathbb{D}(v)$. The set of generalized stochastic processes $u \in X \otimes(S)_{-\rho}$ which satisfy (8) constitutes the domain of the Malliavin derivative, denoted by $\operatorname{Dom}_{-}^{\rho}(\mathbb{D})$. Thus the domain of the Malliavin derivative is given by

$$
\operatorname{Dom}_{-}^{\rho}(\mathbb{D})=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-p}^{\rho}(\mathbb{D})=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{-\rho}: \sum_{\alpha \in I}|\alpha|^{1+\rho} \alpha 1^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\} .
$$

A process $u \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ is called a Malliavin differentiable process.
Theorem 3.2. The Malliavin derivative of a process $u \in X \otimes(S)_{-\rho}$ is a linear and continuous mapping
$\mathbb{D}: \quad \operatorname{Dom}_{-p}^{\rho}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$,
for $l>p+1$ and $p \in \mathbb{N}_{0}$.

Proof. Let $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$. Then,

$$
\begin{aligned}
\|\mathbb{D} u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}} & =\sum_{\alpha \in I}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2}\left\|u_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}(2 k)^{-l}\right) \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{2}\left\|u_{\beta}\right\|_{X}^{2}(2 k)^{-l}\left(\frac{\beta!}{\beta_{k}}\right)^{1-\rho}(2 k)^{p}\right)(2 \mathbb{N})^{-p \beta} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{1+\rho}(2 k)^{-(l-p)}\right)\left\|u_{\beta}\right\|_{X}^{2}(\beta!)^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq \sum_{\beta \in I}\left(\sum_{k=1}^{\infty} \beta_{k}\right)^{1+\rho}\left(\sum_{k=1}^{\infty}(2 k)^{-(l-p)}\right)\left\|u_{\beta}\right\|_{X}^{2}(\beta!)^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& =c \sum_{\beta \in I}|\beta|^{1+\rho}(\beta!)^{1-\rho}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p \beta}=c\|u\|_{D o m_{-p}^{\rho}(\mathbb{D})}^{2}<\infty,
\end{aligned}
$$

where $c=\sum_{k \in \mathbb{N}}(2 k)^{-(l-p)}<\infty$ for $l-p>1$ and where we used $\left(\alpha-\varepsilon^{(k)}\right)!=\frac{\alpha!}{\alpha_{k}}, \alpha_{k}>0$ and the estimate $\sum_{k \in \mathbb{N}} \alpha_{k}^{1+\rho} \leq\left(\sum_{k \in \mathbb{N}} \alpha_{k}\right)^{1+\rho}=|\alpha|^{1+\rho}$.

For all $\alpha \in \mathcal{I}$ we have $|\alpha|<\alpha!$. Thus, the smallest domain of the spaces $\operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ is obtained for $\rho=0$ and the largest is obtained for $\rho=1$. In particular we have $\operatorname{Dom}_{-}^{0}(\mathbb{D}) \subset \operatorname{Dom}_{-}^{1}(\mathbb{D})$. Moreover if $p \leq q$ then $\operatorname{Dom}_{-p}^{\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{-q}^{\rho}(\mathbb{D})$.

For square integrable stochastic process $u \in X \otimes(L)^{2}$ the domain is given by

$$
\operatorname{Dom}_{0}(\mathbb{D})=\left\{u \in X \otimes(L)^{2}: \sum_{\alpha \in I}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\}
$$

Theorem 3.3. The Malliavin derivative of a process $u \in \operatorname{Dom}_{0}(\mathbb{D})$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{0}(\mathbb{D}) \rightarrow X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}
$$

$\operatorname{Proof}$. Let $u \in \operatorname{Dom}_{0}(\mathbb{D})$, i.e. $\sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty$. Then,

$$
\|\mathbb{D} u\|_{X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}}^{2}=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha_{k}^{2}\left(\alpha-\varepsilon^{(k)}\right)!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha_{k} \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{\alpha \in I}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty .
$$

In general, for $\rho \in[0,1]$ the domain of $\mathbb{D}$ in $X \otimes(S)_{\rho}$ is

$$
\operatorname{Dom}_{+}^{\rho}=\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{p}^{\rho}(\mathbb{D})=\bigcap_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{\rho}: \sum_{\alpha \in I}|\alpha|^{1-\rho}(\alpha!)^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} .
$$

Theorem 3.4. Let $\rho \in[0,1]$. The Malliavin derivative of a test stochastic process $v \in X \otimes(S)_{\rho}$ is a linear and continuous mapping

$$
\mathbb{D}: \quad \operatorname{Dom}_{p}^{\rho}(\mathbb{D}) \rightarrow X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}
$$

for $l<p-1$ and $p \in \mathbb{N}_{0}$.

Proof. Let $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{p}^{\rho}(\mathbb{D})$. Then, from (9) and

$$
\begin{aligned}
\|\mathbb{D} v\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}^{2} & =\sum_{\alpha \in I}\left\|\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) v_{\alpha+\varepsilon^{(k)}} \xi_{k}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{\alpha \in I}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2}\left\|v_{\left.\alpha+\varepsilon^{k}\right)}\right\|_{X}^{2}(2 k)^{l}\right) \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{2}\left\|v_{\beta}\right\|_{X}^{2}(2 k)^{l}\left(\frac{\beta!}{\beta_{k}}\right)^{1+\rho}(2 k)^{-p}\right)(2 \mathbb{N})^{p \beta} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{1-\rho}(2 k)^{-(p-l)}\right)\left\|v_{\beta}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq c^{1-\rho} \sum_{\beta \in I}|\beta|^{1-\rho}(\beta!)^{1+\rho}\left\|v_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta}<\infty,
\end{aligned}
$$

the assertion follows, where we used

$$
\sum_{k \in \mathbb{N}} \beta_{k}^{1-\rho}(2 k)^{l-p} \leq\left(\sum_{k \in \mathbb{N}} \beta_{k}\right)^{1-\rho}\left(\sum_{k \in \mathbb{N}}(2 k)^{\frac{l-p}{1-\rho}}\right)^{1-\rho} \leq|\beta|^{1-\rho} \cdot c^{1-\rho}
$$

and $c=\sum_{k \in \mathbb{N}}(2 k)^{\frac{l-p}{1-p}} \leq \sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$, for $p>l+1$. We also used $\beta_{k}\left(\beta-\varepsilon^{(k)}\right)!=\beta!, \beta \in \mathcal{I}$ and $(2 \mathbb{N})^{\varepsilon^{(k)}}=(2 k)$, $k \in \mathbb{N}$.

Note that $\operatorname{Dom}_{p}^{\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-p}^{\rho}(\mathbb{D})$ for all $p \in \mathbb{N}$. Therefore, $\operatorname{Dom}_{+}^{\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$. Moreover, using the estimate $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ it follows that

$$
\begin{aligned}
& X \otimes(S)_{-\rho,-(p-2)} \subseteq \operatorname{Dom}_{-p}^{\rho}(\mathbb{D}) \subseteq X \otimes(S)_{-\rho,-p}, \quad p>3, \quad \text { and } \\
& X \otimes(S)_{\rho, p+1} \subseteq \operatorname{Dom}_{p}^{\rho}(\mathbb{D}) \subseteq X \otimes(S)_{\rho, p,}, \quad p>0 .
\end{aligned}
$$

Remark 3.5. For $u \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\mathbb{D})$ it is usual to write

$$
\mathbb{D}_{t} u=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k}(t) \otimes H_{\alpha-\varepsilon^{(k)}},
$$

in order to emphasise that the Malliavin derivative takes a random variable into a process, i.e. that $\mathbb{D} u$ is a function of $t$. Moreover, the formula

$$
\mathbb{D}_{t} F(\omega)=\lim _{h \rightarrow 0} \frac{1}{h}\left(F\left(\omega+h \cdot \kappa_{[t, \infty)}\right)-F(\omega)\right), \quad \omega \in S^{\prime}(\mathbb{R})
$$

justifies the name stochastic derivative for the Malliavin operator. Since generalized functions do not have point values, this notation would be somewhat misleading for $u \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$. Therefore, for notational uniformity, we omit the index $t$ in $\mathbb{D}_{t}$ that usually appears in the literature and write $\mathbb{D}$.

The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $(L)^{2}$-sense. In [6] we have extended the definition of the Skorokhod integral from Hilbert space valued processes to the class of $S^{\prime}$-valued generalized processes.

Definition 3.6. Let $F=\sum_{\alpha \in I} f_{\alpha} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$, be a generalized $S^{\prime}(\mathbb{R})$-valued stochastic process and let $f_{\alpha} \in X \otimes S^{\prime}(\mathbb{R})$ be given by the expansion $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k}, f_{\alpha, k} \in X$. If there exist $p \geq 0, l \geq 0$ such that

$$
\sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha!\left(\alpha_{k}+1\right)\right)^{1-\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty
$$

then the Skorokhod integral of $F$ is given by

$$
\begin{equation*}
\delta(F)=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}=\sum_{\alpha>0} \sum_{k \in \mathbb{N}} f_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} . \tag{10}
\end{equation*}
$$

A linear combination of two Skorokhod integrable processes $F, G$ is again Skorokhod integrable process $a F+b G, a, b \in \mathbb{R}$ such that $\delta(a F+b G)=a \delta(F)+b \delta(G)$.

In general, the domain $\operatorname{Dom}_{-}^{\rho}(\delta)$ of the Skorokhod integral is

$$
\operatorname{Dom}_{-}^{\rho}(\delta)=\bigcup_{\substack{(, p) \in \mathbb{N}^{2} \\ p>l+1}} \operatorname{Dom}_{(-l,-p)}^{\rho}(\delta)=\bigcup_{\substack{(, p) \in \mathbb{N}^{2} \\ \text { p>l+1 }}}\left\{F \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}: \sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha!\left(\alpha_{k}+1\right)\right)^{1-\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty\right\} .
$$

Theorem 3.7. Let $\rho \in[0,1]$. The Skorokhod integral $\delta$ of a $S_{-l}(\mathbb{R})$-valued stochastic process is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{(-l,-p)}^{\rho}(\delta) \rightarrow X \otimes(S)_{-\rho,-p}, \quad p>l+1
$$

Proof. This statement follows from

$$
\begin{aligned}
\|\delta(F)\|_{X \otimes(S)_{-\rho,-p}}^{2} & =\sum_{|\alpha| \geq 1} \alpha!^{1-\rho}\left\|\sum_{k \in \mathbb{N}} f_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{|\alpha| \geq 1}\left\|\sum_{k \in \mathbb{N}} \alpha!^{\frac{1-\rho}{2}} f_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{\beta \in I}\left\|\sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!^{\frac{1-\rho}{2}} f_{\beta, k}(2 k)^{-\frac{p}{2}}\right\|_{X}^{2}(2 \mathbb{N})^{-p \beta} \\
& =\sum_{\beta \in I}\left\|\sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!^{\frac{1-\rho}{2}} f_{\beta, k}(2 k)^{-\frac{l}{2}}(2 k)^{-\frac{p-l}{2}}\right\|_{X}^{2}(2 \mathbb{N})^{-p \beta} \\
& \leq \sum_{\beta \in I}\left(\sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!^{1-\rho}\left\|f_{\beta, k}\right\|_{X}^{2}(2 k)^{-l} \sum_{k \in \mathbb{N}}(2 k)^{-(p-l)}\right)(2 \mathbb{N})^{-p \beta} \\
& \leq c \sum_{\beta \in I} \sum_{k \in \mathbb{N}}\left(\beta!\left(\beta_{k}+1\right)\right)^{1-\rho}\left\|f_{\beta, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \beta}=c\|F\|_{D o m_{(-l,-p)}^{\rho}(\delta)}^{2}<\infty
\end{aligned}
$$

where $c=\sum_{k \in \mathbb{N}}(2 k)^{-(p-l)}<\infty$ for $p>l+1$.

Note that for $\rho=1$ it holds that $\operatorname{Dom}_{-}^{1}(\delta)=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$.
Now we characterize the domains $\operatorname{Dom}_{+}^{\rho}(\delta)$ and $\operatorname{Dom}_{0}(\delta)$ of the Skorokhod integral for test processes from $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ and square integrable processes from $X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$. The form of the derivative is in all cases given by the expression (10).

For square integrable stochastic processes $T \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$ of the form $T=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} t_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$, $t_{\alpha, k} \in X$, we define

$$
\operatorname{Dom}_{0}(\delta)=\left\{T \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}: \sum_{\alpha \in I}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{\frac{1}{2}} \alpha!^{\frac{1}{2}}\left\|t_{\alpha, k}\right\|_{X}\right)^{2}<\infty\right\}
$$

Theorem 3.8. The Skorokhod integral $\delta$ of an $L^{2}(\mathbb{R})$-valued stochastic process is a linear and continuous mapping
$\delta: \quad \operatorname{Dom}_{0}(\delta) \rightarrow X \otimes(L)^{2}$.

Proof. Let $T=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} t_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\delta)$. Then,

$$
\begin{aligned}
\|\delta(T)\|_{X \otimes(L)^{2}}^{2} & =\sum_{|\alpha| \geq 1}\left\|\sum_{k \in \mathbb{N}} t_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \alpha!=\sum_{|\alpha| \geq 1}\left\|\sum_{k \in \mathbb{N}} \alpha!^{\frac{1}{2}} t_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \\
& =\sum_{\beta \in I}\left\|\sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!^{\frac{1}{2}} t_{\beta, k}\right\|_{X}^{2} \leq \sum_{\beta \in I}\left(\sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!^{\frac{1}{2}}\left\|t_{\beta, k}\right\|_{X}\right)^{2} \\
& =\sum_{\beta \in I}\left(\sum_{k \in \mathbb{N}} \beta!^{\frac{1}{2}}\left(\beta_{k}+1\right)^{\frac{1}{2}}\left\|t_{\beta, k}\right\|_{X}\right)^{2}=\|T\|_{\text {Domo }(\delta)}^{2}<\infty .
\end{aligned}
$$

In general, for any $\rho \in[0,1]$, the domain $\operatorname{Dom}_{+}^{\rho}(\delta)$ of the Skorokhod integral in $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ is

$$
\operatorname{Dom}_{+}^{\rho}(\delta)=\bigcap_{\substack{(l, p) \in \mathbb{N}^{2} \\ l>p+1}} \operatorname{Dom}_{(l, p)}^{\rho}(\delta)=\bigcap_{\substack{(p,) \in \mathbb{N}^{2} \\ \square>p+1}}\left\{F \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}: \sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{1+\rho} \alpha!^{1+\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha}<\infty\right\}
$$

Theorem 3.9. The Skorokhod integral $\delta$ of an $S_{l}(\mathbb{R})$-valued stochastic test process is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{(l, p)}^{\rho}(\delta) \rightarrow X \otimes(S)_{\rho, p}, \quad l>p+1
$$

Proof. Let $U=\sum_{\alpha \in I} u_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{(l, p)}^{\rho}(\delta), u_{\alpha}=\sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \in X \otimes S_{l}(\mathbb{R}), u_{\alpha, k} \in X$, for $l>p+1$. Then we obtain

$$
\begin{aligned}
\|\delta(U)\|_{X \otimes(S)_{\rho, p}}^{2} & =\sum_{\beta \in I}\left\|\sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!^{\frac{1+\rho}{2}} u_{\beta, k}(2 k)^{\frac{p}{2}}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta} \\
& \leq \sum_{\beta \in I}\left(\sum_{k \in \mathbb{N}}\left(\beta!\left(\beta_{k}+1\right)\right)^{1+\rho}\left\|u_{\beta, k}\right\|_{X}^{2}(2 k)^{l} \sum_{k \in \mathbb{N}}(2 k)^{-(l-p)}\right)(2 \mathbb{N})^{p \beta} \leq c\|U\|_{D o m_{(, p)}^{\rho}(\delta)}^{2}<\infty,
\end{aligned}
$$

where $c=\sum_{k \in \mathbb{N}}(2 k)^{-(l-p)}<\infty$ for $l>p+1$.
Using the estimates $\alpha_{k}+1 \leq 2|\alpha|$, which holds for all $\alpha \in \mathcal{I}$ except for $\alpha=0$, and $|\alpha| \leq(2 \mathbb{N})^{\alpha}, \alpha \in \mathcal{I}$ we obtain

$$
\begin{aligned}
\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha!^{1+\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha} & \leq \sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{1+\rho} \alpha!^{1+\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha} \\
& \leq \sum_{k \in \mathbb{N}}\left\|f_{0, k}\right\|_{X}^{2}(2 k)^{l}+4 \sum_{\alpha>0} \sum_{k \in \mathbb{N}}|\alpha|^{2} \alpha!^{1+\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha} \\
& \leq\left\|f_{0}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}+4 \sum_{\alpha>0} \sum_{k \in \mathbb{N}} \alpha!^{1+\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{(p+2) \alpha} \\
& \leq 4\|F\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p+2}}^{2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p+2} \subseteq \operatorname{Dom}_{(l, p)}^{\rho}(\delta) \subseteq X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}, \quad \text { for } l>p+1 \text { and } \\
& X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-(p-1)} \subseteq \operatorname{Dom}_{(-l,-p)}^{\rho}(\delta) \subseteq X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p,} \quad \text { for } p>l+1 .
\end{aligned}
$$

The third main operator of the Malliavin calculus is the Ornstein-Uhlenbeck operator.
Definition 3.10. The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathcal{R}=\delta \circ \mathbb{D}$ and called the Ornstein-Uhlenbeck operator.

Therefore, for $u \in X \otimes(S)_{-\rho}$ given in the chaos expansion form $u=\sum_{\alpha \in I} u_{\alpha} \otimes H_{\alpha}$, the Ornstein-Uhlenbeck operator is given by

$$
\begin{equation*}
\mathcal{R}(u)=\sum_{\alpha \in I}|\alpha| u_{\alpha} \otimes H_{\alpha} . \tag{11}
\end{equation*}
$$

The Orstein-Uhlenbeck operator is linear, i.e. by (11) $\mathcal{R}(a u+b v)=a \mathcal{R}(u)+b \mathcal{R}(v), a, b \in \mathbb{R}$ holds.
Let

$$
\operatorname{Dom}_{-}^{\rho}(\mathcal{R})=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-p}^{\rho}(\mathcal{R})=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{-\rho}: \sum_{\alpha \in I}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha}<\infty\right\} .
$$

Theorem 3.11. The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \operatorname{Dom}_{-p}^{\rho}(\mathcal{R}) \rightarrow X \otimes(S)_{-\rho,-p}, \quad p \in \mathbb{N}_{0} .
$$

Moreover, $\operatorname{Dom}_{-}^{\rho}(\mathcal{R}) \subseteq \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$.
Proof. Let $v=\sum_{\alpha \in I} v_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{-p}^{\rho}(\mathcal{R})$, for some $p \in \mathbb{N}_{0}$. Then, from (11) it follows that

$$
\|\mathcal{R} v\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in I}|\alpha|^{2}\left\|v_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha}<\infty
$$

For $v \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ we obtain

$$
\sum_{\alpha \in I}|\alpha|^{1+\rho}\left\|v_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in I}|\alpha|^{2}\left\|v_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha}
$$

and the last assertion follows. Note that for $\rho=1, \operatorname{Dom}_{-p}^{1}(\mathcal{R})=\operatorname{Dom}_{-p}^{1}(\mathbb{D})$.
For square integrable processes we define

$$
\operatorname{Dom}_{0}(\mathcal{R})=\left\{w \in X \otimes(L)^{2}: \sum_{\alpha \in I} \alpha!|\alpha|^{2}\left\|w_{\alpha}\right\|_{X}^{2}<\infty\right\}
$$

Theorem 3.12. The operator $\mathcal{R}$ is a linear and continuous operator

$$
\mathcal{R}: \quad \operatorname{Dom}_{0}(\mathcal{R}) \rightarrow X \otimes(L)^{2}
$$

Moreover, $\operatorname{Dom}_{0}(\mathcal{R}) \subseteq \operatorname{Dom}_{0}(\mathbb{D})$.
Proof. Let $w=\sum_{\alpha \in \mathcal{I}} w_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\mathcal{R})$. Then $\mathcal{R}(w)=\sum_{\alpha \in \mathcal{I}}|\alpha| w_{\alpha} \otimes H_{\alpha}$ and

$$
\|\mathcal{R}(w)\|_{X \otimes(L)^{2}}^{2}=\sum_{\alpha \in I}|\alpha|^{2}\left\|w_{\alpha}\right\|_{X}^{2}=\|w\|_{D o m_{0}(\mathcal{R})}^{2}<\infty
$$

Now from $|\alpha| \leq|\alpha|^{2}$ for $\alpha \in \mathcal{I}$ it follows that $\operatorname{Dom}_{0}(\mathcal{R}) \subseteq \operatorname{Dom}_{0}(\mathbb{D})$.
For test processes, we define

$$
\operatorname{Dom}_{+}^{\rho}(\mathcal{R})=\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{p}^{\rho}(\mathcal{R})=\bigcap_{p \in \mathbb{N}_{0}}\left\{v \in X \otimes(S)_{\rho, p}: \sum_{\alpha \in I}(\alpha!)^{1+\rho}|\alpha|^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} .
$$

Theorem 3.13. The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \quad \operatorname{Dom}_{p}^{\rho}(\mathcal{R}) \rightarrow X \otimes(S)_{\rho, p}, \quad p \in \mathbb{N} .
$$

Moreover, $\operatorname{Dom}_{p}^{\rho}(\mathcal{R}) \subseteq \operatorname{Dom}_{p}^{\rho}(\mathbb{D})$.
Proof. Let $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{p}^{\rho}(\mathcal{R})$. Then,

$$
\|\mathcal{R} v\|_{X \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha \in I}\left\|v_{\alpha}\right\|_{X}^{2}|\alpha|^{1+\rho} \alpha!^{2}(2 \mathbb{N})^{p \alpha}=\|v\|_{D o m_{p}^{p}(\mathcal{R})}^{2}<\infty .
$$

From

$$
\sum_{\alpha \in I}|\alpha|^{1-\rho} \alpha!^{1+\rho}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in I}|\alpha|^{2} \alpha!^{1+\rho}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}
$$

follows that $\operatorname{Dom}_{p}^{\rho}(\mathcal{R}) \subseteq \operatorname{Dom}_{p}^{\rho}(\mathbb{D})$.
Note also that

$$
\begin{aligned}
& X \otimes(S)_{\rho, p+2} \subseteq \operatorname{Dom}_{p}^{\rho}(\mathcal{R}) \subseteq X \otimes(S)_{\rho, p}, \quad p \in \mathbb{N}, \quad \text { and } \\
& X \otimes(S)_{-\rho,-(p-2)} \subseteq \operatorname{Dom}_{-p}^{\rho}(\mathcal{R}) \subseteq X \otimes(S)_{-\rho,-p} .
\end{aligned}
$$

In [8] we have proven that the mappings $\delta: \operatorname{Dom}_{-}^{\rho}(\delta) \rightarrow X \otimes(S)_{-\rho}, \mathcal{R}: \operatorname{Dom}_{-}^{\rho}(\mathcal{R}) \rightarrow X \otimes(S)_{-\rho}$, for $\rho=1$, are surjective on the subspace of centered random variables (random variables with zero expectation). In the next section we prove the same type of surjectivity of the mappings for $\rho \in[0,1)$ as well, i.e. that the mappings $\delta: \operatorname{Dom}_{+}^{\rho}(\delta) \rightarrow X \otimes(S)_{\rho}, \mathcal{R}: \operatorname{Dom}_{+}^{\rho}(\mathcal{R}) \rightarrow X \otimes(S)_{\rho}, \delta: \operatorname{Dom}_{0}(\delta) \rightarrow X \otimes(L)^{2}, \mathcal{R}: \operatorname{Dom}_{0}(\mathcal{R}) \rightarrow X \otimes(L)^{2}$ have the corresponding range of centered generalized random variables. The mappings $\mathbb{D}: \operatorname{Dom}_{-}^{\rho}(\mathbb{D}) \rightarrow$ $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}, \mathbb{D}: \operatorname{Dom}_{+}^{\rho}(\mathbb{D}) \rightarrow X \otimes S(\mathbb{R}) \otimes(S)_{\rho}, \mathbb{D}: \operatorname{Dom}_{0}(\mathbb{D}) \rightarrow X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$ are surjective on the subspace of generalized stochastic processes satisfying a certain symmetry condition which will be discussed in detail.

## 4. Range of the Malliavin Operators

Theorem 4.1. (The Ornstein-Uhlenbeck operator) Let $g$ have zero generalized expectation. The equation

$$
\mathcal{R} u=g, \quad E u=\tilde{u}_{0} \in X,
$$

has a unique solution $u$ represented in the form

$$
u=\tilde{u}_{0}+\sum_{\alpha \in T,|\alpha|>0} \frac{g_{\alpha}}{|\alpha|} \otimes H_{\alpha} .
$$

Moreover, the following holds:
$1^{\circ}$ If $g \in X \otimes(S)_{-\rho,-p}, p \in \mathbb{N}$, then $u \in \operatorname{Dom}_{-p}^{\rho}(\mathcal{R})$.
$2^{\circ}$ If $g \in X \otimes(S)_{\rho, p}, p \in \mathbb{N}$, then $u \in \operatorname{Dom}_{p}^{\rho}(\mathcal{R})$.
$3^{\circ}$ If $g \in X \otimes(L)^{2}$, then $u \in \operatorname{Dom}_{0}(\mathcal{R})$.

Proof. Let us seek for a solution in form of $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$. From $\mathcal{R} u=g$ it follows that

$$
\sum_{\alpha \in I}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in I} g_{\alpha} \otimes H_{\alpha}
$$

i.e., $u_{\alpha}=\frac{g_{\alpha}}{|\alpha|}$ for all $\alpha \in \mathcal{I},|\alpha|>0$. From the initial condition we obtain $u_{(0,0,0,0, \ldots)}=E u=\tilde{u}_{0}$.
$1^{\circ}$ Let $g \in X \otimes(S)_{-\rho,-p}$. Then $u \in \operatorname{Dom}_{-p}^{\rho}(\mathcal{R})$ since

$$
\begin{aligned}
\|u\|_{D o m_{-p}^{\rho}(\mathcal{R})}^{2} & =\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0}|\alpha|^{2}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0}|\alpha|^{2}(\alpha!)^{1-\rho} \frac{\left\|g_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 \mathbb{N})^{-p \alpha} \\
& =\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0}(\alpha!)^{1-\rho}\left\|g_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\left\|u_{0}\right\|_{X}^{2}+\|g\|_{X \otimes(S)_{-\rho,-p}^{2}}^{2}<\infty
\end{aligned}
$$

$2^{\circ}$ Assume that $g \in X \otimes(S)_{\rho, p}$. Then $u \in \operatorname{Dom}_{p}^{\rho}(\mathcal{R})$ since

$$
\begin{aligned}
\|u\|_{D o m_{p}^{\rho}(\mathcal{R})}^{2} & =\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0}|\alpha|^{2}(\alpha!)^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0}(\alpha!)^{1+\rho}\left\|g_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \\
& =\left\|u_{0}\right\|_{X}^{2}+\|g\|_{X \otimes(S)_{\rho, p}}^{2}<\infty .
\end{aligned}
$$

$3^{\circ}$ If $g$ is square integrable, then $u \in \operatorname{Dom}_{0}(\mathcal{R})$ since

$$
\|u\|_{D o m_{0}(\mathcal{R})}^{2}=\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0}|\alpha|^{2} \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\left\|u_{0}\right\|_{X}^{2}+\sum_{|\alpha|>0} \alpha!\left\|g_{\alpha}\right\|_{X}^{2}=\|g\|_{X \otimes(L)^{2}}^{2}<\infty
$$

Corollary 4.2. Let $\rho \in[0,1]$. Each process $g \in X \otimes(S)_{ \pm \rho}$, resp. $g \in X \otimes(L)^{2}$ can be represented as $g=E g+\mathcal{R}(u)$, for some $u \in \operatorname{Dom}_{ \pm}^{\rho}(\mathcal{R})$, resp. $u \in \operatorname{Dom}_{0}(\mathcal{R})$.

In [10] we provided one way for solving equation $\mathbb{D} u=h$ : Using the chaos expansion method we transformed equation (15) into a system of infinitely many equations of the form

$$
\begin{equation*}
u_{\alpha+\varepsilon^{(k)}}=\frac{1}{\alpha_{k}+1} h_{\alpha, k}, \quad \text { for all } \quad \alpha \in \mathcal{I}, k \in \mathbb{N}, \tag{12}
\end{equation*}
$$

from which we calculated $u_{\alpha}$, by induction on the length of $\alpha$.
Denote by $r=r(\alpha)=\min \left\{k \in \mathbb{N}: \alpha_{k} \neq 0\right\}$, for a nonzero multi-index $\alpha \in I$, i.e. let $r$ be the position of the first nonzero component of $\alpha$. Then the first nonzero component of $\alpha$ is the $r$ th component $\alpha_{r}$, i.e. $\alpha=\left(0, \ldots, 0, \alpha_{r}, \ldots, \alpha_{m}, 0, \ldots\right)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of $\alpha$, except the $r$ th, which is $\alpha_{r}-1$. With the given notation we call $\alpha_{\varepsilon^{(r)}}$ the representative of $\alpha$ and write $\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}$. For $\alpha \in \mathcal{I},|\alpha|>0$ the set

$$
\mathcal{K}_{\alpha}=\left\{\beta \in \mathcal{I}: \alpha=\beta+\varepsilon^{(j)}, \text { for those } j \in \mathbb{N}, \text { such that } \alpha_{j}>0\right\}
$$

is a nonempty set, because it contains at least the representative of $\alpha$, i.e. $\alpha_{\varepsilon^{(r)}} \in \mathcal{K}_{\alpha}$. Note that, if $\alpha=n \varepsilon^{(r)}$, $n \in \mathbb{N}$ then $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)=1$ and in all other cases $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)>1$. Further, for $|\alpha|>0, \mathcal{K}_{\alpha}$ is a finite set because $\alpha$ has finitely many nonzero components and $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)$ is equal to the number of nonzero components of $\alpha$. For example, the first nonzero component of $\alpha=(0,3,1,0,5,0,0, \ldots)$ is the second one. It follows that $r=2, \alpha_{r}=3$ and the representative of $\alpha$ is $\alpha_{\varepsilon^{(r)}}=\alpha-\varepsilon^{(2)}=(0,2,1,0,5,0,0, \ldots)$. The multi-index $\alpha$ has three nonzero components, thus the set $\mathcal{K}_{\alpha}$ consists of three elements: $\mathcal{K}_{\alpha}=\{(0,2,1,0,5,0, \ldots),(0,3,0,0,5,0, \ldots),(0,3,1,0,4,0, \ldots)\}$.

In [10] we obtained the coefficients $u_{\alpha}$ of the solution of (12) as functions of the representative $\alpha_{\varepsilon^{(r)}}$ of a nonzero multi-index $\alpha \in \mathcal{I}$ in the form

$$
u_{\alpha}=\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon^{(r)}, r}} \quad \text { for }|\alpha| \neq 0, \alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}
$$

Theorem 4.3. ([10]) Let $h=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-p}, p \in \mathbb{N}_{0}$, with $h_{\alpha, k} \in X$ such that

$$
\begin{equation*}
\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon^{(r)}, r}}=\frac{1}{\alpha_{j}} h_{\beta, j}, \tag{13}
\end{equation*}
$$

for the representative $\alpha_{\varepsilon^{(r)}}$ of $\alpha \in \mathcal{I},|\alpha|>0$ and all $\beta \in \mathcal{K}_{\alpha}$, such that $\alpha=\beta+\varepsilon^{(j)}$, for $j \geq r, r \in \mathbb{N}$. Then, equation (15) has a unique solution in $X \otimes(S)_{-1,-2 p}$. The chaos expansion of the generalized stochastic process, which represents the unique solution of equation (15) is given by

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)} \in I} \frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon^{(r)}, r}} \otimes H_{\alpha} . \tag{14}
\end{equation*}
$$

Here we provide another way of solving equation $\mathbb{D} u=h$ using the Skorokod integral operator.
Theorem 4.4. (The Malliavin derivative) Let $h$ have the chaos expansion $h=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$ and assume that condition (13) holds. Then the equation

$$
\begin{equation*}
\mathbb{D} u=h, \quad E u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X, \tag{15}
\end{equation*}
$$

has a unique solution $u$ represented in the form

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha \in T,|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} . \tag{16}
\end{equation*}
$$

## Moreover, the following holds:

```
\(1^{\circ}\) If \(h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q>p+1\), then \(u \in \operatorname{Dom}_{-q}^{\rho}(\mathbb{D})\).
\(2^{\circ}\) If \(h \in X \otimes S_{p}(\mathbb{R}) \otimes(S)_{\rho, q}, p>q+1\), then \(u \in \operatorname{Dom}_{q}^{\rho}(\mathbb{D})\).
\(3^{\circ}\) If \(h \in \operatorname{Dom}_{0}(\delta)\), then \(u \in \operatorname{Dom}_{0}(\mathbb{D})\).
```

Proof. $1^{\circ}$ The proof is similar as for case $2^{\circ}$, so we present the proof of $2^{\circ}$.
$2^{\circ}$ Let $h \in X \otimes S_{p}(\mathbb{R}) \otimes(S)_{\rho, q}$. Then $h \in \operatorname{Dom}_{(p, q-2)}^{\rho}(\delta)$. Now, applying the Skorokhod integral on both sides of (15) one obtains

$$
\mathcal{R} u=\delta(h) .
$$

From the initial condition it follows that the solution $u$ is given in the form $u=\widetilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} u_{\alpha} \otimes H_{\alpha}$ and its coefficients are obtained from the system

$$
\begin{equation*}
|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k,} \quad|\alpha|>0, \tag{17}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$. Condition (13) ensures that $\delta$ is injective i.e. $\delta(\mathbb{D} u)=\delta(h)$ implies $\mathbb{D} u=h$.

It remains to prove that the solution $u \in \operatorname{Dom}_{q}^{\rho}(\mathbb{D})$. Clearly,

$$
\begin{aligned}
\left\|u-\tilde{u}_{0}\right\|_{D o m_{q}^{\rho}(\mathbb{D})}^{2} & =\sum_{\alpha \in I}|\alpha|^{1-\rho}(\alpha!)^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{q \alpha}=\sum_{\alpha \in I,|\alpha|>0}|\alpha|^{1-\rho} \frac{(\alpha!)^{1+\rho}}{|\alpha|^{2}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k), k}}\right\|_{X}^{2}(2 \mathbb{N})^{q \alpha} \\
& =\sum_{\beta \in I}\left\|\sum_{k \in \mathbb{N}} h_{\beta, k} \frac{\left(\beta+\varepsilon^{(k)}\right)!^{\frac{1+\rho}{2}}}{\left|\beta+\varepsilon^{(k)}\right|^{\frac{1+\rho}{2}}}(2 k)^{\frac{q}{2}}\right\|_{X}^{2}(2 \mathbb{N})^{q \beta} \leq \sum_{\beta \in I}\left\|\sum_{k \in \mathbb{N}} h_{\beta, k} \beta!^{\frac{1+\rho}{2}}(2 k)^{\frac{p}{2}}(2 k)^{q-p}\right\|_{X}^{2}(2 \mathbb{N})^{q \beta} \\
& \leq \sum_{\beta \in I}\left(\sum_{k \in \mathbb{N}}\left\|h_{\beta, k}\right\|_{X}^{2} \beta!^{1+\rho}(2 k)^{p} \sum_{k \in \mathbb{N}}(2 k)^{q-p}\right)(2 \mathbb{N})^{q \beta} \leq c \sum_{\beta \in I} \sum_{k \in \mathbb{N}}\left\|h_{\beta, k}\right\|_{X}^{2} \beta!^{1+\rho}(2 k)^{p}(2 \mathbb{N})^{q \beta} \\
& =c\|h\|_{X \otimes S_{p}(\mathbb{R}) \otimes(S)_{\rho, q}}^{2}<\infty,
\end{aligned}
$$

since $c=\sum_{k \in \mathbb{N}}(2 k)^{q-p}<\infty$, for $p>q+1$. In the fourth step of the estimation we used that $\frac{\left(\beta+\varepsilon^{(k)}\right)!}{\left|\beta+\varepsilon^{(k)}\right|} \leq \beta$ !. Thus,

$$
\|u\|_{\text {Dom }_{q}^{p}(\mathbb{D})}^{2} \leq 2\left(\left\|\tilde{u}_{0}\right\|_{X}^{2}+c\|h\|_{X \otimes S_{p}(\mathbb{R}) \otimes\left(S S_{p, q}\right)}^{2}\right)<\infty
$$

$3^{\circ}$ In this case we have that $u$ given in (16) satisfies

$$
\|u\|_{D o m_{0}(\mathbb{D})}^{2}=\sum_{\alpha \in I}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{\alpha \in I,|\alpha|>0} \frac{\alpha!}{|\alpha|}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \leq \sum_{\alpha \in I} \alpha!\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2}=\|h\|_{D o m_{0}(\delta)}^{2}<\infty .
$$

Corollary 4.5. If $\mathbb{D}(u)=0$, then $u=E u$, i.e. $u$ is constant almost surely.
Remark 4.6. The form of the solution (16) can be transformed to the form (14) obtained in [10]. First we express all $h_{\beta, k}$ in condition (13) in terms of $h_{\alpha_{\varepsilon}(t), r}$, i.e.

$$
h_{\beta, k}=\frac{\alpha_{j}}{\alpha_{r}} h_{\alpha_{\varepsilon_{(k)}}, r,}
$$

where $\beta \in \mathcal{K}_{\alpha}$ correspond to the nonzero components of $\alpha$ in the following way: $\beta=\alpha-\varepsilon^{(k)}, k \in \mathbb{N}$, and $r \in \mathbb{N}$ is the first nonzero component of $\alpha$. Note that the set $\mathcal{K}_{\alpha}$ has as many elements as the multi-index $\alpha$ has nonzero components. Therefore, from the form of the coefficients (17) obtained in Theorem 4.4 we have

$$
\frac{1}{|\alpha|} \sum_{\beta \in \mathcal{K}_{\alpha}} h_{\beta, k}=\frac{1}{|\alpha|} \sum_{j \in \mathbb{N}, \alpha_{j} \neq 0} \frac{\alpha_{j}}{\alpha_{r}} h_{\alpha_{(q)}, r}=\frac{1}{|\alpha|} \frac{\sum_{j \in \mathbb{N}} \alpha_{j}}{\alpha_{r}} h_{\alpha_{\varepsilon}(v), r}=\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon(\gamma)}, r} .
$$

Theorem 4.7. (The Skorokhod integral) Let $f$ be a process with zero expectation and chaos expansion representation of the form $f=\sum_{\alpha \in I,|\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}, f_{\alpha} \in X$. Then the integral equation

$$
\begin{equation*}
\delta(u)=f, \tag{18}
\end{equation*}
$$

has a unique solution $u$ in the class of processes satisfying condition (13) given by

$$
\begin{equation*}
u=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} . \tag{19}
\end{equation*}
$$

Moreover, the following holds:
$1^{\circ}$ If $f \in X \otimes(S)_{-\rho,-p}$, then $u \in \operatorname{Dom}_{(-l,-p)}^{\rho}(\delta)$, for $l>p+1$.
$2^{\circ}$ If $f \in X \otimes(S)_{\rho, p}, p \in \mathbb{N}$, then $u \in \operatorname{Dom}_{(l, p)}^{\rho}(\delta)$, for $l<p-1$.
$3^{\circ}$ If $f \in X \otimes(L)^{2}$, then $u \in \operatorname{Dom}_{0}(\delta)$.
Proof. $1^{\circ}$ Since the proof of $1^{\circ}$ and $2^{\circ}$ are analogous, we will conduct only the proof of one of them.
$2^{\circ}$ We seek for the solution in Range ${ }_{+}^{\rho}(\mathbb{D})$. It is clear that $u \in \operatorname{Range}_{+}^{\rho}(\mathbb{D})$ is equivalent to $u=\mathbb{D}(\widetilde{u})$, for some $\widetilde{u}$. Thus, equation (18) is equivalent to the system of equations

$$
u=\mathbb{D}(\widetilde{u}), \quad \mathcal{R}(\widetilde{u})=f .
$$

The solution to $\mathcal{R}(\widetilde{u})=f$ is given by

$$
\widetilde{u}=\widetilde{u}_{0}+\sum_{\alpha \in I,|\alpha| \geq 1} \frac{f_{\alpha}}{|\alpha|} \otimes H_{\alpha}
$$

where $\tilde{u}_{(0,0,0, \ldots)}=\widetilde{u}_{0}$ can be chosen arbitrarily. Now, the solution of the initial equation (18) is obtained after applying the operator $\mathbb{D}$, i.e.

$$
u=\mathbb{D}(\widetilde{u})=\sum_{\alpha \in T,|\alpha| \geq 1} \sum_{k \in \mathbb{N}} \alpha_{k} \frac{f_{\alpha}}{|\alpha|} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} .
$$

One can directly check that this $u$ satisfies (13): Indeed with $u_{\alpha, k}=\left(\alpha_{k}+1\right) \frac{f_{\alpha+t^{(k)}}^{\left|\alpha+\varepsilon^{(k) \mid}\right|}}{}$ we have $\frac{1}{\alpha_{k}} u_{\alpha-\varepsilon^{(k)}, k}=\frac{f_{\alpha}}{|\alpha|}$ for all $k \in \mathbb{N}$.

It remains to prove the convergence of the solution (19) in in the space $\operatorname{Dom}_{(l, p)}^{\rho}(\delta)$. First we prove that $\tilde{u} \in \operatorname{Dom}_{p}^{\rho}(\mathbb{D})$ and then $u \in \operatorname{Dom}_{(l, p)}^{\rho}(\delta)$ for appropriate $l \in \mathbb{N}$. We obtain

$$
\begin{aligned}
\|\widetilde{u}\|_{D o m_{p}^{\rho}(\mathbb{D})}^{2} & =\sum_{\alpha \in I}|\alpha|^{1-\rho}(\alpha!)^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\left\|\tilde{u}_{0}\right\|_{X}^{2}+\sum_{\alpha \in T,|\alpha|>0}|\alpha|^{1-\rho}(\alpha!)^{1+\rho} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 \mathbb{N})^{p \alpha} \\
& \leq\left\|\tilde{u}_{0}\right\|_{X}^{2}+\sum_{\alpha \in T,|\alpha|>0}(\alpha!)^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\left\|\tilde{u}_{0}\right\|_{X}^{2}+\|f\|_{X \otimes(S)_{\rho, p}}^{2}<\infty
\end{aligned}
$$

and thus $\tilde{u} \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$. Now,

$$
\begin{aligned}
\|u\|_{D o m_{(, p)\rangle}^{p}(\delta)}^{2} & =\sum_{\alpha \in I} \sum_{k \in \mathbb{N}}(\alpha!)^{1+\rho}\left(\alpha_{k}+1\right)^{3+\rho} \frac{\| f_{\alpha+\varepsilon^{(k)} \|_{X}^{2}}^{\left|\alpha+\varepsilon^{(k)}\right|^{2}}(2 k)^{l}(2 \mathbb{N})^{p \alpha}=\sum_{\alpha \in I,|\alpha|>0} \sum_{k \in \mathbb{N}}(\alpha!)^{1+\rho} \alpha_{k}^{2} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 k)^{l}(2 \mathbb{N})^{p\left(\alpha-\varepsilon^{(k)}\right)}}{} \\
& \leq \sum_{\alpha \in I,|\alpha|>0}(\alpha!)^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}\left(\sum_{k \in \mathbb{N}} \frac{\alpha_{k}^{2}}{|\alpha|^{2}}(2 k)^{l}(2 k)^{-p}\right) \leq c\|f\|_{X \otimes(S)_{\rho, p}}^{2}<\infty,
\end{aligned}
$$

since $c=\sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$ for $p>l+1$. In the second step we used that $\left(\alpha-\varepsilon^{(k)}\right)!\alpha_{k}=\alpha!$, and in the fourth step we used $\alpha_{k} \leq|\alpha|$.
$3^{\circ}$ In this case we have
$\|\widetilde{u}\|_{D o m_{0}(\mathbb{D})}^{2}=\sum_{\alpha \in I}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\left\|\tilde{u}_{0}\right\|_{X}^{2}+\sum_{\alpha \in I,|\alpha|>0}|\alpha| \alpha!\frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}} \leq\left\|\tilde{u}_{0}\right\|_{X}^{2}+\sum_{\alpha \in I,|\alpha|>0} \alpha!\left\|f_{\alpha}\right\|_{X}^{2}=\left\|\tilde{u}_{0}\right\|_{X}^{2}+\|f\|_{X \otimes(L)^{2}}^{2}<\infty$ and thus $\tilde{u} \in \operatorname{Dom}_{0}(\mathbb{D})$. Also,

$$
\begin{aligned}
\|u\|_{D o m_{0}(\delta)}^{2} & =\sum_{\alpha \in I} \alpha!\| \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{\frac{1}{2}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}^{\left|\alpha+\varepsilon^{(k)}\right|}\left\|_{X}^{2}=\sum_{|\beta| \geq 1}\right\| \sum_{k \in \mathbb{N}} \beta_{k}^{\frac{3}{2}}\left(\beta-\varepsilon^{(k)}\right)!^{\frac{1}{2}} \frac{f_{\beta}}{|\beta|}\left\|_{X}^{2}=\sum_{|\beta| \geq 1}\right\| \sum_{k \in \mathbb{N}} \beta_{k} \beta!^{\frac{1}{2}} \frac{f_{\beta}}{|\beta|} \|_{X}^{2}}{} \\
& =\sum_{\mid \beta \beta \geq 1} \frac{\beta!}{|\beta|^{2}}\left\|f_{\beta}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}} \beta_{k}\right)^{2}=\sum_{|\beta| \geq 1} \beta!\left\|f_{\beta}\right\|_{X}^{2}=\|f\|_{X \otimes(L)^{2}}^{2}<\infty .
\end{aligned}
$$

Corollary 4.8. Each process $f \in X \otimes(S)_{ \pm \rho}$, resp. $f \in X \otimes(L)^{2}$ can be represented as $f=E f+\delta(u)$ for some $u \in X \otimes S(\mathbb{R}) \otimes(S)_{ \pm \rho}$, resp. $u \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$.

The latter result reduces to the celebrated Itô representation theorem (see e.g. [4]) in case when $f$ is a square integrable adapted process.

## 5. Properties of the Malliavin Operators

In the classical $(L)^{2}$ setting it is known that the Skorokhod integral is the adjoint of the Malliavin derivative. We extend this result in the next theorem and prove their duality by pairing a generalized process with a test process. The classical result is revisited in part $3^{\circ}$ of the theorem.

Theorem 5.1. (Duality) Assume that either of the following holds:

```
\(1^{\circ} F \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})\) and \(u \in \operatorname{Dom}_{+}^{\rho}(\delta)\)
\(2^{\circ} F \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})\) and \(u \in \operatorname{Dom}_{-}^{\rho}(\delta)\)
\(3^{\circ} F \in \operatorname{Dom}_{0}(\mathbb{D})\) and \(u \in \operatorname{Dom}_{0}(\delta)\)
```

Then the following duality relationship between the operators $\mathbb{D}$ and $\delta$ holds:

$$
\begin{equation*}
E(F \cdot \delta(u))=E(\langle\mathbb{D} F, u\rangle) \tag{20}
\end{equation*}
$$

where (20) denotes the equality of the generalized expectations of two objects in $X \otimes(S)_{-\rho}$ and $\langle\cdot, \cdot\rangle$ denotes the dual pairing of $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.

Proof. First we show that the duality relationship (20) between $\mathbb{D}$ and $\delta$ holds formally. Let $u \in \operatorname{Dom}(\delta)$ be given in its chaos expansion form $u=\sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} u_{\beta, j} \otimes \xi_{j} \otimes H_{\beta}$. Then $\delta(u)=\sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} u_{\beta, j} \otimes H_{\beta+\varepsilon(j)}$. Let $F \in \operatorname{Dom}(\mathbb{D})$ be given as $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}$. Then $\mathbb{D}(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\left.\alpha+\varepsilon^{k}\right)} \otimes \xi_{k} \otimes H_{\alpha}$. Therefore we obtain

$$
\begin{aligned}
F \cdot \delta(u) & =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\beta, j} \otimes H_{\alpha} \cdot H_{\beta+\varepsilon(\eta)} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\beta, j} \otimes \sum_{\gamma \leq \min \left\{\alpha, \beta+\varepsilon^{(j)}\right\}} \gamma!\cdot\binom{\alpha}{\gamma}\binom{\beta+\varepsilon^{(j)}}{\gamma} H_{\alpha+\beta+\varepsilon(\eta)-2 \gamma} .
\end{aligned}
$$

The generalized expectation of $F \cdot \delta(u)$ is the zeroth coefficient in the previous sum, which is obtained when $\alpha+\beta+\varepsilon^{(j)}=2 \gamma$ and $\gamma \leq \min \left\{\alpha, \beta+\varepsilon^{(j)}\right\}$, i.e. only for the choice $\beta=\alpha-\varepsilon^{(j)}$ and $\gamma=\alpha, j \in \mathbb{N}$. Thus,

$$
E(F \cdot \delta(u))=\sum_{\alpha \in T,|\alpha|>0} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\alpha-\varepsilon^{(j)}, j} \cdot \alpha!=\sum_{\alpha \in I} \sum_{j \in \mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot\left(\alpha+\varepsilon^{(j)}\right)!.
$$

On the other hand,

$$
\begin{aligned}
\langle\mathbb{D}(F), u\rangle & =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} u_{\beta, j}\left\langle\xi_{k}, \xi_{j}\right\rangle H_{\alpha} \cdot H_{\beta} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{j \in \mathbb{N}}\left(\alpha_{j}+1\right) f_{\alpha+\varepsilon^{(j)}} u_{\beta, j} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\cdot\binom{\alpha}{\gamma}\binom{\beta}{\gamma} \cdot H_{\alpha+\beta-2 \gamma}
\end{aligned}
$$

and its generalized expectation is obtained for $\alpha=\beta=\gamma$. Thus

$$
E(\langle\mathbb{D}(F), u\rangle)=\sum_{\alpha \in I} \sum_{j \in \mathbb{N}}\left(\alpha_{j}+1\right) f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot \alpha!=\sum_{\alpha \in I} \sum_{j \in \mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot\left(\alpha+\varepsilon^{(j)}\right)!=E(F \cdot \delta(u)) .
$$

$1^{\circ}$ Let $\rho \in[0,1]$ be fixed. Let $F \in \operatorname{Dom}_{-p}^{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{(r, s)}^{\rho}(\delta)$, for some $p \in \mathbb{N}$ and all $r, s \in \mathbb{N}, r>s+1$. Then $\mathbb{D} F \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$ for $l>p+1$. Since $r$ is arbitrary, we may assume that $r=l$ and denote by $\langle\cdot, \cdot\rangle$ the dual pairing between $S_{-l}(\mathbb{R})$ and $S_{l}(\mathbb{R})$. Moreover, $\langle\mathbb{D} F, u\rangle$ is well defined in $X \otimes(S)_{-p,-p}$. On the other hand, $\delta(u) \in X \otimes(S)_{\rho, s}$ and thus by Theorem 2.12, $F \cdot \delta(u)$ is also defined as an element in $X \otimes(S)_{-\rho,-k}$,
for $k \in[p, s-8], s>p+8$. Since $s$ was arbitrary, one can take any $k \geq p$. This means that both objects, $F \cdot \delta(u)$ and $\langle\mathbb{D} F, u\rangle$ exist in $X \otimes(S)_{-\rho,-k}$, for $k \geq p$. Taking generalized expectations of $\langle\mathbb{D} F, u\rangle$ and $F \cdot \delta(u)$ we showed that the zeroth coefficients of the formal expansions are equal. Therefore the duality formula is valid for this case.
$2^{\circ}$ Let $F \in \operatorname{Dom}_{p}^{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{(-r,-s)}^{\rho}(\delta)$, for some $r, s \in \mathbb{N}, s>r+1$, and all $p \in \mathbb{N}$. Then $\mathbb{D} F \in$ $X \otimes S_{l}(\mathbb{R}) \otimes(S)_{p, p}, l<p-1$, but since $p$ is arbitrary, so is $l$. Now, $\langle\mathbb{D} F, u\rangle$ is a well defined object in $X \otimes(S)_{-p,-s}$. On the other hand, $\delta(u) \in X \otimes(S)_{-\rho,-s}$ and thus by Theorem 2.12,F• $\delta(u)$ is also well defined and belongs to $X \otimes(S)_{-\rho,-k}$, for $k \in[s, p-8], p>s+8$. Thus, both processes $F \cdot \delta(u)$ and $\langle\mathbb{D} F, u\rangle$ belong to $X \otimes(S)_{-\rho,-k}$ for $k \geq s$.
$3^{\circ}$ For $F \in \operatorname{Dom}_{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\delta)$ the dual pairing $\langle\mathbb{D} F, u\rangle$ represents the inner product in $L^{2}(\mathbb{R})$ and the product $F \delta(u)$ is an element in $X \otimes(L)^{1}$ (see Remark 2.13). The classical duality formula is clearly valid for this case.

The next theorem states a higher order duality formula, which connects the $k$ th order iterated Skorokhod integral and the Malliavin derivative operator of $k$ th order, $k \in \mathbb{N}$. For the definition of higher order iterated operators we refer to [8].
Theorem 5.2. Let $f \in \operatorname{Dom}_{+}^{\rho}\left(\mathbb{D}^{(k)}\right)$ and $u \in \operatorname{Dom}_{-}^{\rho}\left(\delta^{(k)}\right)$, or let $f \in \operatorname{Dom}_{-}^{\rho}\left(\mathbb{D}^{(k)}\right)$ and $u \in \operatorname{Dom}_{+}^{\rho}\left(\delta^{(k)}\right), k \in \mathbb{N}$. Then the duality formula

$$
E\left(f \cdot \delta^{(k)}(u)\right)=E\left(\left\langle\mathbb{D}^{(k)}(f), u\right\rangle\right)
$$

holds, where $\langle\cdot, \cdot\rangle$ denotes the duality pairing of $S^{\prime}(\mathbb{R})^{\otimes k}$ and $S(\mathbb{R})^{\otimes k}$.
Proof. The assertion follows by induction and applying Theorem 5.1 successively $k$ times.
Remark 5.3. The previous theorems are special cases of a more general identity. It can be proven, under suitable assumptions that make all the products well defined, that the following formulae hold:

$$
\begin{align*}
& F \delta(u)=\delta(F u)+\langle\mathbb{D}(F), u\rangle,  \tag{21}\\
& F \delta^{(k)}(u)=\sum_{i=0}^{k}\binom{k}{i} \delta^{(k-i)}\left(\left\langle\mathbb{D}^{(i)} F, u\right\rangle\right), \quad k \in \mathbb{N} .
\end{align*}
$$

Taking the expectation in (21) and using the fact that $\delta(F u)=0$, the duality formula (20) follows.
Example 5.4. Let $\psi \in L^{2}(\mathbb{R})$. In [6] we have shown that the stochastic exponentials $\exp ^{\diamond}\{\delta(\psi)\}$ are eigenvalues of the Malliavin derivative, i.e. $\mathbb{D}\left(\exp ^{\diamond}\{\delta(\psi)\}\right)=\psi \cdot \exp ^{\diamond}\{\delta(\psi)\}$. We will prove that they are also eigenvalues of the Ornstein-Uhlenbeck operator. Indeed, using (21) we obtain

$$
\begin{aligned}
\mathcal{R}\left(\exp ^{\diamond}\{\delta(\psi)\}\right) & =\delta\left(\psi \cdot \exp ^{\diamond}\{\delta(\psi)\}\right)=\delta(\psi) \exp ^{\diamond}\{\delta(\psi)\}-\left\langle\mathbb{D}\left(\exp ^{\diamond}\{\delta(\psi)\}\right), \psi\right\rangle \\
& =\delta(\psi) \exp ^{\diamond}\{\delta(\psi)\}-\left\langle\psi \cdot \exp ^{\diamond}\{\delta(\psi)\}, \psi\right\rangle \\
& =\left(\delta(\psi)-\|\psi\|_{L^{2}(\mathbb{R})}^{2}\right) \exp ^{\diamond}\{\delta(\psi)\} .
\end{aligned}
$$

In the next theorem we prove a weaker type of duality instead of (20) which holds if $F \in \operatorname{Dom}_{-}^{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-}^{0}(\delta)$ are both generalized processes. Recall that $\ll, \cdot, \cdot>_{r}$ denotes the scalar product in $(S)_{0, r}$.
Lemma 5.5. Let $u \in \operatorname{Dom}_{-q}^{0}(\mathbb{D})$ and $\varphi \in S_{-n}(\mathbb{R}), n<q-1$. Then $u \cdot \varphi \in \operatorname{Dom}_{(-n,-q)}^{0}(\delta)$.
Proof. Let $u=\sum_{\alpha \in I} u_{\alpha} H_{\alpha}$ and $\varphi=\sum_{k \in \mathbb{N}} \varphi_{k} \xi_{k}$. Then, $u \cdot \varphi=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} u_{\alpha} \varphi_{k} \xi_{k} H_{\alpha}$ and

$$
\begin{aligned}
\|u \cdot \varphi\|_{\operatorname{Dom}_{(-n,-\eta)}^{0}(\delta)}^{2} & =\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha!\left(\alpha_{k}+1\right)\left\|u_{\alpha}\right\|_{X}^{2} \varphi_{k}^{2}(2 k)^{-n}(2 \mathbb{N})^{-q \alpha}=\sum_{\alpha \in I} \alpha!\left\|u_{\alpha}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)(2 k)^{-n} \varphi_{k}^{2}\right)(2 \mathbb{N})^{-q \alpha} \\
& \leq\left(\left\|u_{0}\right\|_{X}^{2}+2 \sum_{|\alpha|>0} \alpha!\mid \alpha\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}\right) \cdot \sum_{k \in \mathbb{N}} \varphi_{k}^{2}(2 k)^{-n}=\left(\left\|u_{0}\right\|_{X}^{2}+2\|u\|_{D o m_{-q}^{0}(\mathbb{D})}^{2}\right) \cdot\|\varphi\|_{-n}^{2}<\infty .
\end{aligned}
$$

We used the estimate $\alpha_{k}+1 \leq 2|\alpha|$, for $|\alpha|>0, k \in \mathbb{N}$.

Theorem 5.6. (Weak duality) Let $\rho=0$ and consider the Hida spaces. Let $F \in \operatorname{Dom}_{-p}^{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-q}^{0}(\mathbb{D})$ for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R}), n<q-1$, it holds that

$$
\ll\langle\mathbb{D} F, \varphi\rangle_{-r}, u>_{0,-r}=\ll F, \delta(\varphi u)>_{0,-r},
$$

for $r>\max \{q, p+1\}$.
Proof. Let $F=\sum_{\alpha \in I} f_{\alpha} H_{\alpha} \in \operatorname{Dom}_{-p}^{0}(\mathbb{D}), u=\sum_{\alpha \in I} u_{\alpha} H_{\alpha} \in \operatorname{Dom}_{-q}^{0}(\mathbb{D})$ and $\varphi=\sum_{k \in \mathbb{N}} \varphi_{k} \xi_{k} \in S_{-n}(\mathbb{R})$. Then, for $k>p+1, \mathbb{D} F \in X \otimes S_{-k}(\mathbb{R}) \otimes(S)_{0,-p} \subseteq X \otimes S_{-r}(\mathbb{R}) \otimes(S)_{0,-r}$ if $r>p+1$. Also, by Lemma 5.5 it follows that $\varphi u \in \operatorname{Dom}_{(-n,-q)}^{0}(\delta)$ and since $q>n+1$, this implies that $\delta(\varphi u) \in X \otimes(S)_{0,-q} \subseteq X \otimes(S)_{0,-r}$, for $r \geq q$. Therefore we let $r>\max \{p+1, q\}$. Thus,

$$
\begin{aligned}
\langle\mathbb{D} F, \varphi\rangle_{-r} & =\left\langle\sum_{k \in \mathbb{N}} \sum_{\alpha \in I}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} H_{\alpha} \otimes \xi_{k}, \sum_{k \in \mathbb{N}} \varphi_{k} \xi_{k}\right\rangle_{-r} \\
& =\sum_{k \in \mathbb{N}} \varphi_{k} \sum_{\alpha \in I}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} H_{\alpha}(2 k)^{-r},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\ll\langle\mathbb{D} F, \varphi\rangle_{-r}, u \gg 0,-r & =\ll \sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \varphi_{k}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}}(2 k)^{-r} H_{\alpha}, \sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \gg 0,-r \\
& =\sum_{\alpha \in I} \alpha!u_{\alpha} \sum_{k \in \mathbb{N}} \varphi_{k}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}}(2 k)^{-r}(2 \mathbb{N})^{-r \alpha} .
\end{aligned}
$$

On the other hand, $\varphi u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha} \varphi_{k} \xi_{k} \otimes H_{\alpha}$ and $\delta(\varphi u)=\sum_{\alpha>0} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_{k} H_{\alpha}$. Thus,

$$
\begin{aligned}
\ll F, \delta(\varphi u) \ggg_{0,-r} & =\ll \sum_{\alpha \in I} f_{\alpha} H_{\alpha} \sum_{\alpha>0} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_{k} H_{\alpha} \gg 0,-r \\
& =\sum_{\alpha>0} \alpha!f_{\alpha} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_{k}(2 \mathbb{N})^{-r \alpha} \\
& =\sum_{\beta \in I} \sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!f_{\beta+\varepsilon^{(k)}} u_{\beta} \varphi_{k}(2 \mathbb{N})^{-r\left(\beta+\varepsilon^{(k)}\right)} \\
& =\sum_{\beta \in I} \sum_{k \in \mathbb{N}} \beta!\left(\beta_{k}+1\right) f_{\beta+\varepsilon^{(k)}} u_{\beta} \varphi_{k}(2 k)^{-r}(2 \mathbb{N})^{-r \beta},
\end{aligned}
$$

which completes the proof.
The following theorem states the product rule for the Ornstein-Uhlenbeck operator. Its special case for $F, G \in \operatorname{Dom}_{0}(\mathcal{R})$ and $F \cdot G \in \operatorname{Dom}_{0}(\mathcal{R})$ states that (22) holds (see e.g. [2]). We extend the classical $(L)^{2}$ case to multiplying a generalized process with a test process. The product rule also holds if we multiply two generalized processes, but in this case the ordinary product has to be replaced by the Wick product.

Theorem 5.7. (Product rule for $\mathcal{R}$ )
$1^{\circ}$ Let $F \in \operatorname{Dom}_{+}^{\rho}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R})$, or vice versa. Then $F \cdot G \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R})$ and

$$
\begin{equation*}
\mathcal{R}(F \cdot G)=F \cdot \mathcal{R}(G)+G \cdot \mathcal{R}(F)-2 \cdot\langle\mathbb{D} F, \mathbb{D} G\rangle, \tag{22}
\end{equation*}
$$

holds, where $\langle\cdot, \cdot\rangle$ is the dual pairing between $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.
$2^{\circ}$ Let $F, G \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R})$. Then $F \cdot G \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R})$ and

$$
\begin{equation*}
\mathcal{R}(F \diamond G)=F \diamond \mathcal{R}(G)+\mathcal{R}(F) \diamond G . \tag{23}
\end{equation*}
$$

Proof. $1^{\circ}$ Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{+}^{\rho}(\mathcal{R})$ and $G=\sum_{\beta \in \mathcal{I}} g_{\beta} \otimes H_{\beta} \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R})$. Then, $\mathcal{R}(F)=\sum_{\alpha \in \mathcal{I}}|\alpha| f_{\alpha} \otimes H_{\alpha}$ and $\mathcal{R}(G)=\sum_{\beta \in I}|\beta| g_{\beta} \otimes H_{\beta}$.

The left hand side of (22) can be written in the form

$$
\begin{aligned}
\mathcal{R}(F \cdot G) & =\mathcal{R}\left(\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}\right) \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha+\beta-2 \gamma| H_{\alpha+\beta-2 \gamma} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(|\alpha|+|\beta|-2|\gamma|) H_{\alpha+\beta-2 \gamma} .
\end{aligned}
$$

On the other hand, the first two terms on the right hand side of (22) are

$$
\begin{equation*}
\mathcal{R}(F) \cdot G=\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha| H_{\alpha+\beta-2 \gamma} \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
F \cdot \mathcal{R}(G)=\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\beta| H_{\alpha+\beta-2 \gamma} . \tag{25}
\end{equation*}
$$

Since $F \in \operatorname{Dom}_{+}^{\rho}(\mathcal{R}) \subseteq \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$ and $G \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R}) \subseteq \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ we have $\mathbb{D}(F)=\sum_{\alpha \in I} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes$ $H_{\alpha-\varepsilon^{(k)}}$ and $\mathbb{D}(G)=\sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} \beta_{j} g_{\beta} \otimes \xi_{j} \otimes H_{\beta-\varepsilon^{(k)} \text {. Thus, the third term on the right hand side of (22) is }}$

$$
\begin{aligned}
\langle\mathbb{D}(F), \mathbb{D}(G)\rangle & =\left\langle\sum_{\alpha \in T,|\alpha| \gg 0} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)},} \sum_{\beta \in T,|\beta|>0} \sum_{j \in \mathbb{N}} \beta_{j} g_{\beta} \otimes \xi_{j} \otimes H_{\beta-\varepsilon}(\eta)\right\rangle \\
& =\sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} \alpha_{k} \beta_{j} f_{\alpha} g_{\beta}\left\langle\xi_{k}, \xi_{j}\right\rangle \otimes H_{\alpha-\varepsilon^{(k)}} \cdot H_{\beta-\varepsilon^{(j)}} \\
& =\sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \alpha_{k} \beta_{k} f_{\alpha} g_{\beta} \otimes \sum_{\left.\gamma \leq \min \mid \alpha-\varepsilon^{(k)}, \beta-\varepsilon^{(k)}\right\}} \gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} H_{\alpha+\beta-2 \varepsilon^{(k)}-2 \gamma,}
\end{aligned}
$$

where we used the fact that $\left\langle\xi_{k}, \xi_{j}\right\rangle=0$ for $k \neq j$ and $\left\langle\xi_{k}, \xi_{j}\right\rangle=1$ for $k=j$. Now we put $\theta=\gamma+\varepsilon^{(k)}$ and use the identities

$$
\alpha_{k} \cdot\binom{\alpha-\varepsilon^{(k)}}{\gamma}=\alpha_{k} \cdot\binom{\alpha-\varepsilon^{(k)}}{\theta-\varepsilon^{(k)}}=\theta_{k} \cdot\binom{\alpha}{\theta}, \quad k \in \mathbb{N},
$$

and $\theta_{k} \cdot\left(\theta-\varepsilon^{(k)}\right)!=\theta!$. Thus we obtain

$$
\begin{aligned}
\langle\mathbb{D}(F), \mathbb{D}(G)\rangle & =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}} \theta_{k}^{2}\left(\theta-\varepsilon^{(k)}\right)!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}} \theta_{k} \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}}\left(\sum_{k \in \mathbb{N}} \theta_{k}\right) \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}}|\theta| \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta .}
\end{aligned}
$$

Combining all previously obtained results we now have

$$
\begin{aligned}
\mathcal{R}(F \cdot G)= & \sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(|\alpha|+|\beta|-2|\gamma|) H_{\alpha+\beta-2 \gamma} \\
= & \sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha| H_{\alpha+\beta-2 \gamma}+\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\beta| H_{\alpha+\beta-2 \gamma} \\
& -2 \sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}}|\gamma| \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}
\end{aligned}
$$

$$
=\mathcal{R}(F) \cdot G+F \cdot \mathcal{R}(G)-2 \cdot\langle\mathbb{D}(F), \mathbb{D}(G)\rangle
$$

and thus (22) holds.
Assume that $F \in \operatorname{Dom}_{-p}^{\rho}(\mathcal{R})$ and $G \in \operatorname{Dom}_{q}^{\rho}(\mathcal{R})$. Then $\mathcal{R}(F) \in X \otimes(S)_{-\rho,-p}$ and $\mathcal{R}(G) \in X \otimes(S)_{\rho, q}$. From Theorem 2.12 it follows that $F \cdot \mathcal{R}(G)$ and $G \cdot \mathcal{R}(F)$ are both well defined and belong to $X \otimes(S)_{-p,-s}$, for $s \in[p, q-8], q-p>8$. Similarly, $\langle\mathbb{D}(F), \mathbb{D}(G)\rangle$ belongs to $X \otimes(S)_{-\rho_{,}-p}$, since $\mathbb{D}(F) \in X \otimes S_{-l_{1}}(\mathbb{R}) \otimes(S)_{-\rho,-p}$, where $l_{1}>p+1$ and $\mathbb{D}(G) \in X \otimes S_{l_{2}}(\mathbb{R}) \otimes(S)_{\rho, q}$, where $l_{2}<q-1$ and the dual pairing is obtained for any $l \in\left[l_{1}, l_{2}\right]$. Thus, the right hand side of (22) is in $X \otimes(S)_{-\rho,-s} s \geq p$. Hence, $F \cdot G \in \operatorname{Dom}_{-s}^{\rho}(\mathcal{R})$.

## $2^{\circ}$ From

$$
G \diamond \mathcal{R}(F)=\sum_{\gamma \in I} \sum_{\alpha+\beta=\gamma}|\alpha| f_{\alpha} g_{\beta} H_{\gamma} \quad \text { and } \quad F \diamond \mathcal{R}(G)=\sum_{\gamma \in I} \sum_{\alpha+\beta=\gamma} f_{\alpha}|\beta| g_{\beta} H_{\gamma}
$$

it follows that

$$
G \diamond \mathcal{R}(F)+F \diamond \mathcal{R}(G)=\sum_{\gamma \in I}|\gamma| \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} H_{\gamma}=\mathcal{R}(F \diamond G) .
$$

If $F \in \operatorname{Dom}_{-p}^{\rho}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-q}^{\rho}(\mathcal{R})$, then $\mathcal{R}(F) \in X \otimes(S)_{-\rho,-p}$ and $\mathcal{R}(G) \in X \otimes(S)_{-\rho,-q}$. From Theorem 2.9 it follows that $\mathcal{R}(F) \diamond G \in X \otimes(S)_{-p,-(p+q+4)}$ and $\mathcal{R}(G) \diamond F \in X \otimes(S)_{-\rho,-(p+q+4)}$. Thus, the right hand side of (23) is in $X \otimes(S)_{-\rho,-(p+q+4)}$, i.e. $F \diamond G \in \operatorname{Dom}_{-r}^{\rho}(\mathcal{R})$ for $r=p+q+4$.

Corollary 5.8. Let $F \in \operatorname{Dom}_{+}^{\rho}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-}^{\rho}(\mathcal{R})$, or vice versa (including also the possibility $F, G \in \operatorname{Dom}_{0}(\mathcal{R})$ ). Then the following property holds:

$$
E(F \cdot \mathcal{R}(G))=E(\langle\mathbb{D} F, \mathbb{D} G\rangle)
$$

Proof. From the chaos expansion form of $\mathcal{R}(F \cdot G)$ it follows that $E \mathcal{R}(F \cdot G)=0$. Moreover, taking the expectations on both sides of (24) and (25) we obtain

$$
E(\mathcal{R}(F) \cdot G)=E(F \cdot \mathcal{R}(G)) .
$$

Now, from Theorem 5.7 it follows that

$$
0=2 E(F \cdot \mathcal{R}(G))-2 E(\langle\mathbb{D} F, \mathbb{D} G\rangle),
$$

and the assertion follows.
In the classical literature $([2,15])$ it is proven that the Malliavin derivative satisfies the product rule with respect to ordinary multiplication, i.e. if $F, G \in \operatorname{Dom}_{0}(\mathbb{D})$ such that $F \cdot G \in \operatorname{Dom}_{0}(\mathbb{D})$ then (26) holds. The following theorem recapitulates this result and extends it for multiplication of a generalized process with a test processes, and extends it also for Wick multiplication.

Theorem 5.9. (Product rule for $\mathbb{D}$ )
$1^{\circ}$ Let $F \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ and $G \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$ or vice versa. Then $F \cdot G \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ and the product rule

$$
\begin{equation*}
\mathbb{D}(F \cdot G)=F \cdot \mathbb{D} G+\mathbb{D} F \cdot G \tag{26}
\end{equation*}
$$

holds.
$2^{\circ}$ Let $F, G \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$. Then $F \diamond G \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ and

$$
\mathbb{D}(F \diamond G)=F \diamond \mathbb{D} G+\mathbb{D} F \diamond G
$$

Proof. $1^{\circ}$

$$
\begin{aligned}
\mathbb{D}(F \cdot G) & =\mathbb{D}\left(\sum_{\alpha \in I} f_{\alpha} H_{\alpha} \cdot \sum_{\beta \in I} g_{\beta} H_{\beta}\right) \\
& =\mathbb{D}\left(\sum_{\alpha \in I} \sum_{\beta \in I} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}\right) \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\left(\alpha_{k}+\beta_{k}-2 \gamma_{k}\right) \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
\end{aligned}
$$

On the other side we have

$$
\begin{aligned}
F \cdot \mathbb{D}(G) & =\sum_{\alpha \in I} f_{\alpha} H_{\alpha} \cdot \sum_{\beta \in I} \sum_{k \in \mathbb{N}} \beta_{k} g_{\beta} \xi_{k} H_{\beta-\varepsilon^{(k)}} \\
& =\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \left\{\alpha, \beta-\varepsilon^{(k)}\right\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} \beta_{k} \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
\end{aligned}
$$

and

$$
G \cdot \mathbb{D}(F)=\sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \left\{\alpha-\varepsilon^{(k)}, \beta\right\}} \gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta}{\gamma} \alpha_{k} \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}} .
$$

Summing up the chaos expansions for $F \cdot \mathbb{D}(G)$ and $G \cdot \mathbb{D}(F)$ and applying the identities

$$
\alpha_{k}\binom{\alpha-\varepsilon^{(k)}}{\gamma}=\alpha_{k} \cdot \frac{\left(\alpha-\varepsilon^{(k)}\right)!}{\gamma!\left(\alpha-\varepsilon^{(k)}-\gamma\right)!}=\frac{\alpha!}{\gamma!(\alpha-\gamma)!} \cdot\left(\alpha_{k}-\gamma_{k}\right)=\binom{\alpha}{\gamma}\left(\alpha_{k}-\gamma_{k}\right)
$$

and

$$
\beta_{k}\binom{\beta-\varepsilon^{(k)}}{\gamma}=\binom{\beta}{\gamma}\left(\beta_{k}-\gamma_{k}\right)
$$

for all $\alpha, \beta \in I, k \in \mathbb{N}$ and $\gamma \in \mathcal{I}$ such that $\gamma \leq \min \{\alpha, \beta\}$ and the expression $\left(\alpha_{k}-\gamma_{k}\right)+\left(\beta_{k}-\gamma_{k}\right)=\alpha_{k}+\beta_{k}-2 \gamma_{k}$ we obtain (26).

Assume that $F \in \operatorname{Dom}_{-p}^{\rho}(\mathbb{D}), G \in \operatorname{Dom}_{q}^{\rho}(\mathbb{D})$. Then $\mathbb{D}(F) \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}, l>p+1$, and $\mathbb{D}(G) \in$ $X \otimes S_{k}(\mathbb{R}) \otimes(S)_{\rho, q}, k<q-1$. From Theorem 2.12 it follows that all products on the right hand side of (26) are well defined, moreover $F \cdot \mathbb{D}(G) \in X \otimes S_{k}(\mathbb{R}) \otimes(S)_{-\rho,-r}, \mathbb{D}(F) \cdot G \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-r}$, for $r \in[p, q-8], q>p+8$. Thus the right hand sifde of (26) can be embedded into $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-r}, r \geq p$. Thus, $F \cdot G \in \operatorname{Dom}_{-r}^{\rho}(\mathbb{D})$.
$2^{\circ}$ By definition of the Malliavin derivative and the Wick product it can be easily verified that

$$
\begin{aligned}
\mathbb{D}(F) \diamond G+F \diamond \mathbb{D}(G) & =\sum_{\gamma \in I} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \alpha_{k} f_{\alpha} g_{\beta} H_{\gamma}+\sum_{\gamma \in I} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \beta_{k} f_{\alpha} g_{\beta} H_{\gamma} \\
& =\sum_{\gamma \in I} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=\gamma} \gamma_{k} f_{\alpha} g_{\beta} H_{\gamma-\varepsilon^{(k)}}=\mathbb{D}(F \diamond G) .
\end{aligned}
$$

If $F \in \operatorname{Dom}_{-p}^{\rho}(\mathbb{D})$ and $G \in \operatorname{Dom}_{-q}^{\rho}(\mathbb{D})$, then $\mathbb{D}(F) \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}, l>p+1$, and $\mathbb{D}(G) \in X \otimes S_{-k}(\mathbb{R}) \otimes$ $(S)_{-\rho,-q}, k>q+1$. From Theorem 2.9 it follows that $\mathbb{D}(F) \diamond G$ and $F \diamond \mathbb{D}(G)$ both belong to $X \otimes S_{-m}(\mathbb{R}) \otimes$ $(S)_{-\rho,-(p+q+4)}, m=\max \{l, k\}$. Thus, $F \diamond G \in \operatorname{Dom}_{-r}^{\rho}(\mathbb{D})$ for $r=p+q+4$.

A generalization of Theorem 5.9 for higher order derivatives, i.e. the Leibnitz formula is given in the next theorem.

Theorem 5.10. Let $F, G \in \operatorname{Dom}_{-}^{\rho}\left(\mathbb{D}^{(k)}\right), k \in \mathbb{N}$, then $F \diamond G \in \operatorname{Dom}_{-}^{\rho}\left(\mathbb{D}^{(k)}\right)$ and the Leibnitz rule holds:

$$
\mathbb{D}^{(k)}(F \diamond G)=\sum_{i=0}^{k}\binom{k}{i} \mathbb{D}^{(i)}(F) \diamond \mathbb{D}^{(k-i)}(G)
$$

where $\mathbb{D}^{(0)}(F)=F$ and $\mathbb{D}^{(0)}(G)=G$.
Moreover, if $G \in \operatorname{Dom}_{+}^{\rho}\left(\mathbb{D}^{(k)}\right)$, then $F \cdot G \in \operatorname{Dom}_{-}^{\rho}\left(\mathbb{D}^{(k)}\right)$ and

$$
\begin{equation*}
\mathbb{D}^{(k)}(F \cdot G)=\sum_{i=0}^{k}\binom{k}{i} \mathbb{D}^{(i)}(F) \cdot \mathbb{D}^{(k-i)}(G) \tag{27}
\end{equation*}
$$

Proof. The Leibnitz rule (27) follows by induction and applying Theorem 5.9. Clearly, (27) holds also if $F, G \in \operatorname{Dom}_{0}\left(\mathbb{D}^{(k)}\right)$ and $F \cdot G \in \operatorname{Dom}_{0}\left(\mathbb{D}^{(k)}\right)$.

Theorem 5.11. Assume that either of the following hold:

$$
\begin{aligned}
& 1^{\circ} F \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D}), G \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{+}^{\rho}(\delta), \\
& 2^{\circ} F, G \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{-}^{\rho}(\delta) \\
& 3^{\circ} F, G \in \operatorname{Dom}_{0}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{0}(\delta) .
\end{aligned}
$$

Then the second integration by parts formula holds:

$$
E(F\langle\mathbb{D} G, u\rangle)+E(G\langle\mathbb{D} F, u\rangle)=E(F G \delta(u)) .
$$

Proof. The assertion follows directly from the duality formula (20) and the product rule (26). Assume the first case holds when $F \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D}), G \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{+}^{\rho}(\delta)$. Then $F \cdot G \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$, too, and we have

$$
\begin{aligned}
E(F G \delta(u)) & =E(\langle\mathbb{D}(F \cdot G), u\rangle)=E(\langle F \cdot \mathbb{D}(G)+G \cdot \mathbb{D}(F), u\rangle) \\
& =E(F\langle\mathbb{D}(G), u\rangle)+E(G\langle\mathbb{D}(F), u\rangle) .
\end{aligned}
$$

The second and third case can be proven in an analogous way.
The next theorem states the chain rule for the Malliavin derivative. The classical $(L)^{2}$-case has been known throughout the literature as a direct consequence of the definition of Malliavin derivatives as Fréchet derivatives. Here we provide an alternative proof suited to the setting of chaos expansions.

Theorem 5.12. (Chain rule) Let $\phi$ be a twice continuously differentiable function with bounded derivatives.
$1^{\circ}$ If $F \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$, resp. $F \in \operatorname{Dom}_{0}(\mathbb{D})$, then $\phi(F) \in \operatorname{Dom}_{+}^{\rho}(\mathbb{D})$, resp. $\phi(F) \in \operatorname{Dom}_{0}(\mathbb{D})$, and the chain rule holds:

$$
\begin{equation*}
\mathbb{D}(\phi(F))=\phi^{\prime}(F) \cdot \mathbb{D}(F) \tag{28}
\end{equation*}
$$

$2^{\circ}$ If $F \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ and $\phi$ is analytic, then $\phi^{\diamond}(F) \in \operatorname{Dom}_{-}^{\rho}(\mathbb{D})$ and

$$
\begin{equation*}
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\phi^{\prime \diamond}(F) \diamond \mathbb{D}(F) \tag{29}
\end{equation*}
$$

Proof. $1^{\circ}$ First we prove that (28) holds true when $\phi$ is a polynomial of degree $n, n \in \mathbb{N}$. Then we use the Stone-Weierstrass theorem and approximate a continuously differentiable function $\phi$ by a polynomial $\widetilde{p}_{n}$ of degree $n$, and since we assumed that $\phi$ is regular enough, $\bar{p}_{n}^{\prime}$ will also approximate $\phi^{\prime}$.

By Theorem 5.9 we obtain by induction on $k \in \mathbb{N}$ that

$$
\begin{aligned}
\mathbb{D}\left(F^{k+1}\right) & =\mathbb{D}\left(F \cdot F^{k}\right) \\
& =\mathbb{D}(F) \cdot F^{k}+F \cdot \mathbb{D}\left(F^{k}\right)=\mathbb{D}(F) \cdot F^{k}+F \cdot k F^{k-1} \cdot \mathbb{D}(F) \\
& =(k+1) F^{k} \cdot \mathbb{D}(F)
\end{aligned}
$$

Since $\mathbb{D}$ is a linear operator, we have for any polynomial $p_{n}(x)=\sum_{k=0}^{n} a_{k} x^{k}$ with real coefficients $a_{k}, k \in \mathbb{N}$ :

$$
\begin{equation*}
\mathbb{D}\left(p_{n}(F)\right)=\sum_{k=0}^{n} a_{k} \mathbb{D}\left(F^{k}\right)=\sum_{k=1}^{n} a_{k} k F^{(k-1)} \cdot \mathbb{D}(F)=p_{n}^{\prime}(F) \cdot \mathbb{D}(F) \tag{30}
\end{equation*}
$$

Let $\phi \in C^{2}(\mathbb{R})$ and $F \in \operatorname{Dom}_{p}^{\rho}(\mathbb{D}), p \in \mathbb{N}$. Then, by the Stone-Weierstrass theorem, there exists a polynomial $\widetilde{p_{n}}$ such that

$$
\left\|\phi(F)-\widetilde{p_{n}}(F)\right\|_{X \otimes(S)_{\rho, p}}=\left\|\phi(F)-\sum_{k=0}^{n} a_{k} F^{k}\right\|_{X \otimes(S)_{\rho, p}} \rightarrow 0
$$

and

$$
\left\|\phi^{\prime}(F)-{\widetilde{p_{n}}}^{\prime}(F)\right\|_{X \otimes(S)_{\rho, p}}=\left\|\phi^{\prime}(F)-\sum_{k=1}^{n} a_{k} k F^{k-1}\right\|_{X \otimes(S)_{\rho, \eta}} \rightarrow 0
$$

as $n \rightarrow \infty$.
From (30) and the fact that $\mathbb{D}$ is a bounded operator, Theorem 3.2, we obtain (for $l<p-1$ )

$$
\begin{aligned}
&\left\|\mathbb{D}(\phi(F))-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}=\left\|\mathbb{D}(\phi(F))-\mathbb{D}\left(\widetilde{p_{n}}(F)\right)+\mathbb{D}\left(\widetilde{p_{n}}(F)\right)-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}} \\
& \leq\left\|\mathbb{D}(\phi(F))-\mathbb{D}\left(\widetilde{p_{n}}(F)\right)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}+\left\|\mathbb{D}\left(\widetilde{p_{n}}(F)\right)-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes\left(S S_{\rho, p}\right.} \\
&=\left\|\mathbb{D}\left(\phi(F)-\widetilde{p_{n}}(F)\right)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes\left(S S_{\rho, p}\right.}+\left\|\widetilde{p_{n}^{\prime}}(F) \cdot \mathbb{D}(F)-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes\left(S S_{\rho, p}\right.} \\
& \leq\|\mathbb{D}\| \cdot\left\|\left(\phi(F)-\widetilde{p_{n}}(F)\right)\right\|_{X \otimes\left(S S_{\rho, p}\right.}+\left\|\widetilde{p_{n}^{\prime}}(F)-\phi^{\prime}(F)\right\| \cdot\|\mathbb{D}(F)\|_{X \otimes(S)_{\rho, p}} \rightarrow 0,
\end{aligned}
$$

as $n \rightarrow \infty$. From this follows (28) as well as the estimate

$$
\|\mathbb{D}(\phi(F))\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}} \leq\left\|\phi^{\prime}(F)\right\|_{X \otimes(S)_{\rho, p}} \cdot\|\mathbb{D}(F)\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}<\infty,
$$

and thus $\phi(F) \in \operatorname{Dom}_{p}^{\rho}(\mathbb{D})$.
$2^{\circ}$ The proof of (29) for the Wick version can be conducted in a similar manner. According to Theorem 5.9 we have

$$
\mathbb{D}\left(F^{\diamond k}\right)=k F^{\diamond(k-1)} \diamond \mathbb{D}(F) .
$$

If $\phi$ is an analytic function given by $\phi(x)=\sum_{k=0}^{\infty} a_{k} x^{k}$, then $\phi^{\prime}(x)=\sum_{k=1}^{\infty} a_{k} k x^{k-1}$, and consequently

$$
\phi^{\diamond}(F)=\sum_{k=0}^{\infty} a_{k} F^{\diamond k}, \quad \phi^{\prime \diamond}(F)=\sum_{k=1}^{\infty} a_{k} k F^{\diamond(k-1)} .
$$

Thus,

$$
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\sum_{k=0}^{\infty} a_{k} \mathbb{D}\left(F^{\diamond k}\right)=\sum_{k=0}^{\infty} a_{k} k F^{\diamond(k-1)} \diamond \mathbb{D}(F)=\phi^{\diamond \diamond}(F) \diamond \mathbb{D}(F)
$$

and the identity (29) follows.

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# CHAOS EXPANSION METHODS IN MALLIAVIN CALCULUS: A SURVEY OF RECENT RESULTS 

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Dedicated to Professor Bogoljub Stanković on the occasion of his 90th birthday and to Professor James Vickers on the occasion of his 60 th birthday


#### Abstract

We present a review of the most important historical as well as recent results of Malliavin calculus in the framework of the Wiener-Itô chaos expansion.

AMS Mathematics Subject Classification (2010): 60H40, 60H07, 60H10, 60G20 Key words and phrases: White noise space; series expansion; Malliavin derivative; Skorokhod integral; Ornstein-Uhlenbeck operator; Wick product; Gaussian process; density; Stein's method


## 1. Introduction

The Malliavin derivative $\mathbb{D}$, the Skorokhod integral $\delta$ and the OrnsteinUhlenbeck operator $\mathcal{R}$ are three operators that play a crucial role in the stochastic calculus of variations, an infinite-dimensional differential calculus on white noise spaces $[2,7,35,41,42,47]$. These operators correspond respectively to the annihilation, the creation and the number operator in quantum operator theory.

- The Malliavin derivative, as a modification of Gâteaux derivatives, represents a stochastic gradient in direction of the white noise process [3, 35,42 ]. Originally, it was invented by Paul Malliavin in order to provide a probabilistic proof of Hörmander's sum of squares theorem for hypoelliptic operators and to study the existence and regularity of density of the solution of stochastic differential equations [28], but nowadays it has found significant applications in stochastic control and mathematical finance [8, 29, 46].
- The Skorokhod integral, as the adjoint operator of the Malliavin derivative, is a standard tool in classical $(L)^{2}$ theory of non-adapted stochastic

[^1]differential equations. It represents an extension of the Itô integral from the space of adapted processes to the space of non-anticipative processes $[6,12,15]$. Sometimes it is referred to as the stochastic divergence operator.

- The Ornstein-Uhlenbeck operator, as the composition of the stochastic gradient and divergence, is a stochastic analogue of the Laplacian.

It is of great importance to manage solving different classes of equations which involve the operators of Malliavin calculus. In particular, we consider the following basic equations involving the operators of Malliavin calculus:

$$
\begin{equation*}
\mathcal{R} u=g, \quad \mathbb{D} u=h, \quad \delta u=f . \tag{1.1}
\end{equation*}
$$

In the classical setting, the domain of these operators is a strict subset of the set of processes with finite second moments [7, 26, 35] leading to Sobolev type normed spaces. A more general characterization of the domain of these operators in Kondratiev generalized function spaces has been derived in [18, 22, 23], while in [24] we considered their domains within Kondratiev test function spaces. The three equations in (1.1), that have been considered in [20] and [24] provide a full characterization of the range of all three operators. Moreover, the solutions to equations (1.1) are obtained in an explicit form, which is highly useful for computer modelling that involves polynomial chaos expansion simulation methods used in numerical stochastic analysis [9, 30, 48].

After a short review of the results on uniqueness of the solutions to equations (1.1) (Theorem 3.1, Theorem 4.1, Theorem 5.1) obtained in [20] and [24], we proceed to prove some properties such as the duality relationship between the Malliavin derivative and the Skorokhod integral (Theorem 6.1) and the chain rule (Theorem 6.11), as well as many others such as the product rule (Theorem 6.6, Theorem 6.8), partial integration etc.

A special emphasis is put on the characterization of Gaussian processes and Gaussian solutions of equations (1.1). As an important consequence and application of our results we obtain a connection between the Wick product and the ordinary product (Theorem 4.6 and Theorem 5.10). We also provide several illustrative examples to facilitate comprehension of our results. These examples can be considered as supplementary material to [20] and [24].

A recent discovery made in [32]-[34] made a nice connection between the Malliavin calculus and Stein's method, which is used to measure the distance to Gaussian distributions. In Theorem 7.10 we review this relationship using the chaos expansion method.

The method of chaos expansions is used to illustrate several known results in Malliavin calculus and thus provide a comprehensive insight into its capabilities. For example, we prove using the chaos expansion method some well-known results such as the commutator relationship between $\mathbb{D}$ and $\delta$ (Theorem 5.8), the relation between Itô integration and Riemann integration (Remark 5.9) as well as the Itô representation theorem (Corollary 5.3).

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We strongly emphasize the methodology of the chaos expansion technique for solving singular SDEs. This method has been applied successfully to several classes of SPDEs (e.g. [19, 21, 25, 26, 27, 39, 45]) to obtain an explicit form of the solution. Therefore, we have chosen to write an expository survey with detailed step-by-step proofs and comprehensive examples that illustrate the full advantage of this technique. Some advantages of the chaos expansion technique are the following:

- It provides an explicit form of the solution. The solution is obtained in the form of a series expansion.
- It is easy to apply, since it uses orthogonal bases and series expansions, applying the method of undetermined coefficients. Note that we avoid using the Hermite transform [13] or the $S$-transform [12], since these methods depend on the ability to apply their inverse transforms. Our method requires only finding an appropriate weight factor to make the resulting series convergent.
- It can be adapted to create numerical approximations and model simulations (e.g. by stochastic Galerkin methods). Polynomial chaos expansion approximations are known to be more efficient than Monte Carlo methods. Moreover, for non-Gaussian processes, convergence can be easily improved by changing the Hermite basis to another family of orthogonal polynomials (Charlier, Laguerre, Meixner, etc.).


## 2. Preliminaries

Consider the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel $\sigma$-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to the characteristic function

$$
\begin{equation*}
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \phi \in S(\mathbb{R}) \tag{2.1}
\end{equation*}
$$

given by the Bochner-Minlos theorem.
Denote by $h_{n}(x)=(-1)^{n} e^{\frac{x^{2}}{2}} \frac{d^{n}}{d x^{n}}\left(e^{-\frac{x^{2}}{2}}\right), n \in \mathbb{N}_{0}, \mathbb{N}_{0}=\mathbb{N} \cup\{0\}$, the family of Hermite polynomials and $\xi_{n}(x)=\frac{1}{\sqrt[4]{\pi} \sqrt{(n-1)!}} e^{-\frac{x^{2}}{2}} h_{n-1}(\sqrt{2} x), n \in \mathbb{N}$, the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^{2}(\mathbb{R})$. For a complete preview of properties of $h_{n}$ and $\xi_{n}$ a comprehensive reference is [10]. We follow the characterization of the Schwartz spaces in terms of the Hermite basis: The space of rapidly decreasing functions as a projective limit space $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$ and the space of tempered distributions as an inductive limit space $S^{\prime}(\mathbb{R})=\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$ where

$$
S_{l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k}:\|f\|_{l}^{2}=\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{Z}, \mathbb{Z}=-\mathbb{N} \cup \mathbb{N}_{0}
$$

Note that $S_{p}(\mathbb{R})$ is a Hilbert space endowed with the scalar product $\langle\cdot, \cdot\rangle_{p}$ given by

$$
\left\langle\xi_{k}, \xi_{l}\right\rangle_{p}=\left\{\begin{array}{rl}
0, & k \neq l \\
\left\|\xi_{k}\right\|_{p}^{2}=(2 k)^{p}, & k=l .
\end{array}, \quad p \in \mathbb{Z}\right.
$$

Moreover, the functions $\tilde{\xi}_{k}=\xi_{k}(2 k)^{-\frac{p}{2}}, k \in \mathbb{N}$, constitute an orthonormal basis for $S_{p}(\mathbb{R})$. Indeed,

$$
\left\langle\tilde{\xi}_{k}, \tilde{\xi}_{l}\right\rangle_{p}=\left\{\begin{array}{rl}
0, & k \neq l \\
\left\|\tilde{\xi}_{k}\right\|_{p}^{2}=\left\|\xi_{k}\right\|_{(L)^{2}}^{2}=1, & k=l .
\end{array}, \quad p \in \mathbb{Z} .\right.
$$

### 2.1. The Wiener chaos spaces

Let $\mathcal{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ denote the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right), \alpha_{i} \in$ $\mathbb{N}_{0}, i=1,2, \ldots, m, m \in \mathbb{N}$. The $k$ th unit vector $\varepsilon^{(k)}=(0, \cdots, 0,1,0, \cdots), k \in \mathbb{N}$ is the sequence of zeros with the only entry 1 as its $k$ th component. The multiindex $\mathbf{0}=(0,0,0,0, \ldots)$ has all zero entries. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$.

Operations with multi-indices are carried out componentwise e.g. $\alpha+\beta=$ $\left(\alpha_{1}+\beta_{1}, \alpha_{2}+\beta_{2}, \ldots\right), \alpha!=\alpha_{1}!\alpha_{2}!\alpha_{3}!\cdots,\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$. Note that $\alpha>\mathbf{0}$ (equivalently $|\alpha|>0$ ) if there is at least one component $\alpha_{k}>0$. We adopt the convention that $\alpha-\beta$ exists only if $\alpha-\beta>\mathbf{0}$ and otherwise it is not defined.

Let $(2 \mathbb{N})^{\alpha}=\prod_{k=1}^{\infty}(2 k)^{\alpha_{k}}$. Note that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ for $p>1$ (see e.g. [13]).

Let $(L)^{2}=L^{2}\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$ be the Hilbert space of random variables with finite second moments. We define by

$$
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in \mathcal{I}
$$

the Fourier-Hermite orthogonal basis of $(L)^{2}$ such that $\left\|H_{\alpha}\right\|_{(L)^{2}}^{2}=\alpha!$. In particular, for the $k$ th unit vector $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$.

The prominent Wiener-Itô chaos expansion theorem states that each element $F \in(L)^{2}$ has a unique representation of the form

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha}(\omega)
$$

$\omega \in S^{\prime}(\mathbb{R}), c_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that $\|F\|_{(L)^{2}}^{2}=\sum_{\alpha \in \mathcal{I}} c_{\alpha}^{2} \alpha!<\infty$.
Definition 2.1. The spaces

$$
\mathcal{H}_{k}=\left\{F \in(L)^{2}: F=\sum_{\alpha \in \mathcal{I},|\alpha|=k} c_{\alpha} H_{\alpha}\right\}, \quad k \in \mathbb{N}_{0}
$$

that are obtained by closing the linear span of the $k$ th order Hermite polynomials in $(L)^{2}$ are called the Wiener chaos spaces of order $k$.

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For example, $\mathcal{H}_{0}$ is the set of constant random variables, $\mathcal{H}_{1}$ is the set of Gaussian random variables, $\mathcal{H}_{2}$ is the space of quadratic Gaussian random variables and so on. We will show that $\mathcal{H}_{1}$ contains only Gaussian random variables and that the most important processes, Brownian motion and white noise, belong to $\mathcal{H}_{1}$.

Each $\mathcal{H}_{k}, k \in \mathbb{N}_{0}$ is a closed subspace of $(L)^{2}$. Moreover, the Wiener-Itô chaos expansion theorem can be stated in the form:

$$
(L)^{2}=\bigoplus_{k=0}^{\infty} \mathcal{H}_{k}
$$

Hence, every $F \in(L)^{2}$ can be represented in the form $F(\omega)=\sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathcal{I} \\|\alpha|=k}} c_{\alpha} H_{\alpha}(\omega)$, $\omega \in S^{\prime}(\mathbb{R})$, where $\sum_{|\alpha|=k} c_{\alpha} H_{\alpha}(\omega) \in \mathcal{H}_{k}, k=0,1,2, \ldots$.
Theorem 2.2. All random variables which belong to $\mathcal{H}_{1}$ are Gaussian random variables.

Proof. Random variables that belong to the space $\mathcal{H}_{1}$ are linear combinations of elements $\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}, \omega \in S^{\prime}(\mathbb{R})$. From the definition of the Gaussian measure (2.1) it follows that $E_{\mu}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=0$ and $\operatorname{Var}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=E_{\mu}\left(\left\langle\omega, \xi_{k}\right\rangle^{2}\right)=$ $\left\|\xi_{k}\right\|_{L^{2}(\mathbb{R})}^{2}=1$. Thus, from the form of the characteristic function we conclude that $\left\langle\omega, \xi_{k}\right\rangle: \mathcal{N}(0,1), k \in \mathbb{N}$. Thus, every finite linear combination of Gaussian random variables $\sum_{k=1}^{n} a_{k}\left\langle\omega, \xi_{k}\right\rangle$ is a Gaussian random variable and the limit of Gaussian random variables $\sum_{k=1}^{\infty} a_{k}\left\langle\omega, \xi_{k}\right\rangle=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} a_{k}\left\langle\omega, \xi_{k}\right\rangle$ is also Gaussian.

After Example 2.13 it will be also clear that $\mathcal{H}_{1}$ is the closed Gaussian space generated by the random variables $B_{t}(\omega), t \geq 0$, where $B_{t}$ is Brownian motion (see also [41]).

Remark 2.3. We note the following important facts:

1) Although the space $(L)^{2}$ is constructed with respect to Gaussian measure, it contains all (square integrable) random variables, not just those with Gaussian distribution but also all absolutely continuous, singularly continuous, discrete and mixed type distributions.
2) All Gaussian random variables belong to $\mathcal{H}_{0} \oplus \mathcal{H}_{1}$ and thus their chaos expansion is given in terms of multi-indices of length at most one (those with zero expectation are strictly in $\mathcal{H}_{1}$ ). Quadratic Gaussian random variables belong to $\mathcal{H}_{0} \oplus \mathcal{H}_{1} \oplus \mathcal{H}_{2}$ and by linearity so does the Chi-square distribution, too. In general, the $n$th power of a Gaussian random variable belongs to $\bigoplus_{k=0}^{n} \mathcal{H}_{k}$, for $n \in \mathbb{N}$, and thus its chaos expansion is given in terms of multi-indices of lengths from zero to $n$.
3) Discrete random variables (with finite variance) belong to $\bigoplus_{k=0}^{\infty} \mathcal{H}_{k}$, i.e. their chaos expansions forms consist of multi-indices of all lengths.
4) All finite sums i.e. partial sums of a chaos expansion correspond to absolutely continuous distributions or almost surely constant distributions. There is no possibility to obtain discrete random variables by using finite sums in the Wiener-Itô expansion. This is a consequence of Theorem 7.8.
In the next section we introduce suitable spaces, called Kondratiev spaces, that will contain random variables with infinite variances.

### 2.2. Kondratiev spaces

The stochastic analogue of Schwartz spaces as generalized function spaces are the Kondratiev spaces of generalized random variables.

Definition 2.4. The space of the Kondratiev test random variables $(S)_{1}$ consists of elements $f=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha} \in(L)^{2}, c_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|f\|_{1, p}^{2}=\sum_{\alpha \in \mathcal{I}} c_{\alpha}^{2}(\alpha!)^{2}(2 \mathbb{N})^{p \alpha}<\infty, \quad \text { for all } p \in \mathbb{N}_{0}
$$

The space of the Kondratiev generalized random variables $(S)_{-1}$ consists of formal expansions of the form $F=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}, b_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|F\|_{-1,-p}^{2}=\sum_{\alpha \in \mathcal{I}} b_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty, \quad \text { for some } p \in \mathbb{N}_{0}
$$

Definition 2.5. The space of the Hida test random variables $(S)_{0}^{+}$consists of elements $f=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha} \in(L)^{2}, c_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|f\|_{0, p}^{2}=\sum_{\alpha \in \mathcal{I}} c_{\alpha}^{2} \alpha!(2 \mathbb{N})^{p \alpha}<\infty, \quad \text { for all } p \in \mathbb{N}_{0}
$$

The space of the Hida generalized random variables $(S)_{0}^{-}$consists of formal expansions of the form $F=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}, b_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|F\|_{0,-p}^{2}=\sum_{\alpha \in \mathcal{I}} b_{\alpha}^{2} \alpha!(2 \mathbb{N})^{-p \alpha}<\infty, \quad \text { for some } p \in \mathbb{N}_{0}
$$

This provides a sequence of spaces $(S)_{\rho, p}=\left\{f \in(L)^{2}:\|f\|_{\rho, p}<\infty\right\}$, $\rho \in\{-1,0,1\}, p \in \mathbb{Z}$, such that

$$
\begin{aligned}
& (S)_{1, p} \subseteq(S)_{0, p} \subseteq(L)^{2} \subseteq(S)_{0,-p} \subseteq(S)_{-1,-p} \\
& (S)_{1, p} \subseteq(S)_{1, q} \subseteq(L)^{2} \subseteq(S)_{-1,-q} \subseteq(S)_{-1,-p}
\end{aligned}
$$

for all $p \geq q \geq 0$ and the inclusions denote continuous embeddings and $(S)_{0,0}=$ $(L)^{2}$. Thus, $(S)_{1}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{1, p}$ and $(S)_{0}^{+}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{0, p}$ can be equipped with the projective topology and $(S)_{-1}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-1,-p},(S)_{0}^{-}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{0,-p}$ as

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their duals with the inductive topology. Note that $(S)_{1},(S)_{0}^{+}$are nuclear and the following Gel'fand triples

$$
(S)_{1} \subseteq(L)^{2} \subseteq(S)_{-1}, \quad(S)_{0}^{+} \subseteq(L)^{2} \subseteq(S)_{0}^{-}
$$

are obtained.
From the estimate $\alpha!\leq(2 \mathbb{N})^{\alpha}$ it follows that

$$
(2 \mathbb{N})^{-p \alpha} \leq \alpha!(2 \mathbb{N})^{-p \alpha} \leq(2 \mathbb{N})^{-(p-1) \alpha}
$$

thus

$$
(S)_{-1,-(p-1)} \subseteq(S)_{0,-p} \subseteq(S)_{-1,-p}, \quad \text { for all } p \in \mathbb{N}
$$

and similarly

$$
(S)_{1, p+1} \subseteq(S)_{0, p} \subseteq(S)_{1, p}, \quad \text { for all } p \in \mathbb{N}_{0}
$$

We will denote by $\ll \cdot \cdot \gg$ the dual pairing between $(S)_{0,-p}$ and $(S)_{0, p}$. Its action is given by $\ll A, B \gg=\ll \sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}, \sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha} \gg=\sum_{\alpha \in \mathcal{I}} \alpha!a_{\alpha} b_{\alpha}$. In case of random variables with finite variance it reduces to the scalar product $\ll A, B>{ }_{(L)^{2}}=E(A B)$. For any fixed $p \in \mathbb{Z},(S)_{0, p}, p \in \mathbb{Z}$, is a Hilbert space (we identify the case $p=0$ with $(L)^{2}$ ) endowed with the scalar product

$$
\ll H_{\alpha}, H_{\beta}>_{p}=\left\{\begin{aligned}
0, & \alpha \neq \beta, \\
\alpha!(2 \mathbb{N})^{p \alpha}, & \alpha=\beta,
\end{aligned} \quad \text { for } p \in \mathbb{Z},\right.
$$

extended by linearity and continuity to

$$
\ll A, B \gg{ }_{p}=\sum_{\alpha \in \mathcal{I}} \alpha!a_{\alpha} b_{\alpha}(2 \mathbb{N})^{p \alpha}, \quad p \in \mathbb{Z} .
$$

In the framework of white noise analysis, the problem of pointwise multiplication of generalized functions is overcome by introducing the Wick product. It is well defined in the Kondratiev spaces of test and generalized stochastic functions $(S)_{1}$ and $(S)_{-1}$; see for example $[12,13]$.

Definition 2.6. Let $F, G \in(S)_{-1}$ be given by their chaos expansions $F(\omega)=$ $\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega)$ and $G(\omega)=\sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta}(\omega)$, for unique $f_{\alpha}, g_{\beta} \in \mathbb{R}$. The Wick product of $F$ and $G$ is the element denoted by $F \diamond G$ and defined by

$$
\begin{aligned}
F \diamond G(\omega) & =\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) H_{\gamma}(\omega) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} H_{\alpha+\beta}(\omega) .
\end{aligned}
$$

The same definition is provided for the Wick product of test random variables belonging to $(S)_{1}$.

Note that the Kondratiev spaces $(S)_{1}$ and $(S)_{-1}$ are closed under the Wick multiplication [13], while the space $(L)^{2}$ is not closed under it.

Example 2.7. The random variable defined by the chaos expansion $F=$ $\sum_{n=1}^{\infty} \frac{1}{n \sqrt{n!}} H_{n \varepsilon^{(n)}}$ belongs to $(L)^{2}$ since $\|F\|_{(L)^{2}}^{2}=\sum_{n=1}^{\infty} \frac{1}{n^{2}}<\infty$, but $F \diamond F$ is not in $(L)^{2}$. Clearly,

$$
\begin{aligned}
\|F \diamond F\|_{(L)^{2}}^{2} & =\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \frac{1}{k(n-k) \sqrt{k!(n-k)!}}\right)^{2} n! \\
& \geq \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \frac{1}{k(n-k)}\right)^{2}=\infty
\end{aligned}
$$

The most important property of the Wick multiplication is its relation to the Itô-Skorokhod integration [12, 13], since it reproduces the fundamental theorem of calculus. This fact will be revisited in Remark 5.9.

In the sequel we will need the notion of Wick-versions of analytic functions. For this purpose note that the $n$th Wick power is defined by $F^{\diamond n}=F^{\diamond(n-1)} \diamond F$, $F^{\diamond 0}=1$. Note that $H_{n \varepsilon_{k}}=H_{\varepsilon_{k}}^{\diamond n}$ for $n \in \mathbb{N}_{0}, k \in \mathbb{N}$.

Definition 2.8. If $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a real analytic function at the origin represented by the power series

$$
\varphi(x)=\sum_{n=0}^{\infty} a_{n} x^{n}, \quad x \in \mathbb{R},
$$

then its Wick version $\varphi^{\diamond}:(S)_{-1} \rightarrow(S)_{-1}$ is given by

$$
\varphi^{\diamond}(F)=\sum_{n=0}^{\infty} a_{n} F^{\diamond n}, \quad F \in(S)_{-1}
$$

### 2.3. Generalized stochastic processes

Let $\tilde{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\tilde{X}}$ and let $\tilde{X}^{\prime}$ denote its dual space. In this section we describe $\tilde{X}$-valued random variables. Most notably, if $\tilde{X}$ is a space of functions on $\mathbb{R}$, e.g. $\left.\tilde{X}=C^{k}([a, b])\right),-\infty<a<b<$ $\infty$ or $\tilde{X}=L^{2}(\mathbb{R})$, we obtain the notion of a stochastic process. We will also define processes where $\tilde{X}$ is not a normed space, but a nuclear space topologized by a family of seminorms, e.g. $\tilde{X}=S(\mathbb{R})$ (see e.g. [38]).

Definition 2.9. Let $f$ have the formal expansion $f=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}$, where $f_{\alpha} \in X, \alpha \in \mathcal{I}$. Define the following spaces:

$$
\begin{aligned}
X \otimes(S)_{1, p} & =\left\{f:\|f\|_{X \otimes(S)_{1, p}}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} \\
X \otimes(S)_{-1,-p} & =\left\{f:\|f\|_{X \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}, \\
X \otimes(S)_{0, p} & =\left\{f:\|f\|_{X \otimes(S)_{0, p}}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} \\
X \otimes(S)_{0,-p} & =\left\{f:\|f\|_{X \otimes(S)_{0,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\},
\end{aligned}
$$

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where $X$ denotes an arbitrary Banach space (allowing both possibilities $X=\tilde{X}$, $X=\tilde{X}^{\prime}$ ).

Especially, for $p=0, X \otimes(S)_{0,0}$ will be denoted by

$$
X \otimes(L)^{2}=\left\{f:\|f\|_{X \otimes(S)_{0,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|f_{\alpha}\right\|_{X}^{2}<\infty\right\}
$$

We will denote by $E(F)=f_{(0,0,0, \ldots)}$ the generalized expectation of the process $F$.

Definition 2.10. Generalized stochastic processes and test stochastic processes in Kondratiev sense are elements of the spaces

$$
X \otimes(S)_{-1}=\bigcup_{p \in \mathbb{N}} X \otimes(S)_{-1,-p}, \quad X \otimes(S)_{1}=\bigcap_{p \in \mathbb{N}} X \otimes(S)_{1, p}
$$

respectively.
Generalized stochastic processes and test stochastic processes in Hida sense are elements of the spaces

$$
X \otimes(S)_{0}^{-}=\bigcup_{p \in \mathbb{N}} X \otimes(S)_{0,-p}, \quad X \otimes(S)_{0}^{+}=\bigcap_{p \in \mathbb{N}} X \otimes(S)_{0, p}
$$

respectively.
Remark 2.11. In this case the symbol $\otimes$ denotes the projective tensor product of two spaces i.e. $\tilde{X}^{\prime} \otimes(S)_{-1}$ is the completion of the tensor product with respect to the $\pi$-topology.

The Kondratiev space $(S)_{1}$ is nuclear and thus $\left(\tilde{X} \otimes(S)_{1}\right)^{\prime} \cong \tilde{X}^{\prime} \otimes(S)_{-1}$. Note that $\tilde{X}^{\prime} \otimes(S)_{-1}$ is isomorphic to the space of linear bounded mappings $\tilde{X} \rightarrow(S)_{-1}$, and it is also isomporphic to the space of linear bounded mappings $(S)_{+1} \rightarrow \tilde{X}^{\prime}$. The same holds for the Hida spaces, too.

In [43] and [44] a general setting of $S^{\prime}$-valued generalized stochastic process is provided (we restrict our attention to the Kondratiev setting): $S^{\prime}(\mathbb{R})$-valued generalized stochastic processes are elements of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ and they are given by chaos expansions of the form

$$
\begin{equation*}
f=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k} \tag{2.2}
\end{equation*}
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathcal{I}} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-1}$ and $a_{\alpha, k} \in X$. Thus,

$$
\begin{aligned}
X & \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p} \\
& =\left\{f:\|f\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}, k \in \mathbb{N}}\left\|a_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
\end{aligned}
$$

and

$$
X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}=\bigcup_{p, l \in \mathbb{N}} X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}
$$

The generalized expectation of an $S^{\prime}$-valued stochastic process $f$ is given by $E(f)=\sum_{k \in \mathbb{N}} a_{(0,0, \ldots), k} \otimes \xi_{k}=b_{(0,0, \ldots)}$.

In an analogous way, we define $S$-valued test processes as elements of $X \otimes S(\mathbb{R}) \otimes(S)_{1}$, which are given by chaos expansions of the form (2.2), where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathcal{I}} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{1}$ and $a_{\alpha, k} \in X$. Thus,
$X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}=\left\{f:\|f\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}}^{2}=\sum_{\alpha \in \mathcal{I}, k \in \mathbb{N}} \alpha!^{2}\left\|a_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha}<\infty\right\}$
and

$$
X \otimes S(\mathbb{R}) \otimes(S)_{1}=\bigcap_{p, l \in \mathbb{N}} X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}
$$

One can define the Hida spaces in a similar way. Especially, for $p=l=0$, one obtains the space of processes with finite second moments and square integrable trajectories $X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$. It is isomorphic to $X \otimes L^{2}(\mathbb{R} \times \Omega)$ and if $X$ is a separable Hilbert space, then it is also isomorphic to $L^{2}(\mathbb{R} \times \Omega ; X)$.
Remark 2.12. In the sequel we will use the notation $\mathcal{H}_{k}, k \in \mathbb{N}_{0}$, to denote not just $(L)^{2}$-random variables, but also generalized stochastic processes and test processes which have a chaos expansion of the form (2.2) only with multi-indices of length $|\alpha|=k$.
Example 2.13. Brownian motion as an element of $S^{\prime}(\mathbb{R}) \otimes(L)^{2}$, is defined by

$$
B_{t}(\omega):=\left\langle\omega, \kappa_{[0, t]}\right\rangle, \quad \omega \in S^{\prime}(\mathbb{R})
$$

where $\kappa_{[0, t]}$ is the characteristic function of the interval $[0, t], t>0$. It is a Gaussian process with zero expectation and covariance function $E_{\mu}\left(B_{t}(\omega) B_{s}(\omega)\right)=$ $\min \{t, s\}$. The chaos expansion of Brownian motion is given by

$$
B_{t}(\omega)=\sum_{k=1}^{\infty}\left(\int_{0}^{t} \xi_{k}(s) d s\right) H_{\varepsilon^{(k)}}(\omega)
$$

For all $k \in \mathbb{N}$, its coefficients $\int_{0}^{t} \xi_{k}(s) d s$ are in $C^{\infty}(\mathbb{R})$.
Singular white noise is defined by the chaos expansion

$$
W_{t}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega)
$$

and it is an element of the space $S_{k}(\mathbb{R}) \otimes(S)_{-1,-p}$ for $k, p \geq 1$. $\frac{d}{d t} B_{t}=W_{t}$ holds with weak derivatives in the $(S)_{-1}$ sense. Both Brownian motion and singular white noise belong to the Wiener chaos space of order one.

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### 2.4. Multiplication of stochastic processes

We generalize the definition of the Wick product of random variables to the set of generalized stochastic processes in the way as it is done in [19, 39] and [40]. From now on we assume that $X$ is closed under multiplication, i.e. $x \cdot y \in X$ for all $x, y \in X$.

Definition 2.14. Let $F, G \in X \otimes(S)_{ \pm 1}$ be generalized (resp. test) stochastic processes given by chaos expansions $f=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}, g=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha}$, where $f_{\alpha}, g_{\alpha} \in X, \alpha \in \mathcal{I}$. Then the Wick product $F \diamond G$ is defined by

$$
\begin{equation*}
F \diamond G=\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) \otimes H_{\gamma} \tag{2.3}
\end{equation*}
$$

Theorem 2.15. Let the stochastic processes $F$ and $G$ be given in their chaos expansion forms $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}$ and $G=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha}$.

1. If $F \in X \otimes(S)_{-1,-p_{1}}$ and $G \in X \otimes(S)_{-1,-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{-1,-q}$, for $q \geq p_{1}+p_{2}+2$.
2. If $F \in X \otimes(S)_{1, p_{1}}$ and $G \in X \otimes(S)_{1, p_{2}}$ for $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{1, q}$, for $q \leq \min \left\{p_{1}, p_{2}\right\}-2$.

Proof. 1. By the Cauchy-Schwartz inequality, the following holds

$$
\begin{aligned}
\| F & \diamond G \|_{X \otimes(S)_{-1,-q}}^{2} \\
& =\sum_{\gamma \in \mathcal{I}}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-q \gamma} \\
& \leq \sum_{\gamma \in \mathcal{I}}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-\left(p_{1}+p_{2}+2\right) \gamma} \\
& \leq \sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p_{1} \gamma}\right)\left(\sum_{\alpha+\beta=\gamma}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p_{2} \gamma}\right)(2 \mathbb{N})^{-2 \gamma} \\
& \leq\left(\sum_{\gamma \in \mathcal{I}}(2 \mathbb{N})^{-2 \gamma}\right)\left(\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p_{1} \alpha}\right)\left(\sum_{\beta \in \mathcal{I}}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p_{2} \beta}\right) \\
& \left.=M \cdot\|F\|_{X \otimes(S)_{-1,-p_{1}}^{2}}^{2} \cdot\|G\|_{X \otimes(S)_{-1,-p_{2}}^{2}<\infty,}^{<\infty}\right)
\end{aligned}
$$

since $M=\sum_{\gamma \in \mathcal{I}}(2 \mathbb{N})^{-2 \gamma}<\infty$ by the nuclearity of $(S)_{-1}$.
2. Let now $F \in X \otimes(S)_{1, p_{1}}$ and $G \in X \otimes(S)_{1, p_{2}}$ for all $p_{1}, p_{2} \in \mathbb{N}_{0}$. Then the chaos expansion form of $F \diamond G$ is given by (2.3) and

$$
\begin{aligned}
\| F & \diamond G \|_{X \otimes(S)_{1, q}}^{2} \\
& =\sum_{\gamma \in \mathcal{I}} \gamma!^{2}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{q \gamma} \cdot(2 \mathbb{N})^{2 \gamma}(2 \mathbb{N})^{-2 \gamma} \\
& =\sum_{\gamma \in \mathcal{I}}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} \gamma!f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2} \gamma}\right\|_{X}^{2} \\
& \leq \sum_{\gamma \in \mathcal{I}}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} \alpha!\beta!(2 \mathbb{N})^{\alpha+\beta} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \leq M\left(\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{2\left(\frac{q+2}{2}+1\right) \alpha}\right)\left(\sum_{\beta \in \mathcal{I}} \beta!^{2}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{2\left(\frac{q+2}{2}+1\right) \beta}\right) \\
& \leq M\left(\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p_{1} \alpha}\right)\left(\sum_{\beta \in \mathcal{I}} \beta!^{2}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p_{2} \beta}\right) \\
& =M \cdot\|F\|_{X \otimes(S)_{1, p_{1}}^{2}}^{2} \cdot\|G\|_{X \otimes(S)_{1, p_{2}}^{2}}^{2}<\infty,
\end{aligned}
$$

if $q \leq p_{1}-2$ and $q \leq p_{2}-2$. We used beside the Cauchy-Schwartz inequality the estimate $(\alpha+\beta)!\leq \alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}$, for all $\alpha, \beta \in \mathcal{I}$.

Applying the well-known formula for the Fourier-Hermite polynomials (see [13])

$$
\begin{equation*}
H_{\alpha} \cdot H_{\beta}=\sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \tag{2.4}
\end{equation*}
$$

one can define the ordinary product $F \cdot G$ of two stochastic processes $F$ and $G$. Thus, by applying formally (2.4) we obtain

$$
\begin{aligned}
F \cdot G & =\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \cdot \sum_{\beta \in \mathcal{I}} g_{\beta} \otimes H_{\beta} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \otimes H_{\alpha} \cdot H_{\beta} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \otimes \sum_{0 \leq \gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \\
& =F \diamond G+\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \otimes \sum_{0<\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \\
& =F \diamond G+\sum_{\tau \in \mathcal{I}} \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\substack{\gamma>0, \delta \leq \tau \\
\gamma+\tau-\delta=\beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau} .
\end{aligned}
$$

For example, for Brownian motion we have

$$
B_{t_{1}} \cdot B_{t_{2}}=B_{t_{1}} \diamond B_{t_{2}}+\min \left\{t_{1}, t_{2}\right\}, \quad B_{t}^{2}=B_{t}^{\diamond 2}+t
$$

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Note also that, $E(F \diamond G)=f_{0} g_{0}=E F \cdot E G$, without the assumption of independence of $F$ and $G$ as opposed to $E(F \cdot G) \neq E F \cdot E G$.

Particularly, it is clear that the following identities hold for the FourierHermite polynomials:

$$
H_{\varepsilon^{(k)}} \cdot H_{\varepsilon^{(l)}}=\left\{\begin{array}{ccc}
H_{2 \varepsilon^{(k)}}+1 & , & k=l \\
H_{\varepsilon^{(k)}+\varepsilon^{(l)}} & , \quad k \neq l
\end{array}=\left\{\begin{array}{ccc}
H_{\varepsilon^{(k)}}^{\diamond 2}+1 & , & k=l \\
H_{\varepsilon^{(k)} \diamond H_{\varepsilon^{(l)}}} & , \quad k \neq l
\end{array} .\right.\right.
$$

In Section 4 we will use the Malliavin derivative operator to express the difference between the ordinary product and the Wick product of a generalized stochastic process from $X \otimes(S)_{-1}$ and singular white noise $W_{t}$ (Theorem 4.6). Here we state some general cases when the ordinary product is well defined.

Theorem 2.16. The following holds:

1. If $F, G \in X \otimes(S)_{1}$ then the product $F \cdot G$ is a well defined element in $X \otimes(S)_{1}$. Moreover, for every $m \in \mathbb{N}_{0}$ there exist $r, s \in \mathbb{N}_{0}$ and $C(m)>0$ such that

$$
\|F \cdot G\|_{X \otimes(S)_{1, m}} \leq C(m)\|F\|_{X \otimes(S)_{1, r}}\|G\|_{X \otimes(S)_{1, s}}
$$

holds.
2. If $F \in X \otimes(S)_{1}$ and $G \in X \otimes(S)_{-1}$ then their product $F \cdot G$ is well defined and belongs to $X \otimes(S)_{-1}$.

The proof is similar to the one for multiplication of Schwartz test functions and multiplication of tempered distributions with test functions.

Note, for $F, G \in X \otimes(L)^{2}$ the ordinary product $F \cdot G$ does not have to belong to $X \otimes(L)^{2}$.

### 2.5. Operators of the Malliavin calculus

In $[2,7,25,26,35,42]$ the Malliavin derivative and the Skorokhod integral are defined on a subspace of $(L)^{2}$ so that the resulting process after application of these operators necessarily remains in $(L)^{2}$. We will recall of these classical results and denote the corresponding domains with a "zero" in order to retain a nice symmetry between test and generalized processes. In [18, 19, 22, 23] we allowed values in $(S)_{-1}$ and thus obtained larger domains for all operators. These domains will be denoted by a "minus" sign to reflect the fact that they correspond to generalized processes. In [24] we introduced also domains for test processes. These domains will be denoted by a "plus" sign.

Definition 2.17. Let a generalized stochastic process $u \in X \otimes(S)_{-1}$ be of the form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$. If there exists $p \in \mathbb{N}$ such that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{2.5}
\end{equation*}
$$

then the Malliavin derivative of $u$ is defined by

$$
\begin{equation*}
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}} \tag{2.6}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$, i.e.

$$
H_{\alpha-\varepsilon}(k)=\left\{\begin{array}{cc}
0, & \alpha_{k}=0 \\
H_{\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right)}, & \alpha_{k} \geq 1
\end{array} .\right.
$$

for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$.
The set of generalized stochastic processes $u \in X \otimes(S)_{-1}$ which satisfy (2.5) constitutes the domain of the Malliavin derivative, denoted by $\operatorname{Dom}_{-}(\mathbb{D})$. Thus the domain of the Malliavin derivative is given by

$$
\begin{aligned}
\operatorname{Dom}_{-}(\mathbb{D}) & =\bigcup_{p \in \mathbb{N}} \operatorname{Dom}_{-p}(\mathbb{D}) \\
& =\bigcup_{p \in \mathbb{N}}\left\{u \in X \otimes(S)_{-1}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
\end{aligned}
$$

A process $u \in \operatorname{Dom}_{-}(\mathbb{D}) \subset X \otimes(S)_{-1}$ is called a Malliavin differentiable process. Note that (2.6) can also be expressed in the form

$$
\begin{equation*}
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha} \tag{2.7}
\end{equation*}
$$

For stochastic test processes from $X \otimes(S)_{1}$, the Malliavin derivative is always defined, i.e.

$$
\operatorname{Dom}_{p}(\mathbb{D})=\left\{u \in X \otimes(S)_{1}: \sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}=X \otimes(S)_{1, p}
$$

In order to retain symmetry in notation, we denote

$$
\operatorname{Dom}_{+}(\mathbb{D})=\bigcap_{p \in \mathbb{N}} \operatorname{Dom}_{p}(\mathbb{D})=\bigcap_{p \in \mathbb{N}}\left(X \otimes(S)_{1, p}\right)=X \otimes(S)_{1}
$$

In the classical literature it is usual to define the Malliavin derivative only for the $(L)^{2}$ case:

Definition 2.18. Let a square integrable stochastic process $u \in X \otimes(L)^{2}$ be of the form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$. If the condition

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty \tag{2.8}
\end{equation*}
$$

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holds, then $u$ is a Malliavin differentiable process and the Malliavin derivative of $u$ is defined by (2.6). All processes $u$ satisfying the condition (2.8) belong to the domain of $\mathbb{D}$ denoted by $\operatorname{Dom}_{0}(\mathbb{D})$, i.e. the domain is given by

$$
\operatorname{Dom}_{0}(\mathbb{D})=\left\{u \in X \otimes(L)^{2}: \sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\}
$$

Theorem 2.19. ([18, 24])
a) The Malliavin derivative of a generalized process $u \in X \otimes(S)_{-1}$ is a linear and continuous mapping

$$
\mathbb{D}: \quad \operatorname{Dom}_{-p}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}
$$

for $l>p+1$ and $p \in \mathbb{N}$.
b) The Malliavin derivative of a test stochastic process $v \in X \otimes(S)_{1}$ is a linear and continuous mapping

$$
\mathbb{D}: \quad \operatorname{Dom}_{p}(\mathbb{D}) \rightarrow X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}
$$

for $l<p-1$ and $p \in \mathbb{N}$.
c) The Malliavin derivative of a square integrable process $u \in \operatorname{Dom}_{0}(\mathbb{D})$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{0}(\mathbb{D}) \rightarrow X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}
$$

Proof. a) Let $u$ be as in Definition 2.17. Then,

$$
\begin{aligned}
\|\mathbb{D} u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}}^{2} & =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k=1}^{\infty} \alpha_{k} u_{\alpha} \otimes \xi_{k}\right\|_{X \otimes S_{-l}}^{2}(2 \mathbb{N})^{-p\left(\alpha-\varepsilon^{(k)}\right)} \\
& =\sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}} \alpha_{k}^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 k)^{-l}\right)(2 k)^{p}(2 \mathbb{N})^{-p \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \sum_{k \in \mathbb{N}}(2 k)^{-l+p} \\
& \leq C \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{aligned}
$$

where $\sum_{k \in \mathbb{N}}(2 k)^{-l+p}=C<\infty$ for $l>p+1$. We also used the generalized Minkowski inequality to obtain that

$$
\sum_{k \in \mathbb{N}} \alpha_{k}^{2}(2 k)^{p-l} \leq\left(\sum_{k \in \mathbb{N}} \alpha_{k}^{4}\right)^{\frac{1}{2}} \cdot \sum_{k \in \mathbb{N}}(2 k)^{p-l}
$$

and the fact that $\left(\sum_{k \in \mathbb{N}} \alpha_{k}^{4}\right)^{\frac{1}{2}} \leq \sum_{k \in \mathbb{N}} \alpha_{k}^{2} \leq\left(\sum_{k \in \mathbb{N}} \alpha_{k}\right)^{2}=|\alpha|^{2}$.
b) Let $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{1, p}$ for all $p \geq 0$, i.e. let the condition $\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{2}(2 \mathbb{N})^{p \alpha}<\infty$ hold. Then, from (2.6) and

$$
\begin{aligned}
\|\mathbb{D} v\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}}^{2} & =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} \alpha_{k} v_{\alpha} \otimes \xi_{k}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}\left(\alpha-\varepsilon^{(k)}\right)!^{2}(2 \mathbb{N})^{p\left(\alpha-\varepsilon^{(k)}\right)} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k}^{2}\left(\alpha-\varepsilon^{(k)}\right)!^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p\left(\alpha-\varepsilon^{(k)}\right)} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha!^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 k)^{l-p}(2 \mathbb{N})^{p \alpha} \\
& \leq C \sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=C\|v\|_{X \otimes(S)_{1, p}}^{2}<\infty
\end{aligned}
$$

the assertion follows, where $C=\sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$ for $p>l+1$. We also used $\alpha_{k}\left(\alpha-\varepsilon^{(k)}\right)!=\alpha!, k \in \mathbb{N}, \alpha \in \mathcal{I}$ and $(2 \mathbb{N})^{\varepsilon^{(k)}}=(2 k), k \in \mathbb{N}$.
c) Let $u \in \operatorname{Dom}_{0}(\mathbb{D})$, i.e. $\sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty$. Then,

$$
\begin{aligned}
\|\mathbb{D} u\|_{X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k}^{2}\left(\alpha-\varepsilon^{(k)}\right)!\left\|w_{\alpha}\right\|_{X}^{2} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} \alpha!\left\|w_{\alpha}\right\|_{X}^{2}=\sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|w_{\alpha}\right\|_{X}^{2}<\infty .
\end{aligned}
$$

Note that $\operatorname{Dom}_{p}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-p}(\mathbb{D})$ for all $p \in \mathbb{N}$. Therefore

$$
\operatorname{Dom}_{+}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-}(\mathbb{D})
$$

Moreover, using the estimate $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ it follows that

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(p-2) \alpha}, \text { i.e., } \\
X \otimes(S)_{-1,-(p-2)} \subseteq \operatorname{Dom}_{-p}(\mathbb{D}) \subseteq X \otimes(S)_{-1,-p}, \quad p>3
\end{gathered}
$$

Remark 2.20. For $u \in \operatorname{Dom}_{+}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\mathbb{D})$ it is usual to write

$$
\mathbb{D}_{t} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k}(t) \otimes H_{\alpha-\varepsilon^{(k)}}
$$

in order to emphasise that the Malliavin derivative takes a random variable into a process i.e. that $\mathbb{D} u$ is a function of $t$. Moreover, the formula

$$
\mathbb{D}_{t} F(\omega)=\lim _{h \rightarrow 0} \frac{1}{h}\left(F\left(\omega+h \cdot \kappa_{[t, \infty)}\right)-F(\omega)\right), \quad \omega \in S^{\prime}(\mathbb{R})
$$

justifies the name stochastic derivative for the Malliavin operator. Since generalized functions do not have point values, this notation would be somewhat misleading for $u \in \operatorname{Dom}_{-}(\mathbb{D})$. Therefore, for notational uniformity, we omit the index $t$ in $\mathbb{D}_{t}$ that usually appears in the literature and write $\mathbb{D}$.

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The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $(L)^{2}$-sense. In [18] we have extended the definition of the Skorokhod integral from Hilbert space valued processes to the class of $S^{\prime}$-valued generalized processes.

Definition 2.21. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$, be a generalized $S^{\prime}(\mathbb{R})$-valued stochastic process and let $f_{\alpha} \in X \otimes S^{\prime}(\mathbb{R})$ be given by the expansion $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k}, f_{\alpha, k} \in X$. Then the process $F$ is integrable in the Skorokhod sense and the chaos expansion of its stochastic integral is given by

$$
\begin{equation*}
\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}} \tag{2.9}
\end{equation*}
$$

In [20] we proved that the domain $\operatorname{Dom}_{-}(\delta)$ of the Skorokhod integral is

$$
\begin{aligned}
\operatorname{Dom}_{-}(\delta) & =X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1} \\
& =\bigcup_{(l, p) \in \mathbb{N}^{2}} \operatorname{Dom}_{(-l,-p)}(\delta)=\bigcup_{(l, p) \in \mathbb{N}^{2}}\left(X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}\right) .
\end{aligned}
$$

In [24] we characterized the domains $\operatorname{Dom}_{+}(\delta)$ and $\operatorname{Dom}_{0}(\delta)$ of the Skorokhod integral for test processes from $X \otimes S(\mathbb{R}) \otimes(S)_{1}$ and square integrable processes from $X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$. The form of the derivative is in all cases given by the expression (2.9).

The domain $\operatorname{Dom}_{+}(\delta)$ of the Skorokhod integral is

$$
\operatorname{Dom}_{+}(\delta)=\bigcap_{(l, p) \in \mathbb{N}^{2}} \operatorname{Dom}_{(l, p)}(\delta)
$$

$\operatorname{Dom}_{(l, p)}(\delta)=\left\{F \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}: \sum_{\alpha \in \mathcal{I} k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2} \alpha!^{2}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha}<\infty\right\}$.
For square integrable stochastic processes $T \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$ of the form $T=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} t_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}, t_{\alpha, k} \in X$, the classical definition is:
$\operatorname{Dom}_{0}(\delta)=\left\{T \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}: \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \alpha!\left\|t_{\alpha, k}\right\|_{X}^{2}<\infty\right\}$.
Theorem 2.22. ([18, 24])
a) The Skorokhod integral $\delta$ of an $S_{-l}(\mathbb{R})$-valued generalized stochastic process is a linear and continuous mapping

$$
\delta: X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p} \rightarrow X \otimes(S)_{-1,-q}, \quad q \geq p, q>l+1, l \in \mathbb{N}
$$

b) The Skorokhod integral $\delta$ of a $S_{l}(\mathbb{R})$-valued stochastic test process is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{(l, p)}(\delta) \rightarrow X \otimes(S)_{1, q}, \quad q \leq \min \{l, p\}
$$

for all $l, p \in \mathbb{N}$.
c) The Skorokhod integral $\delta$ of an $L^{2}(\mathbb{R})$-valued stochastic process is a linear and continuous mapping

$$
\delta: \quad \operatorname{Dom}_{0}(\delta) \rightarrow X \otimes(L)^{2} .
$$

Proof. a) Let $F$ be as in Definition 2.21. Clearly,

$$
\begin{aligned}
\|\delta(F)\|_{X \otimes(S)_{-1,-q}}^{2} & =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} f_{\alpha, k}\right\|_{X}^{2}(2 \mathbb{N})^{-q\left(\alpha+\varepsilon^{(k)}\right)} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} f_{\alpha, k}(2 k)^{-\frac{q}{2}}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left|f_{\alpha, k}\right|(2 k)^{-\frac{l}{2}}(2 k)^{-\frac{(q-l)}{2}}\right)^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left|f_{\alpha, k}\right|^{2}(2 k)^{-l} \sum_{k \in \mathbb{N}}(2 k)^{-(q-l)}\right)(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{-l}^{2}(2 \mathbb{N})^{-p \alpha} \cdot \sum_{k \in \mathbb{N}}(2 k)^{-(q-l)} \\
& \leq M\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}^{2}}^{2}<\infty,
\end{aligned}
$$

for $q \geq p$, where we used the Cauchy-Schwarz inequality and the fact that $M=\sum_{k \in \mathbb{N}}(2 k)^{-(q-l)}<\infty$, for $q>l+1$.
b) Let $U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}, u_{\alpha}=\sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \in$ $X \otimes S_{l}(\mathbb{R}), u_{\alpha, k} \in X$, for $p, l \geq 1$. Then we obtain

$$
\begin{aligned}
\|\delta(U)\|_{X \otimes(S)_{1, q}}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|u_{\alpha, k}\right\|_{X}^{2}\left(\alpha+\varepsilon^{(k)}\right)!^{2} \|(2 \mathbb{N})^{q\left(\alpha+\varepsilon^{(k)}\right)} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|u_{\alpha, k}\right\|_{X}^{2} \alpha!^{2}\left(\alpha_{k}+1\right)^{2}(2 k)^{q}(2 \mathbb{N})^{q \alpha} \\
& \leq\|U\|_{D_{o m}^{(l, p)}(\delta)}^{2}<\infty,
\end{aligned}
$$

for $q \leq p, q \leq l$.
c) Let $T=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} t_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\delta)$. Then,

$$
\begin{aligned}
\|\delta(T)\|_{X \otimes(L)^{2}}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|t_{\alpha, k}^{2}\right\|_{X}^{2}\left(\alpha+\varepsilon^{(k)}\right)! \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|t_{\alpha, k}\right\|_{X}^{2}\left(\alpha_{k}+1\right) \alpha!<\infty
\end{aligned}
$$

where we used $\left(\alpha+\varepsilon^{(k)}\right)!=\left(\alpha_{k}+1\right) \alpha!$, for $\alpha \in \mathcal{I}, k \in \mathbb{N}$.

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Using the estimate $\alpha_{k}+1 \leq 2|\alpha|$, which holds for all $\alpha \in \mathcal{I}$ except for $\alpha=\mathbf{0}$, we obtain

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha!^{2}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2} \alpha!^{2}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha} \\
& \leq \sum_{k \in \mathbb{N}}\left\|f_{0, k}\right\|_{X}^{2}(2 k)^{l}+4 \sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}}|\alpha|^{2} \alpha!^{2}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha} \\
& \leq\left\|f_{\mathbf{0}}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}+4 \sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}} \alpha!^{2}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{(p+2) \alpha} \\
& \leq 4\|F\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p+2}}^{2}
\end{aligned}
$$

Thus,

$$
X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p+2} \subseteq \operatorname{Dom}_{(l, p)}(\delta) \subseteq X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}, \quad p \in \mathbb{N}
$$

The third main operator of the Malliavin calculus is the Ornstein-Uhlenbeck operator. We describe the domain of the Ornstain-Uhlenbeck operator for different classes of generalized stochastic processes.
Definition 2.23. The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathcal{R}=\delta \circ \mathbb{D}$ and called the Ornstein-Uhlenbeck operator.

Therefore, for a generalized process $u \in X \otimes(S)_{-1}$ given in the chaos expansion form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$, the Ornstein-Uhlenbeck operator is given by

$$
\begin{equation*}
\mathcal{R}(u)=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} . \tag{2.10}
\end{equation*}
$$

Let

$$
\begin{aligned}
\operatorname{Dom}_{-}(\mathcal{R}) & =\bigcup_{p \in \mathbb{N}} \operatorname{Dom}_{-p}(\mathcal{R}) \\
& =\bigcup_{p \in \mathbb{N}}\left\{u \in X \otimes(S)_{-1}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
\end{aligned}
$$

For test processes, we define

$$
\begin{aligned}
\operatorname{Dom}_{+}(\mathcal{R}) & =\bigcap_{p \in \mathbb{N}} \operatorname{Dom}_{p}(\mathcal{R}) \\
& =\bigcap_{p \in \mathbb{N}}\left\{v \in X \otimes(S)_{1}: \sum_{\alpha \in \mathcal{I}} \alpha!^{2}|\alpha|^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} .
\end{aligned}
$$

For square integrable processes the classical definition is:

$$
\operatorname{Dom}_{0}(\mathcal{R})=\left\{w \in X \otimes(L)^{2}: \sum_{\alpha \in \mathcal{I}} \alpha!|\alpha|^{2}\left\|w_{\alpha}\right\|_{X}^{2}<\infty\right\} .
$$

Theorem 2.24. ([22, 24])
a) The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \operatorname{Dom}_{-p}(\mathcal{R}) \rightarrow X \otimes(S)_{-1,-p}, p \in \mathbb{N}
$$

In this case the domains of $\mathbb{D}$ and $\mathcal{R}$ coincide, i.e. $\operatorname{Dom}_{-}(\mathcal{R})=\operatorname{Dom}_{-}(\mathbb{D})$.
b) The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \quad \operatorname{Dom}_{p}(\mathcal{R}) \rightarrow X \otimes(S)_{1, p}, \quad p \in \mathbb{N} .
$$

In this case the domains of the operators $\mathbb{D}$ and $\mathcal{R}$ do not coincide, i.e. $\operatorname{Dom}_{+}(\mathbb{D}) \supsetneq \operatorname{Dom}_{+}(\mathcal{R})$.
c) The operator $\mathcal{R}$ is a linear and continuous operator

$$
\mathcal{R}: \quad \operatorname{Dom}_{0}(\mathcal{R}) \rightarrow X \otimes(L)^{2}
$$

In this case the domains of the operators $\mathbb{D}$ and $\mathcal{R}$ also do not coincide and $\operatorname{Dom}_{0}(\mathbb{D}) \supsetneq \operatorname{Dom}_{0}(\mathcal{R})$.

Proof. a) Let $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-1,-p}$, for some $p \in \mathbb{N}$. Clearly,

$$
\|\mathcal{R} u\|_{X \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}|\alpha|^{2}(2 \mathbb{N})^{-p \alpha}=\|u\|_{D^{-p} m_{-p}(\mathcal{R})}^{2}<\infty .
$$

b) Let a stochastic process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{1, p}$, for all $p \in \mathbb{N}$, i.e. $\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{2}(2 \mathbb{N})^{p \alpha}<\infty$, for all $p \in \mathbb{N}$. Then,

$$
\|\mathcal{R} v\|_{X \otimes(S)_{1, p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{X}^{2}|\alpha|^{2} \alpha!^{2}(2 \mathbb{N})^{p \alpha}=\|v\|_{D o m_{p}(\mathcal{R})}^{2}<\infty,
$$

and the statement follows.
c) Let $w=\sum_{\alpha \in \mathcal{I}} w_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\mathcal{R})$. Then $\mathcal{R}(w)=\sum_{\alpha \in \mathcal{I}}|\alpha| w_{\alpha} \otimes H_{\alpha}$ and

$$
\|\mathcal{R}(w)\|_{X \otimes(L)^{2}}^{2}=\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|w_{\alpha}\right\|_{X}^{2}<\infty
$$

by the assumption $w \in \operatorname{Dom}_{0}(\mathcal{R})$.
Note also that

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in \mathcal{I}} \alpha!^{2}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{(p+2) \alpha}, \\
\text { i.e., } X \otimes(S)_{1, p+2} \subseteq \operatorname{Dom}_{p}(\mathcal{R}) \subseteq X \otimes(S)_{1, p}, \quad p \in \mathbb{N} .
\end{gathered}
$$

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Remark 2.25. Note that $\mathbb{D}: \mathcal{H}_{k} \rightarrow \mathcal{H}_{k-1}$ reduces the Wiener chaos space order and therefore Malliavin differentiation corresponds to the annihilation operator, while $\delta: \mathcal{H}_{k} \rightarrow \mathcal{H}_{k+1}$ increases the chaos order and thus Skorokhod integration corresponds to the creation operator. Clearly, $\mathcal{R}: \mathcal{H}_{k} \rightarrow \mathcal{H}_{k}$ and the OrnsteinUhlenbeck operator corresponds to the number operator in quantum theory.

In the following sections we prove that the mappings $\mathbb{D}: \operatorname{Dom}_{ \pm}(\mathbb{D}) \rightarrow$ $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{ \pm 1}, \delta: \operatorname{Dom}_{ \pm}(\delta) \rightarrow X \otimes(S)_{ \pm 1}, \mathcal{R}: \operatorname{Dom}_{ \pm}(\mathcal{R}) \rightarrow X \otimes(S)_{ \pm 1}$, given in this section are surjective.

## 3. The Ornstein-Uhlenbeck operator

Theorem 3.1. ([20, 24]) Let $g$ have zero generalized expectation. The equation

$$
\mathcal{R} u=g, \quad E u=\tilde{u}_{0} \in X,
$$

has a unique solution $u$ represented in the form

$$
u=\tilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{g_{\alpha}}{|\alpha|} \otimes H_{\alpha} .
$$

Moreover, the following holds:

1. If $g \in X \otimes(S)_{-1,-p}, p \in \mathbb{N}$, then $u \in \operatorname{Dom}_{-p}(\mathcal{R})$.
2. If $g \in X \otimes(S)_{1, p}, p \in \mathbb{N}$, then $u \in \operatorname{Dom}_{p}(\mathcal{R})$.
3. If $g \in X \otimes(L)^{2}$, then $u \in \operatorname{Dom}_{0}(\mathcal{R})$.

Proof. Let us seek for a solution in form of $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$. From $\mathcal{R} u=g$ it follows that

$$
\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha}
$$

i.e., $u_{\alpha}=\frac{g_{\alpha}}{|\alpha|}$ for all $\alpha \in \mathcal{I},|\alpha|>0$. From the initial condition we obtain $u_{(0,0,0,0, \ldots)}=E u=\tilde{u}_{0}$.

1. Assume that $g \in X \otimes(S)_{-1,-p}$. Then, $u \in \operatorname{Dom}_{-p}(\mathcal{R})$ since

$$
\begin{aligned}
\|u\|_{D o m_{-p}(\mathcal{R})}^{2} & =\sum_{|\alpha|>0}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{|\alpha|>0}\left\|g_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =\|g\|_{X \otimes(S)_{-1,-p}}^{2}<\infty
\end{aligned}
$$

2. In this case $u \in \operatorname{Dom}_{p}(\mathcal{R})$ since

$$
\begin{aligned}
\|u\|_{D_{o m}^{p}(\mathcal{R})}^{2} & =\sum_{|\alpha|>0} \alpha!^{2}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{|\alpha|>0} \alpha!^{2}\left\|g_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\|f\|_{X \otimes(S)_{1, p}}^{2}<\infty .
\end{aligned}
$$

3. If $g$ is square integrable, then $u \in \operatorname{Dom}_{0}(\mathcal{R})$ since

$$
\|u\|_{D o m_{0}(\mathcal{R})}^{2}=\sum_{|\alpha|>0}|\alpha|^{2} \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{|\alpha|>0} \alpha!\left\|h_{\alpha}\right\|_{X}^{2}=\|h\|_{X \otimes(L)^{2}}^{2}<\infty .
$$

Remark 3.2. Note that $\mathcal{R} u=u$ if and only if $u \in \mathcal{H}_{1}$ i.e. Gaussian processes with zero expectation are the only fixed points for the Ornstein-Uhlenbeck operator. For example, $\mathcal{R}\left(B_{t}\right)=B_{t}$ and $\mathcal{R}\left(W_{t}\right)=W_{t}$.

Also, it is clear that $\mathcal{H}_{m}$ is the eigenspace corresponding to the eigenvalue $m(m \in \mathbb{N})$ of the Ornstein-Uhlenbeck operator.
Remark 3.3. If $E u=0$, one can define the pseudo-inverse $\mathcal{R}^{-1}$ as in [32, 35], given by

$$
\mathcal{R}^{-1} u=\mathcal{R}^{-1}\left(\sum_{\alpha \in \mathcal{I},|\alpha|>0} u_{\alpha} \otimes H_{\alpha}\right)=\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{u_{\alpha}}{|\alpha|} \otimes H_{\alpha} .
$$

Thus,

$$
\begin{equation*}
\mathcal{R} \mathcal{R}^{-1}(u)=u \quad \text { and } \quad \mathcal{R}^{-1} \mathcal{R}(u)=u \tag{3.1}
\end{equation*}
$$

In the general case, for $E u \neq 0$, we have

$$
\mathcal{R} \mathcal{R}^{-1}(u-E u)=u \quad \text { and } \quad \mathcal{R}^{-1} \mathcal{R}(u)=u
$$

Corollary 3.4. Each process $g \in X \otimes(S)_{ \pm 1}$, resp. $g \in X \otimes(L)^{2}$, can be represented as

$$
g=E g+\mathcal{R}(u)
$$

for some $u \in \operatorname{Dom}_{ \pm}(\mathcal{R})$, resp. $u \in \operatorname{Dom}_{0}(\mathcal{R})$.
Proof. The assertion follows for $u=\mathcal{R}^{-1}(g-E g)$.
Remark 3.5. We note that if a stochastic process $f$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathcal{H}_{i}$ for some $m \in \mathbb{N}$, then the solution $u$ of the equation $\mathcal{R} u=f$ belongs also to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathcal{H}_{i}$.

## 4. The Malliavin derivative

Theorem 4.1. ([20, 24]) Let a process $h$ have the chaos expansion representation $h=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$. Then the equation

$$
\left\{\begin{array}{l}
\mathbb{D} u=h,  \tag{4.1}\\
E u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X,
\end{array}\right.
$$

has a unique solution $u$ represented in the form

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} . \tag{4.2}
\end{equation*}
$$

Moreover, the following holds:

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1. If $h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-q}, p, q \in \mathbb{N}$, then $u \in \operatorname{Dom}_{-q}(\mathbb{D})$.
2. If $h \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$, then $u \in \operatorname{Dom}_{0}(\mathbb{D})$.
3. If $h \in X \otimes S_{p}(\mathbb{R}) \otimes(S)_{1, q}, p, q \in \mathbb{N}$, then $u \in \operatorname{Dom}_{q}(\mathbb{D})$.

Proof. 1. Applying the Skorokhod integral on both sides of (4.1) one obtains

$$
\mathcal{R} u=\delta(h)
$$

for a given $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}=\operatorname{Dom}_{-}(\delta)$. From the initial condition it follows that the solution $u$ is given in the form $u=\widetilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} u_{\alpha} \otimes H_{\alpha}$ and its coefficients are obtained from the system

$$
\begin{equation*}
|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}, \quad|\alpha|>0 \tag{4.3}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$. Hence, the solution $u$ is given in the form (4.2). Now, we prove that the solution $u$ belongs to the space $\operatorname{Dom}_{-q}(\mathbb{D})$. Clearly,

$$
\begin{aligned}
\|u\|_{D o m_{-q}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \\
& =\sum_{\alpha \in \mathcal{I},|\alpha|>0}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k), k}}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha, k}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}(2 \mathbb{N})^{-q \varepsilon^{(k)}} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left\|h_{\alpha, k}\right\|_{X}(2 k)^{-\frac{p}{2}}(2 k)^{-\frac{(q-p)}{2}}\right)^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|h_{\alpha, k}\right\|_{X}^{2}(2 k)^{-p}(2 \mathbb{N})^{-q \alpha} \sum_{k \in \mathbb{N}}(2 k)^{-(q-p)} \\
& =C\|h\|_{X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-q}}^{2}<\infty
\end{aligned}
$$

since $C=\sum_{k \in \mathbb{N}}(2 k)^{-(q-p)}<\infty$, for $q>p+1$.
2. In this case we have that

$$
\begin{aligned}
\|u\|_{\operatorname{Dom}_{0}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{\alpha!}{|\alpha|}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha, k}\right\|_{X}^{2} \frac{\left(\alpha+\varepsilon^{(k)}\right)!}{\left|\alpha+\varepsilon^{(k)}\right|} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|h_{\alpha, k}\right\|_{X}^{2} \alpha! \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!\left\|f_{\alpha}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2}=\|f\|_{X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}}^{2}<\infty .
\end{aligned}
$$

We have made use of the fact $\frac{\left(\alpha+\varepsilon^{(k)}\right)!}{\left|\alpha+\varepsilon^{(k)}\right|} \leq \alpha!$.
3. Clearly, $\delta$ can again be applied onto $h$, since $h \in X \otimes S_{p}(\mathbb{R}) \otimes(S)_{1, q} \subseteq$ $\operatorname{Dom}_{(p, q-2)}(\delta)$. It remains to prove that the solution $u$ given in the form (4.2) belongs to $\operatorname{Dom}_{q}(\mathbb{D})$. Clearly,

$$
\begin{aligned}
\|u\|_{D o m_{q}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{q \alpha} \\
& =\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{\alpha!^{2}}{|\alpha|^{2}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2}(2 \mathbb{N})^{q \alpha} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha, k}\right\|_{X}^{2} \frac{\left(\alpha+\varepsilon^{(k)}\right)!^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}(2 \mathbb{N})^{q \alpha}(2 \mathbb{N})^{q \varepsilon^{(k)}} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha, k}\right\|_{X}^{2} \alpha!^{2}(2 \mathbb{N})^{q \alpha}(2 k)^{q} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k=1}^{\infty} h_{\alpha, k}(2 k)^{\frac{q}{2}}\right\|_{X}^{2} \alpha!^{2}(2 \mathbb{N})^{q \alpha} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k=1}^{\infty} h_{\alpha, k}(2 k)^{\frac{p}{2}}(2 k)^{\frac{q-p}{2}}\right\|_{X}^{2} \alpha!^{2}(2 \mathbb{N})^{q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty}\left\|h_{\alpha, k}\right\|_{X}^{2}(2 k)^{p} \sum_{k=1}^{\infty}(2 k)^{q-p} \alpha!^{2}(2 \mathbb{N})^{q \alpha} \\
& =C \cdot\|h\|_{X \otimes S_{p}(\mathbb{R}) \otimes(S)_{1, q}^{2}<\infty}^{2},
\end{aligned}
$$

since $C=\sum_{k \in \mathbb{N}}(2 k)^{q-p}<\infty$, for $p>q+1$. In the fourth step of the estimation we used again that $\frac{\left(\alpha+\varepsilon^{(k)}\right)!}{\left|\alpha+\varepsilon^{(k)}\right|} \leq \alpha$ !.

Corollary 4.2. If $\mathbb{D}(u)=0$, then $u=E u$ i.e. $u$ is constant almost surely.
In other words, the kernel of the operator $\mathbb{D}$ is $\mathcal{H}_{0}$.
Corollary 4.3. For every $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{ \pm 1}$, resp. $h \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$, there exists a unique $u \in \operatorname{Dom}_{ \pm}(\mathbb{D})$, resp. $u \in \operatorname{Dom}_{0}(\mathbb{D})$, such that $E u=0$ and $h=\mathbb{D}(u)$ holds.

Proof. The assertion follows for $u=\mathcal{R}^{-1}(\delta(h))$.
Example 4.4. Let $t \geq 0$. Consider now the following examples which illustrate the results of Theorem 4.1.

1. Denote by $\kappa_{\left[0, t_{0}\right]}=\left\{\begin{array}{ll}1, & t \in\left[0, t_{0}\right] \\ 0, & t \notin\left[0, t_{0}\right]\end{array}\right.$ the characteristic function of the interval $\left[0, t_{0}\right]$. It is an element of $L^{2}(\mathbb{R})$ and thus its expansion representation is $\kappa_{\left[0, t_{0}\right]}(t)=\sum_{k=1}^{\infty}\left(\int_{0}^{t_{0}} \xi_{k}(t) d t\right) \xi_{k}(t)$. Consider the initial value

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$$
\begin{equation*}
\mathbb{D} u=\kappa_{\left[0, t_{0}\right]}(t), \quad E u=\widetilde{u}_{0} \tag{4.4}
\end{equation*}
$$

Recall that $H_{(0,0,0, \ldots)}=1$ and then we may regard $h$ in (4.1) as $h=\kappa_{\left[0, t_{0}\right]}(t) \in L^{2}(\mathbb{R}) \otimes \mathcal{H}_{0}$. Therefore, $u \in L^{2}(\mathbb{R}) \otimes\left(\mathcal{H}_{0} \oplus \mathcal{H}_{1}\right)$. From (4.3) we obtain the form of the coefficients of the solution $u_{\varepsilon(k)}=h_{0, k}=\int_{0}^{t_{0}} \xi_{k}(t) d t$. Then the solution of the equation (4.4) is of the form

$$
u\left(t_{0}, \omega\right)=\widetilde{u}_{0}+\sum_{k=1}^{\infty} \int_{0}^{t_{0}} \xi_{k}(t) d t H_{\varepsilon^{(k)}}(\omega)=\widetilde{u}_{0}+B_{t_{0}}(\omega)
$$

i.e. it is Brownian motion with drift parameter $\tilde{u}_{0}$.
2. Consider the equation

$$
\begin{equation*}
\mathbb{D} u=d_{t_{0}}(t), \quad E u=\widetilde{u}_{0} \tag{4.5}
\end{equation*}
$$

where $d_{t_{0}}(t)$ denotes the Dirac delta function concentrated at $t_{0}$, represented in the chaos expansion form

$$
d_{t_{0}}(t)=\sum_{k=1}^{\infty} \xi_{k}\left(t_{0}\right) \xi_{k}(t)=\sum_{k=1}^{\infty} \xi_{k}\left(t_{0}\right) \xi_{k}(t) H_{0}(\omega)
$$

The solution to (4.5) belongs to the space $S^{\prime}(\mathbb{R}) \otimes\left(\mathcal{H}_{0} \oplus \mathcal{H}_{1}\right)$ because $d_{t_{0}}(t) \in S^{\prime}(\mathbb{R}) \otimes \mathcal{H}_{0}$. The chaos expansion form of the solution is given by

$$
u=\widetilde{u}_{0}+\sum_{k=1}^{\infty} \xi_{k}\left(t_{0}\right) H_{\varepsilon^{(k)}}(\omega)=\widetilde{u}_{0}+W_{t_{0}}(\omega)
$$

i.e. it represents singular white noise.
3. Consider now an equation with singular white noise

$$
\mathbb{D} u=W_{t}(\omega), \quad E u=0
$$

$W_{t}$ belongs to the Wiener chaos space of order one and (since we assumed $E u=0$ ) the solution $u$ will belong to the Wiener chaos space of order two. From $W_{t}=\sum_{k=1}^{\infty} \xi_{k} H_{\varepsilon^{(k)}}$ it follows that $h_{\alpha, k}=1$ only for $\alpha=\varepsilon^{(k)}$ and $h_{\alpha, k}=0$ for all $\alpha \neq \varepsilon^{(k)}$. Thus, $h_{\alpha-\varepsilon^{(k)}, k}=h_{\varepsilon^{(k)}, k}=1$ only for $\alpha=2 \varepsilon^{(k)}$ and is equal to zero for all other $\alpha \in \mathcal{I}$. Thus, with $|\alpha|=2$ we obtain $u_{\alpha}$ from (4.3), and the form of the solution is

$$
u(\omega)=\frac{1}{2} \sum_{k=1}^{\infty} H_{2 \varepsilon^{(k)}}(\omega)
$$

4. Consider the equation

$$
\mathbb{D} u=B_{t_{0}}(\omega) \kappa_{\left[0, t_{0}\right]}(t), \quad E u=0
$$

The chaos expansion of the right hand side is

$$
\begin{aligned}
B_{t_{0}}(\omega) \kappa_{\left[0, t_{0}\right]}(t) & =\sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(\int_{0}^{t_{0}}\left(\int_{0}^{t_{0}} \xi_{k}(s) d s\right) \xi_{j}(t) d t\right) \xi_{j}(t) H_{\varepsilon^{(k)}}(\omega) \\
& =\sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(\int_{0}^{t_{0}} \xi_{k}(s) d s\right)\left(\int_{0}^{t_{0}} \xi_{j}(s) d s\right) \xi_{j}(t) H_{\varepsilon^{(k)}}(\omega)
\end{aligned}
$$

This implies $h_{\varepsilon^{(k), j}}=\frac{1}{2}\left(\int_{0}^{t_{0}} \xi_{k}(s) d s\right)\left(\int_{0}^{t_{0}} \xi_{j}(s) d s\right)$. Again, $h_{\alpha-\varepsilon^{(l), l}}$ is nonzero only for $\alpha$ of the form $\alpha=\varepsilon^{(l)}+\varepsilon^{(k)}$ and in this case we have with $|\alpha|=2$ that

$$
u_{\varepsilon^{(k)}, l}=\frac{1}{2} h_{\varepsilon^{(k)}, l}=\frac{1}{2}\left(\int_{0}^{t_{0}} \xi_{k}(s) d s\right)\left(\int_{0}^{t_{0}} \xi_{l}(s) d s\right) .
$$

Thus, the solution belongs to the space $L^{2}(\mathbb{R}) \otimes \mathcal{H}_{2}$ and is of the form

$$
u=\frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty}\left(\int_{0}^{t_{0}} \xi_{k}(t) d t\right)\left(\int_{0}^{t_{0}} \xi_{l}(s) d s\right) H_{\varepsilon^{(k)}+\varepsilon^{(l)}}(\omega) .
$$

Note that the solution can be represented in terms of the Wick product

$$
u=\frac{1}{2} B_{t_{0}}(\omega)^{\diamond 2} .
$$

5. Consider now the equation

$$
\mathbb{D}(u)=B_{t_{1}} \kappa_{\left[0, t_{2}\right]}(t), \quad E u=0
$$

Similarly as in the previous case it can be shown by symmetry of $t_{1}$ and $t_{2}$ that it is equivalent to the equation

$$
\mathbb{D}(u)=B_{t_{2}} \kappa_{\left[0, t_{1}\right]}(t), \quad E u=0
$$

and that both equations have the solution

$$
u=\frac{1}{2} B_{t_{1}} \diamond B_{t_{2}}=\frac{1}{2}\left(B_{t_{1}} B_{t_{2}}-\min \left\{t_{1}, t_{2}\right\}\right) .
$$

6. Similarly as in the previous cases, $u=\frac{1}{2} W_{t_{1}} \diamond W_{t_{2}}$ solves the equation

$$
\mathbb{D} u=W_{t_{1}}(\omega) d_{t_{2}}(t)=W_{t_{2}}(\omega) \delta_{t_{1}}(t)
$$

while $u=\frac{1}{2} W_{t_{0}}(\omega)^{\diamond 2}$ is the solution to the equation

$$
\mathbb{D} u=W_{t_{0}}(\omega) d_{t_{0}}(t)
$$

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Remark 4.5. If a stochastic process $h$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathcal{H}_{i}$ for some $m \in \mathbb{N}$, then the unique solution $u$ of the equation (4.1) belongs to the Wiener chaos space $\bigoplus_{i=0}^{m+1} \mathcal{H}_{i}$. In particular, if the input function $h$ is a constant random variable i.e. an element of $\mathcal{H}_{0}$, then the solution $u$ to (4.1) is a Gaussian process.

Theorem 4.6. ([20]) Let $h \in X \otimes(S)_{-1}$ and $W_{t}$, $B_{t}$ denote white noise and Brownian motion, respectively. Then,

$$
h \cdot W_{t}-h \diamond W_{t}=\mathbb{D}(h),
$$

i.e. $\frac{d}{d t}\left(h \cdot B_{t}-h \diamond B_{t}\right)=\mathbb{D}(h)$ in the weak $S^{\prime}(\mathbb{R})$-sense.

Proof. Let $h$ be of the form $h=\sum_{\alpha \in \mathcal{I}} h_{\alpha} H_{\alpha}$ and $W_{t}=\sum_{n=1}^{\infty} \xi_{n}(t) H_{\varepsilon^{(n)}}$. Then,

$$
h \diamond W_{t}=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\varepsilon^{(n)}=\gamma} h_{\alpha} \xi_{n}(t) H_{\gamma}=\sum_{\gamma \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\gamma-\varepsilon^{(n)}} \xi_{n}(t) H_{\gamma}
$$

and

$$
h \cdot W_{t}=\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_{n}(t) H_{\alpha-\varepsilon^{(n)}} H_{\varepsilon^{(n)}} .
$$

Now, applying the well-known formula (2.4) for Hermite polynomials one obtains

$$
H_{\alpha-\varepsilon^{(n)}} \cdot H_{\varepsilon^{(n)}}=H_{\alpha}+\left(\alpha-\varepsilon^{(n)}\right)_{n} H_{\alpha-2 \varepsilon^{(n)}},
$$

where we used $\left(\underset{\varepsilon^{(k)}}{\alpha}\right)=\alpha_{k}, k \in \mathbb{N}$. Hence,

$$
h \cdot W_{t}=\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_{n}(t)\left(H_{\alpha}+\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}}\right),
$$

which implies

$$
\begin{aligned}
h \cdot W_{t}-h \diamond W_{t} & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_{n}(t)\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} \xi_{n}(t)\left(\alpha_{n}+1\right) H_{\alpha} \\
& =\mathbb{D}(h),
\end{aligned}
$$

using (2.7). Thus the assertion follows.

Remark 4.7. Note that if $h \in X \otimes(S)_{-1,-p}$, then $\mathbb{D}(h) \in X \otimes S_{-l}(\mathbb{R}) \otimes$ $(S)_{-1,-(p+2)}, l>p+1$. Thus, apart from the Wick product $h \diamond W_{t}$ being welldefined, the ordinary product is also well-defined in the generalized sense as an element of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$, and it is given by $h \cdot W_{t}=h \diamond W_{t}+\mathbb{D}(h)$.

Example 4.8. Let $X=S^{\prime}(\mathbb{R})$ and $h=W_{t_{0}}$. Then

$$
\begin{equation*}
W_{t_{0}} \cdot W_{t}=W_{t_{0}} \diamond W_{t}+\mathbb{D}\left(W_{t_{0}}\right)=W_{t_{0}} \diamond W_{t}+d_{t_{0}}(t) \tag{4.6}
\end{equation*}
$$

holds in $S^{\prime}(\mathbb{R}) \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. Note that (4.6) is well defined for all $\left(t, t_{0}\right) \in \mathbb{R}^{2}$, except for $t=t_{0}$, where the Dirac delta distribution $d_{t_{0}}(t)=d\left(t-t_{0}\right) \in$ $S^{\prime}(\mathbb{R}) \otimes S^{\prime}(\mathbb{R})$ has its singularity. It is possible to give meaning to $d_{t_{0}}\left(t_{0}\right)=$ $\sum_{n=1}^{\infty} \xi_{n}\left(t_{0}\right)^{2}$ as the point value of a distribution in the sense of Colombeau generalized numbers. Thus, in Colombeau sense, it will be possible to define $W_{t}^{2}=W_{t}^{\diamond 2}+d_{t}(t)$. For the Colombeau theory, we refer the reader to $[5,11]$.

The previous theorem states that the Malliavin derivative indicates the speed of change between the ordinary product and the Wick product.

A generalization of Theorem 4.6 can be obtained by replacing white noise with an arbitrary process of first chaos order, i.e. considering $f \in \mathcal{H}_{1}$ and comparing the difference between $h \cdot f$ and $h \diamond f$. This will be done in Theorem 5.10 in the next section.

Remark 4.9. Note that if a stochastic process $h$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathcal{H}_{i}$ for some $m \in \mathbb{N}$, then the unique solution $u$ of the equation $\mathbb{D}(u)=h$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m+1} \mathcal{H}_{i}$.
Remark 4.10. It is easy to check that if $\psi \in S_{-l}(\mathbb{R})$ is given by $\psi=\sum_{i=1}^{\infty} \psi_{i} \xi_{i}$, then $\delta(\psi) \in(S)_{-1,-l}$ and it is given by $\delta(\psi)=\sum_{i=1}^{\infty} \psi_{i} H_{\varepsilon^{(i)}}$. Moreover, one can define the Wick version of the stochastic exponential:

$$
\exp ^{\diamond} \delta(\psi)=\sum_{k=0}^{\infty} \frac{\delta(\psi)^{\diamond k}}{k!}=\sum_{\alpha \in \mathcal{I}} \frac{\psi^{\alpha}}{\alpha!} H_{\alpha}, \quad \text { where } \quad \psi^{\alpha}=\prod_{i=1}^{\infty} \psi_{i}^{\alpha_{i}} .
$$

In [18] we have proven that the stochastic exponentials are eigenvectors of the Malliavin derivative corresponding to the eigenvalue $\psi$, i.e. the process $u=\tilde{u}_{0} \otimes \exp ^{\diamond} \delta(\psi) \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-l}$ is the unique solution to the equation

$$
\left\{\begin{array}{cl}
\mathbb{D} u=\psi \otimes u, & \psi \in S^{\prime}(\mathbb{R}) \\
E u=\widetilde{u}_{0}, & \widetilde{u}_{0} \in X
\end{array}\right.
$$

## 5. The Skorokhod integral

Theorem 5.1. ([20,24]) Let $f$ be a process with zero expectation and chaos expansion representation of the form $f=\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}, f_{\alpha} \in X$. Then the integral equation

$$
\begin{equation*}
\delta(u)=f \tag{5.1}
\end{equation*}
$$

has a unique solution $u$ given by

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} . \tag{5.2}
\end{equation*}
$$

Moreover, the following holds:

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1. If $f \in X \otimes(S)_{-1,-p}, p \in \mathbb{N}$, then $u \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}$, for $l>p+1$.
2. If $f \in X \otimes(S)_{1, p}, p \in \mathbb{N}$, then $u \in \operatorname{Dom}_{(l, p)}(\delta)$, for $l<p-1$.
3. If $f \in X \otimes(L)^{2}$, then $u \in \operatorname{Dom}_{0}(\delta)$.

Proof. 1. We seek for the solution in Range_( $\mathbb{D}$ ). It is clear that $u \in$ Range $_{-}(\mathbb{D})$ is equivalent to $u=\mathbb{D}(\widetilde{u})$, for some $\widetilde{u}$. This approach is general enough, since according to Theorem 4.1, for all $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ there exists $\widetilde{u} \in$ Dom_ $_{-}(\mathbb{D})$ such that $u=\mathbb{D}(\widetilde{u})$ holds. Thus, equation (5.1) is equivalent to the system of equations

$$
\left\{\begin{aligned}
u & =\mathbb{D}(\widetilde{u}), \\
\mathcal{R}(\widetilde{u}) & =f .
\end{aligned}\right.
$$

The solution to $\mathcal{R}(\widetilde{u})=f$ is given by

$$
\widetilde{u}=\widetilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} \frac{f_{\alpha}}{|\alpha|} \otimes H_{\alpha},
$$

where $\tilde{u}_{(0,0,0, \ldots)}=\widetilde{u}_{0}$ can be chosen arbitrarily. Now, the solution of the initial equation (5.1) is obtained after applying the operator $\mathbb{D}$, i.e.

$$
\begin{aligned}
u=\mathbb{D}(\widetilde{u}) & =\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} \sum_{k \in \mathbb{N}} \alpha_{k} \frac{f_{\alpha}}{|\alpha|} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha}
\end{aligned}
$$

It remains to prove the convergence of the solution (5.2) in $X \otimes S^{\prime}(\mathbb{R}) \otimes$ $(S)_{-1}$. Under the assumption $f \in X \otimes(S)_{-1,-p}$, for some $p>0$, we prove first that $\widetilde{u} \in \operatorname{Dom}_{-p}(\mathbb{D})$. Clearly,

$$
\begin{aligned}
\|\widetilde{u}\|_{D o m_{-p}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|\widetilde{u}_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{\alpha \in \mathcal{I},|\alpha|>0}|\alpha|^{2} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{\alpha \in \mathcal{I},|\alpha|>0}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty .
\end{aligned}
$$

Hence, the convergence of the solution $u$ in the space $X \otimes S_{-l} \otimes(S)_{-1,-p}$, for $l>p+1$ follows from

$$
\begin{aligned}
\|u\|_{X \otimes S_{-l} \otimes(S)_{-1,-p}}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{\left(\alpha_{k}+1\right)^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}\left\|\xi_{k}\right\|_{-l}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I},|\alpha|>0} \sum_{k \in \mathbb{N}}\left\|f_{\alpha}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p\left(\alpha-\varepsilon^{(k)}\right)} \\
& \leq M \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{aligned}
$$

since $M=\sum_{k \in \mathbb{N}}(2 k)^{p-l}$ is finite for $l>p+1$.
2. The form of the solution (5.2) is obtained in a similar way as in the previous case. We prove the convergence of the solution $u$ in the space $\operatorname{Dom}_{(l, p)}(\delta)$. First we prove that $\widetilde{u} \in \operatorname{Dom}_{p}(\mathbb{D})$ and then $u \in \operatorname{Dom}_{(l, p)}(\delta)$ for appropriate $l \in \mathbb{N}$. We obtain

$$
\begin{aligned}
\|\widetilde{u}\|_{D_{o m_{p}(\mathbb{D})}^{2}}^{2} & =\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{\alpha \in \mathcal{I},|\alpha|>0} \alpha!^{2} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 \mathbb{N})^{p \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I},|\alpha|>0} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\|f\|_{X \otimes(S)_{1, p}}^{2}<\infty
\end{aligned}
$$

and thus $\widetilde{u} \in \operatorname{Dom}_{+}(\mathbb{D})$. Now,

$$
\begin{aligned}
\|u\|_{\operatorname{Dom}_{(l, p)}(\delta)}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha!^{2}\left(\alpha_{k}+1\right)^{4} \frac{\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}(2 k)^{l}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{\alpha \in \mathcal{I},|\alpha|>0} \sum_{k \in \mathbb{N}} \alpha!^{2} \alpha_{k}^{2} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 k)^{l}(2 \mathbb{N})^{p\left(\alpha-\varepsilon^{(k)}\right)} \\
& \leq \sum_{\alpha \in \mathcal{I},|\alpha|>0} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \sum_{k \in \mathbb{N}} \alpha_{k}^{2} \frac{1}{|\alpha|^{2}}(2 k)^{l}(2 k)^{-p} \\
& \leq C\|f\|_{X \otimes(S)_{1, p}}^{2}<\infty,
\end{aligned}
$$

since $C=\sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$ for $p>l+1$. In the second step we used that $\left(\alpha-\varepsilon^{(k)}\right)!\alpha_{k}^{2}=\alpha!\alpha_{k}$, and in the fourth step we used $\alpha_{k} \leq|\alpha|$.
3. In this case we have

$$
\begin{aligned}
\|\widetilde{u}\|_{D_{0 o m_{0}}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{\alpha \in \mathcal{I},|\alpha|>0}|\alpha| \alpha!\frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}} \\
& \leq \sum_{\alpha \in \mathcal{I},|\alpha|>0} \alpha!\left\|f_{\alpha}\right\|_{X}^{2}=\|f\|_{X \otimes(L)^{2}}^{2}<\infty
\end{aligned}
$$

and thus $\widetilde{u} \in \operatorname{Dom}_{0}(\mathbb{D})$. Also,

$$
\begin{aligned}
& \|u\|_{D o m_{0}(\delta)}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha!\left(\alpha_{k}+1\right)^{3} \frac{\| f_{\alpha+\varepsilon^{(k)} \|_{X}^{2}}^{\left|\alpha+\varepsilon^{(k)}\right|^{2}}=\sum_{\alpha \in \mathcal{I},|\alpha|>0} \sum_{k \in \mathbb{N}} \alpha!\alpha_{k}^{2} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}}{}=\sum_{\alpha \in \mathcal{I},|\alpha|>0} \alpha!\left(\sum_{k \in \mathbb{N}} \frac{\alpha_{k}^{2}}{|\alpha|^{2}}\right)\left\|f_{\alpha}\right\|_{X}^{2} \leq\|f\|_{X \otimes(L)^{2}}^{2}<\infty \\
& \text { since for }|\alpha|>0 \text { the estimate } \frac{\sum_{k \in \mathbb{N}} \alpha_{k}^{2}}{|\alpha|^{2}} \leq \frac{\left(\sum_{k \in \mathbb{N}} \alpha_{k}\right)^{2}}{|\alpha|^{2}}=1 . \text { holds. }
\end{aligned}
$$

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Remark 5.2. If a stochastic process $f$ belongs to the Wiener chaos space $\bigoplus_{i=1}^{m} \mathcal{H}_{i}$ for some $m \in \mathbb{N}$, then the solution $u$ of the equation (5.1) belongs to the Wiener chaos space $\bigoplus_{i=0}^{m-1} \mathcal{H}_{i}$. Especially, if $f$ is a quadratic Gaussian random process, i.e. an element of $\mathcal{H}_{2}$, then the solution $u$ to (5.1) is a Gaussian process.

Corollary 5.3. Each process $f \in X \otimes(S)_{ \pm 1}$, resp. $f \in X \otimes(L)^{2}$, can be represented as

$$
f=E f+\delta(u)
$$

for some $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{ \pm 1}$, resp. $u \in X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$.
Proof. The assertion follows for $u=\mathbb{D}\left(\mathcal{R}^{-1}(f-E f)\right)$.
Note that the latter result reduces to the celebrated Itô representation theorem (see e.g. [13, 41]) in the case when $f$ is a square integrable adapted process.
Remark 5.4. In [47] a more general formula appears for the $f \in(L)^{2}$ case, which is equivalent to the classical Wiener-Itô chaos expansion. For $f \in(L)^{2}$ there exist $u_{k}, k \in \mathbb{N}$, such that each $u_{k}$ is a square integrable function symmetric in all arguments,

$$
f=E f+\sum_{k=1}^{\infty} \delta^{(k)}\left(u_{k}\right)
$$

and $u_{k}$ are given by

$$
u_{k}=\frac{1}{k!} E\left(\mathbb{D}^{(k)} f\right)
$$

Moreover, if $f$ is given by the chaos expansion $f=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$, then $u_{k}=\sum_{|\alpha|=k} f_{\alpha} \xi^{\hat{\otimes} \alpha}$, where $\xi^{\hat{\otimes} \alpha}=\xi_{1}^{\hat{\otimes} \alpha_{1}} \hat{\otimes} \xi_{2}^{\hat{\otimes} \alpha_{2}} \hat{\otimes} \cdots$ and $\hat{\otimes}$ denotes the symmetric tensor product.

Remark 5.5. Since Gaussian processes play an important role in white noise analysis, we elaborate the explicit form of solutions in special cases for $m=2$ and for $m=3$.

1. First, assuming that the process $f$ has zero expectation and a chaos expansion in the Wiener chaos space of maximal order two, i.e.

$$
f=\sum_{\alpha \in \mathcal{I}, 1 \leq|\alpha| \leq 2} f_{\alpha} \otimes H_{\alpha} \in \mathcal{H}_{1} \oplus \mathcal{H}_{2}, \quad f_{\alpha} \in X
$$

the solution $u$ of the equation (5.1) belongs to the Wiener chaos space of order one $u \in \mathcal{H}_{0} \oplus \mathcal{H}_{1}$, i.e. it is a Gaussian process. Clearly, from (5.2) we obtain the coefficients $u_{\alpha, k}$, for lengths $|\alpha| \leq 1$ and $k \in \mathbb{N}$. Therefore, for $\alpha=(0,0, \ldots$.$) the coefficients are$

$$
\begin{equation*}
u_{(0,0, \ldots), k}=f_{\varepsilon(k)}, \tag{5.3}
\end{equation*}
$$

and for $\alpha=\varepsilon^{(j)}, j \in \mathbb{N}$ the coefficients are

$$
u_{\varepsilon^{(j)}, k}=\left\{\begin{array}{ll}
f_{2 \varepsilon^{(j)}}, & k=j  \tag{5.4}\\
\frac{1}{2} f_{\varepsilon^{(j)}+\varepsilon^{(k)}}, & k \neq j
\end{array} .\right.
$$

Note that the coefficients of the solution are symmetric, i.e. $u_{\varepsilon^{(j)}, k}=u_{\varepsilon^{(k), j}}=\frac{1}{2} f_{\varepsilon^{(j)}+\varepsilon^{(k)}}, k \neq j, k, j \in \mathbb{N}$. Thus the solution of (5.1) is given by

$$
u=u_{0}+\sum_{k=1}^{\infty} f_{2 \varepsilon^{(j)}} \otimes \xi_{j} \otimes H_{\varepsilon^{(j)}}+\frac{1}{2} \sum_{j=1}^{\infty} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} f_{\varepsilon^{(j)}+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\varepsilon^{(j)}}
$$

with the generalized expectation

$$
u_{0}=\sum_{k=1}^{\infty} f_{\varepsilon(k)} \otimes \xi_{k} .
$$

2. If $f \in \mathcal{H}_{1} \oplus \mathcal{H}_{2} \oplus \mathcal{H}_{3}$, then the solution $u$ belongs to the Wiener chaos space of maximal order two $\mathcal{H}_{0} \oplus \mathcal{H}_{1} \oplus \mathcal{H}_{2}$, i.e. can be expressed in terms of multi-indices of length zero, one and two. The coefficients of the constant part of the solution (the generalized expectation), obtained for $|\alpha|=0$, are given by (5.3) and of the Gaussian part of the solution, obtained for $|\alpha|=1$, are represented in the form (5.4). For $|\alpha|=2$ two cases may occur, $\alpha=2 \varepsilon^{(i)}, i \in \mathbb{N}$ or $\alpha=\varepsilon^{(i)}+\varepsilon^{(j)}, i \neq j$. Then, the coefficients are represented by

$$
\begin{gathered}
u_{2 \varepsilon^{(i)}, k}=\left\{\begin{array}{ll}
f_{3 \varepsilon^{(i)}}, & k=i \\
\frac{2}{3} f_{2 \varepsilon^{(i)}+\varepsilon^{(k)}}, & k \neq i
\end{array}, \quad k \in \mathbb{N}, \quad\right. \text { and } \\
u_{\varepsilon^{(i)}+\varepsilon^{(j)}, k}=\left\{\begin{array}{ll}
\frac{2}{3} f_{2 \varepsilon^{(i)}+\varepsilon^{(j)}}, & k=i \\
\frac{2}{3} f_{\varepsilon^{(i)}+2 \varepsilon^{(j)}}, & k=j \\
\frac{1}{3} f_{\varepsilon^{(i)}+\varepsilon^{(j)}+\varepsilon^{(k)},}, & k \neq i, k \neq j
\end{array}, \quad k \in \mathbb{N} .\right.
\end{gathered}
$$

3. In general, for any $\alpha \in \mathcal{I},|\alpha|=n$ the coefficients are given in the form

$$
\begin{gathered}
u_{(n-1) \varepsilon^{(k)}, k}=f_{n \varepsilon^{(k)}}, \quad \text { and } \\
= \begin{cases}\frac{1}{n} f_{\varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{2}\right)}+\ldots+\varepsilon^{\left(i_{n-1}\right)}, k}= \\
\frac{2}{n} f_{2 \varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{2}\right)}+\ldots+\varepsilon^{\left(i_{3}\right)}+\ldots+\varepsilon^{\left(i_{n-1}\right)}+\varepsilon^{(k)}}, k \notin\left\{i_{1}, i_{2}, \ldots, i_{n-1}\right\} \\
\frac{3}{n} f_{3 \varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{4}\right)}+\ldots+\varepsilon^{\left(i_{n-1}\right)}+\varepsilon^{(k)}} & , k=i_{1} \notin\left\{i_{2}, \ldots, i_{n-1}\right\} \\
\vdots & , k=i_{1}=i_{2} \notin\left\{i_{3}, \ldots, i_{n-1}\right\} \\
\frac{n-1}{n} f_{(n-1) \varepsilon^{\left(i_{1}\right)}+\varepsilon^{k}} & , k=i_{1}=i_{2}=\ldots=i_{n-2} \neq i_{n-1}\end{cases}
\end{gathered}
$$

for $k, i_{1}, i_{2}, \ldots, i_{n-1}, n \in \mathbb{N}$.
Example 5.6. We provide some examples as illustrations for the integral equation (5.1).

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1. The solution of the equation

$$
\delta u=B_{t_{0}}(\omega)
$$

belongs to the Wiener chaos space of order zero and it is obtained in the form

$$
u(t)=\sum_{k \in \mathbb{N}} \int_{0}^{t_{0}} \xi_{k}(t) d t \xi_{k}(t)=\kappa_{\left[0, t_{0}\right]}(t)
$$

i.e. it is the characteristic function of the interval $\left[0, t_{0}\right]$.
2. Consider the equation with singular white noise

$$
\begin{equation*}
\delta u=W_{t_{0}}(\omega) \tag{5.5}
\end{equation*}
$$

where $W_{t_{0}}(\omega)=\sum_{k=1}^{\infty} \xi_{k}\left(t_{0}\right) H_{\varepsilon(k)}$. It is clear that $W_{t_{0}}$ belongs to the Wiener chaos space of order one. Hence the solution of (5.5) belongs to the Wiener chaos space of order zero. From (5.2) we obtain the chaos expansion form of the solution

$$
\begin{aligned}
u(t) & =\sum_{k \in \mathbb{N}} u_{(0,0, \ldots), k} \xi_{k}(t) H_{0}(\omega) \\
& =\sum_{k \in \mathbb{N}} \xi_{k}\left(t_{0}\right) \xi_{k}(t)=d_{t_{0}}(t),
\end{aligned}
$$

which is the Dirac delta function concentrated at $t_{0}$.
3. Let $\delta u=\sum_{j=1}^{\infty} H_{2 \varepsilon^{(j)}}(\omega)$. The solution belongs to the Wiener chaos space of order one. From (5.2) we obtain the form of the coefficients

$$
u_{\varepsilon^{(j)}, k}=\left\{\begin{array}{cc}
0, & j \neq k \\
f_{2 \varepsilon^{(j)},}, & j=k
\end{array}=\left\{\begin{array}{cc}
0, & j \neq k \\
1, & j=k
\end{array} .\right.\right.
$$

Thus the solution is obtained in the form

$$
u(t, \omega)=\sum_{j \in \mathbb{N}} u_{\varepsilon^{(j)}, j} \xi_{j}(t) H_{\varepsilon^{(j)}}=\sum_{j \in \mathbb{N}} \xi_{j}(t) H_{\varepsilon^{(j)}}=W_{t}(\omega)
$$

and represents singular white noise.
4. Consider the equation

$$
\delta u=\frac{1}{2} B_{t_{0}}^{\diamond 2}(\omega),
$$

with right hand side $\frac{1}{2} B_{t_{0}}^{\diamond 2}(\omega)=\frac{1}{2}\left(B_{t_{0}}^{2}(\omega)-t_{0}\right)$ in the Wiener chaos space of order two. The solution will belong to the chaos space of order one, i.e. it will be a Gaussian process. Since

$$
\frac{1}{2} B_{t_{0}}^{\diamond 2}=\frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty}\left(\int_{0}^{t_{0}} \xi_{k}(t) d t\right)\left(\int_{0}^{t_{0}} \xi_{l}(s) d s\right) H_{\varepsilon^{(k)}+\varepsilon^{(l)}}
$$

by symmetry of the coefficients it follows that

$$
f_{\varepsilon^{(l)}+\varepsilon^{(k)}}=f_{\varepsilon^{(k)}+\varepsilon^{(l)}}=\frac{1}{2}\left(\int_{0}^{t_{0}} \xi_{k}(t) d t\right)\left(\int_{0}^{t_{0}} \xi_{l}(s) d s\right) .
$$

By partial integration we obtain

$$
\begin{aligned}
\int_{0}^{t_{0}}\left(\int_{0}^{t} \xi_{k}(s) d s\right) \xi_{l}(t) d t & =\left(\int_{0}^{t_{0}} \xi_{k}(s) d s\right)\left(\int_{0}^{t_{0}} \xi_{l}(s) d s\right) \\
& -\int_{0}^{t_{0}}\left(\int_{0}^{t} \xi_{l}(s) d s\right) \xi_{k}(t) d t
\end{aligned}
$$

i.e. by symmetry of $k$ and $l$ :

$$
\begin{aligned}
\int_{0}^{t_{0}}\left(\int_{0}^{t} \xi_{k}(s) d s\right) \xi_{l}(t) d t & =\int_{0}^{t_{0}}\left(\int_{0}^{t} \xi_{l}(s) d s\right) \xi_{k}(t) d t \\
& =\frac{1}{2}\left(\int_{0}^{t_{0}} \xi_{k}(s) d s\right)\left(\int_{0}^{t_{0}} \xi_{l}(s) d s\right)
\end{aligned}
$$

Now, for each $j \in \mathbb{N}$, by (5.4) we obtain

$$
\begin{aligned}
u_{\varepsilon^{(j)}, k}=\frac{1}{2}\left(f_{\varepsilon^{(j)}+\varepsilon^{(k)}}+f_{\varepsilon^{(k)}+\varepsilon^{(j)}}\right) & =\frac{1}{2}\left(\int_{0}^{t_{0}} \xi_{k}(t) d t\right)\left(\int_{0}^{t_{0}} \xi_{j}(s) d s\right) \\
& =\int_{0}^{t_{0}}\left(\int_{0}^{t} \xi_{j}(s) d s\right) \xi_{j}(t) d t
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u(t, \omega) & =\sum_{j=1}^{\infty} \sum_{k=1}^{\infty}\left(\int_{0}^{t_{0}}\left(\int_{0}^{t} \xi_{j}(s) d s\right) \xi_{k}(t) d t\right) \otimes \xi_{k}(t) \otimes H_{\varepsilon^{(j)}}(\omega) \\
& =\sum_{j=1}^{\infty}\left(\int_{0}^{t} \xi_{j}(s) d s\right) \kappa_{\left[0, t_{0}\right]}(t) \otimes H_{\varepsilon^{(j)}}(\omega) \\
& =B_{t}(\omega) \kappa_{\left[0, t_{0}\right]}(t)
\end{aligned}
$$

Note that the Skorokhod integral coincides with the Itô integral for which it is well-known that $\int_{0}^{t_{0}} B_{t} d B_{t}=\frac{1}{2}\left(B_{t_{0}}^{2}(\omega)-t_{0}\right)$.
5. Similarly to the previous case, the equation

$$
\delta u=\frac{1}{2} W_{t_{0}}^{\diamond 2}(\omega)
$$

has the solution

$$
u(t, \omega)=W_{t}(\omega) \delta_{t_{0}}(t)
$$

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Remark 5.7. Note that the operators $\mathbb{D}$ and $\delta$ are not inverse operators. From the previous examples we have seen, e.g. that for $Z=\sum_{k=1}^{\infty} H_{2 \varepsilon(k)}$ we have $\mathbb{D}\left(\frac{1}{2} Z\right)=W_{t}$, while $\delta\left(W_{t}\right)=Z$. Also, $\mathbb{D}\left(\frac{1}{2} B_{t_{0}}^{\diamond 2}\right)=B_{t_{0}} \kappa_{\left[0, t_{0}\right]}$, while $\delta\left(B_{t_{0}} \kappa_{\left[0, t_{0}\right]}\right)$ $=B_{t_{0}} \delta\left(\kappa_{\left[0, t_{0}\right]}\right)=B_{t_{0}}^{2}=B_{t_{0}}^{\diamond 2}+t_{0}$. The "disturbing" factor $\frac{1}{2}$ is a consequence of the fact that $Z$ and $B_{t}^{\diamond 2}$ belong to the Wiener chaos space $\mathcal{H}_{2}$.

It is also clear that $\mathcal{R}\left(\frac{1}{2} Z\right)=\delta\left(\mathbb{D}\left(\frac{1}{2} Z\right)=\delta\left(W_{t}\right)=Z\right.$ and $\mathcal{R}\left(\frac{1}{2} B_{t_{0}}^{\diamond 2}\right)=$ $\delta\left(\mathbb{D}\left(\frac{1}{2} B_{t_{0}}^{\diamond 2}\right)\right)=\delta\left(B_{t_{0}} \kappa_{\left[0, t_{0}\right]}\right)=B_{t_{0}}^{2}=B_{t_{0}}^{\diamond 2}+t_{0}$, which are both in compliance with $\mathcal{R}\left(H_{\alpha}\right)=|\alpha| H_{\alpha}$ and Theorem 3.1.

The operators $\mathbb{D}$ and $\delta$ do not commute, which can easily be seen from $\mathbb{D}\left(\delta\left(W_{t}\right)\right)=\mathbb{D}(Z)=2 W_{t}$ and $\delta\left(\mathbb{D}\left(W_{t}\right)\right)=\delta\left(d_{t}\right)=W_{t}$.

Theorem 5.8. Let $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. If $u \in \operatorname{Dom}_{-}(\mathbb{D})$, then $\delta(u) \in$ Dom_( $\mathbb{D}$ ) and the following relation holds:

$$
\begin{equation*}
\mathbb{D}(\delta u)=u+\delta(\mathbb{D} u) \tag{5.6}
\end{equation*}
$$

Proof. Let $u$ be of the form $u=\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$. Then $\delta(u)=$ $\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}$, and consequently

$$
\begin{aligned}
\mathbb{D}(\delta(u)) & =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \sum_{i=1}^{\infty}\left(\alpha+\varepsilon^{(k)}\right)_{i} \otimes \xi_{i} \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k}\left(\left(\alpha_{k}+1\right) \otimes \xi_{k} \otimes H_{\alpha}+\sum_{i \neq k} \alpha_{i} \otimes \xi_{i} \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}+\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{i} u_{\alpha, k} \otimes \xi_{i} \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \\
& =u+\delta(\mathbb{D}(u)) .
\end{aligned}
$$

The latter equality follows from $\mathbb{D}(u)=\sum_{\alpha \in \mathcal{I}} \sum_{i=1}^{\infty} \alpha_{i}\left(\sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k}\right) \otimes \xi_{i} \otimes$ $H_{\alpha-\varepsilon^{(i)}} \in X \otimes S^{\prime}(\mathbb{R}) \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ which implies

$$
\delta(\mathbb{D}(u))=\sum_{\alpha \in \mathcal{I}} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{i} u_{\alpha, k} \otimes \xi_{i} \otimes H_{\alpha-\varepsilon^{(i)}+\varepsilon^{(k)}}
$$

Since $u \in \operatorname{Dom}_{-}(\mathbb{D})$, from Theorem 2.19 it follows that $\mathbb{D} u \in \operatorname{Dom}_{-}(\delta)$. Theorem 2.22 ensures that the result remains in $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$, thus the right hand side of (5.6) is well defined and belongs to $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. This means that the left hand side is also an element in $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$, thus $\delta(u)$ must be in the domain of $\mathbb{D}$.

Remark 5.9. Note that if $u \in X \otimes L^{2}(\mathbb{R}) \otimes(S)_{-1}$, then

$$
\delta(u)=\int_{\mathbb{R}} u \diamond W_{t} d t
$$

where the right hand side is interpreted as the $X$-valued Bochner integral in the Riemann sense. This is in accordance with the known fact that Itô-Skorokhod integration with the rules of Itô integration (Itô's calculus) generates the same results as integration interpreted in the classical Riemann sense following the rules of ordinary calculus, if the integrand is interpreted as the Wick product with white noise. For example,

$$
\begin{aligned}
\int_{\left[0, t_{0}\right]} B_{t} d B_{t} & =\delta\left(\kappa_{\left[0, t_{0}\right]}(t) B_{t}\right)=\int_{\left[0, t_{0}\right]} B_{t} \diamond W_{t} d t=\int_{\left[0, t_{0}\right]} B_{t} \diamond B_{t}^{\prime} d t \\
& =\frac{1}{2} B_{t_{0}}^{\diamond 2}=\frac{1}{2}\left(B_{t_{0}}^{2}-t_{0}\right) .
\end{aligned}
$$

The general case follows easily from the definition of the Skorokhod integral. If $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$ is in $X \otimes L^{2}(\mathbb{R}) \otimes(S)_{-1}$ then $u_{\alpha, k}=\int_{\mathbb{R}} u_{\alpha}(t) \xi_{k}(t) d t$ for all $\alpha \in \mathcal{I}, k \in \mathbb{N}$. Thus,

$$
\begin{aligned}
\delta(u) & =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}=\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \int_{\mathbb{R}} u_{\alpha}(t) \xi_{k}(t) d t \otimes H_{\alpha+\varepsilon^{(k)}} \\
& =\int_{\mathbb{R}}\left(\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha}(t) \xi_{k}(t) \otimes H_{\alpha+\varepsilon^{(k)}}\right) d t \\
& =\int_{\mathbb{R}}\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) \otimes H_{\alpha}\right) \diamond\left(\sum_{k=1}^{\infty} \xi_{k}(t) \otimes H_{\varepsilon^{(k)}}\right) d t \\
& =\int_{\mathbb{R}} u \diamond W_{t} d t .
\end{aligned}
$$

The following theorem extends the result of Theorem 4.6 and reflects a nice connection between the Wick product and the ordinary product if one of the multiplicands is a Gaussian process.

Theorem 5.10. ([20])
(a) Let $f \in X \otimes(S)_{-1}$ be a Gaussian process, i.e. an element of $\mathcal{H}_{0} \bigoplus \mathcal{H}_{1}$ of the form $f=\sum_{k=0}^{\infty} f_{k} H_{\varepsilon^{(k)}}$. Then, for any $h \in X \otimes(S)_{-1}$ of the form $h=\sum_{\alpha \in \mathcal{I}} h_{\alpha} H_{\alpha}$,

$$
\begin{equation*}
h \cdot f-h \diamond f=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha+\varepsilon^{(k)}} f_{k}\left(\alpha_{k}+1\right) H_{\alpha} \tag{5.7}
\end{equation*}
$$

holds, where the right hand side is an element in $X \otimes(S)_{1}$ if only finitely many of its coefficients are nonzero, otherwise it is understood as a formal (not necessarily convergent) expansion. Some special cases under which it is a convergent expansion in $X \otimes(S)_{-1}$ are provided below:

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(b) In particular, if $g \in X \otimes S(\mathbb{R})$, where $g$ denotes the unique solution to $\delta(g)=f$, then

$$
h \cdot \delta(g)-h \diamond \delta(g)=\langle\mathbb{D}(h), g\rangle
$$

holds in $X \otimes(S)_{-1}$.
(c) In particular, if $h \in X \otimes(S)_{1}$ and $g \in X \otimes S^{\prime}(\mathbb{R})$, where $g$ denotes the unique solution to $\delta(g)=f$, then

$$
h \cdot \delta(g)-h \diamond \delta(g)=\langle g, \mathbb{D}(h)\rangle
$$

holds in $X \otimes(S)_{-1}$.
(d) In case $g \in X \otimes S(\mathbb{R})$ and $\mathbb{D}(h) \in X \otimes L^{2}(\mathbb{R}) \otimes(S)_{-1}$, as well as in the case $g \in X \otimes L^{2}(\mathbb{R})$ and $\mathbb{D}(h) \in X \otimes L^{2}(\mathbb{R}) \otimes(S)_{1}$, formula (5.7) reduces to

$$
h \cdot \delta(g)-h \diamond \delta(g)=\int_{\mathbb{R}} g(t) \cdot \mathbb{D}(h)(t) d t
$$

Proof. (a) Assume $E(f)=f_{0}=0$. Then, according to Theorem 5.1 there exists a unique $g$ such that $\delta(g)=f$ and moreover this $g$ is given by $g=\sum_{k=1}^{\infty} f_{k} \xi_{k}$ as an element of $X \otimes S^{\prime}(\mathbb{R})$. Thus,

$$
h \diamond f=h \diamond \delta(g)=\sum_{\gamma \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\gamma-\varepsilon^{(n)}} f_{n} H_{\gamma}
$$

and

$$
\begin{aligned}
h \cdot \delta(g) & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} f_{n} H_{\alpha-\varepsilon^{(n)}} H_{\varepsilon(n)} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} f_{n}\left(H_{\alpha}+\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}}\right) .
\end{aligned}
$$

This implies

$$
\begin{aligned}
h \cdot \delta(g)-h \diamond \delta(g) & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} f_{n}\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} f_{n}\left(\alpha_{n}+1\right) H_{\alpha} .
\end{aligned}
$$

Now, for arbitrary $f$ let $\tilde{f}=f-E(f)$ and $\tilde{g}$ such that $f=E(f)+\delta(\tilde{g})$. Since for constants the Wick product and the ordinary product coincide, we have

$$
\begin{aligned}
h \cdot f-h \diamond f & =h \cdot E(f)+h \cdot \delta(\tilde{g})-h \diamond E(f)-h \diamond \delta(\tilde{g})=h \cdot \delta(\tilde{g})-h \diamond \delta(\tilde{g}) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} f_{n}\left(\alpha_{n}+1\right) H_{\alpha} .
\end{aligned}
$$

Convergence of the series on the right hand side of (5.7) can be proven only in the special cases (b), (c) and (d). For example, if (b) holds, then $g=\sum_{k=1}^{\infty} f_{k} \xi_{k}$ and $f_{k}=\left\langle\xi_{k}, g\right\rangle, k \in \mathbb{N}$, which reduces to $f_{k}=\int_{\mathbb{R}} g(t) \xi_{k}(t) d t$ in case of $g \in L^{2}(\mathbb{R})$ and since $\mathbb{D}(h)=\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}\left(\alpha_{n}+1\right) \xi_{n} H_{\alpha}$, we may write the right hand side of (5.7) as

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}\left(\alpha_{n}+1\right)\left\langle\xi_{n}, g\right\rangle H_{\alpha} & =\left\langle\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}\left(\alpha_{n}+1\right) \xi_{n} H_{\alpha}, g\right\rangle \\
& =\langle\mathbb{D}(h), g\rangle .
\end{aligned}
$$

Assume that $h \in X \otimes(S)_{-1,-p}$ for some $p>0$ and that $g \in X \otimes S_{l}(\mathbb{R})$ for all $l>0$. Then $h \cdot \delta(g)-h \diamond \delta(g)=\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha} f_{n} \alpha_{n} H_{\alpha-\varepsilon^{(n)}}$ is well defined in $X \otimes(S)_{-1,-q}$ for $q \geq p+2$. This follows from the fact that $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ and thus

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{I} n=1} \sum^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}\left|\alpha_{n}\right|^{2}(2 \mathbb{N})^{-q\left(\alpha-\varepsilon^{(n)}\right)} \\
& \quad=\sum_{\alpha \in \mathcal{I} n=1} \sum_{\alpha}^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}\left|\alpha_{n}\right|^{2}(2 \mathbb{N})^{-q \alpha}(2 n)^{q} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(q-2) \alpha} \sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{q} \\
& \quad \leq \sum_{\alpha \in \mathcal{I}}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{l}<\infty
\end{aligned}
$$

for $q-2 \geq p$ and $q \leq l$. Since $l$ is arbitrary this holds for all $q \geq p+2$.
The proofs of (c) and (d) are similar.
Remark 5.11. Especially, if $f_{1}, f_{2}$ are both Gaussian processes such that $f_{i}=\delta\left(g_{i}\right), g_{i} \in X \otimes L^{2}(\mathbb{R}), i=1,2$, then

$$
\delta\left(g_{1}\right) \cdot \delta\left(g_{2}\right)-\delta\left(g_{1}\right) \diamond \delta\left(g_{2}\right)=\int_{\mathbb{R}} g_{1}(t) g_{2}(t) d t
$$

This is in compliance with the $(L)^{2}$-result from [13].
Example 5.12. For example if $g=d_{t}$ (the Dirac delta distribution) we have $f=\delta\left(d_{t}\right)=W_{t},\left\langle d_{t}, \mathbb{D}(h)\right\rangle=\mathbb{D}(h)(t)$ and thus retrieve the result of Theorem 4.6.

From (5.7) it follows that

$$
B_{t}^{2}-B_{t}^{\diamond 2}=\int_{\mathbb{R}} \kappa_{[0, t]}(s) \mathbb{D}\left(B_{t}\right)(s) d s=\int_{\mathbb{R}} \kappa_{[0, t]}(s) \kappa_{[0, t]}(s) d s=t
$$

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Remark 5.13. One might define a new type of "scalarized" Wick product containing in itself an integral operator, i.e. the scalar product in $L^{2}(\mathbb{R})$ or the dual pairing $\langle\cdot, \cdot\rangle$ of a distribution in $\mathcal{S}^{\prime}(\mathbb{R})$ and a test function in $\mathcal{S}(\mathbb{R})$. Thus, if $a=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha} \in L^{2}(\mathbb{R}) \otimes(S)_{-1}, b=\sum_{\beta \in \mathcal{I}} b_{\beta} H_{\beta} \in L^{2}(\mathbb{R}) \otimes(S)_{-1}$, then $a \in(S)_{-1}$ is defined by

$$
a b=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma}\left\langle a_{\alpha}, b_{\beta}\right\rangle H_{\gamma} .
$$

Similarly, if $a \in \mathcal{S}^{\prime}(\mathbb{R}) \otimes(S)_{-1}, b \in \mathcal{S}(\mathbb{R}) \otimes(S)_{-1}$, the result will be $a b \in(S)_{-1}$.
Now, the right hand side of (5.7) can be rewritten as $\mathbb{D}(h) \mathbb{D}(f)$. Clearly,

$$
\begin{aligned}
\mathbb{D}(h) \boxtimes \mathbb{D}(f) & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha+\varepsilon^{(k)}}\left(\alpha_{k}+1\right) \xi_{k} H_{\alpha} \sum_{k \in \mathbb{N}} f_{k} \xi_{k} H_{(0,0,0, \ldots)} \\
& =\sum_{\gamma \in \mathcal{I}}\left\langle\sum_{k \in \mathbb{N}} h_{\alpha+\varepsilon^{(k)}}\left(\alpha_{k}+1\right) \xi_{k}, \sum_{l \in \mathbb{N}} f_{l} \xi_{l}\right\rangle H_{\gamma} \\
& =\sum_{\gamma \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha+\varepsilon^{(k)}}\left(\alpha_{k}+1\right) f_{k} H_{\gamma}
\end{aligned}
$$

since $\left\langle\xi_{k}, \xi_{l}\right\rangle=1$ only for $k=l$ and $\left\langle\xi_{k}, \xi_{l}\right\rangle=0$ for $k \neq l$.
Thus, Theorem 5.10 b) - d) state that

$$
h \cdot f=h \diamond f+\mathbb{D}(h) \boxtimes \mathbb{D}(f) .
$$

In $[14,31]$ a more general formula appears in the $f, g \in X \otimes(L)^{2}$ case, where the Wick product scalarizes through the $n$-fold integral:

$$
\begin{equation*}
h \cdot f=h \diamond f+\sum_{n \in \mathbb{N}} \frac{1}{n!}\left(\mathbb{D}^{(n)}(h) \diamond \mathbb{D}^{(n)}(f)\right)=\sum_{n \in \mathbb{N}_{0}} \frac{1}{n!}\left(\mathbb{D}^{(n)}(h) \diamond \mathbb{D}^{(n)}(f)\right), \tag{5.8}
\end{equation*}
$$

under suitable conditions that ensure the convergence of the latter sum.
In a very similar manner to (5.8) it is possible to express the Wick product through the ordinary product. This has been proved in [14] and used also in [37]. For $f, g \in X \otimes(L)^{2}$ it holds that

$$
\begin{equation*}
h \diamond f=\sum_{n \in \mathbb{N}_{0}} \frac{(-1)^{n}}{n!}\left\langle\mathbb{D}^{(n)}(h), \mathbb{D}^{(n)}(f)\right\rangle, \tag{5.9}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ denotes the scalar product in $L^{2}(\mathbb{R})^{\otimes n}$.
Both identities: (5.8) and (5.9) can be generalized to the case when $f \in \operatorname{Dom}_{+}(\mathbb{D})$ and $h \in \operatorname{Dom}_{-}(\mathbb{D})$ or vice versa. In this case we interpret $\langle\cdot, \cdot\rangle$ as the dual pairing between $S^{\prime}(\mathbb{R})^{\otimes n}$ and $S(\mathbb{R})^{\otimes n}$.

## 6. Properties of the Malliavin operators

The following theorem states the duality between the Malliavin derivative and the Skorokhod integral in form of (6.1), which is also called the integration by parts formula.

Theorem 6.1. (Duality) Assume that either of the following holds:
(a) $F \in \operatorname{Dom}_{-}(\mathbb{D})$ and $u \in \operatorname{Dom}_{+}(\delta)$
(b) $F \in \operatorname{Dom}_{+}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-}(\delta)$
(c) $F \in \operatorname{Dom}_{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\delta)$.

Then the following duality relationship between the operators $\mathbb{D}$ and $\delta$ holds:

$$
\begin{equation*}
E(F \cdot \delta(u))=E(\langle\mathbb{D} F, u\rangle) \tag{6.1}
\end{equation*}
$$

where (6.1) denotes the equality of the generalized expectations of two objects in $X \otimes(S)_{-1}$ and $\langle\cdot, \cdot\rangle$ denotes the dual paring of $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.

Proof. First we note that Theorem 2.16 implies that in all three cases (a), (b) and (c), the product on the left hand side of (6.1) is well defined and $F \cdot \delta(u)$ is an element in $X \otimes(S)_{-1}$. Also, the application of the dual pairing in $S^{\prime}(\mathbb{R})$ will make $\langle\mathbb{D}, u\rangle$ also an element in $X \otimes(S)_{-1}$. Now we prove that both objects have the same expectation.

Let $u \in \operatorname{Dom}(\delta)$ be given in its chaos expansion form $u=\sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} u_{\beta, j} \otimes$ $\xi_{j} \otimes H_{\beta}$. Then $\delta(u)=\sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} u_{\beta, j} \otimes H_{\beta+\varepsilon^{(j)}}$. Let $F \in \operatorname{Dom}(\mathbb{D})$ be given as $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}$. Then $\mathbb{D}(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha}$. Therefore we obtain

$$
\begin{aligned}
F \cdot \delta(u) & =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\beta, j} \otimes H_{\alpha} \cdot H_{\beta+\varepsilon^{(j)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\beta, j} \otimes \sum_{\gamma \leq \min \left\{\alpha, \beta+\varepsilon^{(j)}\right\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta+\varepsilon^{(j)}}{\gamma} H_{\alpha+\beta+\varepsilon^{(j)}-2 \gamma .} .
\end{aligned}
$$

The generalized expectation of $F \cdot \delta(u)$ is the zeroth coefficient in the previous sum, which is obtained when $\alpha+\beta+\varepsilon^{(j)}=2 \gamma$ and $\gamma \leq \min \left\{\alpha, \beta+\varepsilon^{(j)}\right\}$, i.e. only for the choice $\beta=\alpha-\varepsilon^{(j)}$ and $\gamma=\alpha, j \in \mathbb{N}$. Thus,

$$
E(F \cdot \delta(u))=\sum_{\alpha \in \mathcal{I},|\alpha|>0} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\alpha-\varepsilon^{(j)}, j} \cdot \alpha!=\sum_{\alpha \in \mathcal{I}} \sum_{j \in \mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot\left(\alpha+\varepsilon^{(j)}\right)!.
$$

On the other hand,

$$
\begin{aligned}
\langle\mathbb{D}(F), u\rangle & =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} u_{\beta, j}\left\langle\xi_{k}, \xi_{j}\right\rangle H_{\alpha} \cdot H_{\beta} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}}\left(\alpha_{j}+1\right) f_{\alpha+\varepsilon^{(j)}} u_{\beta, j} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}
\end{aligned}
$$

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and its generalized expectation is obtained for $\alpha=\beta=\gamma$. Thus

$$
\begin{aligned}
E(\langle\mathbb{D}(F), u\rangle) & =\sum_{\alpha \in \mathcal{I}} \sum_{j \in \mathbb{N}}\left(\alpha_{j}+1\right) f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot \alpha! \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{j \in \mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot\left(\alpha+\varepsilon^{(j)}\right)! \\
& =E(F \cdot \delta(u)) .
\end{aligned}
$$

The next theorem states a higher order duality formula, which connects the $k$ th order iterated Skorokhod integral and the Malliavin derivative operator of $k$ th order, $k \in \mathbb{N}$.
Theorem 6.2. Let $f \in \operatorname{Dom}_{+}\left(\mathbb{D}^{(k)}\right)$ and $u \in \operatorname{Dom}_{-}\left(\delta^{(k)}\right)$, or alternatively let $f \in \operatorname{Dom}_{-}\left(\mathbb{D}^{(k)}\right)$ and $u \in \operatorname{Dom}_{+}\left(\delta^{(k)}\right), k \in \mathbb{N}$. Then the duality formula

$$
E\left(f \cdot \delta^{(k)}(u)\right)=E\left(\left\langle\mathbb{D}^{(k)}(f), u\right\rangle\right)
$$

holds, where $\langle\cdot, \cdot\rangle$ denotes the duality pairing of $S^{\prime}(\mathbb{R})^{\otimes k}$ and $S(\mathbb{R})^{\otimes k}$.
Remark 6.3. The previous theorems are special cases of a more general identity. It can be proven that, under suitable assumptions which make all the products well defined, the following formulae hold:

$$
\begin{equation*}
F \delta(u)=\delta(F u)+\langle\mathbb{D}(F), u\rangle \tag{6.2}
\end{equation*}
$$

$$
\begin{equation*}
F \delta^{(k)}(u)=\sum_{i=0}^{k}\binom{k}{i} \delta^{(k-i)}\left(\left\langle\mathbb{D}^{(i)} F, u\right\rangle\right), \quad k \in \mathbb{N} . \tag{6.3}
\end{equation*}
$$

The special case of (6.3) when $u \in \operatorname{Dom}_{0}(\delta)$ i.e. when $u$ is square integrable has been proven in [33]. Taking the expectation in (6.2) and using the fact that $\delta(F u)=0$, the duality formula (6.1) follows.

Example 6.4. Let $\psi \in L^{2}(\mathbb{R})$. In Remark 4.10 we have shown that the stochastic exponentials $\exp ^{\diamond}\{\delta(\psi)\}$ are eigenvalues of the Malliavin derivative i.e. $\mathbb{D}\left(\exp ^{\diamond}\{\delta(\psi)\}\right)=\psi \cdot \exp ^{\diamond}\{\delta(\psi)\}$. We will prove that they are also eigenvalues of the Ornstein-Uhlenbeck operator. Indeed, using (6.2) we obtain

$$
\begin{aligned}
\mathcal{R}\left(\exp ^{\diamond}\{\delta(\psi)\}\right) & =\delta\left(\psi \cdot \exp ^{\diamond}\{\delta(\psi)\}\right)=\delta(\psi) \exp ^{\diamond}\{\delta(\psi)\}-\left\langle\mathbb{D}\left(\exp ^{\diamond}\{\delta(\psi)\}\right), \psi\right\rangle \\
& =\delta(\psi) \exp ^{\diamond}\{\delta(\psi)\}-\left\langle\psi \cdot \exp ^{\diamond}\{\delta(\psi)\}, \psi\right\rangle \\
& =\left(\delta(\psi)-\|\psi\|_{L^{2}(\mathbb{R})}^{2}\right) \exp ^{\diamond}\{\delta(\psi)\} .
\end{aligned}
$$

In the next theorem we prove a weaker type of duality instead of (6.1) which holds if $F \in \operatorname{Dom}_{-}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-}(\delta)$ are both generalized processes. Recall that $\ll, \cdot, \cdot>_{r}$ denotes the scalar product in $(S)_{0, r}$.

Theorem 6.5. (Weak duality, [24]) Let $F \in \operatorname{Dom}_{-p}(\mathbb{D})$ and $u \in X \otimes(S)_{-1,-q}$ for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R}), n \in \mathbb{N}$, it holds that

$$
\ll\langle\mathbb{D} F, \varphi\rangle_{-r}, u>_{-r}=\ll F, \delta(\varphi u) \ggg_{-r}
$$

for $r>1+\max \{q, p+1, n+1\}$.
Proof. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in X \otimes(S)_{-1,-p}, u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in X \otimes(S)_{-1,-q}$ and $\varphi=\sum_{k \in \mathbb{N}} \varphi_{k} \xi_{k} \in S_{-n}(\mathbb{R})$. Let $r>1+\max \{q, p+1, n+1\}$. Then, for $k>p+1, \mathbb{D} F \in X \otimes S_{-k}(\mathbb{R}) \otimes(S)_{-1,-p} \subseteq X \otimes S_{-(r-1)}(\mathbb{R}) \otimes(S)_{-1,-(r-1)} \subseteq$ $X \otimes S_{-r}(\mathbb{R}) \otimes(S)_{0,-r}$. Also, $\varphi u \in X \otimes S_{-n}(\mathbb{R}) \otimes(S)_{-1,-q}$ implies that for $w>\max \{q, n+1\}, \delta(\varphi u) \in X \otimes(S)_{-1,-w} \subseteq X \otimes(S)_{-1,-(r-1)} \subseteq X \otimes(S)_{0,-r}$. Clearly, $\varphi \in S_{-n}(\mathbb{R}) \subseteq S_{-r}(\mathbb{R})$. Thus,

$$
\begin{aligned}
\langle\mathbb{D} F, \varphi\rangle_{-r} & =\left\langle\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathcal{I}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} H_{\alpha} \otimes \xi_{k}, \sum_{k \in \mathbb{N}} \varphi_{k} \xi_{k}\right\rangle_{-r} \\
& =\sum_{k \in \mathbb{N}} \varphi_{k} \sum_{\alpha \in \mathcal{I}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} H_{\alpha}(2 k)^{-r},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\ll & \langle\mathbb{D} F, \varphi\rangle_{-r}, u \gg_{-r} \\
& =\ll \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \varphi_{k}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}}(2 k)^{-r} H_{\alpha}, \sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}>_{-r} \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!u_{\alpha} \sum_{k \in \mathbb{N}} \varphi_{k}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}}(2 k)^{-r}(2 \mathbb{N})^{-r \alpha} .
\end{aligned}
$$

On the other hand,

$$
\varphi u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha} \varphi_{k} \xi_{k} \otimes H_{\alpha}
$$

and

$$
\delta(\varphi u)=\sum_{\alpha>0} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_{k} H_{\alpha} .
$$

Thus,

$$
\begin{aligned}
\ll F, \delta(\varphi u) \gg-r & =\ll \sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}, \sum_{\alpha>0} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_{k} H_{\alpha} \gg-r \\
& =\sum_{\alpha>\mathbf{0}} \alpha!f_{\alpha} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon(k)} \varphi_{k}(2 \mathbb{N})^{-r \alpha} \\
& =\sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\beta+\varepsilon^{(k)}\right)!f_{\beta+\varepsilon^{(k)}} u_{\beta} \varphi_{k}(2 \mathbb{N})^{-r\left(\beta+\varepsilon^{(k)}\right)} \\
& =\sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} \beta!\left(\beta_{k}+1\right) f_{\beta+\varepsilon^{(k)}} u_{\beta} \varphi_{k}(2 k)^{-r}(2 \mathbb{N})^{-r \beta},
\end{aligned}
$$

which completes the proof.

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The following theorem states the product rule for the Ornstein-Uhlenbeck operator. Its special case for $F, G \in \operatorname{Dom}_{0}(\mathcal{R})$ states that $F \cdot G$ is also in $\operatorname{Dom}_{0}(\mathcal{R})$ and (6.4) holds; the proof can be found e.g. in [16].

Theorem 6.6. (Product rule for $\mathcal{R}$ )
a) Let $F \in \operatorname{Dom}_{+}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-}(\mathcal{R})$, or vice versa. Then $F \cdot G \in$ Dom_( $\mathcal{R}$ ) and

$$
\begin{equation*}
\mathcal{R}(F \cdot G)=F \cdot \mathcal{R}(G)+G \cdot \mathcal{R}(F)-2 \cdot\langle\mathbb{D} F, \mathbb{D} G\rangle \tag{6.4}
\end{equation*}
$$

holds, where $\langle\cdot, \cdot\rangle$ is the dual paring between $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.
b) Let $F, G \in \operatorname{Dom}_{-}(\mathcal{R})$. Then $F \cdot G \in \operatorname{Dom}_{-}(\mathcal{R})$ and

$$
\begin{equation*}
\mathcal{R}(F \diamond G)=F \diamond \mathcal{R}(G)+\mathcal{R}(F) \diamond G \tag{6.5}
\end{equation*}
$$

Proof. a) First let us note that according to Theorem 2.16, $F \cdot \mathcal{R}(G)$ and $G \cdot \mathcal{R}(F)$ are both well defined and belong to $X \otimes(S)_{-1}$. Similarly, $\langle\mathbb{D}(F), \mathbb{D}(G)\rangle$ belongs to $X \otimes(S)_{-1}$, thus the right hand side of (6.4) is in $X \otimes(S)_{-1}$, which means that $F \cdot G \in \operatorname{Dom}_{-}(\mathcal{R})$ according to Theorem 3.1.

Now let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{+}(\mathcal{R})$ and $G=\sum_{\beta \in \mathcal{I}} g_{\beta} \otimes H_{\beta} \in \operatorname{Dom}_{-}(\mathcal{R})$. Then, $\mathcal{R}(F)=\sum_{\alpha \in \mathcal{I}}|\alpha| f_{\alpha} \otimes H_{\alpha}$ and $\mathcal{R}(G)=\sum_{\beta \in \mathcal{I}}|\beta| g_{\beta} \otimes H_{\beta}$.

The left hand side of (6.4) can be written in the form

$$
\begin{aligned}
\mathcal{R}(F \cdot G) & =\mathcal{R}\left(\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha+\beta-2 \gamma| H_{\alpha+\beta-2 \gamma} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(|\alpha|+|\beta|-2|\gamma|) H_{\alpha+\beta-2 \gamma} .
\end{aligned}
$$

On the other hand, the first two terms on the right hand side of (6.4) are

$$
\begin{equation*}
\mathcal{R}(F) \cdot G=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha| H_{\alpha+\beta-2 \gamma} \tag{6.6}
\end{equation*}
$$

and

$$
\begin{equation*}
F \cdot \mathcal{R}(G)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\beta| H_{\alpha+\beta-2 \gamma} \tag{6.7}
\end{equation*}
$$

Since $F \in \operatorname{Dom}_{+}(\mathcal{R}) \subset \operatorname{Dom}_{+}(\mathbb{D})$ and $G \in \operatorname{Dom}_{-}(\mathcal{R})=\operatorname{Dom}_{-}(\mathbb{D})$ we have $\mathbb{D}(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}$ and $\mathbb{D}(G)=\sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} \beta_{j} g_{\beta} \otimes \xi_{j} \otimes$
$H_{\beta-\varepsilon^{(k)}}$. Thus, the third term on the right hand side of (6.4) is

$$
\begin{aligned}
& \langle\mathbb{D}(F), \mathbb{D}(G)\rangle=\left\langle\sum_{|\alpha|>0} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}, \sum_{|\beta|>0} \sum_{j \in \mathbb{N}} \beta_{j} g_{\beta} \otimes \xi_{j} \otimes H_{\beta-\varepsilon^{(j)}}\right\rangle \\
& =\sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} \alpha_{k} \beta_{j} f_{\alpha} g_{\beta}\left\langle\xi_{k}, \xi_{j}\right\rangle \otimes H_{\alpha-\varepsilon^{(k)}} \cdot H_{\beta-\varepsilon^{(j)}} \\
& =\sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \alpha_{k} \beta_{k} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \left\{\alpha-\varepsilon^{(k)}, \beta-\varepsilon^{(k)}\right\}} \underset{\gamma}{ }\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} H_{\alpha+\beta-2 \varepsilon^{(k)}-2 \gamma},
\end{aligned}
$$

where we used the fact that $\left\langle\xi_{k}, \xi_{j}\right\rangle=0$ for $k \neq j$ and $\left\langle\xi_{k}, \xi_{j}\right\rangle=1$ for $k=j$. Now we put $\theta=\gamma+\varepsilon^{(k)}$ and use the identities

$$
\alpha_{k} \cdot\binom{\alpha-\varepsilon^{(k)}}{\gamma}=\alpha_{k} \cdot\binom{\alpha-\varepsilon^{(k)}}{\theta-\varepsilon^{(k)}}=\theta_{k} \cdot\binom{\alpha}{\theta}, \quad k \in \mathbb{N},
$$

and $\theta_{k} \cdot\left(\theta-\varepsilon^{(k)}\right)!=\theta$ !. Thus we obtain

$$
\begin{aligned}
\langle\mathbb{D}(F), \mathbb{D}(G)\rangle & =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}} \theta_{k}^{2}\left(\theta-\varepsilon^{(k)}\right)!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}} \theta_{k} \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}}\left(\sum_{k \in \mathbb{N}} \theta_{k}\right) \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}}|\theta| \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta .}
\end{aligned}
$$

Combining all previously obtained results we now have

$$
\begin{aligned}
\mathcal{R}(F \cdot G) & =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(|\alpha|+|\beta|-2|\gamma|) H_{\alpha+\beta-2 \gamma} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha| H_{\alpha+\beta-2 \gamma} \\
& +\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\beta| H_{\alpha+\beta-2 \gamma} \\
& -2 \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}}|\gamma| \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \\
& =\mathcal{R}(F) \cdot G+F \cdot \mathcal{R}(G)-2 \cdot\langle\mathbb{D}(F), \mathbb{D}(G)\rangle
\end{aligned}
$$

and thus (6.4) holds.

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b) If $F, G \in \operatorname{Dom}_{-}(\mathcal{R})$, then $\mathcal{R}(F), \mathcal{R}(G) \in X \otimes(S)_{-1}$. From Theorem 2.15 it follows that $\mathcal{R}(F) \diamond G, \mathcal{R}(G) \diamond F \in X \otimes(S)_{-1}$. Thus, the right hand side of (6.5) is in $X \otimes(S)_{-1}$ i.e. $F \diamond G \in \operatorname{Dom}_{-}(\mathcal{R})$.

From

$$
\begin{aligned}
& G \diamond \mathcal{R}(F)=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma}|\alpha| f_{\alpha} g_{\beta} H_{\gamma} \\
& F \diamond \mathcal{R}(G)=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} f_{\alpha}|\beta| g_{\beta} H_{\gamma}
\end{aligned}
$$

it follows that

$$
G \diamond \mathcal{R}(F)+F \diamond \mathcal{R}(G)=\sum_{\gamma \in \mathcal{I}}|\gamma| \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} H_{\gamma}=\mathcal{R}(F \diamond G)
$$

Corollary 6.7. Let $F \in \operatorname{Dom}_{+}(\mathcal{R})$ and $G \in \operatorname{Dom}_{-}(\mathcal{R})$, or vice versa (including the possibility $F, G \in \operatorname{Dom}_{0}(\mathcal{R})$ ). Then the following property holds:

$$
E(F \cdot \mathcal{R}(G))=E(\langle\mathbb{D} F, \mathbb{D} G\rangle)
$$

Proof. From the chaos expansion form of $\mathcal{R}(F \cdot G)$ it follows that $E \mathcal{R}(F \cdot G)=0$. Moreover, taking the expectations on both sides of (6.6) and (6.7) we obtain

$$
E(\mathcal{R}(F) \cdot G)=E(F \cdot \mathcal{R}(G))
$$

Now, from Theorem 6.6 it follows that

$$
0=2 E(F \cdot \mathcal{R}(G))-2 E(\langle\mathbb{D} F, \mathbb{D} G\rangle)
$$

and the assertion follows.
In the classical literature $([29,35])$ it is proven that the Malliavin derivative satisfies the product rule (with respect to ordinary multiplication) i.e. if $F, G \in$ $\operatorname{Dom}_{0}(\mathbb{D})$, then $F \cdot G \in \operatorname{Dom}_{0}(\mathbb{D})$ and (6.8) holds. The following theorem recapitulates this result and extends it for generalized and test processes, and extends it also for Wick multiplication [1].

Theorem 6.8. (Product rule for $\mathbb{D}$ )
a) Let $F \in \operatorname{Dom}_{-}(\mathbb{D})$ and $G \in \operatorname{Dom}_{+}(\mathbb{D})$ or vice versa. Then $F \cdot G \in$ $\operatorname{Dom}_{-}(\mathbb{D})$ and

$$
\begin{equation*}
\mathbb{D}(F \cdot G)=F \cdot \mathbb{D} G+\mathbb{D} F \cdot G \tag{6.8}
\end{equation*}
$$

b) Let $F, G \in \operatorname{Dom}_{-}(\mathbb{D})$. Then $F \diamond G \in \operatorname{Dom}_{-}(\mathbb{D})$ and

$$
\mathbb{D}(F \diamond G)=F \diamond \mathbb{D} G+\mathbb{D} F \diamond G
$$

Proof. a)

$$
\begin{aligned}
& \mathbb{D}(F \cdot G)=\mathbb{D}\left(\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \cdot \sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta}\right)= \\
& \mathbb{D}\left(\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}\right)= \\
& \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\left(\alpha_{k}+\beta_{k}-2 \gamma_{k}\right) \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
\end{aligned}
$$

On the other side we have

$$
\begin{aligned}
& F \cdot \mathbb{D}(G)=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \cdot \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} \beta_{k} g_{\beta} \xi_{k} H_{\beta-\varepsilon^{(k)}}= \\
& \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \left\{\alpha, \beta-\varepsilon^{(k)}\right\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} \beta_{k} \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
\end{aligned}
$$

and
$G \cdot \mathbb{D}(F)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \left\{\alpha-\varepsilon^{(k)}, \beta\right\}} \gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta}{\gamma} \alpha_{k} \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}$.
Summing up chaos expansions for $F \cdot \mathbb{D}(G)$ and $G \cdot \mathbb{D}(F)$ and applying the identities

$$
\begin{aligned}
\alpha_{k}\binom{\alpha-\varepsilon^{(k)}}{\gamma} & =\alpha_{k} \cdot \frac{\left(\alpha-\varepsilon^{(k)}\right)!}{\gamma!\left(\alpha-\varepsilon^{(k)}-\gamma\right)!}=\frac{\alpha!}{\gamma!(\alpha-\gamma)!} \cdot\left(\alpha_{k}-\gamma_{k}\right) \\
& =\binom{\alpha}{\gamma}\left(\alpha_{k}-\gamma_{k}\right)
\end{aligned}
$$

and

$$
\beta_{k}\binom{\beta-\varepsilon^{(k)}}{\gamma}=\binom{\beta}{\gamma}\left(\beta_{k}-\gamma_{k}\right),
$$

for all $\alpha, \beta \in \mathcal{I}, k \in \mathbb{N}$ and $\gamma \in \mathcal{I}$ such that $\gamma \leq \min \{\alpha, \beta\}$ and the expression $\left(\alpha_{k}-\gamma_{k}\right)+\left(\beta_{k}-\gamma_{k}\right)=\alpha_{k}+\beta_{k}-2 \gamma_{k}$ we obtain (6.8).

From Theorem 2.16 it follows that all products on the right hand side of (6.8) are well defined, thus the right hand side of (6.8) is an element of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. Thus, $F \cdot G \in \operatorname{Dom}_{-}(\mathbb{D})$.
b) By definition of the Malliavin derivative and the Wick product it can be easily verified that

$$
\begin{aligned}
\mathbb{D}(F) \diamond G+F \diamond \mathbb{D}(G) & =\sum_{\gamma \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \alpha_{k} f_{\alpha} g_{\beta} H_{\gamma}+\sum_{\gamma \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \beta_{k} f_{\alpha} g_{\beta} H_{\gamma} \\
& =\sum_{\gamma \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=\gamma} \gamma_{k} f_{\alpha} g_{\beta} H_{\gamma-\varepsilon^{(k)}}=\mathbb{D}(F \diamond G) .
\end{aligned}
$$

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If $F, G \in$ Dom_ $_{-}(\mathbb{D})$, then $\mathbb{D}(F), \mathbb{D}(G) \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. From Theorem 2.15 follows that $\mathbb{D}(F) \diamond G$ and $F \diamond \mathbb{D}(G)$ both belong to $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. Thus, $F \diamond G \in$ Dom_ $_{-}(\mathbb{D})$.

A generalization of Theorem 6.8 for higher order derivatives, i.e. the Leibnitz formula is given in the next theorem.

Theorem 6.9. Let $F, G \in \operatorname{Dom}_{-}\left(\mathbb{D}^{(k)}\right), k \in \mathbb{N}$, then $F \diamond G \in \operatorname{Dom}_{-}\left(\mathbb{D}^{(k)}\right)$ and the Leibnitz rule holds:

$$
\mathbb{D}^{(k)}(F \diamond G)=\sum_{i=0}^{k}\binom{k}{i} \mathbb{D}^{(i)}(F) \diamond \mathbb{D}^{(k-i)}(G),
$$

where $\mathbb{D}^{(0)}(F)=F$ and $\mathbb{D}^{(0)}(G)=G$.
Moreover, if $G \in \operatorname{Dom}_{+}\left(\mathbb{D}^{(k)}\right)$, then $F \cdot G \in \operatorname{Dom}_{-}\left(\mathbb{D}^{(k)}\right)$ and

$$
\begin{equation*}
\mathbb{D}^{(k)}(F G)=\sum_{i=0}^{k}\binom{k}{i} \mathbb{D}^{(i)}(F) \mathbb{D}^{(k-i)}(G) \tag{6.9}
\end{equation*}
$$

Proof. The Leibnitz rule (6.9) follows by induction and applying Theorem 6.8. Clearly, (6.9) holds also if $F, G \in \operatorname{Dom}_{0}\left(\mathbb{D}^{(k)}\right)$.

Theorem 6.10. Assume that either of the following hold:

- $F \in \operatorname{Dom}_{-}(\mathbb{D}), G \in \operatorname{Dom}_{+}(\mathbb{D})$ and $u \in \operatorname{Dom}_{+}(\delta)$,
- $F, G \in \operatorname{Dom}_{+}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-}(\delta)$,
- $F, G \in \operatorname{Dom}_{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\delta)$.

Then the second integration by parts formula holds:

$$
\begin{equation*}
E(F\langle\mathbb{D} G, u\rangle)+E(G\langle\mathbb{D} F, u\rangle)=E(F G \delta(u)) \tag{6.10}
\end{equation*}
$$

Proof. The assertion (6.10) follows directly from the duality formula (6.1) and the product rule (6.8). Assume the first case holds when $F \in \operatorname{Dom}_{-}(\mathbb{D})$, $G \in \operatorname{Dom}_{+}(\mathbb{D})$ and $u \in \operatorname{Dom}_{+}(\delta)$. Then $F \cdot G \in \operatorname{Dom}_{-}(\mathbb{D})$, too, and we have

$$
\begin{aligned}
E(F G \delta(u)) & =E(\langle\mathbb{D}(F \cdot G), u\rangle)=E(\langle F \cdot \mathbb{D}(G)+G \cdot \mathbb{D}(F), u\rangle) \\
& =E(F\langle\mathbb{D}(G), u\rangle)+E(G\langle\mathbb{D}(F), u\rangle)
\end{aligned}
$$

The second and third case can be proven in an analogous way.
The next theorem states the chain rule for the Malliavin derivative. The classical $(L)^{2}$-case has been known throughout the literature and its Wickversion was introduced in [1].

Theorem 6.11. (Chain rule) Let $\phi$ be a twice continuously differentiable function with bounded derivatives.

1. If $F \in \operatorname{Dom}_{+}(\mathbb{D})$ (resp. $F \in \operatorname{Dom}_{0}(\mathbb{D})$ ), then $\phi(F) \in \operatorname{Dom}_{+}(\mathbb{D})$ (resp. $\phi(F) \in \operatorname{Dom}_{0}(\mathbb{D})$ ) and the chain rule holds:

$$
\begin{equation*}
\mathbb{D}(\phi(F))=\phi^{\prime}(F) \cdot \mathbb{D}(F) . \tag{6.11}
\end{equation*}
$$

2. If $F \in \operatorname{Dom}_{-}(\mathbb{D})$ and $\phi$ is analytic, then $\phi^{\diamond}(F) \in \operatorname{Dom}_{-}(\mathbb{D})$ and

$$
\begin{equation*}
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\phi^{\prime \diamond}(F) \diamond \mathbb{D}(F) \tag{6.12}
\end{equation*}
$$

Proof. First we prove that (6.11) holds true when $\phi$ is a polynomial of degree $n, n \in \mathbb{N}$. Then we use the Stone-Weierstrass theorem and approximate a continuously differentiable function $\phi$ by a polynomial $\widetilde{p}_{n}$ of degree $n$, and since we assumed that $\phi$ is regular enough, $\widetilde{p}_{n}^{\prime}$ will also approximate $\phi^{\prime}$.
(i) Denote by $q_{n}(x)=x^{n}, n \in \mathbb{N}$ and let $p(x)=\sum_{k=0}^{n} a_{k} q_{k}(x)=\sum_{k=0}^{n} a_{k} x^{k}$ be a polynomial of degree $n$ with real coefficients $a_{0}, a_{1}, \ldots, a_{n}$, and $a_{n} \neq 0$. By induction on $n$, we prove the chain rule for $q_{n}$, i.e. we prove

$$
\begin{equation*}
\mathbb{D}\left(p_{n}(F)\right)=p_{n}^{\prime}(F) \cdot \mathbb{D}(F), \quad n \in \mathbb{N} \tag{6.13}
\end{equation*}
$$

For $n=1, q_{1}(x)=x$ and (6.13) holds since

$$
\mathbb{D}\left(q_{1}(F)\right)=\mathbb{D}(F)=1 \cdot \mathbb{D}(F)=q_{1}^{\prime}(F) \cdot \mathbb{D}(F)
$$

Assume (6.13) holds for $k \in \mathbb{N}$. Then, for $q_{k+1}=x^{k+1}$ by Theorem 6.8 we have

$$
\begin{aligned}
\mathbb{D}\left(q_{k+1}(F)\right) & =\mathbb{D}\left(F^{k+1}\right)=\mathbb{D}\left(F \cdot F^{k}\right) \\
& =\mathbb{D}(F) \cdot F^{k}+F \cdot \mathbb{D}\left(F^{k}\right)=\mathbb{D}(F) \cdot F^{k}+F \cdot k F^{k-1} \cdot \mathbb{D}(F) \\
& =(k+1) F^{k} \cdot \mathbb{D}(F)=q_{k+1}^{\prime}(F) \cdot \mathbb{D}(F) .
\end{aligned}
$$

Thus, (6.13) holds for every $n \in \mathbb{N}$.
Since $\mathbb{D}$ is a linear operator, (6.13) holds for any polynomial $p_{n}$, i.e.

$$
\mathbb{D}\left(p_{n}(F)\right)=\sum_{k=0}^{n} a_{k} \mathbb{D}\left(q_{k}(F)\right)=\sum_{k=0}^{n} a_{k} q_{k}^{\prime}(F) \cdot \mathbb{D}(F)=p_{n}^{\prime}(F) \cdot \mathbb{D}(F)
$$

(ii) Let $\phi \in C^{2}(\mathbb{R})$ and $F \in \operatorname{Dom}_{p}(\mathbb{D}), p \in \mathbb{N}$. Then, by the Stone-Weierstrass theorem, there exists a polynomial $\widetilde{p_{n}}$ such that

$$
\left\|\phi(F)-\widetilde{p_{n}}(F)\right\|_{X \otimes(S)_{1, p}}=\left\|\phi(F)-\sum_{k=0}^{n} a_{k} F^{k}\right\|_{X \otimes(S)_{1, p}} \rightarrow 0
$$

and

$$
\left\|\phi^{\prime}(F)-{\widetilde{p_{n}}}^{\prime}(F)\right\|_{X \otimes(S)_{1, p}}=\left\|\phi^{\prime}(F)-\sum_{k=1}^{n} a_{k} k F^{k-1}\right\|_{X \otimes(S)_{1, p}} \rightarrow 0
$$

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as $n \rightarrow \infty$.
We denote by $\mathcal{X}_{l p}=X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}$. From (6.13) and the fact that $\mathbb{D}$ is a bounded operator (Theorem 2.19) we obtain (for $l<p-1$ )

$$
\begin{aligned}
& \left\|\mathbb{D}(\phi(F))-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}}=\left\|\mathbb{D}(\phi(F))-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{\mathcal{X}_{l p}} \\
& =\left\|\mathbb{D}(\phi(F))-\mathbb{D}\left(\widetilde{p_{n}}(F)\right)+\mathbb{D}\left(\widetilde{p_{n}}(F)\right)-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{\mathcal{X}_{l p}} \\
& \leq\left\|\mathbb{D}(\phi(F))-\mathbb{D}\left(\widetilde{p_{n}}(F)\right)\right\|_{\mathcal{X}_{l_{p}}}+\left\|\mathbb{D}\left(\widetilde{p_{n}}(F)\right)-\phi^{\prime}(F) \mathbb{D}(F)\right\|_{\mathcal{X}_{l_{p}}} \\
& =\left\|\mathbb{D}\left(\phi(F)-\widetilde{p_{n}}(F)\right)\right\|_{\mathcal{X}_{l p}}+\left\|{\widetilde{p_{n}}}^{\prime}(F) \mathbb{D}(F)-\phi^{\prime}(F) \mathbb{D}(F)\right\|_{\mathcal{X}_{l p}} \\
& \leq\|\mathbb{D}\| \cdot\left\|\left(\phi(F)-\widetilde{p_{n}}(F)\right)\right\|_{X \otimes(S)_{1, p}}+\left\|{\widetilde{p_{n}}}^{\prime}(F)-\phi^{\prime}(F)\right\| \cdot\|\mathbb{D}(F)\|_{X \otimes(S)_{1, p}} \rightarrow 0,
\end{aligned}
$$

as $n \rightarrow \infty$. From this follows (6.11) as well as the estimate

$$
\|\mathbb{D}(\phi(F))\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}} \leq\left\|\phi^{\prime}(F)\right\|_{X \otimes(S)_{1, p}} \cdot\|\mathbb{D}(F)\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{1, p}}<\infty
$$

and thus $\phi(F) \in \operatorname{Dom}_{p}(\mathbb{D})$.
(iii) The proof of (6.12) for the Wick version can be conducted in a similar manner. According to Theorem 6.8 we have

$$
\mathbb{D}\left(F^{\diamond k}\right)=k F^{\diamond(k-1)} \diamond \mathbb{D}(F) .
$$

If $\phi$ is an analytic function given by $\phi(x)=\sum_{k=0}^{\infty} a_{k} x^{k}$, then $\phi^{\prime}(x)=\sum_{k=1}^{\infty} a_{k} k x^{k-1}$, and thus

$$
\phi^{\diamond}(F)=\sum_{k=0}^{\infty} a_{k} F^{\diamond k}, \quad \phi^{\diamond}(F)=\sum_{k=1}^{\infty} a_{k} k F^{\diamond(k-1)} .
$$

Thus,

$$
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\sum_{k=0}^{\infty} a_{k} \mathbb{D}\left(F^{\diamond k}\right)=\sum_{k=0}^{\infty} a_{k} k F^{\diamond(k-1)} \diamond \mathbb{D}(F)=\phi^{\wedge}(F) \diamond \mathbb{D}(F)
$$

Example 6.12. For example, $\mathbb{D}\left(B_{t_{0}}^{2}\right)=2 B_{t_{0}} \cdot \mathbb{D}\left(B_{t_{0}}\right)=2 B_{t_{0}} \cdot \kappa_{\left[0, t_{0}\right]}(t)$, $\mathbb{D}\left(B_{t_{0}}^{\diamond 2}\right)=2 B_{t_{0}} \cdot \kappa_{\left[0, t_{0}\right]}(t)$ and $\mathbb{D}\left(W_{t_{0}}^{\diamond 2}\right)=2 W_{t_{0}} \diamond \mathbb{D}\left(W_{t_{0}}\right)=2 W_{t_{0}} \cdot d_{t_{0}}(t)$, since the Wick product reduces to the ordinary product if one of the multiplicands is deterministic. This is in compliance with Example 4.4 and Example 5.6.

Also, $\mathbb{D}\left(\exp ^{\diamond}\left(W_{t_{0}}\right)\right)=\exp ^{\diamond}\left(W_{t_{0}}\right) \cdot d_{t_{0}}(t)$, or more generally $\mathbb{D}\left(\exp ^{\diamond} \delta(h)\right)=$ $\exp ^{\diamond} \delta(h) \cdot h$ for any $h \in S^{\prime}(\mathbb{R})$, which verifies once again that the stochastic exponentials are eigenvectors of the Malliavin derivative (see Remark 4.10).
Example 6.13. Geometric Brownian motion is defined by

$$
G_{t_{0}}=G_{0} \cdot e^{\left(\mu-\frac{1}{2} \sigma^{2}\right) t_{0}+\sigma B_{t_{0}}}
$$

for some constants $\mu, \sigma>0$. Then,

$$
\begin{aligned}
\mathbb{D} G_{t_{0}} & =G_{0} \cdot e^{\left(\mu-\frac{1}{2} \sigma^{2}\right) t_{0}} \cdot \mathbb{D}\left(e^{\sigma B_{t_{0}}}\right)=G_{0} \cdot e^{\left(\mu-\frac{1}{2} \sigma^{2}\right) t_{0}} \cdot \sigma \cdot e^{\sigma B_{t_{0}}} \cdot \mathbb{D}\left(B_{t_{0}}\right) \\
& =G_{0} \cdot e^{\left(\mu-\frac{1}{2} \sigma^{2}\right) t_{0}} \cdot \sigma \cdot e^{\sigma B_{t_{0}} \cdot \kappa_{\left[0, t_{0}\right]}(t)=\sigma \cdot G_{t_{0}} \cdot \kappa_{\left[0, t_{0}\right]}(t)} \\
& =\left\{\begin{array}{rr}
\sigma \cdot G_{t_{0}}, & t \in\left[0, t_{0}\right] \\
0, & t \notin\left[0, t_{0}\right] .
\end{array}\right.
\end{aligned}
$$

## 7. Applications of the Malliavin calculus

One of the first and most important applications of the Malliavin calculus concerns the existence and smoothness of a density for the probability law of random variables. Other, more recent applications in finance ( $[2,29,36]$ ) have been developed for option pricing and computing greeks (greeks measure the stability of the option price under variations of the parameters) via the ClarkOcone formula. A few years ago it was also discovered that Malliavin calculus is in a close relationship with Stein's method and can be used for estimating the distance of a random variable from Gaussian variables.

In this section we provide an overwiev of some applications and capabilities of the Malliavin calculus.

For simplicity we will assume that $X=\mathbb{R}$.

### 7.1. Measurability and densities

Let $A \in \mathcal{B}$ be a Borel set in $S^{\prime}(\mathbb{R})$. Denote by $\kappa_{A}$ its indicator function i.e. the random variable $\kappa_{A}(\omega)=1$ for $\omega \in A$ and $\kappa_{A}(\omega)=0$ for $\omega \in A^{c}$. Then $\kappa_{A}=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}$, where $a_{\alpha}=E\left(\kappa_{A} \cdot H_{\alpha}\right), \alpha \in \mathcal{I}$. Especially, $a_{\mathbf{0}}=E\left(\kappa_{A}\right)=$ $P(A)$.

Proposition 7.1. ([35]) $\kappa_{A} \in \operatorname{Dom}_{0}(\mathbb{D})$ if and only if $P(A)=0$ or $P(A)=1$.
Proof. Since $E\left(\kappa_{A}\right)=P(A)$, the chaos expansion of the indicator function is $\kappa_{A}=P(A)+\sum_{\alpha>0} a_{\alpha} H_{\alpha}, a_{\alpha}=E\left(H_{\alpha} \kappa_{A}\right)$

Assume first that $P(A) \in\{0,1\}$. Then $\kappa_{A}=$ const a.e. (it is either 0 or 1 a.e.), thus $a_{\alpha}=0$ for all $\alpha>\mathbf{0}$. Clearly, (2.8) is satisfied and $\kappa_{A} \in \operatorname{Dom}_{0}(\mathbb{D})$.

It remains to prove the other direction, that $\sum_{\alpha>\mathbf{0}}|\alpha| \alpha!\left|a_{\alpha}\right|^{2}$ cannot be finite unless $a_{\alpha}=0$ for all $\alpha>\mathbf{0}$.

Assume $\kappa_{A} \in \operatorname{Dom}_{0}(\mathbb{D})$. Let $\phi \in C_{0}^{\infty}(\mathbb{R})$ be such that $\phi(t)=t^{2}$ for $t \in[-1,1]$. According to Theorem 6.11 we have

$$
\mathbb{D}\left(\phi\left(\kappa_{A}\right)\right)=\phi^{\prime}\left(\kappa_{A}\right) \mathbb{D}\left(\kappa_{A}\right) .
$$

Since $\phi\left(\kappa_{A}\right)=\kappa_{A}^{2}=\kappa_{A}$, it follows that

$$
\mathbb{D}\left(\kappa_{A}\right)=2 \cdot \kappa_{A} \cdot \mathbb{D}\left(\kappa_{A}\right) .
$$

Thus both for $\omega \in A$ and for $\omega \in A^{c}$ we obtain $\mathbb{D}\left(\kappa_{A}\right)=0$. Now, from Corollary 4.2 it follows that $\kappa_{A}(\omega)=$ const for almost all $\omega \in \Omega$. For the chaos expansion of $\kappa_{A}$ this means that $\kappa_{A}=E\left(\kappa_{A}\right)=P(A)$ a.e. and $a_{\alpha}=0$ for all $\alpha>\mathbf{0}$ and const $=P(A)$. This implies that $P(A)$ is either zero or one.

Remark 7.2. If $P(A) \in(0,1)$, then $\kappa_{A} \notin \operatorname{Dom}_{0}(\mathbb{D})$. For example, $f(\omega)=$ $\kappa_{\left\{B_{t}(\omega)>0\right\}} \notin \operatorname{Dom}_{0}(\mathbb{D})$ since $P\left\{B_{t}>0\right\} \in(0,1)$.

On the other hand, $\kappa_{A} \in \operatorname{Dom}_{-}(\mathbb{D})$ regardless of the value of $P(A)$. This follows from $a_{\alpha}=E\left(\kappa_{A} H_{\alpha}\right) \leq E\left(H_{\alpha}\right) \leq 1$, thus

$$
\left\|\kappa_{A}\right\|_{D o m_{-p}(\mathbb{D})}^{2} \leq \sum_{\alpha>\mathbf{0}}|\alpha|^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha>\mathbf{0}}(2 \mathbb{N})^{-(p-2) \alpha}<\infty, \quad p>3 .
$$

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Remark 7.3. Let $A$ be a closed subspace of $S^{\prime}(\mathbb{R})$. Denote by $\sigma[A]$ the sub- $\sigma$ algebra of $\mathcal{B}$ generated by $A$. A random variable $f$ is measurable with respect to $\sigma[A]$ if and only if $\mathbb{D}(f)=0$ a.e. on $A^{c}$.

In particular, it can be proven $([3,17,35])$ that if a stochastic process $f_{t}$ is adapted to the Brownian filtration $A_{t}=\sigma\left[B_{s}: s \leq t\right]$, then supp $\mathbb{D}\left(f_{t}\right)=[0, t]$ i.e. $\mathbb{D} f_{t}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha}(t) \otimes \xi_{k}(s) \otimes H_{\alpha-\varepsilon^{(k)}}=0$ for $s>t$.

Remark 7.4. Let $h \in L^{2}(\mathbb{R})$ and let

$$
M(s)=\exp ^{\diamond} \delta\left(h \kappa_{[0, s]}\right)=\exp \left(\int_{0}^{s} h(t) d B_{t}-\frac{1}{2} \int_{0}^{s} h^{2}(t) d t\right), \quad s \geq 0
$$

be the stochastic exponential of $h \kappa_{[0, s]}$. According to Remark 4.10 it is an eigenvector of the Malliavin derivative, thus $\mathbb{D}(M(s))=h(t) \kappa_{[0, s]}(t) M(s)$, i.e.

$$
\mathbb{D}(M(s))=h(t) M(s), \quad \text { for } \quad t \in[0, s] .
$$

It is known $([2,35])$ that $M(s)$ is a martingale with respect to the Brownian filtration, thus for $0 \leq t \leq s$ we have

$$
E\left(\mathbb{D} M(s) \mid A_{t}\right)=E\left(h(t) M(s) \mid A_{t}\right)=h(t) E\left(M(s) \mid A_{t}\right)=h(t) M(t) .
$$

On the other hand, from Corollary 5.3 it follows that $M(s)=E(M)+\delta(u)$ for $u=\mathbb{D}\left(\mathcal{R}^{-1}(M-E M)\right)$. Since $\delta\left(h(t) \kappa_{[0, s]} M(s)\right)=\delta(\mathbb{D}(M(s)))=\mathcal{R}(M(s))$, it follows that $u=h(t) \kappa_{[0, s]} M(s)$, i.e.

$$
\begin{aligned}
M(s) & =E(M)+\int_{0}^{s} h(t) M(t) d B_{t} \\
& =E(M)+\int_{0}^{s} E\left(\mathbb{D} M(s) \mid A_{t}\right) d B_{t}
\end{aligned}
$$

Since the stochastic exponentials are dense in $(L)^{2}$ it follows that the latter formula can be extended to all $M \in \operatorname{Dom}_{0}(\mathbb{D})$. This result is known as the Clark-Ocone formula.

Theorem 7.5. (Clark-Ocone formula) Let $F \in \operatorname{Dom}_{0}(\mathbb{D})$ be adapted to the Brownian filtration. Then,

$$
F(s)=E(F)+\int_{0}^{s} E\left(\mathbb{D} F(s) \mid A_{t}\right) d B_{t} .
$$

## Example 7.6.

$B_{T}^{2}=T+\int_{0}^{T} E\left(\mathbb{D} B_{T}^{2} \mid A_{t}\right) d B_{t}=T+\int_{0}^{T} E\left(2 B_{T} \kappa_{[0, T]} \mid A_{t}\right) d B_{t}=T+2 \int_{0}^{T} B_{t} d B_{t}$,
by the martingale property of Brownian motion.

Remark 7.7. For the stochastic exponential $M$ it also holds that

$$
\begin{aligned}
\mathbb{D}\left(E\left(M(s) \mid A_{t}\right)\right) & =\mathbb{D}(M(t))=h(x) \kappa_{[0, t]} M(t) \\
& =h(x) \kappa_{[0, t]} E\left(M(s) \mid A_{t}\right)=\kappa_{[0, t]} E\left(h(x) \kappa_{[0, s]} M(s) \mid A_{t}\right) \\
& =\kappa_{[0, t]} E\left(\mathbb{D} M(s) \mid A_{t}\right),
\end{aligned}
$$

for $0 \leq t \leq s$. This result extends to all adapted processes: if $F \in \operatorname{Dom}_{0}(\mathbb{D})$ is adapted, then $E\left(F \mid A_{t}\right) \in \operatorname{Dom}_{0}(\mathbb{D})$ and

$$
\mathbb{D}\left(E\left(F(s) \mid A_{t}\right)\right)=\kappa_{[0, t]} E\left(\mathbb{D} F(s) \mid A_{t}\right)
$$

In the sequel we are going to show that absolutely continuous distributions can be characterized via the Malliavin derivative and there exists an explicit formula for the density of the distribution. For this purpose we note that $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}=\langle\mathbb{D} F, \mathbb{D} F\rangle_{L^{2}(\mathbb{R})}$ is an element in $(L)^{2}$. If $F$ is of the form $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$, then $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}=\sum_{k \in \mathbb{N}}\left(\sum_{\alpha \in \mathcal{I}} f_{\alpha+\varepsilon^{(k)}}\left(\alpha_{k}+1\right) H_{\alpha}\right)^{2}$.
Theorem 7.8. ([17]) Let $F \in \operatorname{Dom}_{0}(\mathbb{D})$ be such that $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})} \neq 0$ a.e. and $\frac{\mathbb{D} F}{\|\mathbb{D} F\|^{2}} \in \operatorname{Dom}_{0}(\delta)$. Then for every $\phi \in C_{0}^{2}(\mathbb{R})$,

$$
\begin{equation*}
E\left(\phi^{\prime}(F)\right)=E\left(\phi(F) \cdot \delta\left(\frac{\mathbb{D} F}{\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}}\right)\right) . \tag{7.1}
\end{equation*}
$$

Moreover, $F$ is an absolutely continuous random variable and its density $\varphi$ is given by

$$
\begin{equation*}
\varphi(t)=E\left(\kappa_{\{F>t\}} \cdot \delta\left(\frac{\mathbb{D} F}{\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}}\right)\right) \tag{7.2}
\end{equation*}
$$

Proof. Using the chain rule (Theorem 6.11) and the duality relationship (Theorem 6.1) we obtain

$$
\begin{aligned}
E\left(\phi^{\prime}(F)\right) & =E\left(\frac{\phi^{\prime}(F)}{\langle u, \mathbb{D} F\rangle} \cdot\langle u, \mathbb{D} F\rangle\right)=E\left(\left\langle\frac{u}{\langle u, \mathbb{D} F\rangle}, \phi^{\prime}(F) \mathbb{D} F\right\rangle\right) \\
& =E\left(\left\langle\frac{u}{\langle u, \mathbb{D} F\rangle}, \mathbb{D}(\phi(F))\right\rangle\right)=E\left(\delta\left(\frac{u}{\langle u, \mathbb{D} F\rangle}\right) \cdot \phi(F)\right)
\end{aligned}
$$

holds for any $u \in \operatorname{Dom}_{0}(\delta)$. Especially, for $u=\mathbb{D} F$ we obtain (7.1).
Putting $\phi(x)=\int_{-\infty}^{x} \kappa_{(a, b)}(s) d s, \phi^{\prime}(x)=\kappa_{(a, b)}(x)$ into (7.1) (in fact, we approximate $\kappa_{(a, b)}$ with a sequence of smooth functions) we obtain by Fubini's theorem that

$$
\begin{aligned}
P\{a<F<b\} & =E\left(\int_{-\infty}^{F} \kappa_{(a, b)}(s) d s \cdot \delta\left(\frac{\mathbb{D} F}{\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}}\right)\right) \\
& =\int_{a}^{b}\left(\kappa_{\{F>s\}} \cdot \delta\left(\frac{\mathbb{D} F}{\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}}\right)\right) d s,
\end{aligned}
$$

Chaos expansion methods in Malliavin calculus: A survey of recent results 97 which proves (7.2).

Example 7.9. Let $F \in(L)^{2}$ be a standardized Gaussian random variable i.e. an element of $\mathcal{H}_{1}$ with chaos expansion $F=\sum_{j=1}^{\infty} f_{j} H_{\varepsilon^{(j)}}, \sum_{j=1}^{\infty}\left|f_{j}\right|^{2}=1$. Then $\mathbb{D} F=\sum_{j=1}^{\infty} f_{j} \xi_{j} \in \mathcal{H}_{0}$ and $\|\mathbb{D} F\|_{L^{2}(\mathbb{R})}^{2}=1$. Also, $\delta(\mathbb{D} F)=\mathcal{R}(F)=F$ since Gaussian variables are fixed points of the Ornstein-Uhlenbeck operator. Thus, by (7.2) the density is given by $\varphi(t)=E\left(\kappa_{\{F>t\}} F\right)$. Indeed, it is easy to verify that $\int_{t}^{\infty} x \frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}} d x=\frac{1}{\sqrt{2 \pi}} e^{-\frac{t^{2}}{2}}$.

### 7.2. Gaussian approximations

In this section we present some results obtained by combining the Malliavin calculus with Stein's method as recently investigated in [34]. It is well-known that a random variable $N$ has $\mathcal{N}(0,1)$ distribution if and only if

$$
E\left(N \cdot F(N)-F^{\prime}(N)\right)=0,
$$

for every smooth function $F$. Thus, according to Stein's lemma [4], one can measure the distance to $N \sim \mathcal{N}(0,1)$, for an arbitrary random variable $Z$ by measuring the expectation of $Z \cdot F(Z)-F^{\prime}(Z)$. We will show using Malliavin calculus that

$$
E(Z \cdot F(Z))=E\left(F^{\prime}(Z)\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right)
$$

holds for every $F \in C^{2}(\mathbb{R})$. Thus, in order to measure the distance to $N \sim \mathcal{N}(0,1)$, one needs to estimate

$$
\begin{equation*}
E\left|1-\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right| \tag{7.3}
\end{equation*}
$$

where $E\left|1-\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right|=0$ if and only if $Z \sim \mathcal{N}(0,1)$.
Theorem 7.10. Let $f \in \operatorname{Dom}_{+}(\mathbb{D})$ or $f \in \operatorname{Dom}_{0}(\mathbb{D})$ such that $E(f)=0$ and let $F \in C^{2}(\mathbb{R})$. Then

$$
E(f \cdot F(f))=E\left(F^{\prime}(f) \cdot\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle\right)
$$

Proof. Since $E f=0$ from (3.1) it follows that $\mathcal{R}^{-1} f=f$. Therefore, by the duality formula (6.1) and Theorem 6.11 we have

$$
\begin{aligned}
E(f \cdot F(f)) & =E\left(\mathcal{R} \mathcal{R}^{-1}(f) \cdot F(f)\right)=E\left(\delta \mathbb{D} \mathcal{R}^{-1}(f) \cdot F(f)\right) \\
& =E\left(\left\langle\mathbb{D} F(f), \mathbb{D} \mathcal{R}^{-1} f\right\rangle\right)=E\left(F^{\prime}(f) \cdot\left\langle\mathbb{D}(f), \mathbb{D} \mathcal{R}^{-1}(f)\right\rangle\right)
\end{aligned}
$$

An immediate consequence of Theorem 7.10 and Stein's lemma is the following corollary.

Corollary 7.11. Let $f \in \operatorname{Dom}_{+}(\mathbb{D})$ or $f \in \operatorname{Dom}_{0}(\mathbb{D})$ such that $E(f)=0$. Then $f \sim \mathcal{N}(0,1)$ if and only if $\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle=1$.

Theorem 7.12. A random variable $f$ has $\mathcal{N}(0,1)$ distribution if and only if $f \in(L)^{2} \cap \mathcal{H}_{1}$ and $\|f\|_{(L)^{2}}^{2}=1$, i.e. if it is of the form $f=\sum_{j=1}^{\infty} f_{j} H_{\varepsilon^{(j)}}$ and $\sum_{j=1}^{\infty}\left|f_{j}\right|^{2}=1$ holds.

Proof. Let $f \in(L)^{2} \cap \mathcal{H}_{1}$ and $\|f\|_{(L)^{2}}^{2}=1$. According to Theorem 2.2, $f$ must be Gaussian. Since $E(f)=f_{0}=0$ and $\operatorname{Var}(f)=E\left(f^{2}\right)=\|f\|_{(L)^{2}}^{2}=1$, it follows that the underlying distribution is the standardized Gaussian one.

Vice versa, assume that $f$ has $\mathcal{N}(0,1)$ distribution. From Corollary 7.11 it follows that $\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle=1$. Assume that $f$ has chaos expansion representation $f=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$.

From Theorem 5.1 follows that the equation $\delta(u)=f$, for $E f=0$ has a unique solution $u=\mathbb{D} \mathcal{R}^{-1} f$ and it is of the form (5.2).

Thus,

$$
\begin{aligned}
1 & =\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle \\
& =\left\langle\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} \xi_{k} \otimes H_{\alpha}, \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}}\left(\beta_{j}+1\right) \frac{f_{\beta+\varepsilon^{(j)}}}{\mid \beta+\varepsilon^{(j) \mid}} \xi_{j} \otimes H_{\beta}\right\rangle \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} \cdot \frac{f_{\beta+\varepsilon^{(k)}}^{\left|\beta+\varepsilon^{(k)}\right|}\left(\beta_{k}+1\right) \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} .}{} .
\end{aligned}
$$

The latter expression can be equal to one if and only if its expectation is equal to one, and all higher order coefficients in the chaos expansion are equal to zero.

Thus, $E\left(\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle\right)=1$ implies (for $\left.\alpha=\beta=\gamma\right)$ that

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{\left(\alpha_{k}+1\right)^{2}}{\left|\alpha+\varepsilon^{(k)}\right|} \cdot f_{\alpha+\varepsilon^{(k)}}^{2} \cdot \alpha! & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{\left(\alpha_{k}+1\right)}{|\alpha|+1} \cdot\left(\alpha+\varepsilon^{(k)}\right)!f_{\alpha+\varepsilon^{(k)}}^{2} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{\alpha_{k}}{|\alpha|} \cdot \alpha!f_{\alpha}^{2} \\
& =\sum_{\alpha \in \mathcal{I}} \frac{\alpha!}{|\alpha|}\left(\sum_{k \in \mathbb{N}} \alpha_{k}\right) f_{\alpha}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!f_{\alpha}^{2} \\
& =\|f\|_{(L)^{2}}^{2}=1
\end{aligned}
$$

On the other hand, all higher order coefficients have to be equal to zero, which leaves only the possibility that

$$
f_{\alpha+\varepsilon^{(k)}}=0, \quad \text { for all }|\alpha|>0
$$

i.e. $f_{\alpha}=0$ for all $|\alpha| \geq 2$. Thus, $f \in \mathcal{H}_{1}$.

Corollary 7.13. A random variable $f$ has $\mathcal{N}\left(m, \sigma^{2}\right)$ distribution if and only if $f \in(L)^{2} \cap \mathcal{H}_{0} \oplus \mathcal{H}_{1}$ and $\|f\|_{(L)^{2}}^{2}=\sigma^{2}$, i.e. if it is of the form $f=\sum_{j=0}^{\infty} f_{j} H_{\varepsilon^{(j)}}$ and $\sum_{j=1}^{\infty}\left|f_{j}\right|^{2}=\sigma^{2}$ holds.

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We extend the previous theorem also for generalized random variables (e.g. the white noise process at a fixed time point). These processes have an infinite variance (infinite $(L)^{2}$ norm) and they can be regarded as elements of the Kondratiev spaces. Recall that $\langle\cdot, \cdot\rangle_{-p}$ denotes the scalar product in the Schwartz space $S_{-p}(\mathbb{R})$.

Theorem 7.14. Let $f \in \operatorname{Dom}_{-p}(\mathbb{D})$ and $E(f)=0$. The following statements are equivalent:

- $f$ has a generalized Gaussian distribution,
- $f \in \mathcal{H}_{1}$,
- $\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle_{-p}=\|f\|_{(S)_{-1,-p}}^{2}<\infty$.

Proof. Similarly as in the proof of Theorem 7.12 we assume that $f$ is of the form $f=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$. From

$$
\begin{aligned}
& \text { const }=\left\langle\mathbb{D} f, \mathbb{D} \mathcal{R}^{-1} f\right\rangle_{-p} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} H_{\alpha} \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}}\left(\beta_{j}+1\right) \frac{f_{\beta+\varepsilon^{(j)}}}{\mid \beta+\varepsilon^{(j) \mid}} H_{\beta}\left\langle\xi_{k}, \xi_{j}\right\rangle_{-p} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)} \frac{f_{\beta+\varepsilon^{(k)}}}{\mid \beta+\varepsilon^{(k) \mid}}\left(\beta_{k}+1\right)(2 k)^{-p} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}}
\end{aligned}
$$

follows that

$$
\begin{aligned}
\text { const } & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{\left(\alpha_{k}+1\right)^{2}}{\left|\alpha+\varepsilon^{(k)}\right|} f_{\alpha+\varepsilon^{(k)}}^{2} \alpha!(2 k)^{-p} \\
& =\sum_{\alpha \in \mathcal{I}} \frac{\alpha!}{|\alpha|} f_{\alpha}^{2} \sum_{k=1}^{\infty} \alpha_{k}(2 k)^{-p}
\end{aligned}
$$

and $f_{\alpha}=0$ for all $|\alpha| \geq 2$ i.e. $f \in \mathcal{H}_{1}$. Thus,

$$
\begin{aligned}
\text { const } & =\sum_{j=1}^{\infty} \frac{\varepsilon^{(j)}!}{\left|\varepsilon^{(j)}\right|} f_{\varepsilon^{(j)}}^{2} \sum_{k=1}^{\infty} \delta_{k j}(2 k)^{-p}=\sum_{j=1}^{\infty} f_{\varepsilon^{(j)}}^{2}(2 j)^{-p} \\
& =\sum_{j=1}^{\infty} f_{\varepsilon^{(j)}}^{2}(2 \mathbb{N})^{-p \varepsilon^{(j)}}=\|f\|_{(S)_{-1,-p}}^{2}
\end{aligned}
$$

where $\delta_{k j}=0, k \neq j$ and $\delta_{k j}=1, k=j$ is the Kronecker symbol.

Example 7.15. White noise is a generalized Gaussian process. For each fixed time point $t_{0}$ we have $\left\|W_{t_{0}}\right\|_{(S)_{-1,-p}}^{2}=\sum_{j=1}^{\infty}\left|\xi_{j}\left(t_{0}\right)\right|^{2}(2 j)^{-p}<\infty$, for $p \geq 1$ by boundedness of the Hermite functions: $\sup _{t \in \mathbb{R}}\left|\xi_{n}(t)\right| \leq C n^{-\frac{1}{12}}, n \in \mathbb{N}$.

Remark 7.16. Theorem 7.12 and Theorem 7.14 together with Theorem 2.2 provide a complete characterization of Gaussian processes (classical and generalized processes): All Gaussian processes belong to $\mathcal{H}_{1}$ and $\mathcal{H}_{1}$ contains nothing else apart from Gaussian processes.

Theorem 7.17. ([32]) Let $Z \in \operatorname{Dom}_{+}(\mathbb{D})$ or $Z \in \operatorname{Dom}_{0}(\mathbb{D})$ be such that $E(Z)=0$ and $\operatorname{Var}(Z)=1$. Then the expectation (7.3) satisfies

$$
E\left(\left|1-\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right|\right) \leq \sqrt{\operatorname{Var}\left(\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right)} .
$$

Proof. The assertion follows directly from $E(Y)^{2} \leq E\left(Y^{2}\right)$, i.e. $E(Y) \leq$ $\sqrt{\operatorname{Var}(Y)}$ and from $\operatorname{Var}(1-U)=\operatorname{Var}(U)$.

Thus, in order to measure how close is $Z$ to being normally distributed, one has to estimate how close is $\operatorname{Var}\left(\left\langle\mathbb{D} Z, \mathbb{D} \mathcal{R}^{-1} Z\right\rangle\right)$ to zero. This quantity is larger than the Kolmogorov distance, but nevertheless still a good approximation.

## Acknowledgement

The paper was supported by the project Modeling and harmonic analysis methods and PDEs with singularities, No. 174024, and partially supported by the project Modeling and research methods of operational control of traffic based on electric traction vehicles optimized by power consumption criterion, No. TR36047, both financed by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

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Received by the editors January 31, 2015

# CORRIGENDUM AND ADDENDUM TO "CHAOS EXPANSION METHODS IN MALLIAVIN CALCULUS: <br> A SURVEY OF RECENT RESULTS" 

## Tijana Levajković ${ }^{1}$, Stevan Pilipović ${ }^{2}$ and Dora Seleši ${ }^{3}$

The estimate $\alpha!\leq(2 \mathbb{N})^{\alpha}$ on page 51 in [1], as well as the inclusions $(S)_{-1,-(p-1)} \subseteq(S)_{0,-p}$ and $(S)_{0, p} \subseteq(S)_{1, p}, p \in \mathbb{N}$, are not correct. The correct inclusions are: $(S)_{1, p} \subseteq(S)_{0, p}$ and $(S)_{0,-p} \subseteq(S)_{-1,-p}, p \in \mathbb{N}_{0}$.

Consequently, the statement and proof of Theorem 6.5 will hold only for the Hida spaces but not for the Kondratiev spaces. For this purpose we note that we may define $\operatorname{Dom}_{0,-p}(\mathbb{D})=\left\{u \in X \otimes(S)_{0,-p}: \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}|\alpha| \alpha!(2 \mathbb{N})^{-p \alpha}<\right.$ $\infty\}$, and by the proof of Theorem $2.19[1], \mathbb{D}: \operatorname{Dom}_{0,-p}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes$ $(S)_{0,-p}, l>p+1$. Similarly, we define $\operatorname{Dom}_{0,-l,-q}(\delta)=\left\{u \in X \otimes S_{-l}(\mathbb{R}) \otimes\right.$ $\left.(S)_{0,-q}: \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty}\left\|u_{\alpha, k}\right\|_{X}^{2} \alpha!\left(\alpha_{k}+1\right)(2 k)^{-l}(2 \mathbb{N})^{-q \alpha}<\infty\right\}$ and by the proof of Theorem $2.22[1], \delta: \operatorname{Dom}_{0,-l,-q}(\delta) \rightarrow X \otimes(S)_{0,-q}, q>l+1, l \in \mathbb{N}$.

The statement and proof of Theorem 6.5 on page 86 now have to be modified as follows.

Theorem 6.5. (Weak duality) Let $F \in \operatorname{Dom}_{0,-p}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0,-q}(\mathbb{D})$ for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R}), n<q-1$, it holds that

$$
\ll\langle\mathbb{D} F, \varphi\rangle_{-r}, u>_{-r}=\ll F, \delta(\varphi u) \gg_{-r},
$$

for $r>\max \{q, p+1\}$.
Proof. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in \operatorname{Dom}_{0,-p}(\mathbb{D}), u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in \operatorname{Dom}_{0,-q}(\mathbb{D})$ and $\varphi=\sum_{k \in \mathbb{N}} \varphi_{k} \xi_{k} \in S_{-n}(\mathbb{R})$. Then, for $k>p+1, \mathbb{D} F \in X \otimes S_{-k}(\mathbb{R}) \otimes$ $(S)_{0,-p} \subseteq X \otimes S_{-r}(\mathbb{R}) \otimes(S)_{0,-r}$ if $r>p+1$. Also, one can easily check that $\varphi u \in \operatorname{Dom}_{0,-n,-q}(\delta)$ and since $q>n+1$, this implies that $\delta(\varphi u) \in$ $X \otimes(S)_{0,-q} \subseteq X \otimes(S)_{0,-r}$, for $r \geq q$. Therefore we let $r>\max \{p+1, q\}$. The rest of the proof is conducted as in [1].

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Received by the editors May 7, 2016

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An International Journal of Probability and Stochastic Reports

# Fundamental equations with higher order Malliavin operators 

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To cite this article: Tijana Levajković, Stevan Pilipović \& Dora Seleši (2016) Fundamental equations with higher order Malliavin operators, Stochastics, 88:1, 106-127, DOI: 10.1080/17442508.2015.1036434

To link to this article: http://dx.doi.org/10.1080/17442508.2015.1036434

Published online: 22 Jun 2015.

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# Fundamental equations with higher order Malliavin operators 

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(Received 9 July 2014; accepted 29 March 2015)
We consider three fundamental equations of the Malliavin calculus: the equation involving the Malliavin derivative, the Skorokhod integral and the Ornstein-Uhlenbeck operator of order $k, k \in \mathbb{N}$. These equations provide a complete characterization of the domain and range of the aforementioned operators. Applying the chaos expansion method in white noise spaces we solve these equations and obtain an explicit form of the solutions in the space of Kondratiev generalized stochastic processes.
Keywords: generalized stochastic processes; white noise; chaos expansion; Malliavin derivative; Skorokhod integral; Ornstein-Uhlenbeck operator

AMS Subject Classification: 60H40; 60H07; 60H10

## 1. Introduction

Three operators: the Malliavin derivative $\mathbb{D}$, the Skorokhod integral $\delta$ and the OrnsteinUhlenbeck (OU) operator $\mathcal{R}$, play a crucial role in the stochastic calculus of variations. Especially, the Skorokhod integral is of great importance in the study of non-adapted stochastic differential equations (SDEs). Some excellent references on Malliavin calculus and stochastic integration have been written by Nualart [19], Sanz-Solé [24], Dalang [2] and their coworkers. Since the pioneer work of Itô [8] in characterizing stochastic integrals in terms of Hermite polynomials, another important keystone was the development of white noise analysis made by Hida [6] who set up an appropriate functional analytical framework using nuclear operators to characterize Gaussian processes. His approach is closely connected to modern quantum theory, where the Malliavin derivative is known as the annihilation operator, the Skorokhod integral as the creation operator and the OU operator as the number operator. Second quantization operator techniques refer to weakening the topology of $(L)^{2}$ spaces in order to obtain weighted spaces of generalized stochastic processes such as the Hida spaces, Kondratiev spaces, etc. Along the line of infinite dimensional analysis, with a more probabilistical approach are the works of Da Prato [3], Øksendal [7], Rozovsky [14] and of their coworkers.

It is of great importance to manage solving different classes of equations which involve the operators of Malliavin calculus, but so far all proofs have been on the line of pure existence/uniqueness and it has been slightly neglected in the literature to explicitly solve equations of this kind. In particular, we consider the following basic equations

[^3]involving $k$ th order iterated operators $(k \in \mathbb{N})$ :
\[

$$
\begin{align*}
\mathcal{R}^{(k)} u & =g,  \tag{1}\\
\mathbb{D}^{(k)} u & =h  \tag{2}\\
\delta^{(k)} u & =f . \tag{3}
\end{align*}
$$
\]

All three operators (Malliavin derivative, Skorokhod integral and OU operator) are considered in a generalized Kondratiev space setting rather than in the usual $(L)^{2}$ setting. As a generalization of Hilbert space-valued stochastic processes, we define $S^{\prime}(\mathbb{R})$-valued stochastic processes, which allows further generalizations of the operators.

We provide a new characterization of the domain of all three operators, a more general one than in the usual $(L)^{2}$ setting, a characterization we have adopted also in [9,11-13]. We show an appropriate embedding of these domains into the Kondratiev-type spaces, which makes them convenient to study higher order iterated operators of the Malliavin calculus. On the other hand, Equations (1)-(3) we considered in this paper will provide (for $k=1$ ) a full characterization of the range of all three operators. Moreover, we obtain explicit forms of the solutions of the general $k$ th order Equations (1)-(3), which is highly useful for computer modelling that involves polynomial chaos expansion (PCE) simulation methods used in numerical stochastic analysis. Some excellent applications of the PCE method are made in the papers of Karniadakis [28], Matthies [17], Ernst [4] and many others with a growing tendency to apply PCE methods in industry.

The main purpose of this paper is to prove the existence and uniqueness of Equations (1)-(3). We present the methodology of chaos expansions on Equation (1), which is the most representative to get familiar with its idea. We also correct the estimate obtained in [9] for the domain of the Skorokhod integral in Theorem 2.8. The first main result of the paper is the proof of the existence and uniqueness of a solution of Equation (2), which will be provided in Theorems 4.1 and 4.5. The second main result of the paper is to present an explicit form of the solution of the integral Equation (3), which will be done in Theorems 5.1 and 5.3. As a consequence, representation forms via $k$ th order integrals and $k$ th order OU operators follow for singular stochastic processes (Corollaries 3.2 and 5.4).

In [11] we proved that the Malliavin derivative indicates the rate of change in time between the ordinary product and the Wick product, i.e. $h \cdot W_{t}-h \diamond W_{t}=\mathbb{D}(h)$ holds. In this paper we go one step further and prove a similar result for stochastic processes other than white noise $W_{t}$ (see Theorem 5.2). Hence, as a consequence one can define the ordinary product in a generalized sense. This result is closely related to that in [18], where the authors study nonlinear SDEs by replacing polynomial nonlinearities with Wick type nonlinearities and estimate the error by a Taylor series involving Wick products and Malliavin derivatives. We also compare the ordinary derivative with the Malliavin derivative in Theorem 4.4.

The chaos expansion method we are using to solve Equations (1)-(3) can also be used to solve equations involving generalized Malliavin operators defined in [16], but this will be the topic of a future paper.

## 2. Basic notions

Let $(\Omega, \mathcal{F}, P)$ be the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel sigma-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to
the characteristic function

$$
\begin{equation*}
\int_{S^{\prime}(\mathbb{R})} \mathrm{e}^{i\langle\omega, \phi\rangle} \mathrm{d} \mu(\omega)=\mathrm{e}^{-(1 / 2)\|\phi\|_{L^{2}(\mathbb{R})}^{2}, \quad \phi \in S(\mathbb{R}),, ~, ~ . ~} \tag{4}
\end{equation*}
$$

given by the Bochner-Minlos theorem.
Denote by $h_{n}(x)=(-1)^{n} \mathrm{e}^{x^{2} / 2} \mathrm{~d}^{n} / \mathrm{d} x^{n}\left(\mathrm{e}^{-\left(x^{2} / 2\right)}\right), n \in \mathbb{N}_{0}, \mathbb{N}_{0}=\mathbb{N} \bigcup\{0\}$, the family of Hermite polynomials and $\xi_{n}(x)=1 / \sqrt[4]{\pi} \sqrt{(n-1)!} \mathrm{e}^{-\left(x^{2} / 2\right)} h_{n-1}(\sqrt{2} x), n \in \mathbb{N}$, the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^{2}(\mathbb{R})$. We follow the characterization of the Schwartz spaces in terms of the Hermite basis: the space of rapidly decreasing functions as a projective limit space $S(\mathbb{R})=$ $\cap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$ and the space of tempered distributions as an inductive limit space $S^{\prime}(\mathbb{R})=$ $\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$ where

$$
\begin{equation*}
S_{l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k}:\|f\|_{l}^{2}=\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, \quad l \in \mathbb{Z}, \mathbb{Z}=-\mathbb{N} \bigcup \mathbb{N}_{0} \tag{5}
\end{equation*}
$$

### 2.1 The Wiener-Itô chaos expansion

Let $\mathcal{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ denote the set of sequences of non-negative integers which have only finitely many non-zero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right), \alpha_{i} \in \mathbb{N}_{0}, i=1,2, \ldots, m$, $m \in \mathbb{N}$. The $k$ th unit vector $\varepsilon^{(k)}=(0, \ldots, 0,1,0, \ldots), k \in \mathbb{N}$ is the sequence of zeros with the number 1 as the $k$ th component. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. Let $(2 \mathbb{N})^{\alpha}=\prod_{k=1}^{\infty}(2 k)^{\alpha_{k}}$. Note that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ for $p>1$ (see, e.g. [7]).

Let $(L)^{2}=L^{2}\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$ be the Hilbert space of random variables with finite second moments. We define by

$$
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in \mathcal{I}
$$

the Fourier-Hermite orthogonal basis of $(L)^{2}$ such that $\left\|H_{\alpha}\right\|_{(L)^{2}}^{2}=\alpha$ !. In particular, for the $k$ th unit vector $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$.

The prominent Wiener-Itô chaos expansion theorem states that each element $F \in(L)^{2}$ has a unique representation of the form

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega)
$$

$\omega \in S^{\prime}(\mathbb{R}), a_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that $\|F\|_{(L)^{2}}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2} \alpha!<\infty$.

### 2.2 Spaces of generalized random variables

The stochastic analogue of Schwartz spaces as generalized function spaces is the Kondratiev spaces of generalized random variables.

Definition 2.1. The space of the Kondratiev test random variables $(S)_{1}$ consists of elements $f=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha} \in(L)^{2}, a_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|f\|_{1, p}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2}(\alpha!)^{2}(2 \mathbb{N})^{p \alpha}<\infty, \quad \text { for all } \quad p \in \mathbb{N}_{0}
$$

The space of the Kondratiev generalized random variables $(S)_{-1}$ consists of formal expansions of the form $F=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}, a_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that

$$
\|F\|_{-1,-p}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty, \quad \text { for some } \quad p \in \mathbb{N}_{0}
$$

This provides a sequence of spaces $(S)_{\rho, p}=\left\{f=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}:\|f\|_{\rho, p}<\infty\right\}$, $\rho \in\{-1,1\}, p \in \mathbb{Z}$. Thus, $(S)_{1}=\cap_{p \in \mathbb{N}_{0}}(S)_{1, p}$ can be equipped with the projective topology and $(S)_{-1}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-1,-p}$ as its dual with the inductive topology. It holds that $(S)_{1}$ is a nuclear space and the following Gel'fand triple is obtained

$$
(S)_{1} \subseteq(L)^{2} \subseteq(S)_{-1}
$$

### 2.3 Generalized processes

Let $X$ be a Banach space of functions on $\mathbb{R}$ endowed with $\|\cdot\|_{X}$ and $X^{\prime}$ its dual. Alternatively, $X$ can be taken as a nuclear space $X=\cap_{k=0}^{\infty} X_{k}$ endowed with a family of seminorms $\left\{\|\cdot\|_{k} ; k \in \mathbb{N}_{0}\right\}$ and $X^{\prime}=\bigcup_{k=0}^{\infty} X_{-k}$ its topological dual. The most common examples used in this paper for $X$ will be the Schwartz spaces $S(\mathbb{R}), S^{\prime}(\mathbb{R})$ and $C^{\infty}(\mathbb{R})$.

Definition 2.2. Generalized stochastic processes are elements of the tensor product space $X \otimes(S)_{-1}$ or $X^{\prime} \otimes(S)_{-1}$.

The Kondratiev space $(S)_{1}$ is nuclear and thus $\left(X \otimes(S)_{1}\right)^{\prime} \cong X^{\prime} \otimes(S)_{-1}$. Note that $X^{\prime} \otimes(S)_{-1}$ is isomorphic to the space of linear bounded mappings $X \rightarrow(S)_{-1}$.

Theorem 2.3. ([20]) Let $X=\cap_{k=0}^{\infty} X_{k}$ be a nuclear space endowed with a family of seminorms $\left\{\|\cdot\|_{k} ; k \in \mathbb{N}_{0}\right\}$ and let $X^{\prime}=\bigcup_{k=0}^{\infty} X_{-k}$ be its topological dual. Generalized stochastic processes as elements of $X^{\prime} \otimes(S)_{-1}$ have a chaos expansion of the form

$$
\begin{equation*}
F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}, \quad f_{\alpha} \in X_{-k}, \quad \alpha \in \mathcal{I} \tag{6}
\end{equation*}
$$

where $k \in \mathbb{N}_{0}$ does not depend on $\alpha \in \mathcal{I}$, and there exists $p \in \mathbb{N}_{0}$ such that

$$
\begin{equation*}
\|F\|_{X^{\prime} \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{-k}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{7}
\end{equation*}
$$

The same holds for processes which are elements of $X \otimes(S)_{-1}$, where $X$ is a Banach space. In this case the norm $\|F\|_{X \otimes(S)_{-1,-p}}^{2}$ is defined via (7) where $\|\cdot\|_{-k}$ should be replaced by $\|\cdot\|_{X}$.

With the same notation as in (6) we will denote by $E F=f_{(0,0,0, \ldots)}$ the generalized expectation of the process $F$.

In $[25,26]$ a general setting of $S^{\prime}$-valued generalized stochastic process is provided: $S^{\prime}(\mathbb{R})$-valued generalized stochastic processes are elements of $\tilde{X} \otimes(S)_{-1}$, where
$\tilde{X}=X \otimes S^{\prime}(\mathbb{R})$, and are given by chaos expansions of the form (recall (5))

$$
f=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k}
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathcal{I}} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-1}$ and $a_{\alpha, k} \in X$. Thus, for some $p, l \in \mathbb{N}_{0}$,

$$
\|f\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|a_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty
$$

The generalized expectation of an $S^{\prime}$-valued stochastic process $f$ is given by

$$
E f=\sum_{k \in \mathbb{N}} a_{(0,0, \ldots), k} \otimes \xi_{k}=b_{(0,0, \ldots)}
$$

We generalize the definition of the Wick product of random variables to the set of generalized stochastic processes in the way as it is done in [10,21,22,27].

Definition 2.4. Let $F, G \in X \otimes(S)_{-1}$ be generalized stochastic processes with chaos expansions of the form (6). Assume $X$ to be a space closed under the multiplication $f_{\alpha} g_{\beta}$, for $f_{\alpha}, g_{\beta} \in X$. Then the Wick product $F \diamond G$ is defined by

$$
F \diamond G=\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) \otimes H_{\gamma}
$$

### 2.4 Operators of the Malliavin calculus

We provide now the definitions of the Malliavin derivative, the Skorokhod integral and the OU operator, which are extensions of the classical definitions of these operators to the space of generalized stochastic processes. In $[1,3,14,15,19,24]$ the Malliavin derivative and the Skorokhod integral are defined on a subspace of $(L)^{2}$ so that the resulting process after application of these operators always remains in $(L)^{2}$. In $[9,10,12]$ we allowed values in $(S)_{-1}$ and thus obtained a larger domain for all operators.

Definition 2.5. Let a generalized stochastic process $u \in X \otimes(S)_{-1}$ be of the form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$. If there exists $p \in \mathbb{N}_{0}$ such that $\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$, then the Malliavin derivative of $u$ is defined by

$$
\begin{equation*}
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}} \tag{8}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$, i.e.

$$
H_{\alpha-\varepsilon^{(k)}}=\left\{\begin{array}{cc}
0, & \alpha_{k}=0 \\
H_{\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right)}, & \alpha_{k} \geq 1
\end{array}\right.
$$

for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$.

In the white noise setting the operator $\mathbb{D}$ is also called the stochastic gradient of a generalized stochastic process $u$. The domain of the Malliavin derivative is given by

$$
\operatorname{Dom}(\mathbb{D})=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{p}(\mathbb{D})=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \bigotimes(S)_{-1}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
$$

A process $u \in \operatorname{Dom}(\mathbb{D}) \subset X \otimes(S)_{-1}$ is called a Malliavin differentiable process.
Theorem 2.6. ([9]) The Malliavin derivative is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{p}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}, \quad l>p+1, \quad p \in \mathbb{N}_{0}
$$

The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $(L)^{2}$-sense. In [9] we have extended the definition of the Skorokhod integral from Hilbert space-valued processes to the class of $S^{\prime}$-valued generalized processes.

DEFINITION 2.7. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes v_{\alpha} \otimes H_{\alpha} \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-r}, p, r \in \mathbb{N}_{0}$, be a generalized $S_{-p}(\mathbb{R})$-valued stochastic process and let $v_{\alpha} \in S_{-p}(\mathbb{R})$ be given by the expansion $v_{\alpha}=\sum_{k \in \mathbb{N}} v_{\alpha, k} \xi_{k}, v_{\alpha, k} \in \mathbb{R}$. Then the process $F$ is integrable in the Skorokhod sense and the chaos expansion of its stochastic integral is given by

$$
\begin{equation*}
\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} f_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} \tag{9}
\end{equation*}
$$

The following theorem estimates the domain and range of the Skorokhod integral in a more precise manner than provided in [9] where a minor error occurred.

TheOrem 2.8. The Skorokhod integral $\delta$ is a linear and continuous mapping

$$
\delta: X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-r} \rightarrow X \otimes(S)_{-1,-q}, \quad q \geq r, \quad q>p+1
$$

Proof. Clearly,

$$
\begin{aligned}
\|\delta(F)\|_{X \otimes(S)_{-1,-q}}^{2} & =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} v_{\alpha, k} f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q\left(\alpha+\varepsilon^{(k)}\right)} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} v_{\alpha, k} f_{\alpha}(2 k)^{-(q / 2)}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}}\left|v_{\alpha, k}\right|(2 k)^{-(p / 2)}(2 k)^{-((q-p) / 2)}\right)^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}}\left|v_{\alpha, k}\right|^{2}(2 k)^{-p} \cdot \sum_{k \in \mathbb{N}}(2 k)^{-(q-p)}\right)(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}\left\|v_{\alpha}\right\|_{-p}^{2}(2 \mathbb{N})^{-r \alpha} \cdot \sum_{k \in \mathbb{N}}(2 k)^{-(q-p)} \\
& \leq M\|F\|_{X \otimes S_{-p}\left(\mathbb{R}^{n}\right) \otimes(S)_{-1,-r}}^{2}<\infty,
\end{aligned}
$$

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for $q \geq r$, where we used the Cauchy-Schwarz inequality and the fact that

$$
M=\sum_{k \in \mathbb{N}}(2 k)^{-(q-p)}<\infty, \quad q>p+1
$$

It follows that the domain of the Skorokhod integral is

$$
\operatorname{Dom}(\delta)=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}=\bigcup_{(p, r) \in \mathbb{N}_{0}^{2}} \operatorname{Dom}_{(p, r)}(\delta)=\bigcup_{(p, r) \in \mathbb{N}_{0}^{2}}\left(X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-r}\right)
$$

Definition 2.9. The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathcal{R}=\delta \circ \mathbb{D}$ and called the $O U$ operator.

Thus, for $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-1}$ the OU operator is given by

$$
\mathcal{R}(u)=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} .
$$

Let

$$
\operatorname{Dom}(\mathcal{R})=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{p}(\mathcal{R})=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \bigotimes(S)_{-1}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
$$

Theorem 2.10. ([12]) The operator $\mathcal{R}$ is a linear and continuous mapping

$$
\mathcal{R}: \operatorname{Dom}_{p}(\mathcal{R}) \rightarrow X \otimes(S)_{-1,-p}, \quad p \in \mathbb{N}_{0}
$$

Note that in this setting the domains of $\mathbb{D}$ and $\mathcal{R}$ coincide, i.e. $\operatorname{Dom}(\mathcal{R})=\operatorname{Dom}(\mathbb{D})$.
In the following sections we will prove that the mappings $\mathcal{R}: \operatorname{Dom}(\mathcal{R}) \rightarrow X \otimes(S)_{-1}$, $\mathbb{D}: \operatorname{Dom}(\mathbb{D}) \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ and $\delta: \operatorname{Dom}(\delta) \rightarrow X \otimes(S)_{-1}$, given in Theorems 2.6, 2.8 and 2.10 , are surjective, i.e. the range of the operators are, respectively,

$$
\begin{aligned}
\operatorname{Range}(\mathcal{R}) & =X \otimes(S)_{-1} \\
\operatorname{Range}(\mathbb{D}) & =X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1} \\
\operatorname{Range}(\delta) & =X \otimes(S)_{-1}
\end{aligned}
$$

## 3. Equation with the OU operator

We define iteratively $\mathcal{R}^{(k)}=\mathcal{R} \circ \mathcal{R}^{(k-1)}, k \in \mathbb{N}$, where $\mathcal{R}^{0}=I d$ is the identity operator. Using the fact that $\mathcal{R}^{(k)}\left(H_{\alpha}\right)=|\alpha|^{k} H_{\alpha}, \alpha \in \mathcal{I}$ it follows that
$\operatorname{Dom}\left(\mathcal{R}^{(k)}\right)=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{p}\left(\mathcal{R}^{(k)}\right)=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{-1}: \sum_{\alpha \in \mathcal{I}}|\alpha|^{2 k}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}$.
Note that actually for each $k \in \mathbb{N}, \operatorname{Dom}\left(\mathcal{R}^{(k)}\right) \cong X \otimes(S)_{-1}$. Since $|\alpha| \leq(2 \mathbb{N})^{\alpha}$, it follows that

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2 k}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(p-2 k) \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}<\infty
$$

$p-2 k \geq q$. This means that if $u \in X \otimes(S)_{-1,-q}$ for some $q \geq 0$, then $u \in \operatorname{Dom}_{p}\left(\mathcal{R}^{(k)}\right)$ for $p \geq q+2 k$.

Theorem 3.1. Let $g \in X \otimes(S)_{-1,-p}, p \in \mathbb{N}_{0}$, have zero generalized expectation. Then for each $k \in \mathbb{N}$ the equation

$$
\mathcal{R}^{(k)} u=g, \quad E u=\tilde{u}_{0} \in X
$$

has a unique solution $u \in \operatorname{Dom}_{p}\left(\mathcal{R}^{(k)}\right)$ represented in the form

$$
u=\tilde{u}_{0}+\sum_{\alpha \in \mathcal{T},|\alpha|>0} \frac{g_{\alpha}}{|\alpha|^{k}} \otimes H_{\alpha} .
$$

Proof. If we seek for a solution in the form of $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$, then from $\mathcal{R}^{(k)} u=g$ it follows that

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{k} u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha},
$$

i.e. $u_{\alpha}=g_{\alpha} /|\alpha|^{k}$ for all $\alpha \in \mathcal{I},|\alpha|>0$. Clearly, $g$ must have zero expectation for the equation to make sense, and therefore $u_{(0,0,0,0, \ldots)}$ can be chosen arbitrarily. On the other hand, if we have an initial condition $E(u)=\tilde{u}_{0}$, then $u_{(0,0,0,0, \ldots)}=E u=\tilde{u}_{0}$. Also, $u \in$ $\operatorname{Dom}_{p}\left(\mathcal{R}^{(k)}\right)$ since

$$
\sum_{|\alpha|>0}|\alpha|^{2 k}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{|\alpha|>0}\left\|g_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty .
$$

Corollary 3.2. For every $k \in \mathbb{N}$, each process $g \in X \otimes(S)_{-1}$ can be represented in the form

$$
g=E(g)+\mathcal{R}^{(k)}(u)
$$

for a certain $u \in X \otimes(S)_{-1}$ given in terms of $g$.

Proof. Assume $k=1$. If $E(g)=0$, this is the statement of Theorem 3.1 Otherwise, for $E(g) \neq 0$, let $\tilde{g}=g-E(g), E(\tilde{g})=0$, and apply the previous case to obtain $\tilde{u}$ such that $\mathcal{R}(\tilde{u})=\tilde{g}$. Now, since $\tilde{u}=\mathcal{R}^{-1}(\tilde{g})=\mathcal{R}^{-1}(g-E(g))$ and $g-E(g)=\mathcal{R}\left(\mathcal{R}^{-1}(g-E(g))\right)$, it follows that $g=E(g)+\mathcal{R}(u)$ for $u=\mathcal{R}^{-1}(g-\mathcal{R}(g))$.

Similarly, for arbitrary $k \in \mathbb{N}, g=E(g)+\mathcal{R}^{(k)}(u)$ for $u=\mathcal{R}^{(-k)}(g-E(g))$, where $\mathcal{R}^{(-k)}$ denotes the inverse operator of $\mathcal{R}^{(k)}$ and is well defined according to Theorem 3.1

## 4. Equation with the Malliavin derivative

We consider the initial value problem involving the Malliavin derivative operator

$$
\begin{cases}\mathbb{D} u=h, & h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}  \tag{10}\\ E u=\tilde{u}_{0}, & \tilde{u}_{0} \in X\end{cases}
$$

and prove existence and uniqueness of its solution. We will solve the equation by applying the integral operator on both sides of the equation and by using Theorem 3.1

TheOrem 4.1. Let a process $h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-q}, p \in \mathbb{N}_{0}, q>p+1$, have a chaos expansion representation $h=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$. Then Equation (10) has a unique solution in $\operatorname{Dom}_{q}(\mathbb{D})$ represented in the form

$$
\begin{equation*}
u=\tilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} \tag{11}
\end{equation*}
$$

Proof. From the assumption $h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-q}$, for some $p \geq 0, q>p+1$, it follows that $h$ is integrable in the Skorokhod sense and its integral $\delta(h)$ is of the form (9). Note that the assumption $q>p+1$ does not reduce generality of the theorem, since every process from $X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-k}$ for $k \leq p+1$ can be embedded into the larger space $X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-\tilde{k}}$, where $\tilde{k}>p+1$.

We are looking for a solution of (10) in $\operatorname{Dom}_{q}(\mathbb{D})$ in its explicit form

$$
u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}
$$

First we apply the operator $\delta$ on both sides of Equation (10) and thus obtain the equation $\delta(\mathbb{D} u)=\delta(h)$. Putting $\delta \circ \mathbb{D}=\mathcal{R}$ we transform the initial Equation (10) into its equivalent form

$$
\mathcal{R} u=\delta(h)
$$

for a given $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. Thus, the solution $u$ is calculated from

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\delta\left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}\right) \\
\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}} \\
\sum_{\alpha \in \mathcal{I},|\alpha|>0}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} .
\end{gathered}
$$

Due to the uniqueness of the chaos expansion of a process represented in the orthogonal basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, it follows that

$$
|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}, \quad|\alpha|>0
$$

Thus, for $|\alpha|>0$ the coefficients of the solution are represented by

$$
\begin{equation*}
u_{\alpha}=\frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}, \quad \text { for }|\alpha|>0 \tag{12}
\end{equation*}
$$

From the initial condition $E u=\tilde{u}_{0}$ it follows that

$$
u_{(0,0, \ldots)}=\tilde{u}_{0}
$$

Now, we prove that the solution $u$ belongs to the space $\operatorname{Dom}_{q}(\mathbb{D})$. Clearly,

$$
\begin{aligned}
\|u\|_{D o m_{q}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha, k}\right\|_{X}^{2}(2 \mathbb{N})^{-q\left(\alpha+\varepsilon^{(k)}\right)} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left\|h_{\alpha, k}\right\|_{X}(2 k)^{-(p / 2)}(2 k)^{-((q-p) / 2)}\right)^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|h_{\alpha, k}\right\|_{X}^{2}(2 k)^{-p}(2 \mathbb{N})^{-q \alpha} \sum_{k \in \mathbb{N}}(2 k)^{-(q-p)} \\
& =C\|h\|_{X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-q}}^{2}<\infty
\end{aligned}
$$

since $C=\sum_{k \in \mathbb{N}}(2 k)^{-(q-p)}<\infty$, for $q>p+1$.
The following theorem serves as a motivation to consider SDEs with the Malliavin derivative.

Theorem 4.2. ([11]) Let $h \in X \otimes(S)_{-1}$ and $W_{t}, B_{t}$ denote white noise and Brownian motion, respectively. Then,

$$
h \cdot W_{t}-h \diamond W_{t}=\mathbb{D}(h)
$$

i.e. $(\mathrm{d} / \mathrm{d} t)\left(h \cdot B_{t}-h \diamond B_{t}\right)=\mathbb{D}(h)$ in weak $S^{\prime}(\mathbb{R})$-sense.

Remark 1. Note that if $h \in X \otimes(S)_{-1,-p}$, then $\mathbb{D}(h) \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-(p+2)}, l>p+1$. Thus, apart from the Wick product $h \diamond W_{t}$ being well defined, the ordinary product is also well defined in the generalized sense as an element of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ and it is given by $h \cdot W_{t}=h \diamond W_{t}+\mathbb{D}(h)$.

Example 4.3. Let $X=S^{\prime}(\mathbb{R})$ and $h=W_{t_{0}}$, where $W_{t}$ is singular white noise. Then

$$
\begin{equation*}
W_{t_{0}} \cdot W_{t}=W_{t_{0}} \diamond W_{t}+\mathbb{D}\left(W_{t_{0}}\right)=W_{t_{0}} \diamond W_{t}+\mathrm{d}_{t_{0}}(t) \tag{13}
\end{equation*}
$$

holds in $S^{\prime}(\mathbb{R}) \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ (see Section 6). Note that (13) is well defined for all $\left(t, t_{0}\right) \in$ $\mathbb{R}^{2}$ except for $t=t_{0}$ where the Dirac delta distribution $\mathrm{d}_{t_{0}}(t)=\mathrm{d}\left(t-t_{0}\right) \in S^{\prime}(\mathbb{R}) \otimes S^{\prime}(\mathbb{R})$ has its singularity. It is possible to give a meaning to $d_{t_{0}}\left(t_{0}\right)=\sum_{n=1}^{\infty} \xi_{n}\left(t_{0}\right)^{2}$ as the point value of a distribution in the sense of Colombeau generalized numbers, but this exceeds the scope of this article and will be the topic of an upcoming paper. Thus, in Colombeau sense, it will be possible to define $W_{t}^{2}=W_{t}^{\diamond 2}+\mathrm{d}_{t}(t)$. For Colombeau theory we refer to [5].

The previous theorem stating that the Malliavin derivative indicates the speed of change in time between the ordinary product and the Wick product motivates us to
consider equations of the type $\mathbb{D}(u)=(\mathrm{d} / \mathrm{d} t) f$, i.e. to compare the Malliavin derivative with the ordinary derivative.

Theorem 4.4. Let $f \in X \otimes S_{-k}(\mathbb{R}) \otimes(S)_{-1,-p}, p>k+1$, and $\tilde{u}_{0} \in X$. Assume $f$ is of the form $f=\sum_{\alpha \in \mathcal{I}} \sum_{j \in \mathbb{N}} f_{\alpha, j} \otimes \xi_{j} \otimes H_{\alpha}$, where $f_{\alpha, j} \in X, \alpha \in \mathcal{I}, j \in \mathbb{N}$. The equation

$$
\mathbb{D}(u)=\frac{\mathrm{d}}{\mathrm{~d} t} f, \quad E u=\tilde{u}_{0},
$$

has a unique solution $u \in \operatorname{Dom}_{p}(\mathbb{D})$, given by

$$
u=\tilde{u}_{0}+\sum_{\alpha \in \mathcal{T},|\alpha|>0} \frac{1}{|\alpha|}\left(f_{\alpha-\varepsilon^{(1)}, 2}+\sum_{j=2}^{\infty}\left(f_{\alpha-\varepsilon^{(1)}, j+1} \sqrt{\frac{j+1}{2}}-f_{\alpha-\varepsilon^{(),}, j-1} \sqrt{\frac{j}{2}}\right)\right) \otimes H_{\alpha} .
$$

Proof. By differentiating $f$ component wise in weak $S^{\prime}(\mathbb{R})$ sense we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} f(t) & =\sum_{\alpha \in \mathcal{I}}\left(\sum_{j=1}^{\infty} f_{\alpha, j} \frac{\mathrm{~d}}{\mathrm{~d} t} \xi_{j}(t)\right) H_{\alpha} \\
& =\sum_{\alpha \in \mathcal{I}}\left(\sum_{j=1}^{\infty} f_{\alpha, j}\left(\sqrt{\frac{j}{2}} \xi_{j-1}(t)-\sqrt{\frac{j+1}{2}} \xi_{j+1}(t)\right)\right) H_{\alpha},
\end{aligned}
$$

by the well-known identity formula for derivatives of Hermite functions [7]. This is further equal to

$$
\sum_{\alpha \in \mathcal{I}}\left(f_{\alpha, 2} \xi_{1}(t)+\sum_{j=2}^{\infty}\left(f_{\alpha, j+1} \sqrt{\frac{j+1}{2}}-f_{\alpha, j-1} \sqrt{\frac{j}{2}}\right) \xi_{j}(t)\right) H_{\alpha}
$$

Note that if $f_{\alpha}=\sum_{j=1}^{\infty} f_{\alpha, j} \xi_{j} \in X \otimes S_{-k}(\mathbb{R}), \alpha \in \mathcal{I}$ then $(\mathrm{d} / \mathrm{d} t) f_{\alpha} \in X \otimes S_{-(k+1)}(\mathbb{R})$ since

$$
\begin{aligned}
\left\|\frac{\mathrm{d}}{\mathrm{~d} t} f_{\alpha}\right\|_{X \otimes S_{-(k+1)}(\mathbb{R})}^{2} & \leq\left\|f_{\alpha, 2}\right\|_{X}^{2}+\sum_{j=1}^{\infty}\left(\left\|f_{\alpha, j+1}\right\|_{X}^{2}(j+1)+\left\|f_{\alpha, j-1}\right\|_{X}^{2}(j)\right)(2 j)^{-(k+1)} \\
& \leq C \sum_{j=1}^{\infty}\left\|f_{\alpha, j}\right\|_{X}^{2}(2 j)^{-k}=C\left\|f_{\alpha}\right\|_{X \otimes S_{-k}(\mathbb{R})}^{2}<\infty
\end{aligned}
$$

Thus, applying Theorem 4.1 to the equation

$$
\mathbb{D}(u)=\sum_{\alpha \in \mathcal{I}}\left(f_{\alpha, 2} \xi_{1}(t)+\sum_{j=2}^{\infty}\left(f_{\alpha, j+1} \sqrt{\frac{j+1}{2}}-f_{\alpha, j-1} \sqrt{\frac{j}{2}}\right) \xi_{j}(t)\right) H_{\alpha},
$$

where the right hand side is an element of $X \otimes S_{-(k+1)}(\mathbb{R}) \otimes(S)_{-1,-p}, p>k+1$, we obtain a unique solution of the form

$$
u=\tilde{u}_{0}+\sum_{\alpha \in \mathcal{T},|\alpha|>0} \frac{1}{|\alpha|}\left(f_{\alpha-\varepsilon^{(1)}, 2}+\sum_{j=2}^{\infty}\left(f_{\alpha-\varepsilon^{(1)}, j+1} \sqrt{\frac{j+1}{2}}-f_{\alpha-\varepsilon^{(j)}, j-1} \sqrt{\frac{j}{2}}\right)\right) H_{\alpha}
$$

that satisfies

$$
\begin{aligned}
\|u\|_{\text {Dom }_{p}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq\left\|\tilde{u}_{0}\right\|_{X}^{2}+C \sum_{|\alpha|>0} \sum_{j=1}^{\infty}\left\|f_{\alpha j}\right\|_{X}^{2}(2 j)^{-k}(2 \mathbb{N})^{-p \alpha} \\
& =\left\|\tilde{u}_{0}\right\|_{X}^{2}+C\|f\|_{X \otimes S_{-k}(\mathbb{R}) \otimes(S)_{-1,-p}^{2}}^{2}<\infty .
\end{aligned}
$$

Now we turn to the case of equations involving higher orders of the Malliavin derivative. Define $\mathbb{D}^{0}=I d, \mathbb{D}^{(k)}=\mathbb{D} \circ \mathbb{D}^{(k-1)}, \quad k=1,2,3, \ldots$ and recall that $\mathbb{D}: \operatorname{Dom}_{p}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}$, for $l>p+1$. For higher order derivatives to be well defined, it is necessary that each result of the application of the operator $\mathbb{D}$ remains in its domain. For this purpose we note that if $u \in X \otimes(S)_{-1,-q}$ for some $q \geq 0$, then there always exists $p>q$ such that $u \in \operatorname{Dom}_{p}(\mathbb{D})$. This follows from the fact that $|\alpha| \leq(2 \mathbb{N})^{\alpha}$, and thus

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(p-2) \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}<\infty,
$$

for $p \geq q+2$.
Thus, e.g. for $\mathbb{D}^{(2)}$ :

$$
\operatorname{Dom}_{p}(\mathbb{D}) \xrightarrow{\mathbb{D}} X \otimes S_{-l_{1}} \otimes(S)_{-1,-p} \subseteq S_{-l_{1}} \otimes \operatorname{Dom}_{p+2}(\mathbb{D}) \xrightarrow{\mathbb{D}} S_{-l_{1}} \otimes S_{-l_{2}} \otimes X \otimes(S)_{-1,-(p+2)}
$$

where $l_{1}>p+1$ and $l_{2}>p+3$.
Similarly, for any $k \in \mathbb{N}$,

$$
\mathbb{D}^{(k)}: X \otimes(S)_{-1,-(p-2)} \subset \operatorname{Dom}_{p}(\mathbb{D}) \rightarrow X \otimes S_{-l_{1}} \otimes S_{-l_{2}} \otimes \cdots \otimes S_{-l_{k}} \otimes(S)_{-1,-(p+2 k)}
$$

where $l_{j}>p+1+2(j-1), j=1,2, \ldots, k$.
Theorem 4.5. Let $h \in X \otimes S_{-p_{1}}(\mathbb{R}) \otimes S_{-p_{2}}(\mathbb{R}) \cdots \otimes S_{-p_{k}}(\mathbb{R}) \otimes(S)_{-1,-q}$ be of the form

$$
h=\sum_{\alpha \in \mathcal{I}} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} h_{\alpha, i_{1}, i_{2}, \ldots, i_{k}} \otimes \xi_{i_{1}} \otimes \xi_{i_{2}} \otimes \ldots \otimes \xi_{i_{k}} \otimes H_{\alpha} .
$$

The equation

$$
\begin{equation*}
D^{(k)} u=h \tag{14}
\end{equation*}
$$

together with the initial conditions

$$
\begin{equation*}
E u=\tilde{u}_{0} \in X, \quad E(\mathbb{D} u)=\tilde{u}_{1} \in X \otimes S^{\prime}(\mathbb{R}), \ldots, \quad E\left(\mathbb{D}^{(k-1)} u\right)=\tilde{u}_{k-1} \in X \otimes S^{\prime}(\mathbb{R})^{\otimes(k-1)} \tag{15}
\end{equation*}
$$

where $\tilde{u}_{j} \in X \otimes S_{-p_{(k-i+1)}}(\mathbb{R}) \otimes \cdots \otimes S_{-p_{k}}(\mathbb{R}), j=1,2, \ldots, k-1$, is of the form

$$
\tilde{u}_{j}=\sum_{i_{(k-j+1)}=1}^{\infty} \sum_{i_{(k-j+2)}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{j, i_{(k-j+1)}, i_{(k-j+2)}, \ldots, i_{k}} \otimes \tilde{\xi}_{(k-j+1)} \otimes \tilde{\xi}_{(k-j+2)} \otimes \ldots \otimes \xi_{i_{k}},
$$

has a unique solution $u \in \operatorname{Dom}_{q}(\mathbb{D}), q>1+\max \left\{p_{1}, \ldots, p_{k}\right\}$, of the form

$$
\begin{align*}
u= & \tilde{u}_{0}+\sum_{i_{k}=1}^{\infty} \tilde{u}_{1, i_{k}} \otimes H_{\varepsilon^{\left(i_{k}\right)}}+\frac{1}{2} \sum_{i_{k-1}=1}^{\infty} \sum_{i_{k}=1}^{\infty} \tilde{u}_{2, i_{k-1}, i_{k}} \otimes H_{\varepsilon^{\left(i_{k-1}\right)}+\varepsilon^{\left(i_{k}\right)}} \\
& +\frac{1}{3} \sum_{i_{k-2}=1}^{\infty} \sum_{i_{k-1}=1}^{\infty} \sum_{i_{k}=1}^{\infty} \tilde{u}_{3, i_{k-2}, i_{k-1}, i_{k}} \otimes H_{\varepsilon^{\left(i_{k-2}\right)+\varepsilon^{\left(i_{k-1}\right)}+\varepsilon^{\left(i_{k}\right)}}}  \tag{16}\\
& +\cdots+\frac{1}{k} \sum_{i_{2}=1}^{\infty} \sum_{i_{3}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{k-1, i_{2}, \ldots, i_{k}} \otimes H_{\varepsilon^{\left(i_{2}\right)}+\varepsilon^{\left(i_{3}\right)}+\cdots+\varepsilon^{\left(i_{k}\right)}} \\
& +\sum_{|\alpha| \geq k} \frac{1}{|\alpha|^{k}} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} h_{\alpha-\varepsilon^{\left(i_{1}\right)-\varepsilon^{(i 2)}-\cdots-\varepsilon^{\left(i_{k}, i_{1}, i_{2}, \cdots, i_{k}\right.}} \otimes H_{\alpha} .} .
\end{align*}
$$

Proof. The proof follows by induction on $k$ and Theorem 4.1.
Applying Theorem 4.1 to the equation

$$
\mathbb{D}\left(\mathbb{D}^{(k-1)}(u)\right)=h,
$$

where $h=\sum_{\alpha \in \mathcal{I}} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} h_{\alpha, i_{1}, i_{2}, \ldots, i_{k}} \otimes \xi_{i_{1}} \otimes \xi_{i_{2}} \otimes \ldots \otimes \xi_{i_{k}} \otimes H_{\alpha}$ we obtain the solution in form of

$$
\mathbb{D}^{(k-1)}(u)=\tilde{u}_{k-1}+\sum_{|\alpha| \geq 1} \frac{1}{|\alpha|} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} h_{\alpha-\varepsilon} \varepsilon_{\left.i i_{1}\right), i_{1}, i_{2}, \ldots, i_{k}} \otimes \xi_{i_{2}} \otimes \xi_{i_{3}} \otimes \cdots \otimes \xi_{i_{k}} \otimes H_{\alpha} .
$$

Applying Theorem 4.1 once again and using that

$$
\tilde{u}_{k-1}=\sum_{i_{2}=1}^{\infty} \sum_{i_{3}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{k-1, i_{2}, \ldots, i_{k}} \otimes \xi_{i_{2}} \otimes \xi_{i_{3}} \otimes \ldots \otimes \xi_{i_{k}}
$$

we obtain

$$
\begin{aligned}
\mathbb{D}^{(k-2)}(u)= & \tilde{u}_{k-2}+\sum_{i_{2}=1}^{\infty} \sum_{i_{3}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{k-1, i_{2}, \ldots, i_{k}} \otimes \xi_{i_{3}} \otimes \cdots \otimes \xi_{i_{k}} \otimes H_{\varepsilon^{(i 2)}} \\
& +\sum_{|\alpha| \geq 2} \frac{1}{|\alpha|^{2}} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} h_{\alpha-\varepsilon^{\left(i_{1}\right)}-\varepsilon^{\left(i_{2}\right), i_{1}, i_{2}, \ldots, i_{k}}} \otimes \xi_{i_{3}} \otimes \cdots \otimes \xi_{i_{k}} \otimes H_{\alpha} .
\end{aligned}
$$

Following the same procedure with

$$
\tilde{u}_{k-2}=\sum_{i_{3}=1}^{\infty} \sum_{i_{4}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{k-2, i_{3}, \ldots, i_{k}} \otimes \xi_{i_{3}} \otimes \xi_{i_{4}} \otimes \ldots \otimes \xi_{i_{k}}
$$

we obtain

$$
\begin{aligned}
\mathbb{D}^{(k-3)}(u)= & \tilde{u}_{k-3}+\sum_{i_{3}=1}^{\infty} \sum_{i_{4}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{k-2, i_{3} \ldots, i_{k}} \otimes \xi_{i_{4}} \otimes \ldots \otimes \xi_{i_{k}} \otimes H_{\varepsilon^{\left(i_{3}\right)}} \\
& +\frac{1}{2} \sum_{i_{2}=1}^{\infty} \sum_{i_{3}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} \tilde{u}_{k-1, i_{2}, \ldots, i_{k}} \otimes \xi_{i_{4}} \otimes \cdots \otimes \xi_{i_{k}} \otimes H_{\varepsilon^{\left(i_{2}\right)}+\varepsilon^{\left(i_{3}\right)}} \\
& +\sum_{|\alpha| \geq 3} \frac{1}{|\alpha|^{3}} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty} h_{\alpha-\varepsilon^{\left(i_{1}\right)}-\varepsilon^{\left(i_{2}\right)}-\varepsilon^{\left(i_{3}\right), i_{1}, i_{2}, \ldots, i_{k}}} \otimes \xi_{i_{4}} \otimes \cdots \otimes \xi_{i_{k}} \otimes H_{\alpha} .
\end{aligned}
$$

After another $k-3$ steps we obtain (16).
Convergence of the series given in (16) follows from

$$
\begin{gathered}
\|u\|_{D o m_{q}(\mathbb{D})}^{2} \leq C(k)\left(\left\|\tilde{u}_{0}\right\|_{X}^{2}+\left\|\tilde{u}_{1}\right\|_{X \otimes S_{-p}(\mathbb{R})}^{2}+\cdots+\left\|\tilde{u}_{k-1}\right\|_{X \otimes S_{-p}(\mathbb{R})^{\otimes(k-1)}}^{2}\right. \\
\left.+\|h\|_{X \otimes S_{-p}(\mathbb{R})^{\otimes k} \otimes\left(S S_{-1,-q}\right.}^{2}\right)<\infty,
\end{gathered}
$$

where $C(k)$ is a constant depending only on $k, p=\max \left\{p_{1}, \ldots, p_{k}\right\}$ and $q>p+1$ according to the assumption.

## 5. Equation with the Skorokhod integral

We consider now the integral equation

$$
\begin{equation*}
\delta(u)=f, \tag{17}
\end{equation*}
$$

where $\delta$ denotes the Skorokhod integral. We look for the solution in Range( $\mathbb{D}$ ). It is clear that $u \in \operatorname{Range}(\mathbb{D})$ is equivalent to $u=\mathbb{D}(\tilde{u})$, for some $\tilde{u}$. This approach is general enough, since according to Theorem 4.1, for all $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ there exists $\tilde{u} \in$ $X \otimes(S)_{-1}$ such that $u=\mathbb{D}(\tilde{u})$ holds, i.e. Range $(\mathbb{D})=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$.

Theorem 5.1. Let $f \in X \otimes(S)_{-1,-p}, p \in \mathbb{N}_{0}$, with zero expectation, have the chaos expansion representation of the form

$$
f=\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}, \quad f_{\alpha} \in X .
$$

Then the integral Equation (17) has a unique solution $u$ in $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}$, for $l>p+1$, given by

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} . \tag{18}
\end{equation*}
$$

Proof. Equation (17) is equivalent to the system of equations

$$
\left\{\begin{array} { l } 
{ u = \mathbb { D } ( \tilde { u } ) } \\
{ \delta ( \mathbb { D } ( \tilde { u } ) ) = f , }
\end{array} \quad \text { i.e. } \quad \left\{\begin{array}{l}
u=\mathbb{D}(\tilde{u}) \\
\mathcal{R}(\tilde{u})=f .
\end{array}\right.\right.
$$

First we solve the equation

$$
\begin{equation*}
\mathcal{R}(\tilde{u})=f \tag{19}
\end{equation*}
$$

We are looking for the solution in the form $\tilde{u}=\sum_{\alpha \in \mathcal{I}} \tilde{u} \otimes H_{\alpha}$, where $\tilde{u}_{\alpha} \in X$ are the unknown coefficients. Therefore, from

$$
\mathcal{R}(\tilde{u})=\sum_{\alpha \in \mathcal{I}}|\alpha| \tilde{u}_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{T},|\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}
$$

it follows the form of the coefficients

$$
\tilde{u}_{\alpha}=\frac{f_{\alpha}}{|\alpha|}, \quad \alpha \in \mathcal{I}, \quad|\alpha| \geq 1 .
$$

Hence, the solution of Equation (19) is represented in the form

$$
\begin{equation*}
\tilde{u}=\tilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} \frac{f_{\alpha}}{|\alpha|} \otimes H_{\alpha}, \tag{20}
\end{equation*}
$$

where $\tilde{u}_{(0,0,0, \ldots)}=\tilde{u}_{0}$ can be chosen arbitrarily. Now, the solution (18) of the initial Equation (17) is obtained after applying the operator $\mathbb{D}$ to the solution (20), i.e. from

$$
\begin{aligned}
u=\mathbb{D}(\tilde{u}) & =\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} \sum_{k \in \mathbb{N}} \alpha_{k} \frac{f_{\alpha}}{|\alpha|} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} .
\end{aligned}
$$

Therefore if we are looking for the solution in the form

$$
u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha},
$$

then the coefficients of the solution are

$$
\begin{equation*}
u_{\alpha, k}=\frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|}\left(\alpha_{k}+1\right), \quad \alpha \in \mathcal{I}, k \in \mathbb{N} . \tag{21}
\end{equation*}
$$

It remains to prove the convergence of the solution (18) in $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. Under the assumption $f \in X \otimes(S)_{-1,-p}$, for some $p \geq 0$, it follows from Theorem 3 that $\tilde{u} \in \operatorname{Dom}_{p}(\mathbb{D})$.

Hence, the convergence of the solution $u$ in the space $X \otimes S_{-l} \otimes(S)_{-1,-p}$, for $l>p+1$, follows from

$$
\begin{aligned}
\|u\|_{X \otimes S_{-l} \otimes(S)_{-1,-p}}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|u_{\alpha, k}\right\|_{X}^{2}\left\|\xi_{k}\right\|_{-l}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{\left(\alpha_{k}+1\right)^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}\left\|\xi_{k}\right\|_{-l}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}\left\|\xi_{k}\right\|_{-l}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I},|\alpha|>0} \sum_{k \in \mathbb{N}}\left\|f_{\alpha}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p\left(\alpha-\varepsilon^{(k)}\right)} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|f_{\alpha}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}(2 k)^{p} \\
& \leq M \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty,
\end{aligned}
$$

since $M=\sum_{k \in \mathbb{N}}(2 k)^{p-l}$ is finite for $l>p+1$.

Remark 1. It is well known that $\mathbb{D}$ and $\delta$ do not commute and the following holds. If $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$, then $\mathbb{D}(\delta u)=u+\delta(\mathbb{D} u)$.

Theorem 5.2.
(a) Let $f \in X \otimes(S)_{-1}$ be of the form $f=\sum_{k=0}^{\infty} f_{k} H_{\varepsilon^{(k)}}$. Then, for any $h \in X \otimes(S)_{-1}$ of the form $h=\sum_{\alpha \in \mathcal{I}} h_{\alpha} H_{\alpha}$,

$$
\begin{equation*}
h \cdot f-h \diamond f=\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} h_{\alpha+\varepsilon^{(k)}} f_{k}\left(\alpha_{k}+1\right) H_{\alpha}, \tag{22}
\end{equation*}
$$

where the right-hand side is understood as a formal (not necessarily convergent) expansion in $X \otimes(S)_{-1}$.
(b) Especially, if $g \in X \otimes S(\mathbb{R})$, where $g$ denotes the unique solution to $\delta(g)=f$, then

$$
h \cdot \delta(g)-h \diamond \delta(g)=\langle\mathbb{D}(h), g\rangle
$$

holds in $X \otimes(S)_{-1}$.
(c) Especially, if $h \in X \otimes(S)_{1}$ and $g \in X \otimes S^{\prime}(\mathbb{R})$, where $g$ denotes the unique solution to $\delta(g)=f$, then

$$
h \cdot \delta(g)-h \diamond \delta(g)=\langle g, \mathbb{D}(h)\rangle
$$

holds in $X \otimes(S)_{-1}$.
(d) In case $g \in X \otimes S(\mathbb{R})$ and $\mathbb{D}(h) \in X \otimes L^{2}(\mathbb{R}) \otimes(S)_{-1}$, as well as in the case $g \in$ $X \otimes L^{2}(\mathbb{R})$ and $\mathbb{D}(h) \in X \otimes L^{2}(\mathbb{R}) \otimes(S)_{1}$, formula (22) reduces to

$$
h \cdot \delta(g)-h \diamond \delta(g)=\int_{\mathbb{R}} g(t) \cdot \mathbb{D}(h)(t) \mathrm{d} t .
$$

Proof. (a) Assume $E(f)=f_{0}=0$. Then, according to Theorem 5.1 there exists a unique $g$ such that $\delta(g)=f$ and moreover this $g$ is given by $g=\sum_{k=1}^{\infty} f_{k} \xi_{k}$ as an element of $X \otimes S^{\prime}(\mathbb{R})$. Thus,

$$
h \diamond f=h \diamond \delta(g)=\sum_{\gamma \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\gamma-\varepsilon^{(n)}} f_{n} H_{\gamma}
$$

and

$$
\begin{aligned}
h \cdot \delta(g) & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} f_{n} H_{\alpha-\varepsilon^{(n)}} H_{\varepsilon^{(n)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} f_{n}\left(H_{\alpha}+\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}}\right) .
\end{aligned}
$$

This implies

$$
\begin{aligned}
h \cdot \delta(g)-h \diamond \delta(g) & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} f_{n}\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} f_{n}\left(\alpha_{n}+1\right) H_{\alpha} .
\end{aligned}
$$

Now, for arbitrary $f$ let $\tilde{f}=f-E(f)$ and $\tilde{g}$ such that $f=E(f)+\delta(\tilde{g})$. Since for constants the Wick product and the ordinary product coincide, we have

$$
\begin{aligned}
h \cdot f-h \diamond f & =h \cdot E(f)+h \cdot \delta(\tilde{g})-h \diamond E(f)-h \diamond \delta(\tilde{g})=h \cdot \delta(\tilde{g})-h \diamond \delta(\tilde{g}) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} f_{n}\left(\alpha_{n}+1\right) H_{\alpha} .
\end{aligned}
$$

Convergence of the series on the right-hand side of (22) will be proven only in the special cases (b), (c) and (d).
(b) Since $g=\sum_{k=1}^{\infty} f_{k} \xi_{k}$ and $f_{k}=\left\langle\xi_{k}, g\right\rangle, k \in \mathbb{N}$, which reduces to $f_{k}=\int_{\mathbb{R}} g(t) \xi_{k}(t) \mathrm{d} t$ in case of $g \in L^{2}(\mathbb{R})$ and since $\mathbb{D}(h)=\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}\left(\alpha_{n}+1\right) \xi_{n} H_{\alpha}$, we may write the right-hand side of (22) as

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}\left(\alpha_{n}+1\right)\left\langle\xi_{n}, g\right\rangle H_{\alpha} & =\left\langle\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}\left(\alpha_{n}+1\right) \xi_{n} H_{\alpha}, g\right\rangle \\
& =\langle\mathbb{D}(h), g\rangle
\end{aligned}
$$

Assume that $h \in X \otimes(S)_{-1,-p}$ for some $p>0$, i.e. $\sum_{\alpha \in \mathcal{I}}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$ and that $g \in X \otimes S_{l}(\mathbb{R})$ for all $l>0$ (equivalently $\left.f=\delta(g) \in X \otimes(S)_{1, l}\right)$, i.e. $\sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{l}<\infty$. Then $h \cdot \delta(g)-h \diamond \delta(g)=\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha} f_{n} \alpha_{n} H_{\alpha-\varepsilon^{(n)}}$ is well defined in $X \otimes(S)_{-1,-q}$ for
$q \geq p+2$. This follows from the fact that $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ and thus

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}\left|\alpha_{n}\right|^{2}(2 \mathbb{N})^{-q\left(\alpha-\varepsilon^{(n)}\right)} & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}\left|\alpha_{n}\right|^{2}(2 \mathbb{N})^{-q \alpha}(2 n)^{q} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(q-2) \alpha} \sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{q} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{l}<\infty
\end{aligned}
$$

for $q-2 \geq p$ and $q \leq l$. Since $l$ is arbitrary this holds for all $q \geq p+2$.
(c) Since $h \in X \otimes(S)_{1}, \quad \sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty$ for all $p>0$. Assume $g \in$ $X \otimes S_{-l}(\mathbb{R})$ for some $l>0$, i.e. $\sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{-l}<\infty$. Similarly as in (b) we can show that the right-hand side of (22) is equal to $\langle g, \mathbb{D}(h)\rangle$ and $h \cdot \delta(g)-h \diamond \delta(g)=$ $\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha} f_{n} \alpha_{n} H_{\alpha-\varepsilon^{(n)}}$ is well defined in $X \otimes(S)_{-1,-q}$ for $q=l$. Indeed, for all $n \in \mathbb{N}, \quad q>0$ and $\alpha \in \mathcal{I}$ such that $\alpha_{n} \neq 0$, we have $(2 n)^{q} \leq(2 \mathbb{N})^{q \alpha}$ and $(2 \mathbb{N})^{-q \alpha} \leq(2 n)^{-q}$. Since $\alpha-\varepsilon^{(n)}$ is not defined if $\alpha_{n}=0$, we now obtain

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}\left|\alpha_{n}\right|^{2}(2 \mathbb{N})^{-q\left(\alpha-\varepsilon^{(n)}\right)} & =\sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}\left|\alpha_{n}\right|^{2}(2 \mathbb{N})^{-q \alpha}(2 n)^{q} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty}\left\|h_{\alpha}\right\|_{X}^{2}\left\|f_{n}\right\|_{X}^{2}(2 \mathbb{N})^{2 \alpha}(2 n)^{-q}(2 \mathbb{N})^{q \alpha} \\
& \leq \sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|h_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{(2+q) \alpha} \sum_{n=1}^{\infty}\left\|f_{n}\right\|_{X}^{2}(2 n)^{-q}
\end{aligned}
$$

which is finite for $q=l$.
(d) The proof is similar to those in (b) and (c).

Now we turn to the case of equations involving higher orders of the Skorokhod integral. Define $\delta^{0}=I d, \delta^{(k)}=\delta \circ \delta^{(k-1)}, k \in \mathbb{N}$ and recall that $\delta: X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-r} \rightarrow$ $X \otimes(S)_{-1,-p}, p \geq r, p>l+1$.

Thus, for any $k \in \mathbb{N}$,

$$
\delta^{(k)}: X \otimes S_{-l_{1}} \otimes S_{-l_{2}} \otimes \cdots \otimes S_{-l_{k}} \otimes(S)_{-1,-r} \rightarrow X \otimes(S)_{-1,-p}
$$

for $p \geq r, p>\max \left\{l_{1}, l_{2}, \ldots, l_{k}\right\}+1$.
TheOrem 5.3. Let $f \in X \otimes(S)_{-1,-p}, p \in \mathbb{N}_{0}$, with zero expectation and zero terms in the chaos subspaces spanned by $H_{\alpha},|\alpha|=1,2, \ldots, k-1$, have the chaos expansion form $f=\sum_{|\alpha| \geq k} f_{\alpha} \otimes H_{\alpha}, f_{\alpha} \in X, \alpha \in \mathcal{I}$. Then the integral equation

$$
\begin{equation*}
\delta^{(k)} u=f \tag{23}
\end{equation*}
$$

has a unique solution $u$ in $X \otimes S_{-l_{1}} \otimes S_{-l_{2}} \otimes \ldots \otimes S_{-l_{k}} \otimes(S)_{-1,-p}$, where $\max \left\{l_{1}, l_{2}, \ldots, l_{k}\right\}>$ $p+1$, and it is given by

$$
\begin{gather*}
u=\sum_{\alpha \in \mathcal{I}} \sum_{i_{1}=1}^{\infty} \sum_{i_{2}=1}^{\infty} \cdots \sum_{i_{k}=1}^{\infty}\left(\alpha_{i_{1}}+1\right)\left(\alpha_{i_{2}}+1\right) \cdots\left(\alpha_{i_{k}}+1\right) f_{\alpha+\varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{2}\right)}+\cdots+\varepsilon^{\left(i_{k}\right)}}  \tag{24}\\
\left(\left|\alpha+\varepsilon^{\left(i_{k}\right)} \| \alpha+\varepsilon^{\left(i_{k-1}\right)}+\varepsilon^{\left(i_{k}\right)}\right| \cdots\left|\alpha+\varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{2}\right)}+\cdots+\varepsilon^{\left(i_{k}\right)}\right|\right)^{-1} \otimes \xi_{i_{1}} \otimes \cdots \otimes \xi_{i_{k}} \otimes H_{\alpha} .
\end{gather*}
$$

Proof. The proof follows by induction on $k$ and Theorem 5.1.
The equation $\delta\left(\delta^{(k-1)} u\right)=f, E(f)=0$, has according to Theorem 5.1 the solution

$$
\delta^{(k-1)} u=\sum_{\alpha \in \mathcal{I}} \sum_{i_{1}=1}^{\infty}\left(\alpha_{i_{1}}+1\right) \frac{f_{\alpha+\varepsilon^{\left(i_{1}\right)}}}{\left|\alpha+\varepsilon^{\left(i_{1}\right)}\right|} \otimes \xi_{i_{1}} \otimes H_{\alpha}
$$

and by assumption $f_{\beta}=0$ for $|\beta|=0,1$, thus $E\left(\delta^{(k-1)} u\right)=0$. Applying Theorem 5.1 once again we obtain the solution to the equation

$$
\delta\left(\delta^{(k-2)} u\right)=\sum_{\alpha \in \mathcal{I}} \sum_{i_{1}=1}^{\infty}\left(\alpha_{i_{1}}+1\right) \frac{f_{\alpha+\varepsilon^{\left(i_{1}\right)}}}{\left|\alpha+\varepsilon^{\left(i_{1}\right)}\right|} \otimes \xi_{i_{1}} \otimes H_{\alpha}
$$

in form of

$$
\delta^{(k-2)} u=\sum_{\alpha \in \mathcal{I}} \sum_{i_{2}=1}^{\infty} \sum_{i_{1}=1}^{\infty}\left(\alpha_{i_{2}}+1\right)\left(\alpha_{i_{1}}+1\right) \frac{f_{\alpha+\varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{2}\right)}}}{\left|\alpha+\varepsilon^{\left(i_{2}\right)} \| \alpha+\varepsilon^{\left(i_{1}\right)}+\varepsilon^{\left(i_{2}\right)}\right|} \otimes \dot{\xi}_{i_{1}} \otimes \dot{\xi}_{i_{2}} \otimes H_{\alpha} .
$$

Since $f_{\beta}=0$ for $|\beta|=0,1,2$, we have $E\left(\delta^{(k-2)} u\right)=0$ and we may apply Theorem 5.1 again to obtain an explicit form for $\delta^{(k-3)} u$. Altogether after $k$ steps one obtains the solution $u$ in form (24).

Convergence of the series follows from

$$
\begin{aligned}
&\|u\|_{X \otimes S_{-l_{1}} \otimes \cdots \otimes S_{-l_{k}} \otimes(S)_{-1,-p}} \leq \sum_{\alpha \in \mathcal{I}} \sum_{i_{1}, \ldots, i_{k}} \| f_{\alpha+\varepsilon^{\left(i_{1}\right)}+\cdots+\varepsilon^{\left(i_{k}\right.} \|_{X}^{2}\left(2 i_{1}\right)^{-l_{1}} \cdots\left(2 i_{k}\right)^{-l_{k}}(2 \mathbb{N})^{-p \alpha}} \\
& \leq \sum_{\alpha \in I} \sum_{i_{1}, \ldots, i_{k}}\left\|f_{\alpha}\right\|_{X}^{2}\left(2 i_{1}\right)^{-l_{1}} \cdots\left(2 i_{k}\right)^{-l_{k}}(2 \mathbb{N})^{-p\left(\alpha-\varepsilon^{\left(i_{1}\right)}-\cdots-\varepsilon^{\left(i_{k}\right)}\right)} \\
& \leq \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \sum_{i_{1} \in \mathbb{N}}\left(2 i_{1}\right)^{-\left(l_{1}-p\right)} \cdots \sum_{i_{k} \in \mathbb{N}}\left(2 i_{k}\right)^{-\left(l_{k}-p\right)}<\infty,
\end{aligned}
$$

for $l_{i}-p>1, i=1,2, \ldots, k$.

Corollary 5.4. For every $k \in \mathbb{N}$, each process $f \in X \otimes(S)_{-1}$ can be represented as

$$
f=E(f)+\sum_{j=1}^{k} \delta^{(j)}\left(u_{j}\right),
$$

for some $u_{j} \in X \otimes \underbrace{S^{\prime}(\mathbb{R}) \otimes \ldots \otimes S^{\prime}(\mathbb{R})}_{j} \otimes(S)_{-1}, j=1,2, \ldots, k$.

Proof. Assume first that $k=1$. If $E(f)=0$, then by Theorem 5.1 follows that there exists $u$ such that $f=\delta(u)$. Otherwise, let $\tilde{f}=f-E(f), E(\tilde{f})=0$, and apply the previous case to obtain $\tilde{u}$ such that $\tilde{f}=\delta(\tilde{u})$, where $\tilde{u}=\mathbb{D}(\tilde{v}), \tilde{v}=\mathcal{R}^{-1}(\tilde{f})$, i.e. $\tilde{u}=\mathbb{D}\left(\mathcal{R}^{-1}(\tilde{f})\right)$. Thus, $f-E(f)=\delta\left(\mathbb{D}\left(\mathcal{R}^{-1}(f-E(f))\right)\right.$, i.e.

$$
\begin{equation*}
f=E(f)+\delta(u) \quad \text { for } u=\mathbb{D}\left(\mathcal{R}^{-1}(f-E(f))\right. \tag{25}
\end{equation*}
$$

Now, for $k=2$, it holds that $f=E(f)+\delta\left(\tilde{u}_{1}\right)=E(f)+\delta\left(E\left(\tilde{u}_{1}\right)+\delta\left(u_{2}\right)\right)=E(f)+$ $\delta\left(u_{1}\right)+\delta^{(2)}\left(u_{2}\right), \tilde{u}_{1}=\left(\mathbb{D} \circ \mathcal{R}^{-1}\right)(f-E(f)), u_{2}=\left(\mathbb{D} \circ \mathcal{R}^{-1}\right)\left(\tilde{u}_{1}-E\left(\tilde{u}_{1}\right)\right), u_{1}=E\left(\tilde{u}_{1}\right)$.

For arbitrary $k \in \mathbb{N}$ we define recursively $\tilde{u}_{1}=\left(\mathbb{D} \circ \mathcal{R}^{-1}\right)(f-E(f)), \tilde{u}_{j}=\left(\mathbb{D} \circ \mathcal{R}^{-1}\right) \times$ $\left(\tilde{u}_{j-1}-E\left(\tilde{u}_{j-1}\right)\right)$ for $j=2,3, \ldots, k-1$, let $u_{j}=E\left(\tilde{u}_{j}\right), \quad j=1,2, \ldots, k-1 \quad$ and $u_{k}=\left(\mathbb{D} \circ \mathcal{R}^{-1}\right)\left(\tilde{u}_{k-1}-E\left(\tilde{u}_{k-1}\right)\right)$. With this choice of the integrands $u_{j}$ we obtain

$$
f=E(f)+\delta\left(u_{1}\right)+\delta^{(2)}\left(u_{2}\right)+\cdots+\delta^{(k)}\left(u_{k}\right)
$$

Remark 2. Note that the statement of Corollary 5.4 reduces to the celebrated Itô representation theorem (see, e.g. $[7,23]$ ) in case when $f$ is a square integrable adapted process.

## 6. Examples

(1) The following table provides some illustrative examples to Theorems 3.1, 4.1 and 5.1. In all examples, $\kappa_{[0, t]}$ denotes the characteristic function of the interval $[0, t]$, $\mathrm{d}_{t}$ denotes the Dirac delta function concentrated at the point $t, W_{t}=$ $\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon^{(k)}}$ denotes singular white noise, $B_{t}=\sum_{k=1}^{\infty}\left(\int_{0}^{t} \xi_{k}(s) \mathrm{d} s\right) H_{\varepsilon^{(k)}}$ denotes Brownian motion and $Z=\sum_{k=1}^{\infty} H_{2 \varepsilon^{(k)}}$ is a Kondratiev generalized random variable. Complete explanations and calculations can be found in [11].

| Equations | Solutions |
| :--- | :--- |
| $\mathcal{R}(u)=B_{t_{0}}, E u=\tilde{u}_{0}$ | $u=\tilde{u}_{0}+B_{t_{0}}$ |
| $\mathcal{R}(u)=W_{t_{0}}, E u=\tilde{u}_{0}$ | $u=\tilde{u}_{0}+W_{t_{0}}$ |
| $\mathcal{R}(u)=B_{t_{0}, 2}^{\diamond 2}, E u=0$ | $u=(1 / 2) B_{t_{0}}^{\diamond 2}$ |
| $\mathcal{R}(u)=W_{t_{0}}, E u=0$ | $u=(1 / 2) W_{t_{0}}^{\diamond_{2}}$ |
| $\mathcal{R}(u)=Z, E u=0$ | $u=(1 / 2) Z$ |
| $\mathbb{D}(u)=\kappa_{\left[0, t_{0}\right]}, E u=\tilde{u}_{0}$ | $u=\tilde{u}_{0}+B_{t_{0}}$ |
| $\mathbb{D}(u)=\mathrm{d}_{t_{0}}, E u=\tilde{u}_{0}$ | $u=\tilde{u}_{0}+W_{t_{0}}$ |
| $\mathbb{D}(u)=W_{t}, E u=0$ | $u=(1 / 2) Z$ |
| $\mathbb{D}(u)=B_{t_{0}} \kappa_{\left[0, t_{0}\right]}, E u=0$ | $u=(1 / 2) B_{t_{0}}^{\diamond 2}$ |
| $\mathbb{D}(u)=W_{t_{0}} \mathrm{~d}_{t_{0}}, E u=0$ | $u=(1 / 2) W_{t_{0}}^{\diamond_{2}}$ |
| $\delta(u)=B_{t_{0}}$ | $u=\kappa_{\left[0, t_{0}\right]}$ |
| $\delta(u)=W_{t_{0}}$ | $u=d_{t_{0}}$ |
| $\delta(u)=(1 / 2) B_{t_{0}}^{\diamond 2}=(1 / 2)\left(B_{t_{0}}^{2}-t_{0}\right)$ | $u=B_{t} \kappa \kappa_{\left[0, t_{0}\right]}$ |
| $\delta(u)=(1 / 2) W_{t_{0}}^{\diamond 2}$ | $u=W_{t} \mathrm{~d}_{t_{0}}$ |
| $\delta(u)=Z$ | $u=W_{t}$ |

(2) Examples of equations with second-order iterated operators, which are illustrations of Theorems 3.1, 4.5 and 5.3, are given in the following table.

| Equations | Solutions |
| :--- | :--- |
| $\mathcal{R}^{(2)} u=Z$ | $u=(1 / 4) Z$ |
| $\mathbb{D}^{(2)} u=\kappa_{\left[0, t_{0}\right]} \otimes \kappa_{\left[0, t_{0}\right]}, E u=0, E(\mathbb{D} u)=0$ | $u=(1 / 2) B_{t_{0}}^{\diamond 2}$ |
| $\delta^{(2)} u=Z$ | $u=\mathrm{d}_{t_{0}}$ |

## Disclosure statement

No potential conflict of interest was reported by the authors.

## Funding

The paper was supported by the projects Modeling and harmonic analysis methods and PDEs with singularities [grant number 174024], and Modeling and research methods of operational control of traffic based on electric traction vehicles optimized by power consumption criterion [grant number TR36047], both financed by the Ministry of Science, Republic of Serbia.

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Electron. J. Probab. 20 (2015), no. 19, 1-23. ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3696

# Stochastic evolution equations with multiplicative noise 

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#### Abstract

We study parabolic stochastic partial differential equations (SPDEs), driven by two types of operators: one linear closed operator generating a $C_{0}$-semigroup and one linear bounded operator with Wick-type multiplication, all of them set in the infinite dimensional space framework of white noise analysis. We prove existence and uniqueness of solutions for this class of SPDEs. In particular, we also treat the stationary case when the time-derivative is equal to zero.


Keywords: generalized stochastic process; chaos expansion; stochastic evolution equation; Wick product; white noise; $C_{0}$-semigroup; infinitesimal generator; stochastic operator.
AMS MSC 2010: 60H30; 60H40; 60G20; 60H07; 47D06; 46N30.
Submitted to EJP on July 25, 2014, final version accepted on February 14, 2015.

## 1 Introduction and definitions

We consider a stochastic Cauchy problem of the form

$$
\begin{align*}
\frac{d}{d t} U(t, x, \omega) & =\mathbf{A} U(t, x, \omega)+\mathbf{B} \diamond U(t, x, \omega)+F(t, x, \omega)  \tag{1.1}\\
U(0, x, \omega) & =U^{0}(x, \omega),
\end{align*}
$$

where $t \in(0, T], \omega \in \Omega$, and $U(t, \cdot, \omega)$ belongs to some Banach space $X$. The operator $\mathbf{A}$ is densely defined, generating a $C_{0}$-semigroup and $\mathbf{B}$ is a linear bounded operator which combined with the Wick product $\diamond$ introduces convolution-type perturbations into the equation. All stochastic processes are considered in the setting of Wiener-Itô chaos expansions. A comprehensive explanation of the action of the operators $\mathbf{A}$ and $\mathbf{B}$ in this framework will be provided in Section 2.

Our investigations in this paper are inspired by [12] where the authors provide a comprehensive analysis of equations of the form

$$
\frac{d}{d t} u(t, x, \omega)=\mathbf{A} u(t, x, \omega)+\delta(\mathbf{M} u(t, x, \omega))=\mathbf{A} u(t, x, \omega)+\int \mathbf{M} u(t, x, \omega) \diamond W(x, \omega) d x
$$

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where $\delta$ denotes the Skorokhod integral and $W$ denotes the spatial white noise process. In Proposition 2.8 we prove that for every operator $\mathbf{M}$ there exists a corresponding operator $\mathbf{B}$ such that $\mathbf{B} \triangleleft u=\delta(\mathbf{M} u)$. On the other hand, the class of operators $\mathbf{B}$ is much larger. This holds also for the class of operators A we consider (a comprehensive analysis of all operators is given in Section 2.1). Thus, we extend the results of [12] and [13] to a more general class of stochastic differential equations which are driven by two linear multiplicative operators: A acting with ordinary multiplication, while $\mathbf{B} \diamond$ is acting with the convolution-type Wick product.

We have studied elliptic SPDEs, in particular the stochastic Dirichlet problem of the form $\mathbf{L} \diamond u+f=0$ in our previous papers [11], [18], [19]. As a conclusion to this series of papers we study parabolic SPDEs of the form (2.1). Such equations also include as a special case equations of the form $\frac{d}{d t} u=\mathbf{L} u+f$ and $\frac{d}{d t} u=\mathbf{L} \diamond u+f$, where $L$ is a strictly elliptic second order partial differential operator. These equations describe the heat conduction in random media (inhomogeneous and anisotropic materials), where the properties of the material are modeled by a positively definite stochastic matrix.

Other special cases of (2.1) include the heat equation with random potential $\frac{d}{d t} u=$ $\Delta u+\mathbf{B} \triangleleft u$, the Schrödinger equation $(i \hbar) \frac{d}{d t} u=\Delta u+\mathbf{B} \diamond u+f$, the transport equation $\frac{d}{d t} u=$ $\frac{d^{2}}{d x^{2}} u+W \diamond \frac{d}{d x} u$ driven by white noise as in [20], the generalized Langevin equation $\frac{d}{d t} u=$ $\mathbf{J} u+\mathbf{C}\left(Y^{\prime}\right)$, where $Y$ is a Lévy process, $\mathbf{J}$ the infinitesimal generator of a $C_{0}$-semigroup and $\mathbf{C}$ a bounded operator, which was studied in [1], as well as the equation $\frac{d}{d t} u=$ $\mathbf{L} u+W \diamond u$, where $\mathbf{L}$ is a strictly elliptic partial differential operator as studied in [3] and [8].

Equations of the form $\frac{d}{d t} u=\mathbf{A} u+\mathbf{B} W$ were also studied in [14] and [15], where A is not necessarily generating a $C_{0}-$ semigroup, but an $r$-integrated or a convolution semigroup.

In order to solve (2.1) we apply the method of Wiener-Itô chaos expansions, also known as the propagator method. With this method we reduce the SPDE to an infinite triangular system of PDEs, which can be solved by induction. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial SPDE.

We also consider the case of stationary equations $\mathbf{A} U+\mathbf{B} \boxtimes U+F=0$. In particular, elliptic SPDEs have been studied in [11], [13], [18] and [19]. With the method of chaos expansions one can also treat hyperbolic SPDEs [9] and SPDEs with singularities [21]. One of its advantages is that it provides explicit solutions in terms of a series expansion, which can be easily implemented also to numerical approximations and computational simulations.

## $1.1 C_{0}$-semigroups

We recall some well-known facts which will be used in the sequel (see [16]). Let $X$ be a Banach space. If $B$ is a bounded linear operator on $X$ and $A$ is the infinitesimal generator of a $C_{0}$-semigroup $\left\{T_{t}\right\}_{t \geq 0}$ satisfying $\left\|T_{t}\right\|_{L(X)} \leq M e^{w t}, t \geq 0$, for some $M, w>0$, then $A+B$ is the infinitesimal generator of a $C_{0}-$ semigroup $\left\{S_{t}\right\}_{t \geq 0}$, on $X$ satisfying

$$
\left\|S_{t}\right\|_{L(X)} \leq M e^{\left(w+M\|B\|_{L(X)}\right) t}, t \geq 0
$$

Let $u(0)=u^{0} \in D=\operatorname{Dom}(A)$ and $f \in C([0, \infty), X)$. Recall that $u:[0, T] \rightarrow X$ is a (classical) solution on $[0, T]$ to

$$
\begin{equation*}
\frac{d}{d t} u(t)=A u(t)+f(t), t \in(0, T], \quad u(0)=u^{0} \tag{1.2}
\end{equation*}
$$

if $u$ is continuous on $[0, T]$, continuously differentiable on $(0, T], u(t) \in D, t \in(0, T]$ and the equation is satisfied on $(0, T]$. If $f \in L^{1}((0, T), X)$, then $u(t)=T_{t} u^{0}+\int_{0}^{t} T_{t-s} f(s) d s, t \in$

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$[0, T]$ belongs to $C([0, T], X)$, and it is called a mild solution. Clearly, a mild solution that is continuously differentiable on $(0, T]$ is a classical solution.

Let $f \in L^{1}((0, T), X) \cap C((0, T], X)$ and $v(t)=\int_{0}^{t} T_{t-s} f(s) d s, t \in[0, T]$. The initial value problem has a solution $u$ for every $u^{0} \in D$ if one of the following conditions is satisfied (see [16]):
(i) $v$ is continuously differentiable on $(0, T)$.
(ii) $v(t) \in D$ for $0<t \leq T$ and $A v(t)$ is continuous on ( $0, T]$.

If the initial value problem has a solution on $[0, T]$ for some $u^{0} \in D$, then $v(t)$ satisfies both (i) and (ii). Note that if $f \in C^{1}([0, T], X)$ then conditions (i) and (ii) are fulfilled. Moreover, if $f \in C^{1}([0, T], X)$ and $u^{0} \in D(A)$, then for the solution $u$ of (1.2) we have that $u \in C^{1}([0, T], X)$ and $\frac{d}{d t} u(0)=A u^{0}+f(0)$.

### 1.2 Generalized stochastic processes

Denote by $(\Omega, \mathcal{F}, P)$ the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel sigma-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to the characteristic function

$$
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=\exp \left[-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}\right], \quad \phi \in S(\mathbb{R})
$$

given by the Bochner-Minlos theorem.
We recall the notions related to $L^{2}(\Omega, \mu)$ (see [7]) where $\Omega=S^{\prime}(\mathbb{R})$ and $\mu$ is Gaussian white noise measure. Define the set of multi-indices $\mathcal{I}$ to be $\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$, i.e. the set of sequences of non-negative integers which have only finitely many nonzero components. Especially, we denote by $\mathbf{0}=(0,0,0, \ldots)$ the multi-index with all entries equal to zero. The length of a multi-index is $|\alpha|=\sum_{i=1}^{\infty} \alpha_{i}$ for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots\right) \in \mathcal{I}$, and it is always finite. Similarly, $\alpha!=\prod_{i=1}^{\infty} \alpha_{i}$ !, and all other operations are also carried out componentwise. We will use the convention that $\alpha-\beta$ is defined if $\alpha_{n}-\beta_{n} \geq 0$ for all $n \in \mathbb{N}$, i.e., if $\alpha-\beta \geq \mathbf{0}$, and leave $\alpha-\beta$ undefined if $\alpha_{n}<\beta_{n}$ for some $n \in \mathbb{N}$.

The Wiener-Itô theorem (sometimes also referred to as the Cameron-Martin theorem) states that one can define an orthogonal basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ of $L^{2}(\Omega, \mu)$, where $H_{\alpha}$ are constructed by means of Hermite orthogonal polynomials $h_{n}$ and Hermite functions $\xi_{n}$,

$$
H_{\alpha}(\omega)=\prod_{n=1}^{\infty} h_{\alpha_{n}}\left(\left\langle\omega, \xi_{n}\right\rangle\right), \quad \alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \ldots\right) \in \mathcal{I}, \quad \omega \in \Omega=S^{\prime}(\mathbb{R})
$$

Then, every $F \in L^{2}(\Omega, \mu)$ can be represented via the so called chaos expansion

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega), \quad \omega \in S^{\prime}(\mathbb{R}), \quad \sum_{\alpha \in \mathcal{I}}\left|f_{\alpha}\right|^{2} \alpha!<\infty, \quad f_{\alpha} \in \mathbb{R}, \quad \alpha \in \mathcal{I}
$$

Denote by $\varepsilon_{k}=(0,0, \ldots, 1,0,0, \ldots), k \in \mathbb{N}$ the multi-index with the entry 1 at the $k$ th place. Denote by $\mathcal{H}_{1}$ the subspace of $L^{2}(\Omega, \mu)$, spanned by the polynomials $H_{\varepsilon_{k}}(\cdot), k \in \mathbb{N}$. The subspace $\mathcal{H}_{1}$ contains Gaussian stochastic processes, e.g. Brownian motion is given by the chaos expansion $B(t, \omega)=\sum_{k=1}^{\infty} \int_{0}^{t} \xi_{k}(s) d s H_{\varepsilon_{k}}(\omega)$.

Denote by $\mathcal{H}_{m}$ the $m$ th order chaos space, i.e. the closure of the linear subspace spanned by the orthogonal polynomials $H_{\alpha}(\cdot)$ with $|\alpha|=m, m \in \mathbb{N}_{0}$. Then the Wiener-Itô chaos expansion states that $L^{2}(\Omega, \mu)=\bigoplus_{m=0}^{\infty} \mathcal{H}_{m}$, where $\mathcal{H}_{0}$ is the set of constants in $L^{2}(\Omega, \mu)$.

It is well-known that the time-derivative of Brownian motion (white noise process) does not exist in the classical sense. However, changing the topology on $L^{2}(\Omega, \mu)$ to

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a weaker one, T. Hida [6] defined spaces of generalized random variables containing the white noise as a weak derivative of the Brownian motion. We refer to [6], [7] for white noise analysis (as an infinite dimensional analogue of the Schwartz theory of deterministic generalized functions).

Let $(2 \mathbb{N})^{\alpha}=\prod_{n=1}^{\infty}(2 n)^{\alpha_{n}}, \quad \alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \ldots\right) \in \mathcal{I}$. We will often use the fact that the series $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}$ converges for $p>1$. Define the Banach spaces

$$
(S)_{1, p}=\left\{F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in L^{2}(\Omega, \mu):\|F\|_{(S)_{1, p}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{2}\left|f_{\alpha}\right|^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}, \quad p \in \mathbb{N}_{0}
$$

Their topological dual spaces are given by

$$
(S)_{-1,-p}=\left\{F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}:\|F\|_{(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left|f_{\alpha}\right|^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}, \quad p \in \mathbb{N}_{0} .
$$

The Kondratiev space of generalized random variables is $(S)_{-1}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-1,-p}$ endowed with the inductive topology. It is the strong dual of $(S)_{1}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{1, p}$, called the Kondratiev space of test random variables which is endowed with the projective topology. Thus,

$$
(S)_{1} \subseteq L^{2}(\Omega, \mu) \subseteq(S)_{-1}
$$

forms a Gelfand triplet.
The time-derivative of the Brownian motion exists in the generalized sense and belongs to the Kondratiev space $(S)_{-1,-p}$ for $p \geq \frac{5}{12}$. We refer to it as to white noise and its formal expansion is given by $W(t, \omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon_{k}}(\omega)$.

We extended in [17] the definition of stochastic processes also to processes of the chaos expansion form $U(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega)$, where the coefficients $u_{\alpha}$ are elements of some Banach space $X$. We say that $U$ is an $X$-valued generalized stochastic process, i.e. $U(t, \omega) \in X \otimes(S)_{-1}$ if there exists $p>0$ such that $\|U\|_{X \otimes(S)_{-1,-p}}^{2}=$ $\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$.

The Wick product of stochastic processes $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}, G=\sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta} \in X \otimes$ $(S)_{-1}$ is

$$
F \diamond G=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} H_{\gamma}=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} f_{\beta} g_{\alpha-\beta} H_{\alpha},
$$

and the $n$th Wick power is defined by $F^{\diamond n}=F^{\diamond(n-1)} \diamond F, F^{\diamond 0}=1$. Note that $H_{n \varepsilon_{k}}=H_{\varepsilon_{k}}^{\diamond n}$ for $n \in \mathbb{N}_{0}, k \in \mathbb{N}$.

For example, let $X=C^{k}[0, T], k \in \mathbb{N}$. In [18] we proved that differentiation of a stochastic process can be carried out componentwise in the chaos expansion, i.e. due to the fact that $(S)_{-1}$ is a nuclear space it holds that $C^{k}\left([0, T],(S)_{-1}\right)=C^{k}[0, T] \otimes(S)_{-1}$. This means that a stochastic process $U(t, \omega)$ is $k$ times continuously differentiable if and only if all of its coefficients $u_{\alpha}(t), \alpha \in \mathcal{I}$ are in $C^{k}[0, T]$.

The same holds for Banach space valued stochastic processes i.e. elements of $C^{k}([0, T], X) \otimes(S)_{-1}$, where $X$ is an arbitrary Banach space. By the nuclearity of $(S)_{-1}$, these processes can be regarded as elements of the tensor product space

$$
C^{k}\left([0, T], X \otimes(S)_{-1}\right)=C^{k}([0, T], X) \otimes(S)_{-1}=\bigcup_{p=0}^{\infty} C^{k}([0, T], X) \otimes(S)_{-1,-p}
$$

## 2 Stochastic operators

Definition 2.1. Let $X$ be a Banach space and $\mathbf{O}: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ an operator acting on the space of stochastic processes. We will call $\mathbf{O}$ to be a coordinatewise operator if there exists a family of operators $o_{\alpha}: X \rightarrow X, \alpha \in \mathcal{I}$, such that $\mathbf{O}\left(\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} o_{\alpha}\left(f_{\alpha}\right) H_{\alpha}$ for all $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in X \otimes(S)_{-1}$.

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Clearly, not all operators are coordinatewise, for example $\mathbf{O}(F)=F^{\diamond 2}$ can not be written in this form.
Definition 2.2. The subclass of simple coordinatewise operators consists of operators for which $o_{\alpha}=o_{\beta}=o, \alpha, \beta \in \mathcal{I}$, that is, they can be written in form of $\mathbf{O}\left(\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}\right)=$ $\sum_{\alpha \in \mathcal{I}} o\left(f_{\alpha}\right) H_{\alpha}$ for some operator o : $X \rightarrow X$.

For example, the operator of differentiation [18] and the Fourier transform [21] are simple coordinatewise operators. The Ornstein-Uhlenbeck operator is a coordinatewise operator but it is not a simple coordinatewise operator.

Note that even if all $o_{\alpha}, \alpha \in \mathcal{I}$, are bounded linear operators, the coordinatewise operator $\mathbf{O}$ itself does not need to be bounded. If $o_{\alpha}, \alpha \in \mathcal{I}$, are uniformly bounded by some $C>0$, then $\mathbf{O}$ is also a bounded operator. This follows from

$$
\begin{aligned}
\|\mathbf{O}(F)\|_{X \otimes(S)_{-1,-p}}^{2} & \leq \sum_{\alpha \in \mathcal{I}}\left\|o_{\alpha}\right\|_{L(X)}^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq C^{2} \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=C^{2}\|F\|_{X \otimes(S)_{-1,-p}}^{2}<\infty
\end{aligned}
$$

for $F \in X \otimes(S)_{-1,-p}$.
This condition is sufficient, but not necessary, and can be loosened by the embedding $(S)_{-1,-p} \subseteq(S)_{-1,-q}, q \geq p$.
Lemma 2.3. Let $\mathbf{O}$ be a coordinatewise operator for which all $o_{\alpha}, \alpha \in \mathcal{I}$, are polynomially bounded i.e. $\left\|o_{\alpha}\right\|_{L(X)} \leq R(2 \mathbb{N})^{r \alpha}$ for some $r, R>0$. Then, there exists $q \geq p$ such that $\mathbf{O}: X \otimes(S)_{-1,-p} \rightarrow X \otimes(S)_{-1,-q}$ is bounded.

Proof. Let $q \geq p+2 r$. Then,

$$
\begin{aligned}
\|\mathbf{O}(F)\|_{X \otimes(S)-1,-q}^{2} & \leq R^{2} \sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{2 r \alpha}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}=R^{2} \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(q-2 r) \alpha} \\
& \leq R^{2} \sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=R^{2}\|F\|_{X \otimes(S)-1,-p}^{2}<\infty
\end{aligned}
$$

Thus, $\|\mathbf{O}\|_{L(X) \otimes(S)_{-1}} \leq R$.
Note that the condition $\left\|o_{\alpha}\right\|_{L(X)} \leq R(2 \mathbb{N})^{r \alpha}$ for some $r, R>0$ is actually equivalent to stating that there exists $r>0$ such that $\sum_{\alpha \in \mathcal{I}}\left\|o_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-r \alpha}<\infty$.

Throughout the paper we will consider the equation

$$
\begin{align*}
\frac{d}{d t} U(t, \omega) & =\mathbf{A} U(t, \omega)+\mathbf{B} \diamond U(t, \omega)+F(t, \omega), \quad t \in(0, T], \omega \in \Omega  \tag{2.1}\\
U(0, \omega) & =U^{0}(\omega)
\end{align*}
$$

where both operators $\mathbf{A}$ and $\mathbf{B}$ are assumed to be coordinatewise operators, i.e. composed out of a family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, respectively. The operators $A_{\alpha}$, $\alpha \in \mathcal{I}$, are assumed to be infinitesimal generators of $C_{0}$-semigroups with a common domain $D$ dense in $X$ and the action of $\mathbf{A}$ is given by $\mathbf{A}(U)=\sum_{\alpha \in \mathcal{I}} A_{\alpha}\left(u_{\alpha}\right) H_{\alpha}$, for $U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in \operatorname{Dom}(\mathbf{A}) \subseteq D \otimes(S)_{-1}$, where

$$
\operatorname{Dom}(\mathbf{A})=\left\{U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in D \otimes(S)_{-1}: \exists p_{U}>0, \sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\left(u_{\alpha}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p_{U} \alpha}<\infty\right\}
$$

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The operators $B_{\alpha}, \alpha \in \mathcal{I}$, are assumed to be bounded and linear on $X$, and the action of the operator $\mathbf{B} \diamond: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ is defined by

$$
\mathbf{B} \diamond(U)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} B_{\beta}\left(u_{\alpha-\beta}\right) H_{\alpha}=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} B_{\alpha}\left(u_{\beta}\right) H_{\gamma}
$$

In the next two lemmas we provide two sufficient conditions that ensure the operator $\mathbf{B} \triangleleft$ to be well-defined. Both conditions are actually equivalent to the fact that $B_{\alpha}, \alpha \in \mathcal{I}$, are polynomially bounded, but they provide finer estimates on the stochastic order (Kondratiev weight) of the domain and codomain of $\mathbf{B} \diamond$.
Lemma 2.4. If the operators $B_{\alpha}, \alpha \in \mathcal{I}$, satisfy $\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-r \alpha}<\infty$, then $\mathbf{B} \diamond$ is well-defined as a mapping $\mathbf{B} \diamond: X \otimes(S)_{-1,-p} \rightarrow X \otimes(S)_{-1,-(p+r+m)}, m>1$.

Proof. For $U \in X \otimes(S)_{-1,-p}$ and $q=p+r+m$ we have

$$
\begin{aligned}
& \sum_{\gamma \in \mathcal{I}}\left\|\sum_{\alpha+\beta=\gamma} B_{\alpha}\left(u_{\beta}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-q \gamma} \leq \sum_{\gamma \in \mathcal{I}}\left[\sum_{\alpha+\beta=\gamma}\left\|B_{\alpha}\right\|_{L(X)}\left\|u_{\beta}\right\|_{X}\right]^{2}(2 \mathbb{N})^{-(p+r+m) \gamma} \\
&=\sum_{\gamma \in \mathcal{I}}(2 \mathbb{N})^{-m \gamma}\left(\sum_{\alpha+\beta=\gamma}\left\|B_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-r \gamma}\right)\left(\sum_{\alpha+\beta=\gamma}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}\right) \\
& \leq M\left(\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-r \alpha}\right)\left(\sum_{\beta \in \mathcal{I}}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p \beta}\right)<\infty
\end{aligned}
$$

where $M=\sum_{\gamma \in \mathcal{I}}(2 \mathbb{N})^{-m \gamma}<\infty$, for $m>1$.
Lemma 2.5. If the operators $B_{\alpha}, \alpha \in \mathcal{I}$, satisfy $\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|_{L(X)}(2 \mathbb{N})^{-\frac{r}{2} \alpha}<\infty$, for some $r>0$, then $\mathbf{B} \diamond$ is well-defined as a mapping $\mathbf{B} \diamond: X \otimes(S)_{-1,-r} \rightarrow X \otimes(S)_{-1,-r}$.

Proof. For $U \in X \otimes(S)_{-1,-r}$, we have by the generalized Minkowski inequality that

$$
\begin{aligned}
\sum_{\gamma \in \mathcal{I}}\left\|\sum_{\alpha+\beta=\gamma} B_{\alpha}\left(u_{\beta}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-r \gamma} & \leq \sum_{\gamma \in \mathcal{I}}\left[\sum_{\alpha+\beta=\gamma}\left\|B_{\alpha}\right\|_{L(X)}\left\|u_{\beta}\right\|_{X}\right]^{2}(2 \mathbb{N})^{-r \gamma} \\
& \leq \sum_{\gamma \in \mathcal{I}}\left[\sum_{\alpha+\beta=\gamma}\left\|B_{\alpha}\right\|_{L(X)}(2 \mathbb{N})^{-\frac{r}{2} \alpha}\left\|u_{\beta}\right\|_{X}(2 \mathbb{N})^{-\frac{r}{2} \beta}\right]^{2} \\
& \leq\left(\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|_{L(X)}(2 \mathbb{N})^{-\frac{r}{2} \alpha}\right)^{2} \sum_{\beta \in \mathcal{I}}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-r \beta}<\infty
\end{aligned}
$$

### 2.1 Special cases and relationship to other works

Some of the most important operators of stochastic calculus are the operators of the Malliavin calculus. We recall their definitions in the generalized $S^{\prime}(\mathbb{R})$ setting [10].

- The Malliavin derivative, D , as a stochastic gradient in the direction of white noise, is a linear and continuous mapping $\mathbb{D}: X \otimes(S)_{-1} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ given by

$$
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon_{k}}, \quad \text { for } u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}
$$

In terms of quantum theory it corresponds to the annihilation operator reducing the order of the chaos space ( $\mathbb{D}: \mathcal{H}_{m} \rightarrow \mathcal{H}_{m-1}$ ).

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- The Skorokhod integral, $\delta$, as an extension of the Itô integral to non-anticipating processes, is a linear and continuous mapping $\delta: X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ given by

$$
\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} \otimes v_{\alpha, k} \otimes H_{\alpha+\varepsilon_{k}}, \quad \text { for } F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes\left(\sum_{k \in \mathbb{N}} v_{\alpha, k} \xi_{k}\right) \otimes H_{\alpha}
$$

It is the adjoint operator of the Malliavin derivative and in terms of quantum theory it corresponds to the creation operator increasing the order of the chaos space ( $\delta: \mathcal{H}_{m} \rightarrow \mathcal{H}_{m+1}$ ).

- The Ornstein-Uhlenbeck operator, $\mathcal{R}$, as the composition of the previous ones $\delta \circ \mathbb{D}$, is the stochastic analogue of the Laplacian. It is a linear and continuous mapping $\mathcal{R}: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ given by

$$
\mathcal{R}(u)=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}, \quad \text { for } u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}
$$

In terms of quantum theory it corresponds to the number operator. It is a selfadjoint operator $\mathcal{R}: \mathcal{H}_{m} \rightarrow \mathcal{H}_{m}$ with eigenvectors equal to the basis elements $H_{\alpha}, \alpha \in \mathcal{I}$, i.e. $\mathcal{R}\left(H_{\alpha}\right)=|\alpha| H_{\alpha}, \alpha \in \mathcal{I}$. Thus, Gaussian processes with zero expectation are the only fixed points for the Ornstein-Uhlenbeck operator.

Clearly, the Ornstein-Uhlenbeck operator is a coordinatewise operator, while the Malliavin derivative and the Skorokhod integral are not coordinatewise operators.

The Ornstein-Uhlenbeck operator is the infinitesimal generator of the semigroup $T_{t}=e^{t \mathcal{R}}, t \geq 0$, given by $T_{t}(u)=\sum_{\alpha \in \mathcal{I}} e^{-|\alpha| t} u_{\alpha} \otimes H_{\alpha}$, for $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-1}$.

It is also closely connected to the Ornstein-Uhlenbeck process. The OrnsteinUhlenbeck process is the solution of the SDE $d u(t, \omega)=-u(t, \omega) d t+d B(t, \omega), u(0, \omega)=$ $u_{0}(x, \omega)$, and it is given by $u(t, \omega)=e^{-t} u_{0}(\omega)+\int_{0}^{t} e^{t-s} d B(s, \omega)$. It is a Markov process with transition semigroup $\left\{T_{t}\right\}_{t \geq 0}$ [2]. The solution of the generalized heat equation $\frac{d}{d t} u+\mathcal{R}(u)=0, u(0)=u_{0}$, is given by $u=T_{t}\left(u_{0}\right)$, i.e. $u(t, x)=\left(T_{t} u_{0}\right)(x)$ and $\left(T_{t} \varphi\right)(x)=E\left(\varphi(u(t, x))\right.$ for any $\varphi \in C_{b}(\mathbb{R})$ and $u$ is the Ornstein-Uhlenbeck process.

Now we turn to our equation

$$
\begin{equation*}
\frac{d}{d t} U(t, \omega)=\mathbf{A} U(t, \omega)+\mathbf{B} \diamond U(t, \omega)+F(t, \omega) \tag{2.2}
\end{equation*}
$$

where $\mathbf{A}$ and $\mathbf{B}$ are coordinatewise operators as described in Section 2, composed out of a family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, respectively, where $A_{\alpha}$ are infinitesimal generators on $X$ and $B_{\alpha}$ are bounded linear operators on $X$, both families being polynomially bounded, and their actions given by

$$
\begin{gather*}
\mathbf{A} U=\sum_{\alpha \in \mathcal{I}} A_{\alpha}\left(u_{\alpha}\right) H_{\alpha}, \quad \text { for } U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha},  \tag{2.3}\\
\mathbf{B} \diamond U=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} B_{\beta}\left(u_{\alpha-\beta}\right) H_{\alpha}, \quad \text { for } U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} . \tag{2.4}
\end{gather*}
$$

Some important special cases include the following:
I) Special cases for $\mathbf{A}$ :

1) $\mathbf{A}$ is a simple coordinatewise operator, i.e. $A_{\alpha}=A, \alpha \in \mathcal{I}$, where $A$ is the infinitesimal generator of a $C_{0}$-semigroup on $X$. Such operators are, for

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example the Laplacian $\Delta$ on $X=W^{2,2}\left(\mathbb{R}^{n}\right)$ or any strictly elliptic linear partial differential operator of even order $P(x, D)=\sum_{|\iota| \leq 2 m} a_{\iota}(x) D^{\iota}$. For example, second order elliptic operators can be written in divergence form $L=\nabla \cdot(Q \nabla \cdot+b)+c \nabla \cdot$, where $Q$ is a positively definite function matrix.
2) $A_{\alpha}=A+R_{\alpha}, \alpha \in \mathcal{I}$, where $A$ is as in 1), while $R_{\alpha}, \alpha \in \mathcal{I}$, are bounded linear operators on $X$ so that $\mathbf{R}$ is a coordinatewise operator

$$
\mathbf{R} U(t, \omega)=\sum_{\alpha \in \mathcal{I}} R_{\alpha} u_{\alpha}(t) H_{\alpha}(\omega)
$$

Especially, if we take $A=0$ and $R_{\alpha}$ to be multiplication operators $R_{\alpha}(x)=$ $r_{\alpha} \cdot x, x \in X$, then the resulting operator $\mathbf{R}$ is a self-adjoint operator with eigenvalues $r_{\alpha}$ corresponding to the eigenvectors $H_{\alpha}$ and thus represents a natural generalization of the Ornstein-Uhlenbeck operator. For $r_{\alpha}=|\alpha|, \alpha \in \mathcal{I}$, we retrieve the Ornstein-Uhlenbeck operator $\mathcal{R}$.
Finally, we note that every bounded linear coordinatewise operator $\mathbf{R}$ can be written in the form $\mathbf{R} u=\delta(\mathbf{M} u)$ where $\mathbf{M}$ is a generalization of the Malliavin derivative. This will be done in Proposition 2.6.
II) Special cases for B:

1) $\mathbf{B}$ is an operator acting as a multiplication operator with a deterministic function, i.e. $B_{\alpha}=b$ for $\alpha=(0,0,0,0, \ldots)$ and $B_{\alpha}=0$ for all other $\alpha \in \mathcal{I}$. Its action is thus

$$
\mathbf{B} \diamond U(t, \omega)=\sum_{\alpha \in \mathcal{I}} b \cdot u_{\alpha}(t) H_{\alpha}(\omega)
$$

For example, we may take $X=L^{2}\left(\mathbb{R}^{n}\right)$ and $b=b(x), x \in \mathbb{R}^{n}$, for an essentially bounded function $b$.
2) $\mathbf{B}$ is multiplication with spatial white noise on $X=L^{2}\left(\mathbb{R}^{n}\right)$. Let $B_{k}:=B_{\varepsilon_{k}}=\xi_{k}$, $k \in \mathbb{N}$, and $B_{\alpha}=0$ for $\alpha \neq \varepsilon_{k}$, i.e. $B_{k}(v(x))=\xi_{k}(x) \cdot v(x), k \in \mathbb{N}$. Then,

$$
\mathbf{B} \diamond U(t, \omega)=W(x, \omega) \diamond U(t, \omega)
$$

Clearly,

$$
\begin{aligned}
\mathbf{B} \diamond U & =\sum_{\gamma \in \mathcal{I}} \sum_{k \in \mathbb{N}} B_{k}\left(u_{\alpha-\varepsilon_{k}}\right) H_{\gamma}=\sum_{\gamma \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon_{k}} \xi_{k} H_{\gamma} \\
& =\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\varepsilon_{k}=\gamma} u_{\alpha} \xi_{k} H_{\gamma}=W \diamond U .
\end{aligned}
$$

Multiplication with spatial white noise is important for applications since it describes stationary perturbations.
3) $\mathbf{B}$ is of the form $B_{\varepsilon_{k}}=B_{k}, k \in \mathbb{N}$, and $B_{\alpha}=0$ for $\alpha \neq \varepsilon_{k}$, where $B_{k}: X \rightarrow X$, $k \in \mathbb{N}$, are bounded linear operators.
Note that in this case there is a one-to-one correspondence between operators of the form $\mathbf{B} \triangleleft$ and operators of the form $\delta(\mathbf{M} u)$ where $\mathbf{M}$ is a simple coordinatewise operator. This will be done in Proposition 2.8.
4) B is a simple coordinatewise operator, i.e. $B_{\alpha}=B, \alpha \in \mathcal{I}$, where $B$ is a bounded linear operator on $X$. Alternatively, we may also regard operators as $B: X \rightarrow X^{\prime}$ in order to make them bounded; such operators are for example the divergence $\nabla \cdot$ as a mapping from $X=W^{1,2}\left(\mathbb{R}^{n}\right)$ to $X^{\prime}=W^{-1,2}\left(\mathbb{R}^{n}\right)$.
5) $\mathbf{B} \diamond=\nabla \cdot \diamond(Q \diamond \nabla \cdot+b \diamond)+c \diamond \nabla \cdot$ as a strictly elliptic second order operator with random coefficients. This operator is obtained for $B_{\alpha}=\nabla \cdot\left(Q_{\alpha} \nabla \cdot+b_{\alpha}\right)+c_{\alpha} \nabla \cdot$, $\alpha \in \mathcal{I}$, and was studied in [18] and [19].

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Proposition 2.6. Let $\mathbf{R}: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ be a bounded linear coordinatewise operator defined by $\mathbf{R} u(t, \omega)=\sum_{\alpha \in \mathcal{I}} R_{\alpha} u_{\alpha}(t) H_{\alpha}(\omega)$.

1. There exists an operator $\mathbf{M}: X \otimes(S)_{-1} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ of the form

$$
\mathbf{M} u=\sum_{k=1}^{\infty} \mathbf{M}_{k} u \otimes \xi_{k}, \quad u \in X \otimes(S)_{-1}
$$

for some coordinatewise operators $\mathbf{M}_{k}: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}, k \in \mathbb{N}$, such that

$$
\mathbf{R} u=\delta(\mathbf{M} u)
$$

holds.
2. Especially, if $\mathbf{R}$ is a selfadjoint operator, then $\mathbf{M}$ is a generalization of the Malliavin derivative.

Proof. a) In [10] we proved that the Skorokhod integral is invertible, i.e. there exists a unique solution to equations of the form $\delta(v)=f$. Considering the equation $\delta(\mathbf{M} u)=$ $\sum_{\alpha \in \mathcal{I}} R_{\alpha} u_{\alpha} H_{\alpha}$ and applying the result from [10], we obtain $\mathbf{M} u$ in the form

$$
\mathbf{M} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{R_{\alpha+\varepsilon_{k}}\left(u_{\alpha+\varepsilon_{k}}\right)}{\left|\alpha+\varepsilon_{k}\right|} \otimes \xi_{k} \otimes H_{\alpha}
$$

By defining

$$
\mathbf{M}_{k} u=\sum_{\alpha \in \mathcal{I}}\left(\alpha_{k}+1\right) \frac{R_{\alpha+\varepsilon_{k}}\left(u_{\alpha+\varepsilon_{k}}\right)}{\left|\alpha+\varepsilon_{k}\right|} \otimes H_{\alpha}, \quad k \in \mathbb{N}
$$

we obtain the assertion.
b) Let $\mathbf{R}$ be a self-adjoint operator with eigenvalues $r_{\alpha}$ and eigenvectors $H_{\alpha}, \alpha \in \mathcal{I}$, i.e., an operator of the form $\mathbf{R} u=\sum_{\alpha \in \mathcal{I}} r_{\alpha} u_{\alpha} H_{\alpha}$. Assume that $r_{\alpha}=\sum_{k \in \mathbb{N}} r_{k, \alpha}$ for some $r_{k, \alpha} \in \mathbb{R}, k \in \mathbb{N}, \alpha \in \mathcal{I}$, is an arbitrary decomposition of the value $r_{\alpha}$.

Define

$$
\mathbf{M}_{k} u=\sum_{\alpha \in \mathcal{I}} r_{k, \alpha} u_{\alpha} \otimes H_{\alpha-\varepsilon_{k}}
$$

Then $\mathbf{M} u=\sum_{k \in \mathbb{N}} \mathbf{M}_{k} u \otimes \xi_{k}=\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathcal{I}} r_{k, \alpha} u_{\alpha} \otimes H_{\alpha-\varepsilon_{k}} \otimes \xi_{k}$ and

$$
\delta(\mathbf{M} u)=\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathcal{I}} r_{k, \alpha} u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} r_{\alpha} u_{\alpha} \otimes H_{\alpha}
$$

Remark 2.7. The converse is not true. Even if each $\mathbf{M}_{k}, k \in \mathbb{N}$, is a simple coordinatewise operator (and so is $\mathbf{M}$ ), $\mathbf{R}:=\delta \circ \mathbf{M}$ does not need to be a coordinatewise operator. This would require that the system $R_{\alpha}\left(u_{\alpha}\right)=\sum_{k \in \mathbb{N}} m_{k}\left(u_{\alpha-\varepsilon_{k}}\right), \alpha \in \mathcal{I}$, is solvable for $R_{\alpha}(\cdot)$ given the functions $m_{k}(\cdot), k \in \mathbb{N}$, which is not true in general.
Proposition 2.8. Let $\mathbf{M}: X \otimes(S)_{-1} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ be of the form

$$
\begin{equation*}
\mathbf{M} u=\sum_{k=1}^{\infty} \mathbf{M}_{k} u \otimes \xi_{k}, \quad u \in X \otimes(S)_{-1} \tag{2.5}
\end{equation*}
$$

for some simple coordinatewise operators $\mathbf{M}_{k}: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}, k \in \mathbb{N}$. Then, there exists a coordinatewise operator $\mathbf{B}$ such that $B_{\alpha}=0$ for $\alpha \neq \varepsilon_{k}, k \in \mathbb{N}$, and

$$
\delta(\mathbf{M} u)=\mathbf{B} \diamond u
$$

holds.
Conversely, for any coordinatewise operator $\mathbf{B}$ such that $B_{\alpha}=0$ for $\alpha \neq \varepsilon_{k}, k \in$ $\mathbb{N}$, there exists an operator $\mathbf{M}$ of the form $\mathbf{M} u=\sum_{k=1}^{\infty} \mathbf{M}_{k} u \otimes \xi_{k}$ for some simple coordinatewise operators $\mathbf{M}_{k}, k \in \mathbb{N}$, such that $\delta(\mathbf{M} u)=\mathbf{B} \triangleleft u$ holds.

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Proof. Let M be an operator as stated above and since $\mathrm{M}_{k}$ are simple coordinatewise operators, we can write them as

$$
\mathbf{M}_{k}(u)=\sum_{\alpha \in \mathcal{I}} m_{k}\left(u_{\alpha}\right) H_{\alpha}, \quad u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}
$$

for some operators $m_{k}: X \rightarrow X, k \in \mathbb{N}$. Thus,

$$
\mathbf{M} u=\sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{I}} m_{k}\left(u_{\alpha}\right) H_{\alpha} \otimes \xi_{k}
$$

which further implies

$$
\begin{equation*}
\delta(\mathbf{M} u)=\sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{I}} m_{k}\left(u_{\alpha}\right) H_{\alpha+\varepsilon_{k}}=\sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{I}} m_{k}\left(u_{\alpha-\varepsilon_{k}}\right) H_{\alpha} \tag{2.6}
\end{equation*}
$$

On the other hand, if $\mathbf{B}$ is such that $B_{\alpha}=0$ for $\alpha \neq \varepsilon_{k}, k \in \mathbb{N}$, and we denote by $B_{k}:=B_{\varepsilon_{k}}, k \in \mathbb{N}$, the operators acting on $X$, then

$$
\begin{equation*}
\mathbf{B} \triangleleft u=\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} B_{k}\left(u_{\alpha-\varepsilon_{k}}\right) H_{\alpha} . \tag{2.7}
\end{equation*}
$$

From (2.6) and (2.7) it follows that $\delta(\mathbf{M} u)=\mathbf{B} \triangleleft u$ if and only if $m_{k}=B_{k}$ for all $k \in \mathbb{N}$. Thus, there is a one-to-one correspondence between the operators $\mathbf{B} \diamond$ and $\delta \circ \mathbf{M}$.

Remark 2.9. In [12] and [13] Rozovskii and Lototsky considered the equation $\frac{d}{d t}=$ $\mathbf{A} u+\delta(\mathbf{M} u)+f$, where $\mathbf{M}$ is of the form (2.5). They implicitly assumed that all their operators $\mathbf{A}$ and $\mathbf{M}_{k}, k \in \mathbb{N}$, belong to our class of simple coordinatewise operators. This corresponds to our special cases I-1) and II-3).

Some special cases of stochastic differential equations covered by (2.2) include the following:

- The heat equation with random potential

$$
\frac{d}{d t} u=\Delta u+\mathbf{B} \triangleleft u
$$

In particular, if the random potential is modeled by stationary perturbations, we may take spatial white noise as a model and obtain $\frac{d}{d t} u=\Delta u+W \diamond u$. This corresponds to the special choice of operators I-1) and II-2).

- The heat equation in random (inhomogeneous and anisotropic) media, where the physical properties of the medium are modeled by a stochastic matrix $Q$. This corresponds to the case I-1) with $\mathbf{A}=0$ and II-5) leading to an equation of the form

$$
\frac{d}{d t} u=\nabla \cdot \diamond(Q \diamond \nabla \cdot u+b \diamond u)+c \diamond \nabla \cdot u+f
$$

- Taking $\mathbf{A}=0$ and $B_{k}:=B_{\varepsilon_{k}}=\xi_{k} \nabla \cdot, k \in \mathbb{N}$, (see special cases II-2) and II-4)) we obtain the transport equation driven by white noise

$$
\frac{d}{d t} u=\Delta u+W \diamond \nabla \cdot u
$$

- The Langevin equation

$$
\frac{d}{d t} u=-\lambda u+W(t)
$$

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$\lambda>0$, corresponding to the case $\mathrm{I}-1$ ) with $A=-\lambda, f=W$ and $\mathbf{B}=0$. Its solution is the Ornstein-Uhlenbeck process describing the spatial position of a Brownian particle in a fluid with viscosity $\lambda$.
In [1] the authors considered the generalized Langevin equation leading to generalized Ornstein-Uhlenbeck operators driven by Lévy processes

$$
\frac{d}{d t} u=J u+C\left(\frac{d}{d t} Y\right),
$$

where $Y$ is a Lévy process, $J$ the infinitesimal generator of a $C_{0}$-semigroup and $C$ a bounded operator. All processes are Hilbert space valued. This corresponds to our case with $X$ being this Hilbert space, $\mathbf{A}=J, \mathbf{B}=0$ and $f=C\left(Y^{\prime}\right)$.

- The equation $\frac{d}{d t}=\mathbf{A} u+\delta(\mathbf{M} u)+f$, that was extensively studied in [12] and [13]. This corresponds to our special cases I-1) and II-3).
- The equation

$$
\frac{d}{d t} u=L u+W \diamond u
$$

where $L$ is a strictly elliptic partial differential operator as studied in [3] and [8]. This corresponds to the special case I-1) and II-2).

## 3 Stochastic evolution equations

Now we turn to the general case of stochastic Cauchy problems of the form $\frac{d}{d t} U(t, \omega)=$ $\mathbf{A} U(t, \omega)+\mathbf{B} \diamond U(t, \omega)+F(t, \omega), t \in(0, T], \omega \in \Omega$, with initial value $U(0, \omega)=U^{0}(\omega), \omega \in \Omega$, and all processes are $X$-valued for a Banach space $X$.
Definition 3.1. It is said that $U$ is a solution to (2.1) if $U \in C([0, T], X) \otimes(S)_{-1} \cap$ $C^{1}((0, T], X) \otimes(S)_{-1}$ and $U$ satisfies (2.1).
Theorem 3.2. Let A be a coordinatewise operator of the form (2.3), where the operators $A_{\alpha}, \alpha \in \mathcal{I}$, defined on the same domain $D$ dense in $X$, are infinitesimal generators of $C_{0}$-semigroups $\left(T_{t}\right)_{\alpha}, t \geq 0, \alpha \in \mathcal{I}$, uniformly bounded by

$$
\begin{equation*}
\left\|\left(T_{t}\right)_{\alpha}\right\|_{L(X)} \leq M e^{w t}, t \geq 0, \quad \text { for some } M, w>0 \tag{3.1}
\end{equation*}
$$

Let $\mathbf{B} \diamond$ be of the form (2.4), where $B_{\alpha}, \alpha \in \mathcal{I}$, are bounded linear operators on $X$ so that there exists $p>0$ such that

$$
\begin{equation*}
K:=\sum_{\alpha \in \mathcal{I}}\left\|B_{\alpha}\right\|(2 \mathbb{N})^{-p \frac{\alpha}{2}}<\infty \tag{3.2}
\end{equation*}
$$

Let the initial value $U^{0} \in X \otimes(S)_{-1}$ be such that $U^{0} \in \operatorname{Dom}(\mathbf{A})$ i.e.

$$
\begin{equation*}
U^{0}(\omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha}(\omega) \in X \otimes(S)_{-1,-p}, \text { satisfies } \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{A} U^{0}(\omega)=\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha}^{0} H_{\alpha}(\omega) \in X \otimes(S)_{-1,-p}, \text { satisfies } \sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha} u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{3.4}
\end{equation*}
$$

Moreover, let
$F(t, \omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega) \in C^{1}([0, T], X) \otimes(S)_{-1}, \quad t \mapsto f_{\alpha}(t) \in C^{1}([0, T], X), \alpha \in \mathcal{I}$,
so that $\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}+\sup _{t \in[0, T]}\left\|f_{\alpha}^{\prime}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha}<\infty$.

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Then, the stochastic Cauchy problem (2.1) has a unique solution $U$ in $C^{1}([0, T], X) \otimes$ $(S)_{-1,-p}$.

Proof. We seek for the solution in form of $U(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega)$. Then, the Cauchy problem (2.1) is equivalent to the infinite system:

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =A_{\alpha} u_{\alpha}(t)+\sum_{\beta \leq \alpha} B_{\beta} u_{\alpha-\beta}(t)+f_{\alpha}(t), \quad t \in(0, T]  \tag{3.6}\\
u_{\alpha}(0) & =u_{\alpha}^{0} \in D, \quad \alpha \in \mathcal{I} .
\end{align*}
$$

Let $\mathbf{0}$ be the multi-index $\mathbf{0}=(0,0, \ldots)$. We rewrite (3.6) as

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =\left(A_{\alpha}+B_{0}\right) u_{\alpha}(t)+\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta} u_{\alpha-\beta}(t)+f_{\alpha}(t), \quad t \in(0, T]  \tag{3.7}\\
u_{\alpha}(0) & =u_{\alpha}^{0} \in D, \quad \alpha \in \mathcal{I}
\end{align*}
$$

Next, $A_{\alpha}+B_{0}$ are infinitesimal generators of $C_{0}-$ semigroups $\left(S_{t}\right)_{\alpha}$ in $X$ such that

$$
\begin{equation*}
\left\|\left(S_{t}\right)_{\alpha}\right\| \leq M e^{\left(w+M\left\|B_{0}\right\|\right) t}, t \geq 0, \alpha \in \mathcal{I} \tag{3.8}
\end{equation*}
$$

According to Subsection 1.1, if $f_{\alpha}, \alpha \in \mathcal{I}$, fulfills condition (i) or (ii), the inhomogeneous initial value problem (3.7) has a solution $u_{\alpha}(t) \in C([0, T], X) \cap C^{1}((0, T], X), \alpha \in \mathcal{I}$, given by

$$
\begin{align*}
& u_{\mathbf{0}}(t)=\left(S_{t}\right)_{\mathbf{0}} u_{\mathbf{0}}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\mathbf{0}} f_{\mathbf{0}}(s) d s, \quad t \in[0, T] \\
& u_{\alpha}(t)=\left(S_{t}\right)_{\alpha} u_{\alpha}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta} u_{\alpha-\beta}(s)+f_{\alpha}(s)\right) d s, \quad t \in[0, T] . \tag{3.9}
\end{align*}
$$

Since $f_{\alpha} \in C^{1}([0, T], X)$ it follows by induction on $\alpha$ that

$$
\sum_{0<\beta \leq \alpha} B_{\beta} u_{\alpha-\beta}(s)+f_{\alpha}(s) \in C^{1}([0, T], X), \quad \text { for all } \quad \alpha \in \mathcal{I} .
$$

Thus, $u_{\alpha} \in C^{1}([0, T], X)$ and $\frac{d}{d t} u_{\alpha}(0)=\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0}+\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta} u_{\alpha-\beta}^{0}+f_{\alpha}(0), \alpha \in \mathcal{I}$.
Note that for each fixed $\alpha \in \mathcal{I}, u_{\alpha}(t)$ exists for all $t \in[0, T]$ and it is the unique (classical) solution on the whole interval $[0, T]$. It remains to prove that $\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega)$ converges in $C^{1}([0, T], X) \otimes(S)_{-1,-p}$.

First, we show that $U(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \in C^{1}\left(\left[0, T_{0}\right], X\right) \otimes S_{-1,-p}$ for appropriate $T_{0} \in(0, T]$, i.e. we show that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{C^{1}\left(\left[0, T_{0}\right], X\right)}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}+\sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{3.10}
\end{equation*}
$$

Later on we will prove that the same holds if we take in (3.10) supremums over the intervals $\left[T_{0}, 2 T_{0}\right],\left[2 T_{0}, 3 T_{0}\right], \ldots$ etc. Since $[0, T]$ can be covered by finitely many intervals of the form $\left[k T_{0},(k+1) T_{0}\right], k \in \mathbb{N}_{0}$, we conclude that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}+\sup _{t \in[0, T]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{3.11}
\end{equation*}
$$

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In order to do this, we introduce a notation for subsets of multi-indices

$$
\mathcal{I}_{n, m}=\{\alpha \in \mathcal{I}:|\alpha| \leq n \wedge \operatorname{Index}(\alpha) \leq m\}, n, m \in \mathbb{N},
$$

where, for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$, we have $|\alpha|=\alpha_{1}+\alpha_{2}+\cdots+\alpha_{m}$ and $\operatorname{Index}(\alpha)$ is last coordinate where $\alpha$ has a nonzero entry. For later reference, we introduce the function

$$
\begin{equation*}
C(t)=\frac{M^{2}}{\left(w+M\left\|B_{\mathbf{0}}\right\|\right)^{2}}\left(e^{\left(w+M\left\|B_{\mathbf{0}}\right\|\right) t}-1\right)^{2} \tag{3.12}
\end{equation*}
$$

and fix $T_{0} \in(0, T]$ such that $C\left(T_{0}\right)<\frac{1}{5 K^{2}}$.
First, we show that

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}(t)\right\|_{C\left(\left[0, T_{0}\right], X\right)}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

by proving that partial sums $\sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}, n, m \in \mathbb{N}$, are bounded from above.

Using (3.9) we obtain

$$
\begin{aligned}
\frac{1}{3} \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} & \leq \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}_{n, m}}\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\| \sum_{\mathbf{0}<\beta \leq \alpha}\left\|B_{\beta} u_{\alpha-\beta}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}_{n, m}}\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\|\left\|f_{\alpha}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} .
\end{aligned}
$$

The first term on the right-hand side, for all $t \in\left[0, T_{0}\right]$, having in mind (3.3) and (3.8), satisfies

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} & \leq \sum_{\alpha \in \mathcal{I}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq M^{2} e^{2\left(w+M\left\|B_{0}\right\|\right) T_{0}} \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}:=Q_{1}<\infty . \tag{3.13}
\end{align*}
$$

Similarly, for all $t \in\left[0, T_{0}\right]$, using (3.5) and (3.8), the third term satisfies

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}_{n, m}} & {\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\|\left\|f_{\alpha}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\|\left\|f_{\alpha}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} } \\
& \leq\left[\int_{0}^{t} M e^{\left(w+M\left\|B_{0}\right\|\right)(t-s)} d s\right]^{2} \sum_{\alpha \in \mathcal{I}} \sup _{s \in[0, t]}\left\|f_{\alpha}(s)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq \frac{M^{2}}{\left(w+M\left\|B_{\mathbf{0}}\right\|\right)^{2}}\left(e^{\left(w+M\left\|B_{0}\right\|\right) T_{0}}-1\right)^{2} \sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}:=G<\infty . \tag{3.14}
\end{align*}
$$

Note that in (3.14) we took the supremum over the whole interval $[0, T]$.
For the second term, using (3.2), (3.8), (3.12) and the generalized Minkowski inequal-

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ity, we obtain

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}_{n, m}} & {\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\| \sum_{\beta+\gamma=\alpha}\left\|B_{\beta}\right\|\left\|u_{\gamma}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} } \\
& \leq \frac{M^{2}}{\left(w+M\left\|B_{0}\right\|\right)^{2}}\left(e^{\left(w+M\left\|B_{0}\right\|\right) t}-1\right)^{2} \sum_{\alpha \in \mathcal{I}_{n, m}}\left[\sum_{\beta+\gamma=\alpha} \sup _{s \in[0, t]}\left\|B_{\beta}\right\|\left\|u_{\gamma}(s)\right\|_{X}\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq C\left(T_{0}\right)\left(\sum_{\beta \in \mathcal{I}_{n, m}}\left\|B_{\beta}\right\|(2 \mathbb{N})^{-p \frac{\beta}{2}}\right)^{2}\left(\sum_{\gamma \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\gamma}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}\right) \\
& \leq C\left(T_{0}\right) K^{2} \sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} . \tag{3.15}
\end{align*}
$$

Finally, for all $n, m \in \mathbb{N}$, we obtain

$$
\frac{1}{3} \sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq Q_{1}+G+C\left(T_{0}\right) K^{2} \sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}
$$

Since $\frac{1}{3}-C\left(T_{0}\right) K^{2}>\frac{1}{5}-C\left(T_{0}\right) K^{2}>0$, we have

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q_{1}+G}{\frac{1}{3}-C\left(T_{0}\right) K^{2}} \tag{3.16}
\end{equation*}
$$

Let $\left(m_{n}\right)_{n \in \mathbb{N}}$ be an arbitrary sequence of positive integers tending to infinity. Then,

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\lim _{n \rightarrow \infty} \sum_{\alpha \in \mathcal{I}_{n, m_{n}}} \sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q_{1}+G}{\frac{1}{3}-C\left(T_{0}\right) K^{2}}
$$

since it is a series of positive numbers and thus does not depend on the order of summation.

Now we show that

$$
\sum_{\alpha \in \mathcal{I}}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{C\left(\left[0, T_{0}\right], X\right)}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

In order to acomplish that, we differentiate (3.9) with respect to $t$, and obtain

$$
\begin{align*}
\frac{d}{d t} u_{\mathbf{0}}(t) & =\left(S_{t}\right)_{\mathbf{0}}\left(A_{\mathbf{0}}+B_{\mathbf{0}}\right) u_{\mathbf{0}}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\mathbf{0}} \frac{d}{d s} f_{\mathbf{0}}(s) d s+\left(S_{t}\right)_{\mathbf{0}} f(0), \quad t \in[0, T] \\
\frac{d}{d t} u_{\alpha}(t) & =\left(S_{t}\right)_{\alpha}\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta} \frac{d}{d s} u_{\alpha-\beta}(s)+\frac{d}{d s} f_{\alpha}(s)\right) d s  \tag{3.17}\\
& +\left(S_{t}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta} u_{\alpha-\beta}(0)+f_{\alpha}(0)\right), \quad t \in[0, T], \quad \alpha \in \mathcal{I} .
\end{align*}
$$

In the sequel we estimate partial sums of $\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}$. So,

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$$
\begin{aligned}
\frac{1}{5} \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} & \leq \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}_{n, m}}\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\| \sum_{\mathbf{0}<\beta \leq \alpha}\left\|B_{\beta} \frac{d}{d s} u_{\alpha-\beta}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}_{n, m}}\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\|\left\|\frac{d}{d s} f_{\alpha}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left[\sum_{\mathbf{0}<\beta \leq \alpha}\left\|B_{\beta} u_{\alpha-\beta}(0)\right\|_{X}\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|f_{\alpha}(0)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}
\end{aligned}
$$

According to (3.3) and (3.4), we obtain $\sum_{\alpha \in \mathcal{I}}\left(A_{\alpha}+B_{0}\right) u_{\alpha}^{0} H_{\alpha}(\omega) \in X \otimes(S)_{-1,-p}$. So the first term on the right-hand side can be evaluated by

$$
\begin{align*}
& \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \quad \leq M^{2} e^{2\left(w+M\left\|B_{\mathbf{0}}\right\|\right) T_{0}} \sum_{\alpha \in \mathcal{I}}\left\|\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}:=Q_{1}^{\prime}<\infty . \tag{3.18}
\end{align*}
$$

The third term, for all $t \in\left[0, T_{0}\right]$, satisfies

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}_{n, m}} & {\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\|\left\|\frac{d}{d s} f_{\alpha}(s)\right\| d s\right]^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\|\left\|\frac{d}{d s} f_{\alpha}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} } \\
& \leq \frac{M^{2}}{\left(w+M\left\|B_{0}\right\|\right)^{2}}\left(e^{\left(w+M\left\|B_{0}\right\|\right) T_{0}}-1\right)^{2} \sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|\frac{d}{d s} f_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}:=G^{\prime}<\infty \tag{3.19}
\end{align*}
$$

The fourth term, using (3.2), (3.3), (3.8) and the generalized Minkowski inequality, can be estimated by

$$
\begin{align*}
& \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left[\sum_{\mathbf{0}<\beta \leq \alpha}\left\|B_{\beta} u_{\alpha-\beta}(0)\right\|_{X}\right]^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left[\sum_{\beta+\gamma=\alpha}\left\|B_{\beta} u_{\gamma}^{0}\right\|_{X}\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& \quad \leq M^{2} e^{2\left(w+M\left\|B_{0}\right\|\right) t} \sum_{\alpha \in \mathcal{I}}\left[\sum_{\beta+\gamma=\alpha}\left\|B_{\beta}\right\|\left\|u_{\gamma}^{0}\right\|_{X}\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& \quad \leq M^{2} e^{2\left(w+M\left\|B_{0}\right\|\right) T_{0}}\left(\sum_{\beta \in \mathcal{I}}\left\|B_{\beta}\right\|(2 \mathbb{N})^{-p \frac{\beta}{2}}\right)^{2}\left(\sum_{\gamma \in \mathcal{I}}\left\|u_{\gamma}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}\right):=H_{1}^{\prime}<\infty . \tag{3.20}
\end{align*}
$$

For the fifth term, using (3.5) and (3.8), we have

$$
\begin{align*}
& \sum_{\alpha \in \mathcal{I}_{n, m}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|f_{\alpha}(0)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathcal{I}}\left\|\left(S_{t}\right)_{\alpha}\right\|^{2}\left\|f_{\alpha}(0)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq M^{2} e^{2\left(w+M\left\|B_{0}\right\|\right) T_{0}} \sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}:=N^{\prime}<\infty . \tag{3.21}
\end{align*}
$$

Finally, for the second term, using (3.2), (3.8), (3.12) and the generalized Minkowski

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inequality, we obtain

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}_{n, m}} & {\left[\int_{0}^{t}\left\|\left(S_{t-s}\right)_{\alpha}\right\| \sum_{\beta+\gamma=\alpha}\left\|B_{\beta}\right\|\left\|\frac{d}{d s} u_{\gamma}(s)\right\|_{X} d s\right]^{2}(2 \mathbb{N})^{-p \alpha} } \\
& \leq \frac{M^{2}}{\left(w+M\left\|B_{0}\right\|\right)^{2}}\left(e^{\left(w+M\left\|B_{0}\right\|\right) t}-1\right)^{2} \sum_{\alpha \in \mathcal{I}_{n, m}}\left[\sum_{\beta+\gamma=\alpha} \sup _{s \in[0, t]}\left\|B_{\beta}\right\|\left\|\frac{d}{d s} u_{\gamma}(s)\right\|_{X}\right]^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq C(t)\left(\sum_{\beta \in \mathcal{I}_{n, m}}\left\|B_{\beta}\right\|(2 \mathbb{N})^{-p \frac{\beta}{2}}\right)^{2}\left(\sum_{\gamma \in \mathcal{I}_{n, m}} \sup _{s \in[0, t]}\left\|\frac{d}{d t} u_{\gamma}(s)\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}\right) \\
& \leq C\left(T_{0}\right) K^{2} \sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \tag{3.22}
\end{align*}
$$

Finally, for all $n, m \in \mathbb{N}$, we obtain

$$
\begin{aligned}
\frac{1}{5} \sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq & Q_{1}^{\prime}+G^{\prime}+H_{1}^{\prime}+N^{\prime} \\
& +C\left(T_{0}\right) K^{2} \sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}
\end{aligned}
$$

Since $\frac{1}{5}-C\left(T_{0}\right) K^{2}>0$, we have

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}_{n, m}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q_{1}^{\prime}+G^{\prime}+H_{1}^{\prime}+N^{\prime}}{\frac{1}{5}-C\left(T_{0}\right) K^{2}} \tag{3.23}
\end{equation*}
$$

Again, taking $\left(m_{n}\right)_{n \in \mathbb{N}}$ to be an arbitrary sequence of positive integers tending to infinity, we have

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\lim _{n \rightarrow \infty} \sum_{\alpha \in \mathcal{I}_{n, m_{n}}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q_{1}^{\prime}+G^{\prime}+H_{1}^{\prime}+N^{\prime}}{\frac{1}{5}-C\left(T_{0}\right) K^{2}}
$$

Therefore, we obtain

$$
\begin{align*}
& U(t, \omega) \in C^{1}\left(\left[0, T_{0}\right], X\right) \otimes(S)_{-1,-p}, \text { i.e. } \\
& \sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}+\sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha} \leq  \tag{3.24}\\
& 2 \sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in\left[0, T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}+\sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}\right)(2 \mathbb{N})^{-p \alpha}<\infty .
\end{align*}
$$

Next, we consider in (3.24) supremums over the interval [ $T_{0}, 2 T_{0}$ ]. On $\left[T_{0}, 2 T_{0}\right]$ one can rewrite the initial value problem (3.6) in the following equivalent form:

$$
\begin{align*}
\frac{d}{d t} v_{\alpha}(t) & =A_{\alpha} v_{\alpha}(t)+\sum_{\beta \leq \alpha} B_{\beta} v_{\alpha-\beta}(t)+f_{\alpha}\left(T_{0}+t\right), \quad t \in\left(0, T_{0}\right]  \tag{3.25}\\
v_{\alpha}(0) & =v_{\alpha}^{0}:=u_{\alpha}\left(T_{0}\right), \quad \alpha \in \mathcal{I} .
\end{align*}
$$

The semigroup corresponding to the generator $A_{\alpha}+B_{0}$ in (3.25) is again the semigroup $\left(S_{t}\right)_{\alpha}, t \geq 0$. Using (3.6) and (3.24), we have that $U(t, \omega) \in \operatorname{Dom}(\mathbf{A})$, for all $t \in\left[0, T_{0}\right]$, and $\mathbf{A} U(t, \omega) \in X \otimes(S)_{-1,-p}, t \in\left[0, T_{0}\right]$. According to this we have that $V^{0}(\omega)=U\left(T_{0}, \omega\right)=$ $\sum_{\alpha \in \mathcal{I}} v_{\alpha}^{0} H_{\alpha}(\omega) \in \operatorname{Dom}(\mathbf{A})$ and $\mathbf{A} V^{0}(\omega) \in X \otimes(S)_{-1,-p}$. Thus,

$$
v_{\alpha}(t)=\left(S_{t}\right)_{\alpha} v_{\alpha}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta} v_{\alpha-\beta}(s)+f_{\alpha}\left(T_{0}+s\right)\right) d s, \quad t \in\left[0, T_{0}\right]
$$

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and clearly $v_{\alpha}(t)=u_{\alpha}\left(T_{0}+t\right), t \in\left[0, T_{0}\right], \alpha \in \mathcal{I}$.
When approximating partial sums of $\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|v_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}$, comparing to the previous calculations for $u_{\alpha}(t)$, only the constant $Q_{1}$ will be different, and here, we denote it by $Q_{2}$, so we again obtain

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|v_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[T_{0}, 2 T_{0}\right]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q_{2}+G}{\frac{1}{3}-C\left(T_{0}\right) K^{2}}
$$

Similarly, for the derivative $\frac{d}{d t} V(t, \omega)$ we obtain

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in\left[0, T_{0}\right]}\left\|\frac{d}{d t} v_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q_{2}^{\prime}+G^{\prime}+H_{2}^{\prime}+N^{\prime}}{\frac{1}{5}-C\left(T_{0}\right) K^{2}}
$$

where, comparing to the estimates of $\frac{d}{d t} U(t, \omega)$, only the constants $Q_{1}^{\prime}$ and $H_{1}^{\prime}$ have changed and we denoted them here by $Q_{2}^{\prime}$ and $H_{2}^{\prime}$.

For arbitrary $T>0$, one can cover the interval $[0, T]$ by intervals of the form $\left[k T_{0},(k+\right.$ 1) $T_{0}$ ], $k \in \mathbb{N}_{0}$, in finitely many steps (say in $l$ steps). So we have

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q+G}{\frac{1}{3}-C\left(T_{0}\right) K^{2}}
$$

where $Q=\max _{1 \leq k \leq l}\left\{Q_{k}\right\}$. Thus,

$$
U(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \in C([0, T], X) \otimes(S)_{-1,-p}
$$

Also,

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \frac{Q^{\prime}+G^{\prime}+H^{\prime}+N^{\prime}}{\frac{1}{5}-C\left(T_{0}\right) K^{2}}
$$

where $Q^{\prime}=\max _{1 \leq k \leq l}\left\{Q_{k}^{\prime}\right\}, H^{\prime}=\max _{1 \leq k \leq l}\left\{H_{k}^{\prime}\right\}$. Since $\frac{d}{d t} u_{\alpha}(t) \in C([0, T], X), \alpha \in \mathcal{I}$, we have

$$
\frac{d}{d t} U(t, \omega)=\sum_{\alpha \in \mathcal{I}} \frac{d}{d t} u_{\alpha}(t) H_{\alpha}(\omega) \in C([0, T], X) \otimes(S)_{-1,-p}
$$

Therefore, $U(t, \omega) \in C^{1}([0, T], X) \otimes(S)_{-1,-p}$ and thus, $U$ is a solution of (2.1) in the sense of Definition 3.1.

The solution $U$ is unique due to the uniqueness of the coordinatewise (classical) solutions $u_{\alpha}$ in (3.9) and due to uniqueness in the Wiener-Itô chaos expansion.

Note that according to the previous theorem the solution $U$ remains in the same stochastic order space $(S)_{-1,-p}$ where the input data $U^{0}, \mathbf{A} U^{0}$ and $F$ belong to.
Example 3.3. We provide three examples of equation (2.1) where $\mathbf{A}$ is a uniformly bounded (not a simple) coordinatewise operator. Consider the Banach space $X=$ $L^{p}(\mathbb{R}), 1 \leq p<\infty$, and the stochastic Cauchy problem

$$
\begin{align*}
\frac{d}{d t} U(t, x, \omega) & =\mathbf{A} U(t, x, \omega)+W \diamond U(t, x, \omega)+F(t, x, \omega)  \tag{3.26}\\
U(0, x, \omega) & =U^{0}(x, \omega)
\end{align*}
$$

where the operator $\mathbf{A}: \operatorname{Dom}(\mathbf{A}) \rightarrow X \otimes(S)_{-1}$ is a coordinatewise operator composed out of a family of closed operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ of the form $A_{\alpha}=a_{\alpha} D, \alpha \in \mathcal{I}$, where the functions $a_{\alpha} \in L^{\infty}(\mathbb{R}), \alpha \in \mathcal{I}$, are uniformly bounded, i.e. $\sup _{x \in \mathbb{R}}\left|a_{\alpha}(x)\right| \leq M, \alpha \in \mathcal{I}$, for

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some $M>0$, and $D$ is one of the following differential operators: $\frac{\partial}{\partial x}, \frac{\partial^{2}}{\partial x^{2}}$ or $\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial}{\partial x}$, and $W=\sum_{k \in \mathbb{N}} \xi_{k} H_{\varepsilon_{k}}$ represents spatial white noise. Then, (3.26) is equivalent to the infinite system

$$
\begin{aligned}
\frac{d}{d t} u_{\alpha}(t, x) & =A_{\alpha} u_{\alpha}(t, x)+\sum_{k \in \mathbb{N}} \xi_{k}(x) u_{\alpha-\varepsilon_{k}}(t, x)+f_{\alpha}(t, x) \\
u_{\alpha}(0, x) & =u_{\alpha}^{0}(x), \quad \alpha \in \mathcal{I} .
\end{aligned}
$$

The $C_{0}$-semigroup that corresponds to the closed operator $D$, denoted by $T_{t}, t \geq 0$, is, respectively,

$$
\begin{aligned}
& T_{t} g(x)=g(t+x), \quad g \in L^{p}(\mathbb{R}), \quad \text { for } D=\frac{\partial}{\partial x}, \\
& T_{t} g(x)=\frac{1}{\sqrt{4 \pi t}} \int_{\mathbb{R}} g(x-y) e^{-\frac{y^{2}}{4 t}} d y, \quad g \in L^{p}(\mathbb{R}), \quad \text { for } D=\frac{\partial^{2}}{\partial x^{2}} \\
& T_{t} g(x)=\frac{1}{\sqrt{4 \pi t}} \int_{\mathbb{R}} g(x-y) e^{-\frac{(y+t)^{2}}{4 t}} d y, \quad g \in L^{p}(\mathbb{R}), \quad \text { for } D=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial}{\partial x} .
\end{aligned}
$$

In all cases, we have, using Young's inequality, that $\left\|T_{t}\right\| \leq 1, t \geq 0$. The $C_{0}$-semigroups corresponding to the operators $A_{\alpha}, \alpha \in \mathcal{I}$, are of the form $\left(S_{t}\right)_{\alpha}=a_{\alpha} T_{t}$. Thus, $\left\|\left(S_{t}\right)_{\alpha}\right\| \leq$ $M, \alpha \in \mathcal{I}$. The operators $B_{\alpha}, \alpha \in \mathcal{I}$, are given by $B_{\varepsilon_{k}}=\xi_{k}, k \in \mathbb{N}$ and $B_{\alpha}=0, \alpha \neq \varepsilon_{k}$. Thus, $\left\|B_{\alpha}\right\| \leq \sup _{k \in \mathbb{N}}\left\|\xi_{k}\right\|_{L^{\infty}(\mathbb{R})} \leq 1, \alpha \in \mathcal{I}$. Now, according to Theorem 3.2, equation (3.26) has a unique solution $U(t, x, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t, x) H_{\alpha}(\omega)$, where

$$
u_{\alpha}(t, x)=\left(S_{t}\right)_{\alpha} u_{\alpha}^{0}(x)+\int_{0}^{t}\left(S_{t-s}\right)_{\alpha}\left(\sum_{k} \xi_{k}(x) u_{\alpha-\varepsilon_{k}}(s, x)+f_{\alpha}(s, x)\right) d s, \alpha \in \mathcal{I}
$$

Example 3.4. Consider the Cauchy problem

$$
\begin{aligned}
\frac{d}{d t} U(t, \omega) & =\mathbf{A} U(t, \omega)+\mathbf{B} \diamond U(t, \omega)+F(t, \omega) \\
U(0, \omega) & =U^{0}(\omega)
\end{aligned}
$$

where $\mathbf{A}$ is a simple coordinatewise operator $A_{\alpha}=A, \alpha \in \mathcal{I}$, generating a $C_{0}$-semigroup, $B_{\alpha} \neq 0$ only for $\alpha=\varepsilon_{k}, k \in \mathbb{N}$, are such that $\sum_{k \in \mathbb{N}}\left\|B_{\varepsilon_{k}}\right\|(2 k)^{-\frac{p}{2}}<\infty$, and $U^{0}$ and $F$ are deterministic functions, i.e. $u_{\alpha}^{0}=0$ and $f_{\alpha}=0$ for all $\alpha \in \mathcal{I} \backslash\{\mathbf{0}\}$.

The solution of this system, according to Theorem 3.2, is

$$
\begin{aligned}
& u_{\mathbf{0}}(t)=T_{t} u_{\mathbf{0}}^{0}+\int_{0}^{t} T_{t-s} f_{\mathbf{0}}(s) d s \\
& u_{\alpha}(t)=\int_{0}^{t} T_{t-s}\left(\sum_{k \in \mathbb{N}} B_{\varepsilon_{k}} u_{\alpha-\varepsilon_{k}}(s)\right) d s, \quad \alpha \in \mathcal{I} \backslash \mathbf{0}
\end{aligned}
$$

the same form as it was obtained in [12].
We provide two generalisations of Theorem 3.2: one possibility is to allow the operators $B_{\alpha}$ to depend on the time variable $t$ (except for $B_{0}$ which must be free of $t$ ). This embraces for example SPDEs driven by space-time noises which have zero expectation (and are thus free of $t$ ). The other possibility is to allow $B_{0}$ to be unbounded but satisfying certain properties so that $A_{\alpha}+B_{0}$ are infinitesimal generators of $C_{0}$-semigroups. For example, if $A_{\alpha}=\frac{\partial^{2}}{\partial x^{2}}$ and $B_{0}=\frac{\partial}{\partial x}$, then although $B_{0}$ is unbounded, $A_{\alpha}+B_{0}$ is the generator of a contraction semigroup. Following [4] we will enlist some sufficient conditions which ensure that $A_{\alpha}+B_{0}$ is the generators of a $C_{0}$-semigroup.

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Remark 3.5. In Theorem 3.2 one can consider operators $B_{\alpha}(t), \alpha \in \mathcal{I} \backslash\{\mathbf{0}\}$, depending on $t$, so that $B_{\alpha} \in C^{1}([0, T], L(X)), \alpha \in \mathcal{I} \backslash\{\mathbf{0}\}, B_{\mathbf{0}}(t)=B_{\mathbf{0}} \in L(X)$, for all $t \in[0, T]$, and

$$
\begin{align*}
K: & =\sum_{\substack{\alpha \in \mathcal{I}, \alpha>0}}\left\|B_{\alpha}\right\|_{C^{1}([0, T], L(X))}(2 \mathbb{N})^{-p \frac{\alpha}{2}} \\
& =\sum_{\substack{\alpha \in \mathcal{I}, \alpha>0}}\left(\sup _{t \in[0, T]}\left\|B_{\alpha}(t)\right\|_{L(X)}+\sup _{t \in[0, T]}\left\|\frac{d}{d t} B_{\alpha}(t)\right\|_{L(X)}\right)(2 \mathbb{N})^{-p \frac{\alpha}{2}}<\infty . \tag{3.27}
\end{align*}
$$

Replacing (3.2) by (3.27) and retaining all other assumptions of Theorem 3.2, one can again obtain a unique solution $U$ in $C^{1}([0, T], X) \otimes(S)_{-1,-p}$ of the corresponding Cauchy problem (2.1).

The solution is $U(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega), u_{\alpha}(t) \in C^{1}([0, T], X), \alpha \in \mathcal{I}$, where (see (3.9))

$$
\begin{align*}
& u_{\mathbf{0}}(t)=\left(S_{t}\right)_{\mathbf{0}} u_{\mathbf{0}}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\mathbf{0}} f_{\mathbf{0}}(s) d s, \quad t \in[0, T] \\
& u_{\alpha}(t)=\left(S_{t}\right)_{\alpha} u_{\alpha}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta}(s) u_{\alpha-\beta}(s)+f_{\alpha}(s)\right) d s, \quad t \in[0, T] \tag{3.28}
\end{align*}
$$

Its derivative is $\frac{d}{d t} U(t, \omega)=\sum_{\alpha \in \mathcal{I}} \frac{d}{d t} u_{\alpha}(t) H_{\alpha}(\omega)$, where (see (3.17))

$$
\begin{align*}
\frac{d}{d t} u_{\mathbf{0}}(t) & =\left(S_{t}\right)_{\mathbf{0}}\left(A_{\mathbf{0}}+B_{\mathbf{0}}\right) u_{\mathbf{0}}^{0}+\int_{0}^{t}\left(S_{t-s}\right)_{\mathbf{0}} \frac{d}{d s} f_{\mathbf{0}}(s) d s+\left(S_{t}\right)_{\mathbf{0}} f(0), \quad t \in[0, T] \\
\frac{d}{d t} u_{\alpha}(t) & =\left(S_{t}\right)_{\alpha}\left(A_{\alpha}+B_{\mathbf{0}}\right) u_{\alpha}^{0} \\
& +\int_{0}^{t}\left(S_{t-s}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha}\left(B_{\beta}(s) \frac{d}{d s} u_{\alpha-\beta}(s)+\frac{d}{d s} B_{\beta}(s) u_{\alpha-\beta}(s)\right)+\frac{d}{d s} f_{\alpha}(s)\right) d s \\
& +\left(S_{t}\right)_{\alpha}\left(\sum_{\mathbf{0}<\beta \leq \alpha} B_{\beta}(0) u_{\alpha-\beta}(0)+f_{\alpha}(0)\right), \quad t \in[0, T], \quad \alpha \in \mathcal{I} . \tag{3.29}
\end{align*}
$$

The proof can be performed in the same manner as in Theorem 3.2, now taking $T_{0} \in(0, T]$ to be small enough so that $C\left(T_{0}\right)<\frac{1}{6 K^{2}}$, since now we have six summands in (3.29) instead of the previous five in (3.17).
Remark 3.6. In Theorem 3.2 one can consider the operator $B_{0}$ to be unbounded, densely defined on $D$ (the same domain which is common for all $A_{\alpha}$ ) so that either of the following holds:
(i) $A_{\alpha}, \alpha \in \mathcal{I}$, are generating contraction semigroups (i.e. $M=1, w=0$ ), and $B_{0}$ is dissipative, $A_{\alpha}$-bounded with $a_{\alpha}^{0}<1$ (i.e. there exist $a_{\alpha}, b_{\alpha}>0$ such that $\left\|B_{\mathbf{0}} x\right\| \leq a_{\alpha}\left\|A_{\alpha} x\right\|+b_{\alpha}\|x\|, x \in D$, and $a_{\alpha}^{0}=\inf \left\{a_{\alpha}>0: \exists b_{\alpha}>0, \forall x \in D,\left\|B_{\mathbf{0}} x\right\| \leq\right.$ $\left.a_{\alpha}\left\|A_{\alpha} x\right\|+b_{\alpha}\|x\|\right\}$ ), for all $\alpha \in \mathcal{I}$,
(ii) $B_{0}$ is closable, dissipative and $A_{\alpha}$-compact (i.e. $B:\left(D,\|\cdot\|_{A_{\alpha}}\right) \rightarrow X$ is compact where $\|\cdot\|_{A_{\alpha}}$ denotes the graph norm), for all $\alpha \in \mathcal{I}$,
(iii) $A_{\alpha}$ are generating analytic semigroups (i.e. $w<0$ ), $\alpha \in \mathcal{I}$, and $B_{0}$ is closable and $A_{\alpha}$-compact .

Then, $A_{\alpha}+B_{0}$ is the infinitesimal generator of a $C_{0}$-semigroup (denote it $\left.\left(S_{t}\right)_{\alpha}\right)$ for all $\alpha \in \mathcal{I}$. If the semigroups $\left(T_{t}\right)_{\alpha}$ corresponding to $A_{\alpha}$ are uniformly bounded in $\alpha$, then so will be $\left(S_{t}\right)_{\alpha}$. Retaining all other assumptions of Theorem 3.2, now we follow the

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same proof pattern with the semigroup $\left(S_{t}\right)_{\alpha},\left\|\left(S_{t}\right)_{\alpha}\right\| \leq \tilde{M} e^{\tilde{w} t}$, for some $\tilde{M} \geq 1, \tilde{w} \in \mathbb{R}$, independent of $\alpha$.

Finally we note that in case (i) and (ii) $A_{\alpha}+B_{0}$ will be generating contraction semigroups, while in case (iii) they will be generating analytic semigroups.

## 4 Stationary equations

In this section we consider stationary equations of the form

$$
\begin{equation*}
\mathbf{A} U+\mathbf{B} \diamond U+F=0 \tag{4.1}
\end{equation*}
$$

where A: $X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ and $\mathbf{B} \diamond: X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ are coordinatewise operators as in (2.3) and (2.4). We assume that $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ are bounded operators and that $A_{\alpha}$ are of the form

$$
A_{\alpha}=\widetilde{A}_{\alpha}+C_{\alpha}, \quad \alpha \in \mathcal{I}
$$

where $B_{0}$ and $\widetilde{A}_{\alpha}, \alpha \in \mathcal{I}$ are compact operators and $C_{\alpha}$ are self adjoint operators for all $\alpha \in \mathcal{I}$. Denote by $r_{\alpha}$ the eigenvalue corresponding to the orthogonal family of eigenvectors $H_{\alpha}$, i.e. $C_{\alpha}\left(H_{\alpha}\right)=r_{\alpha} H_{\alpha}, \alpha \in \mathcal{I}$. Using classical results of elliptic PDEs and the Fredholm alternative (see [5]) we prove existence and uniqueness of the solution to (4.1).

Theorem 4.1. Let $X$ be a Banach space. Let A : $X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ and $\mathbf{B} \diamond$ : $X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ be coordinatewise operators, for which the following assumptions hold:

1. A is of the form $\mathbf{A}=\widetilde{\mathbf{A}}+\mathbf{C}$, where $\widetilde{\mathbf{A}}(U)=\sum_{\alpha \in \mathcal{I}} \widetilde{A}_{\alpha}\left(u_{\alpha}\right) H_{\alpha}$ and $\widetilde{A}_{\alpha}: X \rightarrow X$ are compact operators for all $\alpha \in \mathcal{I}, \mathbf{C}(U)=\sum_{\alpha \in \mathcal{I}} r_{\alpha} u_{\alpha} H_{\alpha}, r_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, and $\mathbf{B}$ is of the form (2.4), where $B_{0}: X \rightarrow X$ is a compact operator. Assume there exists $K>0$ such that:

$$
\begin{equation*}
-\left\|\widetilde{A}_{\alpha}\right\|-\left\|B_{0}\right\|-r_{\alpha} \geq 0, \quad \text { for all } \quad \alpha \in \mathcal{I} \tag{4.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{\alpha \in \mathcal{I}}\left(\frac{1}{-r_{\alpha}-\left\|\widetilde{A}_{\alpha}\right\|-\left\|B_{0}\right\|}\right)<K \tag{4.3}
\end{equation*}
$$

2. $\mathbf{B}$ is of the form (2.4), where $B_{\beta}: X \rightarrow X, \beta \in \mathcal{I} \backslash\{\mathbf{0}\}$, are bounded operators and there exists $p>0$ such that

$$
\begin{equation*}
K \sum_{\substack{\beta \in \mathcal{I} \\ \beta>0}}\left\|B_{\beta}\right\|(2 \mathbb{N})^{\frac{-p \beta}{2}}<\frac{1}{\sqrt{2}} . \tag{4.4}
\end{equation*}
$$

3. For every $\alpha \in \mathcal{I}$

$$
\begin{equation*}
\operatorname{Ker}\left(\widetilde{A}_{\alpha}+\left(1+r_{\alpha}\right) \operatorname{Id}+B_{0}\right)=\{0\} \tag{4.5}
\end{equation*}
$$

Then, for every $F \in X \otimes(S)_{-1,-p}$ there exists a unique solution $U \in X \otimes(S)_{-1,-p}$ to equation (4.1).

Proof. Equation (4.1) is equivalent to $U-(\widetilde{\mathbf{A}}(U)+\mathbf{C} U+U+\mathbf{B} \boxtimes U)=F$ and

$$
\sum_{\gamma \in \mathcal{I}}\left(u_{\gamma}-\widetilde{A}_{\gamma} u_{\gamma}-\left(1+r_{\gamma}\right) u_{\gamma}-\sum_{\alpha+\beta=\gamma} B_{\alpha}\left(u_{\beta}\right)\right) H_{\gamma}=\sum_{\gamma \in \mathcal{I}} f_{\gamma} H_{\gamma}
$$

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Due to uniqueness of the Wiener-Itô chaos expansion this is equivalent to

$$
\begin{equation*}
u_{\gamma}-\left(\widetilde{A}_{\gamma}+\left(1+r_{\gamma}\right) I d+B_{0}\right) u_{\gamma}=f_{\gamma}+\sum_{0<\beta \leq \gamma} B_{\beta}\left(u_{\gamma-\beta}\right), \quad \gamma \in \mathcal{I} . \tag{4.6}
\end{equation*}
$$

By (4.5) it follows that for each $\gamma \in \mathcal{I}$ the homogeneous equation

$$
u_{\gamma}-\left(\widetilde{A}_{\gamma}+\left(1+r_{\gamma}\right) I d+B_{0}\right) u_{\gamma}=0
$$

has only trivial solution $u_{\gamma}=0$. Since the operator $\widetilde{A}_{\gamma}+\left(1+r_{\gamma}\right) I d+B_{0}$ is compact, the classical Fredholm alternative implies that for each $\gamma \in \mathcal{I}$ there exists a unique $u_{\gamma}$ that solves (4.6) and it is of the form

$$
u_{\gamma}=\left(I d-\left(\left(r_{\gamma}+1\right) I d+\widetilde{A}_{\gamma}+B_{0}\right)\right)^{-1}\left(f_{\gamma}+\sum_{\beta>0} B_{\beta}\left(u_{\gamma-\beta}\right)\right), \quad \gamma \in \mathcal{I}
$$

so that

$$
\left\|u_{\gamma}\right\|_{X} \leq \frac{1}{-r_{\gamma}-\left\|\widetilde{A}_{\gamma}\right\|-\left\|B_{0}\right\|} \cdot\left(\left\|f_{\gamma}\right\|_{X}+\sum_{\beta>0}\left\|B_{\beta}\right\|\left\|u_{\gamma-\beta}\right\|_{X}\right), \quad \gamma \in \mathcal{I}
$$

We will prove that $\sum_{\gamma \in \mathcal{I}} u_{\gamma} \otimes H_{\gamma}$ converges in $X \otimes(S)_{-1}$. Indeed,

$$
\begin{aligned}
\sum_{\gamma \in \mathcal{I}}\left\|u_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma} & \leq K^{2} \sum_{\gamma \in \mathcal{I}}\left(\left\|f_{\gamma}\right\|_{X}+\sum_{\gamma=\alpha+\beta, \alpha>\mathbf{0}}\left\|B_{\alpha}\right\|\left\|u_{\beta}\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \gamma} \\
& \leq 2 K^{2}\left(\sum_{\gamma \in \mathcal{I}}\left\|f_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}+\sum_{\gamma \in \mathcal{I}}\left(\sum_{\gamma=\alpha+\beta, \alpha>\mathbf{0}}\left\|B_{\alpha}\right\|\left\|u_{\beta}\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \gamma}\right) \\
& \leq 2 K^{2}\left(\sum_{\gamma \in \mathcal{I}}\left\|f_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}+\left(\sum_{\alpha>\mathbf{0}}\left\|B_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2} \sum_{\beta \in \mathcal{I}}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p \beta}\right) .
\end{aligned}
$$

Therefore,

$$
\left(1-2 K^{2}\left(\sum_{\alpha>\mathbf{0}}\left\|B_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2}\right) \cdot \sum_{\gamma \in \mathcal{I}}\left\|u_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma} \leq 2 K^{2} \sum_{\gamma \in \mathcal{I}}\left\|f_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma} .
$$

By assumption (4.4) we have that $M=1-2 K^{2}\left(\sum_{\alpha>0}\left\|B_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2}>0$. This implies

$$
\sum_{\gamma \in \mathcal{I}}\left\|u_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma} \leq \frac{2 K^{2}}{M} \sum_{\gamma \in \mathcal{I}}\left\|f_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma}<\infty .
$$

Example 4.2. We provide some special cases of equation (4.1).

1. If $A_{\alpha}=0$ for all $\alpha \in \mathcal{I}$ and $B_{\alpha}, \alpha \in \mathcal{I}$ are second order strictly elliptic partial differential operators in divergent form

$$
\begin{equation*}
B_{\alpha}=\sum_{i=1}^{n} D_{i}\left(\sum_{j=1}^{n} a_{\alpha}^{i j}(x) D_{j}+b_{\alpha}^{i}(x)\right)+\sum_{i=1}^{n} c_{\alpha}^{i}(x) D_{i}+d_{\alpha}(x) \tag{4.7}
\end{equation*}
$$

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with essentially bounded coefficients, then equation (4.1) reduces to the elliptic equation

$$
\mathbf{B} \diamond U=F,
$$

which was solved in [18] and [19].
2. Let $\widetilde{A}_{\alpha}=0$ for all $\alpha \in \mathcal{I}$ and let $B_{\alpha}, \alpha \in \mathcal{I}$, be second order strictly elliptic partial differential operators in divergent form (4.7). Let $\mathbf{C}=c P(\mathcal{R})$, for some $c \in \mathbb{R}$, where $\mathcal{R}$ is the Ornstein-Uhlenbeck operator, $P$ a polynomial of degree $m$ with real coefficients and $P(\mathcal{R})$ the differential operator $P(\mathcal{R})=p_{m} \mathcal{R}^{m}+p_{m-1} \mathcal{R}^{m-1}+\ldots+$ $p_{1} \mathcal{R}+p_{0} I d$. Then, the corresponding eigenvalues are $r_{\alpha}=c P(|\alpha|), \alpha \in \mathcal{I}$. Hence, equation (4.1) transforms to the elliptic equation with a perturbation term driven by the polynomial of the Ornstein-Uhlenbeck operator

$$
\mathbf{B} \diamond U+c P(\mathcal{R}) U=F
$$

that was solved in [11].

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Acknowledgments. The paper was supported by the projects Modeling and harmonic analysis methods and PDEs with singularities, No. 174024, and Modeling and research methods of operational control of traffic based on electric traction vehicles optimized by power consumption criterion, No. TR36047, both financed by the Ministry of Science, Republic of Serbia and project No. 114-451-3605/2013 financed by the Provincial Secretariat for Science of Vojvodina.

# Equations Involving Malliavin Derivative: A Chaos Expansion Approach 

Tijana Levajković and Hermann Mena


#### Abstract

We study equations involving the Malliavin derivative operator and the Wick product with a Gaussian process. In particular, we solve an equation with first-order Malliavin derivative operator by the chaos expansion method in white noise spaces. We prove necessary and sufficient conditions for existence and uniqueness of the solution and represent it in explicit way. We characterize the domains of the Malliavin operators in spaces of Kondratiev distributions in general form. In addition, as an illustration we apply stochastic Galerkin method for solving numerically a stationary version of the equation we considered.


Mathematics Subject Classification (2010). 60H07, 60H10, 60H40, 60H35, 60G20.
Keywords. Generalized stochastic process, chaos expansion, Malliavin derivative, Wick product, stochastic differential equation, Galerkin method.

## 1. Introduction

The Malliavin derivative $\mathbb{D}$, the divergence operator $\delta$ and the Ornstein-Uhlenbeck $\mathcal{R}$ operator are main operators of infinite-dimensional stochastic calculus of variations, also known as the Malliavin calculus. These operators play a key role in the study of non-adapted stochastic differential equations. In white noise setting, the Skorokhod integral is an extension of the stochastic Itô integral of anticipating processes to the class of non-anticipating processes and the Malliavin derivative appears as its adjoint operator; the composition of these two operators, called the Ornstein-Uhlenbeck operator, is a linear, unbounded and self-adjoint operator. In quantum theory these operators correspond respectively to the annihilation, the creation and the number operator.

On white noise spaces, a generalized stochastic process has the Wiener-Itô chaos expansion form, i.e., it can be represented in terms of orthogonal polynomial
basis of a Hilbert space of processes with finite second moments. In [12, 15] operators of Malliavin calculus are considered only on spaces of random variables. In this paper we characterize the domains of these operators for generalized stochastic processes which are represented in their chaos expansion form having values in a certain weight space of stochastic distributions. Part of the contribution of this paper is this characterization, the theorems in Section 3 improve the results from [5]-[8]. On the other hand, in Section 4, we study classes of stochastic differential equations which involve the Malliavin operator $\mathbb{D}$ and the Wick product $\diamond$ with a Gaussian process $\mathbf{G}$

$$
\mathbb{D} u=\mathbf{G} \diamond(\mathbf{A} u)+h, \quad E u=\widetilde{u}_{0}
$$

where $\mathbf{A}$ is a coordinatewise operator on space of generalized stochastic processes and $E$ is the generalized expectation. For solving the equation, we apply the chaos expansion method, also known as the propagator method. With this method the problem is reduced to an infinite triangular system of deterministic equations. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial equation. As a case of study, in Theorem 4.1 we prove the existence and uniqueness of the solution, in the Kondratiev type space of generalized processes, for homogeneous problem

$$
\begin{equation*}
\mathbb{D} u=G \diamond u, \quad E u=\widetilde{u}_{0} \tag{1}
\end{equation*}
$$

for a Gaussian process $G$ of a special form. The study of equation (1) is motivated by [9] where it was shown that Malliavin derivative indicates the rate of change in time between ordinary product and the Wick product, i.e., for a stochastic process $h$ in a weight space of distributions and $W_{t}$ being white noise, the following

$$
h \cdot W_{t}-h \diamond W_{t}=\mathbb{D}(h)
$$

holds. Therefore, the ordinary product is well defined in the generalized sense. In this paper, we deal with Gaussian processes in a more general form than white noise. This paper contributes to the study of equations with generalized operators of Malliavin calculus, we refer to previous results [5]-[10]. Wick product and the Malliavin derivative play an important role in nonlinear problems. For instance, in [18] the authors proved that in random fields, random polynomial nonlinearity can be expanded in a Taylor series involving Wick products and Malliavin derivatives, the so-called Wick-Malliavin series expansion. Since the Malliavin derivative represents a stochastic gradient in the direction of white noise, one can consider similar equations that include a stochastic gradient in the direction of more general stochastic process, like the ones defined in [11].

The chaos expansion method is a very useful technique for solving many types of stochastic differential equations. In $[6,17]$ the Dirichlet problem of elliptic stochastic equations was studied and in [10] parabolic equations with the Wick-type convolution operators. Another type of equations have been investigated in $[4,14,11,12,16]$. Moreover, numerical methods for stochastic differential equations and uncertainty quantification based on the polynomial chaos approach
become very popular in recent years. They are highly efficient in practical computations providing fast convergence and high accuracy. For instance, in order to apply the stochastic Galerkin method the derivation of explicit equations for the polynomial chaos coefficients is required. This is, as in the general chaos expansion, highly nontrivial and sometimes impossible. On the other hand, having an analytical representation of the solution all statistical information can be retrieved directly, e.g, mean, covariance function, variance and even sensitivity coefficients, see $[13,20]$ and references therein for a detailed explanation. The major challenge in stochastic simulations is the high dimensionality, which is even higher solving stochastic control problems, e.g., the stochastic linear quadratic regulator problem, as the computational cost increase in the same order as for the simulation but compared to the deterministic control problem [1]. As an illustration, in Section 5 , we solve numerically the stationary form of nonhomogeneous equation (1) with the Laplace operator by the stochastic Galerkin method.

## 2. Spaces and processes

Let $(\Omega, \mathcal{F}, P)$ be the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the sigma-algebra generated by the weak topology on $\Omega$. The existence of the Gaussian white noise measure $\mu$ is guaranteed by the Bochner-Minlos theorem

$$
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \phi \in \mathcal{S}(\mathbb{R})
$$

where $\langle\omega, \phi\rangle$ denotes the dual paring between a tempered distribution $\omega$ and a rapidly decreasing function $\phi$. Let $\left\{\xi_{k}, k \in \mathbb{N}\right\}$ be the family of Hermite functions and $\left\{h_{k}, k \in \mathbb{N}_{0}\right\}$ the family of Hermite polynomials. Recall, the space of rapidly decreasing functions $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$, where $S_{l}(\mathbb{R})=\left\{\varphi=\sum_{k=1}^{\infty} a_{k} \xi_{k}\right.$ : $\left.\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{N}_{0}$, and the space of tempered distributions $S^{\prime}(\mathbb{R})=$ $\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$, where $S_{-l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} b_{k} \xi_{k}: \sum_{k=1}^{\infty} b_{k}^{2}(2 k)^{-l}<\infty\right\}, l \in \mathbb{N}_{0}$. We have a Gel'fand triplet $S(\mathbb{R}) \subseteq L^{2}(\mathbb{R}) \subseteq S^{\prime}(\mathbb{R})$.

The white noise analysis was constructed as an infinite-dimensional analogue of the Schwartz theory of deterministic generalized functions, for more details we refer to $[2,3]$. Denote by $\mathcal{I}$ the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right)$, where $m=$ $\max \left\{i \in \mathbb{N}: \alpha_{i} \neq 0\right\}$. The $k$ th unit vector is denoted by $\varepsilon^{(k)}=(0, \ldots, 0,1,0, \ldots)$, $k \in \mathbb{N}$. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. Let $a=$ $\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1, a^{\alpha}=\prod_{k=1}^{\infty} a_{k}^{\alpha_{k}}, \alpha!=\prod_{k=1}^{\infty} \alpha_{k}!$ and $(2 \mathbb{N} a)^{\alpha}=\prod_{k=1}^{\infty}\left(2 k a_{k}\right)^{\alpha_{k}}$. Note that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ if $p>0$ and $\sum_{\alpha \in \mathcal{I}} a^{-p \alpha}<\infty$ if $p>1$.

Let $(L)^{2}=L^{2}\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$ be a space of random variables and $H_{\alpha}(\omega)=$ $\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \alpha \in \mathcal{I}$ be the Fourier-Hermite orthogonal basis of $(L)^{2}$, where $\left\|H_{\alpha}\right\|_{(L)^{2}}^{2}=\alpha!$. Particularly, $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$. From the Wiener-Itô chaos expansion theorem it follows that every $F \in(L)^{2}$ can be represented in the
form $F(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega), a_{\alpha}=E_{\mu}\left(F H_{\alpha}\right) \in \mathbb{R}, \omega \in \Omega$ such that $\|F\|_{(L)^{2}}^{2}=$ $\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2} \alpha!<\infty$.

Denote by $\mathcal{H}_{1}$ the first-order chaos space, i.e., the closure of the linear subspace of $(L)^{2}$ spanned by the polynomials $H_{\varepsilon_{k}}(\cdot), k \in \mathbb{N}$. We proved in [9] that the subspace $\mathcal{H}_{1}$ contains Gaussian stochastic processes, e.g., Brownian motion and singular white noise. The $k$ th-order Wiener chaos spaces $\mathcal{H}_{k}$ are obtained by closing in $(L)^{2}$ the linear span of the $k$ th-order Hermite polynomials and $(L)^{2}=\bigoplus_{k=0}^{\infty} \mathcal{H}_{k}$.

### 2.1. Kondratiev type spaces

Let $\rho \in[0,1]$ and let sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$. The space of Kondratiev stochastic test functions modified by $a$, denoted by

$$
(S a)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}}(S a)_{\rho, p}, \quad p \in \mathbb{N}_{0}
$$

is the projective limit of spaces

$$
(S a)_{\rho, p}=\left\{f=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha} \in(L)^{2}: \sum_{\alpha \in \mathcal{I}}(\alpha!)^{1+\rho} b_{\alpha}^{2}(2 \mathbb{N} a)^{p \alpha}<\infty\right\}
$$

The space of Kondratiev stochastic generalized functions modified by $a$,

$$
(S a)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}}(S a)_{-\rho,-p}, \quad p \in \mathbb{N}_{0}
$$

is the inductive limit of the spaces

$$
(S a)_{-\rho,-p}=\left\{F=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha}: \sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho} c_{\alpha}^{2}(2 \mathbb{N} a)^{-p \alpha}<\infty\right\}
$$

The action of $F \in(S a)_{-\rho}$ onto a test function $f \in(S a)_{\rho}$ is given by $\langle\langle F, f\rangle\rangle=$ $\sum_{\alpha \in \mathcal{I}} \alpha!c_{\alpha} b_{\alpha}$. The generalized expectation of $F$ is defined as $E_{\mu}(F)=\langle\langle F, 1\rangle\rangle=$ $c_{0}$, the zero coefficient in formal chaos expansion of $F$. For all $\rho \in[0,1]$ we have a Gel'fand triplet $(S a)_{\rho} \subseteq(L)^{2} \subseteq(S a)_{-\rho}$. For $a_{k}=1, k \in \mathbb{N}$ these spaces reduces to the Kondratiev spaces $(S)_{-\rho}$. Furthermore, the largest space of the Kondratiev distributions is $(S)_{-1}$ and the smallest is $(S)_{-0}$, also called the Hida space of stochastic generalized functions. In [5] we constructed the Kondratiev space $(S a)_{-1}$.

### 2.2. Generalized stochastic processes

Let $X$ be a Banach space of functions on $\mathbb{R}$ endowed with $\|\cdot\|_{X}$ and $\rho \in[0,1]$. Following [16], we define stochastic processes (of the Kondratiev type) as elements of tensor product space $X \otimes(S a)_{-\rho}$, as processes having the chaos expansion form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \tag{2}
\end{equation*}
$$

where $u_{\alpha} \in X$, such that $\|u\|_{X \otimes(S a)_{-\rho,-p}^{2}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N} a)^{-p \alpha}<\infty$, for some $p>0$. We denote by $E u=u_{(0,0,0, \ldots)}$ the generalized expectation of the
process $u$. Clearly, stochastic processes of Kondratiev type can be seen as linear and continuous mappings from $X$ into the space of stochastic distributions $(S a)_{-\rho}$.
Example 2.1. Singular white noise is defined by the chaos expansion $W_{t}(\omega)=$ $\sum_{k=1}^{\infty} \xi_{k}(t) H_{\epsilon^{(k)}}(\omega)$, and it is an element of the space $C^{\infty}(\mathbb{R}) \otimes(S)_{-0,-p}$ for $p>\frac{5}{12}$ and for all $t$.

Now we adapt a general setting of $S^{\prime}$-valued generalized stochastic process provided in [16]. $S^{\prime}(\mathbb{R})$-valued generalized stochastic processes are elements of $Y \otimes$ $(S a)_{-\rho}$, where $Y=X \otimes S^{\prime}(\mathbb{R})$, and are given by chaos expansions of the form $f=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} d_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k}$, where $b_{\alpha}=$ $\sum_{k \in \mathbb{N}} d_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathcal{I}} d_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S a)_{-\rho}$ and $d_{\alpha, k} \in X$. Thus, for some $p, l \in \mathbb{N}_{0}$ it holds

$$
\|f\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S a)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|d_{\alpha, k}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 k)^{-l}(2 \mathbb{N} a)^{-p \alpha}<\infty .
$$

### 2.3. Wick product

We generalize the definition of the Wick product of random variables to the set of generalized stochastic processes in the same way as in $[6,7,17]$. Let $F, G \in$ $X \otimes(S)_{-1}$ be generalized stochastic processes given in chaos expansions of the form (2). Assume $X$ to be a space closed under the product $f_{\alpha} g_{\beta}$, for $f_{\alpha}, g_{\beta} \in X$. Then, the Wick product $F \diamond G$ is defined by

$$
F \diamond G=\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) \otimes H_{\gamma} .
$$

## 3. Characterization of domains of operators of Malliavin calculus

In $[5,7]$ we provided the definitions of the main operators of the Malliavin calculus: the Malliavin derivative $\mathbb{D}$, the Skorokhod integral $\delta$ and the Ornstein-Uhlenbeck $\mathcal{R}$ which are extensions of the classical definitions of these operators in $(L)^{2}$ setting to generalized Kondratiev space of stochastic processes [15].

### 3.1. Malliavin derivative $\mathbb{D}$

Let $u \in X \otimes(S)_{-\rho}$ be of the form (2). We say that $u \in \operatorname{Dom}(\mathbb{D})_{-\rho}$ if there exists $p \in \mathbb{N}_{0}$ such that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{3}
\end{equation*}
$$

is satisfied. Then, the Malliavin derivative, i.e., its stochastic gradient, is defined by

$$
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\epsilon(k)},
$$

where $\alpha-\varepsilon^{(k)}=\left(\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0, \ldots\right)$ is defined for $\alpha_{k} \geq 1$. All processes $u$ that belong to the domain $\operatorname{Dom}(\mathbb{D})_{-\rho}$ are called differentiable in Malliavin sense.

Now, we characterize the domains of the Malliavin derivative of generalized stochastic processes which are elements of spaces $X \otimes(S)_{-\rho}$.

Theorem 3.1. The Malliavin derivative of a process $u \in X \otimes(S)_{-\rho}$ is a linear and continuous mapping $\mathbb{D}: \operatorname{Dom}(\mathbb{D})_{-\rho,-p} \cap X \otimes(S)_{-\rho,-p} \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$, for $l>p+1$ and $p \in \mathbb{N}_{0}$.

Proof. We use the property $\left(\alpha-\varepsilon^{(k)}\right)!=\frac{\alpha!}{\alpha_{k}}$, for $k \in \mathbb{N}$ in the proof of this theorem. Assume that a generalized process $u$ is of the form (2) such that it satisfies (3) for some $p \geq 0$. Then we have

$$
\begin{aligned}
\|\mathbb{D} u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2} & =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-p \alpha+p \varepsilon^{(k)}}\left(\alpha-\varepsilon^{(k)}\right)^{1-\rho} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \alpha_{k}^{2}\left(\alpha-\varepsilon^{(k)}\right)!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p\left(\alpha-\varepsilon^{(k)}\right)}(2 k)^{-l} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \alpha_{k}^{2}\left(\frac{\alpha!}{\alpha_{k}}\right)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}(2 k)^{-(l-p)} \\
& \leq C \sum_{\alpha \in \mathcal{I}}\left(\sum_{k=1}^{\infty} \alpha_{k}\right)^{1+\rho}(\alpha!)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =C \sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
\end{aligned}
$$

where $C=\sum_{k=1}^{\infty}(2 k)^{-(l-p)}<\infty$ for $l>p+1$.
When $\rho=1$ the result of the previous theorem reduces to the corresponding one in [5].

For all $\alpha \in \mathcal{I}$ we have $|\alpha|=\sum_{k \in \mathbb{N}} \alpha_{k}<\alpha!=\prod_{k \in \mathbb{N}} \alpha_{k}, \quad \alpha_{k} \in \mathbb{N}$. Thus, the smallest domain of the spaces $\operatorname{Dom}(\mathbb{D})_{-\rho}$ is obtained for $\rho=0$ and the largest is obtained for $\rho=1$. In particular we have inclusions $\operatorname{Dom}(\mathbb{D})_{-0} \subset \operatorname{Dom}(\mathbb{D})_{-1}$. Moreover if $p<q$ then $\operatorname{Dom}(\mathbb{D})_{-\rho,-p} \subseteq \operatorname{Dom}(\mathbb{D})_{-\rho,-q}$. Note for $u \in \operatorname{Dom}(\mathbb{D})_{-\rho}$ it follows that $u \in \operatorname{Dom}(\mathbb{D} a)_{-\rho}$, for a given sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$, for all $k \in \mathbb{N}$. Indeed, there exists $p>1$ such that
$\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha} \leq C \cdot \sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$, where $C=\sum_{\alpha \in \mathcal{I}} a^{-p \alpha}<\infty$.

### 3.2. Skorokhod integral $\boldsymbol{\delta}$

In [8] we extended the definition of the Skorokhod integral from Hilbert spacevalued generalized random variables to to the class of generalized processes. As an adjoint operator of the Malliavin derivative the Skorokhod integral is defined as follows.

Let $\rho \in[0,1]$. Let $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes v_{\alpha} \otimes H_{\alpha} \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-\rho,-p}, p \in \mathbb{N}_{0}$ be a generalized $S_{-p}(\mathbb{R})$-valued stochastic process and let $v_{\alpha} \in S_{-p}(\mathbb{R})$ be given by the expansion $v_{\alpha}=\sum_{k \in \mathbb{N}} v_{\alpha, k} \xi_{k}, v_{\alpha, k} \in \mathbb{R}$. Then, the process $F$ is integrable in the Skorokhod sense and the chaos expansion of its stochastic integral is given by

$$
\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} f_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} .
$$

Theorem 3.2. Let $\rho \in[0,1]$. The Skorokhod integral $\delta$ of a $S_{-q}(\mathbb{R})$-valued stochastic process is a linear and continuous mapping $\delta: X \otimes S_{-q}(\mathbb{R}) \otimes(S)_{-\rho,-p} \rightarrow X \otimes$ $(S)_{-\rho,-(q+1-\rho)}$, for $q-p>1$.
Proof. This statement follows from $\left(\alpha+\varepsilon^{(k)}\right)!=\left(\alpha_{k}+1\right) \alpha!$, the Cauchy-Schwarz inequality and inequalities $\left(\alpha_{k}+1\right) \leq\left|\alpha+\varepsilon^{(k)}\right| \leq(2 \mathbb{N})^{\alpha+\varepsilon^{(k)}}$, when $\alpha \in \mathcal{I}, k \in \mathbb{N}$. Clearly, we have

$$
\begin{aligned}
\|\delta(F)\|_{X \otimes(S)_{-\rho,-(l+1-\rho)}^{2}} & =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} v_{\alpha, k} f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(l+1-\rho)\left(\alpha+\varepsilon^{(k)}\right)}\left(\alpha+\varepsilon^{(k)}\right)!^{1-\rho} \\
& \leq \sum_{\alpha \in \mathcal{I}} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}} v_{\alpha, k}(2 k)^{-\frac{l}{2}}\right)^{2}(2 \mathbb{N})^{-l \alpha} \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}} v_{\alpha, k}(2 k)^{-\frac{q}{2}}(2 k)^{-\frac{1}{2}(l-q)}\right)^{2}(2 \mathbb{N})^{-l \alpha} \\
& \leq C \sum_{\alpha \in \mathcal{I}} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}\left(\sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}(2 k)^{-q}\right)(2 \mathbb{N})^{-l \alpha} \\
& \leq C \sum_{\alpha \in \mathcal{I}} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}\left\|v_{\alpha}\right\|_{-q}^{2}(2 \mathbb{N})^{-p \alpha}<\infty,
\end{aligned}
$$

because $F \in X \otimes S_{-q}(\mathbb{R}) \otimes(S)_{-\rho,-p}$ and $C=\sum_{k \in \mathbb{N}}(2 k)^{-(l-q)}$ is a finite constant for $l>q+1$.

### 3.3. Ornstein-Uhlenbeck operator $\mathcal{R}$

The image of the Malliavin derivative is included in the domain of the Skorokhod integral and thus we can define their composition, the Ornstein-Uhlenbeck operator denoted by $\mathcal{R}=\delta \circ \mathbb{D}$. We define the domain $\operatorname{Dom}(\mathcal{R})_{-\rho}$ to be the set of all processes $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$ such that the condition

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

is satisfied for some $p \geq 0$. If $u \in X \otimes(S)_{-\rho} \cap \operatorname{Dom}(\mathcal{R})_{-\rho}$ then

$$
\mathcal{R} u=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} .
$$

Recall, Gaussian processes with zero expectation are the only fixed points of the Ornstein-Uhlenbeck operator [9].

Note that for $\rho \in[0,1]$ the inclusion $\operatorname{Dom}(\mathcal{R})_{-\rho} \subseteq \operatorname{Dom}(\mathbb{D})_{-\rho}$ holds. For $\rho=1$ spaces $\operatorname{Dom}(\mathcal{R})_{-\rho}$ and $\operatorname{Dom}(\mathbb{D})_{-\rho}$ coincide $[7]$. The domain $\operatorname{Dom}(\mathcal{R} a)_{-\rho}$, where $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$, is $\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha}<\infty$. For $p>1$ from

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha} & <C \cdot \sum_{\alpha \in \mathcal{I}}|\alpha|^{2}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& <\infty, \text { for } C=\sum_{\alpha \in \mathcal{I}} a^{-p \alpha}
\end{aligned}
$$

it follows that if $u \in \operatorname{Dom}(\mathcal{R})_{-\rho}$ then $u \in \operatorname{Dom}(\mathcal{R} a)_{-\rho}$.

## 4. Wick-type equations involving Malliavin derivative

We consider a nonhomogeneous first-order equation involving the Malliavin derivative operator and the Wick product with a Gaussian process G

$$
\begin{equation*}
\mathbb{D} u=\mathbf{G} \diamond \mathbf{A} u+h, \quad E u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X, \tag{4}
\end{equation*}
$$

where $h$ is a $S^{\prime}$-valued generalized stochastic process and $\mathbf{A}$ is a coordinatewise operator. We assume that a Gaussian process $\mathbf{G}$ belongs to $S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$, for some $l, p>0$, i.e., it can be represented in the chaos expansion form

$$
\begin{equation*}
\mathbf{G}=\sum_{k \in \mathbb{N}} g_{k} \otimes H_{\varepsilon^{(k)}}=\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n} \xi_{n} \otimes H_{\varepsilon^{(k)}}, \quad g_{k n} \in \mathbb{R}, \tag{5}
\end{equation*}
$$

such that $\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n}^{2}(2 n)^{-l}(2 k)^{-p}<\infty$. We also assume $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow$ $X \otimes(S)_{-\rho}$ to be a coordinatewise operator, i.e., a linear operator defined by $\mathbf{A}(f)=\sum_{\alpha \in \mathcal{I}} A_{\alpha}\left(f_{\alpha}\right) \otimes H_{\alpha}$, for $f=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$, where $A_{\alpha}:$ $X \rightarrow X, \alpha \in \mathcal{I}$ are polynomially bounded for all $\alpha$, i.e., there exists $r>0$ such that $\sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\right\|^{2}(2 \mathbb{N})^{-r \alpha}<\infty$. If we assume $A_{\alpha}=A$, for all $\alpha \in \mathcal{I}$ then an operator $\mathbf{A}$ is called a simple coordinatewise operator, according to the classification from [10]. Especially, for a simple coordinatewise operator A such that $A_{\alpha}=\mathbf{0}$ the equation (4) reduce to the initial value problem solved in [8].

As a case of study, in this paper we prove existence and uniqueness of a solution for a special form of (4), providing its solution explicitly. Particularly, we assume $A_{\alpha}=I d, \alpha \in \mathcal{I}$ being the identity operator and a Gaussian process $G \in$ $S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$ obtained from $\mathbf{G}$ by choosing $g_{k n}=\left\{\begin{array}{cc}g_{k}, & k=n \\ 0, & k \neq n\end{array}, k, n \in \mathbb{N}\right.$.
Clearly, we consider $G$ to be of the form

$$
\begin{equation*}
G=\sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes H_{\varepsilon^{(k)}}, \tag{6}
\end{equation*}
$$

such that its coefficients $g_{k} \in \mathbb{R}, k \in \mathbb{N}$ satisfy the convergence condition

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} g_{k}^{2}(2 n)^{-q}<\infty, \quad \text { for some } \quad q>0 \tag{7}
\end{equation*}
$$

Therefore, we are interested to solve

$$
\begin{equation*}
\mathbb{D} u=G \diamond u, \quad E u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X \tag{8}
\end{equation*}
$$

i.e., to find a Malliavin differentiable process whose derivative coincides with its Wick product with a certain Gaussian process.
Theorem 4.1. Let $\rho \in[0,1]$. Let $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}, p, l>0$ be a Gaussian process of the form (6) satisfying (7). If $g_{k} \geq \frac{1}{2 k}$, for all $k \in \mathbb{N}$ then there exists a unique solution $u$ in $X \otimes(S g)_{-\rho} \cap \operatorname{Dom}(\mathbb{D})_{-\rho,-p}$ of the initial value problem (8) given by

$$
\begin{equation*}
u=\sum_{\substack{\alpha=\left(2 \beta_{1}, 2 \beta_{2}, \ldots, 2 \beta_{m}, 0, \ldots\right) \in \mathcal{I} \\ \beta_{1}, \beta_{2}, \ldots, \beta_{m} \in \mathbb{N}_{0}}} \frac{C_{\alpha}}{|\alpha|!!}\left(\prod_{k=1}^{\infty} g_{k}^{\beta_{k}}\right) \widetilde{u}_{0} \otimes H_{\alpha}=\widetilde{u}_{0} \otimes \sum_{\beta \in \mathcal{I}} C_{2 \beta} \frac{g^{\beta}}{|2 \beta|!!} H_{\beta} \tag{9}
\end{equation*}
$$

where $C_{\alpha}$ represents the number of all possible decomposition chains connecting multi-indices $\alpha$ and $\alpha_{1}$, such that $\alpha_{1}$ is the first successor of $\alpha$ having only one nonzero component that is obtained by substractions $\alpha-2 \varepsilon^{\left(p_{1}\right)}-\cdots-2 \varepsilon^{\left(p_{s}\right)}=\alpha_{1}$, for $p_{1}, \ldots, p_{s} \in \mathbb{N}, s \geq 0$.

Proof. We are looking for a solution of (8) in the chaos expansion representation form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}, \quad u_{\alpha} \in X \tag{10}
\end{equation*}
$$

which is Malliavin differentiable and which admits the Wick multiplication with a Gaussian process of the form (6). This means that we are seeking for unknown coefficients $u_{\alpha} \in X$ such that the condition $\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} g)^{-p \alpha}<\infty$ is satisfied for some $p>0$. Wick product of a process $u$ and a Gaussian process $G$, represented in their chaos expansion forms (10) and (6) respectively, is a welldefined element $G \diamond u$ given by

$$
G \diamond u=\sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes H_{\varepsilon^{(k)}} \diamond \sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}}
$$

Clearly, assuming (7) and $u \in X \otimes(S)_{-\rho,-p}$ then $G \diamond u \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$, $l, p>0$, because

$$
\begin{aligned}
\|G \diamond u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}(\alpha!)^{1-\rho} g_{k}^{2}(2 k)^{-l}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p\left(\alpha+\varepsilon^{k}\right)} \\
& \leq \sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \cdot \sum_{k \in \mathbb{N}} g_{k}^{2}(2 k)^{-q} \\
& =\|u\|_{X \otimes(S)_{-\rho,-p}^{2}}^{2} \cdot\|G\|_{S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}<\infty
\end{aligned}
$$

where $q=l+p$. Previous estimates are also valid for processes in the Kondratiev space modified with a sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}$. Both, the Wick product $G \diamond u$ and the action of the Malliavin derivative on $u$, belong to the domain of the Skorokhod integral and therefore we can apply the operator $\delta$ on both sides of (8). Thus, we
obtain $\delta(\mathbb{D} u)=\delta(G \diamond u)$. Substituting the composition $\delta \circ \mathbb{D}$ with the OrnsteinUhlenbeck operator $\mathcal{R}$, the initial equation (8) transforms to equivalent one written in terms of the Skorokhod integral $\delta$ and the Ornstein-Uhlenbeck operator $\mathcal{R}$

$$
\begin{equation*}
\mathcal{R} u=\delta(G \diamond u) \tag{11}
\end{equation*}
$$

We replace all the processes in (11) with their chaos expansion expressions, apply operators $\mathcal{R}$ and $\delta$ and obtain unknown coefficients of a process $u$.

$$
\begin{aligned}
\mathcal{R}\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}\right) & =\delta\left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha+\varepsilon^{(k)}}\right) \\
\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} u_{\alpha} \otimes H_{\alpha+2 \varepsilon^{(k)}}
\end{aligned}
$$

We select terms which correspond to multi-indices of length zero and one and obtain

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} u_{\varepsilon^{(k)}} \otimes H_{\varepsilon^{(k)}}+\sum_{\alpha \in \mathcal{I},|\alpha| \geq 2}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I},|\alpha| \geq 2} \sum_{k \in \mathbb{N}} g_{k} u_{\alpha-2 \varepsilon^{(k)}} \otimes H_{\alpha} \tag{12}
\end{equation*}
$$

Due to the uniqueness of chaos expansion representations in the orthogonal Fourier-Hermite basis, we equalize corresponding coefficients on both sides of (12) and obtain the triangular system of deterministic equations

$$
\begin{array}{rlrl}
u_{\varepsilon^{(k)}} & =0, & k \in \mathbb{N} \\
|\alpha| u_{\alpha} & =\sum_{k \in \mathbb{N}} g_{k} u_{\alpha-2 \varepsilon^{(k)}}, & & |\alpha| \geq 2 \tag{14}
\end{array}
$$

where by convention $\alpha-2 \varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$ or $\alpha_{k}=1$, thus $u_{\alpha-2 \varepsilon^{(k)}}=0$ for $\alpha_{k} \leq 1$. We solve the system of equations (13) and (14) by induction with respect to the length of multi-indices $\alpha$ and thus obtain coefficients $u_{\alpha},|\alpha| \geq 1$ of a solution of (8) in explicit form. First, from (14) it follows that $u_{\alpha}$ are represented in terms of $u_{\beta}$ such that $|\beta|=|\alpha|-2$, where $u_{\beta}$ are obtained in the previous step of the induction procedure.

From the initial condition $E u=\widetilde{u}_{0}$ it follows that $u_{(0,0,0, \ldots)}=\widetilde{u}_{0}$ and from (13) we obtain coefficients $u_{\alpha}=0$ for all $|\alpha|=1$. For $|\alpha|=2$ there are two possibilities: $\alpha=2 \varepsilon^{(k)}, k \in \mathbb{N}$ and $\alpha=\varepsilon^{(k)}+\varepsilon^{(j)}, k \neq j, k, j \in \mathbb{N}$. From (14) it follows that

$$
u_{\alpha}= \begin{cases}\frac{1}{2} g_{k} \widetilde{u}_{0}, & \alpha=2 \varepsilon^{(k)} \\ 0, & \alpha=\varepsilon^{(k)}+\varepsilon^{(j)}, k \neq j\end{cases}
$$

Note $\alpha=2 \varepsilon^{(k)}, k \in \mathbb{N}$ has only one nonzero component, so $\alpha=\alpha_{1}$, thus only one term appears in the sum (14) and $C_{\alpha}=1$.

We point out here that $u_{\alpha}=0$ for $|\alpha|=3$, because these coefficients are represented through the coefficients of the length one, which are zero. Moreover, for all $\alpha \in \mathcal{I}$ of odd length, i.e., for all $\alpha \in \mathcal{I}$ such that $|\alpha|=2 n+1, n \in \mathbb{N}$ the coefficients $u_{\alpha}=0$.

Our goal is to obtain a general form of the coefficients $u_{\alpha}$ for $\alpha \in \mathcal{I}$ of even length, i.e., for $|\alpha|=2 n, n \in \mathbb{N}$. Now, for $|\alpha|=4$ there are five different types of $\alpha$. Without loss of generality we consider $\alpha \in\{(4,0,0, \ldots),(3,1,0,0, \ldots)$, $(2,1,1,0, \ldots),(1,1,1,1,0,0, \ldots),(2,2,0,0, \ldots)\}$.

From (14) it follows $u_{(4,0,0, \ldots)}=\frac{1}{4} g_{1} u_{(2,0,0,0, \ldots)}$. Using the forms of $u_{\alpha}$ obtained in the previous steps we get $u_{(4,0,0, \ldots)}=\frac{1}{4} \frac{1}{2} g_{1}^{2} \widetilde{u}_{0}$. We also obtain

$$
u_{(3,1,0, \ldots)}=u_{(2,1,1,0, \ldots)}=u_{(1,1,1,1,0,0, \ldots)}=0
$$

and

$$
u_{(2,2,0,0 \ldots)}=\frac{1}{4}\left(g_{1} u_{(0,2,0, \ldots)}+g_{2} u_{(2,0,0, \ldots)}\right)=\frac{1}{4} \frac{1}{2} g_{1} g_{2} \cdot \widetilde{u}_{0} \cdot 2
$$

It follows that only nonzero coefficients are obtained for multi-indices of forms $\alpha=4 \varepsilon^{(k)}, k \in \mathbb{N}$ and $\alpha=2 \varepsilon^{(k)}+\varepsilon^{(j)}, k \neq j, k, j \in \mathbb{N}$. Thus, for $|\alpha|=4$

$$
u_{\alpha}= \begin{cases}\frac{1}{4!!} g_{k}^{2} \widetilde{u}_{0}, & \alpha=4 \varepsilon^{(k)} \\ 2 \cdot \frac{1}{4!!} g_{k} g_{j} \widetilde{u}_{0}, & \alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}, k \neq j \\ 0, & \text { otherwise }\end{cases}
$$

Note $\alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}$, for $k \neq j$ has two nonzero components, thus there are two terms in the sum (14) and $C_{\alpha}=2$. For example, $\alpha=(2,2,0,0, \ldots)$ can be decomposed in one of two following ways $\alpha=2 \varepsilon^{(1)}+(0,2,0,0, \ldots)$ or $\alpha=$ $2 \varepsilon^{(2)}+(2,0,0,0, \ldots)$, therefore $C_{(2,2,0,0, \ldots)}=2$.

For $|\alpha|=6$ we consider only multi-indices which have all their components even. For the rest $u_{\alpha}=0$. For example, from (14) and from the forms of the coefficients obtained in the previous steps it follows $u_{(6,0,0, \ldots)}=\frac{1}{6} g_{1} u_{(4,0,0, \ldots)}=$ $\frac{1}{6} \frac{1}{4} \frac{1}{2} g_{1}^{3} \widetilde{u}_{0}$. Next, $u_{(4,2,0,0, \ldots)}=\frac{1}{6}\left(g_{1} u_{(2,2,0,0, \ldots)}+g_{2} u_{(4,0,0, \ldots)}\right)=3 \cdot \frac{1}{6} \frac{1}{4} \frac{1}{2} g_{1}^{2} g_{2} \widetilde{u}_{0}$. Finally, $u_{(2,2,2,0, \ldots)}=g_{1} u_{(0,2,2,0, \ldots)}+g_{2} u_{(2,0,2,0, \ldots)}+g_{3} u_{(2,2,0,0, \ldots)}=6 \cdot \frac{1}{6} \frac{1}{4} \frac{1}{2} g_{1} g_{2} g_{3} \widetilde{u}_{0}$. The later coefficient, $C_{\alpha}=6$, meaning that there are six chain decompositions of $\alpha=(2,2,2,0,0, \ldots)$ of the form $\alpha=2 \varepsilon^{\left(p_{1}\right)}+2 \varepsilon^{\left(p_{2}\right)}+\cdots+2 \varepsilon^{\left(p_{s}\right)}+\alpha_{1}$, with $\alpha_{1}$ having only one nonzero component. This case is illustrated in Figure 1(b). For $\alpha=(4,2,0,0, \ldots)$ we have $C_{\alpha}=3$, where all decomposing possibilities are described in Figure 1(a). Thus,

$$
u_{\alpha}= \begin{cases}\frac{1}{6!!} g_{k}^{3} \widetilde{u}_{0}, & \alpha=6 \varepsilon^{(k)}, \\ 3 \cdot \frac{1}{6!!} g_{k}^{2} g_{j} \widetilde{u}_{0}, & \alpha=4 \varepsilon^{(k)}+2 \varepsilon^{(j)}, k \neq j, \\ 6 \cdot \frac{1}{6!!} g_{k} g_{j} g_{i} \widetilde{u}_{0}, & \alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}+2 \varepsilon^{(i)}, k \neq i, j, i \neq j \\ 0, & \text { otherwise. }\end{cases}
$$

We proceed by the same procedure for all even multi-index lengths to obtain $u_{\alpha}$ in the form

$$
u_{\alpha}=\left\{\begin{array}{cl}
\frac{C_{\alpha}}{|\alpha|!!} \cdot g_{1}^{\beta_{1}} g_{2}^{\beta_{2}} \cdots g_{m}^{\beta_{m}} \widetilde{u}_{0}, & \left\{\begin{array}{l}
\alpha=\left(2 \beta_{1}, 2 \beta_{2}, \ldots, 2 \beta_{m}, 0,0\right) \in \mathcal{I} \\
|\alpha|=2 n, n \in \mathbb{N}
\end{array}\right.  \tag{15}\\
0, & |\alpha|=2 n-1, n \in \mathbb{N}
\end{array}\right.
$$

where $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, 0,0, \ldots\right) \in \mathcal{I}, \beta_{1}, \ldots, \beta_{m} \in \mathbb{N}_{0}$ and $C_{\alpha}$ represents the number of decompositions of $\alpha$ in the way $\alpha=2 \varepsilon^{\left(p_{1}\right)}+\cdots+2 \varepsilon^{\left(p_{s}\right)}+\alpha_{1}$,


Figure 1. $\alpha$ values
for all possible $p_{1}, \ldots, p_{s}$, i.e., all the branches paths that connect $\alpha$ and $\alpha_{1}=$ $\left(0,0, \ldots, \widetilde{\alpha}_{i}, 0,0, \ldots\right)$, for some $\widetilde{\alpha}_{i} \neq 0$.

Note, for $\alpha=2 \beta=\left(2 \beta_{1}, 2 \beta_{2}, \ldots, 2 \beta_{m}, 0, \ldots\right) \in \mathcal{I}$ the coefficient $1 \leq C_{\alpha} \leq$ $m$ !, i.e., $C_{\alpha}$ is maximal when all nonzero components of $\alpha$ are equal two.

Summing up all the coefficients in (15) we obtain the form of solution (9). It remains to prove the convergence of the solution $u$ in the space $X \otimes(S g)_{-\rho,-p} \cap$ $\operatorname{Dom}(\mathbb{D})_{-\rho}$, i.e.,

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} g)^{-p \alpha}<\infty
$$

We use inequalities $|\alpha| \leq \alpha!\leq(2 \mathbb{N})^{\alpha}$ for $\alpha \in \mathcal{I}$ and that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N} g)^{-p \alpha}<\infty$ if $p>0$ for a sequence $g$ that satisfies the assumption $g_{k} \geq \frac{1}{2 k}, k \in \mathbb{N}$. Thus, there exists $s>1$ large enough so $m!\left(\frac{\alpha!!}{|\alpha|!!}\right)^{2} \leq(2 \mathbb{N})^{s \alpha}$, for $\alpha \in \mathcal{I}$ and $m=\max \{i \in \mathbb{N}$ : $\left.\alpha_{i} \neq 0\right\}$. For $p>\max \{2, s\}$ we have
$\|u\|_{X \otimes(S g)_{-\rho,-p} \cap \operatorname{Dom}(\mathbb{D})_{-\rho}}^{2}$

$$
\begin{aligned}
& =\sum_{\alpha=\left(2 \beta_{1}, \ldots, 2 \beta_{m}, 0,0, \ldots\right) \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|\widetilde{u}_{0}\right\|_{X}^{2} C_{\alpha}^{2} \frac{g^{2 \beta}}{(|\alpha|!!)^{2}}(2 \mathbb{N} g)^{-p \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha=\left(2 \beta_{1}, \ldots, 2 \beta_{m}, 0,0, \ldots\right) \in \mathcal{I}} \frac{(\alpha!)^{2} m!}{(|\alpha|!!)^{2}} g^{\alpha}(2 \mathbb{N})^{-p \alpha} g^{-p \alpha}
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{s \alpha}(2 \mathbb{N})^{-p \alpha} g^{\alpha} g^{-p \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathcal{I}} g^{-(p-1) \alpha} \sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-(p-s) \alpha}<\infty
\end{aligned}
$$

Example 4.1. For $g_{k}=1, k \in \mathbb{N}$ in (6), a Gaussian process $G$ represents a singular white noise $W$ and equation (8) transforms to the equation

$$
\begin{equation*}
\mathbb{D} u=W \diamond u, \quad E u=\widetilde{u}_{0} . \tag{16}
\end{equation*}
$$

Since the coefficients of $W$ satisfy assumptions of Theorem 4.1, then the equation (16) has a unique solution in $X \otimes(S)_{-\rho}$ represented in the form

$$
u=\widetilde{u}_{0} \otimes \sum_{\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, 0, \ldots\right) \in \mathcal{I}} \frac{C_{2 \beta}}{|2 \beta|!!} H_{2 \beta},
$$

where $C_{2 \beta}$ is the number of all possible chain decompositions of $2 \beta \in \mathcal{I}$ described in Theorem 4.1.

Remark 4.1. a) The same procedure, described in the proof of Theorem 4.1 can be applied for solving equations with Gaussian processes in general form (5). Hence, in order to obtain the coefficients $u_{\alpha}, \alpha \in \mathcal{I}$ of a solution (10) of a homogeneous equation $\mathbb{D} u=\mathbf{G} \triangleleft u, E u=\widetilde{u}_{0}, \widetilde{u}_{0} \in X$ one has to solve the system of deterministic equations

$$
u_{\varepsilon^{(k)}}=0, \text { for } k \in \mathbb{N} \quad \text { and } \quad|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n} u_{\alpha-\varepsilon^{(k)}-\varepsilon^{(n)}}, \quad|\alpha| \geq 2,
$$

that corresponds to the system (13) and (14).
b) Considering a nonhomogeneous problem $\mathbb{D} u=\mathbf{G} \diamond u+h, E u=\widetilde{u}_{0}, \widetilde{u}_{0} \in X$, for $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ the unknown coefficients $u_{\alpha}, \alpha \in \mathcal{I}$ of a solution $u \in$ $X \otimes(S g)_{-\rho}$ are determined from the system of deterministic equations $u_{\varepsilon^{(k)}}=f_{0, k}$, for $k \in \mathbb{N}$ and

$$
|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k} u_{\alpha-\varepsilon^{(k)}-\varepsilon^{(n)}}+\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}, \quad|\alpha| \geq 2 .
$$

The solution $u$ belongs to the Kondratiev space of distributions modified by a sequence $g$ and it can be represented as a sum of the solution that corresponds to homogeneous part of equation $u_{h}$ and a nonhomogeneous part $u_{n h}$ which depends on $f$. The proof is rather technical and we omit it in this paper.
c) Consider equation

$$
\begin{equation*}
\mathbb{D} u=\mathcal{B}(G \diamond u)+h, \quad E u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X, \tag{17}
\end{equation*}
$$

where $\mathcal{B}$ is a coordinatewise operator, i.e., $\mathcal{B}: X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho} \rightarrow X \otimes$ $S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ is a linear operator defined by $\mathcal{B}(f)=\sum_{\alpha \in \mathcal{I}} \mathbf{B}_{\alpha}\left(f_{\alpha}\right) H_{\alpha}$, for $f=$ $\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$, where $\mathbf{B}_{\alpha}: X \otimes S^{\prime}(\mathbb{R}) \rightarrow X \otimes S^{\prime}(\mathbb{R}), \alpha \in \mathcal{I}$ are linear and of the form $\mathbf{B}_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} B_{\alpha, k}\left(\xi_{k}\right), \alpha \in \mathcal{I}$, such that $B_{\alpha, k}: S^{\prime}(\mathbb{R}) \rightarrow$ $S^{\prime}(\mathbb{R}), k \in \mathbb{N}$. We also assume $\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|B_{\alpha, k}\right\|^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty$, for some
$p, l>0$. Especially, if operator $\mathcal{B}$ is a simple coordinatewise operator of the form $B_{\alpha, k}=B=-\Delta+x^{2}+1, \alpha \in \mathcal{I}, k \in \mathbb{N}$ then, in order to solve (17) we can apply the same procedure explained in Theorem 4.1. Recall, the domain of $B$ contains $S^{\prime}(\mathbb{R})$ and the Hermite functions are eigenvectors of $B$ with $B \xi_{k}=2 k \xi_{k}, k \in \mathbb{N}$. We set $h=0$. Clearly,

$$
\begin{aligned}
\mathcal{B}(G \diamond u) & =\mathcal{B}\left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} B_{\alpha, k}\left(\xi_{k}\right) \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} B \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{k} 2 k \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} .
\end{aligned}
$$

Therefore, after applying operator $\delta$ we obtain

$$
\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} 2 k g_{k} u_{\alpha} \otimes H_{\alpha+2 \varepsilon^{(k)}}
$$

The coefficients of the solution are obtained by induction from the system of equations

$$
u_{\varepsilon^{(k)}}=0, \text { for all } k \in \mathbb{N}, \quad \text { and } \quad|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} 2 k g_{k} u_{\alpha-2 \varepsilon^{(k)},},|\alpha| \geq 2
$$

Under assumptions of Theorem 4.1 it can be proven that there exists a unique solution of equation in the space $X \otimes(S g)_{-\rho,-p} \cap \operatorname{Dom}(\mathbb{D})_{-\rho}$, for $p>\max \{3, s\}$ given in the form

$$
u=\widetilde{u}_{0} \otimes \sum_{2 \beta=\left(2 \beta_{1}, \ldots, \beta_{m}, 0,0, \ldots\right) \in \mathcal{I}} \frac{C_{2 \beta}}{|2 \beta|!!!}\left(\prod_{k=1}^{\infty}(2 k) g_{k}^{\beta_{k}}\right) H_{2 \beta} .
$$

## 5. A numerical example

In this section we consider a stationary equation

$$
\begin{equation*}
G \diamond \mathbf{A} u=h, \quad E u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X \tag{18}
\end{equation*}
$$

obtained from (4), for $\mathbb{D} u=0$. Particularly, by applying the stochastic Galerkin method we solve numerically (18) for a simple coordinatewise operator $\mathbf{A}$ with $A_{\alpha}=\Delta, \alpha \in \mathcal{I}$, the Laplace operator in two spatial dimensions and $G$ being a Gaussian random variable. Thus, (18) reduce to

$$
\begin{equation*}
G(\omega) \diamond \sum_{\alpha \in \mathcal{I}} \Delta u(x, y) H_{\alpha}(\omega)=h(\omega), \quad(x, y) \in \mathcal{D}, \omega \in \Omega \tag{19}
\end{equation*}
$$

Note that in the stochastic Galerkin method a finite-dimensional approximation of Fourier-Hermite orthogonal polynomials $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ is used [4, 20]. The main steps are sketched in Algorithm 5.1.

Algorithm 5.1 Main steps of the stochastic Galerkin method
1: Choose finite set of polynomials $H_{\alpha}$ and truncate the random series to a finite random sum.
Solve numerically the deterministic triangular system of equations by a suitable method.
Compute the approximate statistics of the solutions from obtained coefficients. Generate $H_{\alpha}$ and compute the approximate solutions.

Let $\mathcal{D}=\{(x, y):-1 \leq x \leq 1,-1 \leq y \leq 1\}$ be the spatial domain and let $G=g_{0}+\sum_{k \in \mathbb{N}} g_{k} H_{\varepsilon^{(k)}}$ be a Gaussian random variable with mean $E G=g_{0}=10$ and variance $\operatorname{Var} G=\sum_{k \in \mathbb{N}} g_{k}^{2}-g_{0}^{2}=3.3^{2}$. We denote by $\mathcal{I}_{m, p}$ the set of $\alpha=$ $\left(\alpha_{1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$ with $m=\max \left\{i \in \mathbb{N}: \alpha_{i} \neq 0\right\}$ such that $|\alpha| \leq p$. As a first step, we represent $u$ in its truncated polynomial chaos expansion form $\widetilde{u}$, i.e., we approximate solution with the chaos expansion in $\oplus_{k=0}^{p} \mathcal{H}_{k}$ with $m$ random variables $\widetilde{u}(x, y, \omega)=\sum_{\alpha \in \mathcal{I}_{m, p}} \widetilde{u}_{\alpha}(x, y) H_{\alpha}(\omega)$; the previous sum has $P=\frac{(m+p)!}{m!p!}$ terms. Hence, (19) transforms to

$$
\begin{aligned}
g_{0} \cdot & \sum_{\alpha \in \mathcal{I}_{m, p}} \Delta \widetilde{u}_{\alpha}(x, y) H_{\alpha}(\omega)+\sum_{\alpha \in \mathcal{I}_{m, p}} \sum_{k=1}^{m} g_{k} \Delta \widetilde{u}_{\alpha}(x, y) H_{\alpha+\varepsilon^{(k)}}(\omega) \\
& =\sum_{\alpha \in \mathcal{I}_{m, p}} h_{\alpha} H_{\alpha}(\omega)
\end{aligned}
$$

The unknown coefficients $\widetilde{u}_{\alpha}, \alpha \in \mathcal{I}_{m, p}$ are obtained by the projection onto each element of the Fourier-Hermite basis $\left\{H_{\gamma}\right\}, \gamma \in \mathcal{I}_{m, p}$, i.e., by taking the expectations for all $\gamma \in \mathcal{I}_{m, p}$

$$
\begin{aligned}
& E_{\mu}\left(H_{\gamma} \cdot\left(g_{0} \sum_{\alpha \in \mathcal{I}_{m, p}} \Delta \widetilde{u}_{\alpha}(x, y) H_{\alpha}+\sum_{\alpha \in \mathcal{I}_{m, p}} \sum_{k=1}^{m} g_{k} \Delta \widetilde{u}_{\alpha}(x, y) H_{\alpha+\varepsilon^{(k)}}\right)\right) \\
& \quad=E_{\mu}\left(H_{\gamma} \cdot \sum_{\alpha \in \mathcal{I}_{m, p}} h_{\gamma} H_{\alpha}\right) .
\end{aligned}
$$

From the formula $H_{\alpha}(\omega) \cdot H_{\beta}(\omega)=\sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}(\omega)$ for Hermite polynomials [3] and the orthogonality of the polynomial basis, it follows that the initial equation reduces to a system of $P$ deterministic equations for coefficients $\widetilde{u}_{\alpha}$. Particularly, we take $m=15, p=3$ and then obtain $P=816$ deterministic equations in the system. We assume $h_{\alpha}=1$ for $|\alpha| \leq 3$ and $h_{\alpha}=0$ for $|\alpha|>3$. We use central differencing to discretize in the spatial dimensions and 170 grid cells in each spacial direction. Then, we solve numerically the resulting system.


Figure 2. Expected value (left) and variance (right) of the solution

Once the coefficients of the expansion $\widetilde{u}$ are obtained, we are able to compute all the moments of the random field. Particularly the expectation $E u=u_{0}$ and the variance of the solution $\operatorname{Var} u=\sum_{\alpha \in \mathcal{I}_{m, p}} \alpha!\widetilde{u}_{\alpha}^{2}$ are plotted in Figure 2, on $z$-axes over the domain $\mathcal{D}$. We can observe that the variance of the solution is relatively high. In general, this behaviour is related to singularities.

We would like to underline that Wiener chaos expansion converges quite fast, i.e., even small values of $p$ may lead to very accurate approximation. The error generated by the truncation of the Wiener chaos expansion, in $X \otimes(L)^{2}$ is

$$
\begin{aligned}
\mathcal{E}^{2} & =\|u(x, y, \omega)-\widetilde{u}(x, y, \omega)\|_{X \otimes(L)^{2}}^{2} \\
& =E_{\mu}(u(x, y, \omega)-\widetilde{u}(x, y, \omega)) \\
& =\sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, p}} \alpha!\left\|u_{\alpha}(x, y)\right\|_{X}^{2}
\end{aligned}
$$

for $(x, y) \in \mathcal{D}$. Note that if instead of a Gaussian random variable, a stochastic generalized function is considered, i.e., when the coefficients are singular, the error $\mathcal{E}^{2} \rightarrow 0$ converge in the space of Kondratiev distributions.

## Acknowledgement

The paper was partially supported by the project Modeling and research methods of operational control of traffic based on electric traction vehicles optimized by power consumption criterion, No. TR36047, financed by the Ministry of Science, Republic of Serbia.

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# Nonhomogeneous First-order Linear Malliavin Type Differential Equation 

Tijana Levajković and Dora Seleši


#### Abstract

In this paper we solve a nonhomogeneous first-order linear equation involving the Malliavin derivative operator with stochastic coefficients by use of the chaos expansion method. We prove existence and uniqueness of a solution in a certain weighted space of generalized stochastic distributions and represent the obtained solution in the Wiener-Itô chaos expansion form.


Mathematics Subject Classification (2010). Primary: $60 \mathrm{H} 15,60 \mathrm{H} 40,60 \mathrm{H} 10$, 60H07, 60G20.
Keywords. Generalized stochastic process, chaos expansion, Malliavin derivative, stochastic differential equation, nonhomogeneous linear equation.

## 1. Introduction

This paper is devoted to study of generalized stochastic processes which have series expansion representation form given in terms of an orthogonal polynomial basis, defined on an infinite-dimensional space. In particular, we focus on a Hilbert space of square integrable processes defined on a Gaussian white noise probability space where the orthogonal basis is constructed using the Hermite polynomials and the Hermite functions. We provide definition of stochastic generalized random variable spaces over a space of square integrable random variables by adding certain weights in the convergence condition of the series expansion. Introduced by Hida (see [1]) and further developed by many authors (see [2], [3], [7], [10], [12] and references therein), white noise analysis was applied to solving different classes of stochastic differential equations ([5], [8], [14]).

This paper deals with the Malliavin derivative, one of three main operators of the Malliavin stochastic calculus, an infinite-dimensional differential calculus of variations in the white noise setting. Recall, the Skorokhod integral represents an extension of the Itô integral from a set of adapted processes to a set of nonanticipating processes. Its adjoint operator is known as the Malliavin derivative. Operators
of Malliavin calculus are widely used in solving stochastic differential equations. In particular, Malliavin differential operator found place in stochastic differential equations connected to optimal control problems and problems in financial mathematics. We give a more general definition of the Malliavin derivative than in [10], [11]. We allow values in the Kondratiev space of stochastic distributions $(S)_{-\rho}$, $\rho \in[0,1]$ and thus obtain a larger domain for the derivative operator. For basic results related to the Malliavin derivative we refer to [2], [7], [10], [12] and for its applications we refer to [3], [4], [9], [11], [13].

Furthermore, as a description of the chaos expansion method, we solve a nonhomogeneous linear stochastic differential equations involving the Malliavin derivative. We provide a general method of solving, using the Wiener-Itô chaos decomposition form, also known as the propagator method. This method gives good framework and opportunity for solving many classes of stochastic equations (see [7], [8], [9]). The problem is based on finding an appropriate, large enough space of generalized functions where a solution of a considered equation exists.

The paper is organized in the following manner: In Section 2 we provide the basic notation used throughout the paper, followed by the survey on chaos expansions of generalized stochastic processes and $S^{\prime}$-valued generalized stochastic processes. The Malliavin derivative is defined on a set of generalized stochastic processes and the characterization of its domain is stated. In Section 3 we apply the chaos expansion method in order to solve a nonhomogeneous first-order linear Malliavin type differential equation with singular coefficients, represented in the form

$$
\mathbb{D} u=c \otimes u+h, \quad E u=\widetilde{u}_{0}
$$

for $c \in \mathcal{S}^{\prime}(\mathbb{R}), h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}, \widetilde{u}_{0} \in X$ and $E$ is the expectation.

## 2. Notions and notations

Let the basic probability space $(\Omega, \mathcal{F}, P)$ be the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the sigma-algebra generated by the weak topology on $\Omega$ and $\mu$ denotes the white noise measure given by the Bochner-Minlos theorem. The Bochner-Minlos theorem states the existence of a Gaussian probability measure given by the integral transform of the characteristic function

$$
C(\phi)=\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \phi \in \mathcal{S}(\mathbb{R})
$$

where $\langle\omega, \phi\rangle$ denotes the usual dual paring between a tempered distribution $\omega$ and a rapidly decreasing function $\phi$.

Let $\left\{\xi_{k}, k \in \mathbb{N}\right\}$ be the family of Hermite functions and $\left\{h_{k}, k \in \mathbb{N}_{0}\right\}$ the family of Hermite polynomials. It is well known that the space of rapidly decreasing functions $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$, where $S_{l}(\mathbb{R})=\left\{\varphi=\sum_{k=1}^{\infty} a_{k} \xi_{k}:\|\varphi\|_{l}^{2}=\right.$ $\left.\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{N}_{0}$, and the space of tempered distributions $S^{\prime}(\mathbb{R})=$
$\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$, where $S_{-l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} b_{k} \xi_{k}: \quad\|f\|_{-l}^{2}=\sum_{k=1}^{\infty} b_{k}^{2}(2 k)^{-l}<\right.$ $\infty\}, l \in \mathbb{N}_{0}$.

Let $(L)^{2}=L^{2}\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, and $H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \alpha \in \mathcal{I}$ be the Fourier-Hermite orthogonal basis of $(L)^{2}$, where $\mathcal{I}$ denotes the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right)$. In particular, for the $k$ th unit vector $\varepsilon^{(k)}=$ $(0, \ldots, 0,1,0, \ldots)$, the sequence of zeros with the number 1 as the $k$ th component, $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. Let $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1, a^{\alpha}=\prod_{k=1}^{\infty} a_{k}^{\alpha_{k}}, \frac{a^{\alpha}}{\alpha!}=\prod_{k=1}^{\infty} \frac{a_{k}^{\alpha_{k}}}{\alpha_{k}!}$ and $(2 \mathbb{N} a)^{\alpha}=\prod_{k=1}^{\infty}\left(2 k a_{k}\right)^{\alpha_{k}}$. Note that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ if $p>0$ and $\sum_{\alpha \in \mathcal{I}} a^{-p \alpha}<\infty$ if $p>1$. Let $\rho \in[0,1]$.

The space of Kondratiev stochastic test functions modified by the sequence a, denoted by $(S a)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}}(S a)_{\rho, p}, p \in \mathbb{N}_{0}$, is the projective limit of spaces

$$
(S a)_{\rho, p}=\left\{f=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha} \in L^{2}(\mu):\|f\|_{(S a)_{\rho, p}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1+\rho} b_{\alpha}^{2}(2 \mathbb{N} a)^{p \alpha}<\infty\right\}
$$

The space of Kondratiev stochastic generalized functions modified by the sequence $a$, denoted by $(S a)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}}(S a)_{-\rho,-p}, p \in \mathbb{N}_{0}$, is the inductive limit of the spaces

$$
(S a)_{-\rho,-p}=\left\{F=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha}:\|F\|_{(S a)_{-\rho,-p}^{2}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho} c_{\alpha}^{2}(2 \mathbb{N} a)^{-p \alpha}<\infty\right\}
$$

The action of a generalized function $F \in(S a)_{-\rho}$ onto a test function $f \in(S a)_{\rho}$ is given by $\ll F, f \gg=\sum_{\alpha \in \mathcal{I}} \alpha!c_{\alpha} b_{\alpha}$. The generalized expectation of $F$ is defined as $E_{\mu}(F)=\ll F, 1 \gg=c_{0}$. It is considered to be the zero coefficient in the chaos expansion of a generalized function $F$ in orthogonal basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. In particular, if $F \in L^{2}(\mu)$ it coincides with usual expectation.

For $a_{k}=1, k \in \mathbb{N}$ these spaces reduce to the spaces of Kondratiev stochastic test functions $(S)_{\rho}$ and the Kondratiev stochastic generalized functions $(S)_{-\rho}$ respectively. For all $\rho \in[0,1]$ we have a Gel'fand triplet

$$
(S a)_{\rho} \subseteq L^{2}(\mu) \subseteq(S a)_{-\rho}
$$

In particular, the largest space of the Kondratiev stochastic distributions modified by the sequence $a$ is obtained for $\rho=1$ and is denoted by $(S a)_{-1}$. In [4] we introduced the Gaussian type of these spaces and solve equations related to them.

### 2.1. Generalized stochastic processes

Let $I \subset \mathbb{R}$ and $X$ be a Banach space of functions on $I$ endowed with $\|\cdot\|_{X}$ and $X^{\prime}$ its dual. The most common examples used in applications are Schwartz spaces $S(\mathbb{R})$ and $S^{\prime}(\mathbb{R})$, the Sobolev spaces $X=W_{0}^{1,2}(\mathbb{R})$ and $X^{\prime}=W^{-1,2}(\mathbb{R})$.

Definition 2.1. Generalized stochastic processes are elements of tensor product space $X \otimes(S)_{-\rho}$.

Theorem 2.1 ([14]). Let $X$ be a Banach space endowed with $\|\cdot\|_{X}$. Generalized stochastic processes as elements of $X \otimes(S)_{-\rho}$ have a chaos expansion of the form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}, \quad f_{\alpha} \in X, \alpha \in \mathcal{I} \tag{1}
\end{equation*}
$$

and there exists $p \in \mathbb{N}_{0}$ such that

$$
\|u\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha}<\infty
$$

Remark 2.1. Generalized stochastic processes as elements of $X \otimes(S a)_{-1}$ have a chaos expansion of the form (1) and there exists $p \in \mathbb{N}_{0}$ such that

$$
\|u\|_{X \otimes(S a)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha}<\infty
$$

Recall that $(S)_{-1}$ is nuclear and thus $\left(X \otimes(S)_{1}\right)^{\prime} \cong X^{\prime} \otimes(S)_{-1}$. In a similar manner one can consider processes as elements of $X^{\prime} \otimes(S)_{-1}$. Note that $X^{\prime} \otimes(S)_{-1}$ is isomorphic to the space of linear bounded mappings $X \rightarrow(S)_{-1}$.

Definition 2.2. Singular generalized stochastic processes are linear and continuous mappings from $X$ into the space of generalized stochastic functions $(S)_{-1}$, i.e., elements of $\mathcal{L}\left(X,(S)_{-1}\right)$.

Theorem 2.2 ([14]). Let $X=\bigcap_{k=0}^{\infty} X_{k}$ be a nuclear space endowed with a family of seminorms $\left\{\|\cdot\|_{k} ; k \in \mathbb{N}_{0}\right\}$ and let $X^{\prime}=\bigcup_{k=0}^{\infty} X_{-k}$ be its topological dual. Singular generalized stochastic processes as elements of $X^{\prime} \otimes(S)_{-\rho}$ have a chaos expansion of the form

$$
u=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}, \quad f_{\alpha} \in X_{-k}, \alpha \in \mathcal{I}
$$

where $k \in \mathbb{N}_{0}$ does not depend on $\alpha \in \mathcal{I}$, and there exists $p \in \mathbb{N}_{0}$ such that

$$
\|u\|_{X^{\prime} \otimes(S)_{-\rho,-p}^{2}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{-k}^{2}(\alpha!)^{1-\rho}(2 \mathbb{N})^{-p \alpha}<\infty
$$

With the same notation as in (1) we will denote by $E u=f_{(0,0,0, \ldots)}$ the generalized expectation of the process $u$.

Example 2.1. Brownian motion is an element of $C^{\infty}(\mathbb{R}) \otimes(L)^{2}$ and it is defined by the chaos expansion $B_{t}(\omega)=\sum_{k=1}^{\infty} \int_{0}^{t} \xi_{k}(s) d s H_{\varepsilon^{(k)}}(\omega)$. Singular white noise $W_{t}(\cdot)$ is defined by the chaos expansion $W_{t}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\epsilon(k)}(\omega)$, and it is an element of the space $C^{\infty}(\mathbb{R}) \otimes(S)_{-1,-p}$ for $p>\frac{5}{12}$ and for all $t$. It is integrable and the relation $\frac{d}{d t} B_{t}=W_{t}$ holds in the $(S)_{-1}$ sense (see [2]).

### 2.2. Schwartz space-valued generalized random processes

In [15] and [16] a general setting of $S^{\prime}$-valued generalized random process is provided. $S^{\prime}(\mathbb{R})$-valued generalized random processes are elements of $\widetilde{X} \otimes(S)_{-\rho}$, where $\widetilde{X}=X \otimes S^{\prime}(\mathbb{R})$, and are given by chaos expansions of the form

$$
f=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k},
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathcal{I}} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$ and $a_{\alpha, k} \in X$. Thus, for some $p, l \in \mathbb{N}_{0}$,

$$
\|f\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|a_{\alpha, k}\right\|_{X}^{2}(\alpha!)^{1-\rho}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty
$$

### 2.3. The Malliavin derivative within chaos expansion

We provide now the definition of the Malliavin derivative which is an extension of the classical definition of this operator from the space of random processes to the space of generalized stochastic processes ([8], [10], [12]).

Definition 2.3. Let a generalized stochastic process $u \in X \otimes(S)_{-\rho}$ be of the form (1). If there exists $p \in \mathbb{N}_{0}$ such that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{2}
\end{equation*}
$$

then the Malliavin derivative of $u$ is defined by

$$
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\epsilon^{(k)}}
$$

Operator $\mathbb{D}$ is also called the stochastic gradient of a generalized stochastic process $u$. The set of processes $u$ such that (2) is satisfied is the domain of the Malliavin derivative and is denoted by $\operatorname{Dom}(\mathbb{D})_{-\rho}$. A process $u \in \operatorname{Dom}(\mathbb{D})_{-\rho}$ is called Malliavin differentiable process.

Theorem 2.3. The Malliavin derivative of a process $u \in X \otimes(S)_{-\rho}$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}(\mathbb{D})_{-\rho,-p} \subseteq X \otimes(S)_{-\rho,-p} \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}
$$

for $l>p+1$ and $p \in \mathbb{N}_{0}$.
Proof. We use the property $\left(\alpha-\varepsilon^{(k)}\right)!=\frac{\alpha!}{\alpha_{k}}$, for $k \in \mathbb{N}$ in the proof of this theorem. Assume that a generalized process $u$ is of the form (1) such that it satisfies (2) for
some $p \geq 0$. Then we have

$$
\begin{aligned}
& \|\mathbb{D} u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k}\right\|_{X \otimes(S)_{-\iota(\mathbb{R})}^{2}}(2 \mathbb{N})^{-p \alpha+p \varepsilon^{(k)}}\left(\alpha-\varepsilon^{(k)}\right)!^{1-\rho} \\
& \leq \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \alpha_{k}^{2}\left(\alpha-\varepsilon^{(k)}\right)!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p\left(\alpha-\varepsilon^{(k)}\right)}(2 k)^{-l} \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \alpha_{k}^{2}\left(\frac{\alpha!}{\alpha_{k}}\right)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}(2 k)^{-(l-p)} \\
& \leq C \sum_{\alpha \in \mathcal{I}}\left(\sum_{k=1}^{\infty} \alpha_{k}\right)^{1+\rho}(\alpha!)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =C \sum_{\alpha \in \mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty,
\end{aligned}
$$

where $C=\sum_{k=1}^{\infty}(2 k)^{-(l-p)}<\infty$ for $l>p+1$.
When $\rho=1$ the result of the previous theorem reduces to the corresponding one in [4].

## 3. Nonhomogeneous first-order linear equation

We consider now a nonhomogeneous linear Malliavin differential type equation

$$
\left\{\begin{array}{l}
\mathbb{D} u=c \otimes u+h,  \tag{3}\\
E u=\widetilde{u}_{0},
\end{array}\right.
$$

where $c \in S^{\prime}(\mathbb{R}), h$ is a $S^{\prime}$-valued generalized stochastic process and $\widetilde{u}_{0} \in X$.
Note that in a special case for $h=0$ the equation (3) reduces to the corresponding homogeneous equation $\mathbb{D} u=c \otimes u$ satisfying condition $E u=\widetilde{u}_{0}$. To be precise, in this case we obtain the generalized eigenvalue problem for the Malliavin derivative operator, which was solved in [4]. Moreover, it was proved that in a special case, obtained solution coincide with the stochastic exponential. Additionally, putting $c=0$, the initial equation (3) transforms into the first-order differential equation with the Malliavin derivative operator $\mathbb{D} u=h, E u=\widetilde{u}_{0}$, which was recently solved in [6].

The method we will use to solve this equation is a very general and useful tool of Wiener-Itô chaos expansions, also known as the propagator method. With this method we reduce the stochastic differential equation to an infinite system of deterministic equations, which can be solved by induction on length of multi-index. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial stochastic differential equation. This method is applied in several papers: [4], [5], [6], [7], [8], [9], [14], [15].

Denote by $r=r(\alpha)=\min \left\{k \in \mathbb{N}: \alpha_{k} \neq 0\right\}$, for nonzero multi-index $\alpha \in \mathcal{I}$. Then the first nonzero component of $\alpha$ is the $r$ th component $\alpha_{r}$, i.e., $\alpha=\left(0,0, \ldots, 0, \alpha_{r}, \ldots, \alpha_{m}, 0,0, \ldots\right)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of $\alpha$, except the $r$ th, which is $\alpha_{r}-1$. We call $\alpha_{\varepsilon^{(r)}}$ the representative of $\alpha$ and write

$$
\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}, \quad \alpha \in \mathcal{I},|\alpha|>0
$$

Note that $\alpha_{\varepsilon(r)}$ is of the length $|\alpha|-1$.
For example, the first nonzero component of $\alpha=(0,0,2,1,0,5,0,0, \ldots)$ is its third component. It follows that $r=3, \alpha_{r}=2$ and the representative of $\alpha$ is $\alpha_{\varepsilon(r)}=\alpha-\varepsilon^{(3)}=(0,0,1,1,0,5,0,0, \ldots)$.

The set $\mathcal{K}_{\alpha}=\left\{\beta \in \mathcal{I}: \alpha=\beta+\varepsilon^{(j)}\right.$, for some $\left.j \in \mathbb{N}\right\}, \alpha \in \mathcal{I},|\alpha|>0$ is a nonempty set, because $\alpha_{\varepsilon^{(r)}} \in \mathcal{K}_{\alpha}$. Moreover, if $\alpha=n \varepsilon^{(r)}, n \in \mathbb{N}$ then $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)=1$. In all other cases $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)>1$. For example if $\alpha=(0,1,3,0,0,5,0, \ldots)$, then the set $\mathcal{K}_{\alpha}$ has three elements

$$
\mathcal{K}_{\alpha}=\left\{\alpha_{\varepsilon^{(2)}}=(0,0,3,0,0,5,0, \ldots),(0,1,2,0,0,5,0, \ldots),(0,1,3,0,0,4,0, \ldots)\right\}
$$

For $\alpha \in \mathcal{I}$ such that $\operatorname{Card}\left(\mathcal{K}_{\alpha}\right)>1$, we denote by $r_{1}$ the smallest $k \in \mathbb{N}$ such that $\alpha_{\varepsilon^{(r)}}=\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}}$, i.e., $\alpha_{\varepsilon^{\left(r_{1}\right)}}$ is the representative of $\alpha_{\varepsilon^{(r)}}$ and is of length $|\alpha|-2$. Then $\mathcal{K}_{\alpha_{\varepsilon}(r)}=\left\{\beta_{1} \in \mathcal{I}: \alpha_{\varepsilon^{(r)}}=\beta_{1}+\varepsilon^{\left(k_{1}\right)}\right.$, for some $\left.k_{1} \in \mathbb{N}\right\}$. Further on if, $\operatorname{Card}\left(\mathcal{K}_{\alpha_{\varepsilon}(r)}\right)>1$ then we denote by $r_{2}$ the smallest $k \in \mathbb{N}$ such that $\alpha_{\varepsilon^{\left(r_{1}\right)}}=\varepsilon^{\left(r_{2}\right)}+\alpha_{\varepsilon^{\left(r_{2}\right)}}$ and so on. Note that $\mathcal{K}_{\alpha_{\varepsilon}\left(r_{1}\right)}=\left\{\beta_{2} \in \mathcal{I}: \alpha_{\varepsilon^{\left(r_{1}\right)}}=\beta_{2}+\right.$ $\varepsilon^{\left(k_{2}\right)}$, for some $\left.k_{2} \in \mathbb{N}\right\}$. With such a procedure we decompose $\alpha \in \mathcal{I}$ recursively by new representatives of previous representatives and we obtain sequence of $\mathcal{K}$ sets. Thus, for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I},|\alpha|=s+1$ there exists an increasing family of integers $1 \leq r \leq r_{1} \leq r_{2} \leq \cdots \leq r_{s} \leq m, s \in \mathbb{N}$ such that $\alpha_{\varepsilon\left(r_{s}\right)}=(0,0, \ldots)$ and every multi-index $\alpha$ is decomposed by recurrent sum of representatives

$$
\begin{align*}
& \alpha= \varepsilon^{(r)}+\alpha_{\varepsilon^{(r)}} \\
&=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}} \\
& \quad \quad \vdots  \tag{4}\\
&=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\cdots+\varepsilon^{\left(r_{s}\right)}+\alpha_{\varepsilon^{\left(r_{s}\right)}}, \quad \alpha_{\varepsilon^{\left(r_{s}\right)}}=(0,0, \ldots) .
\end{align*}
$$

For example, if $\alpha=(0,0,2,0,0,1,0, \ldots)$, then $r=3, \alpha_{r}=2, \alpha_{\varepsilon_{(r)}}=(0,0,1,0,0$, $1,0, \ldots), r_{1}=3, \alpha_{r_{1}}=1, \alpha_{\varepsilon\left(r_{1}\right)}=(0,0,0,0,0,1,0, \ldots), r_{2}=6, \alpha_{r_{2}}=1, \alpha_{\varepsilon\left(r_{2}\right)}=$ $(0,0,0, \ldots)$, and thus $\alpha=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\varepsilon^{\left(r_{2}\right)}+\alpha_{\varepsilon^{\left(r_{2}\right)}}$. Clearly, $s=|\alpha|-1=2$.

Theorem 3.1. Let $c=\sum_{k=1}^{\infty} c_{k} \xi_{k} \in S^{\prime}(\mathbb{R})$ and let $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ having the representation $h=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$, such that the coefficients $h_{\alpha, k} \in X$
satisfy

$$
\begin{cases}\text { for }|\alpha|=1 &  \tag{5}\\ \text { for }|\alpha|=2 & \frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}=\frac{1}{\beta_{k}} h_{\beta, k}, \\ & \frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}+\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}}=\frac{1}{\beta_{k}} h_{\beta, k}+\frac{1}{\beta_{k} \beta_{k_{1}}} c_{k} h_{\beta_{1}, k_{1}} \\ & \beta \in \mathcal{K}_{\alpha}, \quad \beta_{1} \in \mathcal{K}_{\alpha} \\ \vdots & \end{cases}
$$

for all possible decompositions of $\alpha$.
If $c_{k} \geq \frac{1}{2 k}$, for all $k \in \mathbb{N}$ then equation (3) has a unique solution in $X \otimes(S c)_{-1}$. The chaos expansion of the generalized stochastic process, which represents the unique solution of (3) is given in the form

$$
\begin{align*}
u= & u^{\mathrm{hom}}+u^{\mathrm{nhom}} \\
= & \sum_{\alpha \in \mathcal{I}} u_{\alpha}^{\mathrm{hom}} \otimes H_{\alpha}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} u_{\alpha}^{\mathrm{nhom}} \otimes H_{\alpha}  \tag{6}\\
= & \widetilde{u}_{0} \otimes \sum_{\alpha \in \mathcal{I}} \frac{c^{\alpha}}{\alpha!} H_{\alpha}+\sum_{\substack{\alpha \in \mathcal{I} \\
|\alpha|>0}}\left(\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}+\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}}\right. \\
& \left.\quad+\frac{1}{\alpha_{r} \alpha_{r_{1}} \alpha_{r_{2}}} c_{r} c_{r_{1}} h_{\alpha_{\varepsilon\left(r_{2}\right)}, r_{2}}+\cdots+\frac{1}{\alpha!} c_{r} c_{r_{1}} \ldots c_{r_{s-1}} h_{0, r_{s}}\right) \otimes H_{\alpha}
\end{align*}
$$

i.e., as a superposition of a homogeneous part, denoted by $u^{\mathrm{hom}}$, and its nonhomogeneous part denoted by $u^{\mathrm{nhom}}$. The second sum on right-hand side of (6) runs through $\alpha \in \mathcal{I}$ represented in the recursive form (4).

Proof. We are looking for the solution $u$ in the form $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$, where the coefficients $u_{\alpha} \in X, \alpha \in \mathcal{I}$ are to be found. From $E u=\widetilde{u}_{0}$ it follows $u_{(0,0, \ldots)}=$ $\widetilde{u}_{0}$ and thus, $u=\widetilde{u}_{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} u_{\alpha} \otimes H_{\alpha}$. We use the chaos expansion method and transform the initial equation (3) to an equivalent system of deterministic equations. Thus,

$$
\begin{aligned}
& \mathbb{D}\left(\widetilde{u}_{0}+\sum_{\substack{\alpha \in \mathcal{I} \\
|\alpha|>0}} u_{\alpha} \otimes H_{\alpha}\right) \\
&=\left(\sum_{k \in \mathbb{N}} c_{k} \xi_{k}\right) \otimes\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}\right)+\sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k}\right) \otimes H_{\alpha} \\
& \sum_{\substack{\alpha \in \mathcal{I} \\
|\alpha|>0}}\left(\sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k}\right) \otimes H_{\alpha-\varepsilon(k)}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}} c_{k} u_{\alpha} \otimes \xi_{k}\right) \otimes H_{\alpha}+\sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k}\right) \otimes H_{\alpha} \\
& \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon(k)} \otimes \xi_{k}\right) \otimes H_{\alpha} \\
= & \sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left(c_{k} u_{\alpha}+h_{\alpha, k}\right) \otimes \xi_{k}\right) \otimes H_{\alpha}
\end{aligned}
$$

Due to the uniqueness of the chaos expansion of a generalized process in the orthogonal basis $\xi_{k} \otimes H_{\alpha}, \alpha \in \mathcal{I}$ and $k \in \mathbb{N}$, we transform (3) into a family of deterministic equations

$$
\begin{equation*}
\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}}=c_{k} u_{\alpha}+h_{\alpha, k}, \quad \text { for all } \alpha \in \mathcal{I}, k \in \mathbb{N} . \tag{7}
\end{equation*}
$$

The solution $u_{\alpha}, \alpha \in \mathcal{I}$ is obtained by induction with respect to the length of multi-indices $\alpha$. Recall, from $E u=\widetilde{u}_{0}$ we obtained $u_{(0,0, \ldots)}=\widetilde{u}_{0}$.

Starting with $|\alpha|=0$, i.e., $\alpha=(0,0, \ldots)$, the equations in (7) reduce to

$$
\begin{equation*}
u_{\varepsilon^{(k)}}=c_{k} u_{(0,0, \ldots)}+h_{(0,0, \ldots), k}, \quad k \in \mathbb{N} \tag{8}
\end{equation*}
$$

and we obtain the coefficients $u_{\alpha}$ for $\alpha$ of length one. In particular we have the system

$$
\left\{\begin{align*}
u_{(1,0,0,0, \ldots)} & =c_{1} \widetilde{u}_{0}+h_{0,1}  \tag{9}\\
u_{(0,1,0,0, \ldots)} & =c_{2} \widetilde{u}_{0}+h_{0,2} \\
u_{(0,0,1,0, \ldots)} & =c_{3} \widetilde{u}_{0}+h_{0,3} \\
u_{(0,0,0,1,0, \ldots)} & =c_{4} \widetilde{u}_{0}+h_{0,4} \\
& \vdots
\end{align*}\right.
$$

Note, $u_{\alpha}$ for $|\alpha|=1$ are obtained as a superposition of a homogeneous part, represented in terms of $\widetilde{u}_{0}$, and a nonhomogeneous part, expressed in terms of $h_{\alpha_{\varepsilon(r)}, r}=h_{(0,0, \ldots), r}, r \in \mathbb{N}$.

Next, for $|\alpha|=1$ multi-indices are of the form $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$ and several cases occur. For $k=1, \alpha=\varepsilon^{(1)}=(1,0,0, \ldots)$ the system (7) transforms into

$$
\left\{\begin{array}{rl}
u_{(2,0,0,0, \ldots)} & =\frac{1}{2} c_{1} u_{(1,0,0, \ldots)}+\frac{1}{2} h_{(1,0,0, \ldots), 1} \\
u_{(1,1,0,0, \ldots)} & =c_{2} u_{(1,0,0, \ldots)}+h_{(1,0,0, \ldots), 2} \\
u_{(1,0,1,0, \ldots)} & =c_{3} u_{(1,0,0, \ldots)}+h_{(1,0,0, \ldots), 3} \\
u_{(1,0,0,1,0, \ldots)} & =c_{4} u_{(1,0,0, \ldots)}+h_{(1,0,0, \ldots), 4} \\
& \vdots
\end{array} .\right.
$$

Now we replace expressions $u_{\varepsilon^{(k)}}, k \in \mathbb{N}$ with equalities (8), obtained in the previous step, and receive

$$
\left\{\begin{array}{rl}
u_{(2,0,0,0, \ldots)} & =\frac{1}{2} c_{1}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{1} h_{0,1}+\frac{1}{2} h_{(1,0,0, \ldots), 1}  \tag{10}\\
u_{(1,1,0,0, \ldots)} & =c_{1} c_{2} \widetilde{u}_{0}+c_{2} h_{0,1}+h_{(1,0,0, \ldots), 2} \\
u_{(1,0,1,0, \ldots)} & =c_{1} c_{3} \widetilde{u}_{0}+c_{3} h_{0,1}+h_{(1,0,0, \ldots), 3} \\
u_{(0,0,0,1,0, \ldots)} & =c_{1} c_{4} \widetilde{u}_{0}+c_{4} h_{0,1}+h_{(1,0,0, \ldots), 4} \\
& \vdots
\end{array} .\right.
$$

Continuing, for $k=2, \alpha=\varepsilon^{(2)}=(0,1,0,0, \ldots)$ the equations in (7) transform to

$$
\left\{\begin{align*}
u_{(1,1,0,0, \ldots)} & =c_{1} u_{(0,1,0,0, \ldots)}+h_{(0,1,0,0, \ldots), 1}  \tag{11}\\
u_{(0,2,0,0, \ldots)} & =\frac{1}{2} c_{2} u_{(0,1,0,0, \ldots)}+\frac{1}{2} h_{(0,1,0,0, \ldots), 2} \\
u_{(0,1,1,0, \ldots)} & =c_{3} u_{(0,1,0,0, \ldots)}+h_{(0,1,0,0, \ldots), 3} \\
u_{(0,1,0,1,0, \ldots)} & =c_{4} u_{(0,1,0,0, \ldots)}+h_{(0,1,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

and then after substitution (9) for (11) we obtain

$$
\left\{\begin{align*}
u_{(1,1,0,0, \ldots)} & =c_{1} c_{2} \widetilde{u}_{0}+c_{1} h_{0,2}+h_{(0,1,0,0, \ldots), 1}  \tag{12}\\
u_{(0,2,0,0, \ldots)} & =\frac{1}{2} c_{2}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{2} h_{0,2}+\frac{1}{2} h_{(0,1,0,0, \ldots), 2} \\
u_{(0,1,1,0, \ldots)} & =c_{2} c_{3} \widetilde{u}_{0}+c_{3} h_{0,2}+h_{(0,1,0,0, \ldots), 3} \\
u_{(0,1,0,1,0, \ldots)} & =c_{2} c_{4} \widetilde{u}_{0}+c_{4} h_{0,2}+h_{(0,1,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

For $k=3, \alpha=\varepsilon^{(3)}=(0,0,1,0,0, \ldots)$ the system (7) reduces to

$$
\left\{\begin{align*}
u_{(1,0,1,0,0, \ldots)} & =c_{1} c_{3} \widetilde{u}_{0}+c_{1} h_{0,3}+h_{(0,0,1,0,0, \ldots), 1}  \tag{13}\\
u_{(0,1,1,0,0, \ldots)} & =c_{2} c_{3} \widetilde{u}_{0}+c_{2} h_{0,3}+h_{(0,0,1,0,0, \ldots), 2} \\
u_{(0,0,2,0, \ldots)} & =\frac{1}{2} c_{3}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{3} h_{0,3}+\frac{1}{2} h_{(0,0,1,0,0, \ldots), 3} \\
u_{(0,0,1,1,0, \ldots)} & =c_{3} c_{4} \widetilde{u}_{0}+c_{4} h_{0,3}+h_{(0,0,1,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

Continuing with the same procedure we obtain the unknown coefficients $u_{\alpha}$ of length two. Further on, we will express all multi-indices, which have two different representations of the form $\alpha=\varepsilon^{(k)}+\varepsilon^{\left(k_{1}\right)}$, for $k \neq k_{1}, k, k_{1} \in \mathbb{N}$ in terms of their representatives.

For example, multi-index

$$
(1,1,0,0, \ldots)=\varepsilon^{(1)}+(0,1,0, \ldots)=\varepsilon^{(2)}+(1,0,0, \ldots)
$$

has two different representations of the form $\alpha=\varepsilon^{(k)}+\alpha_{\varepsilon^{(k)}}$ and thus the coefficient $u_{(1,1,0,0, \ldots)}$ appears in systems (10) and (12). Thus, the additional condition

$$
c_{2} h_{0,1}+h_{(1,0,0, \ldots), 2}=c_{1} h_{0,1}+h_{(0,1,0, \ldots), 1}
$$

has to hold in order to have a solvable system.
Moreover, we express $\alpha=(1,1,0,0, \ldots)$ in terms of a sequence of successive representatives, i.e., $r=1, \alpha_{\mathcal{E}(r)}=(0,1,0,0, \ldots), r_{1}=2$ and $\alpha_{\varepsilon_{\left(r_{1}\right)}}=$ $(0,0,0, \ldots)$. Thus, the element $u_{(1,1,0,0, \ldots)}$ is given in the form $u_{(1,1,0,0, \ldots)}=$ $c_{1} c_{2} u_{0}+h_{(0,1,0, \ldots), 1}+c_{1} h_{0,2}$ obtained in (12). Also, the element $u_{(1,0,1,0,0, \ldots)}$ appears in equalities (10) and (13) and we obtained additional condition

$$
c_{3} h_{0,1}+h_{(1,0,0, \ldots), 3}=c_{1} h_{0,3}+h_{(0,0,1,0, \ldots), 1}
$$

which need to be satisfied in order to have a unique $u_{\alpha}$. Multi-index $\alpha=(1,0,1$, $0,0, \ldots)$ can be decomposed in terms of a sequence of successive representatives as follows $\alpha=\varepsilon^{(r)}+\alpha_{\varepsilon^{(r)}}$, where $r=1, \alpha_{\varepsilon^{(r)}}=(0,0,1,0, \ldots)$ and $\alpha_{\varepsilon^{(r)}}=\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}}$, for $r_{1}=3$ and $\alpha_{\varepsilon\left(r_{1}\right)}=(0,0,0, \ldots)$. We use the form (13) to represent $u_{(1,0,1,0,0, \ldots)}$ in terms of its representatives decomposition. Moreover, the element $u_{(0,1,1,0,0, \ldots)}$ appears in equalities (12) and (13), and it follows that also the condition

$$
c_{3} h_{0,2}+h_{(0,1,0,0, \ldots), 3}=c_{2} h_{0,3}+h_{(0,0,1,0, \ldots), 2}
$$

has to be satisfied, and so on.
In this step we obtained forms of the coefficients $u_{\alpha}$ of length two, with validity of the additional condition

$$
\begin{equation*}
h_{\alpha_{\varepsilon}(r), r}+c_{r} h_{(0,0, \ldots), r_{1}}=h_{\beta, j}+c_{j} h_{(0,0, \ldots), k}, \tag{14}
\end{equation*}
$$

where $\alpha=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+(0,0, \ldots), 1 \leq r \leq r_{1}, r, r_{1} \in \mathbb{N}$ and all $\beta \in \mathcal{I}$ such that $\alpha=\beta+\varepsilon^{(j)}$ for $j \geq r$, and $\beta=(0,0, \ldots)+\varepsilon^{(k)}$, for some $k \in \mathbb{N}$. Note that condition (14) corresponds to condition (5) for $|\alpha|=2$. The coefficients $u_{\alpha}$ of length two are represented as a superposition of a homogeneous part, expressed in terms of $\widetilde{u}_{0}$ and a nonhomogeneous part expressed as a linear combination of $h_{\alpha, k}$ for $\alpha$ of length one and product of $c_{k}, k \in \mathbb{N}$ and $h_{\alpha, k}$ for $\alpha$ of length zero, i.e., in terms of representatives $\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}+\frac{1}{\alpha!} c_{r} h_{0, r_{1}}$ for $\alpha=\varepsilon^{(r)}+\alpha_{\varepsilon^{(r)}}, \alpha_{\varepsilon^{(r)}}=\varepsilon^{\left(r_{1}\right)}$, $1 \leq r \leq r_{1}$.

For $|\alpha|=2$ from system of equations (7) and results (10), (12), (13),..., calculated in the previous step, we obtain $u_{\alpha}$, for $\alpha$ of length three. Different combinations for multi-indices of length two occur. If we choose $\alpha=(1,1,0,0, \ldots)$ then the system (7) transforms into the system

$$
\left\{\begin{aligned}
u_{(2,1,0,0, \ldots)} & =\frac{1}{2} c_{1} u_{(1,1,0,0, \ldots)}+\frac{1}{2} h_{(1,1,0,0, \ldots), 1} \\
u_{(1,2,0,0, \ldots)} & =\frac{1}{2} c_{2} u_{(1,1,0,0, \ldots)}+\frac{1}{2} h_{(1,1,0,0, \ldots), 2} \\
u_{(1,1,1,0, \ldots)} & =c_{3} u_{(1,1,0,0, \ldots)}+h_{(1,1,0,0, \ldots), 3} \\
u_{(1,1,0,1,0, \ldots)} & =c_{4} u_{(1,1,0,0, \ldots)}+h_{(1,1,0,0, \ldots), 4} \\
& \vdots
\end{aligned}\right.
$$

We substitute equalities for $u_{\alpha}$ of length two, obtained in the previous step in terms of their representatives decomposition, and transform the system to a more elegant one. In particular, we use expression in (12) for the element $u_{(1,1,0,0, \ldots)}$ and obtain the system of equations

$$
\left\{\begin{align*}
u_{(2,1,0,0, \ldots)} & =\frac{1}{2} c_{1}^{2} c_{2} \widetilde{u}_{0}+\frac{1}{2} c_{1}^{2} h_{0,2}+\frac{1}{2} c_{1} h_{(0,1,0,0, \ldots), 1}+\frac{1}{2} h_{(1,1,0,0, \ldots), 1}  \tag{15}\\
u_{(1,2,0,0, \ldots)} & =\frac{1}{2} c_{1} c_{2}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{1} c_{2} h_{0,2}+\frac{1}{2} c_{2} h_{(0,1,0, \ldots), 1}+\frac{1}{2} h_{(1,1,0,0, \ldots), 2} \\
u_{(1,1,1,0, \ldots)} & =c_{1} c_{2} c_{3} \widetilde{u}_{0}+c_{1} c_{3} h_{0,2}+c_{3} h_{(0,1,0,0, \ldots), 1}+h_{(1,1,0,0, \ldots), 3} \\
u_{(1,1,0,1,0, \ldots)} & =c_{1} c_{2} c_{4} \widetilde{u}_{0}+c_{1} c_{4} h_{0,2}+c_{4} h_{(0,1,0,0, \ldots), 1}+h_{(1,1,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

For $\alpha=(1,0,1, \ldots)$ the system (7) transforms to the system

$$
\left\{\begin{align*}
u_{(2,0,1,0,0, \ldots)} & =\frac{1}{2} c_{1}^{2} c_{3} \widetilde{u}_{0}+\frac{1}{2} c_{1}^{2} h_{0,3}+\frac{1}{2} c_{1} h_{(0,0,1,0,0, \ldots), 1}+\frac{1}{2} h_{(1,0,1,0, \ldots), 1}  \tag{16}\\
u_{(1,1,1,0,0, \ldots)} & =c_{1} c_{2} c_{3} \widetilde{u}_{0}+c_{1} c_{2} h_{0,3}+c_{2} h_{(0,0,1,0, \ldots), 1}+h_{(1,0,1,0,0, \ldots), 2} \\
u_{(1,0,2,0, \ldots)} & =\frac{1}{2} c_{1} c_{3}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{1} c_{3} h_{0,3}+\frac{1}{2} c_{3} h_{(0,0,1,0,0, \ldots), 1}+\frac{1}{2} h_{(1,0,1,0,0, \ldots), 3} \\
u_{(1,0,1,1,0, \ldots)} & =c_{1} c_{3} c_{4} \widetilde{u}_{0}+c_{1} c_{4} h_{0,3}+c_{4} h_{(0,0,1,0,0, \ldots), 1}+h_{(1,0,1,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

and for $\alpha=(1,0,0,1, \ldots)$ we obtain the system

$$
\left\{\begin{aligned}
u_{(2,0,0,1,0,0, \ldots)} & =\frac{1}{2} c_{1}^{2} c_{4} \widetilde{u}_{0}+\frac{1}{2} c_{1}^{2} h_{0,4}+\frac{1}{2} c_{1} h_{(0,0,0,1,0,0, \ldots), 1}+\frac{1}{2} h_{(1,0,1,0, \ldots), 1} \\
u_{(1,1,0,1,0,0, \ldots)} & =c_{1} c_{2} c_{4} \widetilde{u}_{0}+c_{1} c_{2} h_{0,4}+c_{2} h_{(0,0,0,1,0, \ldots), 1}+h_{(1,0,0,1,0,0, \ldots), 2} \\
u_{(1,0,1,1,0, \ldots)} & =c_{1} c_{3} c_{4} \widetilde{u}_{0}+c_{1} c_{3} h_{0,4}+c_{3} h_{(0,0,0,1,0,0, \ldots), 1}+h_{(1,0,0,1,0,0, \ldots), 3} \\
u_{(1,0,0,2,0, \ldots)} & =\frac{1}{2} c_{1} c_{4}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{1} c_{4} h_{0,4}+\frac{1}{2} c_{4} h_{(0,0,0,1,0,0, \ldots), 1}+\frac{1}{2} h_{(1,0,0,1,0,0, \ldots), 4} \\
& \vdots
\end{aligned}\right.
$$

We continue with multi-indices $\alpha=(0,1,1,0,0, \ldots)$ and $\alpha=(0,1,0,1,0, \ldots)$ and transform the system (7) respectively to the systems

$$
\left\{\begin{align*}
u_{(1,1,1,0,0, \ldots)} & =c_{1} c_{2} c_{3} \widetilde{u}_{0}+c_{1} c_{2} h_{0,3}+c_{1} h_{(0,0,1,0,0, \ldots), 2}+h_{(0,1,1,0, \ldots), 1}  \tag{17}\\
u_{(0,2,1,0,0, \ldots)} & =\frac{1}{2} c_{2}^{2} c_{3} \widetilde{u}_{0}+\frac{1}{2} c_{2}^{2} h_{0,3}+\frac{1}{2} c_{2} h_{(0,0,1,0, \ldots), 2}+\frac{1}{2} h_{(0,1,1,0,0, \ldots), 2} \\
u_{(0,1,2,0, \ldots)} & =\frac{1}{2} c_{2} c_{3}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{2} c_{3} h_{0,3}+\frac{1}{2} c_{3} h_{(0,0,1,0,0, \ldots), 1}+h_{(0,1,1,0,0, \ldots), 3} \\
u_{(0,1,1,1,0, \ldots)} & =c_{2} c_{3} c_{4} \widetilde{u}_{0}+c_{2} c_{4} h_{0,3}+c_{4} h_{(0,0,1,0,0, \ldots), 2}+h_{(0,1,1,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

and

$$
\left\{\begin{aligned}
u_{(1,1,0,1,0,0, \ldots)} & =c_{1} c_{2} c_{4} \widetilde{u}_{0}+c_{1} c_{2} h_{0,4}+\frac{1}{2} c_{1} h_{(0,0,0,1,0,0, \ldots), 2}+h_{(0,1,0,1,0, \ldots), 1} \\
u_{(0,2,0,1,0,0, \ldots)} & =\frac{1}{2} c_{2}^{2} c_{4} \widetilde{u}_{0}+\frac{1}{2} c_{2}^{2} h_{0,4}+\frac{1}{2} c_{2} h_{(0,0,0,1,0, \ldots), 2}+\frac{1}{2} h_{(0,1,0,1,0,0, \ldots), 2} \\
u_{(0,1,1,1,0, \ldots)} & =c_{2} c_{3} c_{4} \widetilde{u}_{0}+c_{2} c_{3} h_{0,4}+c_{3} h_{(0,0,0,1,0,0, \ldots), 2}+h_{(0,1,0,1,0,0, \ldots), 3} \\
u_{(0,1,0,2,0, \ldots)} & =\frac{1}{2} c_{2} c_{4}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{2} c_{4} h_{0,4}+\frac{1}{2} c_{4} h_{(0,0,0,1,0,0, \ldots), 2}+\frac{1}{2} h_{(0,1,0,1,0,0, \ldots), 4} \\
& \vdots
\end{aligned}\right.
$$

For multi-indices $\alpha=(2,0,0,0, \ldots)$ and $\alpha=(0,2,0,0, \ldots)$ the system (7) transforms respectively into

$$
\left\{\begin{align*}
u_{(3,0,0,0, \ldots)} & =\frac{1}{6} c_{1}^{3} \widetilde{u}_{0}+\frac{1}{6} c_{1}^{2} h_{0,1}+\frac{1}{6} c_{1} h_{(1,0,0,0, \ldots), 1}+\frac{1}{3} h_{(2,0,0,0, \ldots), 1}  \tag{18}\\
u_{(2,1,0,0, \ldots)} & =\frac{1}{2} c_{1}^{2} c_{2} \widetilde{u}_{0}+\frac{1}{2} c_{1} c_{2} h_{0,1}+\frac{1}{2} c_{2} h_{(1,0,0,0, \ldots), 1}+h_{(2,0,0,0, \ldots), 2} \\
u_{(2,0,1,0, \ldots)} & =\frac{1}{2} c_{1}^{2} c_{3} \widetilde{u}_{0}+\frac{1}{2} c_{1} c_{3} h_{0,1}+\frac{1}{2} c_{3} h_{(1,0,0,0, \ldots), 1}+\frac{1}{2} h_{(2,0,0,0, \ldots), 3} \\
u_{(2,0,0,1,0, \ldots)} & =\frac{1}{2} c_{1}^{2} c_{4} \widetilde{u}_{0}+\frac{1}{2} c_{2} c_{4} h_{0,1}+\frac{1}{2} c_{4} h_{(0,0,0,1,0,0, \ldots), 1}+\frac{1}{2} h_{(2,0,0,0, \ldots), 4} \\
& \vdots
\end{align*}\right.
$$

and

$$
\left\{\begin{aligned}
u_{(1,2,0,0,0, \ldots)} & =\frac{1}{2} c_{1} c_{2}^{2} \widetilde{u}_{0}+\frac{1}{2} c_{1} c_{2} h_{0,2}+\frac{1}{2} c_{1} h_{(0,1,0,0, \ldots), 2}+h_{(0,2,0,0, \ldots), 1} \\
u_{(0,3,0,0,0, \ldots)} & =\frac{1}{6} c_{2}^{3} \widetilde{u}_{0}+\frac{1}{6} c_{2}^{2} h_{0,2}+\frac{1}{6} c_{2} h_{(0,1,0, \ldots), 2}+\frac{1}{3} h_{(0,2,0,0, \ldots), 2} \\
u_{(0,2,1,0,0, \ldots)} & =\frac{1}{2} c_{2}^{2} c_{3} \widetilde{u}_{0}+\frac{1}{2} c_{2} c_{3} h_{0,2}+\frac{1}{2} c_{3} h_{(0,1,0,0, \ldots), 2}+\frac{1}{2} h_{(0,2,0,0, \ldots), 3} \\
u_{(0,2,0,1,0, \ldots)} & =\frac{1}{2} c_{2}^{2} c_{4} \widetilde{u}_{0}+\frac{1}{2} c_{2} c_{4} h_{0,2}+\frac{1}{2} c_{4} h_{(0,1,0,0, \ldots), 2}+\frac{1}{2} h_{(0,2,0,0, \ldots), 4} \\
& \vdots
\end{aligned}\right.
$$

Combining with the previous results, we obtain $u_{\alpha}$, for $|\alpha|=3$. Further on, we will express all multi-indices, which have several different representations of the form $\alpha=\varepsilon^{(k)}+\varepsilon^{\left(k_{1}\right)}+\varepsilon^{\left(k_{2}\right)}$, for $k, k_{1}, k_{2} \in \mathbb{N}$ in terms of theirs representatives. Two different representations of $u_{(2,1,0,0, \ldots)}$ appear in the systems (15) and (18), so the additional condition

$$
\frac{1}{2} c_{1}^{2} h_{0,2}+\frac{1}{2} c_{1} h_{(0,1,0, \ldots), 1}+\frac{1}{2} h_{(1,1,0,0, \ldots), 1}=\frac{1}{2} c_{1} c_{2} h_{0,1}+\frac{1}{2} c_{2} h_{(1,0,0, \ldots), 1}+h_{(2,0,0, \ldots), 2}
$$

follows. We express the element $u_{(2,1,0,0, \ldots)}$ in form of the representatives. Clearly, recursive decomposition of multi-index $(2,1,0,0, \ldots)$ is given by

$$
\begin{aligned}
(2,1,0,0, \ldots) & =\varepsilon^{(r)}+(1,1,0,0, \ldots)=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+(0,1,0,0, \ldots) \\
& =\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\varepsilon^{\left(r_{2}\right)}+(0,0, \ldots)
\end{aligned}
$$

for $r=1, r_{1}=1$ and $r_{2}=2$. Thus, $u_{(2,1,0,0, \ldots)}=\frac{1}{2} c_{1}^{2} c_{2} \widetilde{u}_{0}+\frac{1}{2} h_{(1,1,0,0, \ldots), 1}+$ $\frac{1}{2} c_{1} h_{(0,1,0,0, \ldots), 1}+\frac{1}{2} c_{1}^{2} h_{0,2}$.

Since the coefficient $u_{(1,1,1,0,0, \ldots)}$ appears in three equations (15), (16) and (17), we receive another conditions

$$
\begin{aligned}
& c_{1} c_{3} h_{0,2}+c_{3} h_{(0,1,0,0, \ldots), 1}+h_{(1,1,0,0, \ldots), 3} \\
& \quad=c_{1} c_{2} h_{0,3}+c_{2} h_{(0,0,1,0, \ldots), 1}+h_{(1,0,1,0,0, \ldots), 2} \\
& \quad=c_{1} c_{2} h_{0,3}+c_{1} h_{(0,0,1,0,0, \ldots), 2}+h_{(0,1,1,0, \ldots), 1}
\end{aligned}
$$

and express $\alpha=(1,1,1,0,0, \ldots)$ by its representatives, $r=1, r_{1}=2, r_{3}=2$. The representation of $u_{(1,1,1,0,0, \ldots)}$ is given by (15), i.e., $u_{(1,1,1,0,0, \ldots)}=c_{1} c_{2} c_{3} \widetilde{u}_{0}+$ $h_{(0,1,1,0, \ldots), 1}+c_{1} h_{(0,0,1,0, \ldots), 2}+c_{1} c_{2} h_{0,3}$. Note that previous conditions correspond to conditions (5).

We proceed by the same procedure for all multi-index lengths to obtain $u_{\alpha}$ in the form

$$
\begin{aligned}
u_{\alpha}=\widetilde{u}_{0} \frac{c_{1}^{\alpha_{1}}}{\alpha_{1}!} \cdot \frac{c_{2}^{\alpha_{2}}}{\alpha_{2}!} \cdots+\left(\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon(r)}, r}\right. & +\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}}+\cdots \\
\cdots & \left.+\frac{1}{\alpha_{r} \alpha_{r_{1}} \ldots \alpha_{r_{s}}} c_{r} c_{r_{1}} \ldots c_{r_{s-1}} h_{0, r_{s}}\right)
\end{aligned}
$$

and thus the form of the solution (6).
In general, we decompose multi-index $\alpha$ recurrently by the representatives. To be precise, in the firs step we have $\alpha=\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}}$. Then, in the next step we find the representative of $\alpha_{\varepsilon^{\left(r_{1}\right)}}$, i.e., $\alpha_{\varepsilon^{\left(r_{1}\right)}}=\varepsilon^{\left(r_{2}\right)}+\alpha_{\varepsilon^{\left(r_{2}\right)}}$ and so on...

The coefficients $u_{\alpha}$ are obtained in the form

for the decomposition $\alpha=\varepsilon^{(r)}+\sum_{1 \leq j \leq s} \varepsilon^{\left(r_{j}\right)}+(0,0, \ldots), 1 \leq r \leq r_{1} \leq \cdots \leq r_{s}$, i.e., $\alpha_{\varepsilon^{\left(r_{j}\right)}}=\alpha-\varepsilon^{(r)}-\sum_{1 \leq i \leq j-1} \varepsilon^{\left(r_{i}\right)}, 1 \leq j \leq s$, where $|\alpha|=s+1$.

It remains to prove the convergence of the solution (6) in the space $X \otimes(S c)_{-1}$, i.e., to prove that, for some $p>0$

$$
\|u\|_{X \otimes(S c)_{-1}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} c)^{-p \alpha}<\infty
$$

Let $h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-p}$. Then, there exists $p>0$ such that

$$
\|h\|_{X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|h_{\alpha, k}\right\|_{X}^{2}(2 k)^{-p}(2 \mathbb{N})^{-p \alpha}<\infty
$$

Note that for $\widetilde{u}_{0} \in X$ we have $\left\|\widetilde{u}_{0}\right\|_{X}=\left\|\widetilde{u}_{0}\right\|_{X \otimes(S)_{-1,-q}}$ for all $q>0$. Then, the convergence follows from

$$
\begin{aligned}
\|u\|_{X \otimes(S c)_{-1,-p}}^{2} & =\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} c)^{-p \alpha} \\
& \leq 2 \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{\text {hom }}\right\|_{X}^{2}(2 \mathbb{N} c)^{-p \alpha}+2 \sum_{\substack{\alpha \in \mathcal{I} \\
|\alpha|>0}}\left\|u_{\alpha}^{\text {nhom }}\right\|_{X}^{2}(2 \mathbb{N} c)^{-p \alpha} \\
& =2 A+2 B<\infty .
\end{aligned}
$$

From assumption $c_{k} \geq \frac{1}{2 k}$, for all $k \in \mathbb{N}$, it follows that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N} c)^{-p \alpha}<\infty$ if $p>0$. Then, for $p>3$, we have

$$
\begin{aligned}
A & =\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{\mathrm{hom}}\right\|_{X}^{2}(2 \mathbb{N} c)^{-p \alpha} \\
& =\sum_{\alpha \in \mathcal{I}}\left\|\widetilde{u}_{0}\right\|_{X}^{2} \frac{c^{2 \alpha}}{(\alpha!)^{2}}(2 \mathbb{N} c)^{-p \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathcal{I}} c^{2 \alpha}(2 \mathbb{N})^{-p \alpha} c^{-p \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathcal{I}} c^{-(p-2) \alpha} \sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty
\end{aligned}
$$

For $p>3$ the convergence of the second part $B$ follows from

$$
\begin{aligned}
& B= \sum_{\substack{\alpha \in \mathcal{I} \\
|\alpha|>0}}\left\|u_{\alpha}^{\mathrm{nhom}}\right\|_{X}^{2}(2 \mathbb{N} c)^{-p \alpha} \\
&= \sum_{\substack{\alpha \in \mathcal{I},|\alpha|>0, \alpha=\alpha_{\varepsilon}(r)+\varepsilon^{(r)}}}\left\|\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}+\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}}+\cdots+\frac{1}{\alpha!} c_{r} c_{r_{1}} \ldots c_{r_{s-1}} h_{0, r_{s}}\right\|_{X}^{2} \\
& \times(2 \mathbb{N} c)^{-p\left(\alpha_{\varepsilon}(r)+\varepsilon^{(r)}\right)} \\
& \leq \sum_{\alpha=\alpha_{\varepsilon}(r)+\varepsilon^{(r)}} \frac{|\alpha|}{\alpha_{r}^{2}}\left(\left\|h_{\alpha_{\varepsilon}(r), r}\right\|_{X}^{2}+c_{r}^{2} \| h_{\left.\alpha_{\varepsilon}\left(r_{1}\right), r_{1}\left\|_{X}^{2}+\cdots+c_{r}^{2} c_{r_{1}}^{2} \ldots c_{r_{s-1}}^{2}\right\| h_{0, r_{s}} \|_{X}^{2}\right)}\right. \\
& \quad \times(2 r c)^{-p}(2 \mathbb{N} c)^{-p \alpha_{\varepsilon}(r)} \\
& \leq \sum_{\alpha \in \mathcal{I},|\alpha|>0} c^{2 \alpha}(2 \mathbb{N})^{\alpha}\left(\left\|h_{\alpha_{\varepsilon}(r), r}\right\|_{X}^{2}+\| h_{\left.\alpha_{\varepsilon}\left(r_{1}\right), r_{1}\left\|_{X}^{2}+\cdots+\right\| h_{0, r_{s}} \|_{X}^{2}\right)} \quad \times(2 r)^{-p} c^{-p \alpha}(2 \mathbb{N})^{-p \alpha_{\varepsilon}(r)}\right. \\
& \leq\left(\sum_{\alpha \in \mathcal{I}} c^{-(p-2) \alpha}\right) \cdot \sum_{\alpha \in \mathcal{I}} \sum_{r \in \mathbb{N}}\left\|h_{\alpha, r}\right\|_{X}^{2}(2 r)^{-p}(2 \mathbb{N})^{-(p-1) \alpha}<\infty
\end{aligned}
$$

where we have used the facts that $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ and $(2 \mathbb{N})^{p \varepsilon^{(r)}}(2 \mathbb{N})^{-p \alpha} \leq 1$ for all $\alpha \in \mathcal{I}, r \in \mathbb{N}$. With this statement we complete the proof.

Acknowledgement
This paper was supported by the projects Research and development of the method of operational control of traffic based on electric traction vehicles optimized by power consumption criterion, No. TR36047 and Functional analysis, ODEs and PDEs with singularities, No. 17024, both financed by the Ministry of Science, Republic of Serbia.

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Electron. J. Probab. 0 (2016), no. 0, 1-25.
ISSN: 1083-6489 DOI: 10.1214/EJP.10.1214/YY-TN

# Stochastic evolution equations with Wick-polynomial nonlinearities 

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#### Abstract

We study nonlinear parabolic stochastic partial differential equations with Wickpower and Wick-polynomial type nonlinearities set in the framework of white noise analysis. These equations include the stochastic Fujita equation, the stochastic Fisher-KPP equation and the stochastic FitzHugh-Nagumo equation among many others. By implementing the theory of $C_{0}$-semigroups and evolution systems into the chaos expansion theory in infinite dimensional spaces, we prove existence and uniqueness of solutions for this class of SPDEs. In particular, we also treat the linear nonautonomous case and provide several applications featured as stochastic reactiondiffusion equations that arise in biology, medicine and physics.


Keywords: Hida-Kondratiev spaces ; stochastic nonlinear evolution equations ; Wick product ; $C_{0}$-semigroup ; infinitesimal generator ; Catalan numbers .
AMS MSC 2010: 60H15; 60H40; 60G20; 37L55; 47J35; 11B83.
Submitted to EJP on June 8, 2018, final version accepted on ?.

## 1 Introduction

We study stochastic nonlinear evolution equations of the form

$$
\begin{align*}
& u_{t}(t, \omega)=\mathbf{A} u(t, \omega)+\sum_{k=0}^{n} a_{k} u^{\diamond k}(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{1.1}\\
& u(0, \omega)=u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

where $u(t, \omega)$ is an $X$-valued generalized stochastic process; $X$ is a certain Banach algebra and $\mathbf{A}$ corresponds to a densely defined infinitesimal generator of a $C_{0}$-semigroup. The nonlinear part is given in terms of Wick-powers $u^{\diamond n}=u^{\diamond n-1} \diamond u=u \diamond \ldots \diamond u, n \in \mathbb{N}$, where $\diamond$ denotes the Wick product. The Wick product is involved due to the fact that we allow random terms to be present both in the initial condition $u_{0}$ and the driving force

[^5]$f$. This leads to singular solutions that do not allow to use ordinary multiplication, but require a renormalization of the multiplication, which is done by introducing the Wick product into the equation. The Wick product is known to represent the highest order stochastic approximation of the ordinary product [16].

In our previous paper [14] we treated the case of linear stochastic parabolic equations with Wick-multiplicative noise which includes the case $n=1$. The present paper is an extension of [14] to nonlinear equations, where the nonlinearity is generated by a Wick-polynomial function leading to stochastic versions of Fujita-type equations $u_{t}=$ $\mathbf{A} u+u^{\diamond n}+f$, FitzHugh-Nagumo equations $u_{t}=\mathbf{A} u+u^{\diamond 2}-u^{\diamond 3}+f$, Fisher-KPP equations $u_{t}=\mathbf{A} u+u-u^{\diamond 2}+f$ and Chaffee-Infante equations $u_{t}=\mathbf{A} u+u^{\diamond 3}-u+f$. These equations have found ample applications in ecology, medicine, engineering and physics. For example, the FitzHugh-Nagumo equation is used to study electrical activity of neurons in neurophysiology by modeling the conduction of electric impulses down a nerve axon. The Fisher-Kolmogorov-Petrovsky-Piskunov equation provides a model for the spread of an epidemic in a population or for the distribution of an advantageous gene within a population. Other applications in medicine involve the modeling of cellular reactions to the introduction of toxins, and the process of epidermal wound healing. In plasma physics it has been used to study neutron flux in nuclear reactors, while in ecology it models flame propagation of fire outbreaks. Thus, the study of their stochastic versions, when some of the input factors is disturbed by an external noise factor and hence it becomes randomized, is of immense importance. For instance, a stochastic version of the FitzHugh-Nagumo equation has been studied in [1] and [3], while the stochastic Fisher-KPP equation has been studied in [10] and [19].

We implement the Wiener-Itô chaos expansion method combined with the operator semigroup theory in order to prove the existence and the uniqueness of a solution for (1.1). Using the chaos expansion method any SPDE can be transformed into a lower triangular infinite system of PDEs (also known as the propagator system) that can be solved recursively. Solving this system, one obtains the coefficients of the solution to (1.1). In order to solve the propagator system, we exploit the intrinsic relationship between the Wick product and the Catalan numbers that was discovered in [11] where the authors considered the stochastic Burgers equation. We build upon these ideas in order to solve a general class of stochastic nonlinear equations (1.1).

The plan of exposition is as follows: In the introductory section we recall upon basic notions of $C_{0}$-semigroups, evolution systems and white noise theory including chaos expansions of generalized stochastic processes. In Section 2, which represents the main part of the paper, we prove existence and uniqueness of the solution to (1.1) for the case when $a_{0}=a_{1}=\cdots=a_{n-1}=0$ and $a_{n}=1$. This normalization is made for technical simplicity to illustrate the method of solving and to put out in details all building blocks of the formulae involved. In Section 3 we treat the general case of (1.1) and provide some concrete examples.

### 1.1 Evolution systems

We fix the notation and recall some known facts about evolution systems (see [20, Chapter 5]). Let $X$ be a Banach space. Let $\{A(t)\}_{t \in[s, T]}$ be a family of linear operators in $X$ such that $A(t): D(A(t)) \subset X \rightarrow X, t \in[s, T]$ and let $f$ be an $X$-valued function $f:[s, T] \rightarrow X$. Consider the initial value problem

$$
\begin{align*}
\frac{d}{d t} u(t) & =A(t) u(t)+f(t), \quad 0 \leq s<t \leq T  \tag{1.2}\\
u(s) & =x
\end{align*}
$$

If $u \in C([s, T], X) \cap C^{1}((s, T], X), u(t) \in D(A(t))$ for all $t \in(s, T]$ and $u$ satisfies (1.2), then $u$ is a classical solution of (1.2).

A two parameter family of bounded linear operators $S(t, s), 0 \leq s \leq t \leq T$ on X is called an evolution system if the following two conditions are satisfied:

1. $S(s, s)=I$ and $S(t, r) S(r, s)=S(t, s), \quad 0 \leq s \leq r \leq t \leq T$
2. $(t, s) \mapsto S(t, s)$ is strongly continuous for all $0 \leq s \leq t \leq T$.

Clearly, if $S(t, s)$ is an evolution system associated with the homogeneous evolution problem (1.2), i.e. if $f \equiv 0$, then a classical solution of (1.2) is given by $u(t)=S(t, s) x, t \in$ $[s, T]$.

A family $\{A(t)\}_{t \in[s, T]}$ of infinitesimal generators of $C_{0}$-semigroups on $X$ is called stable if there exist constants $m \geq 1$ and $w \in \mathbb{R}$ (stability constants) such that $(w, \infty) \subseteq$ $\rho(A(t)), t \in[s, T]$ and

$$
\left\|\prod_{j=1}^{k} R\left(\lambda: A\left(t_{j}\right)\right)\right\| \leq \frac{m}{(\lambda-w)^{k}}, \quad \lambda>w
$$

for every finite sequence $0 \leq s \leq t_{1} \leq t_{2} \leq \cdots \leq t_{k} \leq T, k=1,2, \ldots$
Let $\{A(t)\}_{t \in[s, T]}$ be a stable family of infinitesimal generators with stability constants $m$ and $w$. Let $B(t), t \in[s, T]$, be a family of bounded linear operators on $X$. If $\|B(t)\| \leq$ $M, t \in[s, T]$, then $\{A(t)+B(t)\}_{t \in[s, T]}$ is a stable family of infinitesimal generators with stability constants $m$ and $w+M m$.

Let $\{A(t)\}_{t \in[s, T]}$ be a stable family of infinitesimal generators of $C_{0}$-semigroups on $X$ such that the domain $D(A(t))=D$ is independent of $t$ and for every $x \in D, A(t) x$ is continuously differentiable in $X$. If $f \in C^{1}([s, T], X)$ then for every $x \in D$ the evolution problem (1.2) has a unique classical solution $u$ given by

$$
u(t)=S(t, s) x+\int_{s}^{t} S(t, r) f(r) d r, \quad 0 \leq s \leq t \leq T
$$

From the proof of [20, Theorem 5.3, p. 147] one can obtain

$$
\frac{d}{d t} u(t)=A(t) S(t, s) x+A(t) \int_{s}^{t} S(t, r) f(r) d r+f(t), \quad s<t \leq T
$$

Since $t \mapsto A(t)$ is continuous in $B(D, X)$ and $(t, s) \mapsto S(t, s)$ is strongly continuous for all $0 \leq s \leq t \leq T$, we have additionally that the solution $u$ to (1.2) exhibits the regularity property $u \in C^{1}([s, T], X)$ and $\left.\frac{d}{d t} u(t)\right|_{t=s}=A(s) x+f(s)$. Recall that the evolution system $S(t, s)$ satisfies:

1. $\|S(t, s)\| \leq m e^{w(t-s)}, 0 \leq s \leq t \leq T ;$
2. $\left.\frac{\partial^{+}}{\partial t} S(t, s) x\right|_{t=s}=A(s) x, x \in D, 0 \leq s \leq T$ which implies that $\frac{\partial}{\partial t} S(t, s) x=$ $A(t) S(t, s) x$ since $t \mapsto A(t)$ is continuous in $B(D, X)$;
3. $\frac{\partial}{\partial s} S(t, s) x=-S(t, s) A(s) x, x \in D, 0 \leq s \leq t \leq T$;
4. $S(t, s) D \subseteq D$;
5. $S(t, s) x$ is continuous in $D$ for all $0 \leq s \leq t \leq T$ and $x \in D$.

Remark 1.1. Considering infinitezimal generators depending on $t$, we follow the standard approach of Yosida (cf. [24], [12]). We refer to [18] for a method based on an

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equivalent operator extension problem (see also references in [18]). The chaos expansion approach, which is the essence of our paper, requires the existence results for the propagator system i.e. for the coordinate-wise deterministic Cauchy problems. For this purpose we demonstrate the applications of the hyperbolic Cauchy problem given in [20].

### 1.2 Generalized stochastic processes

Denote by $(\Omega, \mathcal{F}, \mu)$ the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$, where $\Omega=S^{\prime}(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel sigma-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to the characteristic function

$$
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=\exp \left[-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}\right], \quad \phi \in S(\mathbb{R})
$$

given by the Bochner-Minlos theorem.
We recall the notions related to $L^{2}(\Omega, \mu)$ (see [9]). The set of multi-indices $\mathcal{I}$ is $\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$, i.e. the set of sequences of non-negative integers which have only finitely many nonzero components. Especially, we denote by $\mathbf{0}=(0,0,0, \ldots)$ the zero multi-index with all entries equal to zero, the length of a multi-index is $|\alpha|=\sum_{i=1}^{\infty} \alpha_{i}$ for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots\right) \in \mathcal{I}$ and $\alpha!=\prod_{i=1}^{\infty} \alpha_{i}!$. We will use the convention that $\alpha-\beta$ is defined if $\alpha_{n}-\beta_{n} \geq 0$ for all $n \in \mathbb{N}$, i.e., if $\alpha-\beta \geq \mathbf{0}$.

The Wiener-Itô theorem (sometimes also referred to as the Cameron-Martin theorem) states that one can define an orthogonal basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ of $L^{2}(\Omega, \mu)$, where $H_{\alpha}$ are constructed by means of Hermite orthogonal polynomials $h_{n}$ and Hermite functions $\xi_{n}$,

$$
H_{\alpha}(\omega)=\prod_{n=1}^{\infty} h_{\alpha_{n}}\left(\left\langle\omega, \xi_{n}\right\rangle\right), \quad \alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \ldots\right) \in \mathcal{I}, \quad \omega \in \Omega
$$

Then, every $F \in L^{2}(\Omega, \mu)$ can be represented via the so called chaos expansion

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega), \quad \omega \in S^{\prime}(\mathbb{R}), \quad \sum_{\alpha \in \mathcal{I}}\left|f_{\alpha}\right|^{2} \alpha!<\infty, \quad f_{\alpha} \in \mathbb{R}, \quad \alpha \in \mathcal{I} .
$$

Denote by $\varepsilon_{k}=(0,0, \ldots, 1,0,0, \ldots), k \in \mathbb{N}$ the multi-index with the entry 1 at the $k$ th place. Denote by $\mathcal{H}_{1}$ the subspace of $L^{2}(\Omega, \mu)$, spanned by the polynomials $H_{\varepsilon_{k}}(\cdot), k \in \mathbb{N}$. The subspace $\mathcal{H}_{1}$ contains Gaussian stochastic processes, e.g. Brownian motion is given by the chaos expansion $B(t, \omega)=\sum_{k=1}^{\infty} \int_{0}^{t} \xi_{k}(s) d s H_{\varepsilon_{k}}(\omega)$.

Denote by $\mathcal{H}_{m}$ the $m$ th order chaos space, i.e. the closure in $L^{2}(\Omega, \mu)$ of the linear subspace spanned by the orthogonal polynomials $H_{\alpha}(\cdot)$ with $|\alpha|=m, m \in \mathbb{N}_{0}$. Then the Wiener-Itô chaos expansion states that $L^{2}(\Omega, \mu)=\bigoplus_{m=0}^{\infty} \mathcal{H}_{m}$, where $\mathcal{H}_{0}$ is the set of constants in $L^{2}(\Omega, \mu)$.

Changing the topology on $L^{2}(\Omega, \mu)$ to a weaker one, T. Hida [8] defined spaces of generalized random variables containing the white noise as a weak derivative of the Brownian motion. We refer to [8], [9] for white noise analysis.

Let $(2 \mathbb{N})^{\alpha}=\prod_{n=1}^{\infty}(2 n)^{\alpha_{n}}, \quad \alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \ldots\right) \in \mathcal{I}$. We will often use the fact that the series $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}$ converges for $p>1$. Using the same technique as in [9, Chapter 2] one can define Banach spaces $(S)_{\rho, p}$ of test functions and their topological duals $(S)_{-\rho,-p}$ of stochastic distributions for all $\rho \geq 0$ and $p \geq 0$.
Definition 1.1. The stochastic test function spaces are defined by

$$
(S)_{\rho, p}=\left\{F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in L^{2}(\Omega, \mu):\|F\|_{(S)_{\rho, p}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1+\rho}\left|f_{\alpha}\right|^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\},
$$

for all $\rho \geq 0, p \geq 0$.
Their topological duals, the stochastic distribution spaces, are given by formal sums:

$$
(S)_{-\rho,-p}=\left\{F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}:\|F\|_{(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho}\left|f_{\alpha}\right|^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
$$

for all $\rho \geq 0, p \geq 0$.
The space of test random variables is $(S)_{\rho}=\bigcap_{p \geq 0}(S)_{\rho, p}, \rho \geq 0$ endowed with the projective topology.
Its dual, the space of generalized random variables is $(S)_{-\rho}=\bigcup_{p \geq 0}(S)_{-\rho,-p}, \rho \geq 0$ endowed with the inductive topology.

The action of $F=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha} \in(S)_{-\rho}$ onto $f=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha} \in(S)_{\rho}$ is given by $\langle F, f\rangle=\sum_{\alpha \in \mathcal{I}}\left(b_{\alpha}, c_{\alpha}\right) \alpha$ !, where $\left(b_{\alpha}, c_{\alpha}\right)$ stands for the inner product in $\mathbb{R}$. Thus, they form a Gelfand triplet

$$
(S)_{\rho} \subseteq L^{2}(\Omega, \mu) \subseteq(S)_{-\rho}, \quad \rho \geq 0
$$

Clearly, the spaces $(S)_{\rho, p}$ and $(S)_{-\rho,-p}$ are separable Hilbert spaces. Moreover, $(S)_{\rho}$ and $(S)_{-\rho}$ are nuclear spaces.

For $\rho=0$ we obtain the space of Hida stochastic distributions $(S)_{-0}$ and for $\rho=1$ the Kondratiev space of generalized random variables $(S)_{-1}$. It holds that

$$
(S)_{1} \hookrightarrow(S)_{0} \hookrightarrow L^{2}(\Omega, \mu) \hookrightarrow(S)_{-0} \hookrightarrow(S)_{-1}
$$

where $\hookrightarrow$ denotes dense inclusions. Usually the values of $\rho$ are restricted to $\rho \in[0,1]$ in order to establish the $S$-transform (see [8], [9]) when solving SPDEs, but in our case values $\rho>1$ may be considered as well.

The time-derivative of the Brownian motion $B(t, \omega)=\sum_{k=1}^{\infty} \int_{0}^{t} \xi_{k}(s) d s H_{\varepsilon_{k}}(\omega)$ exists in a generalized sense and belongs to the Kondratiev space $(S)_{-1,-p}$ for $p \geq \frac{5}{12}$. We refer it as the white noise and its formal expansion is given by $W(t, \omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon_{k}}(\omega)$.

We extended in [21] the definition of stochastic processes to processes with the chaos expansion form $U(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega)$, where the coefficients $u_{\alpha}$ are elements of some Banach space of functions $X$. We say that $U$ is an $X$-valued generalized stochastic process, i.e. $U(t, \omega) \in X \otimes(S)_{-\rho}$ if there exists $p \geq 0$ such that $\|U\|_{X \otimes(S)_{-\rho,-p}^{2}}^{2}=$ $\sum_{\alpha \in \mathcal{I}}(\alpha!)^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$.

For example, let $X=C^{k}[0, T], k \in \mathbb{N}$. We have proved in [22] that the differentiation of a stochastic process can be carried out componentwise in the chaos expansion, i.e. due to the fact that $(S)_{-\rho}$ is a nuclear space it holds that $C^{k}\left([0, T],(S)_{-\rho}\right)=C^{k}[0, T] \hat{\otimes}(S)_{-\rho}$ where $\hat{\otimes}$ denotes the completion of the tensor product which is the same for the $\varepsilon$-completion and $\pi$-completion. In the sequel, we will use the notation $\otimes$ instead of $\hat{\otimes}$. Hence $C^{k}[0, T] \otimes(S)_{-\rho,-p}$ and $C^{k}[0, T] \otimes(S)_{\rho, p}$ denote subspaces of the corresponding completions. We keep the same notation when $C^{k}[0, T]$ is replaced by another Banach space. This means that a stochastic process $U(t, \omega)$ is $k$ times continuously differentiable if and only if all of its coefficients $u_{\alpha}(t), \alpha \in \mathcal{I}$ are in $C^{k}[0, T]$.

The same holds for Banach space valued stochastic processes i.e. elements of $C^{k}([0, T], X) \otimes(S)_{-\rho}$, where $X$ is an arbitrary Banach space. It holds that

$$
C^{k}\left([0, T], X \otimes(S)_{-\rho}\right)=C^{k}([0, T], X) \otimes(S)_{-\rho}=\bigcup_{p \geq 0} C^{k}([0, T], X) \otimes(S)_{-\rho,-p}
$$

In addition, if $X$ is a Banach algebra, then the Wick product of the stochastic processes $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$ and $G=\sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta} \in X \otimes(S)_{-\rho,-p}$ is given by

$$
F \diamond G=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} H_{\gamma}=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} f_{\beta} g_{\alpha-\beta} H_{\alpha},
$$

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and $F \diamond G \in X \otimes(S)_{-\rho,-(p+k)}$ for all $k>1$ (see [9]). The $n$th Wick power is defined by $F^{\diamond n}=F^{\diamond(n-1)} \diamond F, F^{\diamond 0}=1$. Note that $H_{n \varepsilon_{k}}=H_{\varepsilon_{k}}^{\diamond n}$ for $n \in \mathbb{N}_{0}, k \in \mathbb{N}$. Throughout the paper we will assume that $X$ is a Banach algebra.

## 2 Stochastic nonlinear evolution equation of Fujita-type

First we consider the equation (1.1), with $a_{0}=a_{1}=\cdots=a_{n-1}=0$ and $a_{n}=1$, i.e. the equation:

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+u^{\diamond n}(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{2.1}\\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

Let A: $\mathbb{D} \subset X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ be a coordinatewise operator that corresponds to a family of deterministic operators $A_{\alpha}: D_{\alpha} \subset X \rightarrow X, \alpha \in \mathcal{I}$

$$
\mathbf{A} u(t, \omega)=\mathbf{A}\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega)\right)=\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha}(t) H_{\alpha}(\omega), \quad u \in \mathbb{D}
$$

(see [14, Section 2]). We are looking for a solution of (2.1) as an $X$-valued stochastic process $u(t) \in X \otimes(S)_{-1}, t \in[0, T]$ represented in the form

$$
\begin{equation*}
u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega), \quad t \in[0, T], \quad \omega \in \Omega \tag{2.2}
\end{equation*}
$$

The chaos expansion representation of the Wick-square is given by

$$
\begin{align*}
u^{\diamond 2}(t, \omega) & =\sum_{\alpha \in \mathcal{I}}\left(\sum_{\gamma \leq \alpha} u_{\gamma}(t) u_{\alpha-\gamma}(t)\right) H_{\alpha}(\omega)  \tag{2.3}\\
& =u_{\mathbf{0}}^{2}(t) H_{\mathbf{0}}(\omega)+\sum_{|\alpha|>0}\left(2 u_{\mathbf{0}}(t) u_{\alpha}(t)+\sum_{0<\gamma<\alpha} u_{\gamma}(t) u_{\alpha-\gamma}(t)\right) H_{\alpha}(\omega),
\end{align*}
$$

where $t \in[0, T], \omega \in \Omega$. Let $u_{\gamma}^{\diamond m}(t), \gamma \in \mathcal{I}, m \in \mathbb{N}$ denote the coefficients of the chaos expansion of the $m$ th Wick power, i.e. $u^{\diamond m}(t, \omega)=\sum_{\gamma \in \mathcal{I}} u_{\gamma}^{\diamond m}(t) H_{\gamma}(\omega)$, for $m \in \mathbb{N}$. Then, for arbitrary $n \in \mathbb{N}$, it can be shown that the $n$th Wick-power is given by

$$
\begin{aligned}
& u^{\diamond n}(t, \omega)=u^{\diamond n-1}(t, \omega) \diamond u(t, \omega)=\sum_{\alpha \in \mathcal{I}}\left(\sum_{\gamma \leq \alpha} u_{\gamma}^{\diamond n-1}(t) u_{\alpha-\gamma}(t)\right) H_{\alpha}(\omega) \\
& =u_{\mathbf{0}}^{n}(t) H_{\mathbf{0}}(\omega)+\sum_{|\alpha|>0}\left(\binom{n}{1} u_{0}^{n-1}(t) u_{\alpha}(t)+\binom{n}{2} u_{\mathbf{0}}^{n-2} \sum_{0<\gamma_{1}<\alpha} u_{\alpha-\gamma_{1}}(t) u_{\gamma_{1}}(t)\right. \\
& +\binom{n}{3} u_{\mathbf{0}}^{n-3} \sum_{0<\gamma_{1}<\alpha} \sum_{0<\gamma_{2}<\gamma_{1}} u_{\alpha-\gamma_{1}}(t) u_{\gamma_{1}-\gamma_{2}}(t) u_{\gamma_{2}}(t)+\cdots+ \\
& \left.+\binom{n}{n} \sum_{0<\gamma_{1}<\alpha} \sum_{0<\gamma_{2}<\gamma_{1}} \ldots \sum_{0<\gamma_{n-1}<\gamma_{n-2}} u_{\alpha-\gamma_{1}}(t) u_{\gamma_{1}-\gamma_{2}}(t) \ldots u_{\gamma_{n-2}-\gamma_{n-1}}(t) u_{\gamma_{n-1}}(t)\right) H_{\alpha}(\omega) \\
& =u_{\mathbf{0}}^{n}(t) H_{\mathbf{0}}(\omega)+\sum_{|\alpha|>0}\left(n u_{\mathbf{0}}^{n-1}(t) u_{\alpha}(t)+r_{\alpha, n}(t)\right) H_{\alpha}(\omega)
\end{aligned}
$$

where $t \in[0, T], \omega \in \Omega$. The functions $r_{\alpha, n}(t), t \in[0, T], \alpha \in \mathcal{I}, n>1$ contain only the coordinate functions $u_{\beta}, \beta<\alpha$. Moreover, we recall that the Wick power $u^{\diamond n}$ of a stochastic process $u \in X \otimes(S)_{-1,-p}$ is an element of $X \otimes(S)_{-1,-q}$, for $q>p+n-1$, see [9].

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We rewrite all processes that figure in (2.1) in their corresponding Wiener-Itô chaos expansion form and obtain

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \frac{d}{d t} u_{\alpha}(t) H_{\alpha}(\omega) & =\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha}(t) H_{\alpha}(\omega)+\sum_{\alpha \in \mathcal{I}}\left(\sum_{\gamma \leq \alpha} u_{\gamma}^{\diamond n-1}(t) u_{\alpha-\gamma}(t)\right) H_{\alpha}(\omega) \\
& +\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega) \\
\sum_{\alpha \in \mathcal{I}} u_{\alpha}(0) H_{\alpha}(\omega) & =\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha}(\omega) .
\end{aligned}
$$

Due to the orthogonality of the base $H_{\alpha}$ this reduces to the system of infinitely many deterministic Cauchy problems:

$$
1^{\circ} \text { for } \alpha=\mathbf{0}
$$

$$
\begin{equation*}
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+u_{\mathbf{0}}^{n}(t)+f_{\mathbf{0}}(t), \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0}, \quad \text { and } \tag{2.4}
\end{equation*}
$$

$2^{\circ}$ for $\alpha>\mathbf{0}$

$$
\begin{equation*}
\frac{d}{d t} u_{\alpha}(t)=\left(A_{\alpha}+n u_{0}^{n-1}(t) I d\right) u_{\alpha}(t)+r_{\alpha, n}(t)+f_{\alpha}(t), \quad u_{\alpha}(0)=u_{\alpha}^{0} \tag{2.5}
\end{equation*}
$$

with $t \in(0, T]$ and $\omega \in \Omega$.
Let

$$
B_{\alpha, n}(t)=A_{\alpha}+n u_{\mathbf{0}}^{n-1}(t) I d \quad \text { and } \quad g_{\alpha, n}(t)=r_{\alpha, n}(t)+f_{\alpha}(t), \quad t \in[0, T]
$$

for all $\alpha>\mathbf{0}$. Then, the system (2.5) can be written in the form

$$
\begin{equation*}
\frac{d}{d t} u_{\alpha}(t)=B_{\alpha, n}(t) u_{\alpha}(t)+g_{\alpha, n}(t), \quad t \in(0, T] ; \quad u_{\alpha}(0)=u_{\alpha}^{0} \tag{2.6}
\end{equation*}
$$

Note that the inhomogeneous part $g_{\alpha, n}$ in (2.6) does not contain any of the functions $u_{\beta}, \beta<\alpha$ for $|\alpha|=1$, while for $|\alpha|>1$ it involves also $u_{\beta}, \beta<\alpha$. Hence, we distinguish these two cases.
(a) Let $|\alpha|=1$, i.e. $\alpha=\varepsilon_{k}, k \in \mathbb{N}$. Then $g_{\varepsilon_{k}, n}=f_{\varepsilon_{k}}, k \in \mathbb{N}$ and thus (2.6) transforms to

$$
\begin{equation*}
\frac{d}{d t} u_{\varepsilon_{k}}(t)=B_{\varepsilon_{k}, n}(t) u_{\varepsilon_{k}}(t)+f_{\varepsilon_{k}}(t), \quad t \in(0, T] ; \quad u_{\varepsilon_{k}}(0)=u_{\varepsilon_{k}}^{0} . \tag{2.7}
\end{equation*}
$$

(b) Let $|\alpha|>1$. Then

$$
\frac{d}{d t} u_{\alpha}(t)=B_{\alpha, n}(t) u_{\alpha}(t)+g_{\alpha, n}(t), \quad t \in(0, T] ; \quad u_{\alpha}(0)=u_{\alpha}^{0}
$$

Each solution $u$ to (2.1) can be represented in the form (2.2) and hence its coefficients $u_{0}$ and $u_{\alpha}$ for $\alpha>\mathbf{0}$ must satisfy (2.4) and (2.6) respectively. Vice versa, if the coefficients $u_{0}$ and $u_{\alpha}$ for $\alpha>\mathbf{0}$ solve (2.4) and (2.6) respectively, and if the series in (2.2) represented by these coefficients exists in $X \otimes(S)_{-1}$, then it defines a solution to (2.1).
Definition 2.1. An $X$-valued generalized stochastic process $u(t)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha} \in$ $X \otimes(S)_{-1}, t \in[0, T]$ is a coordinatewise classical solution to (2.1) if $u_{0}$ is a classical solution to (2.4) and for every $\alpha \in \mathcal{I} \backslash\{\mathbf{0}\}$, the coefficient $u_{\alpha}$ is a classical solution to (2.6). The coordinatewise solution $u(t) \in X \otimes(S)_{-1}, t \in[0, T]$ is an almost classical solution to (2.1) if $u \in C([0, T], X) \otimes(S)_{-1}$. An almost classical solution is a classical solution if $u \in C([0, T], X) \otimes(S)_{-1} \cap C^{1}((0, T], X) \otimes(S)_{-1}$.

## Stochastic evolution equations with nonlinearities

We assume that the following hold:
(A1) The operators $A_{\alpha}, \alpha \in \mathcal{I}$, are infinitesimal generators of $C_{0}$-semigroups $\left\{T_{\alpha}(s)\right\}_{s \geq 0}$ with a common domain $D_{\alpha}=D, \alpha \in \mathcal{I}$, dense in $X$. We assume that there exist constants $m \geq 1$ and $w \in \mathbb{R}$ such that

$$
\left\|T_{\alpha}(s)\right\| \leq m e^{w s}, s \geq 0 \quad \text { for all } \quad \alpha \in \mathcal{I}
$$

The action of $\mathbf{A}$ is given by

$$
\mathbf{A}(u)=\sum_{\alpha \in \mathcal{I}} A_{\alpha}\left(u_{\alpha}\right) H_{\alpha}
$$

for $u \in \mathbb{D} \subseteq D \otimes(S)_{-1}$ of the form (2.2), where

$$
\mathbb{D}=\left\{u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in D \otimes(S)_{-1}: \exists p_{0} \geq 0, \sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\left(u_{\alpha}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p_{0} \alpha}<\infty\right\}
$$

(A2) The initial value $u^{0}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha} \in \mathbb{D}$, i.e. $u_{\alpha}^{0} \in D$ for every $\alpha \in \mathcal{I}$ and there exists $p \geq 0$ such that

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \\
\sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha}\left(u_{\alpha}^{0}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty .
\end{gathered}
$$

(A3) The inhomogeneous part $f(t, \omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega), t \in[0, T], \omega \in \Omega$ belongs to $C^{1}([0, T], X) \otimes(S)_{-1}$; hence $t \mapsto f_{\alpha}(t) \in C^{1}([0, T], X), \alpha \in \mathcal{I}$ and there exists $p \geq 0$ such that

$$
\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}+\sup _{t \in[0, T]}\left\|f_{\alpha}^{\prime}(t)\right\|_{X}\right)^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

(A4-n) The Cauchy problem

$$
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+u_{\mathbf{0}}^{n}(t)+f_{\mathbf{0}}(t), \quad t \in(0, T] ; \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0}
$$

has a classical solution $u_{\mathbf{0}} \in C^{1}([0, T], X)$.
Remark 2.1. Particularly, if $A_{\mathbf{0}}=\Delta$ is the Laplace operator and $f_{\mathbf{0}} \equiv 0$, then (2.4) belongs to the class of Fujita equations

$$
\begin{equation*}
u_{t}=\Delta u+u^{p}, \quad u(0)=u_{0} \tag{2.8}
\end{equation*}
$$

studied by Fujita, Chen and Watanabe [6, 7]. The authors proved that for a nonnegative initial condition $u^{0} \in C\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$, equation (2.8) has a unique classical solution on some $\left[0, T_{1}\right)$. Moreover, if $p>1+\frac{2}{N}$ then there exist a positive bounded solution. The Fujita equation (2.8) apart from an interest per se also acts as a scaling limit of more general superlinear equations whose nonlinearities exhibit a polynomial growth rate. Originally, it has been developed to describe molecular concentration of a solution subjected to centrifugation and sedimentation.

Remark 2.2. In general, equations of the form (2.4), i.e. the deterministic equation for $\alpha=\mathbf{0}$ can be solved by the Fixed Point Theorem [25]. Thus, in order to check if condition (A4-n) holds, one has to apply fixed point methods or other established methods for deterministic PDEs. The solution to (2.4) will usually blow-up in finite time. Especially
the description of blow-up in the Sobolev supercritical regime poses a challenge that has been tackled in several papers (e.g. [7], [15] for the Fujita equation). We stress that our equation (2.1) and hence also (2.4) is given on a finite time interval, which is assumed to provide a solution on the entire interval (we restrict our considerations form the very start to the interval where no blow-up appears).

Now we focus on solving (2.6) for $\alpha>\mathbf{0}$.
Lemma 2.3. Let the assumptions (A1)-(A4-n) be fulfilled. Then for every $\alpha>\mathbf{0}$ the evolution system (2.6) has a unique classical solution $u_{\alpha} \in C^{1}([0, T], X)$.

Proof. First, for every $\alpha>\mathbf{0}$, we consider the family of operators $B_{\alpha, n}(t)=A_{\alpha}+$ $n u_{0}^{n-1}(t) I d, t \in[0, T]$. According to assumption (A1), the constant family $\left\{A_{\alpha}(t)\right\}_{t \in[0, T]}=$ $\left\{A_{\alpha}\right\}_{t \in[0, T]}$ is a stable family of infinitesimal generators of a $C_{0}-\operatorname{semigroup}\left\{T_{\alpha}(s)\right\}_{s \geq 0}$ on $X$ satisfying $\left\|T_{\alpha}(s)\right\| \leq m e^{w s}$ with stability constants $m \geq 1$ and $w \in \mathbb{R}$. Let

$$
\begin{equation*}
M_{n}=\sup _{t \in[0, T]}\left\|u_{0}(t)\right\|_{X} \tag{2.9}
\end{equation*}
$$

The perturbation $n u_{0}^{n-1}(t) I d: X \rightarrow X, t \in[0, T]$ is a family of uniformly bounded linear operators such that

$$
\left\|n u_{\mathbf{0}}^{n-1}(t) x\right\|_{X}=\left\|n u_{\mathbf{0}}^{n-1}(t)\right\|_{X}\|x\|_{X} \leq \sup _{t \in[0, T]} n\left\|u_{\mathbf{0}}(t)\right\|_{X}^{n-1}\|x\|_{X} \leq n M_{n}^{n-1}\|x\|_{X}
$$

for all $x \in X, t \in[0, T]$, i.e. $\left\|n u_{0}^{n-1}(t) I d\right\| \leq n M_{n}^{n-1}, t \in[0, T]$. Thus, for every $\alpha>\mathbf{0}$, the family $\left\{A_{\alpha}+n u_{0}^{n-1}(t) I d\right\}_{t \in[0, T]}$ is a stable family of infinitesimal generators with stability constants $m$ and $w+n M_{n}^{n-1} m$. By assumption (A4-n) the function $u_{0} \in C^{1}([0, T], X)$ so we obtain continuous differentiability of $\left(A_{\alpha}+n u_{0}^{n-1}(t) I d\right) x, t \in[0, T]$ for every $x \in D$ and for every $\alpha>\mathbf{0}$. Additionally, the domain of the operators $n u_{0}^{n-1}(t) I d$ is the entire space $X$ which implies that all of the operators $B_{\alpha, n}(t), t \in[0, T]$ have a common domain $D\left(B_{\alpha, n}(t)\right)=D\left(A_{\alpha}\right)=D$ not depending on $t$. Notice here that assumption (A1) additionally provides the same domain $D$ of the family $\left\{B_{\alpha, n}(t)\right\}_{t \in[0, T]}$ for all $\alpha>\mathbf{0}$.

Finally, one can associate the unique evolution system $S_{\alpha, n}(t, s)$, for $0 \leq s \leq t \leq T$ for all $\alpha>0$ to the system (2.6) such that

$$
\begin{equation*}
\left\|S_{\alpha, n}(t, s)\right\| \leq m e^{w_{n}(t-s)} \leq m e^{w_{n}(T-s)}, \quad 0 \leq s \leq t \leq T, \quad \alpha>\mathbf{0} \tag{2.10}
\end{equation*}
$$

where $w_{n}=w+n M_{n}^{n-1} m$ see [20, Thm 4.8., p. 145]. Without loss of generality we may assume that $w>0$ and thus will be $w_{n}>0$.

Now one can solve the infinite system of the Cauchy problems (2.6) by induction on the length of the multiindex $\alpha$. Let $|\alpha|=1$. Since $f_{\varepsilon_{k}} \in C^{1}([0, T], X)$, we obtain the unique classical solution $u_{\varepsilon_{k}} \in C^{1}([0, T], X)$ to (2.7) given by

$$
\begin{equation*}
u_{\varepsilon_{k}}(t)=S_{\varepsilon_{k}, n}(t, 0) u_{\varepsilon_{k}}^{0}+\int_{0}^{t} S_{\varepsilon_{k}, n}(t, s) f_{\varepsilon_{k}}(s) d s, \quad t \in[0, T] . \tag{2.11}
\end{equation*}
$$

Now let for every $\beta \in \mathcal{I}$ such that $\mathbf{0}<\beta<\alpha$ the unique classical solution of (2.6) satisfy $u_{\beta} \in C^{1}([0, T], X)$. Then for fixed $|\alpha|>1$ the inhomogeneous part $g_{\alpha, n} \in C^{1}([0, T], X)$ and the solution to (2.6) is of the form

$$
\begin{equation*}
u_{\alpha}(t)=S_{\alpha, n}(t, 0) u_{\alpha}^{0}+\int_{0}^{t} S_{\alpha, n}(t, s) g_{\alpha, n}(s) d s, \quad t \in[0, T] \tag{2.12}
\end{equation*}
$$

where $u_{\alpha} \in C^{1}([0, T], X)$. For more details see [20, Thm 5.3., p. 147].
Now we proceed with four technical lemmas that will be used in the sequel.

Stochastic evolution equations with nonlinearities

Lemma 2.4. Let $\alpha \in \mathcal{I}$. Then

$$
\frac{|\alpha|!}{\alpha!} \leq(2 \mathbb{N})^{2 \alpha}
$$

Proof. This is a direct consequence of [11, Proposition 2.3]. More precisely, in [11] authors proved that $|\alpha|!\leq \mathbf{q}^{\alpha} \alpha!$ if a sequence $\mathbf{q}=\left(q_{k}\right)_{k \in \mathbb{N}}$ satisfies

$$
1<q_{1} \leq q_{2} \leq \ldots \quad \text { and } \quad \sum_{k=1}^{\infty} \frac{1}{q_{k}}<1
$$

Since $\sum_{k=1}^{\infty} \frac{1}{(2 k)^{2}}=\frac{\pi^{2}}{24}<1$, the sequence $(2 \mathbb{N})^{2}=\left((2 k)^{2}\right)_{k \in \mathbb{N}}$ satisfies a required property.

Lemma 2.5. For every $c>0$ there exists $q>1$ such that the following holds

$$
\sum_{\alpha \in \mathcal{I}} c^{|\alpha|}(2 \mathbb{N})^{-q \alpha}<\infty
$$

Proof. Let $c>0$ and choose $s \geq 0$ such that $c \leq 2^{s}$. Then, for $q>s+1$,

$$
\sum_{\alpha \in \mathcal{I}} c^{|\alpha|}(2 \mathbb{N})^{-q \alpha} \leq \sum_{\alpha \in \mathcal{I}} \prod_{i=1}^{\infty}\left(2^{s}\right)^{\alpha_{i}} \prod_{i=1}^{\infty}(2 i)^{-q \alpha_{i}} \leq \sum_{\alpha \in \mathcal{I}} \prod_{i=1}^{\infty}(2 i)^{(s-q) \alpha_{i}}=\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{(s-q) \alpha}<\infty
$$

In the next lemma, for the sake of completeness, we give some useful properties of the well known Catalan numbers, see for example [23].
Lemma 2.6. A sequence $\left\{\mathbf{c}_{n}\right\}_{n \in \mathbb{N}}$ defined by the recurrence relation

$$
\begin{equation*}
\mathbf{c}_{0}=1, \quad \mathbf{c}_{n}=\sum_{k=0}^{n-1} \mathbf{c}_{k} \mathbf{c}_{n-1-k}, \quad n \geq 1 \tag{2.13}
\end{equation*}
$$

is called the sequence of Catalan numbers. The closed formula for $\mathbf{c}_{n}$ is a multiple of the binomial coefficient, i.e. the solution of the Catalan recurrence (2.13) is

$$
\mathbf{c}_{n}=\frac{1}{n+1}\binom{2 n}{n} \quad \text { or } \quad \mathbf{c}_{n}=\binom{2 n}{n}-\binom{2 n}{n+1}
$$

The Catalan numbers satisfy the growth estimate

$$
\begin{equation*}
\mathbf{c}_{n} \leq 4^{n}, n \geq 0 \tag{2.14}
\end{equation*}
$$

Lemma 2.7. [11, p.21] Let $\left\{R_{\alpha}: \alpha \in \mathcal{I}\right\}$ be a set of real numbers such that $R_{0}=$ $0, R_{\varepsilon_{k}}, k \in \mathbb{N}$ are given and

$$
R_{\alpha}=\sum_{\mathbf{0}<\gamma<\alpha} R_{\gamma} R_{\alpha-\gamma}, \quad|\alpha|>1
$$

Then

$$
R_{\alpha}=\frac{1}{|\alpha|}\binom{2|\alpha|-2}{|\alpha|-1} \frac{|\alpha|!}{\alpha!} \prod_{k=1}^{\infty} R_{\varepsilon_{k}}^{\alpha_{k}}, \quad|\alpha|>1
$$

Proof. Let $\alpha \in \mathcal{I},|\alpha|>1$ be given. Then $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}, 0,0, \ldots\right)$ has only finally many non-zero components, so one can associate to it a $d$-dimensional vector $\left(\alpha_{1}, \ldots, \alpha_{d}\right) \in$ $\mathbb{N}_{0}^{d}$. Adopting the proof for the classical Catalan numbers, the authors in [11] considered the function $G(z)=\sum_{\beta \in \mathbb{N}_{0}^{d}} M_{\beta} z^{\beta}, z \in \mathbb{N}_{0}^{d}$, where $M_{\beta}=\sum_{0<\gamma<\beta} M_{\gamma} M_{\beta-\gamma}$ and
$z^{\beta}=z_{1}^{\beta_{1}} \cdots z_{d}^{\beta_{d}}$. The function $G$ satisfies $G^{2}(z)-G(z)+\sum_{k=1}^{d} M_{\varepsilon_{k}} z_{k}=0$, which implies that $G(z)=\sum_{n=1}^{\infty} \frac{1}{n}\binom{2 n-2}{n-1}\left(\sum_{k=1}^{d} M_{\varepsilon_{k}} z_{k}\right)^{n}$. Finally, applying the multinomial formula $\left(\sum_{k=1}^{d} M_{\varepsilon_{k}} z_{k}\right)^{n}=\sum_{\beta \in \mathbb{N}_{0}^{d},|\beta|=n} \frac{n!}{\beta!} \prod_{k=1}^{d}\left(M_{\varepsilon_{k}} z_{k}\right)^{\beta_{k}}$ one obtains

$$
\begin{aligned}
G(z) & =\sum_{\beta \in \mathbb{N}_{0}^{d}} M_{\beta} z^{\beta}=\sum_{n=1}^{\infty} \sum_{\beta \in \mathbb{N}_{0}^{d},|\beta|=n} \frac{1}{n}\binom{2 n-2}{n-1} \frac{n!}{\beta!} \prod_{k=1}^{d} M_{\varepsilon_{k}}^{\beta_{k}} \prod_{k=1}^{d} z_{k}^{\beta_{k}} \\
& =\sum_{\beta \in \mathbb{N}_{0}^{d}}\left(\frac{1}{|\beta|}\binom{2|\beta|-2}{|\beta|-1} \frac{|\beta|!}{\beta!} \prod_{k=1}^{d} M_{\varepsilon_{k}}^{\beta_{k}}\right) z^{\beta} .
\end{aligned}
$$

### 2.1 Proof of the main theorem

The statement of the main theorem is as follows.
Theorem 2.8. Let the assumptions $(A 1)-(A 4-n)$ be fulfilled. Then there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ to (2.1).

Proof. The proof of Theorem 2.8 will be given by induction with respect to $n \in \mathbb{N}$ in Theorems 2.9 and 2.10. We will prove in the first one that the statement of the main theorem holds for $n=2$. Since it is technically pretty challenging to write down the proof of the inductive step for arbitrary $n \in \mathbb{N}$, in Theorem 2.10 the proof is given for $n=3$ by reducing the problem to the case $n=2$. In the same way one can reduce the problem for arbitrary $n \in \mathbb{N}$ to the case $n-1$.

First consider (2.1) for $n=2$, i.e.

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+u^{\diamond 2}(t, \omega)+f(t, \omega), \quad t \in[0, T]  \tag{2.15}\\
u(0, \omega) & =u^{0}(\omega),
\end{align*}
$$

The chaos expansion representation of the Wick-square is given by (2.3). Applying the Wiener-Itô chaos expansion to the nonlinear stochastic equation (2.15) one obtain

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \frac{d}{d t} u_{\alpha}(t) H_{\alpha}(\omega) & =\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha}(t) H_{\alpha}(\omega)+\sum_{\alpha \in \mathcal{I}}\left(\sum_{\gamma \leq \alpha} u_{\gamma}(t) u_{\alpha-\gamma}(t)\right) H_{\alpha}(\omega) \\
& +\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega) \\
\sum_{\alpha \in \mathcal{I}} u_{\alpha}(0) H_{\alpha}(\omega) & =\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha}(\omega)
\end{aligned}
$$

which reduces to the system of infinitely many deterministic Cauchy problems:

$$
\begin{align*}
& 1^{\circ} \text { for } \alpha=\mathbf{0} \\
& \qquad \frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+u_{\mathbf{0}}^{2}(t)+f_{\mathbf{0}}(t), \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0}, \quad \text { and } \tag{2.16}
\end{align*}
$$

$2^{\circ}$ for $\alpha>\mathbf{0}$

$$
\begin{equation*}
\frac{d}{d t} u_{\alpha}(t)=\left(A_{\alpha}+2 u_{\mathbf{0}}(t) I d\right) u_{\alpha}(t)+\sum_{\mathbf{0}<\gamma<\alpha} u_{\gamma}(t) u_{\alpha-\gamma}(t)+f_{\alpha}(t), \quad u_{\alpha}(0)=u_{\alpha}^{0} \tag{2.17}
\end{equation*}
$$

with $t \in(0, T]$ and $\omega \in \Omega$.

## Stochastic evolution equations with nonlinearities

Recall that

$$
B_{\alpha, 2}(t)=A_{\alpha}+2 u_{\mathbf{0}}(t) I d \quad \text { and } \quad g_{\alpha, 2}(t)=\sum_{\mathbf{0}<\gamma<\alpha} u_{\gamma}(t) u_{\alpha-\gamma}(t)+f_{\alpha}(t), \quad t \in[0, T]
$$

for all $\alpha>0$, so the system (2.17) can be written in the form

$$
\begin{equation*}
\frac{d}{d t} u_{\alpha}(t)=B_{\alpha, 2}(t) u_{\alpha}(t)+g_{\alpha, 2}(t), \quad t \in(0, T] ; \quad u_{\alpha}(0)=u_{\alpha}^{0} \tag{2.18}
\end{equation*}
$$

Theorem 2.9. Let the assumptions $(A 1)-(A 4-2)$ be fulfilled. Then there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ to (2.15).

Proof. According to Lemma 2.3 for every $\alpha>0$ the evolution equation (2.18) has an unique classical solution $u_{\alpha} \in C^{1}([0, T], X)$. Thus, the generalized stochastic process $u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega), t \in[0, T], \omega \in \Omega$ has coefficients that are all classical solutions to the corresponding deterministic equation (2.18), hence in order to show that $u$ is an almost classical solution to (2.15) one has to prove that $u \in C([0, T], X) \otimes(S)_{-1}$.

Let $u^{0} \in X \otimes(S)_{-1}$ be an initial condition satisfying assumption (A2) which states that there exist $\tilde{p} \geq 0$ and $\tilde{K}>0$ such that $\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-\tilde{p} \alpha}=\tilde{K}$. Then there also exist $p \geq 0$ and $K \in(0,1)$ such that $\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-2 p \alpha}=K^{2}$, or equivalently

$$
\begin{equation*}
(\exists p \geq 0)(\exists K \in(0,1))(\forall \alpha \in \mathcal{I}) \quad\left\|u_{\alpha}^{0}\right\|_{X} \leq K(2 \mathbb{N})^{p \alpha} \tag{2.19}
\end{equation*}
$$

The inhomogeneous part $f \in C^{1}([0, T], X) \otimes(S)_{-1}$ satisfies assumption (A3) which states that there exists $\tilde{p} \geq 0$ such that $\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-\tilde{p} \alpha}<\infty$. Then there exist $p \geq 0$ and $K \in(0,1)$ such that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X} \leq K(2 \mathbb{N})^{p \alpha}, \quad \alpha \in \mathcal{I} \tag{2.20}
\end{equation*}
$$

The coefficients $u_{\alpha}, \alpha \in \mathcal{I}, \alpha>\mathbf{0}$ of the solution $u$ are given by (2.11) and (2.12) for $n=2$. Denote by

$$
L_{\alpha}:=\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}, \quad \alpha \in \mathcal{I}
$$

First, for $\alpha=\mathbf{0}$ using (2.9) one obtain

$$
\begin{equation*}
L_{\mathbf{0}}=\sup _{t \in[0, T]}\left\|u_{\mathbf{0}}(t)\right\|_{X}=M_{2} \tag{2.21}
\end{equation*}
$$

since the solution to (2.16) satisfies assumption (A4-2). Let $|\alpha|=1$. Then $\alpha=\varepsilon_{k}, k \in \mathbb{N}$ and using (2.11) we have that

$$
\left\|u_{\varepsilon_{k}}(t)\right\|_{X} \leq\left\|S_{\varepsilon_{k}, 2}(t, 0)\right\|\left\|u_{\varepsilon_{k}}^{0}\right\|_{X}+\int_{0}^{t}\left\|S_{\varepsilon_{k}, 2}(t, s)\right\|\left\|f_{\varepsilon_{k}}(s)\right\|_{X} d s, \quad t \in[0, T]
$$

From (2.10) we obtain that

$$
\begin{equation*}
\int_{0}^{t}\left\|S_{\alpha, 2}(t, s)\right\| d s \leq \int_{0}^{t} m e^{w_{2}(t-s)} d s=m \frac{e^{w_{2} t}-1}{w_{2}} \leq \frac{m}{w_{2}} e^{w_{2} T}, \quad t \in[0, T], \quad \alpha>\mathbf{0} \tag{2.22}
\end{equation*}
$$

and now (2.10), (2.19) and (2.20) imply that

$$
\begin{align*}
L_{\varepsilon_{k}} & =\sup _{t \in[0, T]}\left\|u_{\varepsilon_{k}}(t)\right\|_{X} \leq \sup _{t \in[0, T]}\left\{\left\|S_{\varepsilon_{k}, 2}(t, 0)\right\|\left\|u_{\varepsilon_{k}}^{0}\right\|_{X}+\sup _{s \in[0, t]}\left\|f_{\varepsilon_{k}}(s)\right\|_{X} \int_{0}^{t}\left\|S_{\alpha, 2}(t, s)\right\| d s\right\}  \tag{2.23}\\
& \leq m e^{w_{2} T} K(2 \mathbb{N})^{p \varepsilon_{k}}+\frac{m}{w_{2}} e^{w_{2} T} K(2 \mathbb{N})^{p \varepsilon_{k}}=m_{2} e^{w_{2} T} K(2 \mathbb{N})^{p \varepsilon_{k}}, \quad t \in[0, T], \quad k \in \mathbb{N},
\end{align*}
$$

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where $m_{2}=m+\frac{m}{w_{2}}$.
For $|\alpha|>1$ we consider two possibilities for $L_{\alpha}$. First, if $L_{\alpha} \leq \sqrt{K}(2 \mathbb{N})^{p \alpha}$ for all $|\alpha|>1$ then the statement of the theorem follows directly since for $q>2 p+1$ and, having in mind (2.21) and (2.23), we obtain

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}=\sum_{\alpha \in \mathcal{I}} L_{\alpha}^{2}(2 \mathbb{N})^{-q \alpha}=L_{\mathbf{0}}^{2}+\sum_{k \in \mathbb{N}} L_{\varepsilon_{k}}^{2}(2 \mathbb{N})^{-q \varepsilon_{k}}+\sum_{|\alpha|>1} L_{\alpha}^{2}(2 \mathbb{N})^{-q \alpha} \\
\leq M_{2}^{2}+\left(m_{2} e^{w_{2} T} K\right)^{2} \sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(2 p-q) \varepsilon_{k}}+K \sum_{|\alpha|>1}(2 \mathbb{N})^{(2 p-q) \alpha}<\infty
\end{gathered}
$$

i.e. $u \in C([0, T], X) \otimes(S)_{-1,-q}$.

In what follows, we will assume that $L_{\alpha}>\sqrt{K}(2 \mathbb{N})^{p \alpha}$ for some $\alpha \in \mathcal{I},|\alpha|>1$. Denote by $\mathcal{I}_{*}$ the set of all multi-indices $\alpha \in \mathcal{I},|\alpha|>1$, for which $L_{\alpha}>\sqrt{K}(2 \mathbb{N})^{p \alpha}$. Then from (2.12) we obtain

$$
u_{\alpha}(t)=S_{\alpha, 2}(t, 0) u_{\alpha}^{0}+\int_{0}^{t} S_{\alpha, 2}(t, s)\left[\sum_{0<\gamma<\alpha} u_{\alpha-\gamma}(s) u_{\gamma}(s)+f_{\alpha}(s)\right] d s, \quad t \in[0, T] .
$$

From this we have

$$
\begin{aligned}
L_{\alpha} & =\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X} \\
& \leq \sup _{t \in[0, T]}\left\{\left\|S_{\alpha, 2}(t, 0)\right\|\left\|u_{\alpha}^{0}\right\|_{X}+\int_{0}^{t}\left\|S_{\alpha, 2}(t, s)\right\|\left\|\sum_{0<\gamma<\alpha} u_{\alpha-\gamma}(s) u_{\gamma}(s)\right\| d s\right. \\
& \left.+\int_{0}^{t}\left\|S_{\alpha, 2}(t, s)\right\|\left\|f_{\alpha}(s)\right\|_{X} d s\right\} \\
& \leq \sup _{t \in[0, T]}\left\{m e^{w_{2} t}\left\|u_{\alpha}^{0}\right\|_{X}+\sup _{s \in[0, t]} \sum_{0<\gamma<\alpha}\left\|u_{\alpha-\gamma}(s)\right\|_{X}\left\|u_{\gamma}(s)\right\|_{X} \cdot \int_{0}^{t}\left\|S_{\alpha, 2}(t, s)\right\| d s\right. \\
& \left.+\sup _{s \in[0, t]}\|f(s)\|_{X} \int_{0}^{t}\left\|S_{\alpha, 2}(t, s)\right\| d s\right\} .
\end{aligned}
$$

Using (2.22) we obtain

$$
\begin{aligned}
L_{\alpha} & =\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X} \\
& \leq m e^{w_{2} T}\left\|u_{\alpha}^{0}\right\|_{X}+\frac{m}{w_{2}} e^{w_{2} T} \sum_{0<\gamma<\alpha} \sup _{t \in[0, T]}\left\|u_{\alpha-\gamma}(t)\right\|_{X} \sup _{t \in[0, T]}\left\|u_{\gamma}(t)\right\|_{X} \\
& +\frac{m}{w_{2}} e^{w_{2} T} \sup _{s \in[0, t]}\|f(s)\|_{X} \\
& \leq m_{2} e^{w_{2} T} K(2 \mathbb{N})^{p \alpha}+\frac{m}{w_{2}} e^{w_{2} T} \sum_{0<\gamma<\alpha} L_{\alpha-\gamma} L_{\gamma},
\end{aligned}
$$

where again $m_{2}=m+\frac{m}{w_{2}}$. Since $m_{2} \geq \frac{m}{w_{2}}$, one easily obtains

$$
\begin{equation*}
L_{\alpha} \leq m_{2} e^{w_{2} T}\left(K(2 \mathbb{N})^{p \alpha}+\sum_{\mathbf{0}<\gamma<\alpha} L_{\alpha-\gamma} L_{\gamma}\right) \tag{2.24}
\end{equation*}
$$

Let $\tilde{L}_{\alpha}, \alpha>\mathbf{0}, \alpha \in \mathcal{I}_{*}$, be given by

$$
\tilde{L}_{\alpha}:=2 m_{2} e^{w_{2} T}\left(\frac{L_{\alpha}}{\sqrt{K}(2 \mathbb{N})^{p \alpha}}+1\right), \quad \alpha>\mathbf{0}, \alpha \in \mathcal{I}_{*} .
$$

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Thus, from (2.23) we have that for all $k \in \mathbb{N}$

$$
\begin{align*}
\tilde{L}_{\varepsilon_{k}}=2 m_{2} e^{w_{2} T}\left(\frac{L_{\varepsilon_{k}}}{\sqrt{K}(2 \mathbb{N})^{p \varepsilon_{k}}}+1\right) & \leq 2 m_{2} e^{w_{2} T}\left(\frac{m_{2} e^{w_{2} T} K(2 \mathbb{N})^{p \varepsilon_{k}}}{\sqrt{K}(2 \mathbb{N})^{p \varepsilon_{k}}}+1\right)  \tag{2.25}\\
& =2 m_{2} e^{w_{2} T}\left(m_{2} e^{w_{2} T} \sqrt{K}+1\right)
\end{align*}
$$

We proceed with the estimation of the term $\sum_{\mathbf{0}<\gamma<\alpha} \tilde{L}_{\gamma} \tilde{L}_{\alpha-\gamma}$ for given $|\alpha|>1, \alpha \in \mathcal{I}_{*}$.

$$
\begin{aligned}
\sum_{\mathbf{0}<\gamma<\alpha} \tilde{L}_{\gamma} \tilde{L}_{\alpha-\gamma} & =\sum_{\mathbf{0}<\gamma<\alpha}\left(2 m_{2} e^{w_{2} T}\right)^{2}\left(\frac{L_{\gamma}}{\sqrt{K}(2 \mathbb{N})^{p \gamma}}+1\right)\left(\frac{L_{\alpha-\gamma}}{\sqrt{K}(2 \mathbb{N})^{p(\alpha-\gamma)}}+1\right) \\
& \geq\left(2 m_{2} e^{w_{2} T}\right)^{2}\left(\sum_{\mathbf{0}<\gamma<\alpha} \frac{L_{\gamma} L_{\alpha-\gamma}}{K(2 \mathbb{N})^{p \alpha}}+1\right) \\
& =\frac{\left(2 m_{2} e^{w_{2} T}\right)^{2}}{K(2 \mathbb{N})^{p \alpha}} \sum_{\mathbf{0}<\gamma<\alpha} L_{\gamma} L_{\alpha-\gamma}+\left(2 m_{2} e^{w_{2} T}\right)^{2} .
\end{aligned}
$$

Using inequality (2.24) we obtain

$$
\sum_{\mathbf{0}<\gamma<\alpha} \tilde{L}_{\gamma} \tilde{L}_{\alpha-\gamma} \geq \frac{\left(2 m_{2} e^{w_{2} T}\right)^{2}}{K(2 \mathbb{N})^{p \alpha}}\left(\frac{L_{\alpha}}{m_{2} e^{w_{2} T}}-K(2 \mathbb{N})^{p \alpha}\right)+\left(2 m_{2} e^{w_{2} T}\right)^{2}=\frac{4 m_{2} e^{w_{2} T}}{K(2 \mathbb{N})^{p \alpha}} L_{\alpha}
$$

Now since $L_{\alpha}>\sqrt{K}(2 \mathbb{N})^{p \alpha}$ for $\alpha \in \mathcal{I}_{*}$ and since $K<1$ we obtain

$$
\begin{aligned}
\sum_{\mathbf{0}<\gamma<\alpha} \tilde{L}_{\gamma} \tilde{L}_{\alpha-\gamma} & \geq \frac{4 m_{2} e^{w_{2} T}}{\sqrt{K}(2 \mathbb{N})^{p \alpha}} L_{\alpha}=\frac{2 m_{2} e^{w_{2} T}}{\sqrt{K}(2 \mathbb{N})^{p \alpha}} L_{\alpha}+\frac{2 m_{2} e^{w_{2} T}}{\sqrt{K}(2 \mathbb{N})^{p \alpha}} L_{\alpha} \\
& \geq 2 m_{2} e^{w_{2} T}\left(\frac{L_{\alpha}}{\sqrt{K}(2 \mathbb{N})^{p \alpha}}+1\right)=\tilde{L}_{\alpha}
\end{aligned}
$$

Hence, for all $\alpha \in \mathcal{I}_{*},|\alpha|>1$, we have obtained

$$
\sum_{\mathbf{0}<\gamma<\alpha} \tilde{L}_{\gamma} \tilde{L}_{\alpha-\gamma} \geq \tilde{L}_{\alpha}
$$

Let $R_{\alpha}, \alpha>\mathbf{0}$, be defined as follows:

$$
\begin{aligned}
R_{\varepsilon_{k}} & =\tilde{L}_{\varepsilon_{k}}, \quad k \in \mathbb{N}, \\
R_{\alpha} & =\sum_{\mathbf{0}<\gamma<\alpha} R_{\gamma} R_{\alpha-\gamma}, \quad|\alpha|>1 .
\end{aligned}
$$

It is a direct consequence of the definition of the numbers $R_{\alpha}, \alpha>\mathbf{0}$, and it can be shown by induction with respect to the length of the multi-index $\alpha>\mathbf{0}$ that (see [11, Section 5])

$$
\begin{equation*}
\tilde{L}_{\alpha} \leq R_{\alpha}, \quad \alpha>\mathbf{0} \tag{2.26}
\end{equation*}
$$

Lemma 2.7 shows that the numbers $R_{\alpha}, \alpha>\mathbf{0}$ satisfy

$$
R_{\alpha}=\frac{1}{|\alpha|}\binom{2|\alpha|-2}{|\alpha|-1} \frac{|\alpha|!}{\alpha!} \prod_{i=1}^{\infty} R_{\varepsilon_{i}}^{\alpha_{i}}, \quad \alpha>\mathbf{0}
$$

Further on, by (2.25),

$$
\prod_{i=1}^{\infty} R_{\varepsilon_{i}}^{\alpha_{i}}=\prod_{i=1}^{\infty} \tilde{L}_{\varepsilon_{i}}^{\alpha_{i}} \leq \prod_{i=1}^{\infty}\left(2 m_{2} e^{w_{2} T}\left(m_{2} e^{w_{2} T} \sqrt{K}+1\right)\right)^{\alpha_{i}}
$$

Let $c=2 m_{2} e^{w_{2} T}\left(m_{2} e^{w_{2} T} \sqrt{K}+1\right)$. Then

$$
\begin{equation*}
R_{\alpha} \leq \mathbf{c}_{|\alpha|-1} \frac{|\alpha|!}{\alpha!} c^{|\alpha|}, \quad \alpha>\mathbf{0} \tag{2.27}
\end{equation*}
$$

where $\mathbf{c}_{n}=\frac{1}{n+1}\binom{2 n}{n}, n \geq 0$ denotes the $n$th Catalan number (more information on Catalan numbers is provided in Lemma 2.6). Using Lemma 2.4, (2.26), (2.27) and (2.14) we obtain that for $\alpha \in \mathcal{I}_{*},|\alpha|>1$ the estimation

$$
\tilde{L}_{\alpha} \leq R_{\alpha} \leq 4^{|\alpha|-1}(2 \mathbb{N})^{2 \alpha} c^{|\alpha|}
$$

holds. Finally, from the definition of $\tilde{L}_{\alpha}, \alpha>\mathbf{0}$ we obtain

$$
L_{\alpha} \leq\left(\frac{4^{|\alpha|-1}(2 \mathbb{N})^{2 \alpha} c^{|\alpha|}}{2 m_{2} e^{w_{2} T}}-1\right) \sqrt{K}(2 \mathbb{N})^{p \alpha} \leq \frac{\sqrt{K}}{8 m_{2} e^{w_{2} T}}(4 c)^{|\alpha|}(2 \mathbb{N})^{(p+2) \alpha}
$$

Notice that the upper estimate also holds for $|\alpha|>1, \alpha \in \mathcal{I} \backslash \mathcal{I}_{*}$. Indeed, if $L_{\alpha}<$ $\sqrt{K}(2 \mathbb{N})^{p \alpha}$ then also $L_{\alpha}<\frac{\sqrt{K}}{8 m_{2} e^{w_{2} T}}(4 c)^{|\alpha|}(2 \mathbb{N})^{(p+2) \alpha}$, so we obtain

$$
L_{\alpha} \leq \frac{\sqrt{K}}{8 m_{2} e^{w_{2} T}}(4 c)^{|\alpha|}(2 \mathbb{N})^{(p+2) \alpha}, \quad \text { for all } \alpha \in \mathcal{I}, \quad|\alpha|>1
$$

Now we can prove that $u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \in C([0, T], X) \otimes(S)_{-1}$. Denote by $H=\frac{\sqrt{K}}{8 m_{2} e^{w_{2}} T}$. Then

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}=\sup _{t \in[0, T]}\left\|u_{\mathbf{0}}(t)\right\|_{X}^{2}+\sum_{\alpha>\mathbf{0}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \\
&=M_{2}^{2}+\sum_{k \in \mathbb{N}} L_{\varepsilon_{k}}^{2}(2 \mathbb{N})^{-q \varepsilon_{k}}+\sum_{|\alpha|>1} L_{\alpha}^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq M_{2}^{2}+\left(m_{2} e^{w_{2} T} K\right)^{2} \sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(2 p-q) \varepsilon_{k}}+H^{2} \sum_{|\alpha|>1}\left((4 c)^{|\alpha|}(2 \mathbb{N})^{(p+2) \alpha}\right)^{2}(2 \mathbb{N})^{-q \alpha} \\
&=M_{2}^{2}+\left(m_{2} e^{w_{2} T} K\right)^{2} \sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(2 p-q) \varepsilon_{k}}+H^{2} \sum_{|\alpha|>1}\left(16 c^{2}\right)^{|\alpha|}(2 \mathbb{N})^{(2 p+4-q) \alpha} .
\end{aligned}
$$

Taking that $s>0$ is such that $2^{s} \geq 16 c^{2}$, according to Lemma 2.5 , we obtain

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} & \leq M_{2}^{2}+\left(m_{2} e^{w_{2} T} K\right)^{2} \sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(2 p-q) \varepsilon_{k}} \\
& +H^{2} \sum_{|\alpha|>1}(2 \mathbb{N})^{(2 p+4+s-q) \alpha}<\infty
\end{aligned}
$$

for $q>2 p+s+5$.
In the sequel we prove the existence of the almost classical solution of the Cauchy problem

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+u^{\diamond 3}(t, \omega)+f(t, \omega), \quad t \in[0, T]  \tag{2.28}\\
u(0, \omega) & =u^{0}(\omega)
\end{align*}
$$

Note that

$$
\begin{align*}
& u^{\diamond 3}(t, \omega)=u^{\diamond 2}(t, \omega) \diamond u(t, \omega)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t) H_{\alpha}(\omega) \\
& =u_{\mathbf{0}}^{3}(t) H_{\mathbf{0}}(\omega) \\
& +\sum_{|\alpha|>0}\left(3 u_{\mathbf{0}}^{2} u_{\alpha}(t)+3 u_{\mathbf{0}} \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t)+\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)\right) H_{\alpha}(\omega) \tag{2.29}
\end{align*}
$$

## Stochastic evolution equations with nonlinearities

for $t \in[0, T], \omega \in \Omega$. Applying the Wiener-Itô chaos expansion method to the nonlinear stochastic equation (2.28) reduces to the system of infinitely many deterministic Cauchy problems:
$1^{\circ}$ for $\alpha=\mathbf{0}$

$$
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+u_{\mathbf{0}}^{3}(t)+f_{\mathbf{0}}(t), \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0}, \quad \text { and }
$$

$2^{\circ}$ for $\alpha>\mathbf{0}$

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =\left(A_{\alpha}+3 u_{\mathbf{0}}^{2}(t) I d\right) u_{\alpha}(t)+3 u_{\mathbf{0}} \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t)+ \\
& +\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)+f_{\alpha}(t),  \tag{2.30}\\
u_{\alpha}(0) & =u_{\alpha}^{0} .
\end{align*}
$$

with $t \in(0, T]$ and $\omega \in \Omega$.
Let

$$
\begin{align*}
B_{\alpha, 3}(t) & =A_{\alpha}+3 u_{\mathbf{0}}^{2}(t) I d \quad \text { and } \\
g_{\alpha, 3}(t) & =3 u_{\mathbf{0}} \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t)+\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)+f_{\alpha}(t), \quad t \in[0, T] \tag{2.31}
\end{align*}
$$

for all $\alpha>\mathbf{0}$, then, the system (2.30) can be written in the form

$$
\begin{equation*}
\frac{d}{d t} u_{\alpha}(t)=B_{\alpha, 3}(t) u_{\alpha}(t)+g_{\alpha, 3}(t), \quad t \in(0, T] ; \quad u_{\alpha}(0)=u_{\alpha}^{0} \tag{2.32}
\end{equation*}
$$

Theorem 2.10. Let the assumptions $(A 1)-(A 4-3)$ be fulfilled. Then, there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ to (2.28).

Proof. According to Lemma 2.3 for every $\alpha>0$ the evolution equation (2.32) has an unique classical solution $u_{\alpha} \in C^{1}([0, T], X)$ given in the form (2.12). Thus, the generalized stochastic process $u(t, \omega)$, represented in the chaos expansion form (2.2), has coefficients that are all classical solutions to the corresponding deterministic equation (2.32). Hence, in order to show that $u$ is an almost classical solution to (2.28), one has to prove that $u \in C([0, T], X) \otimes(S)_{-1}$.

We assume that the initial condition $u^{0} \in X \otimes(S)_{-1}$ satisfies assumption (A2), i.e. the estimate (2.19) holds true. The inhomogeneous part $f \in C^{1}([0, T], X) \otimes(S)_{-1}$ satisfies assumption (A3), i.e. the estimate (2.20) is true for some $p \geq 0$. Moreover, the coefficients $u_{\alpha}, \alpha \in \mathcal{I}, \alpha>\mathbf{0}$ of the solution $u$ are given by (2.11) and (2.12) for $n=3$. Now, for all $\alpha \in \mathcal{I}$ we are going to estimate

$$
L_{\alpha}=\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}
$$

It is clear that for $\alpha=\mathbf{0}$, by $(A 4-3)$ we have $L_{\mathbf{0}}=\sup _{t \in[0, T]}\left\|u_{\mathbf{0}}(t)\right\|=M_{3}$.
For, $|\alpha|=1$, i.e. for $\alpha=\varepsilon_{k}, k \in \mathbb{N}$ by (2.11) we have that

$$
\left\|u_{\varepsilon_{k}}(t)\right\|_{X} \leq\left\|S_{\varepsilon_{k}, 3}(t, 0)\right\|\left\|u_{\varepsilon_{k}}^{0}\right\|_{X}+\int_{0}^{t}\left\|S_{\varepsilon_{k}, 3}(t, s)\right\|\left\|f_{\varepsilon_{k}}(s)\right\|_{X} d s, \quad t \in[0, T] .
$$

From (2.10) we obtain that

$$
\begin{equation*}
\int_{0}^{t}\left\|S_{\alpha, 3}(t, s)\right\| d s \leq \int_{0}^{t} m e^{w_{3}(t-s)} d s \leq \frac{m}{w_{3}} e^{w_{3} T}, \quad t \in[0, T], \quad \alpha>\mathbf{0} \tag{2.33}
\end{equation*}
$$

Stochastic evolution equations with nonlinearities

By (2.19), (2.20), (2.10) and (2.33) we obtain

$$
\begin{aligned}
L_{\varepsilon_{k}} & =\sup _{t \in[0, T]}\left\|u_{\varepsilon_{k}}(t)\right\|_{X} \leq \sup _{t \in[0, T]}\left\{\left\|S_{\varepsilon_{k}, 3}(t, 0)\right\|\left\|u_{\varepsilon_{k}}^{0}\right\|_{X}+\sup _{s \in[0, t]}\left\|f_{\varepsilon_{k}}(s)\right\|_{X} \int_{0}^{t}\left\|S_{\alpha, 3}(t, s)\right\| d s\right\} \\
& \leq m e^{w_{3} T} K(2 \mathbb{N})^{p \varepsilon_{k}}+\frac{m}{w_{3}} e^{w_{3} T} K(2 \mathbb{N})^{p \varepsilon_{k}},
\end{aligned}
$$

which leads to the estimate

$$
\begin{equation*}
L_{\varepsilon_{k}} \leq m_{3} e^{w_{3} T} K(2 \mathbb{N})^{p \varepsilon_{k}}, \quad k \in \mathbb{N}, \tag{2.34}
\end{equation*}
$$

where $m_{3}=m+\frac{m}{w_{3}}$.
For $|\alpha|=2$ we have two different forms of the multiindex. First, for $\alpha=2 \varepsilon_{k}, k \in \mathbb{N}$ from (2.31) we obtain the form of the inhomogeneous part $g_{2 \varepsilon_{k}, 3}(t)=3 u_{\mathbf{0}}(t) u_{\varepsilon_{k}}^{2}(t)+$ $f_{2 \varepsilon_{k}}(t)$, where

$$
\begin{aligned}
\sup _{s \in[0, t]}\left\|g_{2 \varepsilon_{k}, 3}(s)\right\|_{X} & \leq 3 M_{3} L_{\varepsilon_{k}}^{2}+\sup _{s \in[0, t]}\left\|f_{2 \varepsilon_{k}}(s)\right\|_{X} \\
& \leq 3 M_{3} m_{3}^{2} e^{2 w_{3} T} K^{2}(2 \mathbb{N})^{2 p \varepsilon_{k}}+K(2 \mathbb{N})^{2 p \varepsilon_{k}} \\
& \leq\left(3 M_{3} m_{3}^{2} e^{2 w_{3} T} K^{2}+K\right)(2 \mathbb{N})^{2 p \varepsilon_{k}} .
\end{aligned}
$$

Then, together with (2.12) we obtain

$$
\begin{aligned}
L_{2 \varepsilon_{k}} & =\sup _{t \in[0, T]}\left\|u_{2 \varepsilon_{k}}(t)\right\|_{X} \\
& \leq \sup _{t \in[0, T]}\left\{\left\|S_{2 \varepsilon_{k}, 3}(t, 0)\right\|\left\|u_{2 \varepsilon_{k}}^{0}\right\|_{X}+\sup _{s \in[0, t]}\left\|g_{2 \varepsilon_{k}, 3}(s)\right\|_{X} \int_{0}^{t}\left\|S_{2 \varepsilon_{k}, 3}(t, s)\right\| d s\right\} \\
& \leq m e^{w_{3} T} K(2 \mathbb{N})^{2 p \varepsilon_{k}}+\frac{m}{w_{3}} e^{w_{3} T}\left(3 M_{3} m_{3}^{2} e^{2 w_{3} T} K^{2}+K\right)(2 \mathbb{N})^{2 p \varepsilon_{k}} .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
L_{2 \varepsilon_{k}} \leq a_{1} e^{w_{3} T} K(2 \mathbb{N})^{2 p \varepsilon_{k}}, \quad k \in \mathbb{N}, \tag{2.35}
\end{equation*}
$$

where $a_{1}=m+\frac{m}{w_{3}}\left(3 M_{3} m_{3}^{2} e^{2 w_{3} T} K+1\right)$.
In the second case, for $\alpha=\varepsilon_{k}+\varepsilon_{j}, k \neq j, k, j \in \mathbb{N}$ from (2.31) we obtain the form $g_{\varepsilon_{k}+\varepsilon_{j}, 3}(t)=6 u_{\mathbf{0}}(t) u_{\varepsilon_{k}}(t) u_{\varepsilon_{j}}(t)+f_{\varepsilon_{k}+\varepsilon_{j}}(t)$ of the inhomogeneous part of (2.12). By applying (2.34) and (2.20) it can be estimated as

$$
\begin{aligned}
\sup _{s \in[0, t]}\left\|g_{\varepsilon_{k}+\varepsilon_{j}, 3}(s)\right\|_{X} & \leq 6 M_{3} L_{\varepsilon_{k}} L_{\varepsilon_{j}}+\sup _{s \in[0, t]}\left\|f_{\varepsilon_{k}+\varepsilon_{j}}(s)\right\|_{X} \\
& \leq 6 M_{3} m_{3}^{2} e^{2 w_{3} T} K^{2}(2 \mathbb{N})^{p \varepsilon_{k}}(2 \mathbb{N})^{p \varepsilon_{j}}+K(2 \mathbb{N})^{p \varepsilon_{k}+p \varepsilon_{j}} \\
& \leq\left(6 M_{3} m_{3}^{2} e^{2 w_{3} T} K^{2}+K\right)(2 \mathbb{N})^{p\left(\varepsilon_{k}+\varepsilon_{j}\right)} .
\end{aligned}
$$

Then, (2.12) combined with the previous estimate lead to

$$
\begin{aligned}
L_{\varepsilon_{k}+\varepsilon_{j}} & =\sup _{t \in[0, T]}\left\|u_{\varepsilon_{k}+\varepsilon_{j}}(t)\right\|_{X} \\
& \leq \sup _{t \in[0, T]}\left\{\left\|S_{\varepsilon_{k}+\varepsilon_{j}, 3}(t, 0)\right\|\left\|u_{\varepsilon_{k}+\varepsilon_{j}}^{0}\right\|_{X}+\sup _{s \in[0, t]}\left\|g_{\varepsilon_{k}+\varepsilon_{j}, 3}(s)\right\|_{X} \int_{0}^{t}\left\|S_{\varepsilon_{k}+\varepsilon_{j}, 3}(t, s)\right\| d s\right\} \\
& \leq m e^{w_{3} T} K(2 \mathbb{N})^{p\left(\varepsilon_{k}+\varepsilon_{j}\right)}+\frac{m}{w_{3}} e^{w_{3} T}\left(6 M_{3} m_{3}^{2} e^{2 w_{3} T} K^{2}+K\right)(2 \mathbb{N})^{p\left(\varepsilon_{k}+\varepsilon_{j}\right)} .
\end{aligned}
$$

Then, we obtained

$$
\begin{equation*}
L_{\varepsilon_{k}+\varepsilon_{j}} \leq a_{2} e^{w_{3} T} K(2 \mathbb{N})^{p\left(\varepsilon_{k}+\varepsilon_{j}\right)}, \quad k, j \in \mathbb{N}, k \neq j \tag{2.36}
\end{equation*}
$$

## Stochastic evolution equations with nonlinearities

where $a_{2}=m+\frac{m}{w_{3}}\left(6 M_{3} m_{3}^{2} e^{2 w_{3} T} K+1\right)$. Finaly, from (2.35) and (2.36) we obtain the estimate for all $|\alpha|=2$

$$
L_{\alpha} \leq a_{2} e^{w_{3} T} K(2 \mathbb{N})^{p \alpha}
$$

For $|\alpha|>2$ we deal with general form of the inhomogeneous part of (2.32)

$$
g_{\alpha, 3}(t)=3 u_{\mathbf{0}} \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t)+\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)+f_{\alpha}(t), \quad t \in[0, T] .
$$

The solution to (2.32) is of the form

$$
\begin{aligned}
& u_{\alpha}(t)=S_{\alpha, 3}(t, 0) u_{\alpha}^{0} \\
& +\int_{0}^{t} S_{\alpha, 3}(t, s)\left(3 u_{0} \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t)+\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)+f_{\alpha}(t)\right) d s .
\end{aligned}
$$

We underline that in the previous inductive steps, we obtained the estimates of $L_{\alpha-\theta}=$ $\sup _{t \in[0, T]}\left\|u_{\alpha-\theta}(t)\right\|$ for all $\mathbf{0}<\theta<\alpha$. Then,

$$
\begin{align*}
L_{\alpha}=\sup _{t \in[0, T]} \| & u_{\alpha}(t) \| \leq m e^{\omega_{3} T} K(2 \mathbb{N})^{p \alpha} \\
& +\frac{m}{w_{3}}\left(3 M_{3} \sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta}+\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} L_{\alpha-\beta} L_{\beta-\gamma} L_{\gamma}+K(2 \mathbb{N})^{p \alpha}\right) \\
& \leq m_{3} e^{\omega_{3} T}\left(K(2 \mathbb{N})^{p \alpha}+3 M_{3} \sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta}+\sum_{0<\beta<\alpha} L_{\alpha-\beta} \sum_{0<\gamma<\beta} L_{\beta-\gamma} L_{\gamma}\right), \tag{2.37}
\end{align*}
$$

where $m_{3}=m+\frac{m}{w_{3}}$.
In order to estimate $L_{\alpha}$ for $|\alpha|>2$ we consider two possibilities: (a) $L_{\alpha} \leq \sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta}$, $|\alpha|>2$ and (b) $L_{\alpha}>\sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta},|\alpha|>2$.
(a) Define $R_{\alpha}$ for $|\alpha| \geq 1$ in the following inductive way

$$
\begin{aligned}
R_{\varepsilon_{k}} & =L_{\varepsilon_{k}} \\
R_{\alpha} & =\sum_{0<\beta<\alpha} R_{\alpha-\beta} R_{\beta}, \quad|\alpha| \geq 2,
\end{aligned}
$$

then, using Lemma 2.7, we obtain the estimate

$$
L_{\alpha} \leq R_{\alpha}=\frac{1}{|\alpha|}\binom{2|\alpha|-2}{|\alpha|-1} \frac{|\alpha|!}{\alpha!}\left(\prod_{i=1}^{\infty} R_{\varepsilon_{i}}^{\alpha_{i}}\right)
$$

Moreover, by (2.34) we get

$$
\begin{aligned}
\prod_{i=1}^{\infty} R_{\varepsilon_{i}}^{\alpha_{i}} & =\prod_{i=1}^{\infty} L_{\varepsilon_{i}}^{\alpha_{i}} \leq \prod_{i=1}^{\infty}\left(m_{3} e^{\omega_{3} T} K(2 \mathbb{N})^{p \varepsilon_{k}}\right)^{\alpha_{i}}=\left(m_{3} e^{\omega_{3} T} K\right)^{|\alpha|} \prod_{i=1}^{\infty}(2 i)^{p \alpha_{i}} \\
& =\left(m_{3} e^{\omega_{3} T} K\right)^{|\alpha|}(2 \mathbb{N})^{p \alpha}=c_{3}^{|\alpha|}(2 \mathbb{N})^{p \alpha}
\end{aligned}
$$

where $c_{3}=m_{3} e^{\omega_{3} T} K$. We also used $\prod_{i=1}^{\infty}(2 i)^{p \alpha_{i}}=(2 \mathbb{N})^{p \alpha}$ and $(2 \mathbb{N})^{\varepsilon_{i}}=2 i$. We recall the form of the Catalan numbers $\mathbf{c}_{|\alpha|}=\frac{1}{|\alpha|}\binom{2|\alpha|-2}{|\alpha|-1},|\alpha| \geq 2$. Then, by Lemma 2.4 we obtain

$$
\begin{aligned}
L_{\alpha} & \leq \frac{1}{|\alpha|}\binom{2|\alpha|-2}{|\alpha|-1} \frac{|\alpha|!}{\alpha!} c_{3}^{|\alpha|}(2 \mathbb{N})^{p \alpha} \leq 4^{|\alpha|-1}(2 \mathbb{N})^{2 \alpha} c_{3}^{|\alpha|}(2 \mathbb{N})^{p \alpha} \\
& \leq(2 \mathbb{N})^{p_{3} \alpha}(2 \mathbb{N})^{(2+p) \alpha}
\end{aligned}
$$

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where we used that $4^{|\alpha|-1} c_{3}^{|\alpha|} \leq(2 \mathbb{N})^{p_{3} \alpha}$ for some positive $p_{3}$. Thus, we conclude

$$
L_{\alpha} \leq(2 \mathbb{N})^{\left(p_{3}+p+2\right) \alpha}
$$

Finally, for $q>2 p_{3}+2 p+5$ the statement of the theorem follows from

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} & =\sum_{\alpha \in \mathcal{I}} L_{\alpha}^{2}(2 \mathbb{N})^{-q \alpha} \\
& =L_{\mathbf{0}}^{2}+\sum_{k \in \mathbb{N}} L_{\varepsilon_{k}}^{2}(2 \mathbb{N})^{-q \varepsilon_{k}}+\sum_{|\alpha|>1} L_{\alpha}^{2}(2 \mathbb{N})^{-q \alpha} \\
& \leq M_{3}^{2}+\left(m_{3} e^{w_{3} T} K\right)^{2} \sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(2 p-q) \varepsilon_{k}} \\
& +\sum_{|\alpha|>1}(2 \mathbb{N})^{\left(2\left(p_{3}+p+2\right)-q\right) \alpha}<\infty \tag{2.38}
\end{align*}
$$

i.e. $u \in C([0, T], X) \otimes(S)_{-1,-q}$. Note that in (2.38) the term $\sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(2 p-q) \varepsilon_{k}}$ is finite since $q>2 p+1$ when $q>2 p_{3}+2 p+5$.
(b) We assume, in the second case, that there exists $\alpha \in \mathcal{I},|\alpha| \geq 2$ such that

$$
\begin{equation*}
L_{\alpha}>\sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta} . \tag{2.39}
\end{equation*}
$$

Consider the most complicated case. Then, we would have that the inequality (2.39) is fulfilled for all $\alpha \in \mathcal{I}$. Then, (2.37) reduces to

$$
L_{\alpha} \leq m_{3} e^{w_{3} T}\left(K(2 \mathbb{N})^{p \alpha}+\left(3 M_{3}+1\right) \sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta}\right)
$$

where we used inequality $L_{\beta}>\sum_{0<\gamma<\beta} L_{\beta-\gamma} L_{\gamma}$ for $\beta<\alpha$. Further, we have

$$
L_{\alpha} \leq\left(3 M_{3}+1\right) m_{3} e^{w_{3} T}\left(\frac{K}{3 M_{3}+1}(2 \mathbb{N})^{p \alpha}+\sum_{0<\beta<\alpha} L_{\alpha-\beta} L_{\beta}\right), \quad|\alpha| \geq 2
$$

At this point, we can repeat the proof of Theorem 2.9. Particularly, using the notation $m_{3}^{\prime}=\left(3 M_{3}+1\right) m_{3}$ and $K^{\prime}=\frac{K}{3 M_{3}+1}$, the following inequality

$$
L_{\alpha} \leq m_{3}^{\prime} e^{w_{3} T}\left(K^{\prime}(2 \mathbb{N})^{p \alpha}+\sum_{\mathbf{0}<\beta<\alpha} L_{\alpha-\beta} L_{\beta}\right)
$$

corresponds to the inequality (2.24), since $K^{\prime}<1$, and the proof continues in the same manner as the one from Theorem 2.9, i.e. the proof of solvability of the equation (2.15) with the Wick-square nonlinearity.

Remark 2.11. Note here that if the almost classical solution $u$ to (2.1) satisfies $u \in \mathbb{D}=$ $\operatorname{Dom} \mathbf{A}$ then $u$ is a classical solution to (2.1).

### 2.2 The linear nonautonomous case

Our analysis provides a downright observation for the linear nonautonomous equation

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A}(t) u(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{2.40}\\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

We assume the following:

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(B1) The operator $\mathbf{A}(t): \mathbb{D}^{\prime} \subset X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}, t \in[0, T]$ is a coordinatewise operator depending on $t$ that corresponds to a family of deterministic operators $A_{\alpha}(t): D\left(A_{\alpha}\right) \subset X \rightarrow X, \alpha \in \mathcal{I}$. For every $\alpha \in \mathcal{I}$ the operator family $\left\{A_{\alpha}(t)\right\}_{t \in[0, T]}$ is a stable family of infinitesimal generators of $C_{0}$-semigroups on $X$ with stability constants $m>1$ and $w \in \mathbb{R}$ not depending on $\alpha$, therefore the corresponding evolution systems $S_{\alpha}(t, s)$ satisfy

$$
\left\|S_{\alpha}(t, s)\right\| \leq m e^{w(t-s)} \leq m e^{w T}, \quad 0 \leq s<t \leq T, \quad \alpha \in \mathcal{I} .
$$

The domain $D\left(A_{\alpha}(t)\right)=D$ is independent of $t \in[0, T]$ and $\alpha \in \mathcal{I}$. For every $x \in D$ the function $A_{\alpha}(t) x, t \in[0, T]$ is continuously differentiable in $X$ for each $\alpha \in \mathcal{I}$.
The action of $\mathbf{A}(t), t \in[0, T]$ is given by

$$
\mathbf{A}(t)(u)=\sum_{\alpha \in \mathcal{I}} A_{\alpha}(t)\left(u_{\alpha}\right) H_{\alpha}
$$

for $u \in \mathbb{D}^{\prime} \subseteq D \otimes(S)_{-1}$ of the form (2.2), where
$\mathbb{D}^{\prime}=\left\{u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in D \otimes(S)_{-1}: \exists p_{0} \geq 0, \sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|A_{\alpha}(t)\left(u_{\alpha}\right)\right\|_{X}^{2}(2 \mathbb{N})^{-p_{0} \alpha}<\infty\right\}$.
(B2) The initial value $u^{0}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha} \in \mathbb{D}^{\prime}$, i.e. $u_{\alpha}^{0} \in D$ for every $\alpha \in \mathcal{I}$ and there exists $p \geq 0$ such that

$$
\begin{gathered}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \\
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|A_{\alpha}(t) u_{\alpha}^{0}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty .
\end{gathered}
$$

For the inhomogeneous part $f(t, \omega), \omega \in \Omega, t \in[0, T]$ we assume (A3).
Theorem 2.12. Let the assumptions $(B 1),(B 2)$ and $(A 3)$ be fulfilled. Then there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ to (2.40).

Proof. Applying the Wiener-Itô chaos expansion method to (2.40) we obtain the system of infinitely many deterministic Cauchy problems

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =A_{\alpha}(t) u_{\alpha}(t)+f_{\alpha}(t), \quad t \in(0, T]  \tag{2.41}\\
u_{\alpha}(0) & =u_{\alpha}^{0}, \quad \alpha \in \mathcal{I}
\end{align*}
$$

By virtue of (B1), (B2) and (A3) the Cauchy problem (2.41) fulfills all the assumptions of [20, Theorem 5.3, p. 147] so there exists a unique classical solution $u_{\alpha} \in C^{1}([0, T], X)$ given by

$$
u_{\alpha}(t)=S_{\alpha}(t, 0) u_{\alpha}^{0}+\int_{0}^{t} S_{\alpha}(t, s) f_{\alpha}(s) d s, \quad t \in[0, T]
$$

for all $\alpha \in \mathcal{I}$.
It remains to show that $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha} \in C([0, T], X) \otimes(S)_{-1}$, i.e. that there exists $q>0$ such that $\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}<\infty$.

Without loss of generality, we may assume that the constants $K, p>0$ are such that for all $\alpha \in \mathcal{I}$

$$
\begin{aligned}
\left\|u_{\alpha}^{0}\right\|_{X} & \leq K(2 \mathbb{N})^{p \alpha} \\
\sup _{t \in[0, T]}\left\|f_{\alpha}(t)\right\|_{X} & \leq K(2 \mathbb{N})^{p \alpha} .
\end{aligned}
$$

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Now, for all $\alpha \in \mathcal{I}$, we obtain

$$
\begin{aligned}
\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X} & \leq \sup _{t \in[0, T]}\left\{\left\|S_{\alpha}(t, 0)\right\|\left\|u_{\alpha}^{0}\right\|_{X}+\int_{0}^{t}\left\|S_{\alpha}(t, s)\right\|\left\|f_{\alpha}(s)\right\|_{X} d s\right\} \\
& \leq \sup _{t \in[0, T]}\left\{\left\|S_{\alpha}(t, 0)\right\|\left\|u_{\alpha}^{0}\right\|_{X}+\sup _{s \in[0, t]}\left\|S_{\alpha}(t, s)\right\|\left\|f_{\alpha}(s)\right\|_{X} \int_{0}^{t} d s\right\} \\
& \leq \sup _{t \in[0, T]}\left\{m e^{w t} K(2 \mathbb{N})^{p \alpha}+m e^{w t} K(2 \mathbb{N})^{p \alpha} t\right\} \\
& \leq(1+T) m e^{w T} K(2 \mathbb{N})^{p \alpha} .
\end{aligned}
$$

Finally, for $q>2 p+1$ we obtain

$$
\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \leq\left((1+T) m e^{w T} K\right)^{2} \sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{(2 p-q) \alpha}<\infty
$$

## 3 Extensions and applications

Our results can be extended to a far more general case of stochastic evolution equation of the form

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+p_{n}^{\diamond}(u(t, \omega))+f(t, \omega), \quad t \in(0, T] \\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega \tag{3.1}
\end{align*}
$$

with a Wick-polynomial type of nonlinearity

$$
\begin{equation*}
p_{n}^{\diamond}(u)=\sum_{k=0}^{n} a_{k} u^{\diamond k}=a_{0}+a_{1} u+a_{2} u^{\diamond 2}+a_{3} u^{\diamond 3}+\ldots a_{n} u^{\diamond n}, \tag{3.2}
\end{equation*}
$$

where $a_{n} \neq 0$ and $a_{k}, 0 \leq k \leq n$ are either constants or deterministic functions. Equation (3.1) generalizes equation (2.1) and it can be solved by the very same method presented in the paper, provided that one stipulates that the corresponding deterministic version of (3.1) has a solution and modifies assumption ( $A 4-n$ ) correspondingly. Hence, we replace $(A 4-n)$ with the following assumption:
(A4-pol-n) The Cauchy problem

$$
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+p_{n}\left(u_{\mathbf{0}}(t)\right)+f_{\mathbf{0}}(t), \quad t \in(0, T] ; \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0}
$$

has a classical solution $u_{\mathbf{0}} \in C^{1}([0, T], X)$, where

$$
\begin{equation*}
p_{n}(u)=\sum_{k=0}^{n} a_{k} u^{k}=a_{0}+a_{1} u+a_{2} u^{2}+a_{3} u^{3}+\ldots a_{n} u^{n}, \tag{3.3}
\end{equation*}
$$

is a classical polynomial of degree $n$ corresponding to the Wick-polynomial (3.2).
We extend Theorem 2.8, and for the sake of technical simplicity, present only a procedure for solving (3.1) for $n=3$, but note that the general case may be done mutatis mutandis.

First we note that from the form of the process (2.2) and from the form of its Wickpowers (2.3), as well as from (2.29) we obtain the expansion of the Wick-polynomial

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nonlinearity

$$
\begin{align*}
& p_{3}^{\diamond}(u)=a_{0}+a_{1} u+a_{2} u^{\diamond 2}+a_{3} u^{\diamond 3} \\
& =a_{0} H_{\mathbf{0}}+a_{1}\left(u_{\mathbf{0}} H_{\mathbf{0}}+\sum_{|\alpha|>0} u_{\alpha} H_{\alpha}\right)+a_{2}\left(u_{\mathbf{0}}^{2} H_{\mathbf{0}}+\sum_{|\alpha|>0}\left(2 u_{\mathbf{0}} u_{\alpha}+\sum_{\mathbf{0}<\beta<\alpha} u_{\beta} u_{\alpha-\beta}\right) H_{\alpha}\right)+ \\
& +a_{3}\left(u_{\mathbf{0}}^{3} H_{\mathbf{0}}+\sum_{|\alpha|>0}\left(3 u_{\mathbf{0}}^{2} u_{\alpha}+3 u_{\mathbf{0}} \sum_{0<\beta<\alpha} u_{\alpha-\beta} u_{\beta}+\sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta} u_{\beta-\gamma} u_{\gamma}(t)\right) H_{\alpha}\right) . \tag{3.4}
\end{align*}
$$

When summing up the corresponding coefficients, the expression (3.4) transforms to

$$
\begin{aligned}
p_{3}^{\diamond}(u) & =\left(a_{0}+a_{1} u_{\mathbf{0}}+a_{2} u_{\mathbf{0}}^{2}+a_{3} u_{\mathbf{0}}^{3}\right) H_{\mathbf{0}} \\
& +\sum_{\alpha>\mathbf{0}}\left(\left(3 a_{3} u_{\mathbf{0}}^{2}+2 a_{2} u_{\mathbf{0}}+a_{1}\right) u_{\alpha}+\left(3 a_{3} u_{\mathbf{0}}+a_{2}\right) \sum_{0<\beta<\alpha} u_{\alpha-\beta} u_{\beta}\right. \\
& \left.+a_{3} \sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta} u_{\beta-\gamma} u_{\gamma}\right) H_{\alpha} \\
& =p_{3}\left(u_{\mathbf{0}}\right)+\sum_{\alpha>\mathbf{0}}\left(p_{3}^{\prime}\left(u_{\mathbf{0}}\right) u_{\alpha}+\frac{1}{2!} \cdot p_{3}^{\prime \prime}\left(u_{\mathbf{0}}\right) \sum_{0<\beta<\alpha} u_{\alpha-\beta} u_{\beta}\right. \\
& \left.+\frac{1}{3!} \cdot p_{3}^{\prime \prime \prime}\left(u_{\mathbf{0}}\right) \sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta} u_{\beta-\gamma} u_{\gamma}\right) H_{\alpha}
\end{aligned}
$$

where $p_{3}^{\prime}, p_{3}^{\prime \prime}$ and $p_{3}^{\prime \prime \prime}$ denote the first, the second and the third derivative of the polynomial (3.3), respectively.

Thus, by applying the Wiener-Itô chaos expansion method to the nonlinear stochastic problem (3.1) we obtain the system of infinitely many deterministic Cauchy problems:
$1^{\circ}$ for $\alpha=\mathbf{0}$

$$
\begin{equation*}
\frac{d}{d t} u_{\mathbf{0}}(t)=A_{\mathbf{0}} u_{\mathbf{0}}(t)+p_{3}\left(u_{\mathbf{0}}(t)\right)+f_{\mathbf{0}}(t), \quad u_{\mathbf{0}}(0)=u_{\mathbf{0}}^{0} \tag{3.5}
\end{equation*}
$$

and
$2^{\circ}$ for $\alpha>\mathbf{0}$

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =\left(A_{\alpha}+p_{3}^{\prime}\left(u_{\mathbf{0}}(t)\right) I d\right) u_{\alpha}(t)+\frac{1}{2} p_{3}^{\prime \prime}\left(u_{\mathbf{0}}(t)\right) \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t)+ \\
& +\frac{1}{6} p_{3}^{\prime \prime \prime}\left(u_{\mathbf{0}}(t)\right) \sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)+f_{\alpha}(t)  \tag{3.6}\\
u_{\alpha}(0) & =u_{\alpha}^{0} .
\end{align*}
$$

with $t \in(0, T]$ and $\omega \in \Omega$.
We denote by

$$
\begin{aligned}
B_{\alpha, p_{3}}(t) & =A_{\alpha}+p_{3}^{\prime}\left(u_{\mathbf{0}}(t)\right) I d \quad \text { and } \\
g_{\alpha, p_{3}}(t) & =\frac{1}{2} \cdot p_{3}^{\prime \prime}\left(u_{\mathbf{0}}\right) \sum_{0<\beta<\alpha} u_{\alpha-\beta}(t) u_{\beta}(t) \\
& +\frac{1}{6} \cdot p_{3}^{\prime \prime \prime}\left(u_{\mathbf{0}}\right) \sum_{0<\beta<\alpha} \sum_{0<\gamma<\beta} u_{\alpha-\beta}(t) u_{\beta-\gamma}(t) u_{\gamma}(t)+f_{\alpha}(t),
\end{aligned}
$$

for $t \in(0, T]$ and all $\alpha>\mathbf{0}$. Hence, the problems (3.6) for $\alpha>\mathbf{0}$ can be written in the form

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =B_{\alpha, p_{3}}(t) u_{\alpha}(t)+g_{\alpha, p_{3}}(t), \quad t \in(0, T]  \tag{3.7}\\
u_{\alpha}(0) & =u_{\alpha}^{0}
\end{align*}
$$

Theorem 3.1. Let the assumptions $(A 1)-(A 3)$ and $(A 4-p o l-3)$ be fulfilled. Then, there exists a unique almost classical solution $u \in C([0, T], X) \otimes(S)_{-1}$ to (3.1).
Proof. Under the assumptions $(A 1)-(A 2)$ and the assumption $(A 4-p o l-3)$ that (3.5) has a classical solution in $C^{1}([0, T], X)$, it can be proven (similarly as it was done in Lemma 2.3) that for every $\alpha>\mathbf{0}$ the evolution system (3.7) has a unique classical solution $u_{\alpha} \in C^{1}([0, T], X)$. Then, in order to show that $u$ is an almost classical solution to (3.1), one has to prove that $u \in C([0, T], X) \otimes(S)_{-1}$. Indeed, this can be done in an analogue way as in the proof of Theorem 2.10, with $L_{\mathbf{0}}=\sup _{t \in[0, T]}\left\|u_{\mathbf{0}}(t)\right\|$ and

$$
M_{3}=\max \left\{\sup _{t \in[0, T]}\left\|p_{3}\left(u_{\mathbf{0}}(t)\right)\right\|, \sup _{t \in[0, T]}\left\|p_{3}^{\prime}\left(u_{\mathbf{0}}(t)\right)\right\|, \sup _{t \in[0, T]}\left\|p_{3}^{\prime \prime}\left(u_{\mathbf{0}}(t)\right)\right\|, \sup _{t \in[0, T]}\left\|p_{3}^{\prime \prime \prime}\left(u_{\mathbf{0}}(t)\right)\right\|\right\}
$$

### 3.1 Examples

We present two classes of stochasic reaction-diffusion equations that belong to the class of problems (3.1).

### 3.1.1 Stochastic generalized FitzHugh-Nagumo equation

The nonlinear stochastic evolution equation

$$
\begin{align*}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+u^{\diamond 2}(t, \omega)-u^{\diamond 3}(t, \omega)+f(t, \omega), \quad t \in(0, T]  \tag{3.8}\\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega
\end{align*}
$$

which belongs to the class of generalized FitzHugh-Nagumo equations is an equation of type (3.1). Particularly, for $\mathbf{A}=\triangle$, the corresponding reaction-diffusion deterministic equation

$$
\begin{equation*}
u_{t}=\triangle u(t)+F(u(t)), \quad u(0)=u^{0} \tag{3.9}
\end{equation*}
$$

with a nonlinearity of the form $F(u)=-u(a-u)(b-u)$ is the celebrated FitzHughNagumo equation, which arises in various models of neurophysiology. The equation (3.9) has been introduced by FitzHugh and Nagumo [5, 17] in order to model the conduction of electrical impulses in a nerve axon. A stochastic version of the FitzHugh-Nagumo equation (3.9) was studied in [1], while a control problem for the FitzHugh-Nagumo equation perturbed by coloured Gaussian noise was solved in [3]. Clearly, the equation (3.8) is generalizing (3.9) if we choose $a=0$ and $b=1$ in the form of $F(u)$. For the choice of $a=b=0$ the equation (3.8) reduces to the Fujita type equation (2.1).

Here, by appying Theorem 3.1, we obtain a unique almost classical solution of the equation (3.8).

### 3.1.2 Stochastic generalized Fisher-KPP equation

The deterministic nonlinear equation of the form (3.9) with $F(u)=a u(1-u)$ is called the Fisher equation (also known as the Kolmogorov-Petrovsky-Piskunov equation). Such equations occur in phase transition problems arising in biology, ecology, plasma physics $[4,13]$ etc. Particularly, such an equation provides a deterministic model for the density

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of a population living in an environment with a limited carrying capacity. It also describes the wave progression of an epidemic outbreak or the spread of an advantageous gene within a population. Other applications in medicine involve the modeling of cellular reactions to the introduction of toxins, voltage propagation through a nerve axon, and the process of epidermal wound healing [2]. In other research areas it has been also used to study flame propagation of fire outbreaks, and neutron flux in nuclear reactors.

Stochastic models that include random effects due to some external (enviromental) noise were studied in the framework of white noise analysis [10], where the authors proved the existence of the traveling wave solution. In the same setting, the stochastic KPP equation, i.e. heat equations with semilinear potential and perturbation by a multiplicative noise were considered in [19]. Under suitable assumptions, by applying the Itô calculus, existence of a unique strong traveling wave solution was proven, and an implicit Feyman-Kac-like formula for the solution was presented. Here we consider a generalized Wick-version of the stochastic Fisher-KPP equation

$$
\begin{aligned}
u_{t}(t, \omega) & =\mathbf{A} u(t, \omega)+u(t, \omega)-u^{\diamond 2}(t, \omega)+f(t, \omega), \quad t \in(0, T] \\
u(0, \omega) & =u^{0}(\omega), \quad \omega \in \Omega
\end{aligned}
$$

which can be solved by applying Theorem 3.1.

### 3.2 Conclusion

In this paper we have presented a methodology for solving stochastic evolution equations involving nonlinearities of Wick-polynomial type. However, the applications and extensions of the theory do not stop here. In place of the nonlinearity $u^{\diamond 2}$, one might consider $u \diamond u_{x}$ and with appropriate modifications solve the stochastic Burgers-type equation $u_{t}=u_{x x}+u \diamond u_{x}+f$ or the stochastic KdV equation $u_{t}=u_{x x x}+u \diamond u_{x}+f$, coalesced into the form $u_{t}=\mathbf{A} u+u \diamond u_{x}+f$. One can also replace the nonlinearity $u^{\diamond n}$ by $u \diamond|u|^{n-1}$, where the modulus of a complex-valued stochastic process is understood as $|u|=\sum_{\alpha \in \mathcal{I}}\left|u_{\alpha}\right| H_{\alpha}$, and find explicit solutions to the stochastic nonlinear Schrödinger equation $(i \hbar) u_{t}=\Delta u+u \diamond|u|^{n-1}+f$.

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Acknowledgments. The paper is supported by the following projects and grants: project 174024 financed by the Ministry of Education, Science and Technological Development of the Republic of Serbia, project 451-03-01039/2015-09/26 of the bilateral scientific cooperation between Serbia and Austria, Domus grant 4814/28/2015/HTMT provided by the Hungarian Academy of Sciences, project 142-451-2384 of the Provincial Secretariat for Higher Education and Scientific Research and a research grant for Austrian graduates.

## Chapter 2

## Applications

In this chapter we present applications of the chaos expansion method to the study of optimal control problems. In particular, we consider the stochastic linear quadratic optimal control (SLQR) problem in infinite dimensions. This problem arises naturally in mathematical finance, e.g. in high frequency trading models in option pricing. Solving stochastic optimal control problems is strongly related to the problem of solving backward stochastic differential equations, e.g. if an SLQR problem with random coefficients is considered. The SLQR problem addresses a minimization of a quadratic cost functional subject to a stochastic linear differential state equation. In the finite time horizon case the optimal control is given in a feedback form in terms of the solution of an operator differential Riccati equation, while in the infinite horizon case the optimal control is characterized by the solution of an operator algebraic Riccati equation. In first part of this chapter we present a novel numerical framework for solving SLQR problems using the chaos expansion approach [66, 65]. By applying the method of chaos expansions to the state equation, we obtained a system of deterministic partial differential equations in terms of the coefficients of the state and the control variables. We set up a control problem for each equation, which resulted in a set of deterministic linear quadratic regulator problems. We proved the optimality of the solution expressed in terms of the expansion of these coefficients and compared it to the direct approach. Moreover, we apply this approach to SLQR problems with random coefficients, i.e. the state, control and observation operators are random. We also considered a fractional version of the SLQR problem. By using the fractional isometries defined in Chapter 1, the fractional SLQR problems are transferred to the classical SLQR problems. These results are related to Section 2.1 [66], Section 2.2 [65] and Section 2.3 [68].

The SLQR problem in infinite dimensions was solved by Ichikawa 46 using a dynamic programming approach. Da Prato [25] and Flandoli [30] later considered the SLQR problem for systems driven by analytic semi-
groups with Dirichlet or Neumann boundary controls and with disturbance in the state only. The infinite dimensional SLQR problem with random coefficients has been investigated in 36, 37] along with the associated backward stochastic Riccati equation. We proposed a theoretical framework for the SLQR problem for singular estimates control systems in the presence of noise in the control and in the case of finite time penalization in the performance index [39]. Considering the general setting described in [39, 63], we developed an approximation scheme for solving the control problem and the associated Riccati equation 67]. These results are related to Section 2.4 [39] and Section 2.5 [67].

In addition, we combined the chaos expansion method with splitting methods for solving particular classes of stochastic evolution equations, Section 2.6 [52]. Finally, we present a regularization scheme based on chaos expansions for operator differential algebraic equations with noise disturbances, Section 2.7 3.

## The SLQR problem: a chaos expansion approach

We consider the infinite dimensional stochastic linear quadratic optimal control problem on finite time horizon. The SLQR problem consists of the linear state equation

$$
\begin{equation*}
d y(t)=(\mathbf{A} y(t)+\mathbf{B} u(t)) d t+\mathbf{C} y(t) d B_{t}, \quad y(0)=y^{0}, \quad t \in[0, T], \tag{2.1}
\end{equation*}
$$

with respect to $\mathcal{H}$-valued Brownian motion $B_{t}$ in the classical Gaussian white noise space, and the quadratic cost functional

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{H}}^{2}\right] . \tag{2.2}
\end{equation*}
$$

The operators $\mathbf{A}$ and $\mathbf{C}$ are operators on $\mathcal{H}$ and $\mathbf{B}$ acts from the control space $\mathcal{U}$ to the state space $\mathcal{H}$ and $y^{0}$ is a random variable. Spaces $\mathcal{H}$ and $\mathcal{U}$ are Hilbert spaces. The operators $\mathbf{B}$ and $\mathbf{C}$ are considered to be linear and bounded, while A could be unbounded. The objective is to minimize the quadratic functional (2.2) over all admissible controls $u$ and subject to the condition that $y$ satisfies the state equation (2.1). The operators $\mathbf{R}$ and $\mathbf{G}$ are bounded observation operators taking values in a Hilbert space $\mathcal{H}, \mathbb{E}$ denotes the expectation with respect to the Gaussian measure $\mu$ and $y_{T}=y(T)$. For the class of admissible controls we consider square integrable $\mathcal{U}$-valued adapted controls. The stochastic integration is taken with respect to $\mathcal{H}$-valued Brownian motion and the integral is considered as a Bochner-Pettis type integral [26, 09]. For $\mathbf{C}=0$ the equation (2.1) arises in the deterministic regulator problem and has been well understood in the literature [56, 57, 78. A control process $u^{*}$ is called optimal if it minimizes
the cost functional over all admissible control processes, i.e.,

$$
\min _{u} \mathbf{J}(u)=\mathbf{J}\left(u^{*}\right) .
$$

The corresponding optimal trajectory is denoted by $y^{*}$. Thus, the pair $\left(y^{*}, u^{*}\right)$ is the optimal solution of the considered optimal control problem and is called the optimal pair.

Due to the fundamental theorem of stochastic calculus, for admissible square integrable processes, we consider an equivalent form of the state equation (2.1), its Wick version

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\mathbf{C} y(t) \diamond W_{t}, \quad y(0)=y^{0}, \quad t \in[0, T] . \tag{2.3}
\end{equation*}
$$

We solved the optimal control problem (2.1)-(2.2) by combining the chaos expansion method with the deterministic optimal control theory. The following theorem gives the conditions for the existence of the optimal control in the feedback form using the associated Riccati equation. For more details on existence of mild solutions of (2.1) we refer the reader to [26] and for the optimal control and Riccati feedback synthesis we refer to [46].

Theorem 57 ([26, 46]) Let the following assumptions hold:
(a1) The linear operator $\mathbf{A}$ is an infinitesimal generator of a $C_{0}$-semigroup $\left(e^{\mathbf{A} t}\right)_{t \geq 0}$ on the space $\mathcal{H}$.
(a2) The linear control operator $\mathbf{B}$ is bounded $\mathcal{U} \rightarrow \mathcal{H}$.
(a3) The operators $\mathbf{R}, \mathbf{G}, \mathbf{C}$ are bounded linear operators.
Then, the optimal control $u^{*}$ of the linear quadratic problem (2.1)-(2.2) satisfies the feedback characterization in terms of the optimal state $y^{*}$

$$
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t),
$$

where $\mathbf{P}(t)$ is a positive self-adjoint operator solving the Riccati equation

$$
\begin{array}{r}
\dot{\mathbf{P}}(t)+\mathbf{P}(t) \mathbf{A}+\mathbf{A}^{\star} \mathbf{P}(t)+\mathbf{C}^{\star} \mathbf{P}(t) \mathbf{C}+\mathbf{R}^{\star} \mathbf{R}-\mathbf{P}(t) \mathbf{B B}^{\star} \mathbf{P}(t)=0,  \tag{2.4}\\
\mathbf{P}(T)=\mathbf{G}^{\star} \mathbf{G} .
\end{array}
$$

Here we also invoke the solution of the inhomogeneous deterministic control problem of minimizing the performance index

$$
\begin{equation*}
J(u)=\int_{0}^{T}\left(\|R x\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\|G x(T)\|_{\mathcal{H}}^{2} \tag{2.5}
\end{equation*}
$$

subject to the inhomogeneous differential equation

$$
\begin{equation*}
x^{\prime}(t)=A x(t)+B u(t)+f(t), \quad x(0)=x^{0} . \tag{2.6}
\end{equation*}
$$

Besides the assumptions $(a 1)$ and $(a 2)$, it is enough to assume that $f \in L^{2}((0, T), \mathcal{H})$, to obtain the optimal solution for the state and control $\left(x^{*}, u^{*}\right)$. The feedback form of the optimal control for the inhomogeneous problem 2.5-2.6) is given by

$$
\begin{equation*}
u^{*}(t)=-B^{\star} P_{d}(t) x^{*}(t)-B^{\star} k(t), \tag{2.7}
\end{equation*}
$$

where $P_{d}(t)$ solves the Riccati equation

$$
\left\langle\left(\dot{P}_{d}+P_{d} A+A^{\star} P_{d}+R^{\star} R-P_{d} B B^{\star} P_{d}\right) v, w\right\rangle=0, \quad P_{d}(T) v=G^{\star} G v
$$

for all $v, w$ in $\mathcal{D}(A)$, while $k(t)$ is a solution of the auxiliary differential equation

$$
k^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k(t)+P_{d}(t) f(t)=0
$$

with the boundary conditions $P_{d}(T)=G^{\star} G$ and $k(T)=0$. For the homogeneous problem we refer to [56]. We also refer to [13, 22, 103] for better insight into optimal control theory.

Definition 58 Let $g(t)$ be a $\mathcal{F}_{T}$-predictable Bochner integrable $\mathcal{H}$-valued function.
(1) An $\mathcal{H}$-valued adapted process $y(t)$ is a strong solution of the state equation (2.1) over $[0, T]$ if
(i) $y(t)$ takes values in $D(\mathbf{A}) \cap D(\mathbf{C})$ for almost all $t$ and $\omega$,
(ii) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}}+\|\mathbf{A} y(s)\|_{\mathcal{H}} d s<\infty\right)=1$,
(iii) $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$, and
(iv) for arbitrary $t \in[0, T]$ and $P$-almost surely it satisfies the integral equation

$$
y(t)=y^{0}+\int_{0}^{t} \mathbf{A} y(s) d s+\int_{0}^{t} g(s) d s+\int_{0}^{t} \mathbf{C} y(s) d B_{s}
$$

(2) An $\mathcal{H}$-valued adapted process $y(t)$ is a mild solution of the state equation (2.1) over $[0, T]$ if
$(i)$ the process $y(t)$ takes values in $D(\mathbf{C})$,
(ii) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}} d s<\infty\right)=1$,
(iii) $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$ and
(iv) for arbitrary $t \in[0, T]$ and $P$-almost surely it satisfies the integral equation

$$
y(t)=e^{\mathbf{A} t} y^{0}+\int_{0}^{t} e^{\mathbf{A}(t-s)} g(s) d s+\int_{0}^{t} e^{\mathbf{A}(t-s)} \mathbf{C} y(s) d B_{s}
$$

Note that, under the assumptions of Theorem 57, and given a control process $u \in L^{2}([0, T], \mathcal{U}) \otimes L^{2}(\mu)$, i.e., $g(t)=\mathbf{B} u(t)$, and deterministic initial data, there exits a unique mild solution $y \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\mu)$ of the controlled state equation (2.1), see [26].

The approach developed in [66] combines the method of chaos expansions with the deterministic optimal control theory. We recall that the method of chaos expansions is based on the Wiener-Itô chaos expansion theorem which states that a random variable, respectively a stochastic process, can be expressed as series in terms of an orthogonal basis of stochastic polynomials depending on the probability measure. Particularly, if the underlying probability space is a Gaussian space, then the orthogonal basis of stochastic polynomials is built in terms of the Hermite polynomials and an orthonormal basis of $\mathcal{H}$. The case $\mathcal{H}=\mathcal{L}^{2}(\mathbb{R})$ is very important in applications, where the orthonormal basis $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ can be chosen as the Hermite functions $\left\{\xi_{k}\right\}_{k \in \mathbb{N}}$.

The square integrable processes $y \in \mathcal{L}^{2}([0, T] \times \Omega, \mathcal{H})$ and $u \in \mathcal{L}^{2}([0, T] \times$ $\Omega, \mathcal{U})$ can be represented in their chaos expansion forms

$$
\begin{equation*}
y(t, \omega)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}(t) H_{\alpha}(\omega), \quad u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \tag{2.9}
\end{equation*}
$$

for $t \geq 0, \omega \in \Omega$ and where the coefficients $y_{\alpha} \in \mathcal{L}^{2}([0, T], \mathcal{H})$ and $u_{\alpha} \in$ $\mathcal{L}^{2}([0, T], \mathcal{U})$ for all $\alpha \in \mathcal{I}$. In this way, the deterministic part of a stochastic process is split from its random part. The zero coefficients $y_{0}(t)=\mathbb{E} y(t, \omega)$ and $u_{\mathbf{0}}(t)=E u(t, \omega)$ in (2.9) are the corresponding expectations of $y$ and u. All the operators $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ appearing in the problem (2.1)(2.2) are assumed to be coordinatewise operators, i.e., the action of $\mathbf{A}$ on $y \in \mathcal{L}^{2}([0, T] \times \Omega, \mathcal{H})$ is given by $\mathbf{A} y(t, \omega)=\sum_{\alpha \in \mathcal{I}} A_{\alpha} y_{\alpha}(t) H_{\alpha}(\omega)$,

Theorem 59 ([68]) Let the following assumptions hold:
(A1) The operator $\mathbf{A}: L^{2}([0, T], \mathcal{D}) \otimes L^{2}(\mu) \rightarrow L^{2}([0, T], \mathcal{D}) \otimes L^{2}(\mu)$ is a coordinatewise linear operator that corresponds to the family of deterministic operators $A_{\alpha}: L^{2}([0, T], \mathcal{D}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$, where $A_{\alpha}$ are infinitesimal generators of strongly continuous semigroups $\left(e^{A_{\alpha} t}\right)_{\alpha \in \mathcal{I}}$, $t \geq 0$, defined on a common domain $\mathcal{D}$ that is dense in $\mathcal{H}$, such that for some $m, \theta>0$ and all $\alpha \in \mathcal{I}$

$$
\left\|\left(e^{A_{\alpha} t}\right)_{\alpha}\right\|_{L(\mathcal{H})} \leq m e^{\theta t}, \quad t \geq 0
$$

(A2) The operator $\mathbf{C}: L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\mu) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\mu)$ is a coordinatewise operator corresponding to a family of uniformly bounded deterministic operators $C_{\alpha}: L^{2}([0, T], \mathcal{H}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$.
(A3) The control operator $\mathbf{B}$ is a coordinatewise operator $\mathbf{B}: L^{2}([0, T], \mathcal{U}) \otimes$ $L^{2}(\mu) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\mu)$ that is defined by a family of uniformly
bounded deterministic operators $B_{\alpha}: L^{2}([0, T], \mathcal{U}) \rightarrow L^{2}([0, T], \mathcal{H})$, $\alpha \in \mathcal{I}$.
(A4) Operators $\mathbf{R}$ and $\mathbf{G}$ are bounded coordinatewise operators corresponding to the families of deterministic operators $\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\{G\}_{\alpha \in \mathcal{I}}$ respectively.
$(A 5) \mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$.
Then, the optimal control problem (2.2)-(2.3) has a unique optimal control $u^{*}$ given in the chaos expansion form

$$
\begin{equation*}
u^{*}=-\sum_{\alpha \in \mathcal{I}} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha} \tag{2.10}
\end{equation*}
$$

where $P_{d, \alpha}(t)$ for every $\alpha \in \mathcal{I}$ solves the Riccati equation

$$
\begin{array}{r}
\dot{P}_{d, \alpha}(t)+P_{d, \alpha}(t) A_{\alpha}+A_{\alpha}^{\star} P_{d, \alpha}(t)+R_{\alpha} R_{\alpha}^{\star}-P_{d, \alpha}(t) B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t)=0 \\
P_{d, \alpha}(T)=G_{\alpha}^{\star} G_{\alpha} \tag{2.11}
\end{array}
$$

and $k_{\alpha}(t)$ for each $\alpha \in \mathcal{I}$ solve the auxiliary differential equation
$k_{\alpha}^{\prime}(t)+\left(A_{\alpha}^{\star}-P_{d, \alpha}(t) B_{\alpha} B_{\alpha}^{\star}\right) k_{\alpha}(t)+P_{d, \alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(t) \cdot \mathbf{e}_{i}(t)\right)=0$,
with $k_{\alpha}(T)=0$ and $y^{*}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*} H_{\alpha}$ is the optimal state.
Theorem 59 is an extension of the one from 66], where the case with simple coordinatewise operators was considered. The following theorem gives the characterization of the optimal control

$$
\begin{equation*}
u^{*}(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}=u_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha} \tag{2.13}
\end{equation*}
$$

in terms of the solution of the stochastic Riccati equation.
Theorem 60 ([68]) Let (A1)-(A5) from Theorem 59 hold and let $\mathbf{P}$ be a coordinatewise operator that corresponds to the family of operators $\left\{P_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. Then, the solution of the optimal control problem (2.1)-(2.2) obtained by the chaos expansion approach

$$
\begin{equation*}
u^{*}=-\mathbf{B}^{\star} \mathbf{P}_{d} y^{*}(t)-\mathbf{B}^{\star} \mathcal{K} \tag{2.14}
\end{equation*}
$$

where $\mathbf{P}_{d}(t)$ is a coordinatewise operator corresponding to the deterministic family of operators $\left\{P_{d, \alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\mathcal{K}$ is a stochastic process with coefficients $k_{\alpha}(t)$, i.e., a process of the form $\mathcal{K}=\sum_{\alpha \in \mathcal{I}} k_{\alpha}(t) H_{\alpha}$, with $k_{\mathbf{0}}=0$, is equal to the one obtained by the Riccati approach

$$
\begin{equation*}
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t) \tag{2.15}
\end{equation*}
$$

with a positive self-adjoint operator $\mathbf{P}(t)$ solving the stochastic Riccati equation

$$
\begin{array}{r}
\dot{\mathbf{P}}(t)+\mathbf{P}(t) \mathbf{A}+\mathbf{A}^{\star} \mathbf{P}(t)+\mathbf{C}^{\star} \mathbf{P}(t) \mathbf{C}+\mathbf{R}^{\star} \mathbf{R}-\mathbf{P}(t) \mathbf{B B ^ { \star } \mathbf { P } ( t ) = 0}, \\
\mathbf{P}(T)=\mathbf{G}^{\star} \mathbf{G} \tag{2.16}
\end{array} .
$$

if and only if

$$
\begin{equation*}
C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha} y_{\alpha}^{*}(t)=P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot{ }_{i}(t)\right), \quad|\alpha|>0, k \in \mathbb{N} \tag{2.17}
\end{equation*}
$$

hold for all $t \in[0, T]$.

The condition (2.29) that characterizes the optimality represent the action of the stochastic Riccati operator in each level of the noise. Note that the stochastic Riccati equation (2.16) and the deterministic one 2.8 differ only in the term $C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha}$, i.e., the operator $C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha}, \alpha \in \mathcal{I}$ captures the stochasticity of the equation. Polynomial chaos projects the stochastic part in different levels of singularity, the way that Riccati operator acts in each level is given by 2.29 .

Following the proposed approach the numerical treatment of the SLQR problem relies on solving efficiently Riccati equations arising in the associated deterministic problems. In recent years, numerical methods for solving differential Riccati equations have been proposed [7, 12]. Moreover, the results from 66] were applied also to optimal control problems governing by state equations involving so-called delta noise. Additionally they were extended to SLQR problems with random operators, previously considered by [36, 37].

Although theoretically we have to solve infinitely many control problems, numerically, when approximating the solution by the $p$ th order chaos, we have to solve $\frac{(m+p)!}{m!p!}$ problems in order to achieve the $L^{2}$-convergence. The value of $p$ is in general equal to the number of uncorrelated random variables in the system and $m$ is typically chosen by some heuristic method [50, 102].

Details and a complete study of the SLQR problem with chaos expansion approach are given in Section 2.1 [66].

## The SLQR problem: the infinite horizon case

The infinite dimensional SLQR problem consists of the state equation

$$
\begin{align*}
d y(t) & =(\mathbf{A} y(t)+\mathbf{B} u(t)) d t+\mathbf{C} y(t) d B_{t}, \quad t \geq 0 \\
y(0) & =y^{0} \tag{2.18}
\end{align*}
$$

defined on the state space $\mathcal{H}$, where $\mathbf{A}$ and $\mathbf{C}$ are operators on $\mathcal{H}, \mathbf{B}$ acts from the control space $\mathcal{U}$ to the state space $\mathcal{H}$ and $y^{0}$ is a random variable.

Spaces $\mathcal{H}$ and $\mathcal{U}$ are Hilbert spaces and $\left\{B_{t}\right\}_{t \geq 0}$ is a $\mathcal{H}$-valued Wiener process on a given probability space $(\Omega, \mathcal{F}, \mu)$ in sense of [26]. The operators $\mathbf{B}$ and $\mathbf{C}$ are considered to be linear and bounded, while $\mathbf{A}$ could be unbounded. The objective is to minimize the functional

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{\infty}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t\right] \tag{2.19}
\end{equation*}
$$

over all possible controls $u$ and subject to the condition that $y$ satisfies the state equation (2.18). The operator $\mathbf{R}$ is bounded and takes values in the Hilbert space $\mathcal{H}$. A control process $u^{*}$ is called optimal if it minimizes the cost 2.19 over all admissible control processes $u \in \mathcal{A}$, i.e., for which it holds

$$
\min _{u \in \mathcal{A}} \mathbf{J}(u)=\mathbf{J}\left(u^{*}\right) .
$$

The corresponding trajectory is denoted by $y^{*}$. The pair of stochastic processes $\left(y^{*}, u^{*}\right)$ is called the optimal pair.

The following theorem provides the conditions for the existence of the optimal control in the feedback form by the associated algebraic Riccati equation (ARE). To this approach we are going to refer as standard approach.

Theorem 61 ([27]) Let the following assumptions hold:
(a1) The linear operator $A$ is the infinitesimal generator of a $C_{0}$ semigroup $\left(e^{A t}\right)_{t \geq 0}$ on the space $\mathcal{H}$.
(a2) The linear operator $B$ is bounded $\mathcal{U} \rightarrow \mathcal{H}$.
(a3) The operators $R, C$ are bounded linear operators.
(a4) The system $(A, B, C)$ is stabilizable.
(a5) The system $(A, R, C)$ is detectable.
Then, the optimal control $u^{*}$ of the linear quadratic problem $(2.18)-(2.19)$ satisfies the feedback characterization in terms of the optimal state $y^{*}$

$$
\begin{equation*}
u^{*}(t)=-B^{\star} P y^{*}(t) \tag{2.20}
\end{equation*}
$$

where $P$ is the unique minimal positive self-adjoint operator solving the Riccati equation

$$
\begin{equation*}
P A+A^{\star} P+C^{\star} P C+R^{\star} R-P B B^{\star} P=0 \tag{2.21}
\end{equation*}
$$

We applied the method of chaos expansions for solving 2.18 - 2.19 . The square integrable processes $y \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{H})$ and $u \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{U})$ can be represented in their chaos expansion forms 2.9 for $t \geq 0, \omega \in \Omega$
and where the coefficients $y_{\alpha} \in \mathcal{L}^{2}([0, \infty), \mathcal{H})$ and $u_{\alpha} \in \mathcal{L}^{2}([0, \infty), \mathcal{U})$ for all $\alpha \in \mathcal{I}$. All the operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{R}$ appearing in the problem (2.18)(2.19) are assumed to be simple coordinatewise operators, i.e., the action of $\mathbf{A}$ on $y \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{H})$ is given by $\mathbf{A} y(t, \omega)=\sum_{\alpha \in \mathcal{I}} A y_{\alpha}(t) H_{\alpha}(\omega)$, Hence, by applying the representation forms 2.9) to the equation 2.18) we transform it to a system of deterministic equations. Namely, in a similar way to [45] and [64], the solution of (2.18) can be written in the chaos expansion for 2.9 and its coefficients $y_{\alpha}, \alpha \in \mathcal{I}$ can be computed from

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}} e_{i}(t) \tag{2.22}
\end{equation*}
$$

with $y_{\alpha}(0)=y_{\alpha}^{0}$, where the sum is defined for all $i$ such that the difference of $\alpha-\varepsilon^{(i)}$ is nonnegative. Applying the chaos expansion method to the cost functional (2.19), analogously to [66], one gets a characterization of the optimal control in terms of the expansion coefficients. This is summarized in the following theorem.

Theorem 62 ([65]) Let (a1)-(a5) from Theorem 61 hold. Let $(A, B, R)$ be stabilizable and $\mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$. Then, the following hold:
(a) Solving the problem (2.18)-2.19) is equivalent to solving the deterministic optimal control problems in each $\alpha$-level. Particularly, for $\alpha=\mathbf{0}$ :

$$
\begin{equation*}
\min _{u_{\mathbf{0}}} J\left(u_{\mathbf{0}}\right)=\min _{u_{\mathbf{0}}} \int_{0}^{\infty}\left(\left\|R y_{\mathbf{0}}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\mathbf{0}}(t)\right\|_{\mathcal{U}}^{2}\right) d t \tag{2.23}
\end{equation*}
$$

subject to

$$
\begin{equation*}
y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \tag{2.24}
\end{equation*}
$$

and for $\alpha>\mathbf{0}$ :

$$
\begin{equation*}
\min _{u_{\alpha}} J\left(u_{\alpha}\right)=\min _{u_{\alpha}} \int_{0}^{\infty}\left(\left\|R y_{\alpha}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\alpha}(t)\right\|_{\mathcal{U}}^{2}\right) d t \tag{2.25}
\end{equation*}
$$

subject to 2.22.
(b) The optimal control problem (2.18)-(2.19) has a unique optimal control $u^{*}$ given in the chaos expansion form

$$
\begin{align*}
u^{*}(t) & =-\sum_{\alpha \in \mathcal{I}} B^{\star} P_{d} y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B^{\star} k_{\alpha}(t) H_{\alpha} \\
& =-\mathbf{B}^{\star} \mathbf{P}_{d} y^{*}(t)-\mathbf{B}^{\star} \mathcal{K} \tag{2.26}
\end{align*}
$$

where the operator $\mathbf{P}_{d}$ is the unique minimal positive self-adjoint solution of the $A R E$

$$
\begin{equation*}
\mathbf{P}_{d} \mathbf{A}+\mathbf{A}^{\star} \mathbf{P}_{d}+\mathbf{R R}^{\star}-\mathbf{P}_{d} \mathbf{B} \mathbf{B}^{\star} \mathbf{P}_{d}=0 \tag{2.27}
\end{equation*}
$$

and $\mathcal{K}$ is a stochastic process with the coefficients $k_{\alpha}(t)$ that for all $\alpha \in \mathcal{I}$ solve the auxiliary equations

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+A_{p}^{\star} k_{\alpha}(t)+P_{d}\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) e_{i}(t)\right)=0 \tag{2.28}
\end{equation*}
$$

with the operator $A_{p}^{\star}=A^{\star}-P_{d} B B^{\star}$ and the condition $\lim _{T \rightarrow \infty} k_{\alpha}(T)=0$, and $y^{*}(t)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}$ is the optimal state.

The SLQR problems on finite and infinite horizons are strongly related. In the deterministic setting the infinite horizon problem is studied as a limit of the finite horizon time problem, a similar study holds for the stochastic case and also for the chaos expansion approach. This will be presented somewhere else. The following theorem characterizes the action of the Riccati operator. The recurrence 2.29 can be interpreted as memory property in the noise.

Theorem 63 ([65]) Let the assumptions from Theorem 62 hold. Then, the optimal control $(2.26)$ of $(2.18)-(2.19)$ obtained via the chaos expansion method is equal to the solution (2.20) obtained via the Riccati approach if and only if for all $\alpha>\mathbf{0}$ and $t \geq 0$ it holds

$$
\begin{equation*}
C^{\star} P C y_{\alpha}^{*}(t)=P\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}^{*}(t) e_{i}(t)\right) \tag{2.29}
\end{equation*}
$$

The proposed approach for solving SLQR problems in terms of chaos expansions is not restricted only to problems 2.18 - 2.19 with Gaussian noise, but it can be also applied for more general and non-Gaussian type of noises, e.g. for problems involving colored noise [64]. One needs to replace the base of Hermite polynomials with another class of orthogonal polynomials from the Askey scheme of hypergeometric orthogonal polynomials that corresponds to the specific noise arising in the considered stochastic state equation [102]. More details can be found in Section 2.2 [65].

## The SLQR problem with fractional Brownian motion

We consider a fractional version of the stochastic optimal control problem (2.1)-(2.2). The state equation is linear stochastic differential equation

$$
\begin{equation*}
d \widetilde{y}(t)=(\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)) d t+\widetilde{\mathbf{C}} \widetilde{y}(t) d B_{t}^{(H)} \quad \widetilde{y}(0)=\widetilde{y}^{0}, t \in[0, T] \tag{2.30}
\end{equation*}
$$

with respect to a $\mathcal{H}$-valued fractional Brownian motion in the fractional Gaussian white noise space. The objective is to minimize the quadratic cost
functional

$$
\begin{equation*}
\mathbf{J}^{(H)}(\widetilde{u})=\mathbb{E}_{\mu_{H}}\left[\int_{0}^{T}\left(\|\widetilde{\mathbf{R}} \widetilde{y}\|_{\mathcal{H}}^{2}+\|\widetilde{u}\|_{\mathcal{U}}^{2}\right) d t+\left\|\widetilde{\mathbf{G}} \widetilde{y}_{T}\right\|_{\mathcal{H}}^{2}\right] . \tag{2.31}
\end{equation*}
$$

over all possible controls $\widetilde{u}$ and subject to the condition that $\widetilde{y}$ satisfies 2.30). A control process $\widetilde{u}^{*}$ is called optimal if $\min _{u} \mathbf{J}^{(H)}(\widetilde{u})=\mathbf{J}^{(H)}\left(\widetilde{u}^{*}\right)$. The corresponding trajectory is denoted by $\widetilde{y}^{*}$ and is called optimal. Thus, the pair $\left(\widetilde{y}^{*}, \widetilde{u}^{*}\right)$ is the optimal solution of the problem 2.30 - 2.31 . The operators $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{C}}$ are defined on $\mathcal{H}$ and $\widetilde{\mathbf{B}}$ acts from the control space $\mathcal{U}$ to the state space $\mathcal{H}$ and $\widetilde{y}^{0}$ is a random variable. The operators $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ are considered to be linear and bounded, $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{G}}$ are bounded observation operators taking values in $\mathcal{H}$. Instead of the state equation (2.30), we consider its Wick version

$$
\begin{equation*}
\dot{\widetilde{y}}(t)=\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)+\widetilde{\mathbf{C}} \widetilde{y}(t) \diamond W_{t}^{(H)}, \quad \widetilde{y}(0)=y^{0}, \quad t \in[0, T] \tag{2.32}
\end{equation*}
$$

In Section 2 Theorem 59 we stated conditions under which the stochastic control problem (2.1)-(2.2) has an optimal control given in the feedback form 2.10 . In order to apply this result to the corresponding fractional control problem (2.30)-2.31), we apply the isometry mapping $\mathcal{M}$ [64] to (2.31)-2.32) and transform it to (2.1)-2.2). The solution of the fractional problem is thus obtained from the solution of the corresponding classical problem through the inverse fractional map.
Theorem 64 ([68]) Let the fractional operators $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}, \widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{G}}$ defined on fractional space be coorinatewise operators that correspond to the families $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ respectively. Let the pair $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ be the optimal solution of the fractional stochastic optimal control problem (2.30-2.31). Then, the pair $\left(\mathcal{M} \widetilde{u}^{*}, \mathcal{M} \widetilde{y}^{*}\right)$ is the optimal solution $\left(u^{*}, y^{*}\right)$ of the associated optimal control problem 2.1-(2.2), where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ defined on classical space, are coorinatewise operators that correspond respectively to the same families of deterministic operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. Moreover, if $\left(u^{*}, y^{*}\right)$ is the optimal solution of the stochastic optimal control problem (2.1)-(2.2), then the pair $\left(\mathcal{M}^{-1} u^{*}, \mathcal{M}^{-1} y^{*}\right)$ is the optimal solution $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ of the corresponding fractional optimal control problem 2.30-2.31.

Therefore, the fractional optimal control 2.30-2.31 has an optimal control represented in the feedback form. The optimal solution is obtained from Theorem 59 and Theorem 64 via the inverse fractional mapping $\mathcal{M}^{-1}$. These results are included in Section 2.3 [68].

## The SLQR problem with singular estimates

In this part of the thesis, we consider the stochastic linear quadratic problem in infinite dimensions with state and control dependent noise for the
so-called singular estimate control systems. These systems involve dynamics driven by strongly continuous semigroups and unbounded control actions with the control to state kernel satisfying a singular estimate. Such situation is typical in boundary or point control problems where the action of the control operator is either only densely defined on a control space or its range is outside the state space. In order to quantify the unboundedness of control action-singular estimates play an crucial role. Such estimate describes the amount of blow up of the transfer function. The latter is necessary for a rigorous analysis of control problems and the associated feedback synthesis. We assume that the multiplicative noise operators for the state and the control are bounded. Our study includes the SLQR problem in which disturbance in the control is considered and a final time penalization term is included in the quadratic cost functional, the so-called Bolza problem.

For deterministic systems, the infinite dimensional linear quadratic regulator problem has been studied extensively in the literature [6, 8, 13, 56]. The purpose of the theoretical framework is to address optimal control of systems of partial differential equations. For most systems, the controlling mechanism can only be applied from the interface of the system or at finitely many points or curves [10] which necessitates developing a framework for studying boundary/point control. Such control actions can be captured mathematically using maps which are not bounded with respect to the state space, but take values in a larger dual space. The most natural class of problems where such description has been used are dynamics driven by analytic semigroups. The analyticity property quantifies naturally the blow up of the transfer function when acted upon by an unbounded operator (compatible with fractional powers of the generator). The linear quadratic problem for systems driven by analytic semigroups with these type of control actions were studied by [1, 14, 27, 31, 56]. The situation is much more complicated in the non-analytic case, where there is no natural characterization of singularity other than technical-PDE estimates. However, for some classes of control systems which combine hyperbolic and parabolic dynamics, it has been observed that the control-to-state kernel satisfies a singular estimate which generalizes the case of analytic semigroup dynamics [2, 5, 55, 58, 59. Examples of systems which manifest this type of singular estimate arise frequently in thermo-elastic plate models [11, 18, 60, acoustic-structure interaction equation [5, 9, 60, and fluid-structure interaction models 61. As described above, a deterministic theory of feedback control has been developed for these classes of problems (singular estimate) [54]. However, in the stochastic case the only results available in the literature covering unbounded control actions are the ones dealing with analytic semigroups [25, 36, 30].

The results of this section are related to [39]. There we proved an optimal feedback synthesis along with well-posedness of the Riccati equation. We derived a differential Riccati equation associated with the optimal stochastic linear quadratic control problem, by first showing the existence of a solu-
tion to an expanded system in the integral form of the Riccati equation via a specially crafted fixed point argument. We then proceeded to derive the differential Riccati equation which requires making sense of the weak derivative of the evolution generated by deterministic dynamics with respect to initial time. Here, the obstacle, as in the deterministic case, lies in the fact that the terms of the Riccati equation may not be well defined due to the unboundedness of the control operator. Another difficulty is the finite state penalization which gives rise to possible singularities at the final time and require choosing appropriate spaces to make sense of the quadratic term in the differential Riccati equation [59. Finally, we then used a dynamic programming argument to show that the minimum of the quadratic functional is realized when the control is expressed in feedback form via the solution to the differential Riccati equation. Here, we proceed with the dynamic programming argument on a regularized version of the problem since the Itô formula only applies to $C^{2}$ functions, while the state and control trajectories are not differentiable in the classical sense. For this reason, a forward approach via a maximum principle or a variational method to solve for the optimal control before proceeding to derive the differential Riccati equation is not applicable in this setting.

We consider $(\Omega, \mathcal{F}, P)$ to be a complete probability space. Let $B_{t}$ be a one dimensional real valued stochastic Brownian motion on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_{t}$ the sigma algebra generated by $\left\{B_{\tau}: \tau \leq t\right\}$. We assume that all function spaces are adapted to the filtration $\mathcal{F}_{t}$. We denote by $L_{w}^{2}([s, T], \mathcal{H})$ all stochastic processes $X(t, \omega):[s, T] \times \Omega \rightarrow \mathcal{H}$ such that $\int_{s}^{T}\|X(t)\|_{\mathcal{H}}^{2} d t<\infty$ a.e. in $\Omega$, and $X(t, \cdot)$ is $\mathcal{F}_{t}$-measurable for all $t \in[s, T]$. We also denote by $M_{w}^{2}([s, T], H)$, the space of all strongly measurable square integrable stochastic processes $X:[s, T] \times \Omega \rightarrow \mathcal{H}$ such that $\int_{s}^{T} E\left(\|X(t)\|_{\mathcal{H}}^{2}\right) d t<\infty$, and by $L^{2}\left(\Omega ; H^{1}([s, T], \mathcal{U})\right)$ all strongly measurable square integrable stochastic processes $u:[s, T] \times \Omega \rightarrow \mathcal{U}$ for which it holdes $\int_{s}^{T} E\left(\|u(t)\|_{\mathcal{U}}^{2}\right) d t+$ $\int_{s}^{T} E\left(\left\|u_{t}(t)\right\|_{\mathcal{U}}^{2}\right) d t<\infty$.

We formulate now the optimal control problem in abstract setting. Let the state equation be a stochastic partial differential equation of the form

$$
\begin{align*}
d y(t) & =(A y+B u) d t+(C y+D u) d B_{t}  \tag{2.33}\\
y(s) & =y_{0}
\end{align*}
$$

be defined on a Hilbert state space $\mathcal{H}$, where $A$ and $C$ are operators on $\mathcal{H}$ while $B$ and $D$ are operators acting from the control space $\mathcal{U}$ to the state space $\mathcal{H}$. We take $C$ and $D$ to be bounded operators but $A$ and $B$ could be unbounded. The objective is to minimize the quadratic cost functional

$$
\begin{equation*}
J\left(s, y_{0}, u\right)=\mathbb{E}\left(\int_{s}^{T}\left(\|R y\|_{\mathcal{W}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\|G y(T)\|_{\mathcal{Z}}^{2}\right) \tag{2.34}
\end{equation*}
$$

over all admissible controls $u \in M_{w}^{2}([s, T], \mathcal{U})$, where $R$ and $G$ are bounded linear observation operators taking values in Hilbert spaces $\mathcal{W}$ and $\mathcal{Z}$ respectively. The optimal control and state are denoted by $\left(u^{*}, y^{*}\right)$.

The assumptions we consider are the following:

## Assumptions 1

(1) The linear operator $A$ is an infinitesimal generator of a $C_{0}$-semigroup $e^{A t}$ on the space $\mathcal{H}$.
(2) The linear control operator $B$ acts from $\mathcal{U} \rightarrow\left[\mathcal{D}\left(A^{\star}\right)\right]^{\prime}$ or equivalently $A^{-1} B$ is bounded from $\mathcal{U} \rightarrow \mathcal{H}$.
(3) The noise operator $D: \mathcal{U} \rightarrow \mathcal{H}$ is a bounded linear operator.
(4) There exists a number $\gamma \in(0,1 / 2)$ such that the control to state map kernel $e^{A t} B$ satisfies the singular estimates

$$
\begin{equation*}
\left\|e^{A t} B u\right\|_{\mathcal{H}} \leq \frac{c}{t^{\gamma}}\|u\|_{\mathcal{U}} \tag{2.35}
\end{equation*}
$$

for every $u \in \mathcal{U}$ and $0<t<1$.
(5) The operators $R: \mathcal{H} \rightarrow \mathcal{W}, G: \mathcal{H} \rightarrow \mathcal{Z}$ and $C: \mathcal{H} \rightarrow \mathcal{H}$ are all bounded linear operators.
We first state the result pertaining to existence, regularity and uniqueness of solution to the optimal control problem.
Theorem 65 ([39]) The optimal control problem of minimizing (2.34) subject to the differential equation (2.33) with initial condition $y_{0} \in \mathcal{H}$ has a unique solution $u^{*} \in L^{2}(\Omega, C([s, T], \mathcal{U}))$ and a corresponding optimal state $y^{*} \in L^{2}(\Omega, C([s, T], \mathcal{H}))$.

We next state the result on the feedback form of the optimal control and the associated differential Riccati equation satisfied by the gain operator.

Theorem 66 ([39]) Let Assumptions 1 hold. Then, the optimal control $u^{*}$ has the feedback characterization in terms of the optimal state

$$
u^{*}\left(t, s, y_{0}\right)=-\left(I+D^{\star} P D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) y^{*}(t),
$$

where $P(t) \in C([0, T], \mathcal{L}(\mathcal{H}))$ is a positive self-adjoint operator solving the Riccati equation for every $x, y \in \mathcal{D}(A)$

$$
\begin{array}{r}
\langle\dot{P} x, y\rangle+\langle P A x, y\rangle+\left\langle A^{\star} P x, y\right\rangle+\left\langle C^{\star} P C x, y\right\rangle+\left\langle R^{\star} R x, y\right\rangle \\
-\left\langle\left(B^{\star} P+D^{\star} P C\right)^{\star}\left(I+D^{\star} P D\right)^{-1}\left(B^{\star} P+D^{\star} P C\right) x, y\right\rangle=0,  \tag{2.36}\\
P(T) x=G^{\star} G x,
\end{array}
$$

such that $I+D^{\star} P(t) D>0$.

Specific examples motivating the theory presented above include coupled PDE systems with boundary or point control where hyperbolic and parabolic dynamics are intertwined. These, in particular include thermoelasticity, fluid structure interactions and models arising in structural acoustics [5, 54]. The analysis and results above easily extends to the case $1 / 2 \leq \gamma<1$ when $G=0$. However, for nonzero $G$, this case $1 / 2 \leq \gamma<1$ is more challenging since operator

$$
G L_{T} \equiv G \int_{0}^{T} e^{A(T-\tau)} B d \tau
$$

is no longer bounded $C\left(L^{2}(\Omega), L^{2}([s, T], \mathcal{U})\right) \rightarrow \mathcal{Z}$. In fact, the existence of an optimal control in this case requires closability of $G L_{T}$ [58. Such condition is trivially satisfied when $G$ is bounded and invertible $\mathcal{H} \rightarrow \mathcal{Z}$. In this case, the fixed point argument is no longer applicable.

Moreover, the derivation of the differential Riccati equation 2.36 from the integral Riccati equation involves double singularities at initial and final times in the function $\Phi(t, s) B$, which appears when making sense of the derivative of the evolution with respect to initial time [59]. Note that in the case of deterministic singular estimate control systems, uniqueness of solution to the differential Riccati equation for nonzero $G$ and $\gamma \geq 1 / 2$ in a suitable class of operators [59] is not known, even in the analytic case [58], unless further smoothing properties of $G$ are satisfied.

The results can also be extended to the case when $D$ is unbounded operator satisfying a similar singular estimate condition to that satisfied by $B$, assumption 1. This condition allows the inclusion of systems with noise in the boundary control into the theoretical framework that we developed. In the case when there is no final state penalization, i.e., $G=0$, the value of $\gamma$ in 2.35 could be pushed up to 1 . However, the majority of "non analytic" examples exhibit singularity of the type assumed in 2.35 . For this reason, we focused on this class only. More details are given in Section 2.4 [39].

## The SLQR problem: a numerical approximation framework

In 67] we presented an approximation framework for computing the solution of the stochastic linear quadratic control problem on Hilbert spaces, where we focused on the finite horizon case and the related differential Riccati equations (DREs). Our approximation framework is concerned with singular estimate control systems [55] which model certain coupled systems of parabolic/hyperbolic mixed partial differential equations with boundary or point control. We proved that the solutions of the approximate finitedimensional DREs converge to the solution of the infinite-dimensional DREs. In addition, we proved that the optimal state and control of the approximate finite-dimensional problem converge to the optimal state and control
of the corresponding infinite-dimensional problem. These results are related to Section 2.5 [67.

The deterministic linear quadratic control problem for infinite-dimensional systems has been extensively studied in the literature 13, 14, 56, 57. In particular, approximation schemes for Riccati equations in infinite-dimensional spaces have been proposed in recent years. Chronologically, the first reference is due to Gibson [35, who developed an approximation framework in order to reduce the inherently infinite-dimensional problems to finitedimensional ones using Riccati integral equations. The result proposed by Gibson requires the approximating problems to be defined on the entire original state space which leads to some technical difficulties. Assuming that the dynamics are driven by an analytic semigroup, Banks and Kunisch [8] avoided these difficulties. In the same setting, convergence results for DREs can be found in [7], while results on convergence rates can be found in [51. A complete Riccati theory and convergence analysis for infinite dimensional systems driven by analytic semigroups and a special class of unbounded control operators was developed by Lasiecka and Triggiani in [56]. However, up to our knowledge, convergence results for the stochastic linear quadratic control problem have not been studied in the literature. One of the reasons could be the fact that the computational cost of solving the SLQR problem is much higher compared to the cost in the deterministic case. In this work, we extended the ideas presented in [7, 区, 63] to the SLQR problem. We also avoided technical difficulties related with the fact that Gibson's presentation requires that each of the approximating problems is defined on the whole space.

We consider the infinite dimensional stochastic linear quadratic regulator optimal control problem on Hilbert spaces (2.33)-(2.34) for unbounded control operator $B$, particularly singular estimate control systems, under the Assumptions 1. An optimal feedback synthesis along with well-posedness of the Riccati equation are established in Theorem 65 and Theorem 66 .

We present a general convergence framework developed in [67]. The results given here generalize the deterministic results proposed in [8, 35, 56] to the stochastic case. In particular, the last reference [56] addresses the case of analytic semigroups $e^{A t}$ and unbounded operators $B: \mathcal{U} \rightarrow\left[\mathcal{D}\left(A^{\star}\right)\right]^{\prime}$ satisfying $A^{-\gamma} B: \mathcal{U} \rightarrow \mathcal{H}$, which was generalized by the singular estimate framework 58].

Let $\left(\mathcal{V}^{N}\right)_{N \in \mathbb{N}}$, be a sequence of finite-dimensional linear subspaces of $\mathcal{H} \cap \mathcal{D}\left(B^{\star}\right)$ and let

$$
\Pi^{N}: \mathcal{H} \rightarrow \mathcal{V}^{N}, \quad N \in \mathbb{N}
$$

be the canonical orthogonal projections. Assume that for every $N \in \mathbb{N}$ the operator $A^{N} \in \mathcal{L}\left(\mathcal{V}^{N}\right)$ is an infinitesimal generator of a $C_{0}$-semigroup $e^{A^{N} t}$ on $\mathcal{V}^{N}$ and thus $\left(e^{A^{N} t}\right)_{N \in \mathbb{N}}$ is a sequence of strongly continuous semigroups
on $\mathcal{V}^{N}$. Given operators $B^{N} \in \mathcal{L}\left(\mathcal{U}, \mathcal{V}^{N}\right), G^{N}, Q^{N}, C^{N} \in \mathcal{L}\left(\mathcal{V}^{N}\right)$, we consider the family of finite dimensional stochastic LQR problems on $\mathcal{V}^{N}$

$$
\begin{align*}
d y^{N}(t) & =\left(A^{N} y^{N}(t)+B^{N} u(t)\right) d t+\left(C^{N} y^{N}(t)+D^{N} u(t)\right) d B_{t}, \\
y^{N}(0) & =y_{0}^{N} \tag{2.37}
\end{align*}
$$

and the cost functional

$$
\begin{equation*}
J^{N}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\left\|Q^{N} y^{N}\right\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|G^{N} y^{N}(T)\right\|_{\mathcal{H}}^{2}\right] . \tag{2.38}
\end{equation*}
$$

The optimal control is given in feedback form by

$$
u_{*}^{N}(t)=-\left(I+D^{N^{\star}} P^{N}(t) D^{N}\right)^{-1}\left(B^{N^{\star}} P^{N}(t)+D^{N^{\star}} P^{N}(t) C^{N}\right) y_{*}^{N}(t)
$$

where $P^{N}(t) \in \mathcal{L}\left(\mathcal{V}^{N}\right)$ is the unique self-adjoint solution of the differential Riccati equation:

$$
\begin{array}{r}
\dot{P}^{N}+P^{N} A^{N}+A^{N^{\star}} P^{N}+C^{N^{\star}} P^{N} C^{N}+Q^{N^{\star}} Q^{N} \\
-\left(B^{N^{\star}} P^{N}+D^{N^{\star}} P^{N} C^{N}\right)^{\star}\left(I+D^{N^{\star}} P^{N} D^{N}\right)^{-1}\left(B^{N^{\star}} P^{N}+D^{N^{\star}} P^{N} C^{N}\right)=0, \\
P^{N}(T)=G^{N^{\star}} G^{N} \tag{2.39}
\end{array}
$$

such that $I+D^{N^{\star}} P^{N} D^{N}>0$ and $y_{*}^{N}(t)$ is the optimal state [103].
We impose the following assumptions on the approximation operators:

## Assumptions 2

(1) For all $\varphi \in \mathcal{H}$, the semigroups $e^{A^{N} t} \Pi^{N} \varphi$ converges in $\mathcal{H}$ to $e^{A t} \varphi$ uniformly on $[0, T]$ and in particular there exists $N_{0} \in \mathbb{N}$ such that for $N \geq N_{0}$, we have

$$
\left\|\left(e^{A^{N} t} \Pi^{N}-e^{A t}\right) x\right\|_{\mathcal{H}} \leq \frac{c}{N}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H} .
$$

(2) For all $\varphi \in \mathcal{H}$, the semigroups $e^{A^{N \star} t} \Pi^{N} \varphi$ converge in $\mathcal{H}$ to $e^{A^{\star} t} \varphi$ uniformly on $[0, T]$ and in particular for $N \geq N_{0}$

$$
\left\|\left(e^{A^{N \star} t} \Pi^{N}-e^{A^{\star} t}\right) x\right\|_{\mathcal{H}} \leq \frac{c}{N}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H} .
$$

(3) For all $x \in \mathcal{V}^{N}$ we have for $N \geq N_{0}$

$$
\left\|B^{N \star} \Pi^{N} x\right\|_{\mathcal{U}} \leq c N^{\gamma}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H} .
$$

(4) The projections $\Pi^{N}$ satisfy the convergence estimate for $N \geq N_{0}$

$$
\left\|B^{\star}\left(\Pi^{N}-I\right) x\right\|_{\mathcal{U}} \leq \frac{c}{N}\|x\|_{\mathcal{D}\left(B^{\star}\right)}, \quad \forall x \in \mathcal{D}\left(B^{\star}\right) .
$$

(5) For all $x \in \mathcal{D}\left(B^{\star}\right), B^{N \star} \Pi^{N} x$ converges to $B^{\star} x$ in $\mathcal{U}$ and for $N \geq N_{0}$

$$
\left\|\left(B^{\star}-B^{N \star} \Pi^{N}\right) x\right\|_{\mathcal{U}} \leq \frac{c}{N}\|x\|_{\mathcal{D}\left(B^{\star}\right)}, \quad \forall x \in \mathcal{D}\left(B^{\star}\right)
$$

(6) The approximations $B^{N \star}$ satisfy the uniform singular estimate

$$
\begin{equation*}
\left\|B^{N \star} e^{A^{N \star} t} \Pi^{N} x\right\|_{\mathcal{U}} \leq \frac{c}{t^{\gamma}}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H} \tag{2.40}
\end{equation*}
$$

for $N \geq N_{0}$ and some $\gamma \in\left(0, \frac{1}{2}\right)$.
(7) For all $v \in \mathcal{U}, D^{N} v \rightarrow D v$ in $\mathcal{H}$ and for all $\varphi \in \mathcal{H}$, we have $D^{N \star} \Pi^{N} \varphi \rightarrow D^{\star} \varphi$ in $\mathcal{U}$ such that for $N \geq N_{0}$

$$
\left\|\left(D^{N}-D\right) v\right\|_{\mathcal{H}} \leq \frac{c}{N}\|v\|_{\mathcal{U}}, \quad \forall v \in \mathcal{U}
$$

and

$$
\left\|\left(D^{N \star} \Pi^{N}-D^{\star}\right) \varphi\right\|_{\mathcal{U}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

(8) For all $\varphi \in \mathcal{H}$, we have $C^{N} \Pi^{N} \varphi \rightarrow C \varphi$ and $C^{N \star} \Pi^{N} \varphi \rightarrow C^{\star} \varphi$ in $\mathcal{H}$ such that for $N \geq N_{0}$

$$
\left\|\left(C^{N} \Pi^{N}-C\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H}
$$

and

$$
\left\|\left(C^{N \star} \Pi^{N}-C^{\star}\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H}
$$

(9) For all $\varphi \in \mathcal{H}$, we have $Q^{N} \Pi^{N} \varphi \rightarrow Q \varphi$ and $Q^{N \star} \Pi^{N} \varphi \rightarrow Q^{\star} \varphi$ in $\mathcal{H}$ such that for $N \geq N_{0}$

$$
\left\|\left(Q^{N} \Pi^{N}-Q\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H},
$$

and

$$
\left\|\left(Q^{N \star} \Pi^{N}-Q^{\star}\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

(10) For all $\varphi \in \mathcal{H}$, we have $G^{N} \Pi^{N} \varphi \rightarrow G \varphi$ and $G^{N \star} \Pi^{N} \varphi \rightarrow G^{\star} \varphi$ in $\mathcal{H}$ such that for $N \geq N_{0}$

$$
\left\|\left(G^{N} \Pi^{N}-G\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H}
$$

and

$$
\left\|\left(G^{N \star} \Pi^{N}-G^{\star}\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

The assumption (1) implies that $\Pi^{N} \varphi \rightarrow \varphi$ for all $\varphi \in \mathcal{H}$, which indicates the sense in which the subspaces $\mathcal{V}^{N}$ approximate $\mathcal{H}$.

We next state the main convergence results showing convergence of the solution of the approximate differential Riccati equation 2.39 to the solution of the original Riccati equation 2.36.
Theorem 67 ([67]) Under Assumptions 1 and Assumptions 2, for $\varphi \in \mathcal{H}$, $P^{N}(t) \Pi^{N} \varphi \rightarrow P(t) \varphi$ uniformly on $[0, T]$ in $\mathcal{H}$ as $N \rightarrow \infty$, and in particular

$$
\left\|P^{N}(t) \Pi^{N} \varphi-P(t) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N^{1-\gamma}}\|\varphi\|_{\mathcal{H}}
$$

for $N \geq N_{0}$ and for all $t \in[0, T]$. Moreover, it holds

$$
\left\|B^{N \star} P^{N}(t) \Pi^{N} \varphi-B^{\star} P(t) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N^{1-\gamma}(T-t)^{\gamma}}\|\varphi\|_{\mathcal{H}}
$$

The second result 67) establishes convergence of the optimal pair $u_{*}^{N}$ and $y_{*}^{N}$ of the $N$ problem (2.37) and 2.38) to the optimal pair $u_{*}$ and $y_{*}$ of (2.33) and 2.34).

Theorem 68 ([67]) Under Assumptions 1 and Assumptions 2 and given the condition $\mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right)<\infty$, we have

$$
y_{*}^{N} \rightarrow y_{*} \quad \text { uniformly as } N \rightarrow \infty \text { on }[0, T] \text { in } L^{2}(\Omega, \mathcal{H})
$$

and in particular

$$
\mathbb{E}\left(\left\|y_{*}^{N}\left(t, y_{0}^{N}\right)-y_{*}\left(t, y_{0}\right)\right\|_{\mathcal{H}}^{2}\right) \leq \frac{c}{N^{2(1-\gamma)}} \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right), \quad \forall t \in[0, T]
$$

while

$$
u_{*}^{N} \rightarrow u_{*} \quad \text { uniformly as } N \rightarrow \infty \text { on }[0, T-\epsilon] \text { in } L^{2}(\Omega, \mathcal{U}), \quad \epsilon>0
$$

and in particular

$$
\mathbb{E}\left(\left\|u_{*}^{N}\left(t, y_{0}^{N}\right)-u_{*}\left(t, y_{0}\right)\right\|_{\mathcal{U}}^{2}\right) \leq \frac{c}{N^{2(1-\gamma)}(T-t)^{2 \gamma}} \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right), \quad \forall t \in[0, T]
$$

The approximation framework we have proposed holds with no modification for the case in which only disturbance in the state is considered, i.e. for $D=0$. Our results can be also extended to the non-autonomous case, i.e. the case in which stochastic partial differential equations of the form (2.33) have time-varying coefficients. Approximation results for the deterministic non-autonomous case can be found in [7, 35]. The Riccati equation (2.36) arising in the stochastic linear quadratic control problem is deterministic. Thus, the convergence analysis was developed in the same framework as for Riccati equations arising in the deterministic case.

The proposed approximation scheme could be extended to optimal control problems governed by more general state equations, e.g. when stochastic perturbations are of Wick type within white noise framework. More details are given in Section 2.5 67.

## A splitting/polynomial chaos expansion approach for stochastic evolution equations

Splitting methods are numerical methods for solving differential equations, both ordinary and partial differential equations (PDEs), involving operators that are decomposable into a sum of (differential) operators. Exponential splitting methods are applied in cases when the explicit solution of a splitted equation can be computed. Such computations often rely on applying fast Fourier techniques, see for instant [97]. Resolvent splitting is used in cases when the splitted equation cannot be solved explicitly [41, 91; here we consider this type of methods. Typical examples include time-dependent Schrödinger equation with smooth potential, cubic nonlinear Schrödinger equation (dispersive optical fibers) and nonlinear reaction-diffusion (advection) equations. The splitting methods have been applied to stochastic problems, e.g. for incompressible Stokes equation [20]. In this work we present novel approach for solving stochastic parabolic evolution problems that combines deterministic splitting methods and the chaos expansion method. We consider stochastic evolution equations of the form

$$
\begin{align*}
d u(t) & =((A+B) u(t)+f(t)) d t+(C u(t)+g(t)) d B(t) \\
u(0) & =u^{0}, \tag{2.41}
\end{align*}
$$

where $A, B$ and $C$ are differential operators acting on Hilbert space valued stochastic processes, $\left\{B_{t}\right\}_{t \geq 0}$ is a cylindrical Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f$ and $g$ are deterministic functions. In [79] equation (2.41) involving Gaussian noise terms was solved in an appropriate weighted Wiener chaos space. The deterministic problem that corresponds to (2.41), i.e., the case where $C=0$ and $g=0$, for particular $A u=\partial_{x}\left(a \partial_{x} u\right)$, $B u=\partial_{y}\left(b \partial_{y} u\right)$ and $f$ was studied in [29]. We consider equation (2.41) involving a non-Gaussian noise term. Namely, we consider inhomogeneous parabolic evolution equations involving the operators that can be split in $A+B$ and uniformly distributed random inputs. These equations, can be also written in the form

$$
\begin{align*}
u_{t}(t, x, \omega) & =(A+B) u(t, x, \omega)+G(t, x, \omega) \\
u(0, x, \omega) & =u^{0}(x, \omega), \tag{2.42}
\end{align*}
$$

where $G$ represents the noise term and $u$ is the solution, see e.g. [43, 76, 77, [79]. The existence of a random parameter $\omega$ is due to uncertainties coming from initial conditions and/or a random force term. Therefore, the solution is considered to be a stochastic process.

Stochastic processes with finite second moments on white noise spaces can be represented in series expansion form in terms of a family of orthogonal stochastic polynomials. The classes of orthogonal polynomials are chosen depending on the underlying probability measure [42, 43]. Namely,
the Askey scheme of hypergeometric orthogonal polynomials and the Sheffer system [94, 95 ] can be used to define several discrete and continuous distribution types [102]. We considered problems with non-Gaussian random inputs where the noise term is uniformly distributed. It is known that in order to obtain a square integrable solution of (2.41) with deterministic initial condition, it is enough to assume that the operator $A-\frac{1}{2} C C^{*}$ is elliptic and that the stochastic part (the noise term) is sufficiently regular [26]. In this work, the assumptions on the input data for problem 2.42 will be set such that the existence of a square integrable solution is always established. We do not consider solutions which are generalized stochastic processes as in [76, 79], since our focus is on numerical treatment.

If (2.41) does have a sufficiently regular solution, this solution can be projected on an orthonormal basis in some Hilbert space, resulting in a system of equations for the corresponding Fourier coefficients. Thus, we use the method of chaos expansions to define the solution of (2.41) as a formal Fourier series with the coefficients computed by solving the corresponding system of deterministic PDEs [79]. With this method, the deterministic part of a solution is separated from its random part, i.e., it corresponds to the deterministic method of the separation of variables in PDEs. By construction, the solution is strong in the probabilistic sense. It is uniquely determined by the coefficients, initial condition and the noise term. The coefficients in the Fourier series are uniquely determined by the equation (2.41) and are computed by solving the corresponding lower-triangular system of deterministic parabolic equations.

Practical application of the Wiener polynomial chaos involves two truncations, truncation with respect to the number of the random variables and truncation with respect to the order of the orthogonal Askey polynomials used (in the particular case considered, the Legendre polynomials). More details are given in Section 2.6 [52].

## Stochastic operator differential algebraic equations

In this section we consider stochastic operator differential algebraic equations (ODAEs), i.e. a stochastic differential equation subject to an algebraic constraint

$$
\begin{equation*}
\dot{y}+\mathbf{K} y+\mathbf{B}^{*} u=f, \quad \mathbf{B} y=g \tag{2.43}
\end{equation*}
$$

where the stochastic operator $\mathbf{K}$ is a coordinatewise operator such that the corresponding deterministic operators $\left\{K_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ are densely defined on a given Hilbert space $X$. In the special case when $\mathbf{B}=\mathbb{D}$ and $\mathbf{B}^{*}=\delta$ the system (2.43) transforms to

$$
\begin{equation*}
\dot{y}+\mathbf{K} y+\delta u=f, \quad \mathbb{D} y=g \tag{2.44}
\end{equation*}
$$

with the initial condition $\mathbb{E} y=y^{0}$ and given stochastic processes $f$ and $g$. In the following theorem we consider a more general equation than (2.44). Namely,

$$
\begin{equation*}
\dot{y}=\mathbf{A} y+\mathbf{T} \diamond y+\delta u+f, \quad \mathbb{D} y=g \tag{2.45}
\end{equation*}
$$

which was studied in 64, 68].
Theorem $69([64,68])$ Let $\rho \in[0,1]$. Let A : $X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ be a coordinatewise operator corresponding to a uniformly bounded family of deterministic operators $A_{\alpha}: X \rightarrow X, \alpha \in \mathcal{I}$ and $\mathbf{T}$ be a coordinatewise operator that corresponds to a polynomially bounded family of operators $T_{\alpha}$ : $X \rightarrow X, \alpha \in \mathcal{I}$. Let $g=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{\alpha, k} \xi_{k} H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ such that its coefficients $g_{\alpha k}$ satisfy the condition 1.35 and $f \in X \otimes(S)_{-\rho}$. Let $y^{0} \in X, y^{1} \in X$ be given and the actions $A_{\mathbf{0}} y^{0}$ and $T_{\mathbf{0}} y^{0}$ defined such that $\mathbb{E} f=A_{\mathbf{0}} y^{0}+T_{\mathbf{0}} y^{0}$. Then, the system 2.45 with the initial conditions $\mathbb{E} y=y^{0}$ and $\mathbb{E} \dot{y}=y^{1}$, has unique solution pair $y \in X \otimes(S)_{-\rho}$ and $u \in$ $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ given respectively by

$$
\begin{align*}
y & =y^{0}+\sum_{|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} g_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} \quad \text { and }  \tag{2.46}\\
u & =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{v_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} \tag{2.47}
\end{align*}
$$

where $v=\dot{y}-\mathbf{A} y-\mathbf{T} \diamond y-f$.
A similar result to the one given in Theorem 69 was proved in 3 for semi explicit ODAEs with noise arising in fluid dynamics. Details and a complete study of the regularization of of these equations are given in Section 2.7 [3].

# THE STOCHASTIC LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM IN HILBERT SPACES: A POLYNOMIAL CHAOS APPROACH 

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#### Abstract

We consider the stochastic linear quadratic optimal control problem for state equations of the Itô-Skorokhod type, where the dynamics are driven by strongly continuous semigroup. We provide a numerical framework for solving the control problem using a polynomial chaos expansion approach in white noise setting. After applying polynomial chaos expansion to the state equation, we obtain a system of infinitely many deterministic partial differential equations in terms of the coefficients of the state and the control variables. We set up a control problem for each equation, which results in a set of deterministic linear quadratic regulator problems. Solving these control problems, we find optimal coefficients for the state and the control. We prove the optimality of the solution expressed in terms of the expansion of these coefficients compared to a direct approach. Moreover, we apply our result to a fully stochastic problem, in which the state, control and observation operators can be random, and we also consider an extension to state equations with memory noise.


1. Introduction. Stochastic optimization of infinite dimensional systems arise in many applications, and has become a very active research field in recent years. For finite dimensional systems, extensive results in the field can be found for instance in $[15,63]$. In particular, the linear quadratic regulator problem (LQR) has been well studied in deterministic setting. The stochastic analogue in finite dimensions was first solved by Wonham and Kushner in the 1960's [32, 60, 61]. In the infinite dimensional setting, the stochastic linear quadratic regulator (SLQR) problem was first treated by Ichikawa for systems driven by strongly continuous semigroups and bounded control and noise operators [27], where a full Riccati synthesis of the

[^6]problem analogous to that obtained in finite dimesnions was developed. In later works, Flandoli and Da Prato considered the problem in the analytic semigroups framework for Neumann or Dirichlet type control operators, which represent parabolic PDE with boundary controls [12, 14]. For systems with singular estimates which model a certain class of coupled parabolic/ hyperbolic PDEs, the stochastic linear quadratic problem has been studied by Hafizoglu [22]. In [23], the results were extended to the case including the disturbance in the control and nonzero $G$ (Bolza problem). An approximation scheme for solving the control problem and the associated Riccati equation was also introduced in [39]. Other results have been proposed for systems with stochastic coefficients in [20, 21].

In this work, we consider a polynomial chaos approach for solving the infinite dimensional SLQR problem. The aim is to provide a numerical framework that can be used to obtain efficient numerical solutions to the stochastic linear quadratic problem (or a generalized version of it) which consists of the state equation

$$
\begin{equation*}
d y(t)=(\mathbf{A} y(t)+\mathbf{B} u(t)) d t+\mathbf{C} y(t) d W(t), \quad y(0)=y^{0}, \quad t \in[0, T] \tag{1}
\end{equation*}
$$

defined on Hilbert state space $\mathcal{H}$, where $\mathbf{A}$ and $\mathbf{C}$ are operators on $\mathcal{H}$ and $\mathbf{B}$ acts from the control space $\mathcal{U}$ to the state space $\mathcal{H}$ and $y^{0}$ is a random variable. Spaces $\mathcal{H}$ and $\mathcal{U}$ are Hilbert spaces. Process $W(t)$ is an $\mathcal{H}$-valued Brownian motion. The operators $\mathbf{B}$ and $\mathbf{C}$ are considered to be linear and bounded, while $\mathbf{A}$ could be unbounded. The objective is to minimize the functional

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{H}}^{2}\right] \tag{2}
\end{equation*}
$$

over all possible controls $u$ and subject to the condition that $y$ satisfies the state equation (1). Operators $\mathbf{R}$ and $\mathbf{G}$ are bounded observation operators taking values in $\mathcal{H}, \mathbb{E}$ denotes the expectation and $y_{T}=y(T)$. A control process $u^{*}$ is called optimal if it minimizes the cost (2) over all control processes, i.e. for which it holds

$$
\min _{u} \mathbf{J}(u)=\mathbf{J}\left(u^{*}\right)
$$

The corresponding trajectory is denoted by $y^{*}$. Thus, the pair $\left(y^{*}, u^{*}\right)$ is the optimal solution of the problem (1)-(2) and is called the optimal pair.

First of all, note that state equation (1) can be written in an equivalent abstract form as

$$
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\mathbf{C} y(t) \diamond \dot{W}(t), \quad y(0)=y^{0}, \quad t \in[0, T]
$$

where $\diamond$ denotes the Wick product and $\dot{W}(t)$ an $\mathcal{H}$-valued white noise process. In order to preserve mean dynamics in (1), we represent the random perturbation as a stochastic convolution and obtain the Wick-version of the state equation. Using the Wick product instead of the usual pointwise multiplication we are able to establish a new approach for solving optimal control problems based on the application of the chaos expansion method. Since each square integrable stochastic process $v$ on Gaussian white noise probability space has a unique chaos expansion representation in a Fourier-Hermite orthogonal polynomial basis, $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}$ with deterministic coefficients $v_{\alpha}$, we are able to split the deterministic effects from the randomness and to reduce the original stochastic problem to a family of deterministic ones. The Wick product of two processes $v$ and $h$ is a process given in the
chaos expansion form

$$
v \diamond h=\left(\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}\right) \diamond\left(\sum_{\beta \in \mathcal{I}} h_{\beta} H_{\beta}\right)=\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha \leq \gamma} v_{\alpha} h_{\gamma-\alpha}\right) H_{\gamma} .
$$

Moreover, the relation

$$
\mathbb{E}(v \diamond h)=\mathbb{E}(v) \cdot \mathbb{E}(h)
$$

holds regardless of whether $v$ and $h$ are independent or not. The expectation of the Wick version of the state equation satisfies the corresponding deterministic optimal control problem. Recall, if at least one of the processes $v, h$ is deterministic, then their Wick product and ordinary product coincide, i.e. $v \diamond h=v \cdot h$. Historically, the Wick product first arose in quantum physics, as a renormalization operation, and later played an important role in many problems involving stochastic partial differential equations, in the theory of stochastic integration [19, 25]. By introducing the Wick product $\diamond$ in the considered stochastic problem, one uses an Itô-Skorokhod interpretation of the SPDE. The study of Wick versions of stochastic equations, both linear and nonlinear, together with the study of probabilistic properties of obtained solutions and the comparison with the properties of solutions of corresponding initial equations, can be found in $[8,25,45,48,58]$.

In this work we combine known results of control theory for the SLQR problem with white noise analysis methods. Particularly, in order to characterize the optimal solution in terms of the polynomial chaos, we apply the chaos expansion method to (1)-(2). Since the control operator $\mathbf{B}$ is bounded, we apply the results from [27]. Then, we state the sufficient and necessary condition for the existence of the optimal solution of the considered SLQR problem in terms of the coefficients of the chaos and the solution of the Riccati equation. Theorem 3.1 and Theorem 3.2 are the main contribution of the paper.

Our approach can be generalized to different types of state equations. Always assuming that we are working with linear equations we can consider that operators in the equation are random, see Section 4.2. Another generalization is to consider a different type of noise. In particular, we will discuss in detail how the proposed approach can be extended if we are dealing with noise with memory, which is a special type of noise that is represented in terms of a stochastic integral [11], i.e. we consider the state equation of the form

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\delta(\mathbf{C} y(t)), \quad t \in[0, T] \tag{3}
\end{equation*}
$$

with $y(0)=y^{0}$, where $\delta$ represents the Itô-Skorokhod integral. Moreover, we analyze a problem with an even more general type of noise with memory, which is given by

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\delta_{t}(\mathbf{C} y(t)), \quad t \in[0, T] \tag{4}
\end{equation*}
$$

with $y(0)=y^{0}$, where $\delta_{t}(\mathbf{C} y)$ is the integral process. Optimal control problems involving equations of type (3) and (4) have applications in economics and finance and have been recently studied in [11] using the stochastic maximum principle. Note that, since the argument of the stochastic integral is given as an action of $\mathbf{C}$ on $y$, the evolution equation (3) and (4), each respectively contains a memory property. The disturbance in (3) is a zero mean random variable for all $t \in[0, T]$, while in (4) the perturbation is given via a zero mean stochastic process. We point out that up to our knowledge there is no numerical algorithm for solving these problems. The method proposed in this paper is pioneer in this aspect too.

Polynomial chaos which was first introduced by Wiener in 1938 [59], has recently been used in engineering applications to quantify evolving uncertainty in systems,
see e.g. [18]. Using polynomial chaos, a stochastic system can be represented as a deterministic system with higher dimensionality, but the computational cost is reduced since extensive sampling is no longer required to capture the uncertainty. Only recently, there have been few works on the application of polynomial chaos in stochastic control of engineering systems (finite dimensional) [26, 49, 55]. Some very recent works in particular have been concerned with a polynomial chaos approach to linear control systems modeled by the stochastic LQR, see [16, 17, 57]

White noise analysis was introduced by Hida and furter developed by many authors [25, 45]. In order to build spaces of stochastic test and generalized functions, one has to use series decompositions via orthogonal functions as a basis, with certain weight sequences. Depending on the stochastic measure, this basis can be represented as a family of orthogonal polynomials. The classical Hida approach suggests to start with a nuclear space $\mathcal{E}$ and its dual $\mathcal{E}^{\prime}$, such that

$$
\mathcal{E} \subseteq L^{2}(\mathbb{R}) \subseteq \mathcal{E}^{\prime}
$$

and then take the basic probability space to be $\Omega=\mathcal{E}^{\prime}$ endowed with the Borel sigma algebra of the weak topology and an appropriate probability measure $\mathbb{P}[24,25]$. In this work we deal with a Gaussian white noise space. Thus, the underlying measure is the Gaussian measure. The corresponding orthogonal polynomial basis is constructed using the Hermite polynomials and any orthogonal basis of $L^{2}(\mathbb{R})$. In this case $\mathcal{E}$ and $\mathcal{E}^{\prime}$ are the Schwartz spaces of rapidly decreasing test functions $S(\mathbb{R})$ and tempered distributions $S^{\prime}(\mathbb{R})$ respectively.

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions. They have no point value for $\omega \in \Omega$, only an average value with respect to a test random variable. Following the idea of the construction of $S^{\prime}(\mathbb{R})$ as an inductive limit space over $L^{2}(\mathbb{R})$ with appropriate weights, one can define stochastic generalized random variable spaces over $L^{2}(\Omega)$ by adding certain weights in the convergence condition of the series expansion. Several spaces of this type, weighted by a sequence $q=\left(q_{\alpha}\right)_{\alpha \in \mathcal{I}}$, denoted by $(Q)_{-\rho}$, for $\rho \in[0,1]$ were described in [41]. Thus a Gel' fand triplet

$$
(Q)_{\rho} \subseteq L^{2}(\mathbb{P}) \subseteq(Q)_{\rho}
$$

is obtained, where the inclusions are continuous. The most common weights and spaces appearing in applications are $q_{\alpha}=(2 \mathbb{N})^{\alpha}$ which correspond to the Kondratiev spaces of stochastic test functions $(S)_{\rho}$ and stochastic generalized functions $(S)_{-\rho}$, and exponential weights $q_{\alpha}=e^{(2 \mathbb{N})^{\alpha}}$ linked with the exponential growth spaces of stochastic test functions $\exp (S)_{\rho}$ and stochastic generalized functions $\exp (S)_{-\rho}$. Note that, following ideas from financial mathematics, fractional white noise spaces could be constructed by replacing Brownian motion with fractional Brownian motion [25, 41], or more general with Lévy processes .

The problem of pointwise multiplication of generalized stochastic functions in white noise analysis is overcome by introducing the Wick product. The most important property of the Wick multiplication is its relation to the Ito-Skorokhod integration [25]. In Section 3 we express the diffusion component of (1) in terms of the Wick product as well as in terms of the Itô-Skorokhod integral.

In white noise setting, the Skorokhod integral $\delta$ represents an extension of the Itô integral from a set of adapted processes to a set of non-adaptive processes. They coincide on the set of adapted processes. It is an adjoint operator of the Malliavin derivative $\mathbb{D}$. Their composition is known as the Ornstein-Uhlenbeck operator $\mathcal{R}$ and is a self-adjoint operator on $L^{2}(\Omega)$ that has the elements of the
orthogonal basis as its eigenvalues. These operators are the main operators of an infinite dimensional stochastic calculus of variations called the Malliavin calculus [50]. Classes of elliptic and evolution stochastic differential equations (SDEs) that involve operators of the Malliavin calculus within white noise framework were recently studed in $[46,40,44,38,48,58]$. In $[42,43]$ it was proved that the Malliavin derivative indicates the rate of change in time between the ordinary product and the Wick product. In this paper, we consider stochastic optimal control problems with stochastic perturbations given in an integral form. Moreover, we interpret multiplication as a Wick-type multiplication. By use of the Wiener-Itô chaos expansion representations of integrals we are able to achieve new results.

The chaos expansion methodology is a very useful technique for solving many types of SDEs [40, 44, 45]. The main statistical properties of the solution, its mean, variance, higher moments, can be calculated from the formulas involving only the coefficients of the chaos expansion representation [46, 58]. Moreover, numerical methods for SDEs and uncertainty quantification based on the polynomial chaos approach have become very popular in recent years. They are highly efficient in practical computations providing fast convergence and high accuracy. For instance, in order to apply the stochastic Galerkin method, the derivation of explicit equations for the polynomial chaos coefficients is required. This is, as in the general chaos expansion, highly nontrivial and sometimes impossible. On the other hand, an analytical representation of the solution allows for all statistical information to be retrieved directly, e.g. mean, covariance function and even sensitivity coefficients, see $[47,62]$ and references therein for a detailed explanation.

In order to illustrate our approach, we consider the stochastic linear quadratic problem (1)-(2). In [23, 39], the disturbance in the control and the state is given by a convolution operator. In [44], the authors solve evolution equations involving stochastic convolution operators by combining the chaos expansion approach and the deterministic theory of semigroups in white noise framework. In this paper we will follow the ideas provided in [44] and apply the polynomial chaos expansion to the state equation, and obtain a system of infinitely many deterministic partial differential equations in terms of the coefficients of the state and the control. For each equation we set up a control problem which then gives rise to a system of infinitely many deterministic LQR problems. Solving each control problem, we find optimal coefficients for the state $y$ and the control $u$. Summing up all obtained optimal coefficients in the chaos expansion representations of the state and the control we obtain the pair $\tilde{y}$ and $\tilde{u}$. We investigate the optimality of the solutions $\tilde{y}$ and $\tilde{u}$ and then formulate a necessary and sufficient condition for the existence of the optimal solution of the initial SQLR problem in terms of coefficients, Theorem 3.1 and Theorem 3.2.

In the first part of the paper, we deal with simple coordinatewise operators (deterministic operators) while in the second part of the paper we extend our ideas to the fully stochastic problem, i.e. we allow the operators in the state equation and the cost function to be random. Our approach "chaos expansion+optimization" can be applied to open loop control systems and in general to optimization problems in the same setting.

The paper is organized as follows: In Section 2, we briefly introduce basic concepts, results and notations on the infinite dimensional deterministic and stochastic LQR problems, solutions, white noise analysis and chaos expansions. Then, in Section 3 we apply polynomial chaos methodology to the state equation and set up
linear quadratic control problems in terms of the coefficients and discuss the optimality of the solutions expressed in terms of the expansion of these coefficients. We prove the existence of the optimal control in the feedback form and give the optimality condition. Applications are included in Section 4, an example of a stochastic optimal control involving state equation with memory. We also discuss our approach for a general LQR with random coefficients and provide some application of an infinite dimensional control system from strucuture-acoustics. Finally, in Section 5 we discuss the numerical implementation of the proposed approach.
2. Basic concepts and notations. Let $\mathcal{U}$ and $\mathcal{H}$ be separable Hilbert spaces of controls and states respectively with norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{H}}$, generated by the corresponding scalar products. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\left(w_{t}\right)_{t \geq 0}$ be a real valued one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\left(\overline{\mathcal{F}}_{t}\right)_{t \geq 0}$ be the complete right continuous $\sigma$-algebra generated by $\left(w_{t}\right)_{t \geq 0}$. We assume that all function spaces are adapted to the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$, i.e. we consider only $\mathcal{F}_{t}$-predictable processes. Let $L^{2}(\Omega, \mathbb{P})=L^{2}(\Omega, \mathcal{F}, \mathbb{P})$ be a Hilbert space of square integrable real valued random variables endowed with the norm $\|F\|_{L^{2}(\Omega, \mathbb{P})}^{2}=\mathbb{E}_{\mathbb{P}}\left(F^{2}\right)=\mathbb{E}\left(F^{2}\right)$, for $F \in L^{2}(\Omega, \mathbb{P})$, induced by the scalar product $(F, G)_{L^{2}(\Omega, \mathbb{P})}=\mathbb{E}_{\mathbb{P}}(F G)$, for $F, G \in L^{2}(\Omega, \mathbb{P})$, and $\mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to the measure $\mathbb{P}$. From here onwards, we will omit the measure and write in short $L^{2}(\Omega, \mathbb{P})=L^{2}(\mathbb{P})$ and $\mathbb{E}$ for the expectation.

We denote by $L^{2}(\Omega, \mathcal{U})$ a Hilbert space of $\mathcal{U}$-valued square integrable random variables and by $L^{2}([0, T] \times \Omega, \mathcal{U})$ we denote a Hilbert space of square integrable $\mathcal{F}_{T}$-predictable $\mathcal{U}$-valued stochastic processes $u$ endowed with the norm

$$
\|u\|_{L^{2}([0, T] \times \Omega, \mathcal{U})}^{2}=\int_{0}^{T} \mathbb{E}\left(\|u(t)\|_{\mathcal{U}}^{2}\right) d t
$$

Since $\mathcal{U}$ is a separable Hilbert space, the spaces $L^{2}([0, T] \times \Omega, \mathcal{U})$ and $L^{2}\left([0, T], L^{2}(\Omega\right.$, $\mathcal{U})$ ) are isomorphic [43]. Moreover, an $\mathcal{H}$-valued Brownian motion is denoted by $\left(W_{t}\right)_{t \geq 0}$.

We denote by $L^{2}([0, T] \times \Omega, \mathcal{H})$ all $\mathcal{H}$-valued stochastic processes $X(t, \omega):[0, T] \times$ $\Omega \rightarrow \mathcal{H}$ such that $\int_{0}^{T}\|X(t)\|_{\mathcal{H}}^{2} d t<\infty$ a.e. in $\Omega$ and $X(t, \cdot)$ is $\mathcal{F}_{t}$-measurable $\forall t \in[0, T]$. We also denote by $\mathcal{M}^{2}([0, T] \times \Omega, \mathcal{H})$, the space of all strongly measurable $\mathcal{H}$-valued square integrable stochastic processes $X:[0, T] \times \Omega \rightarrow \mathcal{H}$ such that $\int_{0}^{T} \mathbb{E}\left(\|X(t)\|_{\mathcal{H}}^{2}\right) d t<\infty$. Let $C\left([0, T], L^{2}(\Omega, \mathcal{H})\right)$ be a Hilbert space of $\mathcal{F}_{T^{-}}$ predictable continuous $\mathcal{H}$-valued stochastic processes $y$ endowed with the norm

$$
\|y\|_{C\left([0, T], L^{2}(\Omega, \mathcal{H})\right.}^{2}=\sup _{t \in[0, T]} \mathbb{E}\left(\|y(t)\|_{\mathcal{H}}^{2}\right) .
$$

2.1. The SLQR problem: Existence of solution. The infinite dimensional SLQR optimal control problem on Hilbert spaces is given by the state equation (1), subject to the quadratic cost functional (2). The dynamics of the problem, the operator $\mathbf{A}$, is deterministic and represents an infinitesimal generator of a strongly continuous semigroup $\left(e^{\mathbf{A} t}\right)_{t \geq 0}$ on the state space $\mathcal{H}$. Operators $\mathbf{A}$ and $\mathbf{C}$ are operators on $\mathcal{H}$, while operator $\mathbf{B}$ is the operator acting from the control space $\mathcal{U}$ to the state space $\mathcal{H}$. We take operator $\mathbf{C}$ to be linear and bounded. We assume operators $\mathbf{R}$ and $\mathbf{G}$ to be linear and bounded operators on the space $\mathcal{W}$ and $\mathcal{Z}$ respectively. We denote by $\mathcal{D}(\mathbf{S})$ the domain of a certain operator $\mathbf{S}$, and by $\mathbf{S}^{*}$ the adjoint operator of $\mathbf{S}$.

The aim of the stochastic linear quadratic problem is to minimize the cost functional $\mathbf{J}(u)$ over a set of square integrable controls $u \in L^{2}([0, T] \times \Omega, \mathcal{U})$, which are adapted to the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$.

The following theorem gives the conditions for the existence of the optimal control in the feedback form using the associated Riccati equation. For more details on existence of mild solutions to the SDE (1) we refer to [13] and for the optimal control and Riccati feedback synthesis we refer the reader to [27].

Theorem 2.1. ([13, 27]) Let the following assumptions hold:
(a1) The linear operator $\mathbf{A}$ is the infinitesimal generator of a $C_{0}-\operatorname{semigroup}\left(e^{\mathbf{A} t}\right)_{t \geq 0}$ on the space $\mathcal{H}$.
(a2) The linear control operator $\mathbf{B}$ is bounded $\mathcal{U} \rightarrow \mathcal{H}$.
(a3) The operators $\mathbf{R}, \mathbf{G}, \mathbf{C}$ are bounded linear operators.
Then the optimal control $u^{*}$ of the linear quadratic problem (1)-(2) satisfies the feedback characterization in terms of the optimal state $y^{*}$

$$
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t)
$$

where $\mathbf{P}(t)$ is a positive self-adjoint operator solving the Riccati equation

$$
\begin{array}{r}
\dot{\mathbf{P}}(t)+\mathbf{P}(t) \mathbf{A}+\mathbf{A}^{\star} \mathbf{P}(t)+\mathbf{C}^{\star} \mathbf{P}(t) \mathbf{C}+\mathbf{R}^{\star} \mathbf{R}-\mathbf{P}(t) \mathbf{B B}^{\star} \mathbf{P}(t)=0,  \tag{5}\\
\mathbf{P}(T)=\mathbf{G}^{\star} \mathbf{G} .
\end{array}
$$

2.1.1. Inhomogeneous deterministic $L Q R$ problem. Here we invoke the solution to the inhomogeneous deterministic control problem of minimizing the performance index

$$
\begin{equation*}
J(u)=\int_{0}^{T}\left(\|R x\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\|G x(T)\|_{\mathcal{H}}^{2} \tag{6}
\end{equation*}
$$

subject to the inhomogeneous differential equation

$$
\begin{equation*}
x^{\prime}(t)=A x(t)+B u(t)+f(t), \quad x(0)=x^{0}, \tag{7}
\end{equation*}
$$

under the same assumptions on $A$ and $B$. For the homogeneous problem, case $f=0$, we refer to [34], and we refer to [36] where the inhomogeneous optimal control problem for singular estimate type systems was considered. It is enough to assume that $f \in L^{2}((0, T), \mathcal{H})$, to obtain the solution for the optimal state and control $\left(x^{*}, u^{*}\right)$. The feedback form of the optimal control for the inhomogeneous problem (6)-(7) is given by

$$
u^{*}(t)=-B^{\star} P_{d}(t) x^{*}(t)-B^{\star} k(t)
$$

where $P_{d}(t)$ solves the Riccati equation

$$
\begin{align*}
\left\langle\left(\dot{P}_{d}+P_{d} A+A^{\star} P_{d}+R^{\star} R-P_{d} B B^{\star} P_{d}\right) v, w\right\rangle & =0,  \tag{8}\\
P_{d}(T) v & =G^{\star} G v
\end{align*}
$$

for all $v, w$ in $\mathcal{D}(A)$, while $k(t)$ is a solution to the auxiliary differential equation

$$
k^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k(t)+P_{d}(t) f(t)=0
$$

with the boundary conditions $P_{d}(T)=G^{\star} G$ and $k(T)=0$.
2.1.2. Strong and mild solutions. Let $g(t)$ be a $\mathcal{F}_{T}$-predictable Bochner integrable $\mathcal{H}$-valued function. An $\mathcal{H}$-valued adapted process $y(t)$ is a strong solution of the state equation (1) over [0,T] if:
(1) $y(t)$ takes values in $\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{C})$ for almost all $t$ and $\omega$;
(2) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}}+\|\mathbf{A} y(s)\|_{\mathcal{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$;
(3) for arbitrary $t \in[0, T]$ and $\mathbb{P}$-almost surely, it satisfies the integral equation

$$
y(t)=y^{0}+\int_{0}^{t} \mathbf{A} y(s) d s+\int_{0}^{t} g(s) d s+\int_{0}^{t} \mathbf{C} y(s) d W_{s}
$$

An $\mathcal{H}$-valued adapted process $y(t)$ is a mild solution of the state equation

$$
d y(t)=(\mathbf{A} y(t)+g(t)) d t+\mathbf{C} y(t) d W(t), \quad y(0)=y^{0}
$$

over $[0, T]$ if:
(1) $y(t)$ takes values in $\mathcal{D}(\mathbf{C})$;
(2) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$;
(3) for arbitrary $t \in[0, T]$ and $\mathbb{P}$-almost surely, it satisfies the integral equation

$$
y(t)=e^{\mathbf{A} t} y^{0}+\int_{0}^{t} e^{\mathbf{A}(t-s)} g(s) d s+\int_{0}^{t} e^{\mathbf{A}(t-s)} \mathbf{C} y(s) d W_{s}
$$

Mild solutions are the limits of strong solutions. In the case of a deterministic state equation, i.e. for $C=0$, a mild solution $y \in L^{2}([0, T] ; \mathcal{H})$ can be written in the form

$$
y(t)=e^{\mathbf{A} t} y^{0}+\int_{0}^{t} e^{\mathbf{A}(t-s)} g(s) d s, \quad t \in[0, T]
$$

Note that, under the assumptions of the Theorem 2.1, and given a control $u \in$ $L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)$, i.e. $g(t)=\mathbf{B} u(t)$, and the deterministic initial data $y^{0} \in \mathcal{H}$, there exits a unique mild solution $y \in L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)$ to the controlled state equation (1), cf. [13].
2.2. White noise analysis and chaos expansions. In this section we recall briefly some basic facts from white noise analysis that are needed in our analysis. Denote by $h_{n}(x)=(-1)^{n} e^{\frac{x^{2}}{2}} \frac{d^{n}}{d x^{n}}\left(e^{-\frac{x^{2}}{2}}\right), n \in \mathbb{N}_{0}, \mathbb{N}_{0}=\mathbb{N} \cup\{0\}$, the family of Hermite polynomials and

$$
\xi_{n}(x)=\frac{1}{\sqrt[4]{\pi} \sqrt{(n-1)!}} e^{-\frac{x^{2}}{2}} h_{n-1}(\sqrt{2} x), \quad n \in \mathbb{N}
$$

the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^{2}(\mathbb{R})$ with respect to the Lebesgue measure. These functions are the eigenfunctions for the harmonic oscillator in quantum mechanics. The Hermite functions satisfy the recurrent formula

$$
h_{n+1}(x)=x h_{n}(x)-n h_{n-1}(x), \quad n \in \mathbb{N}, x \in \mathbb{R}
$$

and $h_{n}^{\prime}(x)=n h_{n-1}(x)$, for $n \in \mathbb{N}$ and $h_{0}(x)=1$, while for the Hermite functions the identity formula for derivatives

$$
\xi_{n}^{\prime}(x)=\sqrt{\frac{n}{2}} \xi_{n-1}(x)-\sqrt{\frac{n+1}{2}} \xi_{n+1}(x), \quad x \in \mathbb{R}
$$

holds. Moreover,

$$
\left|\xi_{n}(x)\right| \leq\left\{\begin{array}{c}
c n^{-\frac{1}{2}},|x| \leq 2 \sqrt{n} \\
c e^{-\gamma x},|x|>2 \sqrt{n}
\end{array}\right.
$$

for constants $c$ and $\gamma$ independent of $n$. Clearly, $\xi_{n}, n \in \mathbb{N}$ belong to the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$, i.e. they decay faster than polynomial of any power. The Schwartz spaces can be characterized in terms of the Hermite basis in the following manner: The space of rapidly decreasing functions as a projective limit space $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$, where $S_{l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k} \in L^{2}(\mathbb{R}):\|f\|_{l}^{2}=\right.$ $\left.\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{N}_{0}$ and the space of tempered distributions as an inductive limit space $S^{\prime}(\mathbb{R})=\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$, where $S_{-l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k}:\|f\|_{l}^{2}=\right.$ $\left.\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{-l}<\infty\right\}, l \in \mathbb{N}_{0}$. Also, we have a Gel' fand triple

$$
S(\mathbb{R}) \subseteq L^{2}(\mathbb{R}) \subseteq S^{\prime}(\mathbb{R})
$$

with continuous inclusions.
2.2.1. White noise space. Following the ideas of Hida from [24], we construct white noise probability space. Particularly, we take $\mathcal{E}=S(\mathbb{R})$ the space of rapidly decreasing functions and its dual space $\mathcal{E}^{\prime}=S^{\prime}(\mathbb{R})$ the space of tempered distributions. By $\mathcal{B}$ we denote the Borel sigma algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ the Gaussian white noise measure corresponding to the characteristic function

$$
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \phi \in S(\mathbb{R})
$$

given by the Bochner-Minlos theorem, where $\langle\omega, \phi\rangle$ denotes the dual pairing between a tempered distribution $\omega \in S^{\prime}(\mathbb{R})$ and a test function $\phi \in S(\mathbb{R})$. Thus, the basic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$.

Denote by $\mathcal{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ the set of sequences of non-negative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right), \alpha_{i} \in \mathbb{N}_{0}$, $i=1,2, \ldots, m, m \in \mathbb{N}$. For $k \in \mathbb{N}$, the $k$ th unit vector is $\varepsilon^{(k)}=(0, \cdots, 0,1,0, \cdots)$ and the zero vector is $\mathbf{0}=(0,0, \ldots, 0, .$.$) . The length of a multi-index \alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. We say $\alpha \geq \beta$ if $\alpha_{k} \geq \beta_{k}, k \in \mathbb{N}$. In that case $\alpha-\beta=\left(\alpha_{1}-\beta_{1}, \alpha_{2}-\beta_{2}, \ldots\right)$. For $\alpha<\beta$ the difference $\alpha-\beta$ is not defined. Particularly, we have $\alpha-\varepsilon^{(k)}=\left(\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0, \ldots\right), k \in \mathbb{N}$.

We define by

$$
\begin{equation*}
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in \mathcal{I} \tag{9}
\end{equation*}
$$

the Fourier-Hermite polynomials. They form an orthogonal basis of the separable Hilbert space $L^{2}(\Omega)$ and $\left\|H_{\alpha}\right\|_{L^{2}(\Omega)}^{2}=\alpha$ ! holds. In particular, $H_{0}(\omega)=1$ and for the $k$ th unit vector $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$, see [25].

From the Wiener-Itô chaos expansion theorem it follows that each random variable $F \in L^{2}(\Omega)$ has a unique representation of the form

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega),
$$

$\omega \in \Omega, a_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}$, such that it holds $\|F\|_{L^{2}(\Omega)}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2} \alpha!<\infty$.
The space spanned by $\left\{H_{\alpha}:|\alpha|=k\right\}$ is called the Wiener chaos of order $k$ and is denoted by $\mathcal{H}_{k}, k \in \mathbb{N}_{0}$. Thus, $\mathcal{H}_{0}$ is the set of constant random variables, i.e. for $\alpha=\mathbf{0}$ we obtain the expectation of a certain random variable. The space $\mathcal{H}_{1}$ consists of linear combinations of elements $\langle\omega, \cdot\rangle$ (for example Brownian motion and singular white noise are elements of the Wiener chaos of the first order chaos) and the space $\bigoplus_{j=0}^{k} \mathcal{H}_{j}$ is the set of random variables of the form $p(\langle\omega, \cdot\rangle)$, where $p$ is
a polynomial of degree $n \leq k$ with real coefficients. This implies that each $\mathcal{H}_{k}$ is a finite-dimensional subspace of $L^{2}(\Omega)$. Moreover,

$$
L^{2}(\Omega)=\bigoplus_{k=0}^{\infty} \mathcal{H}_{k},
$$

where the sum is an orthogonal sum [25].
Remark 1. In this paper, the considered white noise space $(\Omega, \mathcal{F}, \mathbb{P})$ is the Gaussian white noise space, where the measure $\mathbb{P}=\mu$ is the Gaussian measure. The Fourier-Hermite polynomials (9) form an orthogonal basis of the Hilbert space $L^{2}(\Omega, \mathbb{P})=L^{2}(\Omega, \mu)$. The further analysis will also hold for other types of white noise spaces or fractional white noise spaces, for which the corresponding Hilbert space $L^{2}(\Omega, \mathbb{P})$ has an orthogonal polynomial basis. For example, for Poisson measure $\mathbb{P}=\nu$, the Charlier polynomials form an orthogonal polynomial basis of the space $L^{2}(\Omega, \nu)$. Note here that there exists a unitary mapping between $L^{2}(\mu)$ and $L^{2}(\nu)$ [41]. In general, one can work with the Askey-scheme of hypergeometric orthogonal polynomials and the Sheffer system [56]. Therefore the presented analysis can be provided in the same manner in all these cases.

Let $\mathcal{H}$ be a real separable Hilbert space with the scalar product $\langle\cdot, \cdot\rangle_{\mathcal{H}}$, and let $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ be one orthonormal basis in $\mathcal{H}$. The space of $\mathcal{H}$-valued square integrable random variables can be represented as $L^{2}(\Omega, \mathcal{H})=\bigoplus_{k=0}^{\infty} \mathcal{H}_{k}(\mathcal{H})$, i.e. each $F \in$ $L^{2}(\Omega, \mathcal{H})$ has a chaos expansion representation of the form

$$
F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}=\sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}} f_{\alpha, k} e_{k}\right) H_{\alpha}
$$

for $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} e_{k} \in \mathcal{H}, \alpha \in \mathcal{I}, f_{\alpha, k} \in \mathbb{R}$, such that it holds

$$
\|F\|_{L^{2}(\Omega, \mathcal{H})}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{\mathcal{H}}^{2} \alpha!=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k}^{2} \alpha!<\infty .
$$

One of the typical complications that arise in solving SDEs is the blowup of $L^{2}$ norms of processes, i.e. their infinite variance. Therefore, the weighted spaces in which the considered equation has a solution have to be introduced. For example, such spaces are the Kondratiev spaces $(S)_{-\rho}, \rho \in[0,1]$ of generalized random variables, which represent the stochastic analogue of Schwartz spaces as generalized function spaces. The largest space of Kondratiev stochastic distributions is $(S)_{-1}$, obtained for $\rho=1$.

Now we introduce the Wick product $\diamond$ of random variables. For $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$ and $G=\sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta}$ the element $F \diamond G$ is called the Wick product of $F$ and $G$ and is given in the form

$$
\begin{equation*}
F \diamond G=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} H_{\alpha+\beta}=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha \leq \gamma} f_{\alpha} g_{\gamma-\alpha} H_{\gamma} . \tag{10}
\end{equation*}
$$

It is well known that the Kondratiev spaces $(S)_{1}$ and $(S)_{-1}$ are closed under the Wick multiplication. The Wick product is a commutative, associative operation, distributive with respect to addition. In particular, for the orthogonal polynomial basis of $L^{2}(\Omega)$ we have $H_{\alpha} \diamond H_{\beta}=H_{\alpha+\beta}$, for $\alpha, \beta \in \mathcal{I}$. Whenever $F, G$ and $F \diamond G$ are integrable it holds $\mathbb{E}(F \diamond G)=\mathbb{E}(F) \cdot \mathbb{E}(G)$, without independence requirement
$[25,43]$. The ordinary product $F \cdot G$ of random variables $F, G \in L^{2}(\Omega)$ is defined by using the multiplication formula

$$
\begin{aligned}
H_{\alpha}(\omega) \cdot H_{\beta}(\omega) & =\sum_{0 \leq \gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}(\omega), \\
& =F \diamond G+\sum_{0<\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}(\omega), \quad \alpha, \beta \in \mathcal{I} .
\end{aligned}
$$

2.2.2. Stochastic processes. Since a square integrable stochastic process is defined as a measurable mapping $[0, T] \rightarrow L^{2}(\Omega)$, then by a generalized stochastic process we consider a measurable mapping from $[0, T]$ into a Kondratiev space $(S)_{-1}$. The chaos expansion representation of generalized stochastic process $F$ follows from the Wiener-Itô chaos expansion theorem. A process $F$ can be represented in the form

$$
\begin{equation*}
F_{t}(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega), \quad t \in[0, T] \tag{11}
\end{equation*}
$$

where $f_{\alpha}, \alpha \in \mathcal{I}$ are measurable real functions and there exists $p \in \mathbb{N}_{0}$ such that for all $t \in[0, T]$

$$
\begin{equation*}
\|F\|_{(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left|f_{\alpha}(t)\right|^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{12}
\end{equation*}
$$

If $\mathcal{H}$ is a real separable Hilbert space, then the expansion (11) holds also for $\mathcal{H}$ valued stochastic processes, for $f_{\alpha} \in \mathcal{H}$. Particularly, for $F \in L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}$ the condition (12) transforms to the following

$$
\|F\|_{L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

for some $p \in \mathbb{N}_{0}$.
For example, one dimensional real valued Brownian motion can be represented in the chaos expansion form $w_{t}(\omega)=\sum_{k=1}^{\infty}\left(\int_{0}^{t} \xi_{k}(s) d s\right) H_{\varepsilon^{(k)}}(\omega), t \geq 0$. For each $t$ it is an element of $L^{2}(\Omega)$. Singular real valued white noise is defined by the formal chaos expansion $\dot{w}_{t}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega)$. From $\left\|\dot{w}_{t}\right\|_{L^{2}(\Omega)}^{2}=\sum_{k=1}^{\infty}\left|\xi_{k}(t)\right|^{2}>$ $\sum_{k=1}^{\infty} \frac{1}{k}=\infty$ and $\left\|\dot{w}_{t}\right\|_{(S)_{-1,-p}}^{2}=\sum_{k=1}^{\infty}\left|\xi_{k}(t)\right|^{2}(2 k)^{-p}<\infty$, for $p>1$ it follows that singular white noise is an element of the space $(S)_{-1}$, for all $t \geq 0$, see [25]. It is integrable and the relation $\frac{d}{d t} w_{t}=\dot{w}_{t}$ holds in the distributional sense. Clearly, both Brownian motion and singular white noise are Gaussian processes.

Let $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then $\mathcal{H}$-valued white noise process is given in the form

$$
\begin{equation*}
\dot{W}_{t}(\omega)=\sum_{k=1}^{\infty} e_{k}(t) H_{\varepsilon^{(k)}}(\omega) . \tag{13}
\end{equation*}
$$

In general, a chaos expansion representation of an $\mathcal{H}$-valued Gaussian process, that belongs to the Wiener chaos space of order one is given in the form

$$
\begin{equation*}
G_{t}(\omega)=\sum_{k \in \mathbb{N}} g_{k}(t) H_{\varepsilon^{(k)}}(\omega)=\sum_{k \in \mathbb{N}}\left(\sum_{i \in \mathbb{N}} g_{k i} e_{i}(t)\right) H_{\varepsilon^{(k)}}(\omega), \tag{14}
\end{equation*}
$$

with real coefficients $g_{k i}$. If the condition

$$
\begin{equation*}
\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{\mathcal{H}}^{2}<\infty \tag{15}
\end{equation*}
$$

is fulfilled, then process $G$ given in the form (14), belongs to the space $L^{2}([0, T] \times$ $\Omega, \mathcal{H})$. If the sum (15) is infinite then the representation (14) is formal, and if additionally

$$
\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \varepsilon} \varepsilon^{(k)}=\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{\mathcal{H}}^{2}(2 k)^{-p}<\infty
$$

holds for some $p \in \mathbb{N}_{0}$, the process $G$, for each $t$, belongs to the Kondratiev space of stochastic distributions, see [41, 45, 54].

Throughout the paper, we work with Hilbert space valued stochastic processes. Thus, an $\mathcal{H}$-valued stochastic process $v$, standard or generalized, has chaos expansion representation of the form

$$
\begin{align*}
v(t, \omega) & =\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega) \\
& =v_{\mathbf{0}}(t)+\sum_{k \in \mathbb{N}} v_{\varepsilon^{(k)}}(t) H_{\varepsilon^{(k)}}(\omega)+\sum_{|\alpha|>1} v_{\alpha}(t) H_{\alpha}(\omega), \quad t \in[0, T], \tag{16}
\end{align*}
$$

where the coefficients $v_{\alpha}$ satisfy a certain convergence condition of the form $\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{\mathcal{H}}^{2} r_{\alpha}<\infty$ for an appropriate family of weights $\left\{r_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. Note that the deterministic part of $v$ in (16) is the coefficient $v_{\mathbf{0}}(t)$, which is the (generalized) expectation of a process $v$.

The Wick product of two stochastic processes is defined in an analogous way as it was defined for random variables and generalized random variables (10), for more details see [40].
2.2.3. Operators. Following [44], we now introduce two classes of operators that we are dealing with, namely coordinatewise and simple coordinatewise operators. An operator $\mathbf{O}$ is called a coordinatewise operator if it is composed of a family of operators $\left\{O_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, such that for a process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}$ it holds

$$
\mathbf{O} v=\sum_{\alpha \in \mathcal{I}} O_{\alpha}\left(v_{\alpha}\right) H_{\alpha}
$$

Moreover, operator $\mathbf{O}$ is a simple coordinatewise operator if $O_{\alpha}=O$ for all $\alpha \in \mathcal{I}$, i.e. if it holds that

$$
\mathbf{O} v=\sum_{\alpha \in \mathcal{I}} O\left(v_{\alpha}\right) H_{\alpha}=O\left(v_{\mathbf{0}}\right)+\sum_{|\alpha|>0} O\left(v_{\alpha}\right) H_{\alpha}
$$

2.2.4. Stochastic integration and Wick multiplication. For a square integrable process $v$ that is adapted in the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ generated by an $\mathcal{H}$-valued Brownian motion $\left(W_{t}\right)_{t \geq 0}$, the corresponding stochastic integral $\int_{0}^{T} v_{t} d W_{t}$ is considered to be the Itô integral $I(v)$. When $v$ is not adapted to the filtration, then the stochastic integral is interpreted as the Itô-Skorokhod integral. From the fundamental theorem of stochastic calculus it follows that the Itô-Skorokhod integral of a $\mathcal{H}$ - valued stochastic process $v=v_{t}(\omega)$ can be represented as a Riemann integral of the Wick product of $v_{t}$ with a singular white noise

$$
\begin{equation*}
\delta(v)=\int_{0}^{T} v d W_{t}(\omega)=\int_{0}^{T} v \diamond \dot{W}_{t}(\omega) d t \tag{17}
\end{equation*}
$$

where the derivative $\dot{W}_{t}=\frac{d}{d t} W_{t}$ is taken in sense of distributions [25].
Thus, for an $\mathcal{H}$-valued adapted processes $v$ the Itô integral and the Skorokhod integral coincide, i.e. $I(v)=\delta(v)$. Note that the Itô integral is an $\mathcal{H}$-valued random
variable, i.e. $I: \mathcal{M}^{2} \rightarrow L^{2}(\Omega)$. From the Wiener-Itô chaos expansion theorem it follows that there exists a unique family $a_{\alpha}, \alpha \in \mathcal{I}$ such that the Itô integral can be represented in the chaos expansion form

$$
\begin{equation*}
I(v)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha} . \tag{18}
\end{equation*}
$$

On the other hand, applying the property (10) to (17) we obtain a chaos expansion representation of the Skorokhod integral. Clearly, for $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}$ we have

$$
\begin{align*}
v \diamond \dot{W}_{t}(\omega) & =\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega) \diamond \sum_{k \in \mathbb{N}} e_{k}(t) H_{\varepsilon^{(k)}}(\omega) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha}(t) e_{k}(t) H_{\alpha+\varepsilon^{(k)}}(\omega) . \tag{19}
\end{align*}
$$

Thus,

$$
\begin{align*}
\delta(v) & =\int_{0}^{T} v_{\alpha}(t) d W_{t}(\omega)=\int_{0}^{T} v_{\alpha}(t) \diamond \dot{W}_{t}(\omega) d t=\int_{0}^{T} \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha}(t) e_{k}(t) H_{\alpha+\varepsilon^{(k)}}(\omega) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\int_{0}^{T} v_{\alpha}(t) e_{k}(t) d t\right) H_{\alpha+\varepsilon^{(k)}}(\omega)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} H_{\alpha+\varepsilon^{(k)}}(\omega), \tag{20}
\end{align*}
$$

where $v_{\alpha}(t)=\sum_{k \in \mathbb{N}} v_{\alpha, k} e_{k}(t)$ is the chaos expansion representation of $v_{\alpha}$ in the orthonormal basis with coefficients $v_{\alpha, k}=<v_{\alpha}, e_{k}>_{\mathcal{H}} \in \mathbb{R}$ and $\omega \in \Omega$. Combining (20) and (18) we obtain the coefficients $a_{\alpha}$, for all $\alpha \in \mathcal{I}$ and $\alpha>\mathbf{0}$ in the form

$$
\begin{equation*}
a_{\alpha}=\sum_{k \in \mathbb{N}} v_{\alpha-\varepsilon^{(k)}, k} . \tag{21}
\end{equation*}
$$

As mentioned in Section 2.2.1, we use the following convention: $v_{\alpha-\varepsilon(k)}$ is not defined if the $k$ th component of $\alpha$, i.e. $\alpha_{k}$ equals zero. For example, for $\alpha=(1,3,0,2,0, \ldots)$ the coefficient $a_{(1,3,0,2,0, \ldots)}$ is expressed as the sum of three coefficients of the process $v$, i.e. from (21) we have

$$
a_{(1,3,0,2,0, \ldots)}=v_{(0,3,0,2,0, \ldots), 1}+v_{(1,2,0,2,0, \ldots), 2}+v_{(1,3,0,1,0, \ldots), 4}
$$

Hence we obtained the chaos expansion representation form of the Itô-Skorokhod integral. Therefore, we are able to represent the stochastic perturbation appearing in equation (1) explicitly. Note also that $\delta(v)$ belongs to the Wiener chaos space of higher order than $v$, see also [25, 42].

Therefore, we say that a square integrable $\mathcal{H}$-valued stochastic process $v$ given in the form $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega)$, with the coefficients $v_{\alpha}(t)=\sum_{k \in \mathbb{N}} v_{\alpha, k} e_{k}(t)$, $v_{\alpha} \in \mathcal{H}, v_{\alpha, k} \in \mathbb{R}$ for all $\alpha \in \mathcal{I}$ is integrable in Itô-Skorokhod sense if the condition

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}|\alpha| \alpha!<\infty \tag{22}
\end{equation*}
$$

holds. Then the Itô-Skorokhod integral of $v$ is of the form (20) and we write $v \in \operatorname{Dom}(\delta)$.

Theorem 2.2. The Skorokhod integral $\delta$ of an $\mathcal{H}$-valued square integrable stochastic process is a linear and continuous mapping

$$
\delta: \quad \operatorname{Dom}(\delta) \rightarrow L^{2}(\Omega)
$$

Proof. Let $v$ satisfies the condition (22). Then we have

$$
\begin{aligned}
\|\delta(v)\|_{L^{2}(\Omega)}^{2} & =\left\|\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} H_{\alpha+\varepsilon^{(k)}}\right\|_{L^{2}(\Omega)}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}\left(\alpha+\varepsilon^{(k)}\right)! \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}\left(\alpha_{k}+1\right) \alpha!\leq 2 \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}|\alpha| \alpha!<\infty,
\end{aligned}
$$

where we used $\left(\alpha+\varepsilon^{(k)}\right)!=\left(\alpha_{k}+1\right) \alpha!$ and the estimate $\alpha_{k}+1 \leq 2|\alpha|$ for all for $\alpha \in \mathcal{I}, k \in \mathbb{N}$.

Detailed analysis of domain and range of operators of the Malliavin calculus in spaces of stochastic distributions can be found in [43].
3. Chaos expansions approach. In this section we study the optimal control problem

$$
\min _{u} \mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{H}}^{2}\right]
$$

subject to the state equation

$$
d y(t)=[\mathbf{A} y(t)+\mathbf{B} u(t)] d t+\mathbf{C} y(t) d W_{t}, \quad y(0)=y^{0}, \quad t \in[0, T]
$$

and provide the main results of the paper.
We assume that all the operators are simple coordinatewise operators and:
(A1) Operator $\mathbf{A}: L^{2}([0, T] \times \Omega, \mathcal{D}(A)) \rightarrow L^{2}([0, T] \times \Omega, \mathcal{H})$ is a simple coordinatewise linear operator that corresponds to the deterministic operator $A: \mathcal{D}(A) \rightarrow \mathcal{H}$, where $A$ is an infinitesimal generator of a $C_{0}$-semigroup $\left(e^{A t}\right)_{t \geq 0}$, defined on a domain $\mathcal{D}(A)$ that is dense in $\mathcal{H}$, such that for some $M, \theta>0$ we have

$$
\left\|e^{A t}\right\|_{L(\mathcal{H})} \leq M e^{\theta t}, \quad t \geq 0
$$

(A2) The operator $\mathbf{C}: L^{2}([0, T] \times \Omega, \mathcal{H}) \rightarrow L^{2}([0, T] \times \Omega, \mathcal{H})$ is a simple coordinatewise operator corresponding to a bounded deterministic operator $C: \mathcal{H} \rightarrow \mathcal{H}$.
(A3) The control operator $\mathbf{B}$ is a simple coordinatewise operator $\mathbf{B}: L^{2}([0, T] \times$ $\Omega, \mathcal{U}) \rightarrow L^{2}([0, T] \times \Omega, \mathcal{H})$ that is defined by a bounded deterministic operator $B: \mathcal{U} \rightarrow \mathcal{H}$.
(A4) Operators $\mathbf{R}$ and $\mathbf{G}$ are bounded simple coordinatewise operators corresponding to the deterministic operators $R$ and $G$ respectively.

Thus, the actions of the operators are given by $\mathbf{A} y(t, \omega)=\sum_{\alpha \in \mathcal{I}} A y_{\alpha}(t) H_{\alpha}(\omega)$, $\mathbf{B} u(t)=\sum_{\alpha \in \mathcal{I}} B u_{\alpha}(t) H_{\alpha}(\omega)$ and $\mathbf{C} y(t, \omega)=\sum_{\alpha \in \mathcal{I}} C y_{\alpha}(t) H_{\alpha}(\omega)$, where

$$
\begin{equation*}
y(t, \omega)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}(t) H_{\alpha}(\omega), \quad u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \tag{23}
\end{equation*}
$$

such that for all $\alpha \in \mathcal{I}$ the coefficients $y_{\alpha} \in L^{2}([0, T], \mathcal{H})$ and $u_{\alpha} \in L^{2}([0, T], \mathcal{U})$.
Since the operator $C$ is a bounded linear operator on $\mathcal{H}$ while $B$ is bounded from $\mathcal{U}$ to $\mathcal{H}$, then $\mathbf{C}$ is a bounded operator on $L^{2}([0, T] \times \Omega, \mathcal{H})$, and $\mathbf{B}$ is bounded from $L^{2}([0, T] \times \Omega, \mathcal{U})$ into $L^{2}([0, T] \times \Omega, \mathcal{H})$.

Theorem 3.1. Let the assumptions (A1)-(A4) hold and let $\mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$. Then, the optimal control problem (1)-(2) has a unique optimal control $u^{*}$ given in the chaos expansion form

$$
u^{*}=-\sum_{\alpha \in \mathcal{I}} B^{*} P_{d}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B^{*} k_{\alpha}(t) H_{\alpha}
$$

where $P_{d}(t)$ solves the Riccati equation (8), i.e.

$$
\begin{aligned}
\dot{P}_{d}(t)+P_{d}(t) A+A^{\star} P_{d}(t)+R R^{*}-P_{d}(t) B B^{\star} P_{d}(t) & =0 \\
P_{d}(T) & =G^{\star} G
\end{aligned}
$$

and $k(t)$ is a solution to the auxiliary differential equation

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k_{\alpha}(t)+P_{d}(t)\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \cdot e_{i}(t)\right)=0 \tag{24}
\end{equation*}
$$

with the terminal condition $k_{\alpha}(T)=0$ and $y^{*}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*} H_{\alpha}$ is the optimal state.
Proof. We divide the proof in several steps. First, we analyze the state equation and apply the chaos expansion method to its equivalent Wick version.

Due to the fundamental theorem of stochastic calculus, an integral of Itô type of an integrable $\mathcal{H}$-valued stochastic process is equal to the Riemann integral of the Wick product of a process and $\mathcal{H}$-valued singular white noise (13), i.e.

$$
\int_{0}^{T} \mathbf{C} y(t) d W(t)=\int_{0}^{T} \mathbf{C} y(t) \diamond \dot{W}(t) d t
$$

where $W(t)$ is a $\mathcal{H}$-valued Brownian motion [25]. Therefore, the state equation can be written in standard differential form, on a class of admissible square integrable processes, as

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\mathbf{C} y(t) \diamond \dot{W}(t), \quad y(0)=y^{0}, \quad t \in[0, T] \tag{25}
\end{equation*}
$$

By applying the chaos expansion method to (25), we obtain a system of deterministic equations. Setting up a control problem for each equation we seek for the optimal control $u$ and the corresponding optimal state $y$ in the form (23). Thus, the goal is to obtain the unknown coefficients $u_{\alpha}$ and $y_{\alpha}$ for all $\alpha \in \mathcal{I}$.

We apply the chaos expansion method to transform the initial condition $y(0)=$ $y^{0}$, for a given $\mathcal{H}$-valued random variable $y^{0}$. Hence we obtain

$$
\sum_{\alpha \in \mathcal{I}} y_{\alpha}(0) H_{\alpha}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{0} H_{\alpha} .
$$

Since the chaos expansion in orthogonal polynomial basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ is unique, we obtain a family of initial conditions for the coefficients of the state

$$
y_{\alpha}(0)=y_{\alpha}^{0}, \quad \text { for all } \quad \alpha \in \mathcal{I}, \quad \text { where } y_{\alpha}^{0} \in \mathcal{H}, \alpha \in \mathcal{I} .
$$

Note that, in case that the initial condition is deterministic $y^{0} \in \mathcal{H}$, then its chaos expansion representation have only one non-zero element, i.e. $y_{0}^{0}$ in the zeroth level.

Next, we apply the chaos expansion method to the state equation (25). The process $y$ is considered to be differentiable if and only of its coordinates are differentiable deterministic functions and

$$
\dot{y}=\frac{d}{d t} y=\sum_{\alpha \in \mathcal{I}} \frac{d}{d t} y_{\alpha}(t) H_{\alpha}(\omega)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{\prime}(t) H_{\alpha}(\omega),
$$

we refer to [44]. From the assumption (A2) and the property (19) for each $y \in \mathcal{D}(\mathbf{C})$ it follows

$$
\mathbf{C} y(t) \diamond \dot{W}_{t}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(C y_{\alpha}\right) e_{k}(t) H_{\alpha+\varepsilon^{(k)}}(\omega),
$$

where $\left\{e_{i}\right\}_{i \in \mathbb{N}}$ denote the orthonormal basis of functions in $\mathcal{H}$. Then, by (A1) and (A3), the equation (25) can be written as

$$
\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{\prime}(t) H_{\alpha}(\omega)=\sum_{\alpha \in \mathcal{I}}\left(A y_{\alpha}(t)+B u_{\alpha}(t)\right) H_{\alpha}(\omega)+\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(C y_{\alpha}\right) e_{k}(t) H_{\alpha+\varepsilon^{(k)}}(\omega) .
$$

Due to the uniqueness of the chaos expansion representations in orthogonal polynomial basis (9), the previous equation reduces to the system of infinitely many deterministic initial value problems:

$$
\begin{align*}
& 1^{\circ} \text { for } \alpha=\mathbf{0} \text { : } \\
& \qquad y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}  \tag{26}\\
& \quad \text { for }|\alpha|>\mathbf{0} \text { : } \\
& \qquad y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \cdot e_{i}(t), \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{27}
\end{align*}
$$

The system of equations (26), (27) is deterministic, and the unknowns correspond to the coefficients of the control and the state variables. It describes how the stochastic state equation propagates chaos through different levels. Note that for $\alpha=\mathbf{0}$, the equation (26) corresponds to the deterministic version of the problem and the state $y_{\mathbf{0}}$ is the expected value of $y$. The terms $y_{\alpha-\varepsilon^{(i)}}(t)$ are obtained recursively with respect to the length of $\alpha$. The sum in (27) goes through all possible decompositions of $\alpha$, i.e., for all $j$ for which $\alpha-\varepsilon^{(j)}$ is defined. Therefore, the sum has as many terms as multi-index $\alpha$ has non-zero components.

Existence and uniqueness of solutions for the systems (26), (27) follows from the assumptions (A1), (A2) and (A3) made on the operators $A, B$ and $C$.

Now we set up optimal control problems for each $\alpha$-level. Considering the deterministic version of the cost function, the problems are defined as:
$1^{\circ}$ for $\alpha=\mathbf{0}$ : the control problem

$$
\begin{equation*}
\min _{u_{\mathbf{0}}} J\left(u_{\mathbf{0}}\right)=\int_{0}^{T}\left(\left\|R y_{\mathbf{0}}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\mathbf{0}}(t)\right\|_{\mathcal{U}}^{2}\right) d t+\left\|G y_{\mathbf{0}}(T)\right\|_{\mathcal{H}}^{2} \tag{28}
\end{equation*}
$$

subject to

$$
y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}, \quad \text { and }
$$

$2^{\circ}$ for $|\alpha|>\mathbf{0}$ : the control problem

$$
\begin{equation*}
J\left(u_{\alpha}\right)=\int_{0}^{T}\left(\left\|R y_{\alpha}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\alpha}(t)\right\|_{\mathcal{U}}^{2}\right) d t+\left\|G y_{\alpha}(T)\right\|_{\mathcal{H}}^{2} \tag{29}
\end{equation*}
$$

subject to

$$
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \cdot e_{i}(t), \quad y_{\alpha}(0)=y_{\alpha}^{0}
$$

which can be solved by induction on the length of multi-index $\alpha \in \mathcal{I}$.
In the next step of the proof we solve the family of deterministic control problems, i.e. we discuss the solution of the deterministic system of control problems (28) and (29).
$1^{\circ}$ For $\alpha=\mathbf{0}$ the state equation (26) is homogeneous, thus the optimal control for $(26),(28)$ is given in the feedback form

$$
\begin{equation*}
u_{\mathbf{0}}^{*}(t)=-B^{\star} P_{d}(t) y_{\mathbf{0}}^{*}(t), \tag{30}
\end{equation*}
$$

where $P_{d}(t)$ solves the Riccati equation (8).
$2^{\circ}$ For each $|\alpha|>\mathbf{0}$ the state equation (27) is inhomogeneous and the optimal control for (29) is given by

$$
\begin{equation*}
u_{\alpha}^{*}(t)=-B^{\star} P_{d}(t) y_{\alpha}^{*}(t)-B^{\star} k_{\alpha}(t), \tag{31}
\end{equation*}
$$

where $P_{d}(t)$ solves the Riccati equation (8), while $k(t)$ is a solution to the auxiliary differential equation (24) with the terminal condition $k_{\alpha}(T)=0$, as discussed in Section 2.1.1.
Summing up all the coefficients we obtain the optimal solution $\left(u^{*}, y^{*}\right)$ represented in terms of chaos expansions. Thus, the optimal state is given in the form

$$
y^{*}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}=y_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} y_{\alpha}^{*}(t) H_{\alpha}
$$

and the corresponding optimal control

$$
\begin{align*}
u^{*} & =\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}=u_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha} \\
& =-B^{\star} P_{d}(t) y_{\mathbf{0}}^{*}-\sum_{|\alpha|>0} B^{\star} P_{d}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B^{\star} k_{\alpha}(t)  \tag{32}\\
& =-\mathbf{B}^{\star} \mathbf{P}_{d} y^{*}(t)-\mathbf{B}^{\star} \mathcal{K},
\end{align*}
$$

where $\mathbf{P}_{d}(t)$ is a simple coordinatewise operator corresponding to the deterministic operator $P_{d}$ and $\mathcal{K}$ is a stochastic process with coefficients $k_{\alpha}(t)$, i.e. of the form $\mathcal{K}=\sum_{\alpha \in \mathcal{I}} k_{\alpha}(t) H_{\alpha}$, with $k_{\mathbf{0}}=0$.

In the next step we prove the optimality of the obtained solution. Under the assumptions of Theorem 2.1, the optimal control problem (1)-(2) is given in feedback form by

$$
\begin{equation*}
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t) \tag{33}
\end{equation*}
$$

with a positive self-adjoint operator $\mathbf{P}(t)$ solving the stochastic Riccati equation (5). Since the state equations (1) and (25) are equivalent, we are going to interpret the optimal solution (33), involving the Riccati operator $\mathbf{P}(t)$ in terms of chaos expansions. Thus, $\mathbf{J}\left(u^{*}\right)=\min _{u} \mathbf{J}(u)$, holds for $u^{*}$ of the form (33).

On the other hand, the stochastic cost function $\mathbf{J}$ is related with the deterministic cost function $J$ by,

$$
\begin{aligned}
\mathbf{J}(u) & =\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{W}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right] \\
& =\mathbb{E}\left(\int_{0}^{T}\|\mathbf{R} y\|_{\mathcal{W}}^{2} d t\right)+\mathbb{E}\left(\int_{0}^{T}\|u\|_{\mathcal{U}}^{2} d t\right) d t+\mathbb{E}\left(\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!\left\|R y_{\alpha}\right\|_{L^{2}([0, T], \mathcal{W})}^{2}+\sum_{\alpha \in \mathcal{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{U})}^{2}+\sum_{\alpha \in \mathcal{I}} \alpha!\left\|G y_{\alpha}(T)\right\|_{\mathcal{H}}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{\alpha \in \mathcal{I}} \alpha!\left(\left\|R y_{\alpha}\right\|_{L^{2}([0, T], \mathcal{W})}^{2}+\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{U})}^{2}+\left\|G y_{\alpha}(T)\right\|_{\mathcal{H}}^{2}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!J\left(u_{\alpha}\right)
\end{aligned}
$$

We used the fact that $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ is an orthogonal basis of the Hilbert space of square integrable random variables, i.e. $\mathbb{E}\left(H_{\alpha} H_{\beta}\right)=\alpha!\delta_{\alpha, \beta}$, where $\delta_{\alpha, \beta}$ is the Kronecker delta symbol and also the fact that the norms

$$
\|u\|_{L^{2}([0, T] \times \Omega, \mathcal{U})}^{2}=\mathbb{E}\left(\int_{0}^{T}\|u(t)\|_{\mathcal{U}}^{2} d t\right)=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{U})}^{2}
$$

and

$$
\|\mathbf{R} y\|_{L^{2}([0, T] \times \Omega, \mathcal{W})}^{2}=\mathbb{E}\left(\int_{0}^{T}\|R y(t)\|_{L^{2}(\Omega, \mathcal{W})}^{2} d t\right)=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|R y_{\alpha}\right\|_{\left(L^{2}([0, T], \mathcal{W})\right.}^{2}
$$

can be represented in terms of the coefficients of processes $y$ and $u$. Thus

$$
\mathbf{J}\left(u^{*}\right)=\min _{u} \mathbf{J}(u)=\min _{u} \sum_{\alpha \in \mathcal{I}} \alpha!J\left(u_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} \alpha!\min _{u_{\alpha}} J\left(u_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} \alpha!J\left(u_{\alpha}^{*}\right) .
$$

and therefore

$$
\begin{equation*}
u^{*}(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega) \tag{34}
\end{equation*}
$$

i.e. the optimal control obtained via direct Riccati approach $u^{*}$ coincides with the optimal control obtained via chaos expansion approach $\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega)$. Moreover, the optimal states are the same and thus the well-posedness of the solution of the optimal state equation obtained via chaos expansion approach follows.

As a final step in the proof, we provide the convergence of the chaos expansions in the optimal state. After applying the chaos expansions to the original state equation we obtained the system of deterministic problems (26) and (27). For each state equation in this system we formulated an optimal control problem for which the solution has the feedback form (30) and (31). The set of optimal controls for the resulting system were then used to determine the set of optimal states via the system of equations

$$
\begin{align*}
& y_{\mathbf{0}}^{\prime}(t)=\left(A-B B^{\star} P_{d}(t)\right) y_{\mathbf{0}}(t) \\
& y_{\alpha}^{\prime}(t)=\left(A-B B^{\star} P_{d}(t)\right) y_{\alpha}(t)-B B^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) e_{i}(t),|\alpha| \geq 1, \tag{35}
\end{align*}
$$

with the initial conditions $y_{\alpha}(0)=y_{\alpha}^{0}$, for all $\alpha \in \mathcal{I}$.
We assumed in (A1) that the operator $A$ is an infinitesimal generator of a strongly continuous semigroup $\left\{S_{t}\right\}_{t \geq 0}=\left(e^{A t}\right)_{t \geq 0}$ such that $\left\|e^{A t}\right\|_{L(\mathcal{H})} \leq M e^{\theta t}$ holds for some positive constants $M$ and $\theta$. Since the operators $B, B^{*}$ and $P_{d}$ are deterministic and bounded, the operator $B B^{*} P_{d}$ is also bounded and thus $A+B B^{*} P_{d}$ is an infinitesimal generator of a strongly evolution $\left(T_{t}\right)_{t \geq 0}$ such that

$$
\left\|T_{t}\right\|_{L(\mathcal{H})} \leq M e^{\theta t+M\left\|B B^{*} P_{d}\right\|_{L(\mathcal{H})} t}, \quad \text { for all } t \geq 0
$$

For more details we refer to [51].
Consider now a small interval $\left[0, T_{0}\right]$, for fixed $T_{0} \in(0, T]$. Denote by

$$
M_{1}(t)=M e^{\theta t+M\left\|B B^{*} P_{d}\right\|_{L(\mathcal{H})} t} \quad \text { and } \quad M_{2}(t)=\frac{M^{2} e^{2\left(\theta+M\left\|B B^{*} P_{d}\right\|_{L(\mathcal{H})}\right) t}}{\left(\theta+M\left\|B B^{*} P_{d}\right\|_{L(\mathcal{H})}\right)^{2}},
$$

for $t \in\left(0, T_{0}\right]$, so that $c T_{0} M_{2}\left(T_{0}\right)\|C\|^{2} \leq 1$. In (A3) we assumed that $C$ is a bounded operator and also that for fixed control $u$ it holds $C y \in \operatorname{Dom}(\delta)$. Thus, the condition (22) holds for $C y$.

Therefore, the mild solution of (35) is given in the form

$$
\begin{aligned}
& y_{\mathbf{0}}(t)=T_{t} y_{\mathbf{0}}^{0} \\
& y_{\alpha}(t)=T_{t} y_{\alpha}^{0}+\int_{0}^{t} T_{t-s}\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(s) e_{i}(s)-B B^{\star} k_{\alpha}(s)\right) d s,|\alpha| \geq 1, t \geq 0 .
\end{aligned}
$$

Since $y^{0} \in L^{2}(\Omega ; \mathcal{H})$, from the initial condition $y(0)=y^{0}$ it follows $\mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}=$ $\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$. Operators $C, B$ and $B^{*}$ are bounded operators, and therefore the inhomogeneity part of (35) belongs to the space $L^{2}(\mathcal{H})$, where functions $k_{\alpha}, \alpha \in \mathcal{I}$ are given in (24). Thus it holds

$$
\begin{aligned}
& \|y\|_{L^{2}(\Omega, \mathcal{H})}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|y_{\alpha}\right\|_{\mathcal{H}}^{2}=\left\|y_{\mathbf{0}}\right\|_{\mathcal{H}}^{2}+\sum_{|\alpha| \geq 1} \alpha!\left\|y_{\alpha}\right\|_{\mathcal{H}}^{2} \\
& \leq 2 M_{1}^{2}\left(T_{0}\right) \cdot\left\|y_{\mathbf{0}}^{0}\right\|_{\mathcal{H}}^{2}+4 M_{1}^{2}\left(T_{0}\right) \cdot \sum_{|\alpha| \geq 1} \alpha!\left\|y_{\alpha}^{0}\right\|_{\mathcal{H}}^{2} \\
& \quad+4 \sum_{|\alpha| \geq 1} \alpha!\int_{0}^{t}\left\|T_{t-s}\right\|^{2}\left\|\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}\right)_{i}-B B^{\star} k_{\alpha}(s)\right\|^{2} d s \\
& \leq 4 M_{1}^{2}\left(T_{0}\right) \cdot\left\|y^{0}\right\|_{L^{2}(\Omega ; \mathcal{H})}^{2} \\
& \quad+c T_{0} M_{2}\left(T_{0}\right)\|C\|^{2}\left(\sum_{\alpha \in \mathcal{I}} \alpha!\left\|y_{\alpha}\right\|_{\mathcal{H}}^{2}+\|B\|^{2}\left\|B^{*}\right\|^{2}\|\mathcal{K}\|_{L^{2}\left(\left[0, T_{0}\right] \times \Omega, \mathcal{H}\right)}^{2}\right),
\end{aligned}
$$

where we used the estimate

$$
\sum_{|\alpha| \geq 1} \sum_{i \in \mathbb{N}}\left(\left(C y_{\alpha-\varepsilon^{(i)}}\right)\right)^{2} \leq\|C\|^{2} \sum_{\alpha \in \mathcal{I}} \alpha!\left\|y_{\alpha}\right\|^{2}=\|C\|^{2}\|y\|_{L^{2}\left(\left[0, T_{0}\right] \times \Omega, \mathcal{H}\right)}^{2} .
$$

It holds $\mathcal{K} \in L^{2}\left(\left[0, T_{0}\right] \times \Omega ; \mathcal{H}\right)$ and also $C y \in \operatorname{Dom}(\delta)$. Therefore, we group all the summands with the term $\|y\|^{2}=\|y\|_{L^{2}\left(\left[0, T_{0}\right] \times \Omega, \mathcal{H}\right)}^{2}$ on the left hand side of the inequality and obtain

$$
\begin{aligned}
\|y\|_{L^{2}\left(\left[0, T_{0}\right] \times \Omega, \mathcal{H}\right)}^{2} \cdot\left(1-c T_{0} M_{2}\left(T_{0}\right)\right. & \left.\|C\|^{2}\right) \leq 4 M_{1}^{2}\left(T_{0}\right)\left\|y^{0}\right\|_{L^{2}(\Omega, \mathcal{H})}^{2} \\
& +c T_{0} M_{2}\left(T_{0}\right)\|C\|^{2}\|B\|^{2}\left\|B^{*}\right\|^{2}\|\mathcal{K}\|_{L^{2}\left(\left[0, T_{0}\right] \times \Omega, \mathcal{H}\right)}^{2} .
\end{aligned}
$$

From the smallness assumption, the boundedness of $y$ on $\left(0, T_{0}\right]$ follows. The interval $(0, T]$ can be covered by the intervals of the form $\left[k T_{0},(k+1) T_{0}\right]$ in finitely many steps. Thus, $y \in L^{2}([0, T] \times \Omega, \mathcal{H})$.

The importance of the convergence result can be seen in its applications for the error analysis that arises in the actual truncation when implementing the algorithm numerically.
3.1. Characterization of optimality. The optimality of our approach (34) can be characterized in terms of the solution of the stochastic Riccati equation (5). The following theorem summarizes our result.

Theorem 3.2. Let conditions (A1)-(A4) hold. Assume that $y^{0}$ is either deterministic or a square integrable $\mathcal{H}$-valued random variable, i.e. it holds $\mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$ and assume $\mathbf{P}$ is a simple coordinatewise operator that corresponds to operator $P$.

Then, the solution of the optimal control of the problem (1)-(2) obtained via chaos expansion (32) is equal to the one obtained via Riccati approach (33) if and only if

$$
\begin{equation*}
C^{\star} P(t) C y_{\alpha}^{*}(t)=P(t)\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot e_{i}(t)\right), \quad|\alpha|>0, k \in \mathbb{N} \tag{36}
\end{equation*}
$$

hold for all $t \in[0, T]$.
Proof. Let us assume first that (32) is equal to (33), then

$$
-\mathbf{B}^{\star} \mathbf{P}_{d} y^{*}(t)=-\mathbf{B}^{\star} \mathbf{P} y^{*}(t)-\mathbf{B}^{\star} \mathcal{K}
$$

we obtain

$$
\left(\mathbf{P}(t)-\mathbf{P}_{d}\right) y^{*}(t)=\mathcal{K}
$$

The difference between $\mathbf{P}(t)$ and $\mathbf{P}_{d}(t)$ is expressed through the stochastic process $\mathcal{K}$, which comes from the influence of inhomogeneities. Assuming that $\mathbf{P}$ is a simple coordinatewise operator that corresponds to operator $P$, we will be able to see the action of stochastic operator $\mathbf{P}$ on the deterministic level, i.e. level of coefficients. Thus, for $y$ given in the chaos expansion form (23) and $\mathbf{P}(t) y^{*}=\sum_{\alpha \in \mathcal{I}} P(t) y_{\alpha}^{*}(t) H_{\alpha}$ it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left(P(t)-P_{d}(t)\right) y_{\alpha}^{*}(t) H_{\alpha}=\sum_{\alpha \in \mathcal{I},|\alpha|>0} P(t) k_{\alpha}(t) H_{\alpha} . \tag{37}
\end{equation*}
$$

Since $k_{\mathbf{0}}(t)=0$ it follows $P(t)=P_{d}(t)$, for $t \in[0, T]$ and for $|\alpha|>0$

$$
\left(P(t)-P_{d}(t)\right) y_{\alpha}^{*}(t)=k_{\alpha}(t)
$$

such that (24) with the condition $k_{\alpha}(T)=0$ holds. We differentiate (37) and substitute (24), together with (5), (8) and (27). Thus, after all calculations we obtain for $|\alpha|=0$

$$
\left(P(t)-P_{d}(t)\right) y_{\mathbf{0}}^{*}(t)=0
$$

and for $|\alpha|>0$

$$
C^{\star} P(t) C y_{\alpha}^{*}(t)=P(t)\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot e_{i}(t)\right), \quad k \in \mathbb{N}
$$

Note that assuming (36) and $\mathbf{P}$ is a simple coordinatewise operator that corresponds to operator $P$, we can go backwards in the analysis and prove that the optimal controls (33) and (32) are the same.

The condition (36) for $|\alpha|=1$, i.e. $\alpha=\varepsilon^{(j)}, j \in \mathbb{N}$ reduces to the condition

$$
C^{\star} P(t) C y_{\varepsilon(j)}^{*}(t)=P(t)\left(C y_{\mathbf{0}}(t) \cdot e_{j}(t)\right),
$$

while for $|\alpha|=2$ it reduces to one of the following situations: for $\alpha=2 \varepsilon^{(j)}, j \in \mathbb{N}$ it becomes

$$
C^{\star} P(t) C y_{2 \varepsilon^{(j)}}^{*}(t)=P(t)\left(C y_{\varepsilon^{(j)}}^{*}(t) \cdot e_{j}(t)\right),
$$

and for $\alpha=\varepsilon^{(j)}+\varepsilon^{(k)}, j, k \in \mathbb{N}, j \neq k$ it becomes

$$
C^{\star} P(t) C y_{2 \varepsilon(j)}^{*}(t)=P(t)\left(C y_{\varepsilon^{(j)}}^{*}(t) \cdot \xi_{k}(t)+C y_{\varepsilon^{(k)}}^{*}(t) \cdot e_{j}(t)\right)
$$

and so on. The recurrence involved in (36), represents a memory property in the noise. This concept has been recently studied in [11]. In the next section we study a control problem with a state equation involving noise with memory.

Remark 2. The assumptions (A1)-(A4) from Theorem 3.2 hold for many applications and they are standard in optimal control [34, 35]. On the other hand, due to the fact that the Riccati equation (5) is deterministic, its solution is naturally related to $\mathbf{P}$, where $\mathbf{P}$ is a simple coordinatewise operator (Section 2.2.3). The latter is not necessary true for stochastic Riccati equations (49) (Section 4.2), there $\mathbf{P}$ might not necessary be a simple coordinstewise operator.

Remark 3. Condition (36) which characterizes optimality represents the action of the stochastic Riccati operator in each level of the noise. Note that the stochastic Riccati equation (5) and the deterministic one (8) differ only in the term $C^{\star} P(t) C$, i.e. the operator $C^{\star} P(t) C$ captures the stochasticity of the equation.
3.2. SLQR problem with disturbance in the state and the control. In general, allowing disturbance in both the state and the control, the state equation can be written as

$$
\begin{equation*}
d y(t)=[\mathbf{A} y(t)+\mathbf{B} u(t)] d t+[\mathbf{C} y(t)+\mathbf{D} u(t)] d W_{t}, \quad y(0)=y^{0} \tag{38}
\end{equation*}
$$

where $\mathbf{D}$ is a simple coordinatewise operator related to a bounded operator $D$. Similar to (25), equation (38) can be written as

$$
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+(\mathbf{C} y(t)+\mathbf{D} u(t)) \diamond \dot{W}(t), \quad y(0)=y^{0}
$$

Therefore, by applying the chaos expansion method, one obtains the following deterministic system of equations:
a) for $|\alpha|=0: \quad y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}$,
b) for $|\alpha|>0$ :

$$
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}} e_{i}(t)+\sum_{i \in \mathbb{N}} D u_{\alpha-\varepsilon^{(i)}} e_{i}(t), y_{\alpha}(0)=y_{\alpha}^{0} .
$$

Then, the optimal states have the form:

$$
1^{\circ} \text { for }|\alpha|=0: y_{\mathbf{0}}^{\prime}(t)=\left(A-B B^{\star} P\right) y_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}
$$

$2^{\circ}$ for $|\alpha|>0$ :

$$
\begin{aligned}
& y_{\alpha}^{\prime}(t)=\left(A-B B^{\star} P\right) y_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C-D B^{\star} P\right) y_{\alpha-\varepsilon^{(i)}} e_{i}(t) \\
&-\sum_{i \in \mathbb{N}} D B^{\star} k e_{i}(t)-B B^{\star} k(t), \quad y_{\alpha}(0)=y_{\alpha}^{0} .
\end{aligned}
$$

Note that, our approach is optimal in this case as well. On the other hand, a direct Riccati approach will lead to an optimal state given by

$$
\begin{aligned}
d y(t)= & \left(A-B\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right)\right) y(t) d t \\
& +\left(C-D\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right)\right) y(t) d W_{t} \\
y(0)= & y^{0}
\end{aligned}
$$

where $P(t)$ is the solution of

$$
\begin{align*}
& \left\langle\left(\dot{P}+P A+A^{\star} P+C^{\star} P C+R^{\star} R\right.\right. \\
& \left.\quad-\left(B^{\star} P+D^{\star} P C\right)^{\star}\left(I+D^{\star} P D\right)^{-1}\left(B^{\star} P+D^{\star} P C\right) v, w\right\rangle=0  \tag{39}\\
& \quad P(T) v=G^{\star} G v
\end{align*}
$$

for all $v, w$ in $\mathcal{D}(A)$.
From the computational point of view, our approach has a lot of potential as it avoids solving (39), and will be explored in future work. Finally, we point out that
a convergence framework for the stochastic linear problem in the general framework of singular estimates has been developed recently in [39].
4. Applications. In this section we extend the results of Section 3 to optimal control problems with state equations involving memory noise. We also consider the state equations with random coefficients following the framework of [20, 21] and give an example of a control system from structure acoustics.
4.1. State equation with memory noise. We apply the introduced method to optimal control problems involving noise with memory. Particularly, we study the SLQR problem with the state equation of the form

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\delta(\mathbf{C} y(t)), \quad y(0)=y^{0}, \quad t \in[0, T] \tag{40}
\end{equation*}
$$

subject to the cost functional $\mathbf{J}(u)$ given by (2). Here $\delta$ denotes the Itô-Skorokhod integral. In the same setting, we can also consider the state equation in more general form

$$
\begin{equation*}
y^{\prime}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\delta_{t}(\mathbf{C} y(t)), \quad y(0)=y^{0}, \quad t \in[0, T] \tag{41}
\end{equation*}
$$

where $\delta_{t}(f)=\int_{0}^{t} f(s) d W_{s}, t \in[0, T]$ is the integral Itô-Skorokhod process. For $t=T, \delta=\delta_{T}$. Note that solving the problem for $\delta$, the problem for $\delta_{t}$ is straight forward since $\delta_{t}(f)=\delta\left(f \chi_{[0, t]}\right), t \in[0, T]$, where $\chi_{[0, t]}$ is the characteristic function on the interval $[0, t]$, i.e. for $t \in[0, T]$

$$
\delta_{t}(\mathbf{C} y)=\int_{0}^{t} \mathbf{C} y(s) d W_{s}=\int_{0}^{T} \mathbf{C} y(s) \chi_{[0, t]}(s) d W_{s}=\delta\left(\mathbf{C} y(s) \chi_{[0, t]}(s)\right) .
$$

As discussed before, the fact that $y$ appears in the stochastic integral implies that the noise contains a memory property [11]. The disturbance $\delta$ is a zero mean random variable for all $t \in[0, T]$, while $\delta_{t}$ is a zero mean stochastic process.

There exists an operator $\tilde{\mathbf{C}}$ such that there is a one to one correspondence between $\tilde{\mathbf{C}} \diamond$ and $\delta \circ \mathbf{C}$, i.e.

$$
\tilde{\mathbf{C}} \diamond y=\delta(\mathbf{C} y) .
$$

Therefore, (40) can be written as

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\tilde{\mathbf{C}} \diamond y, \quad y(0)=y^{0}, \tag{42}
\end{equation*}
$$

i.e. there is a correspondence between the Wick form perturbation and the Skorokhod integral representation [44].

In the following, we apply the chaos expansion approach for solving the SLQR problem related to (40) and compare the solution to the actual solution obtained by a direct Riccati approach applied to equation (42). Since there is no explicit form of $\tilde{\mathbf{C}}$, the suggested polynomial chaos approach for solving the problem is quite promising.

Similarly as in the previous section, we apply the chaos expansion method to (40) and thus transform the equation to a corresponding infinite family of deterministic equations. We look for the optimal coefficients $u_{\alpha}$ and $y_{\alpha}, \alpha \in \mathcal{I}$. Then, we obtain the system of deterministic optimal control problems
$1^{\circ}$ for $\alpha=\mathbf{0}$ : the control problem

$$
\min _{u_{\mathbf{0}}} J\left(u_{\mathbf{0}}\right)=\int_{0}^{T}\left(\left\|R y_{\mathbf{0}}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\mathbf{0}}(t)\right\|_{\mathcal{U}}^{2}\right) d t+\left\|G y_{\mathbf{0}}(T)\right\|_{\mathcal{H}}^{2}
$$

subject to

$$
\begin{equation*}
y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \tag{43}
\end{equation*}
$$

$2^{\circ}$ for $|\alpha|>\mathbf{0}$ : the control problem

$$
J\left(u_{\alpha}\right)=\int_{0}^{T}\left(\left\|R y_{\alpha}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\alpha}(t)\right\|_{\mathcal{U}}^{2}\right) d t+\left\|G y_{\alpha}(T)\right\|_{\mathcal{H}}^{2}
$$

subject to

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}, \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{44}
\end{equation*}
$$

where $\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}$ denotes the $i$ th component of $C y_{\alpha-\varepsilon^{(i)}}$, i.e. a real number, obtained in the previous inductive step. The sum is finite with as many summands as multi-index $\alpha$ has non-zero components.

For $|\alpha|=0$ the state equation in (43) is homogeneous and the optimal control for the state equation is given in the feedback form (30), with positive self adjoint operator $P_{d}$ that satisfies the Riccati equation (8). On the other hand, for each $|\alpha|>0$ the state equation in (44) is inhomogeneous with the inhomogeneity term $\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}\right)_{i}$. Thus, the optimal control is given by (31), where $k_{\alpha}$ are the solutions to the auxiliary differential equations

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k_{\alpha}(t)+P_{d}(t)\left(\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}\right)=0 \tag{45}
\end{equation*}
$$

for $|\alpha|>0$, with the final condition $k_{\alpha}(T)=0$. Summing up all the coefficients, obtained as optimal on each level $\alpha$, the optimal state is then given in the form

$$
y^{*}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}=y_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} y_{\alpha}^{*}(t) H_{\alpha}
$$

and the corresponding optimal control $u^{*}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}=u_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha}$.
The optimal state in each level is given by:

$$
1^{\circ} \text { for }|\alpha|=0, \text { i.e. } \quad \alpha=(0,0, \ldots)=\mathbf{0}
$$

$$
y_{\mathbf{0}}^{\prime}(t)=\left(A-B B^{\star} P_{d}(t)\right) y_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}
$$

$2^{\circ}$ for $|\alpha| \geq 0$ :

$$
y_{\alpha}^{\prime}(t)=\left(A-B B^{\star} P_{d}(t)\right) y_{\alpha}(t)-B B^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}, \quad y_{\alpha}(0)=y_{\alpha}^{0}
$$

where $k_{\alpha}$ are solutions of (24). Thus, the optimal state computed by chaos expansion corresponds to

$$
\begin{equation*}
\dot{y}(t)=\left(\mathbf{A}-\mathbf{B B}^{\star} \mathbf{P}_{d}(t)\right) y(t)+\delta(\mathbf{C} y(t))-\mathbf{B B}^{\star} \mathcal{K}, \quad y(0)=y^{0} \tag{46}
\end{equation*}
$$

where $\mathbf{B B}^{\star} \mathbf{P}_{d}$ is a simple coordinatewise operator given through the deterministic operator $\left(B B^{\star} P_{d}\right)$, where $P_{d}$ is the solution of (8) and $\mathcal{K}$ is a stochastic function given by the expansion

$$
\mathcal{K}=\sum_{\alpha \in \mathcal{I}} k_{\alpha}(t) H_{\alpha}=k_{\varepsilon^{(k)}}(t) H_{\varepsilon^{(k)}}+\sum_{|\alpha|>1} k_{\alpha}(t) H_{\alpha},
$$

where $k_{\mathbf{0}}=0$ and $k_{\alpha}$ are given by (45) respectively. Equation (46) represents the optimal state when we control each level of the chaos expansion. On the other hand, a direct Riccati approach for the SLQR problem related to (42) or (41), up to our knowledge has not been studied in the literature.

Finally, we point out that the convergence of the chaos expansions can be established using a similar argument to the one described in the proof of Theorem 3.1.
4.2. Random coefficients. Let us consider a stochastic linear quadratic control problem of the form

$$
\begin{equation*}
d y(t)=\left[\left(\overline{\mathbf{A}}+\mathbf{A}_{\sharp}\right) y(t)+\mathbf{B} u(t)\right] d t+\mathbf{C} y(t) d W(t), \quad y(0)=y^{0}, \tag{47}
\end{equation*}
$$

subject to the performance index

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right] \tag{48}
\end{equation*}
$$

where $\overline{\mathbf{A}}$ is independent of $\omega$ and is the infinitesimal generator of a $C_{0}$-semigroup, $\mathbf{A}_{\sharp}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ are allowed to be random. The optimal control is given in feedback form in terms of an operator $\mathbf{P}(t)$ solving the backward stochastic Riccati equation

$$
\begin{align*}
&-d \mathbf{P}=\left(\mathbf{R}^{\star} \mathbf{R}+\overline{\mathbf{A}}^{\star} \mathbf{P}+\mathbf{P} \overline{\mathbf{A}}-\mathbf{P} \mathbf{B} \mathbf{B}^{\star} \mathbf{P}+\mathbf{A}_{\sharp}^{\star} \mathbf{P}+\mathbf{P A}_{\sharp}\right) d t  \tag{49}\\
&+\operatorname{Tr}\left(\mathbf{C}^{\star} \mathbf{P} \mathbf{C}+\mathbf{C}^{\star} \mathbf{Q}+\mathbf{Q} \mathbf{C}\right) d t+\mathbf{Q} d W(t),
\end{align*}
$$

with $\mathbf{P}_{\mathbf{0}}(T)=\mathbf{G}^{\star} \mathbf{G}$. The two operators $\mathbf{P}$ and $\mathbf{Q}$ are unknown, and $\mathbf{Q}$ is sometimes referred to as a martingale term, see $[20,21]$ and references therein.

If the operators involved have chaos expansion representations, the same ideas can be applied to fully stochastic problem. Let us consider the operator $\overline{\mathbf{A}}$ to be a coordinatewise operator, i.e. an operator composed of a family of operators $\left\{\bar{A}_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, where $\bar{A}_{\alpha}$ are infinitesimal generators of $C_{0}$-semigroups defined on a common domain that is dense in $\mathcal{H}$ and

$$
\overline{\mathbf{A}}(y)=\sum_{\alpha \in \mathcal{I}} \bar{A}_{\alpha}(y) H_{\alpha} .
$$

For the case when $\overline{\mathbf{A}}$ is independent on randomness, only nonzero operator in the family $\left\{\bar{A}_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ is obtained for $|\alpha|=0$, i.e. $\bar{A}_{\mathbf{0}}=\bar{A}$ and $\bar{A}_{\alpha}=0$ for all $|\alpha|>0$.

Operators $\mathbf{A}_{\sharp}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ are also coordinatewise operators composed by the families of deterministic operators $\left\{A_{\alpha}^{\sharp}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ respectively, and

$$
\begin{gathered}
\mathbf{A}_{\sharp}(F)=\sum_{\alpha \in \mathcal{I}} A_{\alpha}^{\sharp}\left(f_{\alpha}\right) H_{\alpha}, \quad \mathbf{B}(U)=\sum_{\alpha \in \mathcal{I}} B_{\alpha}\left(u_{\alpha}\right) H_{\alpha}, \quad \mathbf{C}(F)=\sum_{\alpha \in \mathcal{I}} C_{\alpha}\left(f_{\alpha}\right) H_{\alpha}, \\
\mathbf{R}(F)=\sum_{\alpha \in \mathcal{I}} R_{\alpha}\left(f_{\alpha}\right) H_{\alpha}, \quad \mathbf{G}(F)=\sum_{\alpha \in \mathcal{I}} G_{\alpha}\left(f_{\alpha}\right) H_{\alpha},
\end{gathered}
$$

for a $\mathcal{H}$-valued process $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}, f_{\alpha} \in \mathcal{H}$ and $\mathcal{U}$-valued process $U=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}, u_{\alpha} \in \mathcal{U}$.

Applying the polynomial chaos method to (47), we obtain:
a) for $|\alpha|=0$, i.e. $\quad \alpha=(0,0, \ldots)=\mathbf{0}$ :

$$
\begin{equation*}
y_{(0,0, \ldots)}^{\prime}(t)=\left(\bar{A}_{\mathbf{0}}+A_{\mathbf{0}}^{\sharp}\right) y_{(0,0, \ldots)}(t)+B_{\mathbf{0}} u_{(0,0, \ldots)}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}, \tag{50}
\end{equation*}
$$

b) for $|\alpha|>0$ :

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=\left(\bar{A}_{\alpha}+A_{\alpha}^{\sharp}\right) y_{\alpha}(t)+B_{\alpha} u_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C_{\alpha} y_{\alpha-\varepsilon^{(i)}}\right)_{i}, \quad y_{\alpha}(0)=y_{\alpha}^{0} . \tag{51}
\end{equation*}
$$

Setting up control problems at each level for (50) and (51), as explained in Section 4.1, in analogy to (46) the optimal state is given by

$$
d y(t)=\left(\left(\overline{\mathbf{A}}+\mathbf{A}_{\sharp}-\mathbf{B B}^{\star} \overline{\mathbf{P}}\right) y(t)\right) d t+\mathbf{C} y(t) \diamond \dot{W}(t)-\mathbf{B B}^{\star} \bar{K}, \quad y(0)=y^{0}
$$

where $\overline{\mathbf{P}}$ is a coordinatewise operator composed by the family $\left\{P_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. The operators $P_{\alpha}$ correspond to the solution of the Riccati equation for the coefficients $\bar{A}_{\alpha}$, $A_{\alpha}^{\sharp}, B_{\alpha}, C_{\alpha}, R_{\alpha}$ and $G_{\alpha}$, i.e. it holds

$$
\begin{align*}
\dot{P}_{\alpha}+P_{\alpha}\left(A_{\alpha}+A_{\alpha}^{\sharp}\right)+\left(A_{\alpha}+A_{\alpha}^{\sharp}\right)^{\star} P_{\alpha}+R_{\alpha}^{\star} R_{\alpha}-\left(P_{\alpha} B_{\alpha} B_{\alpha}^{\star} P_{\alpha}\right) & =0 \\
P_{\alpha}(T) & =G_{\alpha}^{\star} G_{\alpha} \tag{52}
\end{align*}
$$

for each $\alpha \in \mathcal{I}$. Note that (52) is a deterministic Riccati equation for each $\alpha$. Also $\bar{K}$ is a $\mathcal{H}$-valued stochastic process given by

$$
\bar{K}=\sum_{\alpha \in \mathcal{I}} k_{\alpha} H_{\alpha}=k_{\varepsilon^{(i)}} H_{\varepsilon^{(i)}}+\sum_{|\alpha|>1} k_{\alpha} H_{\alpha},
$$

where $k_{\mathbf{0}}=0$ and $k_{\alpha}$, for $|\alpha| \geq 1$ are given by

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+\left(A_{\alpha}^{\star}-P_{\alpha}(t) B_{\alpha} B_{\alpha}^{\star}\right) k_{\alpha}(t)+P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha} x_{\alpha-\varepsilon^{(i)}} e_{i}\right)=0 \tag{53}
\end{equation*}
$$

Equations (53) have a final condition equal to zero. Therefore, in order to control the system (47)-(48) we control each level through the chaos expansion. This implies solving a deterministic control problem at each level. Although theoretically we have to solve all these problems, numerically we can solve $\frac{(m+p)!}{m!p!}$ problems in order to achieve convergence. The value of $p$ is in general equal to the number of uncorrelated random variables in the system and $m$ is typically chosen by some heuristic method [46, 58, 62].
4.3. A specific example from SPDE control. The approach outlined in this paper can be applied to a large class of systems in engineering which are mathematically modeled by partial differential equations. Control problems with stochastic coefficients also arise naturally in mathematical finance. In particular, the linear quadratic optimal control problem with stochastic coefficients and the corresponding backward stochastic Riccati equations (BSREs) have been extensively studied in the finite-horizon and finite-dimensional case [9, 10, 28, 29, 30, 31, 52, 53]. Note that our approach is also valid for finite-dimensional systems since the polynomial chaos method can be applied to systems governed by random matrices.

As an example, we include a control system from structure acoustics which has been well studied in the deterministic setting [2, 3, 4, 37]. The system consists of an acoustic chamber with piezoelectric control mechanism applied to the flexible wall of the chamber. Mathematically, the system is modelled by an open region $\Omega \subset \mathbb{R}^{3}$ with boundary $\partial \Omega=\Gamma_{0} \bigcup \Gamma_{1}$ representing a rigid wall and a flexible wall respectively. The acoustics in the chamber are modelled by a wave equation in the variable $z$ which denotes acoustic pressure

$$
d z_{t}=c^{2} \Delta z d t+\left(\nabla z+z_{t}+w+w_{t}\right) d W_{t} \text { on } \Omega \times[0, T]
$$

where $c$ is the speed of sound and $W_{t}$ is a one dimensional Wiener process on a complete probability space. On the other hand, the dynamics of the elastic wall $\Gamma_{1}$, are modelled by a damped second order equation in the displacement variable $w$

$$
d w_{t}+\Delta^{2} w d t+\rho \Delta w_{t} d t=\rho_{1} z_{t} d t+\sum_{J} a_{j} u_{j} \delta_{\xi_{j}}^{\prime} d t+\left(\nabla w+w_{t}+z+z_{t}\right) d W_{t}
$$

in $\Gamma_{1} \times[0, T]$, where $\rho, \rho_{1}>0$. The piezoelectric control mechanism is mathematically represented by the derivatives of Dirac delta functions supported at curves $\xi_{j}$ with the controls $u \in \mathbb{R}^{J}$ while $a_{j}(x)$ are smooth functions on $\Gamma_{1}$. The acoustic pressure satisfies the boundary conditions

$$
\begin{array}{r}
\frac{\partial}{\partial \nu} z+d_{1} z=0 \text { in } \Gamma_{0} \times[0, T] \\
\frac{\partial}{\partial \nu} z=w_{t} \text { in } \Gamma_{1} \times[0, T],
\end{array}
$$

while the clamped boundary conditions are imposed on the boundary of $\Gamma_{1}$ denoted by $\partial \Gamma_{1}$

$$
w=\frac{\partial}{\partial \nu} w=0 \text { in } \partial \Gamma_{1} \times[0, T] .
$$

We consider the system subject to the initial conditions $z_{0} \in H^{1}(\Omega), z_{1} \in L^{2}(\Omega)$ and $w_{0} \in H^{2}\left(\Gamma_{1}\right) \cap H_{0}^{1}\left(\Gamma_{1}\right)$ and $w_{1} \in L^{2}(\Omega)$.

The multiplicative noise in the system is captured by a bounded operator $C$ on the finite energy space. The control objective is to minimize the functional

$$
\begin{aligned}
& J\left(z, z_{1}, w, w_{1}, u\right)= \\
& \quad \mathbb{E}\left[\int_{0}^{T}\left(\|\Delta w\|_{L^{2}\left(\Gamma_{1}\right)}^{2}+\left\|w_{t}\right\|_{L^{2}\left(\Gamma_{1}\right)}^{2}+\|\nabla z\|_{L^{2}(\Omega)}^{2}+\left\|z_{t}\right\|_{L^{2}(\Omega)}^{2}+\sum_{J}\left|u_{j}(t)\right|^{2}\right) d t\right]
\end{aligned}
$$

over all possible controls $u=\left(u_{1}, u_{2}, \ldots, u_{J}\right) \in L^{2}\left([0, T] ; \mathbb{R}^{J}\right)$. It is well known that the deterministic system is driven by a $C_{0}$ semigroup ( $e^{A t}$ ) with a generator $A$ on the finite energy space $\mathcal{H}$ [2]. Although, the control operator $B$ here is not bounded and takes values in a larger dual space $B: \mathbb{R}^{J} \rightarrow\left[\mathcal{D}\left(A^{\star}\right)\right]^{\prime}$, it exhibits the so called singular estimate condition which is satisfied by the control-to-state map

$$
\left\|e^{A t} B u\right\|_{\mathcal{H}} \leq \frac{c|u|}{t^{3 / 8+\epsilon}}
$$

for all $u \in \mathbb{R}^{J}$ [2]. There has been many works in the literature addressing Riccati feedback synthesis of such control systems known as singular estimate control systems in the deterministic case [36] and references therein, and more recently in the stochastic case [22, 23]. The possible extension and application of the polynomial chaos approach to this class of control systems which typically involve boundary or point control of systems of coupled hyperbolic-parabolic partial differential equations with noise, would be numerically very promising.
5. Numerical approximation. Numerical methods for stochastic differential equations and uncertainty quantification based on the polynomial chaos approach have become popular in recent years. They are known as stochastic Galerkin methods and they are highly efficient in practical computations providing fast convergence and high accuracy [62]. In the following, we summarize the numerical framework proposed in this paper for solving the SLQR problem using polynomial chaos expansion.

First of all, we use a finite dimensional approximation of the Fourier-Hermite orthogonal polynomials $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ [62]. This is standard in the so-called stochastic Galerkin methods. Then, we set up deterministic control problems for each level (28) and (29). We solve the control problem via Riccati approach and compute
the optimal state for each level. We then compute the approximate optimal state and optimal control for the original problem. The main steps are sketched in the following Algorithm:

|  | Main steps of the stochastic Galerkin method for SLQR problems |
| :--- | :--- |
| $1:$ | Choose finite set of polynomials $H_{\alpha}$ and truncate the random series to a finite |
| random sum. |  |
| $2:$ | Set up deterministic control problems for each level of the chaos expansion |
|  | $(26)$ and (27). |
| $3:$ | Compute the optimal control via Riccati approach for each level. |
| $4:$ | Compute the optimal state for each level. |
| $5:$ | Compute the approximate statistics of the solutions from obtained coefficients. |
| $6:$ | Generate $H_{\alpha}$ and compute the approximate optimal state and optimal control. |

We denote by $\mathcal{I}_{m, p}$ the set of $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$ with $m=\max \{i \in \mathbb{N}$ : $\left.\alpha_{i} \neq 0\right\}$ such that $|\alpha| \leq p$. As a first step, we represent $y$ in its truncated polynomial chaos expansion form $\widetilde{y}$, i.e. we approximate the solution with the chaos expansion in $\oplus_{k=0}^{p} \mathcal{H}_{k}$ with $m$ random variables $\tilde{y}(t, \omega)=\sum_{\alpha \in \mathcal{I}_{m, p}} \widetilde{y}_{\alpha}(t) H_{\alpha}(\omega)$; the previous sum has $P=\frac{(m+p)!}{m!p!}$ terms. Once the coefficients of the expansion $\tilde{y}$ are obtained, we are able to compute all the moments of the random field, e.g. the expectation $\mathbb{E} y=y_{0}$ and the variance of the solution $\operatorname{Var}(\tilde{y})=\sum_{\alpha \in \mathcal{I}_{m, p}} \alpha!\tilde{y}_{\alpha}^{2}$.

We would like to underline that the polynomial chaos expansion converges quite fast, i.e even small values of $p$ may lead to very accurate approximation. The error generated by the truncation of the chaos expansion, in $L^{2}(\Omega, \mathcal{H})$ is

$$
\mathcal{E}^{2}=\|y(x, \omega)-\tilde{y}(x, \omega)\|_{L^{2}(\Omega, \mathcal{H})}^{2}=\mathbb{E}\|y(x, \omega)-\tilde{y}(x, \omega)\|_{\mathcal{H}}^{2}=\sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, p}} \alpha!\left\|y_{\alpha}(x)\right\|_{\mathcal{H}}^{2}
$$

for $x \in \mathcal{D}$. Note that if instead of a Gaussian random variable, a stochastic generalized function is considered, i.e. when the coefficients are singular, the error $\mathcal{E}^{2} \rightarrow 0$ converges in a certain space of weighted generalized stochastic functions.

Finally, we would like to point out that efficient solvers for differential Riccati equations have been proposed in recent years $[1,5,6,7,33]$. The potential of this approach is notable. An efficient numerical implementation is work in progress and will be reported somewhere else.

Acknowledgments. The authors would like to thank the referees for their valuable comments. They greatly helped to improve this manuscript. H. Mena was partially supported by the project Numerical methods in Simulation and Optimal Control through the program Nachwuchsförderung 2014 at University of Innsbruck.

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Received November 2015; revised January 2016.

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# Solving Stochastic LQR Problems by Polynomial Chaos 

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#### Abstract

We consider the infinite dimensional stochastic linear quadratic optimal control problem for the infinite horizon case. We provide a numerical framework for solving this problem using a polynomial chaos expansion approach. By applying the method of chaos expansions to the state equation, we obtain a system of deterministic partial differential equations in terms of the coefficients of the state and the control variables. We set up a control problem for each equation, which results in a set of infinite horizon deterministic linear quadratic regulator problems. We prove the optimality of the solution expressed in terms of the expansion of these coefficients compared to the direct approach. We perform numerical experiments which validate our approach and compare the finite and infinite horizon case.


Index Terms-Stochastic optimal control, computational methods.

## I. INTRODUCTION

T1 HE FINITE dimensional stochastic linear quadratic regulator (SLQR) problem has been deeply studied, a complete survey can be found in, e.g., [26]. Several early works in the literature have addressed stochastic optimization in infinite dimensions. A complete Riccati feedback synthesis of the infinite dimensional problem with disturbance in the state in the finite horizon case has been addressed by Da Prato [7]. Recently, a theoretical framework for this problem has been laid for the general case of singular estimates control systems in the presence of noise in the control and considering a finite time penalization in the performance index [13]. Moreover, an approximation scheme for solving the control

Manuscript received March 4, 2018; revised May 16, 2018; accepted May 23, 2018. Date of publication June 7, 2018; date of current version June 24, 2018. The work of T. Levajković was supported by Research Grant for Austrian Graduates at the Office of the Vice Rector for Research, University of Innsbruck. The work of H. Mena and L.-M. Pfurtscheller was supported by the Austrian Science Fund under Project P27926. Recommended by Senior Editor G. Yin. (Corresponding author: Tijana Levajkovic.)
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Digital Object Identifier 10.1109/LCSYS.2018.2844730
problem and the associated differential Riccati equation (DRE) has been proposed in [17]. In this letter we consider a polynomial chaos approach (also known as the method of chaos expansions) for solving infinite dimensional SLQR problems for the infinite horizon case.

The results of this letter can be obtained in a complete analogous way for finite dimensional systems. However, we are interested in applications arising from infinite dimensional systems, e.g., in the optimal control of the stochastic
heat transfer. Moreover, working in the infinite dimensional framework allows one to combine our approach directly with numerical schemes for operator equations, e.g., [5], [10], and [21]. The latter would not be possible by using a finite dimensional setting.

The infinite dimensional SLQR problem consists of the state equation

$$
\begin{align*}
d y(t) & =(A y(t)+B u(t)) d t+C y(t) d W(t), \quad t \geq 0 \\
y(0) & =y^{0} \tag{1}
\end{align*}
$$

defined on the state space $\mathcal{H}$, where $A$ and $C$ are operators on $\mathcal{H}, B$ acts from the control space $\mathcal{U}$ to the state space $\mathcal{H}$ and $y^{0}$ is a random variable. Spaces $\mathcal{H}$ and $\mathcal{U}$ are Hilbert spaces and $\{W(t)\}_{t \geq 0}$ is a $\mathcal{H}$-valued Wiener process on a given probability space ( $\Omega, \mathcal{F}, \mu$ ) in sense of [9]. The operators $B$ and $C$ are considered to be linear and bounded, while $A$ could be unbounded. The objective is to minimize the functional

$$
\begin{equation*}
J(u)=\mathbb{E}\left[\int_{0}^{\infty}\left(\|R y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t\right] \tag{2}
\end{equation*}
$$

over all possible controls $u$ and subject to the condition that $y$ satisfies the state equation (1). The operator $R$ is bounded and takes values in the Hilbert space $\mathcal{H}$ and $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mu$. A control process $u^{*}$ is called optimal if it minimizes the cost (2) over all admissible control processes $u \in \mathcal{A}$, i.e., for which it holds

$$
\min _{u \in \mathcal{A}} J(u)=J\left(u^{*}\right)
$$

The corresponding trajectory is denoted by $y^{*}$. The pair of stochastic processes $\left(y^{*}, u^{*}\right)$ is called the optimal pair.

Polynomial chaos was first introduced by Wiener in 1938 and was further developed by Itô and many other authors. It has recently been applied to solving different types of stochastic (partial) differential equations ( $\mathrm{S}(\mathrm{P}) \mathrm{DEs}$ ),
see [15], [18]. The basic idea is to construct the solution of the considered SPDE as a Fourier series in terms of a Hilbert space basis of orthogonal stochastic polynomials, resulting in a system of deterministic equations for the coefficients. Thus, a stochastic system can be represented as a deterministic system with higher dimensionality, however, the computational cost is reduced since there is no need in extensive sampling to capture the uncertainty. Moreover, the first moments of the optimal solution can be computed easily. This approach has already been applied in [14] and [16] for stochastic optimal control problems. This letter generalizes the results for the finite horizon case presented in [16] to the infinite horizon case and provides numerical examples that validate the proposed approach.

## II. SOLUTION OF THE SLQR PROblem

In this section we discuss the SLQR problem on a Hilbert space $\mathcal{H}$. Let $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$ and let $\mathcal{U}$ be another Hilbert space. We denote by $\mathcal{L}^{2}(\Omega, \mathcal{H})$ the set of $\mathcal{H}$-valued random variables with finite second moments. Let $\mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{H})$ be the set of all $\mathcal{H}$-valued square integrable stochastic processes, i.e., which satisfy $\int_{0}^{\infty} \mathbb{E}\|X(t, \omega)\|_{\mathcal{H}}^{2} d t<$ $\infty$ and let $\mathcal{M}^{2}([0, \infty) \times \Omega, \mathcal{H})$ be the space of all strongly measurable $\mathcal{H}$-valued square integrable stochastic processes such that $\int_{0}^{\infty} \mathbb{E}\|X(t, \omega)\|_{\mathcal{H}}^{2} d t<\infty$. We denote by $D(S)$ the domain, and by $S^{\star}$ the adjoint operator of a certain operator $S$.

Consider the homogeneous stochastic equation

$$
\begin{equation*}
d y(t)=A y(t) d t+C y(t) d W(t), \quad y(0)=y^{0} \tag{3}
\end{equation*}
$$

We call a stochastic process of the form

$$
y(t)=e^{A t} y^{0}+\int_{0}^{t} e^{A(t-s)} C y(s) d W(s)
$$

the mild solution of the equation (3) if $y(t) \in D(C)$, $P\left(\int_{0}^{\infty}\|y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$ and $P\left(\int_{0}^{\infty}\|C y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=$ 1. Then, $(A, C)$ is called stable, if the mild solution of (3) satisfies

$$
\begin{equation*}
\mathbb{E}\left[\|y(t)\|_{\mathcal{H}}^{2}\right] \leq M_{1} e^{-\omega t} \mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}, \quad t \geq 0 \tag{4}
\end{equation*}
$$

for some $M_{1}, \omega>0$ and for all $y^{0} \in \mathcal{L}^{2}(\Omega, \mathcal{H})$.
The system $(A, B, C)$ is called stabilizable, if there exists a bounded operator $K \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ such that $(A-B K, C)$ is stable. Let $D \in \mathcal{L}(\mathcal{H})$ be bounded, then we call $(A, D, C)$ detectable, if there exists a bounded operator $K_{1} \in \mathcal{L}(\mathcal{H})$ such that $\left(A-K_{1} D, C\right)$ is stable, see [4].

## A. Standard Approach

Let us consider the infinite dimensional SLQR optimal control problem (1) - (2). The following theorem provides the conditions for the existence of the optimal control in the feedback form by the associated algebraic Riccati equation (ARE), for details we refer to [8].

Theorem 1 [8]: Let the following assumptions hold:
(a1) The linear operator $A$ is the infinitesimal generator of a $C_{0}$-semigroup $\left(e^{A t}\right)_{t \geq 0}$ on the space $\mathcal{H}$.
(a2) The linear operator $B$ is bounded $\mathcal{U} \rightarrow \mathcal{H}$.
(a3) The operators $R, C$ are bounded linear operators.
(a4) The system $(A, B, C)$ is stabilizable.
(a5) The system $(A, R, C)$ is detectable.
Then, the optimal control $u^{*}$ of the linear quadratic problem (1) - (2) satisfies the feedback characterization in terms of the optimal state $y^{*}$

$$
\begin{equation*}
u^{*}(t)=-B^{\star} P y^{*}(t) \tag{5}
\end{equation*}
$$

where $P$ is the unique minimal positive self-adjoint operator solving the Riccati equation

$$
\begin{equation*}
P A+A^{\star} P+C^{\star} P C+R^{\star} R-P B B^{\star} P=0 \tag{6}
\end{equation*}
$$

## B. Chaos Expansions Approach

In the following we present another approach for solving the control problem (1) - (2), which has a great potential numerically. This approach combines the method of chaos expansions with the deterministic optimal control theory. The method of chaos expansions is based on the Wiener-Itô chaos expansion theorem which states that a random variable, respectively a stochastic process, can be expressed as series in terms of an orthogonal basis of stochastic polynomials depending on the probability measure. Particularly, if the underlying probability space is a Gaussian space, then the orthogonal basis of stochastic polynomials is built in terms of the Hermite polynomials and an orthonormal basis of $\mathcal{H}$. The case $\mathcal{H}=\mathcal{L}^{2}(\mathbb{R})$ is very important in applications, where the orthonormal basis $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ can be chosen as the Hermite functions $\left\{\xi_{k}\right\}_{k \in \mathbb{N}}$. Hence, for $k \in \mathbb{N}_{0}$, we denote by $h_{k}(x)=(-1)^{k} e^{\frac{x^{2}}{2}} \frac{d^{k}}{d x^{k}}\left(e^{-\frac{x^{2}}{2}}\right)$ the family of Hermite polynomials and by

$$
\xi_{k}(x)=\frac{1}{\sqrt[4]{\pi} \sqrt{(k-1)!}} e^{-\frac{x^{2}}{2}} h_{k-1}(\sqrt{2} x), \quad k \in \mathbb{N}
$$

the family of Hermite functions. Let $\mathcal{I}$ be the set of sequences of non-negative integers which have only finitely many nonzero components, i.e., each $\alpha \in \mathcal{I}$ is of the form $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots \alpha_{m}, 0,0, \ldots\right), \alpha_{j} \in \mathbb{N}_{0}, 1 \leq j \leq m, m \in \mathbb{N}$. The $i$-th unit vector is denoted by $\varepsilon^{(i)}$ and the zero vector by $\mathbf{0}$. The sum of all components of $\alpha \in \mathcal{I}$ is its length and is denoted by $|\alpha|$. The Fourier-Hermite polynomials are defined by

$$
H_{\alpha}(\omega)=\prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle\omega, e_{i}\right\rangle\right), \quad \alpha \in \mathcal{I}
$$

Then, the square integrable processes $y \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{H})$ and $u \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{U})$ can be represented in their chaos expansion forms

$$
\begin{align*}
& y(t, \omega)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}(t) H_{\alpha}(\omega) \\
& u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \tag{7}
\end{align*}
$$

for $t \geq 0, \omega \in \Omega$ and where the coefficients $y_{\alpha} \in$ $\mathcal{L}^{2}([0, \infty), \mathcal{H})$ and $u_{\alpha} \in \mathcal{L}^{2}([0, \infty), \mathcal{U})$ for all $\alpha \in \mathcal{I}$. In this way, the deterministic part of a stochastic process is split from its random part. The zeroth coefficients $y_{0}(t)=\mathbb{E} y(t, \omega)$ and
$u_{0}(t)=\mathbb{E} u(t, \omega)$ in (7) are the corresponding expectations of $y$ and $u$.

All the operators $A, B, C$ and $R$ appearing in the problem (1) - (2) are assumed to be coordinatewise operators, i.e., the action of $A$ on $y \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{H})$ is given by $A y(t, \omega)=\sum_{\alpha \in \mathcal{I}} A y_{\alpha}(t) H_{\alpha}(\omega)$, (it acts only on the coefficients $y_{\alpha}$ of the process $y$ ). Hence, by applying the representation forms (7) to the equation (1) we transform it to a system of deterministic equations. Namely, in a similar way to [14] and [15], the solution of (1) can be written in the chaos expansion form (7) and its coefficients $y_{\alpha}, \alpha \in \mathcal{I}$ can be computed from

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}} e_{i}(t), \tag{8}
\end{equation*}
$$

with $y_{\alpha}(0)=y_{\alpha}^{0}$, where the sum is defined for all $i$ such that the difference of $\alpha-\varepsilon^{(i)}$ is nonnegative. Applying the chaos expansion method to the cost functional (2), analogously to [16], one gets a characterization of the optimal control in terms of the expansion coefficients. This is summarized in the following theorem.
Theorem 2: Let (a1) - (a5) from Theorem 1 hold. Let $(A, B, R)$ be stabilizable and $\mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$. Then, the following hold:
(a) Solving the problem (1)-(2) is equivalent to solving the deterministic optimal control problems in each $\alpha$-level. Particularly, for $\alpha=\mathbf{0}$ :

$$
\begin{equation*}
\min _{u_{0}} J\left(u_{\mathbf{0}}\right)=\min _{u_{0}} \int_{0}^{\infty}\left(\left\|R y_{\mathbf{0}}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\mathbf{0}}(t)\right\|_{\mathcal{U}}^{2}\right) d t \tag{9}
\end{equation*}
$$

subject to

$$
\begin{equation*}
y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \tag{10}
\end{equation*}
$$

and for $\alpha>\mathbf{0}$ :

$$
\begin{equation*}
\min _{u_{\alpha}} J\left(u_{\alpha}\right)=\min _{u_{\alpha}} \int_{0}^{\infty}\left(\left\|R y_{\alpha}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\alpha}(t)\right\|_{\mathcal{U}}^{2}\right) d t \tag{11}
\end{equation*}
$$

subject to (8).
(b) The optimal control problem (1) - (2) has a unique optimal control $u^{*}$ given in the chaos expansion form

$$
\begin{align*}
u^{*}(t) & =-\sum_{\alpha \in \mathcal{I}} B^{\star} P_{d} y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B^{\star} k_{\alpha}(t) H_{\alpha} \\
& =-B^{\star} P_{d} y^{*}(t)-B^{\star} \mathcal{K}, \tag{12}
\end{align*}
$$

where the operator $P_{d}$ is the unique minimal positive selfadjoint solution of the ARE

$$
\begin{equation*}
P_{d} A+A^{\star} P_{d}+R R^{\star}-P_{d} B B^{\star} P_{d}=0 \tag{13}
\end{equation*}
$$

and $\mathcal{K}$ is a stochastic process with the coefficients $k_{\alpha}(t)$ that for all $\alpha \in \mathcal{I}$ solve the auxiliary equations

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+A_{p}^{\star} k_{\alpha}(t)+P_{d}\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) e_{i}(t)\right)=0, \tag{14}
\end{equation*}
$$

with $A_{p}^{\star}=A^{\star}-P_{d} B B^{\star}$ and the condition $\lim _{T \rightarrow \infty} k_{\alpha}(T)=0$, and $y^{*}(t)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}$ is the optimal state.

Proof: The proof generalizes the proof from [16] for the finite horizon case. Here we present the main steps. By applying the method of chaos expansions to the problem (1)-(2),
it transforms to the system of deterministic optimal control problems, i.e., the problem (9) - (10) for $|\alpha|=0$ and (8)-(11) for all $|\alpha|>0$. Namely, for each $\alpha \in \mathcal{I}$, we need to solve the deterministic problems minimizing the cost $J\left(u_{\alpha}\right)=\int_{0}^{\infty}\left(\left\|R y_{\alpha}\right\|_{\mathcal{H}}^{2}+\left\|u_{\alpha}\right\|_{\mathcal{U}}^{2}\right) d t$ with respect to

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+f_{\alpha}(t), \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{15}
\end{equation*}
$$

where the inhomogeneous part is of the form $f_{\alpha}=0$ for $|\alpha|=$ 0 and $f_{\alpha}(t)=\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) e_{i}(t)$ for $|\alpha|>0$ and $t>0$. Since the inhomogeneity $f_{\alpha} \in \mathcal{L}^{2}((0, \infty) \mathcal{H})$ and the conditions (a4) - (a5) hold, then for each $\alpha \in \mathcal{I}$ there exists the optimal solution in the feedback form

$$
\begin{equation*}
u_{\alpha}^{*}(t)=-B^{\star} P_{d} y_{\alpha}^{*}(t)-B^{\star} k_{\alpha}(t), \tag{16}
\end{equation*}
$$

where $P_{d}$ solves the algebraic Riccati equation (13), while $k_{\alpha}(t)$ is a solution of the auxiliary differential equation $k_{\alpha}^{\prime}(t)+$ $\left(A^{\star}-P_{d} B B^{\star}\right) k_{\alpha}(t)+P_{d} f_{\alpha}(t)=0$ satisfying $\lim _{T \rightarrow \infty} k_{\alpha}(T)=0$, for $|\alpha|>0$ and $k_{0}=0$, see [4, Part V]. The optimal control for any initial condition $y^{0}$ exists since the system $(A, B, R)$ is stabilizable. Moreover, from (a5) it follows that the feedback operator $A_{p}=A-B B^{\star} P_{d}$ is exponentially stable, and thus the unique solution of (13) is globally attractive. Next, summing up the coefficients (16) into the expansion (7) and applying the linearity properties of the given operators we obtain the form of the optimal control

$$
\begin{aligned}
u^{*}(t) & =u_{\mathbf{0}}(t)+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha} \\
& =-B^{\star} P_{d} y_{\mathbf{0}}^{*}(t)+\sum_{|\alpha|>0}\left(-B^{\star} P_{d y} y_{\alpha}^{*}(t)-B^{\star} k_{\alpha}(t)\right) H_{\alpha} \\
& =-B^{\star} P_{d}\left(\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}\right)-B^{\star}\left(\sum_{\alpha \in \mathcal{I}} k_{\alpha} H_{\alpha}\right),
\end{aligned}
$$

which leads to (12). Finally, a proof that the obtained optimal control is square integrable goes in the similar manner as for the finite horizon case, see [16]. Namely, we include the feedback form (16) of the optimal controls $u_{\alpha}^{*}, \alpha \in \mathcal{I}$ in the state equations (10) and (8) and obtain

$$
\begin{equation*}
y_{\alpha}^{* \prime}(t)=A_{p} y_{\alpha}^{*}(t)+g_{\alpha}(t), y_{\alpha}^{*}(0)=y_{\alpha}^{0} \tag{17}
\end{equation*}
$$

where $g_{\alpha}(t)=-B B^{\star} k_{\alpha}(t)+f_{\alpha}(t)$, for $|\alpha|>0$ and $g_{0}=0$ for $|\alpha|=0$. From the assumption (a1) it follows that $A$ is the infinitesimal generator of a strongly continuous semigroup $\left(e^{A t}\right)_{t \geq 0}$. Since each Hilbert space is a reflexive Banach space, the family $\left(e^{A^{\star} t}\right)_{t \geq 0}$ is a strongly continuous semigroup whose infinitesimal generator is $A^{\star}$, see [4], [22]. The operator $A_{p}$ can be interpreted as a perturbation of $A$ with a bounded operator, and $A_{p}$ is exponentially stable. Hence, we can associate an evolution system $U(t, s)$ to the initial value problems (17) such that the family of solution maps $U(t, s) y_{\alpha}^{0}$ is an evolution in $C([0, \infty), \mathcal{H})$, see [4]. Also, the adjoint operator $A_{p}^{\star}$, is associated to the corresponding adjoint evolution system $U^{\star}(t, s)$, $0 \leq s \leq t$, see [22]. Then, for every $y_{\alpha}^{0} \in D\left(A_{p}\right)$ the mild solution of (17) is given in the form $y_{\mathbf{0}}^{*}(t)=U(t, 0) y_{0}^{0}$ and

$$
y_{\alpha}^{*}(t)=U(t, 0) y_{\alpha}^{0}+\int_{0}^{t} U(t, s) g_{\alpha}(s) d s
$$

for $|\alpha|>0$ and $0 \leq s \leq t$, and $y_{\alpha}$ are continuous functions for all $\alpha \in \mathcal{I}$. Since the inhomogeneity $g_{\alpha} \in \mathcal{L}^{2}([0, \infty), \mathcal{H})$, from $(a 3)-(a 5)$, the estimate of the evolution system and the Grönwall's lemma we obtain that for each for all $\alpha \in \mathcal{I}$, the coefficients $y_{\alpha}^{*}$ satisfiy (4), which together with the assumption $\mathbb{E}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$ lead to $y^{*} \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{H})$.
Note that, for any $T>0$ and $t<T$, the solutions $k_{\alpha}$ of (14) are expressed in terms of the adjoint evolution system

$$
k_{\alpha}(t)=U_{\alpha}^{\star}(T, t) k_{\alpha}(T)+\int_{t}^{T} U_{\alpha}^{\star}(s, t) P_{d} f_{\alpha}(s) d s
$$

for $\alpha \in \mathcal{I}$, such that $\lim _{T \rightarrow \infty} k_{\alpha}(T)=0, \alpha \in \mathcal{I}$. Similarly as for the optimal state process $y^{*}$, it could be shown that the process $\mathcal{K}$ is square integrable, i.e., $\mathcal{K} \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{U})$, which then implies $u^{*} \in \mathcal{L}^{2}([0, \infty) \times \Omega, \mathcal{U})$.

The SLQR problems on finite and infinite horizons are strongly related. In the deterministic setting the infinite horizon problem is studied as a limit of the finite horizon time, a similar study holds for the stochastic case and also for the chaos expansion approach. This will be presented somewhere else. The following theorem characterizes the action of the Riccati operator. The recurrence (18) can be interpreted as memory property in the noise.
Theorem 3: Let the assumptions from Theorem 2 hold. Then, the optimal control (12) of (1)-(2) obtained via the chaos expansion method is equal to the solution (5) obtained via the Riccati approach if and only if for all $\alpha>\mathbf{0}$ and $t \geq 0$ it holds

$$
\begin{equation*}
C^{\star} P C y_{\alpha}^{*}(t)=P\left(\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}^{*}(t) e_{i}(t)\right) . \tag{18}
\end{equation*}
$$

Proof: Similarly as for the finite horizon case [16], we assume that the solutions (12) and (5) are equal. We obtain the difference $P-P_{d}$ expressed in terms of a stochastic process $\mathcal{K}$, whose coefficients are generated by the inhomogeneties $f_{\alpha}$, $\alpha \in \mathcal{I}$ in (15), i.e.,

$$
\begin{equation*}
\left(P-P_{d}\right) y^{*}(t)=\mathcal{K}, \tag{19}
\end{equation*}
$$

where $y^{*}(t)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}$ is the form of the optimal state. After differentiating (19) and substituting the equations (6), (8), (13) and (14), the optimality condition (18) is derived for $|\alpha|>0$.

The proposed approach for solving SLQR problems in terms of chaos expansions is not restricted only to problems (1) - (2) with Gaussian noise, but it can be also applied for more general and non-Gaussian type of noises, e.g., for problems involving colored noise [15]. One needs to replace the base of Hermite polynomials with another class of orthogonal polynomials from the Askey scheme of hypergeometric orthogonal polynomials that corresponds to the specific noise arising in the considered stochastic state equation [25].

## ili. Numerical Simulations

In this section we present an example for the SLQR problem. We consider the infinite horizon problem as well
as the finite horizon problem and compare two approaches to solve these problems.

## A. Stochastic Heat Transfer

As a numerical example we introduce a bilinear controlled heat transfer model, see [2]. On a unit square $\mathcal{D}=[0,1] \times[0,1]$, the heat equation is given with different boundary conditions. On two edges we employ Dirichlet boundary conditions, on the third edge a fixed boundary condition $y=u$ is applied and a stochastic Robin boundary condition $n \cdot \nabla y=0.5(0.5+\dot{w}) y$ is used on the final edge. We discretize the equation in space and use $n=10$ grid points in every direction. Applying central finite differences, we obtain the matrices $A \in \mathbb{R}^{n^{2} \times n^{2}}, C \in \mathbb{R}^{n^{2} \times n^{2}}$ and $B \in \mathbb{R}^{n^{2} \times 1}$. Moreover, $R$ is computed by the mean of the vector $y$, i.e., $R=\frac{1}{n^{2}}(1, \ldots 1)$. Thus, we obtain the SLQR problem (1)-(2). We solve it with two different approaches. The first one, which we will call in the following the standard approach, consists of computing the optimal pair using the results of Theorem 1. Thus, we have to solve the bilinear algebraic Riccati equation (6). Applying Newton's method, one has to solve a bilinear Lyapunov equation in every step. This can be done by a low rank alternating direction iteration (ADI) method, for details we refer to [1]. Then, we apply an implicit Euler-Maruyama scheme to solve the discretized SPDE (1) and compute the optimal state and the optimal control in every time step. The first and the second moments are approximated by Monte-Carlo integration. In the second case, we combine the polynomial chaos approach described in Section II-B, with appropriate deterministic numerical methods. We denote by $\mathcal{I}_{m, p}$ the set of $\alpha=\left(\alpha_{1}, \ldots \alpha_{m}, 0,0, \ldots\right) \in \mathcal{I}$ with $m=\max \left\{i \in \mathbb{N}: \alpha_{i} \neq 0\right\}$ such that $|\alpha| \leq p$. We represent $y$ and $u$ in their chaos expansion forms (7) and truncate the sums after $P=\frac{(m+p)!}{m!p!}$ terms. Particularly, $m$ is the number of uncorrelated random variables used in the approximation and $p$ is the highest order of the stochastic polynomials appearing in the truncated chaos expansions (7). Since the choice of $m$ and $p$ influences the accuracy of the approximation, these parameters can be chosen so that the norm of the approximation remainder is smaller than a prespecified error [15]. The obtained system is in the following step solved by an appropriate numerical method and as outcome the discretized approximation solution of the system is obtained. The global error of the proposed numerical scheme depends on the error generated by the truncation of the chaos expansion and the error influenced by the discretisation method.
We choose $m=10$ and $p=4$, which gives $P=1001$ different levels. For every level $\alpha>\mathbf{0}$ we solve the deterministic differential equation (8) and the auxiliary equation (14) with the implicit Euler method and the corresponding ARE (13) with a solver from the LYAPACK toolbox [23]. Once the coefficients $y_{\alpha}$ are obtained, the moments of the random field, e.g., the expectation $\mathbb{E} y=y_{0}$ and the variance $\operatorname{Var}(y) \approx \sum_{\alpha \in \mathcal{I}_{m, p}} \alpha!y_{\alpha}^{2}$ can be computed. We thus apply the two presented approaches to the example described in this subsection. As initial condition we choose $y\left(0, x_{1}, x_{2}\right)=\exp \left(-\left(x_{1}-x_{2}\right)^{2}\right)$ for $\left(x_{1}, x_{2}\right) \in \mathcal{D}$, as final time $T=\frac{1}{2}$ and as step size $\Delta t=0.0025$. Fig. 1, left


Fig. 1. Left: Mean of the optimal control over time for both approaches. Right: Variance of the optimal control.
shows the mean of the optimal control for both approaches. Using the standard approach, the equation (1) is solved for 10.000 different realizations of the Wiener process and the mean is taken. For the chaos expansion approach, the mean is computed as in the procedure described above. Similarly, the variance of the solution over time is plotted in Fig. 1, right for both schemes. We see that the mean of the solution converges to a steady state quite fast and also the variance decays rapidly after some time steps. The difference between the two different approaches is neglectable and both work well.

As discussed by Mühlpfordt et al. [20] applying the chaos expansion method leads to a truncation error. Hence, we use as reference solution the standard approach with 20.000 simulations and compute the variance of the optimal control. Then, for a different number $m$ of uncorrelated random variables, the relative error of the variance is computed, see Fig. 2, left. The error behaves as expected. For a lower number of polynomials, the error is larger, whereas, using more polynomials yields to a more accurate result. However, even for $m=2$ the relative error is of order $1 e-10$. Fig. 2, right shows the computational costs of both approaches for different grid sizes, i.e., for different sizes of the matrices in the control problem. As the most demanding part of the algorithms is the solution of the ARE, the polynomial chaos approach has great potential as the resulting matrix equation does not include the bilinear term. Note, that in the standard approach the realizations were computed in parallel, which would be doable also for the chaos expansion approach using tensors, however we do not take advantage of it yet. Therefore, we expect to be even more competitive. Moreover, Fig. 2 shows also the possible adaptivity of the algorithm, depending on the desired accuracy, the chaos expansion can be truncated after only a few terms.
As the variance of the optimal control has its peak around 0.05 , we plot the moments of the optimal state at time $T=0.05$. Thus, in Fig. 3, left the mean computed by the standard approach is plotted. Again, the differential equation is solved 10.000 times and we take the mean of the realizations. Fig. 3, right shows the mean of the chaos expansion approach. Similarly as for the mean of the optimal control in Fig. 1, left we observe that the mean of the solution yields to the same result using either the standard scheme or the chaos expansion method. We repeat the same calculation and compute the variance of the optimal state, see Fig. 4. We observe only small differences in the pictures. This is either due to


Fig. 2. Left: Relative error of the optimal control for different number of polynomials in the chaos expansion. Right: Computational costs of the two approaches for different space discretizations.


Fig. 3. Left: Mean of the optimal state at $T=0.05$ obtained by the standard approach. Right: Mean of the optimal state at $\mathrm{T}=0.05$ computed by the chaos expansion method.


Fig. 4. Left: Variance of the optimal state at $T=0.05$ obtained by the standard approach. Right: Variance of the optimal state at $\mathrm{T}=0.05$ computed by the chaos expansion method.
the error in the Monte-Carlo sampling or the truncation of the expansion.

## B. Finite Time Horizon Case

Levajković et al. [16] considered the SLQR problem in finite time horizon. In this section, we integrate the cost functional (2) from 0 to $T$. Using the standard approach, after a numerical discretization one has to solve instead of the matrix ARE a matrix DRE of the form

$$
\begin{aligned}
-\dot{P}(t)= & A^{\top} P(t)+P(t) A+C^{\top} P(t) C \\
& -P(t) B B^{\top} P(t)+R^{\top} R, \quad t \geq 0
\end{aligned}
$$

such that $P(T)=0$. We solve this differential equation based on the splitting schemes proposed in [6]. This method was first introduced for solving DREs arising in deterministic LQR problems [24]. Splitting methods in general show better performance compared to other standard approaches like the ones proposed in [3] and [19]. Thus, for the polynomial chaos approach we have to solve the arising DRE by a splitting scheme with one splitting term less than in the standard approach. The remaining equations are solved by the methods


Fig. 5. Left: Mean of the optimal control over time for both approaches. Right: Variance of the optimal control.


Fig. 6. Norm of the difference between the solution of the DRE and the ARE for both approaches.
introduced in the previous subsection. Then, using the stochastic heat transfer model described in the previous subsection with the same parameters, we compute the first moments of the solution, see Fig. 5. We observe a similar behaviour as in the infinite horizon case. Note that the solution converges to the same steady-state. As the only difference between both problems is given by the related Riccati equations, we compute the absolute difference between the DREs and the AREs for both approaches, see Fig. 6. For this example the solution of the finite horizon problem is very close to the infinite horizon one. This is in accordance to the mentioned remark that the infinite horizon problem can be seen as the limit of the finite horizon problem. From the numerical solution point of view solving the finite horizon case is always more expensive than solving the infinite horizon case as differential Riccati equations have to be solved instead of algebraic ones.

Remark 1: The proposed method is very competitive for solving SLQR problems of the form (1) - (2). Moreover, it has a great potential for solving SLQR problems involving random operators [11], [12], where a backward stochastic Riccati equation has to be solved instead. Also, using the chaos expansion approach allows one to take advantage of state-of-the-art numerical methods for deterministic problems available in the literature.

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# The Stochastic LQR Optimal Control with Fractional Brownian Motion 

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#### Abstract

We consider the stochastic linear quadratic optimal control problem where the state equation is given by a stochastic differential equation of the Itô-Skorokhod type with respect to fractional Brownian motion. The dynamics are driven by strongly continuous semigroups and the cost functional is quadratic. We use the fractional isometry mapping defined between the space of square integrable stochastic processes with respect to fractional Gaussian white noise measure and the space of integrable stochastic processes with respect to the classical Gaussian white noise measure. By this mapping we transform the fractional state equation to a state equation with Brownian motion. Applying the chaos expansion approach, we can solve the optimal control problem with respect to a state equation with the standard Brownian motion. We recover the solution of the original problem by the inverse of the fractional isometry mapping. Finally, we consider a general form of the state equation related to the Gaussian colored noise, we study the control problem, a system with an algebraic constraint and a particular example involving generalized operators from the Malliavin calculus.


## 1. Introduction

The linear quadratic Gaussian control problem for the control of finite-dimensional linear stochastic systems with Brownian motion is well understood, see, e.g., [15]. The case for fractional Brownian motion [10, 11, 12] as well as the infinitedimensional case have been studied recently [9]. A more general problem arises if the noise depends on the state variable, this is the so-called stochastic linear quadratic regulator (SLQR) problem. The SLQR problem in infinite dimensions was solved by Ichikawa in [22] using a dynamic programming approach. Da Prato [8] and Flandoli [14] later considered the SLQR for systems driven by analytic semigroups with Dirichlet or Neumann boundary controls, but with disturbance in the state only. The infinite-dimensional SLQR with random coefficients has been investigated in $[16,17]$ along with the associated backward stochastic Riccati
equation. Recently, a theoretical framework for the SLQR has been laid for singular estimates control systems in the presence of noise in the control and in the case of finite time penalization in the performance index [18]. Considering the general setting described in [18, 26], an approximation scheme for solving the control problem and the associated Riccati equation has been proposed in [28]. In [27], a novel approach for solving the SLQR based on the concept of chaos expansion from white noise analysis is proposed. In this paper we extend the results from [27] to the SLQR problem with fractional Brownian motion.

Fractional Brownian motion $B^{(H)}$ is a one-parameter extension of a standard Brownian motion and the main properties of such a Gaussian process depend on values of the Hurst parameter $H \in(0,1)$. Fractional Brownian motion, as a process with independent increments which have a long-range dependence and self-similarity properties found many applications when modeling wide range of problems in hydrology, telecommunications, queueing theory and mathematical finance [5]. A specific construction of a stochastic integral with respect to a fractional Brownian motion defined for all possible values $H \in(0,1)$, was introduced by Elliot and van der Hoek in [13]. Several different definitions of stochastic integration for fractional Brownian motion appear in literature [5, 13, 39, 42]. In this paper we follow [13] and use the definition of the fractional white noise spaces by use of the fractional transform mapping for all values of $H \in(0,1)$ and the extension of the action of the fractional transform operator to a class of generalized stochastic processes. The main properties of the fractional transform operator and the connection of a fractional Brownian motion with a classical Brownian motion on the classical white noise space were presented in $[5,33]$.

We consider the infinite-dimensional SLQR problem, which consists of the state equation

$$
\begin{equation*}
d \widetilde{y}(t)=(\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)) d t+\widetilde{\mathbf{C}} \widetilde{y}(t) d B^{(H)}(t), \quad \widetilde{y}(0)=\widetilde{y}^{0}, \quad t \in[0, T] \tag{1}
\end{equation*}
$$

defined on Hilbert state space $\mathcal{H}$, where $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{C}}$ are operators on $\mathcal{H}$ and $\widetilde{\mathbf{B}}$ acts from the control space $\mathcal{U}$ to the state space $\mathcal{H}$ and $\widetilde{y}^{0}$ is a random variable. Spaces $\mathcal{H}$ and $\mathcal{U}$ are Hilbert spaces. The operators $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ are considered to be linear and bounded, while A could be unbounded. The objective is to minimize the functional

$$
\begin{equation*}
\mathbf{J}^{(H)}(\widetilde{u})=\mathbb{E}\left[\int_{0}^{T}\left(\|\widetilde{\mathbf{R}} \widetilde{y}\|_{\mathcal{W}}^{2}+\|\widetilde{u}\|_{\mathcal{U}}^{2}\right) d t+\left\|\widetilde{\mathbf{G}} \widetilde{y}_{T}\right\|_{\mathcal{Z}}^{2}\right] \tag{2}
\end{equation*}
$$

over all possible controls $\widetilde{u}$ and subject to the condition that $\widetilde{y}$ satisfies the state equation (1). The operators $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{G}}$ are bounded observation operators taking values in Hilbert spaces $\mathcal{W}$ and $\mathcal{Z}$ respectively, $\mathbb{E}$ denotes the expectation and $y_{T}=y(T)$. A control process $\widetilde{u}^{*}$ is called optimal if it minimizes the cost (2) over all control processes, i.e.,

$$
\min _{u} \mathbf{J}^{(H)}(\widetilde{u})=\mathbf{J}^{(H)}\left(\widetilde{u}^{*}\right)
$$

The corresponding optimal trajectory is denoted by $\widetilde{y}^{*}$. Thus, the pair $\left(\widetilde{y}^{*}, \widetilde{u}^{*}\right)$ is the optimal solution of the problem (1) and (2) and is called the optimal pair.

Following [13] and [33] we construct a fractional isometry in order to transform optimal control problem (1)-(2) from a fractional space to the corresponding optimal control problem with the state equation given with respect to Brownian motion

$$
d y(t)=(\mathbf{A} y(t)+\mathbf{B} u(t)) d t+\mathbf{C} y(t) d B(t), \quad y(0)=y^{0}, \quad t \in[0, T]
$$

and the performance index

$$
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{W}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right]
$$

We combine the chaos expansion method with deterministic theory of optimal control to solve the above optimal control problem. The solution of the initial problem is thus obtained through the inverse fractional map.

Moreover, we also consider a general state equation of the form

$$
\begin{equation*}
\dot{y}=\mathbf{A} y+\mathbf{T} \diamond y+\mathbf{B} u, \quad y(0)=y_{0} \tag{3}
\end{equation*}
$$

where $\mathbf{A}$ is an operator which generates a strongly continuous semigroup, and $\mathbf{T}$ is a linear bounded operator which combined with Wick product $\diamond$ introduces convolution-type perturbations into the equation. Equation (3) is related to Gaussian colored noise. The existence and uniqueness of its generalized solution was proven in [34]. Examples of this type of equations are: the heat equation with random potential, the heat equation in random (inhomogeneous and anisotropic) media, the Langevin equation, etc. The related control problem for (3) will lead to an optimal control defined in a space of generalized processes. A particular case of (3) together with an algebraic constraint arises in fluid dynamics, e.g., Stokes equations. The resulting system is known as semi-explicit operator differential algebraic equation (ODAE) and it has the form

$$
\dot{y}=\mathbf{A} y+\mathbf{B}^{\star} u+\mathbf{T} \diamond y+f, \quad \mathbf{B} y=g
$$

We conclude the paper with the study of an ODAE involving generalized operators of Malliavin calculus. Particularly, we set the operator $\mathbf{B}$ to be the Skorohod integral $\delta$ and $\mathbf{B}^{\star}$ the Malliavin derivative $\mathbb{D}$. Equations involving generalized operators of Malliavin calculus were studied in [29, 30, 31, 34, 35].

The paper is organized as follows. In Section 2 we briefly state the theoretical background needed, then in Section 3 we define the fractional isometry operator $\mathcal{M}$, prove its properties and study the optimal control problem with state equation given in the form of fractional Itô-Skorokhod integral in fractional space. By using the fractional isometry we study the control problem in the standard space, prove the existence and uniqueness of the control and characterize the optimality of our approach. Finally, we extend our results and solve an ODAE involving the operators of Malliavin calculus.

## 2. Theoretical background

Let $\mathcal{U}$ and $\mathcal{H}$ be separable Hilbert spaces of controls and states, respectively, with norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{H}}$, generated by the corresponding scalar products. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\left(b_{t}\right)_{t \geq 0}$ be a real-valued one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ be the complete right continuous $\sigma$-algebra generated by $\left(b_{t}\right)_{t \geq 0}$. We assume that all function spaces are adapted to the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$. Let $L^{2}(\Omega, \mathbb{P})=L^{2}(\Omega, \mathcal{F}, \mathbb{P})$ be the Hilbert space of square integrable real-valued random variables endowed with the norm $\|F\|_{L^{2}(\Omega, \mathbb{P})}^{2}=\mathbb{E}_{\mathbb{P}}\left(F^{2}\right)$, for $F \in L^{2}(\Omega, \mathbb{P})$, induced by the scalar product $(F, G)_{L^{2}(\Omega, \mathbb{P})}=\mathbb{E}_{\mathbb{P}}(F G)$, for $F, G \in L^{2}(\Omega, \mathbb{P})$, and $\mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to the measure $\mathbb{P}$. Throughout the paper, when it is clear which measure $\mathbb{P}$ is used, we will write $\mathbb{E}$ for the expectation and $L^{2}(\Omega)$ for $L^{2}(\Omega, \mathbb{P})$ omitting $\mathbb{P}$. We denote by $L^{2}(\Omega, \mathcal{U})$ the Hilbert space of $\mathcal{U}$-valued square integrable random variables and by $L^{2}([0, T] \times \Omega, \mathcal{U})$ we denote the Hilbert space of square integrable $\mathcal{F}_{T}$-predictable $\mathcal{U}$-valued stochastic processes $u$ endowed with the norm

$$
\|u\|_{L^{2}([0, T] \times \Omega, \mathcal{U})}^{2}=\int_{0}^{T} \mathbb{E}\left(\|u(t)\|_{\mathcal{U}}^{2}\right) d t
$$

Let $C\left([0, T], L^{2}(\Omega, \mathcal{H})\right)$ be the Hilbert space of $\mathcal{F}_{T}$-predictable continuous $\mathcal{H}$-valued stochastic processes $y$ endowed with the norm

$$
\|y\|_{C\left([0, T], L^{2}(\Omega, \mathcal{H})\right)}^{2}=\sup _{t \in[0, T]} \mathbb{E}\left(\|y(t)\|_{\mathcal{H}}^{2}\right)
$$

### 2.1. The SLQR problem: existence of solution

The infinite-dimensional linear quadratic regulator (LQR) stochastic optimal control problem on Hilbert spaces with respect to Brownian motion is given by the state equation

$$
\begin{equation*}
d y(t)=(\mathbf{A} y(t)+\mathbf{B} u(t)) d t+\mathbf{C} y(t) d B(t), \quad y(0)=y^{0}, \quad t \in[0, T] \tag{4}
\end{equation*}
$$

subject to the quadratic cost functional

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{W}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right] \tag{5}
\end{equation*}
$$

The dynamics of the problem, the operator $\mathbf{A}$, is deterministic and represents an infinitesimal generator of a strongly continuous semigroup $\left(e^{\mathbf{A} t}\right)_{t \geq 0}$ on the state space $\mathcal{H}$. The operators $\mathbf{A}$ and $\mathbf{C}$ are operators on $\mathcal{H}$, while $\mathbf{B}$ is the operator acting from the control space $\mathcal{U}$ to the state space $\mathcal{H}$. We take the operator $\mathbf{C}$ to be linear and bounded. We assume the operators $\mathbf{R}$ and $\mathbf{G}$ to be linear and bounded operators acting on the state space $\mathcal{H}$ into Hilbert spaces $\mathcal{W}$ and $\mathcal{Z}$ respectively. For simplicity, we shall assume that $\mathcal{W}=\mathcal{Z}=\mathcal{H}$ from here onwards. We denote by $\mathcal{D}(\mathbf{S})$ the domain of a certain operator $\mathbf{S}$, and by $\mathbf{S}^{\star}$ the adjoint operator of $\mathbf{S}$.

The aim of the stochastic linear quadratic problem is to minimize the cost functional $\mathbf{J}(u)$ over a set of square integrable controls $u \in L^{2}\left([0, T], L^{2}(\Omega, \mathcal{U})\right)$, which are adapted to the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$.

The following theorem gives conditions for the existence of the optimal control in the feedback form using the associated Riccati equation. For more details on existence of mild solutions to the $\operatorname{SDE}$ (4) we refer to [7] and for the optimal control and Riccati feedback synthesis we refer the reader to [22].
Theorem 1 ([7, 22]). Let the following assumptions hold:
(a1) The linear operator $\mathbf{A}$ is the infinitesimal generator of a $C_{0}$-semigroup $\left(e^{\mathbf{A} t}\right)_{t \geq 0}$ on the space $\mathcal{H}$.
(a2) The linear control operator $\mathbf{B}$ is bounded $\mathcal{U} \rightarrow \mathcal{H}$.
(a3) The operators $\mathbf{R}, \mathbf{G}, \mathbf{C}$ are bounded linear operators.
Then the optimal control $u^{*}$ of the linear quadratic problem (4)-(5) satisfies the feedback characterization in terms of the optimal state $y^{*}$

$$
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t)
$$

where $\mathbf{P}(t)$ is a positive self-adjoint operator solving the Riccati equation

$$
\begin{array}{r}
\dot{\mathbf{P}}(t)+\mathbf{P}(t) \mathbf{A}+\mathbf{A}^{\star} \mathbf{P}(t)+\mathbf{C}^{\star} \mathbf{P}(t) \mathbf{C}+\mathbf{R}^{\star} \mathbf{R}-\mathbf{P}(t) \mathbf{B B}^{\star} \mathbf{P}(t)=0 \\
\mathbf{P}(T)=\mathbf{G}^{\star} \mathbf{G} \tag{6}
\end{array}
$$

2.1.1. Inhomogeneous deterministic LQR problem. Here we invoke the solution to the inhomogeneous deterministic control problem of minimizing the performance index

$$
\begin{equation*}
J(u)=\int_{0}^{T}\left(\|R x\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\|G x(T)\|_{\mathcal{H}}^{2} \tag{7}
\end{equation*}
$$

subject to the inhomogeneous differential equation

$$
\begin{equation*}
x^{\prime}(t)=A x(t)+B u(t)+f(t), \quad x(0)=x^{0} \tag{8}
\end{equation*}
$$

Besides the assumptions (a1) and (a2) from Theorem 1 made on $A$ and $B$, it is enough to assume that $f \in L^{2}((0, T), \mathcal{H})$ to obtain the optimal solution for the state and control $\left(x^{*}, u^{*}\right)$. The feedback form of the optimal control for the inhomogeneous problem (7)-(8) is given by

$$
u^{*}(t)=-B^{\star} P_{d}(t) x^{*}(t)-B^{\star} k(t)
$$

where $P_{d}(t)$ solves the Riccati equation

$$
\begin{align*}
\left\langle\left(\dot{P}_{d}+P_{d} A+A^{\star} P_{d}+R^{\star} R-P_{d} B B^{\star} P_{d}\right) v, w\right\rangle & =0 \\
P_{d}(T) v & =G^{\star} G v \tag{9}
\end{align*}
$$

for all $v, w$ in $\mathcal{D}(A)$, while $k(t)$ is a solution to the auxiliary differential equation

$$
k^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k(t)+P_{d}(t) f(t)=0
$$

with the boundary conditions

$$
P_{d}(T)=G^{\star} G \quad \text { and } \quad k(T)=0
$$

For the homogeneous problem we refer to [24], and for the inhomogeneous optimal control problem for singular estimate type systems we refer to [25].
2.1.2. Strong and mild solutions. Let $g(t)$ be an $\mathcal{F}_{T}$-predictable Bochner integrable $\mathcal{H}$-valued function. An $\mathcal{H}$-valued adapted process $y(t)$ is a strong solution to the state equation (4) over $[0, T]$ if:
(1) $y(t)$ takes values in $D(\mathbf{A}) \cap D(\mathbf{C})$ for almost all $t$ and $\omega$;
(2) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}}+\|\mathbf{A} y(s)\|_{\mathcal{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$;
(3) for arbitrary $t \in[0, T]$ and $\mathbb{P}$-almost surely, it satisfies the integral equation

$$
y(t)=y^{0}+\int_{0}^{t} \mathbf{A} y(s) d s+\int_{0}^{t} g(s) d s+\int_{0}^{t} \mathbf{C} y(s) d B_{s}
$$

An $\mathcal{H}$-valued adapted process $y(t)$ is a mild solution to the state equation

$$
d y(t)=(\mathbf{A} y(t)+g(t)) d t+\mathbf{C} y(t) d B(t), \quad y(0)=y^{0}, \quad t \in[0, T]
$$

over $[0, T]$ if:
(1) $y(t)$ takes values in $D(\mathbf{C})$;
(2) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1$;
(3) for arbitrary $t \in[0, T]$ and $\mathbb{P}$-almost surely, it satisfies the integral equation

$$
y(t)=e^{\mathbf{A} t} y^{0}+\int_{0}^{t} e^{\mathbf{A}(t-s)} g(s) d s+\int_{0}^{t} e^{\mathbf{A}(t-s)} \mathbf{C} y(s) d B_{s}
$$

Note that, under the assumptions of Theorem 1, and given a control $u$ from $L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)$, i.e., $g(t)=\mathbf{B} u(t)$, and the deterministic initial data $y^{0} \in \mathcal{H}$, there exits a unique mild solution $y \in L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)$ to the controlled state equation (4), cf. [7].

### 2.2. Fractional Brownian motion

Fractional Brownian motion is one-parameter extension of a Brownian motion. It depends on the Hurst index $H$ which takes values in $(0,1)$. The name is due to the climatologist Hurst, who developed statistical analysis of the early water run-offs of the river Nile. In the framework of Hilbert spaces, fractional Brownian motion was first introduced by Kolmogorov in 1940, where it was called the Wiener Spirals. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who gave a stochastic integral representation of this process in terms of Brownian motion on an infinite interval [38].

Fractional Brownian motion is a process with dependent increments which have long-range dependence and self-similarity properties. For $H>\frac{1}{2}$ fractional Brownian motion has a certain memory feature, which is suitable for modeling weather derivatives, temperature at a specific place as a function of time, water level in a river as a function of time or for describing the values of the $\log$ returns of a stock. On the other hand, for $H<\frac{1}{2}$ fractional Brownian motion has a certain turbulence feature, which is applicable in mathematical finance in the modeling of
financial turbulence, i.e., empirical volatility of a stock or in modeling the prices of electricity in a liberated Nordic electricity market [5, 13, 38, 39, 40].

Definition 1. A one-dimensional real-valued fractional Brownian motion with the Hurst index $H \in(0,1)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian process $b^{(H)}=\left(b^{(H)}(\cdot)\right)_{t \in \mathbb{R}}$ satisfying:
(a) $b_{0}^{(H)}=0$ a.s.,
(b) zero expectation, i.e., $\mathbb{E}\left[b_{t}^{(H)}\right]=0$ for all $t \in \mathbb{R}$, and
(c) the covariance function is of the form

$$
\begin{equation*}
\mathbb{E}\left(b_{s}^{(H)} b_{t}^{(H)}\right)=\frac{1}{2}\left\{|t|^{2 H}+|s|^{2 H}-|t-s|^{2 H}\right\}, s, t \in \mathbb{R} \tag{10}
\end{equation*}
$$

Fractional Brownian motion is a centered Gaussian process with non-independent stationary increments and its dependence structure is modified by the Hurst parameter $H \in(0,1)$. For $H=\frac{1}{2}$ the covariance function can be written in the form $\mathbb{E}\left(b_{t}^{\left(\frac{1}{2}\right)} b_{s}^{\left(\frac{1}{2}\right)}\right)=\min \{s, t\}$ and the process $b_{t}^{\left(\frac{1}{2}\right)}$ becomes a Brownian motion $b_{t}$, which has independent increments. Moreover, for $H \neq \frac{1}{2}$ fractional Brownian motion is neither a semimartingale nor a Markov process. From (10) it follows that

$$
\mathbb{E}\left(b_{t}^{(H)}-b_{s}^{(H)}\right)^{2}=|t-s|^{2 H}
$$

According to the Kolmogorov continuity criterion fractional Brownian motion $b^{(H)}$ has a continuous modification [39]. The parameter $H$ controls the regularity of trajectories. The covariance function (10) is homogeneous of order $2 H$, thus fractional Brownian motion $b^{(H)}$ is an $H$ self-similar process, i.e., $b_{k t}^{(H)}=k^{H} b_{t}^{(H)}, k>0$.

For any $n \in \mathbb{Z}, n \neq 0$ it holds

$$
\begin{aligned}
r(n)=\mathbb{E}\left[b_{1}^{(H)}\left(b_{n+1}^{(H)}-b_{n}^{(H)}\right)\right] & =H(2 H-1) \int_{0}^{1} \int_{n}^{n+1}(u-v)^{2 H-2} d u d v \\
& \sim H(2 H-1)|n|^{2 H-1}, \quad \text { as }|n| \rightarrow \infty
\end{aligned}
$$

Therefore, the increments are positively correlated for $H \in\left(\frac{1}{2}, 1\right)$ and negatively correlated for $H \in\left(0, \frac{1}{2}\right)$. More precisely, for $H \in\left(\frac{1}{2}, 1\right)$ fractional Brownian motion has the long-range dependence property $\sum_{n=1}^{\infty} r(n)=\infty$ and for $H \in$ $\left(0, \frac{1}{2}\right)$ the short-range property $\sum_{n=1}^{\infty}|r(n)|<\infty$. For more details we refer to [5, 20, 39, 41, 46].

### 2.3. White noise analysis and chaos expansions

In this section, we briefly recall some basic facts from white noise analysis. Denote by $h_{n}(x)=(-1)^{n} e^{\frac{x^{2}}{2}} \frac{d^{n}}{d x^{n}}\left(e^{-\frac{x^{2}}{2}}\right), n \in \mathbb{N}_{0}, \mathbb{N}_{0}=\mathbb{N} \cup\{0\}$, the family of

Hermite polynomials and $\xi_{n}(x)=\frac{1}{\sqrt[4]{\pi} \sqrt{(n-1)!}} e^{-\frac{x^{2}}{2}} h_{n-1}(\sqrt{2} x), n \in \mathbb{N}$, the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^{2}(\mathbb{R})$. These functions are the eigenfunctions for the harmonic oscillator in quantum mechanics. Clearly, the elements of $\left\{\xi_{n}\right\}_{n \in \mathbb{N}}$ belong to the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$, i.e., they decay faster than polynomials of any degree. The Schwartz spaces can be characterized in terms of the Hermite basis in the following manner: The space of rapidly decreasing functions as a projective limit space $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$ where $S_{l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k} \in L^{2}(\mathbb{R}):\|f\|_{l}^{2}=\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{N}_{0}$ and the space of tempered distributions as an inductive limit space $S^{\prime}(\mathbb{R})=\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$ where $S_{-l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} a_{k} \xi_{k}:\|f\|_{l}^{2}=\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{-l}<\infty\right\}, l \in \mathbb{N}_{0}$. Also, we have a Gel'fand triple $S(\mathbb{R}) \subseteq L^{2}(\mathbb{R}) \subseteq S^{\prime}(\mathbb{R})$ with continuous inclusions.
2.3.1. Gaussian white noise space. Throughout the paper all analysis is provided on two white noise spaces. Here we introduce the (classical) Gaussian white noise space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$ and later in Section 2.3 .6 we will introduce the fractional Gaussian white noise space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu_{H}\right)$. In both cases, we follow the ideas of Hida from [19]. The underlying space is the space of tempered distributions $S^{\prime}(\mathbb{R})$. By $\mathcal{B}$ we denote the Borel sigma-algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$ and $\mu$ is the Gaussian white noise measure given by the Bochner-Minlos theorem

$$
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \phi \in S(\mathbb{R})
$$

where $\langle\omega, \phi\rangle$ denotes the dual pairing between a tempered distribution $\omega \in S^{\prime}(\mathbb{R})$ and a test function $\phi \in S(\mathbb{R})$.

Denote by $\mathcal{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ the set of sequences of non-negative integers which have only finitely many nonzero components. All multi-indices $\alpha \in \mathcal{I}$ are of the form $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right), \alpha_{i} \in \mathbb{N}_{0}, i=1,2, \ldots, m, m \in \mathbb{N}$. Particularly, $\mathbf{0}=(0,0, \ldots)$ is the zeroth vector and $\varepsilon^{(k)}=(0, \ldots, 0,1,0, \ldots), k \in \mathbb{N}$ is the $k$ th unit vector. The length of a multi-index $\alpha \in \mathcal{I}$ is defined by $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. Let $(2 \mathbb{N})^{\alpha}=\prod_{k=1}^{\infty}(2 k)^{\alpha_{k}}$. It was proven that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ for $p>1$, cf [21]. We say $\alpha \geq \beta$ if $\alpha_{k} \geq \beta_{k}$ for all $k \in \mathbb{N}$. In this case $\alpha-\beta=\left(\alpha_{1}-\beta_{1}, \alpha_{2}-\beta_{2}, \ldots\right)$. For $\alpha<\beta$ the difference $\alpha-\beta$ is not defined.

The space $L^{2}(\mu)=L^{2}(\Omega, \mu)=L^{2}\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$ is the Hilbert space of square integrable random variables with respect to the Gaussian measure $\mu$, i.e., the space of random variables with finite second moments.

Definition 2. The Fourier-Hermite polynomials on $L^{2}(\mu)$ are defined by

$$
\begin{equation*}
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in \mathcal{I} \tag{11}
\end{equation*}
$$

Particularly, $H_{\mathbf{0}}(\omega)=1$ and $H_{\varepsilon^{(k)}}(\omega)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$. The family $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ forms an orthogonal basis of $L^{2}(\mu)$ with $\left\|H_{\alpha}\right\|_{L^{2}(\mu)}^{2}=\alpha!$, see [21].

Theorem 2 (Wiener-Itô chaos expansion theorem). Each element $F \in L^{2}(\mu)$ has a unique representation of the form

$$
F(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega)
$$

with real coefficients $a_{\alpha}, \alpha \in \mathcal{I}, \omega \in \Omega$, such that $\|F\|_{L^{2}(\mu)}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2} \alpha!<\infty$.
The space spanned by $\left\{H_{\alpha}:|\alpha|=k\right\}$ is called the Wiener chaos of order $k$ and is denoted by $\mathcal{H}_{k}, k \in \mathbb{N}_{0}$. Each $\mathcal{H}_{k}$ is an infinite-dimensional subspace of $L^{2}(\mu)$ and

$$
L^{2}(\mu)=\bigoplus_{k=0}^{\infty} \mathcal{H}_{k}
$$

where the sum is an orthogonal sum [21].
Let $\mathcal{H}$ be a real separable Hilbert space. Then each element $F$ of the space of Hilbert-valued square integrable random variables $L^{2}(\Omega, \mathcal{H})=\bigoplus_{k=0}^{\infty} \mathcal{H}_{k}(\mathcal{H})$, can be represented in the form $F(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega)$, for $f_{\alpha} \in \mathcal{H}, \alpha \in \mathcal{I}$, such that

$$
\|F\|_{L^{2}(\Omega, \mathcal{H})}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{\mathcal{H}}^{2} \alpha!<\infty
$$

One of the typical complications that arise in solving stochastic differential equations is the blowup of $L^{2}$-norms of $F$, i.e., infinite variance. Therefore, weighted spaces of random variables in which the considered equation has a solution have to be introduced. For example, such spaces are the Kondratiev spaces $(S)_{-\rho}, \rho \in[0,1]$ of generalized random variables, which represent the stochastic analogue of the Schwartz spaces as generalized function spaces. The largest space of Kondratiev stochastic distributions is $(S)_{-1}$, obtained for $\rho=1$.

The space of the Kondratiev test random variables $(S)_{1}$ can be constructed as the projective limit of the family of spaces

$$
(S)_{1, p}=\left\{f(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega) \in L^{2}(\mu):\|f\|_{1, p}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2}(\alpha!)^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}
$$

$p \in \mathbb{N}_{0}$. The space of the Kondratiev generalized random variables $(S)_{-1}$ can be constructed as the inductive limit of the family of spaces

$$
(S)_{-1,-p}=\left\{F(\omega)=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}(\omega):\|f\|_{-1,-p}^{2}=\sum_{\alpha \in \mathcal{I}} b_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}, p \in \mathbb{N}_{0}
$$

It holds $(S)_{1}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{1, p}$ and $(S)_{-1}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-1, p}$. The action of a generalized random variable $F=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}(\omega) \in(S)_{-1}$ on a test random variable $f=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega) \in(S)_{1}$ is given by $\langle F, f\rangle=\sum_{\alpha \in \mathcal{I}} \alpha!a_{\alpha} b_{\alpha}$. It holds that $(S)_{1}$ is a nuclear space with the Gel'fand triple $(S)_{1} \subseteq L^{2}(\mu) \subseteq(S)_{-1}$ with continuous inclusions [21].

Definition 3. For $F(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega)$ and $G(\omega)=\sum_{\beta \in \mathcal{I}} g_{\beta} H_{\beta}(\omega)$ the element $F \diamond G$ is called the Wick product of $F$ and $G$ and is given in the form

$$
\begin{equation*}
F \diamond G(\omega)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_{\alpha} g_{\beta} H_{\alpha+\beta}(\omega)=\sum_{\gamma \in \mathcal{I}} \sum_{\alpha \leq \gamma} f_{\alpha} g_{\gamma-\alpha} H_{\gamma}(\omega) \tag{12}
\end{equation*}
$$

The Kondratiev spaces $(S)_{1}$ and $(S)_{-1}$ are closed under the Wick multiplication. The Wick product is a commutative, associative operation, and is distributive with respect to addition. In particular, for the orthogonal polynomial basis of $L^{2}(\mu)$ we have $H_{\alpha} \diamond H_{\beta}=H_{\alpha+\beta}$, for all $\alpha, \beta \in \mathcal{I}$. Whenever $F, G$ and $F \diamond G$ are integrable it holds that $\mathbb{E}(F \diamond G)=\mathbb{E}(F) \cdot \mathbb{E}(G)$, without independence requirement $[21,31]$.
2.3.2. Stochastic processes. A square integrable real-valued stochastic process is defined as a measurable mapping $[0, T] \rightarrow L^{2}(\mu)$. A generalized stochastic process is considered to be a measurable mapping from $[0, T]$ into a Kondratiev space $(S)_{-1}$. The chaos expansion representation of generalized stochastic process $F$ follows from Theorem 2. A generalized process $F$ can be represented in the form

$$
F_{t}(\omega)=\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega), \quad t \in[0, T]
$$

where $f_{\alpha}, \alpha \in \mathcal{I}$ are measurable real functions and there exists $p \in \mathbb{N}_{0}$ such that for all $t \in[0, T]$

$$
\sum_{\alpha \in \mathcal{I}}\left|f_{\alpha}(t)\right|^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

If we assume $\mathcal{H}$ to be a real separable Hilbert space, then Theorem 2 can be extended also for $\mathcal{H}$-valued stochastic processes. Particularly, a square integrable $\mathcal{H}$-valued stochastic processes $v$ is an element of $L^{2}([0, T] \times \Omega, \mathcal{H}) \cong L^{2}([0, T], \mathcal{H}) \otimes$ $L^{2}(\Omega, \mu)$ and can be represented in the chaos expansion form

$$
\begin{align*}
v(t, \omega) & =\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega) \\
& =v_{\mathbf{0}}(t)+\sum_{k \in \mathbb{N}} v_{\varepsilon^{(k)}}(t) H_{\varepsilon^{(k)}}(\omega)+\sum_{|\alpha|>1} v_{\alpha}(t) H_{\alpha}(\omega), \quad t \in[0, T] \tag{13}
\end{align*}
$$

where $v_{\alpha} \in L^{2}([0, T], \mathcal{H})$ such that it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \alpha!<\infty \tag{14}
\end{equation*}
$$

A process $v$ with the chaos expansion representation (13) that instead of (14) satisfies the condition

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{15}
\end{equation*}
$$

belongs to $L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}$ and is considered to be a generalized stochastic process. The coefficient $v_{\mathbf{0}}(t)$ is the deterministic part of $v$ in (13) and represents the (generalized) expectation of the process $v$.

Denote by $\left\{\mathbf{e}_{n}(t)\right\}_{n \in \mathbb{N}}$ the orthonormal basis of $L^{2}([0, T], \mathcal{H})$, i.e., the basis obtained by diagonalizing the orthonormal basis $\left\{b_{i}(t) s_{j}\right\}_{i, j \in \mathbb{N}}$, where $\left\{b_{i}(t)\right\}_{i \in \mathbb{N}}$ is the orthonormal basis of $L^{2}([0, T])$ and $\left\{s_{j}\right\}_{j \in \mathbb{N}}$ is the orthonormal basis of $\mathcal{H}$. The coefficients $v_{\alpha}(t) \in L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$ can be represented in the form

$$
v_{\alpha}(t)=\sum_{j \in \mathbb{N}} v_{\alpha, j}(t) s_{j}=\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} v_{\alpha, j, i} b_{i}(t) s_{j}, \quad \alpha \in \mathcal{I}
$$

with $v_{\alpha, j} \in L^{2}([0, T])$ and $v_{\alpha, j, i} \in \mathbb{R}$. Then the chaos expansion (13) of a stochastic process $v \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$ can be written as

$$
v(t, \omega)=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega)=\sum_{\alpha \in \mathcal{I}} \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} v_{\alpha, j, i} s_{j} b_{i}(t) H_{\alpha}(\omega) .
$$

After a diagonalization of $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ it can be rearranged to

$$
v(t, \omega)=\sum_{\alpha \in \mathcal{I}} \sum_{n \in \mathbb{N}} v_{\alpha, n} \mathbf{e}_{n}(t) H_{\alpha}(\omega), \quad v_{\alpha, n} \in \mathbb{R}, \omega \in \Omega, t \in[0, T]
$$

Example 1. (a) A one-dimensional real-valued Brownian motion can be represented in the chaos expansion form $b_{t}(\omega)=\sum_{k=1}^{\infty}\left(\int_{0}^{t} \xi_{k}(s) d s\right) H_{\varepsilon^{(k)}}(\omega), t \geq 0$. For each $t$ it is an element of $L^{2}(\mu)$. A singular real-valued white noise is defined by the formal chaos expansion

$$
\begin{equation*}
w_{t}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega) \tag{16}
\end{equation*}
$$

Since $\sum_{k=1}^{\infty}\left|\xi_{k}(t)\right|^{2}>\sum_{k=1}^{\infty} \frac{1}{k}=\infty$ and $\sum_{k=1}^{\infty}\left|\xi_{k}(t)\right|^{2}(2 k)^{-p}<\infty$ holds for $p>1$, it follows that the singular white noise is an element of the space $(S)_{-1}$, for all $t \geq 0$, see [21]. It is integrable and the relation $\frac{d}{d t} b_{t}=w_{t}$ holds in the distributional sense. Both Brownian motion and singular white noise are Gaussian processes and belong to the Wiener chaos space of order one.
(b) An $\mathcal{H}$-valued white noise process is given in the chaos expansion form

$$
\begin{equation*}
W_{t}(\omega)=\sum_{k=1}^{\infty} \mathbf{e}_{k}(t) H_{\varepsilon^{(k)}}(\omega) \tag{17}
\end{equation*}
$$

Note that the $\mathcal{H}$-valued white noise can be also defined as $\sum_{n \in \mathbb{N}} w_{t}^{n}(\omega) s_{n}$, where $w_{t}^{(n)}(\omega)$ are independent copies of one-dimensional white noise (16) and $\left\{s_{n}\right\}_{n \in \mathbb{N}}$ is the orthonormal basis of $\mathcal{H}$. This definition can be reduced to (17) since

$$
\begin{aligned}
\sum_{n \in \mathbb{N}} w_{t}^{(n)}(\omega) s_{n} & =\sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega) s_{n} \\
& =\sum_{i \in \mathbb{N}} \xi_{i}(t) s_{i} H_{\varepsilon^{(i)}}(\omega) s_{n}=\sum_{i=1}^{\infty} \mathbf{e}_{i}(t) H_{\varepsilon^{(i)}}(\omega)
\end{aligned}
$$

where $\left\{\mathbf{e}_{i}\right\}_{i \in \mathbb{N}}$ is the orthogonal basis of $L^{2}(\mathbb{R}, \mathcal{H})$ obtained by diagonalizing the basis $\left\{\xi_{k}(t) s_{n}\right\}_{k, n \in \mathbb{N}}$.
(c) In general, the chaos expansion representation of an $\mathcal{H}$-valued Gaussian process that belongs to the Wiener chaos space of order one is given in the form

$$
\begin{equation*}
G_{t}(\omega)=\sum_{k \in \mathbb{N}} g_{k}(t) H_{\varepsilon^{(k)}}(\omega)=\sum_{k \in \mathbb{N}}\left(\sum_{i \in \mathbb{N}} g_{k i} \mathbf{e}_{i}(t)\right) H_{\varepsilon^{(k)}}(\omega) \tag{18}
\end{equation*}
$$

with $g_{k} \in L^{2}([0, T], \mathcal{H})$ and $g_{k i}=\left(g_{k}, \mathbf{e}_{i}\right)_{L^{2}([0, T], \mathcal{H})}$ is a real constant. If the condition

$$
\begin{equation*}
\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}<\infty \tag{19}
\end{equation*}
$$

is fulfilled, then $G_{t}$ belongs to the space $L^{2}([0, T] \times \Omega, \mathcal{H}) \cong L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$. If the sum in (19) is infinite then the representation (18) is formal, and if additionally

$$
\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}(2 \mathbb{N})^{-p \varepsilon^{(k)}}=\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}(2 k)^{-p}<\infty
$$

holds for some $p \in \mathbb{N}_{0}$, the process $G_{t}$, for each $t$, belongs to the Kondratiev space of stochastic distributions $(S)_{-1}$, i.e., $G \in L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}$, see $[33,36,44]$.

Note that a Gaussian noise represented in (18) can be interpreted as a colored noise with the representation operator $N$ and the correlation function $\mathcal{C}=N N^{\star}$, such that

$$
\begin{aligned}
\sum_{k \in \mathbb{N}} N^{\star} f_{k}(t) H_{\varepsilon^{(k)}}(\omega) & =\sum_{k \in \mathbb{N}} N^{\star}\left(\sum_{i \in \mathbb{N}} f_{k i} \mathbf{e}_{i}(t)\right) H_{\varepsilon^{(k)}}(\omega) \\
& =\sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{N}} \lambda_{i} f_{k i} \mathbf{e}_{i}(t) H_{\varepsilon^{(k)}}(\omega)
\end{aligned}
$$

with $N^{\star} \mathbf{e}_{i}(t)=\lambda_{i} \mathbf{e}_{i}(t), i \in \mathbb{N}$, [37]. Particularly, we will consider the color noise to be a Gaussian process of the form

$$
\begin{equation*}
L_{t}(\omega)=\sum_{k \in \mathbb{N}} l_{k} \mathbf{e}_{k}(t) H_{\varepsilon^{(k)}}(\omega) \tag{20}
\end{equation*}
$$

with a sequence of real coefficients $\left\{l_{k}\right\}_{k \in \mathbb{N}}$ such that for some $p \in \mathbb{N}$ it holds

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} l_{k}^{2}(2 k)^{-p}<\infty \tag{21}
\end{equation*}
$$

The Wick product of two stochastic processes is defined in an analogous way as it was defined for random variables and generalized random variables (12), for more details see [30].
2.3.3. Operators. Following [34], we define two classes of operators on spaces of stochastic processes, namely coordinatewise and simple coordinatewise operators, that we are going to deal with in the paper.

Definition 4. An operator $\mathbb{O}$ is called a coordinatewise operator if there exists a family of operators $\left\{O_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, such that for a process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}$ it holds

$$
\begin{equation*}
\mathbb{O} v=\sum_{\alpha \in \mathcal{I}} O_{\alpha}\left(v_{\alpha}\right) H_{\alpha} \tag{22}
\end{equation*}
$$

Moreover, operator $\mathbb{O}$ is a simple coordinatewise operator if $O_{\alpha}=O$ for all $\alpha \in \mathcal{I}$, i.e., if it holds that

$$
\mathbb{O} v=\sum_{\alpha \in \mathcal{I}} O\left(v_{\alpha}\right) H_{\alpha}=O\left(v_{\mathbf{0}}\right)+\sum_{|\alpha|>0} O\left(v_{\alpha}\right) H_{\alpha}
$$

Lemma 1. Let $\mathbb{O}: L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$ be a coordinatewise operator that corresponds to a deterministic family of operators $O_{\alpha}: L^{2}([0, T], \mathcal{H}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$. If the operators $O_{\alpha}, \alpha \in \mathcal{I}$ are uniformly bounded by $c>0$ then $\mathbb{O}$ is a bounded operator on $L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$.

Proof. Let $\left\|O_{\alpha}\right\|_{o p} \leq c$ for all $\alpha \in \mathcal{I}$. Then, for $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}$ in $L^{2}([0, T], \mathcal{H}) \otimes$ $L^{2}(\Omega, \mu)$ it holds

$$
\begin{aligned}
& \|\mathbb{O} v\|_{L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)}^{2} \\
& \quad=\sum_{\alpha \in \mathcal{I}}\left\|O_{\alpha} v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \alpha!\leq \sum_{\alpha \in \mathcal{I}}\left\|O_{\alpha}\right\|_{o p}^{2}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \alpha! \\
& \quad \leq c^{2} \sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \alpha!=c^{2}\|v\|_{L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)}^{2} .
\end{aligned}
$$

2.3.4. Stochastic integration and Wick multiplication. For a square integrable process $v$ that is adapted in the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ generated by an $\mathcal{H}$-valued Brownian motion $\left(B_{t}\right)_{t \geq 0}$, the corresponding stochastic integral $\int_{0}^{T} v_{t} d B_{t}$ is considered to be the Itô integral $I(v)$. When $v$ is not adapted to the filtration, then the stochastic integral is interpreted as the Itô-Skorokhod integral. From the fundamental theorem of stochastic calculus it follows that the Itô-Skorokhod integral of an $\mathcal{H}$ valued stochastic process $v=v_{t}(\omega)$ can be represented as a Riemann integral of the Wick product of $v_{t}$ with a singular white noise

$$
\begin{equation*}
\delta(v)=\int_{0}^{T} v d B_{t}(\omega)=\int_{0}^{T} v \diamond W_{t}(\omega) d t \tag{23}
\end{equation*}
$$

where the derivative $W_{t}=\frac{d}{d t} B_{t}$ is taken in sense of distributions [21].
Thus, for an $\mathcal{H}$-valued adapted processes $v$ the Itô integral and the Skorokhod integral coincide, i.e., $I(v)=\delta(v)$. Note that the Itô integral is an $\mathcal{H}$-valued random variable. From the Wiener-Itô chaos expansion theorem, Theorem 2, it follows that there exists a unique family $a_{\alpha}, \alpha \in \mathcal{I}$ such that the Itô integral can be represented in the chaos expansion form

$$
\begin{equation*}
I(v)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha} \tag{24}
\end{equation*}
$$

On the other hand, by (12), (17) and (23) we obtain a chaos expansion representation of the Skorokhod integral, i.e., for $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}$ we have

$$
\begin{align*}
v \diamond W_{t}(\omega) & =\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega) \diamond \sum_{k \in \mathbb{N}} \mathbf{e}_{k}(t) H_{\varepsilon^{(k)}}(\omega) \\
& =\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha}(t) \mathbf{e}_{k}(t) H_{\alpha+\varepsilon^{(k)}}(\omega) \tag{25}
\end{align*}
$$

Thus, from (23) and (25) we obtain

$$
\begin{equation*}
\delta(v)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} H_{\alpha+\varepsilon^{(k)}}(\omega) \tag{26}
\end{equation*}
$$

with real coefficients $v_{\alpha, k}=\left(v_{\alpha}, \mathbf{e}_{k}\right)_{L^{2}([0, T], \mathcal{H})}$ and $\omega \in \Omega$. Combining (26) and (24) we obtain the coefficients $a_{\alpha}$, for all $\alpha \in \mathcal{I}$ and $\alpha>\mathbf{0}$ in the form

$$
\begin{equation*}
a_{\alpha}=\sum_{k \in \mathbb{N}} v_{\alpha-\varepsilon^{(k)}, k} \tag{27}
\end{equation*}
$$

We use the following convention: $v_{\alpha-\varepsilon^{(k)}}$ is not defined if the $k$ th component of $\alpha$, i.e., $\alpha_{k}$ equals zero. For example, for $\alpha=(0,3,0,2,0, \ldots)$ the coefficient $a_{(0,3,0,2,0, \ldots)}$ is expressed as the sum of two coefficients of the process $v$, i.e., from (27) we have $a_{(0,3,0,2,0, \ldots)}=v_{(0,2,0,2,0, \ldots), 2}+v_{(0,3,0,1,0, \ldots), 4}$. The obtained chaos expansion representation form of the Itô-Skorokhod integral (26) will be used in Section 3, where we will be able to represent explicitly the stochastic perturbation in the optimal control problem (4). Note also that $\delta(v)$ belongs to the Wiener chaos space of higher order than $v$, see also $[21,35]$.

Definition 5. A square integrable $\mathcal{H}$-valued stochastic process $v$ given in the form $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}(\omega)$, with the coefficients $v_{\alpha} \in L^{2}([0, T], \mathcal{H})$ such that $v_{\alpha}(t)=\sum_{k \in \mathbb{N}} v_{\alpha, k} \mathbf{e}_{k}(t), v_{\alpha, k} \in \mathbb{R}$ for all $\alpha \in \mathcal{I}$ is integrable in It $\hat{o}-$ Skorokhod sense if the condition

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I},|\alpha|>0} \alpha!\left(\sum_{k \in \mathbb{N}} v_{\alpha-\varepsilon^{(k)}, k}\right)^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left(\sum_{k \in \mathbb{N}} v_{\alpha, k} \sqrt{\alpha_{k}+1}\right)^{2}<\infty \tag{28}
\end{equation*}
$$

holds. Then the chaos expansion form of the Itô-Skorokhod integral of $v$ is given by (26) and we write $v \in \operatorname{Dom}(\delta)$.

Theorem 3. The Skorokhod integral $\delta$ of an $\mathcal{H}$-valued square integrable stochastic process is a linear and continuous mapping

$$
\delta: \quad \operatorname{Dom}(\delta) \rightarrow L^{2}(\Omega)
$$

Proof. Let $u, v \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega)$ be integrable in Itô-Skorokhod sense. Then, for $a, b \in \mathbb{R}$ it holds

$$
\begin{aligned}
\delta(a u+b v) & =\delta\left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(a u_{\alpha, k}+b v_{\alpha, k}\right) \mathbf{e}_{k} H_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(a u_{\alpha, k}+b v_{\alpha, k}\right) H_{\alpha+\varepsilon^{(k)}} \\
& =a \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} H_{\alpha+\varepsilon^{(k)}}+b \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} H_{\alpha+\varepsilon^{(k)}}=a \delta(u)+b \delta(v) .
\end{aligned}
$$

Moreover, from (28) and $\left(\alpha+\varepsilon^{(k)}\right)!=\left(\alpha_{k}+1\right) \alpha!$ for $\alpha \in \mathcal{I}, k \in \mathbb{N}$ we obtain

$$
\begin{aligned}
\|\delta(v)\|_{L^{2}(\Omega)}^{2} & =\left\|\sum_{\alpha \in \mathcal{I},|\alpha|>0} \sum_{k \in \mathbb{N}} v_{\alpha-\varepsilon^{(k)}, k} H_{\alpha}\right\|_{L^{2}(\Omega)}^{2} \\
& =\sum_{|\alpha|>0}\left(\sum_{k \in \mathbb{N}} v_{\alpha-\varepsilon^{(k)}, k}\right)^{2} \alpha!<\infty .
\end{aligned}
$$

From the estimates

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \alpha! & =\sum_{\alpha \in \mathcal{I}} \alpha!\left(\sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}\right) \leq \sum_{\alpha \in \mathcal{I}} \alpha!\left(\sum_{k \in \mathbb{N}} v_{\alpha, k}\right)^{2} \\
& \leq \sum_{\alpha \in \mathcal{I}} \alpha!\left(\sum_{k \in \mathbb{N}} v_{\alpha, k} \sqrt{\alpha_{k}+1}\right)^{2}<\infty
\end{aligned}
$$

we conclude that if $v \in \operatorname{Dom}(\delta)$ then $v \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega)$. Moreover, if the condition

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}|\alpha|\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \alpha!<\infty \tag{29}
\end{equation*}
$$

is fulfilled then $v \in \operatorname{Dom}(\delta)$. This follows from

$$
\sum_{\alpha \in \mathcal{I}} \alpha!\left(\sum_{k \in \mathbb{N}} u_{\alpha, k} \sqrt{\alpha_{k}+1}\right)^{2} \leq c \sum_{\alpha \in \mathcal{I}} \alpha!|\alpha| \sum_{k \in \mathbb{N}} u_{\alpha, k}^{2}<\infty
$$

A detailed analysis of domain and range of operators of the Malliavin calculus in spaces of generalized stochastic processes can be found in $[31,35]$.

Lemma 2. Let $\mathbb{O}: L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega)$ be a coordinatewise operator that corresponds to a uniformly bounded family of linear operators $O_{\alpha}$ : $L^{2}([0, T], \mathcal{H}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$. If a stochastic process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha} \in$ $L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega)$ satisfies the condition $(29)$ then $\mathbb{O} v \in \operatorname{Dom}(\delta)$.

Proof. Since $v \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega)$ satisfies (29) then $v \in \operatorname{Dom}(\delta)$, i.e., (28) holds. Let $\mathbb{O}$ corresponds to the family $O_{\alpha}: L^{2}([0, T], \mathcal{H}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$ such that $\left\|O_{\alpha}\right\|_{\mathcal{L}(\mathcal{H})} \leq c, \alpha \in \mathcal{I}$, where $\mathcal{L}(\mathcal{H})$ denotes the set of linear bounded
operators on $L^{2}([0, T], \mathcal{H})$. From

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}}\left\|O_{\alpha} v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}|\alpha| \alpha! & \leq \sum_{\alpha \in \mathcal{I}}\left\|O_{\alpha}\right\|_{\mathcal{L}(\mathcal{H})}^{2}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}|\alpha| \alpha! \\
& \leq c^{2} \sum_{\alpha \in \mathcal{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}|\alpha| \alpha!<\infty
\end{aligned}
$$

it follows that $\mathbb{O} v \in \operatorname{Dom}(\delta)$.
2.3.5. The fractional transform operator $\boldsymbol{M}^{(\boldsymbol{H})}$. In [13] the authors developed the fractional white noise theory for a Hurst parameter $H \in(0,1)$. They introduced the fractional transform operator $M^{(H)}$, which connects the fractional Brownian motion $b_{t}^{(H)}$ and the standard Brownian motion $b_{t}$ on the white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$. We extend these results for $\mathcal{H}$-valued Brownian motion $B_{t}$ and $\mathcal{H}$-valued white noise $W_{t}$ and their corresponding fractional versions $B_{t}^{(H)}$ and $W_{t}{ }^{(H)}$.

Definition 6 ([13]). Let $H \in(0,1)$. The fractional transform operator $M^{(H)}$ : $S(\mathbb{R}) \rightarrow L^{2}(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ is defined by

$$
\begin{equation*}
\widehat{M^{(H)}} f(y)=|y|^{\frac{1}{2}-H} \widehat{f}(y), \quad y \in \mathbb{R}, \quad f \in S(\mathbb{R}) \tag{30}
\end{equation*}
$$

where $\widehat{f}(y):=\int_{\mathbb{R}} e^{-i x y} f(x) d x$ denotes the Fourier transform of $f$.
Equivalently, the operator $M^{(H)}$ for all $H \in(0,1)$ can be defined as a constant multiple of

$$
\begin{equation*}
-\frac{d}{d x} \int_{\mathbb{R}}(t-x)|t-x|^{H-\frac{3}{2}} f(t) d t \tag{31}
\end{equation*}
$$

such that the constant is chosen so that (30) holds. The operator $M^{(H)}$ has the structure of a convolution operator. Particularly, from (31) it follows that for $H \in\left(0, \frac{1}{2}\right)$ the fractional operator is of the form $M^{(H)} f(x)=C_{H} \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}} d t$, then for $H \in\left(\frac{1}{2}, 1\right)$ it is of the form $M^{(H)} f(x)=C_{H} \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}} d t$ and for $H=\frac{1}{2}$ it reduces to the identity operator, i.e., $M^{\left(\frac{1}{2}\right)} f(x)=f(x)$. The normalizing constant is $C_{H}=\left(2 \Gamma\left(H-\frac{1}{2}\right) \cos \left(\frac{\pi}{2}\left(H-\frac{1}{2}\right)\right)\right)^{-1}$ and $\Gamma$ is the Gamma function.

From (30) we have that the inverse fractional transform operator of the operator $M^{(H)}$ is the operator $M^{(1-H)}$, which is defined by

$$
\widehat{M^{(1-H)}} f(y)=|y|^{H-\frac{1}{2}} \widehat{f}(y), \quad y \in \mathbb{R}, f \in S(\mathbb{R})
$$

Denote by $L_{H}^{2}(\mathbb{R})=\left\{f: \mathbb{R} \rightarrow \mathbb{R}: M^{(H)} f(x) \in L^{2}(\mathbb{R})\right\}$ the closure of $S(\mathbb{R})$ with respect to the norm $\|f\|_{L_{H}^{2}(\mathbb{R})}=\left\|M^{(H)} f\right\|_{L^{2}(\mathbb{R})}$, for $f \in S(\mathbb{R})$, induced by the inner product

$$
(f, g)_{L_{H}^{2}(\mathbb{R})}=\left(M^{(H)} f, M^{(H)} g\right)_{L^{2}(\mathbb{R})}
$$

The operator $M^{(H)}$ is a self-adjoint operator and for $f, g \in L^{2}(\mathbb{R}) \cap L_{H}^{2}(\mathbb{R})$ we have

$$
\begin{aligned}
\left(f, M^{(H)} g\right)_{L_{H}^{2}(\mathbb{R})} & =\left(\widehat{f}, \widehat{M^{(H)}} g\right)_{L^{2}(\mathbb{R})}=\int_{\mathbb{R}}|y|^{\frac{1}{2}-H} \widehat{f}(y) \widehat{g}(y) d y \\
& =\left(\widehat{M^{(H)}} f, \widehat{g}\right)_{L^{2}(\mathbb{R})}=\left(M^{(H)} f, g\right)_{L_{H}^{2}(\mathbb{R})}
\end{aligned}
$$

Remark 1. For fixed $H \in\left(\frac{1}{2}, 1\right)$, define $\phi(s, t)=H(2 H-1)|s-t|^{2 H-2}, s, t \in \mathbb{R}$. Then,

$$
\begin{equation*}
\int_{\mathbb{R}}\left(M^{(H)} f(x)\right)^{2} d x=c_{H} \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) d s d t \tag{32}
\end{equation*}
$$

with $c_{H}$ constant. The property (32) was used in [13, 20, 32] and [38] in order to adapt the classical white noise calculus to the fractional one.

Theorem $4([\mathbf{6}, \mathbf{1 3}])$. Let $M^{(H)}: L_{H}^{2}(\mathbb{R}) \rightarrow L^{2}(\mathbb{R})$ defined by $(30)$ be the extension of the operator $M$ from Definition 6. Then, $M^{(H)}$ is an isometry between the two Hilbert spaces $L^{2}(\mathbb{R})$ and $L_{H}^{2}(\mathbb{R})$. The functions

$$
\begin{equation*}
e_{n}^{(H)}(x)=M^{(1-H)} \xi_{n}(x), \quad n \in \mathbb{N} \tag{33}
\end{equation*}
$$

belong to $S(\mathbb{R})$ and form an orthonormal basis in $L_{H}^{2}(\mathbb{R})$.
From (33) it also follows $e_{n}^{(1-H)}=M^{(H)} \xi_{n}, n \in \mathbb{N}$, where we used the fact that $M^{(1-H)}$ is the inverse operator of the operator $M^{(H)}$. Following [6] and [13] we extend $M^{(H)}$ onto $S^{\prime}(\mathbb{R})$ and define the fractional operator $M^{(H)}: S^{\prime}(\mathbb{R}) \rightarrow$ $S^{\prime}(\mathbb{R})$ by

$$
\left\langle M^{(H)} \omega, f\right\rangle=\left\langle\omega, M^{(H)} f\right\rangle, \quad f \in S(\mathbb{R}), \omega \in S^{\prime}(\mathbb{R})
$$

2.3.6. Fractional Gaussian white noise space. Following [5], for $H \in(0,1)$ we denote by

$$
L^{2}\left(\mu_{H}\right)=L^{2}\left(\mu \circ M^{(1-H)}\right)=\left\{G: \Omega \rightarrow \mathbb{R} ; G \circ M^{(H)} \in L^{2}(\mu)\right\}
$$

the stochastic analogue of $L_{H}^{2}(\mathbb{R})$. It is the space of square integrable functions on $S^{\prime}(\mathbb{R})$ with respect to fractional Gaussian white noise measure $\mu_{H}$. Thus, the space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu_{H}\right)$ denotes the fractional Gaussian white noise space.

Since $G \in L^{2}\left(\mu_{H}\right)$ if and only if $G \circ M^{(H)} \in L^{2}(\mu)$, it follows that $G$ has an expansion of the form

$$
\begin{aligned}
G\left(M^{(H)} \omega\right) & =\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha}(\omega)=\sum_{\alpha \in \mathcal{I}} c_{\alpha} \prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle\omega, \xi_{i}\right\rangle\right) \\
& =\sum_{\alpha \in \mathcal{I}} c_{\alpha} \prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle\omega, M^{(H)} e_{i}\right\rangle\right)=\sum_{\alpha \in \mathcal{I}} c_{\alpha} \prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle M^{(H)} \omega, e_{i}\right\rangle\right) .
\end{aligned}
$$

Definition 7. The family of fractional Fourier-Hermite polynomials is defined by

$$
\begin{equation*}
\widetilde{H}_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, e_{k}\right\rangle\right), \quad \alpha \in \mathcal{I} \tag{34}
\end{equation*}
$$

The family $\left\{\widetilde{H}_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ forms an orthogonal basis of $L^{2}\left(\mu_{H}\right)$ and for all $\alpha \in \mathcal{I}$ it holds $\left\|\widetilde{H}_{\alpha}\right\|_{L^{2}\left(\mu_{H}\right)}^{2}=\alpha!$. Therefore, Theorem 2 can be formulated for fractional square integrable random variables.

Theorem 5. Each $G \in L^{2}\left(\mu_{H}\right)$ can be uniquely represented in the form

$$
G(\omega)=\sum_{\alpha \in \mathcal{I}} c_{\alpha} \widetilde{H}_{\alpha}(\omega), \quad c_{\alpha} \in \mathbb{R}, \alpha \in \mathcal{I}, \omega \in \Omega
$$

such that $\|G\|_{L^{2}\left(\mu_{H}\right)}^{2}=\sum_{\alpha \in \mathcal{I}} c_{\alpha}^{2} \alpha!$ is finite and $\|G\|_{L^{2}\left(\mu_{H}\right)}=\left\|G \circ M^{(H)}\right\|_{L^{2}(\mu)}$.
The fractional Kondratiev spaces $(S)_{1}^{(H)}$ and $(S)_{-1}^{(H)}$ are defined in an analogous way as it was done in Section 2.3.1 for stochastic random variables in the Gaussian white noise case. An $\mathcal{H}$-valued fractional stochastic process $\widetilde{v}$ as element of $L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)$ is uniquely defined by

$$
\begin{equation*}
\widetilde{v}_{t}(\omega)=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) \widetilde{H}_{\alpha}(\omega), \tag{35}
\end{equation*}
$$

where $v_{\alpha} \in L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$ such that (14) holds. Moreover, (35) can be written in the form

$$
\widetilde{v}_{t}(\omega)=\sum_{\alpha \in \mathcal{I}} \sum_{n \in \mathbb{N}} v_{\alpha, n} \mathbf{e}_{n}(t) \widetilde{H}_{\alpha}(\omega), \quad v_{\alpha, n} \in \mathbb{R}, \omega \in \Omega, t \in[0, T] .
$$

The fractional generalized process $\widetilde{v}$ from $L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}^{(H)}$ has a chaos expansion representation of the form (35) such that (15) holds.

The definitions of coordinatewise and simple coordinatewise operators, Section 2.3.3, hold for processes defined on both classical white noise space and fractional white noise space.

## 3. The Stochastic LQR problem with fractional Brownian motion

In order to study the stochastic LQR problem on fractional spaces we introduce an isometry $\mathcal{M}$ between the space of square integrable fractional random variables $L^{2}\left(\mu_{H}\right)$ and the space of integrable random variables $L^{2}(\mu)$. Extending this mapping to stochastic processes we can transform the state equation with fractional Brownian motion to an equation with standard Brownian motion. Therefore, we can solve the optimal control problem with respect to an equation with standard Brownian motion and find the solution of the original problem by applying $\mathcal{M}^{-1}$.

### 3.1. The fractional operator $\mathcal{M}$

Since $M^{(H)}$ is self-adjoint we can connect (11) and (34) for all $\alpha \in \mathcal{I}$

$$
\begin{aligned}
H_{\alpha}(\omega) & =\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, M^{(H)} e_{k}\right\rangle\right)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle M^{(H)} \omega, e_{k}\right\rangle\right) \\
& =\widetilde{H}_{\alpha}\left(M^{(H)} \omega\right)
\end{aligned}
$$

and similarly

$$
\widetilde{H}_{\alpha}(\omega)=H_{\alpha}\left(M^{(1-H)} \omega\right) .
$$

Therefore we define a new (fractional) operator $\mathcal{M}$ which maps the orthogonal basis of $L^{2}\left(\mu_{H}\right)$ into the orthogonal basis of $L^{2}(\mu)$.

Definition 8 ([33]). Let $\mathcal{M}: L^{2}\left(\mu_{H}\right) \rightarrow L^{2}(\mu)$ be defined by

$$
\mathcal{M}\left(\widetilde{H}_{\alpha}(\omega)\right)=H_{\alpha}(\omega), \quad \alpha \in \mathcal{I}, \omega \in \Omega
$$

The operator $\mathcal{M}$ and the fractional operator $M^{(1-H)}$ correspond to each other. For $G=\sum_{\alpha \in \mathcal{I}} c_{\alpha} \widetilde{H}_{\alpha}(\omega) \in L^{2}\left(\mu_{H}\right)$, by linearity and continuity we extend $\mathcal{M}$ to

$$
\begin{equation*}
\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} c_{\alpha} \widetilde{H}_{\alpha}(\omega)\right)=\sum_{\alpha \in \mathcal{I}} c_{\alpha} H_{\alpha}(\omega) \tag{36}
\end{equation*}
$$

Theorem 6 ([33]). The operator $\mathcal{M}$ is an isometry between spaces of classical Gaussian and fractional Gaussian random variables.

Proof. The operator $\mathcal{M}$ is the isometry between $L^{2}\left(\mu_{H}\right)$ and $L^{2}(\mu)$ because it holds $\left\|\mathcal{M}\left(\widetilde{H}_{\alpha}\right)\right\|_{L^{2}(\mu)}=\left\|H_{\alpha}\right\|_{L^{2}(\mu)}=\alpha!=\left\|\widetilde{H}_{\alpha}\right\|_{L^{2}\left(\mu_{H}\right)}$.

The action of $\mathcal{M}$ can be seen as a transformation of the corresponding elements of the orthogonal basis $\left\{\widetilde{H}_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ into $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, see [33]. For every element $F \in L^{2}(\mu)$ there exists a unique $\widetilde{F} \in L^{2}\left(\mu_{H}\right)$ so $F=\mathcal{M} \widetilde{F}$ and also for each $\widetilde{F} \in L^{2}\left(\mu_{H}\right)$ there exists a unique $F \in L^{2}(\mu)$ so $\widetilde{F}=\mathcal{M}^{-1} F$. Further on, such pairs of elements $F$ and $\widetilde{F}$, that are connected via $\mathcal{M}$, will be called the associated pairs. The coefficients of the chaos expansion representations of associated elements $F$ and $\widetilde{F}$ coincide.
Lemma 3. Let $\left.F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} \in L^{2} \underset{\sim}{\mu}\right)$ and $\widetilde{F}=\sum_{\alpha \in \mathcal{I}} \widetilde{f}_{\alpha} \widetilde{H}_{\alpha} \in L^{2}\left(\mu_{H}\right)$. Then $F$ and $\widetilde{F}$ are associated if and only if $\widetilde{f}_{\alpha}=f_{\alpha}$ for all $\alpha \in \mathcal{I}$.
Proof. Let $F$ and $\widetilde{F}$ be associated. Then it holds

$$
\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}=F=\mathcal{M}(\widetilde{F})=\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} \widetilde{f}_{\alpha} \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} \widetilde{f}_{\alpha} H_{\alpha}
$$

Since the chaos expansion representation in the orthogonal basis $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ is unique, it follows that $f_{\alpha}=\widetilde{f}_{\alpha}$ for all $\alpha \in \mathcal{I}$.

The action of the operator $\mathcal{M}$ can be extended to a Kondratiev space of stochastic distributions $\mathcal{M}:(S)_{-1}^{(H)} \rightarrow(S)_{-1}$ by

$$
\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} a_{\alpha} \widetilde{H}_{\alpha}(\omega)\right)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega), \quad a_{\alpha} \in \mathbb{R}
$$

The extension is well defined since there exists $p \in \mathbb{N}$ so $\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$. In an analogous way the action of the operator $\mathcal{M}$ can be extended to stochastic processes and $\mathcal{H}$-valued (generalized) stochastic processes.

Example 2. (a) A real-valued fractional Brownian motion $b_{t}^{(H)}(\omega), H \in(0,1)$ as an element of the fractional Gaussian space $L^{2}\left(\mu_{(1-H)}\right)=L^{2}\left(\mu \circ M^{(H)}\right)$ is given by

$$
b_{t}^{(H)}(\omega)=\sum_{k=1}^{\infty}\left(\int_{0}^{t} \xi_{k}(s) d s\right) \widetilde{H}_{\varepsilon^{(k)}}(\omega)
$$

with help of the property $M^{(H)} \xi_{k}=e_{k}^{(1-H)}$, see [33].
(b) A one-dimensional real-valued fractional singular white noise $w_{t}^{(H)}$ as an element of the fractional Kondratiev space $(S)_{-1}^{(1-H)}$ is defined by the chaos expansion $w_{t}^{(H)}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) \widetilde{H}_{\varepsilon^{(k)}}(\omega)$. It is integrable and the relation $\frac{d}{d t} b_{t}^{(H)}=$ $w_{t}^{(H)}$ holds in the sense of distributions.

Moreover, combining (16) and (36) we obtain

$$
\mathcal{M}^{-1}\left(w_{t}\right)=\mathcal{M}^{-1}\left(\sum_{k=1}^{\infty} \xi_{k} H_{\varepsilon^{(k)}}\right)=\sum_{k=1}^{\infty} \xi_{k} \widetilde{H}_{\varepsilon^{(k)}}(\omega)=w_{t}^{(H)}
$$

(c) An $\mathcal{H}$-valued fractional white noise in the fractional space is given by

$$
\begin{equation*}
W_{t}^{(H)}(\omega)=\sum_{k=1}^{\infty} \mathbf{e}_{k}(t) \widetilde{H}_{\varepsilon^{(k)}}(\omega) \tag{37}
\end{equation*}
$$

where $\left\{\mathbf{e}_{k}\right\}_{k \in \mathbb{N}}$ is an orthonormal basis in $L^{2}([0, T], \mathcal{H})$. By (17) and (37) the relations $\mathcal{M}\left(W_{t}^{(H)}\right)=W_{t}$ and $\mathcal{M}^{-1}\left(W_{t}\right)=W_{t}^{(H)}$ follow.

From here onwards we will keep the following notation: all processes denoted with tilde in subscript will be considered as elements of a fractional space. Therefore, due to Lemma 3, each process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}$ from an $\mathcal{H}$-valued classical space (particularly $L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$ or $\left.L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}\right)$ will be associated to a process $\widetilde{v}=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \widetilde{H}_{\alpha}$ from the corresponding $\mathcal{H}$-valued fractional space (particularly $L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)$ or $\left.L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}^{(H)}\right)$ via the fractional mapping $\mathcal{M}$, i.e., $\mathcal{M}(\widetilde{v})=v$. Since the coefficients of processes $\widetilde{v}$ and $v$ are equal, it also follows

$$
\begin{equation*}
\|\widetilde{v}\|_{L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2}=\|v\|_{L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)}^{2} \tag{38}
\end{equation*}
$$

Theorem 7. The fractional mapping $\mathcal{M}$ satisfies the following properties:
(1) Let the operators $\widetilde{\mathbb{O}}: L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)$ and $\mathbb{O}: L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$ be coordinatewise operators that correspond to the same family of operators $O_{\alpha}: L^{2}([0, T], \mathcal{H}) \rightarrow$ $L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$. Then it holds

$$
\mathcal{M}(\widetilde{\mathbb{O}} \widetilde{v})=\mathbb{O}(\mathcal{M} \widetilde{v})
$$

(2) $\mathcal{M}$ is linear and it also holds $\mathcal{M}(\widetilde{u} \diamond \widetilde{y})=\mathcal{M}(\widetilde{u}) \diamond \mathcal{M}(\widetilde{y})$ and
(3) $\mathcal{M}\left(\mathbb{E}_{\mu_{H}} \widetilde{v}\right)=\mathbb{E}_{\mu}(\mathcal{M} \widetilde{v})$,
for $\widetilde{v} \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)$ and $\widetilde{u}, \widetilde{y} \in L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}^{(H)}$.

Proof. Since $\mathcal{M}$ acts on the orthogonal basis of $L^{2}\left(\Omega, \mu_{H}\right)$ the following is valid:
(1) Let $\widetilde{v} \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)$. From (22) and (36) we obtain

$$
\mathcal{M}(\widetilde{\mathbb{O}} \widetilde{v})=\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} O_{\alpha} v_{\alpha} \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} O_{\alpha} v_{\alpha} H_{\alpha}=\mathbb{O}\left(\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}\right)=\mathbb{O}(\mathcal{M} \widetilde{v})
$$

(2) By definition, the fractional operator $\mathcal{M}$ is linear. It is also homogeneous with respect to the Wick multiplication, i.e., it holds

$$
\begin{aligned}
\mathcal{M}(\widetilde{u} \diamond \widetilde{y}) & =\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} u_{\alpha} y_{\beta} \widetilde{H}_{\alpha+\beta}\right)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} u_{\alpha} y_{\beta} H_{\alpha+\beta} \\
& =\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha} \widetilde{H}_{\alpha}\right) \diamond \mathcal{M}\left(\sum_{\beta \in \mathcal{I}} y_{\beta} \widetilde{H}_{\beta}\right)=\mathcal{M}(\widetilde{u}) \diamond \mathcal{M}(\widetilde{y})
\end{aligned}
$$

(3) For $\widetilde{v} \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}\left(\Omega, \mu_{H}\right)$ an element $\mathbb{E}_{\mu_{H}} \widetilde{v}$ is the zeroth coefficient of fractional expansion of $\widetilde{v}$, i.e., $\mathbb{E}_{\mu_{H}} \widetilde{v}=v_{\mathbf{0}}$. Thus, $\mathcal{M}\left(\mathbb{E}_{\mu_{H}} \widetilde{v}\right)=v_{\mathbf{0}}$. On the other side, $\mathbb{E}_{\mu}(\mathcal{M} \widetilde{v})$ is the zeroth coefficient of the expansion of $\mathcal{M} \widetilde{v}$, which is also equal to $v_{\mathbf{0}}$. Thus, $\mathcal{M}\left(\mathbb{E}_{\mu_{H}} \widetilde{v}\right)=\mathbb{E}_{\mu}(\mathcal{M} \widetilde{v})$.

Theorem 8. For a differentiable $\mathcal{H}$-valued process $\widetilde{z}$ from the fractional space the following holds

$$
\mathcal{M}\left(\frac{d}{d t} \widetilde{z}\right)=\frac{d}{d t}(\mathcal{M} \widetilde{z})
$$

Proof. Differentiation of a stochastic process is a simple coordinatewise operator, i.e., a process is considered to be differentiable if and only if its coordinates are differentiable deterministic functions [34]. The assertion follows by applying $\mathcal{M}$ to $\frac{d}{d t} \widetilde{z}=\sum_{\alpha \in \mathcal{I}} \frac{d}{d t} z_{\alpha}(t) \widetilde{H}_{\alpha}(\omega)=\sum_{\alpha \in \mathcal{I}} z_{\alpha}^{\prime}(t) \widetilde{H}_{\alpha}(\omega)$. We obtain

$$
\begin{aligned}
\mathcal{M}\left(\frac{d}{d t} \widetilde{z}\right) & =\mathcal{M}\left(\sum_{\alpha \in \mathcal{I}} z_{\alpha}^{\prime}(t) \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} z_{\alpha}^{\prime}(t) H_{\alpha} \\
& =\frac{d}{d t}\left(\sum_{\alpha \in \mathcal{I}} z_{\alpha}(t) H_{\alpha}\right)=\frac{d}{d t}(\mathcal{M} \widetilde{z})
\end{aligned}
$$

3.1.1. Fractional integral. The fractional Itô-Skorokhod integral $\delta^{(H)}$ of an $\mathcal{H}$ valued process $\widetilde{u}$ that belongs to $\operatorname{Dom}\left(\delta^{(H)}\right)$ in the fractional space is defined in an analogous way as the Itô-Skorokgod integral (23) in classical space, see Section 2.3.4. Clearly, we say that $\widetilde{u}=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \widetilde{H}_{\alpha} \in \operatorname{Dom}\left(\delta^{(H)}\right)$ if (28) holds. Then the fractional Itô-Skorokhod integral of a process $\widetilde{u}=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \widetilde{H}_{\alpha}$ in fractional space

$$
\delta^{(H)}(\widetilde{u})=\int_{0}^{T} \widetilde{u} d B_{t}^{(H)}=\int_{0}^{T} \widetilde{u} \diamond W_{t}^{(H)} d t
$$

has the chaos expansion representation of the form

$$
\begin{equation*}
\delta^{(H)}(\widetilde{u})=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_{\alpha, k} \widetilde{H}_{\alpha+\varepsilon^{(k)}} \tag{39}
\end{equation*}
$$

The coefficients $u_{\alpha, k}$ are the coefficients of the expansion of the corresponding $\delta(u)$, where $u$ is the associated process to $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}, u_{\alpha} \in L^{2}([0, T], \mathcal{H})$, $\alpha \in \mathcal{I}$.

Theorem 9. For $\widetilde{u} \in \operatorname{Dom}(\delta)$ it holds

$$
\begin{equation*}
\mathcal{M}\left(\delta^{(H)}(\widetilde{u})\right)=\delta(\mathcal{M}(\widetilde{u})) \tag{40}
\end{equation*}
$$

Proof. From (26), (39) and the definition of operator $\mathcal{M}$ the property (40) follows, $\mathcal{M}\left(\delta^{(H)}(\widetilde{u})\right)=\delta(u)=\delta(\mathcal{M}(\widetilde{u}))$ holds for all associated pairs of processes $\widetilde{u}$ and $u=\mathcal{M} \widetilde{u}$. Since $\mathcal{M}$ is an isometry it holds

$$
\begin{aligned}
\|\delta(u)\|_{L^{2}(\Omega, \mu)}^{2} & \left.=\left\|\mathcal{M}\left(\delta^{(H)}(\widetilde{u})\right)\right\|_{L^{2}(\Omega, \mu)}^{2}=\| \delta^{(H)}(\widetilde{u})\right) \|_{L^{2}\left(\Omega, \mu_{H}\right)}^{2} \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!\left(\sum_{k \in \mathbb{N}} u_{\alpha, k} \sqrt{\alpha_{k}+1}\right)^{2}<\infty
\end{aligned}
$$

Remark 2. The definition of the fractional Itô-Skorokhod integral in the classical Gaussian space is given in [5, 6, 39]. In [33] the authors provided a detailed analysis on generalized classical and fractional operators of Malliavin calculus on white noise spaces.

### 3.2. The optimal control problem

We consider the state equation

$$
\begin{equation*}
d \widetilde{y}(t)=[\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)] d t+\widetilde{\mathbf{C}} \widetilde{y}(t) d B_{t}^{(H)}, \quad \widetilde{y}(0)=\widetilde{y}^{0}, \quad t \in[0, T], \tag{41}
\end{equation*}
$$

with respect to an $\mathcal{H}$-valued fractional Brownian motion in the fractional Gaussian white noise space. The objective is to minimize the functional

$$
\begin{equation*}
\mathbf{J}^{(H)}(\widetilde{u})=\mathbb{E}_{\mu_{H}}\left[\int_{0}^{T}\left(\|\widetilde{\mathbf{R}} \widetilde{y}\|_{\mathcal{H}}^{2}+\|\widetilde{u}\|_{\mathcal{U}}^{2}\right) d t+\left\|\widetilde{\mathbf{G}} \widetilde{y}_{T}\right\|_{\mathcal{H}}^{2}\right] \tag{42}
\end{equation*}
$$

over all $\widetilde{u} \in L^{2}([0, T] \times \Omega, \mathcal{U})$.
Due to the fundamental theorem of stochastic calculus, for admissible square integrable processes, the fractional state equation (41) is equivalent to its Wick version

$$
\begin{equation*}
\dot{\widetilde{y}}(t)=\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)+\widetilde{\mathbf{C}} \widetilde{y}(t) \diamond W^{(H)}(t), \quad \widetilde{y}(0)=\widetilde{y}^{0}, \quad t \in[0, T] \tag{43}
\end{equation*}
$$

By using the fractional mapping $\mathcal{M}$ one can transfer the optimal control problem (41)-(42) from the fractional space to the corresponding optimal control problem with the state equation

$$
\begin{equation*}
d y(t)=[\mathbf{A} y(t)+\mathbf{B} u(t)] d t+\mathbf{C} y(t) d B_{t}, \quad y(0)=y^{0}, \quad t \in[0, T] \tag{44}
\end{equation*}
$$

with respect to Brownian motion subject to

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}_{\mu}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{H}}^{2}\right] \tag{45}
\end{equation*}
$$

in the classical Gaussian white noise space. Instead of the state equation (44), on a set of square integrable processes, one can consider its equivalent Wick-type equation

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\mathbf{C} y(t) \diamond W_{t}, \quad y(0)=y^{0}, \quad t \in[0, T] \tag{46}
\end{equation*}
$$

Once the solution of the optimal control problem (44)-(45) is obtained, then using the fractional isometry $\mathcal{M}$ one can also obtain the solution to the initial optimal control problem (41)-(42). That is the statement of the following theorem.
Theorem 10. Let the fractional operators $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}, \widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{G}}$ defined on fractional space be coordinatewise operators that correspond to the families of deterministic operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ respectively. Let the pair $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ be the optimal solution of the fractional stochastic optimal control problem (41)-(42). Then, the pair $\left(\mathcal{M} \widetilde{u}^{*}, \mathcal{M} \widetilde{y}^{*}\right)$ is the optimal solution $\left(u^{*}, y^{*}\right)$ of the associated optimal control problem (44)-(45), where the operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$, $\mathbf{R}$ and $\mathbf{G}$ defined on classical space, are coordinatewise operators that correspond respectively to the same families of deterministic operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, $\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. Moreover, if $\left(u^{*}, y^{*}\right)$ is the optimal solution of the stochastic optimal control problem (44)-(45), then the pair $\left(\mathcal{M}^{-1} u^{*}, \mathcal{M}^{-1} y^{*}\right)$ is the optimal solution $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ to the corresponding fractional optimal control problem (41)-(42).

Proof. Let $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ be the optimal pair of the problem (41)-(42), i.e., its equivalent problem (42)-(43). Then $\min _{u} \mathbf{J}(u)=\mathbf{J}\left(u^{*}\right)$, while $y^{*}$ solves (41) and also (43). Let all operators appearing in the control problem be coordinatewise operators. By applying the chaos expansion method and the properties of the fractional operator $\mathcal{M}$ stated in Theorem 7 and Theorem 8, we transform (43) in fractional space to the corresponding state equation in classical space, i.e.,

$$
\begin{aligned}
\dot{y}(t) & =\mathcal{M}\left(\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)+\widetilde{\mathbf{C}} \widetilde{y}(t) \diamond W^{(H)}(t)\right) \\
& =\mathcal{M}(\widetilde{\mathbf{A}} \widetilde{y})+\mathcal{M}(\widetilde{\mathbf{B}} \widetilde{u})+\mathcal{M}(\widetilde{\mathbf{C}} \widetilde{y}) \diamond \mathcal{M}\left(W_{t}^{(H)}\right) \\
& =\mathbf{A} y+\mathbf{B} u+\mathbf{C} y \diamond W_{t}
\end{aligned}
$$

where $y$ and $u$ are the associated processes to $\widetilde{y}$ and $\widetilde{u}$ respectively. Moreover, by Theorem 7 part (3) and (38) the operator $\mathcal{M}$ transforms the cost functional $\mathbf{J}^{(H)}$ to

$$
\mathcal{M}\left(\mathbf{J}^{(H)}(\widetilde{u})\right)=\mathcal{M}\left(\mathbb{E}_{\mu_{H}}(\widetilde{v})\right)=\mathbb{E}_{\mu}(\mathcal{M} \widetilde{v})=\mathbb{E}_{\mu}(v)=\mathbf{J}(u)
$$

where $\widetilde{v}$ and $v$ are associated elements $\widetilde{v}=\int_{0}^{T}\left(\|\widetilde{\mathbf{R}} \widetilde{y}\|_{\mathcal{H}}^{2}+\|\widetilde{u}\|_{\mathcal{U}}^{2}\right) d t+\left\|\widetilde{\mathbf{G}} \widetilde{y}_{T}\right\|_{\mathcal{H}}^{2}$ and $v=\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{H}}^{2}$.

We will solve the control problem in the classical space (we will generalize the results from [27]) and then, by use of Theorem 10 via the inverse fractional mapping $\mathcal{M}^{-1}$, we obtain the optimal solution for the corresponding fractional problem.

Theorem 11. Let the following assumptions hold:
(A1) The operator $\mathbf{A}: L^{2}([0, T], \mathcal{D}) \otimes L^{2}(\Omega, \mu) \rightarrow L^{2}([0, T], \mathcal{D}) \otimes L^{2}(\Omega, \mu)$ is a coordinatewise linear operator that corresponds to the family of deterministic operators $A_{\alpha}: L^{2}([0, T], \mathcal{D}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$, where $A_{\alpha}$ are infinitesimal generators of strongly continuous semigroups $\left(e^{A_{\alpha} t}\right)_{\alpha \in \mathcal{I}}, t \geq 0$, defined on a common domain $\mathcal{D}$ that is dense in $\mathcal{H}$, such that for some $m, \theta>0$ and all $\alpha \in \mathcal{I}$ we have

$$
\left\|\left(e^{A_{\alpha} t}\right)_{\alpha}\right\|_{L(\mathcal{H})} \leq m e^{\theta t}, \quad t \geq 0
$$

(A2) The operator $\mathbf{C}: L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$ is a coordinatewise operator corresponding to a family of uniformly bounded deterministic operators $C_{\alpha}: L^{2}([0, T], \mathcal{H}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$.
(A3) The control operator $\mathbf{B}$ is a simple coordinatewise operator $\mathbf{B}: L^{2}([0, T], \mathcal{U}) \otimes$ $L^{2}(\Omega, \mu) \rightarrow L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)$ that is defined by a family of uniformly bounded deterministic operators $B_{\alpha}: L^{2}([0, T], \mathcal{U}) \rightarrow L^{2}([0, T], \mathcal{H}), \alpha \in \mathcal{I}$.
(A4) The operators $\mathbf{R}$ and $\mathbf{G}$ are bounded coordinatewise operators corresponding to the families of deterministic operators $\left\{R_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\{G\}_{\alpha \in \mathcal{I}}$ respectively. (A5) $\mathbb{E}_{\mu}\left\|y^{0}\right\|_{\mathcal{H}}^{2}<\infty$.
Then, the optimal control problem (45)-(46) has a unique optimal control $u^{*}$ given in the chaos expansion form

$$
u^{*}=-\sum_{\alpha \in \mathcal{I}} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha}
$$

where $P_{d, \alpha}(t)$ for every $\alpha \in \mathcal{I}$ solves the Riccati equation

$$
\begin{align*}
\dot{P}_{d, \alpha}(t)+P_{d, \alpha}(t) A_{\alpha}+A_{\alpha}^{\star} P_{d, \alpha}(t)+R_{\alpha} R_{\alpha}^{\star}-P_{d, \alpha}(t) B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t) & =0 \\
P_{d, \alpha}(T) & =G_{\alpha}^{\star} G_{\alpha} \tag{47}
\end{align*}
$$

and $k_{\alpha}(t)$ is for each $\alpha \in \mathcal{I}$ a solution to the auxiliary differential equation

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+\left(A_{\alpha}^{\star}-P_{d, \alpha}(t) B_{\alpha} B_{\alpha}^{\star}\right) k_{\alpha}(t)+P_{d, \alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(t) \cdot \mathbf{e}_{i}(t)\right)=0 \tag{48}
\end{equation*}
$$

with the terminal condition $k_{\alpha}(T)=0$ and $y^{*}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*} H_{\alpha}$ is the optimal state.
Proof. Since all the operators $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are coordinatewise, by (22) the actions are given by $\mathbf{A} y(t, \omega)=\sum_{\alpha \in \mathcal{I}} A y_{\alpha}(t) H_{\alpha}(\omega), \mathbf{B} u(t)=\sum_{\alpha \in \mathcal{I}} B u_{\alpha}(t) H_{\alpha}(\omega)$ and $\mathbf{C} y(t, \omega)=\sum_{\alpha \in \mathcal{I}} C y_{\alpha}(t) H_{\alpha}(\omega)$, for

$$
\begin{equation*}
y(t, \omega)=\sum_{\alpha \in \mathcal{I}} y_{\alpha}(t) H_{\alpha}(\omega), \quad u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega) \tag{49}
\end{equation*}
$$

such that for all $\alpha \in \mathcal{I}$ the coefficients $y_{\alpha} \in L^{2}([0, T], \mathcal{H})$ and $u_{\alpha} \in L^{2}([0, T], \mathcal{U})$. From (A2) and (A3) we conclude that the operators $\mathbf{C}$ and $\mathbf{B}$ are bounded and
by Lemma 1 it holds

$$
\begin{aligned}
\|\mathbf{B} u\|_{L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\Omega, \mu)}^{2} & =\sum_{\alpha \in \mathcal{I}} \alpha!\left\|B_{\alpha} u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{H})}^{2} \\
& \leq c^{2} \sum_{\alpha \in \mathcal{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{U})}^{2}=c^{2}\|u\|_{L^{2}([0, T], \mathcal{U}) \otimes L^{2}(\Omega, \mu)}^{2}
\end{aligned}
$$

where $\left\|B_{\alpha}\right\| \leq c$ for all $\alpha \in \mathcal{I}$.
We divide the proof into several steps. First, we consider the Wick version (46) of the state equation (44), we apply the chaos expansion method and obtain a system of deterministic equations. By representing $y$ and $y^{0}$ in their chaos expansion forms, the initial condition $y(0)=y^{0}$, for a given $\mathcal{H}$-valued random variable $y^{0}$, is reduced to a family of initial conditions for the coefficients of the state

$$
y_{\alpha}(0)=y_{\alpha}^{0}, \quad \text { for all } \quad \alpha \in \mathcal{I}, \quad \text { where } y_{\alpha}^{0} \in \mathcal{H}, \alpha \in \mathcal{I}
$$

With the chaos expansion method the state equation (46) transforms to the system of infinitely many deterministic initial value problems:
$1^{\circ}$ for $\alpha=\mathbf{0}$ :

$$
\begin{equation*}
y_{\mathbf{0}}^{\prime}(t)=A_{\mathbf{0}} y_{\mathbf{0}}(t)+B_{\mathbf{0}} u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \tag{50}
\end{equation*}
$$

$2^{\circ}$ for $|\alpha|>0$ :

$$
\begin{array}{r}
y_{\alpha}^{\prime}(t)=A_{\alpha} y_{\alpha}(t)+B_{\alpha} u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(t) \cdot \mathbf{e}_{i}(t)  \tag{51}\\
y_{\alpha}(0)=y_{\alpha}^{0}
\end{array}
$$

where the unknowns correspond to the coefficients of the control and the state variables. It describes how the stochastic state equation propagates chaos through different levels. Note that for $\alpha=\mathbf{0}$, the equation (50) corresponds to the deterministic version of the problem and the state $y_{0}$ is the expected value of $y$. The terms $y_{\alpha-\varepsilon^{(i)}}(t)$ are obtained recursively with respect to the length of $\alpha$. The sum in (51) goes through all possible decompositions of $\alpha$, i.e., for all $j$ for which $\alpha-\varepsilon^{(j)}$ is defined. Therefore, the sum has as many terms as multi-index $\alpha$ has non-zero components. Existence and uniqueness of solutions of (50), (51) follow from the assumptions (A1), (A2) and (A3) for the operators $A_{\alpha}, B_{\alpha}$ and $C_{\alpha}, \alpha \in \mathcal{I}$.

In the second step, we set up optimal control problems for each $\alpha$-level. We seek for the optimal control $u$ and the corresponding optimal state $y$ in the chaos expansion representation form (49), i.e., the goal is to obtain the unknown coefficients $u_{\alpha}$ and $y_{\alpha}$ for all $\alpha \in \mathcal{I}$.

The problems are defined in the following way:
$1^{\circ}$ for $\alpha=\mathbf{0}$ the control problem

$$
\begin{equation*}
\min _{u_{\mathbf{0}}} J\left(u_{\mathbf{0}}\right)=\int_{0}^{T}\left(\left\|R_{\mathbf{0}} y_{\mathbf{0}}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\mathbf{0}}(t)\right\|_{\mathcal{U}}^{2}\right) d t+\left\|G_{\mathbf{0}} y_{\mathbf{0}}(T)\right\|_{\mathcal{H}}^{2} \tag{52}
\end{equation*}
$$

subject to

$$
y_{\mathbf{0}}^{\prime}(t)=A_{\mathbf{0}} y_{\mathbf{0}}(t)+B_{\mathbf{0}} u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}, \quad \text { and }
$$

$2^{\circ}$ for $|\alpha|>0$ the control problem

$$
\begin{equation*}
J\left(u_{\alpha}\right)=\int_{0}^{T}\left(\left\|R_{\alpha} y_{\alpha}(t)\right\|_{\mathcal{H}}^{2}+\left\|u_{\alpha}(t)\right\|_{\mathcal{U}}^{2}\right) d t+\left\|G_{\alpha} y_{\alpha}(T)\right\|_{\mathcal{H}}^{2} \tag{53}
\end{equation*}
$$

subject to

$$
y_{\alpha}^{\prime}(t)=A_{\alpha} y_{\alpha}(t)+B_{\alpha} u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(t) \cdot \mathbf{e}_{i}(t), \quad y_{\alpha}(0)=y_{\alpha}^{0}
$$

and can be solved by the induction on the length of multi-index $\alpha \in \mathcal{I}$. Next we solve the family of deterministic control problems, i.e., we discuss the solution of the deterministic system of control problems (52) and (53):
$1^{\circ}$ For $\alpha=\mathbf{0}$ the state equation (50) is homogeneous, thus the optimal control for (50)-(52) is given in the feedback form

$$
\begin{equation*}
u_{\mathbf{0}}^{*}(t)=-B_{\mathbf{0}}^{\star} P_{d, \mathbf{0}}(t) y_{\mathbf{0}}^{*}(t), \tag{54}
\end{equation*}
$$

where $P_{d, \mathbf{0}}(t)$ solves the Riccati equation (9).
$2^{\circ}$ For each $|\alpha|>0$ the state equation (51) is inhomogeneous and the optimal control for (53) is given by

$$
\begin{equation*}
u_{\alpha}^{*}(t)=-B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t)-B_{\alpha}^{\star} k_{\alpha}(t), \tag{55}
\end{equation*}
$$

where $P_{d, \alpha}(t)$ solves the Riccati equation (47), while $k_{\alpha}(t)$ is a solution to the auxiliary differential equation (48) with the terminal condition $k_{\alpha}(T)=0$, as discussed in Section 2.1.1.
Summing up all the coefficients we obtain the optimal solution $\left(u^{*}, y^{*}\right)$ represented in terms of chaos expansions. Thus, the optimal state is given in the form

$$
y^{*}=\sum_{\alpha \in \mathcal{I}} y_{\alpha}^{*}(t) H_{\alpha}=y_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} y_{\alpha}^{*}(t) H_{\alpha}
$$

and the corresponding optimal control

$$
\begin{align*}
u^{*} & =\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}=u_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha} \\
& =-B_{\mathbf{0}}^{\star} P_{d, \mathbf{0}}(t) y_{\mathbf{0}}^{*}-\sum_{|\alpha|>0} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha}  \tag{56}\\
& =-\sum_{\alpha \in \mathcal{I}} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{\star}(t) H_{\alpha}-\sum_{\alpha \in \mathcal{I}} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha} \\
& =-\mathbf{B}^{\star} \mathbf{P}_{d} y^{*}(t)-\mathbf{B}^{\star} \mathcal{K},
\end{align*}
$$

where $\mathbf{P}_{d}(t)$ is a coordinatewise operator corresponding to the deterministic family of operators $\left\{P_{d, \alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\mathcal{K}$ is a stochastic process with coefficients $k_{\alpha}(t)$, i.e., process of the form $\mathcal{K}=\sum_{\alpha \in \mathcal{I}} k_{\alpha}(t) H_{\alpha}$, with $k_{\mathbf{0}}=0$.

In the following step we prove the optimality of the obtained solution. Assuming (A1)-(A4) it follows that the assumptions of Theorem 1 are fulfilled and thus the optimal control of the problem (4)-(5) is given in the feedback form by

$$
\begin{equation*}
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t) \tag{57}
\end{equation*}
$$

with a positive self-adjoint operator $\mathbf{P}(t)$ solving the stochastic Riccati equation (6). Since the state equations (4) and (46) are equivalent, we are going to interpret the optimal solution (57), involving the Riccati operator $\mathbf{P}(t)$ in terms of chaos expansions. It holds $\mathbf{J}\left(u^{*}\right)=\min _{u} \mathbf{J}(u)$, for $u^{*}$ of the form (57).

On the other hand, the stochastic cost function $\mathbf{J}$ is related with the deterministic cost function $J$ by

$$
\begin{aligned}
\mathbf{J}(u) & =\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathcal{W}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right] \\
& =\mathbb{E}\left(\int_{0}^{T}\|\mathbf{R} y\|_{\mathcal{W}}^{2} d t\right)+\mathbb{E}\left(\int_{0}^{T}\|u\|_{\mathcal{U}}^{2} d t\right)+\mathbb{E}\left(\left\|\mathbf{G} y_{T}\right\|_{\mathcal{Z}}^{2}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!\left\|R_{\alpha} y_{\alpha}\right\|_{L^{2}([0, T], \mathcal{W})}^{2}+\sum_{\alpha \in \mathcal{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{U})}^{2}+\sum_{\alpha \in \mathcal{I}} \alpha!\left\|G_{\alpha} y_{\alpha}(T)\right\|_{\mathcal{Z}}^{2} \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!\left(\left\|R_{\alpha} y_{\alpha}\right\|_{L^{2}([0, T], \mathcal{W})}^{2}+\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathcal{U})}^{2}+\left\|G_{\alpha} y_{\alpha}(T)\right\|_{\mathcal{Z}}^{2}\right) \\
& =\sum_{\alpha \in \mathcal{I}} \alpha!J\left(u_{\alpha}\right) .
\end{aligned}
$$

Thus,

$$
\mathbf{J}\left(u^{*}\right)=\min _{u} \mathbf{J}(u)=\min _{u} \sum_{\alpha \in \mathcal{I}} \alpha!J\left(u_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} \alpha!\min _{u_{\alpha}} J\left(u_{\alpha}\right)=\sum_{\alpha \in \mathcal{I}} \alpha!J\left(u_{\alpha}^{*}\right)
$$

and therefore

$$
\begin{equation*}
u^{*}(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega) \tag{58}
\end{equation*}
$$

i.e., the optimal control obtained via direct Riccati approach $u^{*}$ coincides with the optimal control obtained via chaos expansion approach $\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega)$. Moreover, the optimal states are the same and the existence and uniqueness of the solution of the optimal state equation via chaos expansion approach follows from the direct Riccati approach.

Finally, we prove the convergence of the chaos expansions of the optimal state. We include the feedback forms (54) and (55) of the optimal controls $u_{\alpha}^{*}$, $\alpha \in \mathcal{I}$ in the state equations (50) and (51) and obtain the system

$$
\begin{align*}
& y_{\mathbf{0}}^{\prime}(t)=\left(A_{\mathbf{0}}-B_{\mathbf{0}} B_{\mathbf{0}}^{\star} P_{d, \mathbf{0}}(t)\right) y_{\mathbf{0}}(t) \\
& y_{\alpha}^{\prime}(t)=\left(A_{\alpha}-B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t)\right) y_{\alpha}(t)-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \mathbf{e}_{i}(t) \tag{59}
\end{align*}
$$

for $|\alpha| \geq 1$, with the initial conditions $y_{\alpha}(0)=y_{\alpha}^{0}, \alpha \in \mathcal{I}$.

From the assumption (A1) it follows that $A_{\alpha}, \alpha \in \mathcal{I}$ are infinitesimal generators of strongly continuous semigroups $\left(T_{t}\right)_{\alpha}=\left(e^{A_{\alpha} t}\right)_{\alpha}, t \geq 0$ which are uniformly bounded, i.e., $\left\|e^{A_{\alpha} t}\right\|_{\mathcal{L}(\mathcal{H})} \leq m e^{\theta t}, \alpha \in \mathcal{I}$ holds for some positive constants $m$ and $\theta$, where $\mathcal{L}(\mathcal{H})$ denotes the set of linear bounded mappings on $L^{2}([0, T], \mathcal{H})$. Moreover, the family $\left(T_{t}^{\star}\right)_{\alpha}=\left(e^{A_{\alpha}^{\star} t}\right)_{\alpha}, t \geq 0$ is a family of strongly continuous semigroups whose infinitesimal generators are $A_{\alpha}^{\star}, \alpha \in \mathcal{I}$, the adjoint operators of $A_{\alpha}, \alpha \in \mathcal{I}$. This follows from the fact that each Hilbert space is a reflexive Banach space, see [43].

We denote by $S_{\alpha}(t)=A_{\alpha}-B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t), \alpha \in \mathcal{I}$ and rewrite (59) in simpler form

$$
\begin{array}{ll}
y_{\mathbf{0}}^{\prime}(t)=S_{\mathbf{0}}(t) y_{\mathbf{0}}(t), & y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \\
y_{\alpha}^{\prime}(t)=S_{\alpha}(t) y_{\alpha}(t)+f_{\alpha}(t), & y_{\alpha}(0)=y_{\alpha}^{0}, \tag{60}
\end{array}|\alpha|>1
$$

where $f_{\alpha}(t)=-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \mathbf{e}_{i}(t), \alpha \in \mathcal{I}$.
The operators $S_{\alpha}(t), \alpha \in \mathcal{I}$ can be understood as time dependent continuous perturbations of the operators $A_{\alpha}$. From Theorem 1 it follows that $P_{d, \alpha}(t), \alpha \in \mathcal{I}$ are self adjoint and uniformly bounded operators, i.e., $\left\|P_{d, \alpha}(t)\right\| \leq p, \alpha \in \mathcal{I}$, $t \in[0, T]$. The operators $B_{\alpha}$ and thus $B_{\alpha}^{\star}$ are uniformly bounded, i.e., for all $\alpha \in \mathcal{I}$ we have $\left\|B_{\alpha}\right\| \leq b$ and $\left\|B_{\alpha}^{*}\right\| \leq b, b>0$. Therefore, $B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t), \alpha \in \mathcal{I}$ are uniformly bounded. Hence, we can associate a family of evolution systems $U_{\alpha}(t, s)$, $\alpha \in \mathcal{I}, 0 \leq s \leq t \leq T$ to the initial value problems (60) such that

$$
\left\|U_{\alpha}(t, s)\right\|_{L(\mathcal{H})} \leq e^{\theta_{1} t}, \quad \text { for all } 0 \leq s \leq t \leq T
$$

The family of solution maps $U_{\alpha}(t, s) y_{\alpha}^{0}, \alpha \in \mathcal{I}$ to the non-autonomous system (60) is a family of evolutions which are in $C([0, T], \mathcal{H})$ since $B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}, \alpha \in \mathcal{I}$ are bounded for every $t$, and are for all $\alpha \in \mathcal{I}$ continuous in time, i.e., elements of $C([0, T], \mathcal{L}(\mathcal{H})),[43]$. The adjoint operators $(S(t))_{\alpha}^{\star}=A_{\alpha}^{\star}+P_{d, \alpha}(t) B_{\alpha}^{\star} B_{\alpha}, \alpha \in \mathcal{I}$ are associated to the corresponding adjoint evolution systems $U_{\alpha}^{\star}(t, s), \alpha \in \mathcal{I}$, $0 \leq s \leq t \leq T$, see [43].

The operators $C_{\alpha}, \alpha \in \mathcal{I}$ are uniformly bounded and for all $\alpha \in \mathcal{I}$ it holds $\left\|C_{\alpha}\right\| \leq d, d>0$. For a fixed control $u$ it also holds $\mathbf{C} y \in \operatorname{Dom}(\delta)$, i.e., (28) holds for $\mathbf{C} y$.

Consider a small interval $\left[0, T_{0}\right]$, for fixed $T_{0} \in(0, T]$. Denote by $M_{1}(t)=e^{\theta_{1} t}$ and $M_{2}(t)=\frac{1}{2 \theta_{1}}\left(e^{2 \theta_{1} t}-1\right)^{2}$ for $t \in\left(0, T_{0}\right]$.

For every $y_{\alpha}^{0} \in \operatorname{Dom}(S(t))_{\alpha}$ the mild solution of (60) is given in the form

$$
\begin{aligned}
& y_{\mathbf{0}}(t)=U_{\mathbf{0}}(t, 0) y_{\mathbf{0}}^{0} \\
& y_{\alpha}(t)=U_{\alpha}(t, 0) y_{\alpha}^{0}+\int_{0}^{t} U_{\alpha}(t, s)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(s) \mathbf{e}_{i}(s)-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(s)\right) d s
\end{aligned}
$$

for $|\alpha| \geq 1$ and $0 \leq s \leq t \leq T$ and $y_{\alpha}$ are continuous functions for all $\alpha \in \mathcal{I}$. The operators $C_{\alpha}, B_{\alpha}$ and $B_{\alpha}^{\star}, \alpha \in \mathcal{I}$ are uniformly bounded and therefore the inhomogeneity part of (59) belongs to the space $L^{2}([0, T], \mathcal{H})$, where the functions $k_{\alpha}, \alpha \in \mathcal{I}$ are given in (48). Denote by $X=L^{2}\left(\left[0, T_{0}\right], \mathcal{H}\right)$ and $\mathcal{X}=L^{2}\left(\left[0, T_{0}\right], \mathcal{H}\right) \otimes$
$L^{2}(\mu)$. Thus it holds

$$
\begin{align*}
\|y\|_{\mathcal{X}}^{2}= & \sum_{\alpha \in \mathcal{I}} \alpha!\left\|y_{\alpha}\right\|_{X}^{2}=\left\|y_{0}\right\|_{X}^{2}+\sum_{|\alpha| \geq 1} \alpha!\left\|y_{\alpha}\right\|_{X}^{2} \leq 2 \sum_{\alpha \in \mathcal{I}} \alpha!\left\|U_{\alpha}(t, 0) y_{\alpha}^{0}\right\|_{X}^{2} \\
& +2 \sum_{|\alpha| \geq 1} \alpha!\| \int_{0}^{t}\left(U_{\alpha}(t, s)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(s) \mathbf{e}_{i}(s)-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(s)\right) d s \|_{X}^{2}\right. \\
\leq & 2 M_{1}^{2}\left(T_{0}\right) \sum_{\alpha \in \mathcal{I}} \alpha!\left\|y_{\alpha}^{0}\right\|_{X}^{2} \\
& +8 M_{2}\left(T_{0}\right) d^{2} \sum_{|\alpha| \geq 1} \alpha!|\alpha|\left\|y_{\alpha}\right\|_{X}^{2}+4 M_{2}\left(T_{0}\right) b^{4} \sum_{|\alpha| \geq 1} \alpha!\left\|k_{\alpha}(s)\right\|_{X}^{2} \\
\leq & 2 M_{1}^{2}\left(T_{0}\right)\left\|y^{0}\right\|_{\mathcal{X}}^{2}+4 M_{2}\left(T_{0}\right) d^{2}\|y\|_{\operatorname{Dom}(\delta)}^{2}+4 M_{2}\left(T_{0}\right) b^{4}\|\mathcal{K}\|_{\mathcal{X}}^{2} \tag{61}
\end{align*}
$$

where $\|\mathcal{K}\|_{\mathcal{X}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|k_{\alpha}\right\|_{X}^{2} \alpha$ !. The coefficients $k_{\alpha}$ are the solutions of (48) and are expressed in terms of the adjoint evolution system $U_{\alpha}^{\star}(t, s), \alpha \in \mathcal{I}$. Clearly, the coefficients are of the form

$$
k_{\alpha}(t)=U_{\alpha}^{\star}(T, t) k_{\alpha}(T)+\int_{t}^{T} U_{\alpha}^{\star}(s, t) P_{d, \alpha}(s)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}} \mathbf{e}_{i}(s)\right) d s, t<T
$$

for $\alpha \in \mathcal{I}$. Denote by $\mathcal{X}_{1}=L^{2}\left(\left[T_{0}, T\right]\right)$ and $\left\|U_{\alpha}^{\star}(T, t)\right\| \leq e^{\tilde{\theta} t}=M_{3}(t)$, for $\tilde{\theta}>0$, $\alpha \in \mathcal{I}$ and $M_{4}(t)=\frac{1}{2 \tilde{\theta}}\left(e^{2 \tilde{\theta}(T-t)}-1\right)^{2}$. Since $k_{\alpha}(T)=0$ we obtain

$$
\begin{aligned}
\|\mathcal{K}\|_{\mathcal{X}_{1}}^{2} & =\sum_{\alpha \in \mathcal{I}} \alpha!\left\|\int_{t}^{T} U_{\alpha}^{\star}(s, t) P_{d, \alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}} \mathbf{e}_{i}(s)\right) d s\right\|_{X}^{2} \\
& \leq 2 M_{4}\left(T_{0}\right) p^{2} d^{2} \sum_{\alpha \in \mathcal{I}} \alpha!|\alpha|\left\|u_{\alpha}\right\|_{X}^{2} \leq M_{4}\left(T_{0}\right) p^{2} d^{2}\|y\|_{\operatorname{Dom}(\delta)}^{2}<\infty .
\end{aligned}
$$

Thus, $\|\mathcal{K}\|_{\mathcal{X}}^{2}<\infty$. With this bound we return to (61) and conclude that $\|y\|_{\mathcal{X}}^{2}<\infty$.
The interval $(0, T]$ can be covered by the intervals of the form $\left[k T_{0},(k+1) T_{0}\right]$ in finitely many steps. Thus, $y \in L^{2}([0, T], \mathcal{H}) \otimes L^{2}(\mu)$.

Theorem 11 is an extension of results from [27], where the case with simple coordinatewise operators was considered. The importance of the convergence result can be seen in the error analysis that arises in the actual truncation when implementing the algorithm numerically.

Remark 3. The previous results might be extended for optimal control problems with state equations of the form (3), in spaces of stochastic distributions. By replacing the uniform boundedness conditions on the operators $B_{\alpha}$ and $C_{\alpha}$, $\alpha \in \mathcal{I}$ in (A2) and (A3) with the polynomial growth conditions of the type $\sum_{\alpha \in \mathcal{I}}\left\|C_{\alpha}\right\|^{2}(2 \mathbb{N})^{-s \alpha}<\infty$, for some $s>0$ one can prove that for fixed admissible control, the state equation has a unique solution in the space $L^{2}([0, T], \mathcal{H}) \otimes(S)_{-1}$. A similar theorem to Theorem 11 for the optimal control can be proven. Moreover, the corresponding optimal control problem with fractional noise can be solved.

The following theorem gives the characterization of the optimal solution (58) in terms of the solution of the stochastic Riccati equation (6).

Theorem 12. Let the conditions (A1)-(A5) from Theorem 11 hold and let $\mathbf{P}$ be a coordinatewise operator that corresponds to the family of operators $\left\{P_{\alpha}\right\}_{\alpha \in \mathcal{I}}$. Then, the solution of the optimal control problem (4)-(5) obtained via chaos expansion (56) is equal to the one obtained via Riccati approach (57) if and only if

$$
\begin{equation*}
C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha} y_{\alpha}^{*}(t)=P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot \mathbf{e}_{i}(t)\right), \quad|\alpha|>0, k \in \mathbb{N} \tag{62}
\end{equation*}
$$

hold for all $t \in[0, T]$.
Proof. Let us assume first that (56) is equal to (57), then

$$
-\mathbf{B}^{\star} \mathbf{P} y^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}_{d} y^{*}(t)-\mathbf{B}^{\star} \mathcal{K}
$$

we obtain

$$
\left(\mathbf{P}(t)-\mathbf{P}_{d}\right) y^{*}(t)=\mathcal{K}
$$

The difference between $\mathbf{P}(t)$ and $\mathbf{P}_{d}(t)$ is expressed through the stochastic process $\mathcal{K}$, which comes from the influence of inhomogeneities. Assuming that $\mathbf{P}$ is a coordinatewise operator that corresponds to the family of operators $\{P\}_{\alpha \in \mathcal{I}}$, we will be able to see the action of stochastic operator $\mathbf{P}$ on the deterministic level, i.e., level of coefficients. Thus, for $y$ given in the chaos expansion form (49) and $\mathbf{P}(t) y^{*}=\sum_{\alpha \in \mathcal{I}} P_{\alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}$ it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left(P_{\alpha}(t)-P_{d, \alpha}(t)\right) y_{\alpha}^{*}(t) H_{\alpha}=\sum_{\alpha \in \mathcal{I},|\alpha|>0} k_{\alpha}(t) H_{\alpha} \tag{63}
\end{equation*}
$$

Since $k_{\mathbf{0}}(t)=0$ it follows $P_{\mathbf{0}}(t)=P_{d, \mathbf{0}}(t)$, for $t \in[0, T]$ and for $|\alpha|>0$

$$
\left(P_{\alpha}(t)-P_{d, \alpha}(t)\right) y_{\alpha}^{*}(t)=k_{\alpha}(t)
$$

such that (48) with the condition $k_{\alpha}(T)=0$ holds. We differentiate (63) and substitute (48), together with (6), (9) and (51). Thus, after all calculations we obtain for $|\alpha|=0$

$$
\left(P_{\mathbf{0}}(t)-P_{d, \mathbf{0}}(t)\right) y_{\mathbf{0}}^{*}(t)=0
$$

and for $|\alpha|>0$

$$
C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha} y_{\alpha}^{*}(t)=P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot \mathbf{e}_{i}(t)\right), \quad k \in \mathbb{N}
$$

Note that assuming (62) and $\mathbf{P}$ is a coordinatewise operator that corresponds to operators $P_{\alpha}, \alpha \in \mathcal{I}$ we can go backwards in the analysis and prove that the optimal controls (57) and (56) are the same.

Remark 4. The condition that characterizes the optimality (62) represents the action of the stochastic Riccati operator in each level of the noise. Note that the stochastic Riccati equation (6) and the deterministic one (9) differ only in the term $C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha}$, i.e., the operator $C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha}, \alpha \in \mathcal{I}$ captures the stochasticity
of the equation. Polynomial chaos projects the stochastic part in different levels of singularity, the way that Riccati operator acts in each level is given by (62).

Remark 5. Following our approach the numerical treatment of the SLQR problem relies on solving efficiently Riccati equations arising in the associated deterministic problems. In recent years, numerical methods for solving differential Riccati equations have been proposed, e.g., $[2,3,4,23]$,

### 3.3. Further extensions

We consider now more general form of the state equation

$$
\begin{equation*}
\dot{y}=\mathbf{A} y+\mathbf{B} u+\mathbf{T} \diamond y, \quad y(0)=y^{0} \tag{64}
\end{equation*}
$$

for bounded coordinatewise operators $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{T} \diamond$, where the operator $\mathbf{T} \diamond$ for $y=\sum_{\alpha \in \mathcal{I}} y_{\alpha} H_{\alpha}$ is defined by

$$
\begin{equation*}
\mathbf{T} \diamond(y)=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \leq \alpha} T_{\beta}\left(y_{\alpha-\beta}\right) H_{\alpha} \tag{65}
\end{equation*}
$$

For more details about $\mathbf{T} \diamond$ we refer to [34, 44]. We point out that in [34] the authors proved that (64), for fixed $u$, has a unique solution in space of stochastic generalized processes. Here, we will show that the optimal control problem (45)-(64) for a specific choice of the operator $\mathbf{T}$ can be reduced to the problem (45)-(46), and thus its optimal control can be obtained from Theorem 11. Moreover, one can also consider the corresponding fractional optimal control problem and thus apply Theorem 10 and Theorem 11. This extension is connected to the form of a Gaussian colored noise (20) with the condition (21). We denote $X=L^{2}([0, T], \mathcal{H})$.
Theorem 13. Let $L_{t}$ be of the form (20) such that (21) holds. Let $\mathbf{N}$ be a coordinatewise operator which corresponds to a family of uniformly bounded operators $\left\{N_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and let the operators $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ satisfy the assumptions (A1)-(A4) of Theorem 11. Let the operator $\mathbf{T}$ be a coordinatewise operator defined by a family of operators $\left\{T_{\alpha}\right\}_{\alpha \in \mathcal{I}}, T_{\alpha}: X \rightarrow X, \alpha \in \mathcal{I}$, such that for $|\beta| \leq|\alpha|$

$$
T_{\beta}\left(y_{\alpha-\beta}\right)=\left\{\begin{array}{ll}
N_{\alpha}\left(y_{\alpha}\right) & ,|\beta|=0  \tag{66}\\
l_{k} N_{\alpha-\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right) & ,|\beta|=1, \\
0 & ,|\beta|>1
\end{array} \quad \text { i.e., } \beta=\varepsilon^{(k)}, k \in \mathbb{N}\right.
$$

for $y_{\alpha} \in X, \alpha \in \mathcal{I}$. Then the state equation (64) can be reduced to the state equation (46). Thus, the optimal control problem (45)-(64) has a unique solution.

Proof. By the definition (65) and the chaos expansion method, the state equation (64) reduces to the system:
$1^{\circ}$ for $|\alpha|=0$

$$
\begin{equation*}
\dot{y}_{\mathbf{0}}=\left(A_{\mathbf{0}}+T_{\mathbf{0}}\right) y_{\mathbf{0}}+B_{\mathbf{0}} u_{\mathbf{0}}, \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \tag{67}
\end{equation*}
$$

$2^{\circ}$ for $|\alpha| \geq 1$

$$
\begin{equation*}
\dot{y}_{\alpha}=\left(A_{\alpha}+T_{\mathbf{0}}\right) y_{\alpha}+B_{\alpha} u_{\alpha}+\sum_{\mathbf{0}<\beta \leq \alpha} T_{\beta}\left(y_{\alpha-\beta}\right), \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{68}
\end{equation*}
$$

From (66) it follows

$$
T_{\mathbf{0}}\left(y_{\alpha}\right)=N_{\alpha}\left(y_{\alpha}\right), \alpha \in \mathcal{I} \quad \text { and } \quad T_{\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right)=l_{k} N_{\alpha-\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right)
$$

We define $\hat{A}_{\alpha}=A_{\alpha}+N_{\alpha}, \alpha \in \mathcal{I}$. Since the family $\left\{N_{\alpha}\right\}$ is uniformly bounded and $\left\{A_{\alpha}\right\}$ are infinitesimal generators $C_{0}$-semigroups then the operators $\hat{A}_{\alpha}$ are also infinitesimal generators of $C_{0}$-semigroups and satisfy the condition (A1) of Theorem 11, see [43]. Thus the system (67)-(68) transforms to:
$1^{\circ}$ for $|\alpha|=0$

$$
\dot{y}_{\mathbf{0}}=\hat{A}_{\mathbf{0}} y_{\mathbf{0}}+B_{\mathbf{0}} u_{\mathbf{0}}, \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}
$$

$2^{\circ}$ for $|\alpha| \geq 1$

$$
\dot{y}_{\alpha}=\hat{A}_{\alpha} y_{\alpha}+B_{\alpha} u_{\alpha}+\sum_{k \in \mathbb{N}} l_{k} N_{\alpha-\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right), \quad y_{\alpha}(0)=y_{\alpha}^{0}
$$

Define the operators $\hat{C}_{\mathbf{0}}=N_{\mathbf{0}}$ and $\hat{C}_{\alpha-\varepsilon^{(k)}}=l_{k} N_{\alpha-\varepsilon^{(k)}}$, for $|\alpha| \geq 1, k \in \mathbb{N}$. Therefore, the obtained system corresponds to the state equation of the form

$$
\begin{equation*}
\dot{y}=\hat{\mathbf{A}} y+\mathbf{B} u+\hat{\mathbf{C}} \diamond W_{t} \tag{69}
\end{equation*}
$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ are coordinatewise operators corresponding to the families $\left\{\hat{A}_{\alpha}\right\}$ and $\left\{\hat{C}_{\alpha}\right\}$, respectively. Moreover, the operators $\mathbf{B}$ and $\hat{\mathbf{C}}$ satisfy the assumptions (A2)-(A4) of Theorem 11. Therefore, it can be applied to the optimal control problem (45)-(69).

## 4. An example involving operators from Malliavin calculus

In this section we focus on semi-explicit ODAEs, i.e., systems of a linear semiexplicit equation subject to an algebraic constraint. These systems of equations are motivated by applications, e.g., Stokes equations, linearized Navier-Stokes equations, etc. They are in most cases deterministic and finite-dimensional. However, recently ODAEs with additive noise have been studied in [1]. Here, we consider an ODAE of the form

$$
\dot{y}=\mathbf{A} y+\mathbf{T} \diamond y+\mathbf{B}^{\star} u+f, \quad \mathbf{B} y=g
$$

where the operator $\mathbf{B}$ is the Ito $\hat{-}$-Skorohod integral $\delta$ and $\mathbf{B}^{\star}$ the Malliavin derivative $\mathbb{D}$. The operator $\delta$ is the adjoint operator of $\mathbb{D}$, i.e., the duality relationship

$$
\mathbb{E}(F \cdot \delta(y))=\mathbb{E}(\langle\mathbb{D} F, y\rangle)
$$

holds for stochastic processes $y$ and $F$ belonging to appropriate spaces [41]. Thus, we study the system

$$
\begin{equation*}
\dot{y}=\mathbf{A} y+\delta u+\mathbf{T} \diamond y+f, \quad \mathbb{D} y=g \tag{70}
\end{equation*}
$$

More details on properties of the generalized operators of the Malliavin calculus and the equations involving these operators can be found in $[29,31,34]$. Here we
assume that the space $X$ is the Hilbert space $L^{2}([0, T], \mathcal{H})$. Let $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}$, $u_{\alpha} \in X, \alpha \in \mathcal{I}$ and $F=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \xi_{k} H_{\alpha}, f_{\alpha, k} \in X, \alpha \in \mathcal{I}, k \in \mathbb{N}$ and $\left\{\xi_{k}\right\}_{k \in \mathbb{N}}$ are the Hermite functions. The Malliavin derivative operator $\mathbb{D}$ represents a stochastic gradient in the direction of white noise and is a linear and continuous mapping $\mathbb{D}: X \otimes(S)_{-1} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ given by

$$
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \xi_{k} H_{\alpha-\varepsilon_{k}}
$$

A process $u$ belongs to the domain $\operatorname{Dom}(\mathbb{D})$ if and only if for some $p \in \mathbb{N}_{0}$ it holds

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

The Ito ${ }^{-}$-Skorokhod integral $\delta$ is a linear and continuous mapping $\delta: X \otimes S^{\prime}(\mathbb{R}) \otimes$ $(S)_{-1} \rightarrow X \otimes(S)_{-1}$ and is defined by $\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} H_{\alpha+\varepsilon_{k}}$. Note that the domain $\operatorname{Dom}(\delta)=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. In quantum theory $\mathbb{D}$ corresponds to the annihilation operator and $\delta$ to the creation operator.

We reduce the system (70) to the following two problems: $\mathbb{D} y=g, \mathbb{E} y=y^{0}$ and $\delta(u)=v$ and then apply the results from [29] and [31].
Theorem 14. Let A : $X \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ be a coordinatewise operator corresponding to a uniformly bounded family of deterministic operators $A_{\alpha}: X \rightarrow$ $X, \alpha \in \mathcal{I}$ and $\mathbf{T}$ be a coordinatewise operator that corresponds to a polynomially bounded family of operators $T_{\alpha}: X \rightarrow X, \alpha \in \mathcal{I}$. Let $g=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{\alpha, k} \xi_{k} H_{\alpha} \in$ $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ and $f \in X \otimes(S)_{-1}$, such that $\mathbb{E} f=A_{\mathbf{0}} y^{0}+T_{\mathbf{0}} y^{0}$. Then there exists a unique solution $y \in X \otimes(S)_{-1}$ and $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ of the system (70) with the initial conditions $\mathbb{E} y=y^{0} \in X$ and $\mathbb{E} \dot{y}=y^{1} \in X$ given by

$$
\begin{equation*}
y=y^{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} g_{\alpha-\varepsilon^{(k)}, k} H_{\alpha} \tag{71}
\end{equation*}
$$

and

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{v_{\alpha+\varepsilon^{(k)}}^{\left|\alpha+\varepsilon^{(k)}\right|} \xi_{k} H_{\alpha}, \text {, }, \text {. }}{} \tag{72}
\end{equation*}
$$

where $v=\dot{y}-\mathbf{A} y-\mathbf{T} \diamond y-f$.
Proof. The initial value problem involving the Malliavin derivative operator

$$
\begin{equation*}
\mathbb{D} y=g, \quad \mathbb{E} y=y^{0} \tag{73}
\end{equation*}
$$

can be solved by applying the integral operator on both sides of the equation. Given a process $g \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-1,-q}, p \in \mathbb{N}_{0}, q>p+1$, represented in its chaos expansion form $g=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} g_{\alpha, k} \xi_{k} H_{\alpha}$, the equation (73) has a unique solution in $\operatorname{Dom}(\mathbb{D})$ represented by (71). Additionally, it holds

$$
\|y\|_{X \otimes(S)_{-1,-q}}^{2} \leq\left\|u^{0}\right\|_{X}^{2}+c\|g\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-q}}^{2}<\infty
$$

The operator $\mathbf{A}$ is a coordinatewise operator and it corresponds to an uniformly bounded family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, i.e., it holds $\left\|A_{\alpha}\right\| \leq M, \alpha \in \mathcal{I}$. For
$y \in X \otimes(S)_{-1} \bigcap \operatorname{Dom}(\mathbb{D})$ it holds

$$
\|\mathbf{A} y\|_{X \otimes(S)_{-1,-q}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|A_{\alpha} y_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \leq M\|y\|_{X \otimes(S)_{-1,-q}}^{2}<\infty
$$

and thus $\mathbf{A} y \in X \otimes(S)_{-1,-q}$. The operators $\left\{T_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ are polynomially bounded and it holds $\mathbf{T} \diamond: X \otimes(S)_{-1,-q} \rightarrow X \otimes(S)_{-1,-q}$. Since $g_{\alpha} \in X \otimes S_{-l}(\mathbb{R})$ we can use the formula for derivatives of the Hermite functions [21]. Thus,

$$
\dot{g}_{\alpha}=\sum_{k \in \mathbb{N}} g_{\alpha, k} \otimes \frac{d}{d t} \xi_{k}=\sum_{k \in \mathbb{N}} g_{\alpha, k} \otimes\left(\sqrt{\frac{k}{2}} \xi_{k-1}-\sqrt{\frac{k+1}{2}} \xi_{k+1}\right)
$$

and $\dot{g}_{\alpha} \in X \otimes S_{-l-1}(\mathbb{R})$. We note that the problem $\mathbb{D} \dot{u}=\dot{y}$ with the initial condition $\mathbb{E} \dot{y}=y^{1} \in X$ can be solved as (73). Moreover,

$$
\|\dot{y}\|_{X \otimes(S)_{-1,-q}}^{2} \leq\left\|y^{1}\right\|_{X}^{2}+c\|\dot{g}\|_{X \otimes S_{-l-1}(\mathbb{R}) \otimes(S)_{-1,-q}}^{2}<\infty .
$$

Let $f \in X \otimes(S)_{-1,-q}$ and denote by $v=\dot{y}-\mathbf{A} y-\mathbf{T} \diamond y-f$. From the given assumptions it follows $v \in X \otimes(S)_{-1,-q}$ such that $\mathbb{E} v=0$. Then, $v$ can be represented in the form $v=\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} v_{\alpha} H_{\alpha}$ and the integral equation

$$
\delta(u)=v
$$

has a unique solution $u$ in $X \otimes S_{-l-1}(\mathbb{R}) \otimes(S)_{-1,-q}$, for $l>q$, given in the form (72), see [31, 35]. Moreover, the estimate

$$
\|u\|_{X \otimes(S)_{-1,-q}}^{2} \leq c\left(\|y\|_{X \otimes(S)_{-1,-q}}^{2}+\|f\|_{X \otimes(S)_{-1,-q}}^{2}+\|\dot{y}\|_{X \otimes(S)_{-1,-q}}^{2}\right)
$$

also holds.

## Acknowledgement

The authors would like to thank the referees for their valuable comments. They greatly helped to improve this manuscript. This paper was partially supported by the project Solution of large-scale Lyapunov Differential Equations (P 27926) founded by the Austrian Science Foundation FWF.

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# THE STOCHASTIC LINEAR QUADRATIC CONTROL PROBLEM WITH SINGULAR ESTIMATES* 

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#### Abstract

We study an infinite dimensional finite horizon stochastic linear quadratic control problem in an abstract setting. We assume that the dynamics of the problem are generated by a strongly continuous semigroup, while the control operator is unbounded and the multiplicative noise operators for the state and the control are bounded. We prove an optimal feedback synthesis along with well posedness of the Riccati equation for the finite horizon case. Our results extend the ones proposed in [C. Hafizoglu, Ph.D. Thesis, University of Virginia, Charlottesville, VA, 2006.] to the case in which disturbance in the control is considered and a final time penalization term is included in the quadratic cost functional.


Key words. stochastic linear quadratic control, singular estimate control systems, control of stochastic PDEs

AMS subject classifications. 49N10, 93E20, 93C20, 35M10, 49K20
DOI. 10.1137/16M1056183

1. Introduction. We consider the stochastic linear quadratic problem in infinite dimensions with state and control dependent noise for the so-called singular estimate control systems. These systems involve dynamics driven by $C_{0}$-semigroups and unbounded control actions, with the control to state kernel satisfying a singular estimate. Such a situation is typical in boundary or point control problems where the action of the control operator $B$ is either only densely defined on a control space or its range is outside the state space. In order to quantify the "unboundedness" of control, actionsingular estimates play a pivotal role. Such estimate describes the amount of blowup of the "transfer function." The latter is necessary for a rigorous analysis of control problems and the associated feedback synthesis-be it deterministic or stochastic.

For deterministic systems, the infinite dimensional linear quadratic regulator problem has been studied extensively in the literature [B1, BK, BDDM, LT2]. The purpose of the theoretical framework is to address optimal control of systems of PDEs. For most systems, the controlling mechanism can only be applied from the interface of the system or at finitely many points or curves [BSW] which necessitates developing a framework for studying boundary/point control. Such control actions can be captured mathematically using maps which are not bounded with respect to the state space, but take values in a larger dual space. The most natural class of problems where such

[^7]description has been used are dynamics driven by analytic semigroups. The analyticity property quantifies naturally the blowup of the transfer function when acted upon by an unbounded operator (compatible with fractional powers of the generator). The linear quadratic problem for systems driven by analytic semigroups with these type of control actions was studied by [F2, AT, DI, BDDM, LT2]. The situation is much more complicated in the non-analytic case, where there is no natural characterization of singularity other than technical - often brute force - PDE estimates. However, for some classes of control systems which combine hyperbolic and parabolic dynamics, it has been observed that the control to state kernel satisfies a singular estimate which generalizes the case of analytic semigroup dynamics [AL, ABL, L1, LT1, LTu1]. Examples of systems which manifest this type of singular estimate arise frequently in thermoelastic plate models [BLT, BL, LTu2], acoustic-structure interaction equation [AL, BSS, LTu2], and fluid-structure interaction models [LTu3]. In view of the above, a deterministic theory of feedback control has been developed for these classes of problems (singular estimate); see the references given in [L2]. However, in the stochastic case the only results available in the literature covering unbounded control actions are the ones dealing with analytic semigroups [D, GT1, F1]. The main goal of the present work is to develop a stochastic treatment of unbounded control action problems arising in a general class of dynamical systems which exhibit singular estimates, but are not necessarily analytic. One of the main challenges is to develop an approximation framework which would provide rigorous justification of stochastic estimates. In the analytic case, such a framework is very natural and based on the instant regularizing effect of the dynamics. In the nonanalytic case, a development of regularizing procedures lies at the heart of the problem. This will be accomplished by expanding and building on the results presented in $[\mathrm{H}]$.

The stochastic linear quadratic regulator problem in finite dimensions has been first studied by Kushner (1962) [K] using dynamic programming. The feedback characterization of the optimal control and the derivation of a matrix Riccati equation satisfied by the gain matrix is due to Wonham (1968) [W1, W2]. A complete theory for the stochastic linear quadratic optimal control problem in finite dimensions can be found in [YZ, DMS, FS]. It is notable that the associated Riccati differential equation in the stochastic linear quadratic problem is a deterministic differential equation, and thus the relation between the optimal control and the optimal state which are random variables is purely deterministic. The linear quadratic problem with random coefficients in finite dimension has also been investigated in [CLZ]. In this case, the associated Riccati equation is a backward stochastic equation.

Several early works in the literature have addressed stochastic optimization in infinite dimensions and the application of a semigroup framework to the stochastic setting with bounded inputs [B2, B3, C1, FG, Te1, Te2]. The infinite dimensional ana$\log$ for the stochastic linear quadratic problem and the Riccati equation was treated by Ichikawa [I] via a dynamic programming approach, where he considered dynamics driven by $C_{0}$ semigroups and bounded control and noise operators. In another early work, Curtain [C2] provides a semigroup framework for studying the infinite dimensional linear quadratic Gaussian along with several examples and applications. A complete Riccati feedback synthesis of the infinite dimensional problem with disturbance in the state has been addressed by Da Prato [D] for systems with analytic dynamics and a particular unbounded noise operator which captures the first derivative of the state in a parabolic equation. The analysis was extended to boundary controls by Flandoli [F1] and in particular for analytic systems with Neumann-type controls. In [GRS], the authors consider a more general cost functional and a semilinear state
equation driven by analytic dynamics, and proceed to solve the problem using a Hamilton-Jacobi-Bellman approach. For systems with singular estimates, which is our primary consideration, the stochastic linear quadratic problem has been studied by one of the authors in $[\mathrm{H}]$, but with no disturbance in the control $(D=0)$ and without finite time penalization in the cost functional $(G=0)$. In $[\mathrm{U}]$, the time varying problem has been also addressed for systems driven by strongly continuous evolutions with bounded control and noise operators. In [DMP, D2], the author investigates stochastic linear quadratic differential games involving a stochastic differential equation with fractional Brownian motion with dynamics generated by analytic semigroups. Some recent interesting work has also treated the linear quadratic problem with random coefficients along with the associated backward stochastic Riccati equation [GT1, GT2]. Some recent works have also addressed the question of numerical implementation and finite dimensional approximation schemes of the infinite dimensional stochastic linear quadratic regulator [LM, LMT2, DMSt].

In view of the above the main novel contributions distinguishing this work from other publications are (1) this is the first treatment of stochastic unbounded control systems in the nonanalytic setting and (2) the framework allows for consideration of terminal penalization as well as control action perturbed by noise. Indeed, in the present paper, we consider a more general setting including disturbance in the control, and we also consider the case of the Bolza problem which allows for a finite time penalization in the objective functional whose expected value is to be minimized. This latter aspect of the Bolza-Meyer problem is particularly challenging in the unbounded control case. As shown [F1], the solution to the optimal control problem may not exist, unless a certain closeability hypothesis is introduced. Under such a necessary hypothesis, we provide an optimal feedback synthesis and a Riccati equation for the stochastic linear quadratic optimal control in the context of singular estimate control systems with noise dependence in both state and control.

In the deterministic setting, variational analysis is used to obtain explicit formulas for the optimal control before proceeding to derive the associated Riccati equations [LT1, D]. However, such explicit formulas are not available in the stochastic settingthus preventing applicability of a method of pivotal importance in the deterministic and singular case. Moreover, in our setting, the lack of smoothing does not allow for the application of the stochastic maximum principle or a solution via the Hamilton-Jacobi-Bellman equation unlike the case of analytic dynamics [GRS]. In particular, the state trajectories are mild solutions of the state equations and not necessarily differentiable in the classical sense.

Therefore, in our approach, we derive a differential Riccati equation associated with the optimal stochastic linear quadratic control problem, by first showing the existence of a solution to an expanded system in the integral form of the Riccati equation via a specially crafted fixed point argument. Here we generalize the arguments given in $[\mathrm{H}]$. We then proceed to derive the differential Riccati equation which requires making sense of the weak derivative of the evolution generated by deterministic dynamics with respect to initial time. Here, the obstacle, as in the deterministic case, lies in the fact that the terms of the Riccati equation may not be well defined due to the unboundedness of the control operator. There have been counterexamples in the literature where the Riccati equation is not well posed in the case of unbounded control operators [BLT]. Another difficulty is the finite state penalization which gives rise to possible singularities at the final time and require choosing appropriate spaces to make sense of the quadratic term in the differential Riccati equation [LTu1]. Finally, we then use a dynamic programming argument to show that the minimum of the
quadratic functional is realized when the control is expressed in feedback form via the solution to the differential Riccati equation. Here, we proceed with the dynamic programming argument on a regularized version of the problem since the Itô formula only applies to $C^{2}$ functions, while the state and control trajectories are not differentiable in the classical sense. For this reason, a forward approach via a maximum principle or a variational method to solve for the optimal control before proceeding to derive the differential Riccati equation is not applicable in this setting.

We first formulate the optimal control problem. Let the abstract stochastic differential equation

$$
\begin{align*}
d y(t) & =(A y+B u) d t+(C y+D u) d W_{t},  \tag{1.1}\\
y(s) & =x,
\end{align*}
$$

be defined on a Hilbert state space $H$, where $A$ and $C$ are operators on $H$ while $B$ and $D$ are operators acting from the control Hilbert space $U$ to the state space $H$. We take $C$ and $D$ to be bounded operators but $A$ and $B$ are typically unbounded.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $W_{t}$ a one dimensional real valued stochastic Brownian motion on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_{t}$ the sigma algebra generated by $\left\{W_{\tau}: \tau \leq t\right\}$. We assume that all function spaces are adapted to the filtration $\mathcal{F}_{t}$. We denote by $L_{w}^{2}([s, T] ; H)$ all stochastic processes $X(t, \omega):[s, T] \times \Omega \rightarrow H$ such that 1. $\int_{s}^{T}\|X(t)\|_{H}^{2} d t<\infty$ a.e. in $\Omega$;
2. $X(t, \cdot)$ is $\mathcal{F}_{t}$-measurable $\forall t \in[s, T]$.

We also denote by $M_{w}^{2}([s, T] ; H)$ the space of all strongly measurable square integrable stochastic processes $X:[s, T] \times \Omega \rightarrow H$ such that $\int_{s}^{T} \mathbb{E}\left(\|X(t)\|_{H}^{2}\right) d t<$ $\infty$, and by $L^{2}\left(\Omega ; H^{1}([s, T] ; U)\right)$ all strongly measurable square integrable stochastic processes $u:[s, T] \times \Omega \rightarrow U$ such that $\int_{s}^{T} \mathbb{E}\left(\|u(t)\|_{U}^{2}\right) d t+\int_{s}^{T} \mathbb{E}\left(\left\|u_{t}(t)\right\|_{U}^{2}\right) d t<\infty$. The objective is to minimize the quadratic cost functional

$$
\begin{equation*}
J(s, x, u)=\mathbb{E}\left(\int_{s}^{T}\left(\|R y\|_{W}^{2}+\|u\|_{U}^{2}\right) d t+\|G y(T)\|_{Z}^{2}\right) \tag{1.2}
\end{equation*}
$$

over all $u \in M_{w}^{2}([s, T] ; U)$, where $R$ and $G$ are bounded linear observation operators taking values in Hilbert spaces $W$ and $Z$, respectively. The assumptions we consider are the following.

Assumption 1.1.

1. Operator $A$ is linear and generates a $C_{0}$-semigroup $e^{A t}$ on $H$.
2. The linear operator $B$ acts from $U \rightarrow\left[\mathcal{D}\left(A^{\star}\right)\right]^{\prime}$ or, equivalently, $A^{-1} B$ is bounded from $U \rightarrow H$.
3. The noise operator $D: U \rightarrow H$ is a bounded linear operator.
4. There exists a number $\gamma \in(0,1 / 2)$ such that the control to state map kernel $e^{A t} B$ satisfies the singular estimate

$$
\begin{equation*}
\left\|e^{A t} B u\right\|_{H} \leq \frac{c}{t^{\gamma}}\|u\|_{U} \tag{1.3}
\end{equation*}
$$

for every $u \in U$ and $0<t<1$.
5. The operators $R: H \rightarrow W, G: H \rightarrow Z$, and $C: H \rightarrow H$ are all bounded linear operators.
Remark 1.2. Our framework also allows for $H$-valued Brownian motion $W_{\tau}$, where $(C y+D u) d W_{\tau}$ is interpreted as a Wick product $(C y+D u) \diamond d W_{\tau}$ of generalized random variables on Gaussian white noise probability spaces [HO]. See [LMT1] for chaos
expansion treatment of the abstract stochastic differential equation and the linear quadratic control problem in Hilbert spaces.

Remark 1.3. The singular estimate (1.3) should be interpreted in the following precise sense:

$$
\left|\left\langle e^{A t} A^{-1} B u, A^{*} \phi\right\rangle\right| \leq \frac{c_{T}}{t^{\gamma}}\|u\|_{U}\|\phi\|_{H} \quad \text { for all } \phi \in H
$$

Remark 1.4. The results can also be extended to the case when $D$ is an unbounded operator satisfying a similar singular estimate condition to that satisfied by $B$ in Assumption 1.1(4). This condition allows the inclusion of systems with noise in the boundary control into the theoretical framework developed below, as illustrated by the example included in the last section. However, to spare the reader further technical details, we will just assume $D$ is bounded throughout the paper.

Remark 1.5. In the case when there is no final state penalization, i.e., $(\mathrm{G}=0)$, the value of $\gamma$ in (1.3) could be pushed up to 1 -as in the deterministic case [LTu1]. However, the majority of "nonanalytic" examples exhibit singularity of the type assumed in (1.3). For this reason, we focus on this class only.

In sections 2 and 3, we state our main results and provide some preliminary results on mild solutions to the stochastic abstract differential equation (1.1). In section 4, we prove the existence of a local-in-time solution to the integral Riccati equation via a fixed point argument and we investigate the regularity properties of the Riccati operator. In section 5, we derive the differential Riccati equation from the integral form. In section 6, we show the relation between the solution to the Riccati equation and the optimal control or minimizer of the cost functional (1.2) via dynamic programming, and then extend the result globally in time and show uniqueness of the solution to the Riccati equation in sections 7 and 8 , respectively. We then return to complete the proof of the main results Theorems 2.1 and 2.2 in section 9 . We conclude the paper in section 10 with two examples to illustrate the theory: (1) a hinged thermoelastic plate model with noise and control through Neumann boundary conditions and (2) a linearized fluid-structure interaction model with boundary control which we briefly discuss in the next section.
1.1. Motivating example-fluid-structure interaction. In order to draw the attention of the reader to the significance of the assumptions imposed above on the control problem we provide an example of a fluid-structure interaction control problem with noise which became a motivation for our abstract framework [LTu3]. In the domain $\Omega$, we consider a partition into an interior region $\Omega_{s}$ and an exterior region $\Omega_{f}$, where $\Omega_{f}$ is occupied by a fluid while $\Omega_{s}$ is occupied by a solid body. The interaction between the solid and the fluid takes place on the boundary $\Gamma_{s}$ which separates both regions. The dynamics of the fluid are captured by a linear Stokes equation with multiplicative noise satisfied by fluid velocity $u$ and fluid pressure $p$ :

$$
\begin{array}{rll}
d u-\Delta u d t+\nabla p d t=c_{1} u d W_{t} & \text { in } & \Omega_{f} \times[0, T] \\
\operatorname{div} u=0 & \text { in } & \Omega_{f} \times[0, T] \tag{1.5}
\end{array}
$$

The dynamics of the solid are modeled by a linear second order equation with multiplicative noise

$$
\begin{equation*}
d w_{t}-\operatorname{div} \sigma(w) d t=c_{2} w d W_{t} \quad \text { in } \quad \Omega_{s} \times[0, T] \tag{1.6}
\end{equation*}
$$

in the solid displacement variable $w$, where $\sigma$ is the stress tensor defined by

$$
\sigma_{i j}(w)=\lambda \delta_{i j} \operatorname{div} w+2 \mu \epsilon_{i j}(w)
$$

for $i, j=1,2,3$ and constants $\lambda, \mu>0$, and where $\epsilon$ is the strain tensor defined by

$$
\epsilon_{i j}(w)=\frac{1}{2}\left(\frac{\partial w_{i}}{\partial x_{j}}+\frac{\partial w_{j}}{\partial x_{i}}\right) .
$$

Here, $W_{t}$ is a real Brownian motion on a complete probability space $(\Sigma, \mathcal{F}, P)$.
The interaction between the two bodies at the common interface $\Gamma_{s}$ is captured by the following transmission boundary conditions matching velocities and stresses:

$$
\begin{array}{rll}
u=w_{t} & \text { on } & \Gamma_{s} \times[0, T],  \tag{1.7}\\
\epsilon(u) \nu-p \nu=\sigma(w) \nu+g+g \dot{W}(t) & \text { on } & \Gamma_{s} \times[0, T],
\end{array}
$$

where $\nu$ is the outward unit normal and $g$ is a control function acting as a force. On the outer part of the boundary $\Gamma_{f}$, we prescribe the no slip boundary condition

$$
\begin{equation*}
u=0 \quad \text { on } \quad \Gamma_{f} \times[0, T] . \tag{1.9}
\end{equation*}
$$

When one is given initial conditions in the finite energy space $u_{0} \in H \equiv$ $\left\{L^{2}(\Omega): \operatorname{div} u=0,\left.u \cdot \nu\right|_{\Gamma_{f}}=0\right\}$ and $\left(w_{0}, w_{1}\right) \in H^{1}\left(\Omega_{s}\right) \times L^{2}\left(\Omega_{s}\right)$, the problem is to find a control $g \in L^{2}\left(\Sigma ; L^{2}\left([0, T] ; L^{2}\left(\Gamma_{s}\right)\right)\right)$ to minimize the energy functional

$$
\begin{align*}
J\left(u, w, w_{t}, g\right)=\mathbb{E}( & \int_{0}^{T}\left(\left\|u(t)-u_{T}(t)\right\|_{L_{2}\left(\Omega_{f}\right)}^{2}+\|g(t)\|_{L^{2}\left(\Gamma_{s}\right)}^{2}\right) d t  \tag{1.10}\\
& \left.+\left\|u(T)-u_{D}\right\|_{L_{2}\left(\Omega_{f}\right)}^{2}+\left\|w(T)-w_{D}\right\|_{L_{2}\left(\Omega_{s}\right)}^{2}\right)
\end{align*}
$$

where $u_{D} \in L_{2}\left(\Omega_{f}\right), w_{D} \in L_{2}\left(\Omega_{s}\right), u_{T} \in L_{2}\left(\Omega_{f} \times[0, T]\right)$ are given tracking targets.
2. Main results. We first state the result pertaining to existence, regularity, and uniqueness of the solution to the optimal control problem.

Theorem 2.1. Under Assumption 1.1, there exists a positive self-adjoint operator $P(t) \in C([0, T] ; \mathcal{L}(H))$ satisfying the Riccati equation

$$
\begin{align*}
& \langle\dot{P} x, y\rangle+\langle P A x, y\rangle+\left\langle A^{\star} P x, y\right\rangle+\left\langle C^{\star} P C x, y\right\rangle+\left\langle R^{\star} R x, y\right\rangle \\
& \quad-\left\langle\left(B^{\star} P+D^{\star} P C\right)^{\star}\left(I+D^{\star} P D\right)^{-1}\left(B^{\star} P+D^{\star} P C\right) x, y\right\rangle=0  \tag{2.1}\\
& I+D^{\star} P(t) D>0  \tag{2.2}\\
& P(T) x=G^{\star} G x \tag{2.3}
\end{align*}
$$

for every $x, y \in \mathcal{D}(A)$. Moreover, the following holds:
(i) The minimum of the functional (1.2) is given by

$$
\inf _{u \in M_{w}([s, T] ; U)} J(s, x, u)=\langle P(s) x, x\rangle .
$$

(ii) The solution $P(t)$ is unique in the class of positive self-adjoint operators in $C([0, T] ; \mathcal{L}(H))$.
(iii) The solution $P(t)$ satisfies the estimate

$$
\|P(t) y\|_{H} \leq c\|y\|_{H} \quad \forall t \in[0, T), y \in H
$$

(iv) The operator $B^{\star} P(t)$ satisfies the estimate

$$
\left\|B^{\star} P y\right\|_{H} \leq \frac{c}{(T-t)^{\gamma}}\|y\|_{H} \quad \forall t \in[0, T), y \in H
$$

We next state the result on the feedback form of the optimal control and the associated differential Riccati equation satisfied by the gain operator.

Theorem 2.2. Under Assumption 1.1, the optimal control problem of minimizing (1.2) subject to the differential equation (1.1) with initial condition $x \in H$ has a unique solution $u^{0}(s, \cdot ; x) \in L^{2}(\Omega ; C([s, T) ; U))$ and a corresponding optimal state $y^{0}(s, \cdot ; x) \in L^{2}(\Omega ; C([s, T] ; H))$. Moreover,
(i) the optimal control $u^{0}$ satisfies the estimate

$$
\mathbb{E}\left(\left\|u^{0}(s, t ; x)\right\|_{U}^{2}\right) \leq \frac{c}{(T-t)^{2 \gamma}}\|x\|_{H}^{2} \quad \forall t \in[s, T)
$$

(ii) the optimal control $y^{0}$ satisfies the estimate

$$
\mathbb{E}\left(\left\|y^{0}(s, t ; x)\right\|_{H}^{2}\right) \leq c\|x\|_{H}^{2} \quad \forall t \in[s, T] ;
$$

(iii) the optimal control $u^{0}$ has the feedback characterization in terms of the optimal state

$$
u^{0}(t, s ; x)=-\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) y^{0}(t)
$$

where $P(t)$ is the unique solution to the differential Riccati equation (2.1)-(2.3).

Specific examples motivating the theory presented above include coupled PDE systems with boundary or point control where hyperbolic and parabolic dynamics are interwined. These, in particular include thermoelasticity, fluid-structure interactions, and models arising in structural acoustics [L2, AL].

Remark 2.3. The analysis and result above easily extends to the case $1 / 2 \leq \gamma<1$ when $G=0$. However, for nonzero $G$, this case $1 / 2 \leq \gamma<1$ is more challenging since the operator

$$
G L_{T} \equiv G \int_{0}^{T} e^{A(T-\tau)} B d \tau
$$

is no longer bounded $C\left(L^{2}(\Omega) ; L^{2}([s, T] ; U)\right) \rightarrow Z$. In fact, the existence of an optimal control in this case requires closability of $G L_{T}$ [LT1]. Such a condition is trivially satisfied when $G$ is bounded invertible $H \rightarrow Z$.
3. Preliminaries. Following [DZ1], we say $y(t, s ; x)$ is a mild solution of the stochastic differential equation (1.1) if

1. $y(t, s ; x)=e^{A(t-s)} x+\int_{s}^{t} e^{A(t-\tau)} B u(\tau) d \tau+\int_{s}^{t} e^{A(t-\tau)} C y(\tau) d W_{\tau}+$ $\int_{s}^{t} e^{A(t-\tau)} D u(\tau) d W_{\tau} ;$
2. $y(t, s ; x)$ takes values in $D(C)$;
3. $y(t, s ; x)$ satisfies

$$
P\left(\int_{s}^{T}\|y(\tau)\|_{H} d \tau<\infty\right)=1
$$

and

$$
P\left(\int_{s}^{T}\|C y(\tau)\|_{H}^{2} d \tau<\infty\right)=1
$$

4. $B u$ and $D u$ are $\mathcal{F}_{t}$ measurable Bochner integrable $H$ valued functions.

Results on the existence of mild solutions to (1.1) for a general forcing can be found in [DZ1, HO ]. By strong continuity of the semigroup, we know there exists numbers $\alpha, M>0$ such that $\left\|e^{A t} z\right\|_{H} \leq M e^{\alpha t}\|z\|_{H}$ for all $z \in H$ and $t \in[s, T]$. We start with the existence of a mild solution to (1.1), for which the proof is a standard argument, $[\mathrm{H}]$.

Theorem 3.1. Let $\gamma<1$. Given a function $u \in M_{w}^{2}([s, T] ; U)$ and an initial condition $y(s)=x \in H$, there exists a unique mild solution $y \in M_{w}^{2}([s, T] ; H)$ to the abstract differential equation (1.1). Moreover, if $\gamma<1 / 2$ then $y \in L^{2}(\Omega ; C([s, T] ; H))$.
4. Integral Riccati equation. In this section, we establish the existence of a solution to an integral form of the Riccati equation. The Riccati equation is, by itself, deterministic. However, its form is generated by the underlying stochastic process. This results in several additional terms (with respect to deterministic processes) which require subtle treatment. In fact, the relevant integral form of the differential Riccati equation is

$$
\begin{aligned}
P(t)= & \int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R \Phi(\tau, t) d \tau+\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P(\tau) C \Phi(\tau, t) d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P^{\star}(\tau) D\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) \Phi(\tau, t) d \tau \\
4.1) & +e^{A^{\star}(T-t)} G^{\star} G \Phi(T, t),
\end{aligned}
$$

subject to the condition

$$
\left\langle\left(I+D^{\star} P(t) D\right) x, x\right\rangle>0 \quad \forall x \neq 0 \text { and } x \in U,
$$

where $\Phi(t, s)$ is the solution to the equation

$$
\begin{equation*}
\Phi(t, s) x=e^{A(t-s)} x-\int_{s}^{t} e^{A(t-\tau)} B\left(I+D^{\star} P^{\star}(\tau) D\right)^{-1}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) \Phi(\tau, s) x d \tau \tag{4.2}
\end{equation*}
$$

Our main result in this section is the existence of local-in-time solutions to the above integral equations.

ThEOREM 4.1. The integral equations (4.1) and (4.2) have unique local-in-time solutions $P(t) \in C([s, T] ; H)$ and $\Phi(\cdot, s) \in C([s, T] ; H)$ for $s=T_{\max }<T$ chosen such that $T-T_{\max }$ is sufficiently small. Moreover, the solution $P(t)$ is a positive self-adjoint operator on the space $H$ and satisfies the estimate

$$
\begin{equation*}
\left\|B^{\star} P(t) x\right\|_{H} \leq \frac{c}{(T-t)^{\gamma}}\|x\|_{H} \quad \forall x \in H, t \in[s, T) \tag{4.3}
\end{equation*}
$$

The solutions will be extended to a global solution on the whole interval $[s, T]$ in section 7. One notices that the integral equation (4.1) depends on composition operators $B^{*} P$ and $P B$ which a priori are not defined at all. It is not even clear that $B^{*} P$ can be densely defined (due to the unboundedness of $B$ ). However, the validity
of the singular estimate will enable a rigorous analysis of this equation. We also notice that in the deterministic case one will only have the first and the last term in (4.1). Instead, in the present stochastic case the appearance of the third term provides quadratic dependence on the composition $P B$ and $P$. Classical deterministic methods (either variational or direct) are no longer applicable. In order to tackle the problem of existence, we shall formulate a rather special iteration scheme which enables us to "unscramble" the convoluted dependence on the troublesome operator $B^{*} P$ which a priori has no reason to be even densely defined. After a few preliminaries in section 4.1, the proof will proceed in steps.

Step 1: In section 4.2, we first prove existence of a solution $(\widetilde{P}, \hat{\Phi})$ to the linear integral equation

$$
\begin{aligned}
\widetilde{P}(t)= & \int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R \hat{\Phi}(\tau, t) d \tau+\int_{t}^{T} e^{A^{\star}(\tau-t)} Q^{\star}(\tau) Q(\tau) \hat{\Phi}(\tau, t) d \tau \\
& +\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{C}^{\star}(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{\psi}^{\star}(\tau) B^{\star} \widetilde{P}(\tau) \hat{\Phi}(\tau, t) d \tau+e^{A^{\star}(T-t)} G^{\star} G \hat{\Phi}(T, t) \\
\hat{\Phi}(t, s) x= & e^{A(t-s)} x-\int_{s}^{t} e^{A(t-z)} B \hat{\psi}(z) \hat{\Phi}(z, s) x d z
\end{aligned}
$$

where $Q(t), \hat{C}(t)$, and $\hat{\psi}(t)$ are given bounded operators satisfying the singular estimate (4.6).

Remark 4.2. Note these integral equations formally correspond to the system of linear equations

$$
\begin{aligned}
\frac{d}{d t} \widetilde{P}(t) & =-R^{\star} R-Q^{\star}(t) Q(t)-A^{\star} \widetilde{P}(t)-\widetilde{P}(t) A-\hat{C}^{\star}(t) \widetilde{P}(t) \hat{C}(t)+\hat{\psi}^{\star}(t) B^{\star} \widetilde{P}(t) \\
\frac{d}{d t} \hat{\Phi}(t, s) & =(A-B \hat{\psi}(t)) \hat{\Phi}(t, s) \\
\widetilde{P}(T) & =G^{\star} G, \quad \hat{\Phi}(s, s)=I
\end{aligned}
$$

Step 2: In section 4.3, we next show that the solution $\widetilde{P}$ is a positive self-adjoint operator in $C([s, T] ; \mathcal{L}(H))$ and $\hat{\Phi}(t, s)$ is an evolution while $B^{\star} \widetilde{P}(t)$ satisfies the estimate (4.3).

Step 3: We now define the initial variables

$$
\begin{aligned}
P_{0}(t) & \equiv e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)}, \\
Q_{0}(\tau) & \equiv\left(I+D^{\star} P_{0}(\tau) D\right)^{-1}\left(B^{\star} P_{0}(\tau)+D^{\star} P_{0}(\tau) C\right), \\
\hat{C}_{0}(\tau) & \equiv C-D\left(I+D^{\star} P_{0}(\tau) D\right)^{-1}\left(B^{\star} P_{0}(\tau)+D^{\star} P_{0}(\tau) C\right), \\
\hat{\psi}_{0} & \equiv\left(I+D^{\star} P_{0}(\tau) D\right)^{-1}\left(B^{\star} P_{0}(\tau)+D^{\star} P_{0}(\tau) C\right) .
\end{aligned}
$$

This choice of the positive operator $P_{0}$ guarantees that $B^{\star} P_{0}(t)$ is bounded $H \rightarrow U$ for $t \in[s, T)$ and satisfies (4.3), and that $\left(I+D^{\star} P_{0} D\right)^{-1}$ is well defined and bounded on $U$.

Step 4: We next set up the following iteration scheme on the equation from Step 1:

$$
\begin{aligned}
P_{i+1}(t)= & \int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R \hat{\Phi}_{i}(\tau, t) d \tau+\int_{t}^{T} e^{A^{\star}(\tau-t)} Q_{i}^{\star}(\tau) Q_{i}(\tau) \hat{\Phi}_{i}(\tau, t) d \tau \\
& +\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{C}_{i}^{\star}(\tau) P_{i+1}(\tau) \hat{C}_{i}(\tau) \hat{\Phi}_{i}(\tau, t) d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{\psi}_{i}^{\star}(\tau) B^{\star} P_{i+1}(\tau) \hat{\Phi}_{i}(\tau, t) d \tau+e^{A^{\star}(T-t)} G^{\star} G \hat{\Phi}_{i}(T, t),
\end{aligned}
$$

where $\hat{\Phi}_{i}(t, s) x=e^{A(t-s)} x-\int_{s}^{t} e^{A(t-z)} B \hat{\psi}_{i}(z) \hat{\Phi}_{i}(z, s) x d z$, and

$$
\begin{aligned}
Q_{i}(\tau) & \equiv\left(I+D^{\star} P_{i}(\tau) D\right)^{-1}\left(B^{\star} P_{i}(\tau)+D^{\star} P_{i}(\tau) C\right) \\
\hat{C}_{i}(\tau) & \equiv C-D\left(I+D^{\star} P_{i}(\tau) D\right)^{-1}\left(B^{\star} P_{i}(\tau)+D^{\star} P_{i}(\tau) C\right) \\
\hat{\psi}_{i} & =\left(I+D^{\star} P_{i}(\tau) D\right)^{-1}\left(B^{\star} P_{i}(\tau)+D^{\star} P_{i}(\tau) C\right)
\end{aligned}
$$

Step 1 guarantees the existence of a solution $\left(P_{i+1}, \Phi_{i}\right)$ at each step of the iteration, and that $P_{i+1}$ is a positive self-adjoint operator, such that $B^{\star} P_{i+1}$ is bounded for $t \in$ $[s, T)$ and satisfies (4.3). This in turn gives sense to the operator $\left(I+D^{\star} P_{i+1}(\tau) D\right)^{-1}$ in $\mathcal{L}(U)$ which is needed in the next step of the iteration.

Step 5: Passing through the limit, we finally show that the sequence $P_{i}$ converges to the solution $P$ of the original integral equation (4.1) in $C([s, T] ; \mathcal{L}(H))$.
4.1. Preliminaries. We first introduce the space $C([s, T] ; \mathcal{L}(H))$ of the continuous family $P($.$) of bounded operators on the space H$, where

$$
\|P\|_{C([s, T], \mathcal{L}(H))}=\sup _{s \leq t \leq T}\|P(t)\|_{\mathcal{L}(H)} .
$$

Following $[\mathrm{H}]$, we also introduce the space $C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right)$, where

$$
\mathcal{T}_{s} \equiv\left\{(t, \tau) \in \mathbb{R}^{2}: s \leq \tau \leq t \leq T\right\}
$$

This space $C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right)$ is a Banach space equipped with the norm

$$
\|f\|_{C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right)}=\sup _{(t, \tau) \in \mathcal{T}_{s}}\|f(t, \tau)\|_{\mathcal{L}(H)} .
$$

We also introduce the Banach space $C_{\gamma}([s, T] ; Y)$ (following $[\mathrm{BDDM}]$ ) of continuous functions on $[s, T)$ into a Banach space $Y$, which is equipped with norm

$$
\|f\|_{C_{\gamma}([s, T] ; Y)}=\sup _{t \in[s, T]}(T-t)^{\gamma}\|f(t)\|_{Y}<\infty .
$$

The space accounts for possible singularities at time $T$ of order $\gamma$. We start with the following useful lemmas [L1, L2, LTu1].

Lemma 4.3.
(i) The map $L_{s} \equiv \int_{s}^{t} e^{A(t-\tau)} B d \tau$ is continuous from $C_{\gamma}([s, T] ; U)$ to $C([s, T] ; H)$ for $\gamma<1 / 2$.
(ii) The adjoint map $L_{s}^{\star} \equiv \int_{t}^{T} B^{\star} e^{A^{\star}(\tau-t)} d \tau$ is continuous from $C_{\gamma}([s, T] ; H)$ to $C([s, T] ; U)$ for $\gamma<1 / 2$.
4.2. Linear integral equation. We first consider the linear integral equations

$$
\begin{aligned}
\widetilde{P}(t)= & \int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R \hat{\Phi}(\tau, t) d \tau+\int_{t}^{T} e^{A^{\star}(\tau-t)} Q^{\star}(\tau) Q(\tau) \hat{\Phi}(\tau, t) d \tau \\
& +\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{C}^{\star}(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{\psi}^{\star}(\tau) B^{\star} \widetilde{P}(\tau) \hat{\Phi}(\tau, t) d \tau+e^{A^{\star}(T-t)} G^{\star} G \hat{\Phi}(T, t),
\end{aligned}
$$

and

$$
\begin{equation*}
\hat{\Phi}(t, s) x=e^{A(t-s)} x-\int_{s}^{t} e^{A(t-z)} B \hat{\psi}(z) \hat{\Phi}(z, s) x d z \tag{4.5}
\end{equation*}
$$

In the next lemma, we prove the existence of solutions $\widetilde{P}$ and $\hat{\Phi}(t, s)$ to integral equations (4.4) and (4.5).

Lemma 4.4. Assume $Q(t), \hat{C}(t), \hat{\psi}(t)$ are given bounded operators for every $t \in$ $[s, T)$ satisfying the conditions

$$
\begin{equation*}
\|Q(t) x\|_{H},\|\hat{C}(t) x\|_{H},\|\hat{\psi}(t) x\|_{H} \leq \frac{r\|x\|_{H}}{(T-t)^{\gamma}} \quad \forall x \in H, t \in[s, T) \tag{4.6}
\end{equation*}
$$

for some suitably chosen $r>0$. Then, there exists a unique local-in-time solution $\widetilde{P} \in C\left(\left[T_{0}, T\right] ; \mathcal{L}(H)\right)$ and $\hat{\Phi}(\cdot, \cdot) \in C\left(\mathcal{T}_{T_{0}} ; \mathcal{L}(H)\right)$ to the set of integral equations (4.4) and (4.5) such that

$$
\begin{equation*}
\left\|B^{\star} \widetilde{P}(t) x\right\|_{H} \leq \frac{c}{(T-t)^{\gamma}}\|x\|_{H} . \tag{4.7}
\end{equation*}
$$

To prove the existence of a solution $\widetilde{P}$ and $\hat{\Phi}$, we use a fixed point argument on the map $\Lambda$ defined by

$$
\Lambda\left(\begin{array}{l}
f \\
g \\
h
\end{array}\right)(t)=\left(\begin{array}{c}
\Lambda_{11}(g)(t)+\Lambda_{12}(g)(t)+\Lambda_{13}(f, g)(t)+\Lambda_{14}(g, h)(t)+\Lambda_{15}(g)(t) \\
\Lambda_{2}(g)(t) \\
\Lambda_{31}(g)(t)+\Lambda_{32}(g)(t)+\Lambda_{33}(f, g)(t)+\Lambda_{34}(g, h)(t)+\Lambda_{35}(g)(t)
\end{array}\right)
$$

for $t \in[s, T]$ on the space $X \equiv C([s, T] ; \mathcal{L}(H)) \times C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right) \times C_{\gamma}([s, T] ; \mathcal{L}(H, U))$, where

$$
\begin{aligned}
\Lambda_{11}(g)(t) & \equiv \int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R g(\tau, t) d \tau \\
\Lambda_{12}(g)(t) & \equiv \int_{t}^{T} e^{A^{\star}(\tau-t)} Q^{\star}(\tau) Q(\tau) g(\tau, t) d \tau \\
\Lambda_{13}(f, g)(t) & \equiv \int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{C}^{\star}(\tau) f(\tau) \hat{C}(\tau) g(\tau, t) d \tau \\
\Lambda_{14}(g, h)(t) & \equiv-\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{\psi}^{\star}(\tau) h^{\star}(\tau) g(\tau, t) d \tau \\
\Lambda_{15}(g)(t) & \equiv e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)}-e^{A^{\star}(T-t)} G^{\star} G \int_{t}^{T} e^{A(T-\tau)} B \hat{\psi}(\tau) g(\tau, t) d \tau
\end{aligned}
$$

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and

$$
\Lambda_{2}(f, g, h)=e^{A(t-s)}-L_{s} B \hat{\psi}(\cdot) g(\cdot, \cdot)(t)
$$

while

$$
\begin{aligned}
\Lambda_{31}(g)(t) & \equiv \int_{t}^{T} B^{\star} e^{A^{\star}(\tau-t)} R^{\star} R g(\tau, t) d \tau \\
\Lambda_{32}(g)(t) & \equiv \int_{t}^{T} B^{\star} e^{A^{\star}(\tau-t)} Q^{\star}(\tau) Q(\tau) g(\tau, t) d \tau \\
\Lambda_{33}(f, g)(t) & \equiv \int_{t}^{T} B^{\star} e^{A^{\star}(\tau-t)} \hat{C}^{\star}(\tau) f(\tau) \hat{C}(\tau) g(\tau, t) d \tau \\
\Lambda_{34}(g, h)(t) & \equiv-\int_{t}^{T} B^{\star} e^{A^{\star}(\tau-t)} \hat{\psi}^{\star}(\tau) h^{\star}(\tau) g(\tau, t) d \tau \\
\Lambda_{35}(g)(t) & \equiv B^{\star} e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)}-B^{\star} e^{A^{\star}(T-t)} G^{\star} G \int_{t}^{T} e^{A(T-\tau)} B \hat{\psi}(\tau) g(\tau, t) d \tau .
\end{aligned}
$$

In order to deal with unboundedness of control operator $B$, we seek a fixed point of the system of three equations defined by three variables (operators) which are $f=P, g=\Phi$, and $h=B^{*} P$. All these three quantities will be defined on the space $X$. Clearly we will have $h=B^{*} f$-which then will lead to "hidden" regularity results obtained for the gain operator $B^{*} P$. The fixed point $f, g, h$ here represent the operators $P(t), \Phi(t, s)$, and $B^{\star} P$, respectively.

Lemma 4.5. The map $\Lambda$ maps the ball $B_{r}(0) \subset X$ into itself continuously, and is a contraction on $B_{r}(0)$ for suitably chosen $r>0$ and $s=T_{0}$ such that $T-T_{0}$ is sufficiently small.

Proof. Let $[f, g, h]$ be an element in the ball $B_{r}(0)$. We estimate the norm of $\Lambda[f, g, h]$ in $X$, by considering every component. We spare the reader the technical details of the estimates. Defining $c_{s}$ by

$$
c_{s}=\max \left\{c(T-s), c \frac{(T-s)^{1-\gamma}}{1-\gamma}, c \frac{(T-s)^{1-2 \gamma}}{1-2 \gamma}\right\}
$$

and based on these estimates we impose the condition $6 c M^{2} e^{2 \alpha(T-s)}+6 c_{s} M e^{\alpha(T-s)} \times$ $\left(r^{4}+r^{3}+r^{2}+r\right)<r$ or, equivalently,

$$
\begin{equation*}
c M^{2} e^{2 \alpha(T-s)}+c_{s} M e^{\alpha(T-s)}\left(r^{4}+r^{3}+r^{2}+r\right)-r / 6<0 . \tag{4.8}
\end{equation*}
$$

Let $r=12 c M^{2} e^{2 \alpha T}$ and choose $s$ such that $(T-s)$ is sufficiently small and so that

$$
c_{s}<\frac{c M e^{\alpha T}}{r^{4}+r^{3}+r^{2}+r} .
$$

This guarantees that $\Lambda$ acts from $B_{r}(0)$ into $B_{r}(0)$ in $X$ for our choice of $s$ and $r$. The contraction property can be shown by estimating the norm of the difference of $\Lambda\left[f_{1}, g_{1}, h_{1}\right]^{T}$ and $\Lambda\left[f_{2}, g_{2}, h_{2}\right]^{T}$. Choosing $s=T_{0}$ so that $T-T_{0}$ is sufficiently small we have that $\Lambda$ is a contraction on $B_{r}(X)$ and hence has a unique fixed point $(f, g, h) \in X$.
From the above lemma, we have that the fixed points $(f, g, h)$ represent solutions $\left(\widetilde{P}(t), \hat{\Phi}(t, s), B^{\star} \widetilde{P}(t)\right) \in X$ to (4.4) and (4.5). Estimate (4.7) follows from the membership of $B^{\star} \widetilde{P}$ in $C_{\gamma}([s, T] ; U)$. This proves Lemma 4.4.
4.3. Positivity and self-adjointness of $\widetilde{\boldsymbol{P}}$. Let $s=T_{0}$. In the following lemma, we prove that the solution $\widetilde{P}$ to (4.4) is positive, self-adjoint in addition to the evolution property of $\hat{\Phi}(t, s)$ on the space $C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right)$.

Lemma 4.6.
(i) The operator $\hat{\Phi}(t, s)$, which is defined by (4.5), is an evolution operator on $C([s, T] ; \mathcal{L}(H))$.
(ii) The operator $\widetilde{P}$ solving the integral equation (4.4) is self-adjoint.
(iii) The operator $\widetilde{P}$ solving the integral equation (4.4) is positive.

Proof.
(i) This follows from a standard argument using the evolution property of the semigroup.
(ii) Taking the inner product of (4.4) with $y \in H$ and substituting the expression

$$
e^{A(\tau-t)} y=\hat{\Phi}(\tau, t) y+\int_{t}^{\tau} e^{A(\tau-z)} B \hat{\psi}(z) \hat{\Phi}(z, t) y d z
$$

from (4.5) into the equation, we have

$$
\begin{aligned}
\langle\widetilde{P}(t) x, y\rangle= & \int_{t}^{T}\langle R \hat{\Phi}(\tau, t) x, R \hat{\Phi}(\tau, t) y\rangle d \tau \\
& +\int_{t}^{T}\left\langle R^{\star} R \hat{\Phi}(\tau, t) x, \int_{t}^{\tau} e^{A(\tau-z)} B \hat{\psi}(z) \hat{\Phi}(z, t) y d z\right\rangle d \tau \\
& +\int_{t}^{T}\langle Q(\tau) \hat{\Phi}(\tau, t) x, Q(\tau) \hat{\Phi}(\tau, t) y\rangle d \tau \\
& +\int_{t}^{T}\left\langle Q^{\star}(\tau) Q(\tau) \hat{\Phi}(\tau, t) x, \int_{t}^{\tau} e^{A(\tau-z)} B \hat{\psi}(z) \hat{\Phi}(z, t) y d z\right\rangle d \tau \\
& +\int_{t}^{T}\left\langle\hat{C}^{\star}(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \hat{\Phi}(\tau, t) x\right\rangle d \tau \\
& +\int_{t}^{T}\left\langle\hat{C}^{\star}(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \int_{t}^{\tau} e^{A(\tau-z)} B \hat{\psi}(z) \hat{\Phi}(z, t) y d z\right\rangle d \tau \\
& -\int_{t}^{T}\left\langle\hat{\psi}^{\star}(\tau) B^{\star} \widetilde{P}(\tau) \hat{\Phi}(\tau, t) x, \hat{\Phi}(\tau, t) y\right\rangle d \tau \\
& -\int_{t}^{T}\left\langle\hat{\psi}^{\star}(\tau) B^{\star} \widetilde{P}(\tau) \hat{\Phi}(\tau, t) x, \int_{t}^{\tau} e^{A(\tau-z)} B \hat{\psi}(z) \hat{\Phi}(z, t) y d z\right\rangle d \tau \\
& +\langle G \hat{\Phi}(T, t) x, G \hat{\Phi}(T, t) y\rangle+\left\langle G^{\star} G \hat{\Phi}(T, t) x, \int_{t}^{T} e^{A(T-z)} B \hat{\psi}(z) \hat{\Phi}(z, t) y d z\right\rangle .
\end{aligned}
$$

Changing the order of integration, the second, fourth, sixth, and eighth terms combine into

$$
\begin{aligned}
& \int_{t}^{T} \int_{z}^{T}\left\langle B^{\star} e^{A^{\star}(\tau-z)} R^{\star} R \hat{\Phi}(\tau, t) x, \hat{\psi}(z) \hat{\Phi}(z, t) y\right\rangle d \tau d z \\
& \quad+\int_{t}^{T} \int_{z}^{T}\left\langle B^{\star} e^{A^{\star}(\tau-z)} Q^{\star}(\tau) Q(\tau) \hat{\Phi}(\tau, t) x, \hat{\psi}(z) \hat{\Phi}(z, t) y\right\rangle d \tau d z \\
& \quad+\int_{t}^{T} \int_{z}^{T}\left\langle B^{\star} e^{A^{\star}(\tau-z)} \hat{C}^{\star}(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \hat{\psi}(z) \hat{\Phi}(z, t) y\right\rangle d \tau d z
\end{aligned}
$$

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$$
\begin{aligned}
& -\int_{t}^{T} \int_{z}^{T}\left\langle B^{\star} e^{A^{\star}(\tau-z)} \hat{\psi}^{\star}(\tau) B^{\star} \widetilde{P}(\tau) \hat{\Phi}(\tau, t) x, \hat{\psi}(z) \hat{\Phi}(z, t) y\right\rangle d \tau d z \\
& +\int_{t}^{T}\left\langle B^{\star} e^{A^{\star}(T-z)} G^{\star} G \hat{\Phi}(T, t) x, \hat{\psi}(z) \hat{\Phi}(z, t) y\right\rangle d z \\
= & \int_{t}^{T}\left\langle B^{\star} \widetilde{P}(z) \hat{\Phi}(z, t) x, \hat{\psi}(z) \hat{\Phi}(z, t) y\right\rangle d z,
\end{aligned}
$$

which cancels with the fifth term. Therefore we have

$$
\begin{align*}
\langle\widetilde{P}(t) x, y\rangle= & \int_{t}^{T}\langle R \hat{\Phi}(\tau, t) x, R \hat{\Phi}(\tau, t) y\rangle d \tau+\int_{t}^{T}\langle Q(\tau) \hat{\Phi}(\tau, t) x, Q(\tau) \hat{\Phi}(\tau, t) y\rangle d \tau \\
& +\int_{t}^{T}\left\langle\hat{C}^{\star}(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \hat{\Phi}(\tau, t) y\right\rangle d \tau+\langle G \hat{\Phi}(T, t) x, G \hat{\Phi}(T, t) y\rangle . \tag{4.9}
\end{align*}
$$

On the other hand, we have

$$
\begin{aligned}
\left\langle\widetilde{P}^{\star}(t) x, y\right\rangle= & \int_{t}^{T}\langle R \hat{\Phi}(\tau, t) x, R \hat{\Phi}(\tau, t) y\rangle d \tau+\int_{t}^{T}\langle Q(\tau) \hat{\Phi}(\tau, t) x, Q(\tau) \hat{\Phi}(\tau, t) y\rangle d \tau \\
& +\int_{t}^{T}\left\langle\hat{C}^{\star}(\tau) \widetilde{P}^{\star}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \hat{\Phi}(\tau, t) y\right\rangle d \tau+\langle G \hat{\Phi}(T, t) x, G \hat{\Phi}(T, t) y\rangle .
\end{aligned}
$$

Taking the difference of the two last equations, we get

$$
\left\langle\left[\widetilde{P}-\widetilde{P}^{\star}\right](t) x, y\right\rangle=\int_{t}^{T}\left\langle\hat{C}^{\star}(\tau)\left[\widetilde{P}-\widetilde{P}^{\star}\right](\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \hat{\Phi}(\tau, t) y\right\rangle d \tau
$$

Estimating the left side, and taking the supremum over all $x$ of unit norm and all $y$ in $H$, we obtain

$$
\left\|\widetilde{P}(t)-\widetilde{P}^{\star}(t)\right\|_{\mathcal{L}(H)} \leq c r^{4} \int_{t}^{T}\left\|\widetilde{P}(\tau)-\widetilde{P}^{\star}(\tau)\right\|_{\mathcal{L}(H)} d \tau
$$

Using Gronwall's inequality we conclude that the left-hand side is zero and hence $P(t)=P^{\star}(t)$ for all $t \in[s, T]$.
(iii) To prove positivity, we appeal to (4.9). The operator $\widetilde{P}$ is then the unique fixed point of the map $S$ on $C([s, T] ; \mathcal{L}(H))$ defined by

$$
\begin{aligned}
\langle S(P)(t) x, y\rangle= & \int_{t}^{T}\langle R \hat{\Phi}(\tau, t) x, R \hat{\Phi}(\tau, t) y\rangle d \tau+\int_{t}^{T}\langle Q(\tau) \hat{\Phi}(\tau, t) x, Q(\tau) \hat{\Phi}(\tau, t) y\rangle d \tau \\
& +\int_{t}^{T}\left\langle\hat{C}^{\star}(\tau) P(\tau) \hat{C}(\tau) \hat{\Phi}(\tau, t) x, \hat{\Phi}(\tau, t) y\right\rangle d \tau+\langle G \hat{\Phi}(T, t) x, G \hat{\Phi}(T, t) y\rangle
\end{aligned}
$$

The map $S$ clearly maps positive operators to positive operators. The set of positive operators denoted by $\Sigma_{+}$in $\mathcal{L}(H)$ is a convex set, and the existence of a unique fixed point for $S$ on $C\left(\left[T_{0}, T\right] ; \Sigma_{+}\right)$follows by the contraction mapping theorem, for $T_{0}$ chosen so that $T-T_{0}$ is sufficiently small.

### 4.4. Step 4: Proof of Theorem 4.1.

Proof. To derive the integral equation (4.1), we use the following iteration scheme

$$
\begin{align*}
P_{i+1}(t)= & \int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R \hat{\Phi}_{i}(\tau, t) d \tau+\int_{t}^{T} e^{A^{\star}(\tau-t)} Q_{i}^{\star}(\tau) Q_{i}^{\star}(\tau) \hat{\Phi}_{i}(\tau, t) d \tau \\
& +\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{C}_{i}^{\star}(\tau) P_{i+1}(\tau) \hat{C}_{i}(\tau) \hat{\Phi}_{i}(\tau, t) d \tau \\
0) & -\int_{t}^{T} e^{A^{\star}(\tau-t)} \hat{\psi}_{i}^{\star}(\tau) B^{\star} P_{i+1}(\tau) \hat{\Phi}_{i}(\tau, t) d \tau+e^{A^{\star}(T-t)} G^{\star} G \hat{\Phi}_{i}(T, t), \tag{4.10}
\end{align*}
$$

$$
\begin{aligned}
Q_{i}(\tau) & \equiv\left(I+D^{\star} P_{i}(\tau) D\right)^{-1}\left(B^{\star} P_{i}(\tau)+D^{\star} P_{i}(\tau) C\right) \\
\hat{C}_{i}(\tau) & \equiv C-D\left(I+D^{\star} P_{i}(\tau) D\right)^{-1}\left(B^{\star} P_{i}(\tau)+D^{\star} P_{i}(\tau) C\right), \\
\hat{\psi}_{i} & =\left(I+D^{\star} P_{i}(\tau) D\right)^{-1}\left(B^{\star} P_{i}(\tau)+D^{\star} P_{i}(\tau) C\right), \\
P_{0}(t) & =e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)},
\end{aligned}
$$

and $\hat{\Phi}_{i}$ solves

$$
\begin{equation*}
\hat{\Phi}_{i}(t, s) x=e^{A(t-s)} x-\int_{s}^{t} e^{A(t-z)} B \hat{\psi}_{i}(z) \hat{\Phi}_{i}(z, s) x d z . \tag{4.11}
\end{equation*}
$$

Using the results of Lemmas 4.4 and 4.6 from previous sections, each iteration $P_{i}$ is well defined, positive self-adjoint, and bounded with

$$
\begin{array}{r}
\left\|P_{i}\right\|_{C([s, T] ; \mathcal{L}(H))} \leq r, \\
\left\|B^{\star} P_{i}(t) x\right\|_{H} \leq \frac{r}{(T-t)^{\gamma}}\|x\|_{H}
\end{array}
$$

$\forall x \in H$ and $\forall i \in \mathbb{N}$, while $\Phi_{i} \in C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right)$ such that

$$
\left\|\Phi_{i}\right\|_{C\left(\mathcal{T}_{s} ; \mathcal{L}(H)\right)} \leq r
$$

and this guarantees that the inverse $\left(I+D^{\star} P_{i}(t) D\right)^{-1}$ is well defined and bounded on $H$ at each step. Using standard estimates, it is not difficult to show that the sequence $\left\{P_{i}, \Phi_{i}, B^{\star} P_{i}\right\}$ is Cauchy in $X$ for $s=T_{\max } \geq T_{0}$ chosen such that $T$ $T_{\text {max }}$ is sufficiently small, and thus converging to some $(P(t), \Phi, h(t)) \in X$ with $h(t)=B^{\star} P(t)$. Passing through the limit in (4.10) and (4.11), we obtain (4.1) and (4.2).
5. The differential Riccati equation. In this section, we derive the differential Riccati equation from the integral Riccati equation (4.1). Our main result is then the following.

Theorem 5.1. The Riccati operator $P(t)$ solving the integral Riccati equation (4.1) is a solution to the differential Riccati equation

$$
\begin{align*}
\langle\dot{P}(t) x, y\rangle= & \left.-\langle R x, R y\rangle-\langle A x, P(t) y\rangle-\left\langle A^{\star} P(t) x, y\right\rangle-\left\langle C^{\star} P(t) C x, y\right\rangle\right) \\
& +\left\langle\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) x,\left(B^{\star} P(t)+D^{\star} P(t) C\right) y\right\rangle \tag{5.1}
\end{align*}
$$

for all $x, y \in \mathcal{D}(A)$.

A critical step in this process is to establish a "singular estimate" on the transfer function corresponding to the controlled dynamics. This amounts to the estimate of singularity on the composition operator $\Phi(t, s) B$. To accomplish this we need several preliminary results. To carry out the derivation, we shall need to make sense of the derivative of the evolution $\Phi(t, s)$ with respect to the initial time $s$ (in the weak sense).

### 5.1. Preliminaries. We first define the operator $\mathcal{M}$.

Definition 5.2. Denote by $\mathcal{M}$ the operator

$$
\mathcal{M} \equiv \int_{s}^{t} e^{A(t-\tau)} B\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) d \tau
$$

We also define the space ${ }_{\gamma} C([s, T] ; H)$ following $[\mathrm{BDDM}]$.
Definition 5.3. Let

$$
{ }_{\gamma} C([s, T] ; H) \equiv\left\{f \in C((s, T] ; H): \sup _{t \in[s, T]}(t-s)^{\gamma}\|f(t)\|_{H}<\infty\right\}
$$

The space ${ }_{\gamma} C([s, T] ; H)$ is indeed a Banach space with the norm

$$
\|f\|_{\gamma C}=\sup _{t \in[s, T]}(t-s)^{\gamma}\|f(t)\|_{H}
$$

for $\gamma<1 / 2$. In the following lemma, we establish some of the properties of the operator $\mathcal{M}$.

Lemma 5.4.
(i) The operator $e^{A(\cdot-s)} B x \in{ }_{\gamma} C([s, T] ; H) \quad \forall x \in U$ and satisfies the estimate

$$
\left\|e^{A(t-s)} B x\right\|_{\gamma} C([s, T] ; H) \leq c\|x\|_{U}
$$

(ii) The operator $\mathcal{M}$ is bounded on ${ }_{\gamma} C([s, T] ; H)$ and satisfies the estimate

$$
\|\mathcal{M} g\|_{\gamma C([s, T] ; H)} \leq c(T-s)^{1-\gamma}\|g\|_{\gamma} C([s, T] ; H)
$$

for every $g \in{ }_{\gamma} C([s, T] ; H)$.
(iii) The operator $(I+\mathcal{M})$ is invertible on ${ }_{\gamma} C([s, T] ; H)$ and the inverse satisfies the estimate

$$
\left\|(I+\mathcal{M})^{-1} g\right\|_{\gamma C([s, T] ; H)} \leq c(T-s)\|g\|_{\gamma C([s, T] ; H)}
$$

(iv) The evolution $\Phi(t, s)$ satisfies

$$
\Phi(\cdot, s) x=(I+\mathcal{M})^{-1} e^{A(\cdot-s)} x \quad \forall x \in H
$$

Proof. The proofs are similar to the deterministic case in which $C=D=0$; see [LTu1, Tu].
5.2. Regularity of the "transfer function." We now make sense of the transfer function $\Phi(t, s) B$ and the derivative of the evolution $\Phi(t, s)$ with respect to initial time in an appropriate singular space, which is crucial in the derivation of the differential Riccati equation.

## Proposition 5.5.

(i) For all $x \in U$ and $\gamma<1 / 2$, we have $\Phi(t, s) B x \in{ }_{\gamma} C([s, T] ; H)$ and

$$
\|\Phi(t, s) B x\|_{H} \leq \frac{c}{(t-s)^{\gamma}}\|x\|_{U} \quad \forall x \in U
$$

(ii) For all $x \in \mathcal{D}(A)$, the derivative of the evolution $\Phi(t, s) x$ with respect to initial time in the weak sense is

$$
\begin{aligned}
\frac{\partial}{\partial s} \Phi(\cdot, s) x= & -\Phi(\cdot, s)\left(A-B\left(I+D^{\star} P(s) D\right)^{-1}\left(B^{\star} P(s)\right.\right. \\
& \left.\left.+D^{\star} P(s) C\right)\right) x \in{ }_{\gamma} C([s, T] ; H)
\end{aligned}
$$

and satisfies the estimate

$$
\left\|\frac{\partial}{\partial s} \Phi(t, s) B x\right\|_{H} \leq c\|x\|_{\mathcal{D}(A)}+\frac{c}{(t-s)^{\gamma}}\|x\|_{U}
$$

Proof. The proof follows from Lemma 5.4; see [LT1, Tu].

### 5.3. Proof of Theorem 5.1.

Proof. Let $x, y \in \mathcal{D}(A)$ and consider the integral Riccati equation satisfied by $P(t)$ in (4.1). Taking the derivative with respect to $t$, we have

$$
\begin{aligned}
&\langle\dot{P}(t) x, y\rangle=-\left\langle R^{\star} R x, y\right\rangle-\left\langle C^{\star} P(t) C x, y\right\rangle+\left\langleC ^ { \star } P ( t ) D ( I + D ^ { \star } P ( t ) D ) ^ { - 1 } \left( B^{\star} P(t)\right.\right. \\
&\left.\left.+D^{\star} P(t) C\right) x, y\right\rangle-\left\langle A^{\star} P(t) x, y\right\rangle \\
& \quad+\left\langle\int_{t}^{T} e^{A^{\star}(\tau-t)} R^{\star} R \frac{\partial}{\partial t} \Phi(\tau, t) x, y\right\rangle+\left\langle\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P(\tau) C \frac{\partial}{\partial t} \Phi(\tau, t) x, y\right\rangle \\
& \quad-\left\langle\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P(\tau) D\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) \frac{\partial}{\partial t} \Phi(\tau, t) x, y\right\rangle .
\end{aligned}
$$

We now appeal to Proposition 5.5(ii), where the expression for $\frac{\partial}{\partial t} \Phi(\tau, t)$ was derived so that we obtain

$$
\begin{aligned}
&\langle\dot{P}(t) x, y\rangle=-\left\langle R^{\star} R x, y\right\rangle-\left\langle C^{\star} P(t) C x, y\right\rangle+ \\
&+C^{\star} P(t) D\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)\right. \\
&\left.\left.+D^{\star} P(t) C\right) x, y\right\rangle-\left\langle A^{\star} P(t) x, y\right\rangle \\
&-\left\langle P(t)\left(A-B\left(I+D^{\star} P(t) D\right)^{-1}\right)\left(B^{\star} P(t)+D^{\star} P(t) C\right) x, y\right\rangle
\end{aligned}
$$

where the last term is well defined by boundedness of $P(t) B$ and its adjoint. Rearranging terms, we obtain the differential Riccati equation

$$
\begin{aligned}
\langle\dot{P}(t) x, y\rangle= & -\left\langle R^{\star} R x, y\right\rangle-\left\langle A^{\star} P(t) x, y\right\rangle-\langle P(t) A x, y\rangle-\left\langle C^{\star} P(t) C x, y\right\rangle \\
& +\left\langle\left(P(t) B+C^{\star} P(t) D\right)\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) x, y\right\rangle
\end{aligned}
$$

Remark 5.6. The differential form of the Riccati equation holds for any elements $x, y \in D(A)$. This form will be used for elements $x, y$ resulting from a stochastic process. Since stochastic equations do not posses strong solutions, the applicability of the differential Riccati equation in the stochastic context is questionable. To resolve this issue, we shall introduce an approximation procedure which consists of two steps. Step one: the regularity lemma on page 48 of $[\mathrm{H}]$ allows one to define the derivative of $P$ on a stochastic process which originates in the domain of $A$, with twice differentiable controls and smooth observations $C, D$. In the second step we shall regularize the state $y$ by changing the variable to $v_{n}$. This will allow the application of Itô's formula.

Here we state a regularity lemma and justify the form of the differential Riccati equation when acting on a stochastic process; page 48 in $[\mathrm{H}]$.

Lemma 5.7. If we have the additional assumptions that the operators $A C, A D \in$ $\mathcal{L}(H)$, and $u \in L^{2}\left(\Omega ; H_{0}^{1}([s, T] ; U)\right)$ then given $x \in \mathcal{D}(A)$ we have

$$
\mathbb{E}\left(\langle P(t) X(t), A X(t)\rangle_{H}\right)<\infty
$$

for all $t \in[s, T]$, where $X(t)$ is a solution of the stochastic differential equation

$$
\begin{aligned}
d X & =(A X+B u) d t+(C X+D u) d W_{t} \\
X(s) & =x \in \mathcal{D}(A)
\end{aligned}
$$

Proof. We first write the form of the mild solution to the abstract differential equation as

$$
\begin{aligned}
X(t)= & e^{A(t-s)} x+\int_{s}^{t} e^{A(t-\tau)} B u(\tau) d \tau+\int_{s}^{t} e^{A(t-\tau)} C X(\tau) d W_{\tau} \\
& +\int_{s}^{t} e^{A(t-\tau)} D u(\tau) d W_{\tau}
\end{aligned}
$$

We apply operator $A$ to each side and then split the term $A X(t)$ into two parts $A X(t)=Y_{1}+Y_{2}$, where

$$
Y_{1}(t)=e^{A(t-s)} A x+\int_{s}^{t} e^{A(t-\tau)} A C X(\tau) d W_{\tau}+\int_{s}^{t} e^{A(t-\tau)} A D u(\tau) d W_{\tau}
$$

and $Y_{2}(t)=\int_{s}^{t} e^{A(t-\tau)} A B u(\tau) d \tau$.
We then estimate the norm of $Y_{1}$ in $L^{2}(\Omega ; C([s, T] ; H))$ to obtain

$$
\begin{aligned}
\mathbb{E}\left(\left\|Y_{1}(t)\right\|_{H}^{2}\right) \leq & 3 M^{2} e^{2 \alpha(T-s)}\|A x\|_{H}^{2}+3 M^{2} e^{2 \alpha(T-s)}\|A C\|_{\mathcal{L}(H)}^{2} \int_{s}^{t} \mathbb{E}\left(\|X(\tau)\|_{H}^{2}\right) d \tau \\
& +3 M^{2} e^{2 \alpha(T-s)}\|A D\|_{\mathcal{L}(U, H)}^{2} \int_{s}^{t} \mathbb{E}\left(\|u(\tau)\|_{U}^{2}\right) d \tau
\end{aligned}
$$

where we used the Itô isometry to estimate the stochastic integrals. Since $X(t)$ is the solution to the abstract differential equation, by Theorem 3.1, its norm in $M_{w}^{2}([s, T] ; H)$ is bounded and satisfies

$$
\|X(t)\|_{M_{w}^{2}([s, T] ; H)}^{2} \leq c\|x\|_{H}^{2}+c\|u\|_{M_{w}^{2}([s, T] ; U)}^{2}
$$

Hence, $\mathbb{E}\left(\left\|Y_{1}(t)\right\|_{H}^{2}\right) \leq c Q\left(\|A C\|_{\mathcal{L}(H)},\|A D\|_{\mathcal{L}(H)},\|u\|_{L^{2}\left(\Omega ; H_{0}^{1}([s, T] ; U)\right)},\|A x\|_{H}\right)$, where $Q$ is a polynomial in the indicated norms. We next express $Y_{2}$ as

$$
Y_{2}(t)=-B u(t)+\int_{s}^{t} e^{A(t-\tau)} B u^{\prime}(\tau) d \tau=-B u(t)+I(t)
$$

via integration by parts in time where we used the fact $u(s)=0$ since $\left.u \in H_{0}^{1}([s, T] ; U)\right)$. The second term can be estimated via the singular estimate condition and Hölder's inequality as

$$
\begin{aligned}
\mathbb{E}\left(\left\|\int_{s}^{t} e^{A(t-\tau)} B u^{\prime}(\tau) d \tau\right\|_{H}^{2}\right) & \leq \mathbb{E}\left(\int_{s}^{t} \frac{c}{(t-\tau)^{\gamma}}\left\|u^{\prime}(\tau)\right\|_{U} d \tau\right)^{2} \\
& \leq c(T-s)^{1-2 \gamma} \mathbb{E}\left(\|u\|_{H_{0}^{1}([s, T] ; U)}^{2}\right)
\end{aligned}
$$

We are now ready to estimate the term $\mathbb{E}\left(\langle P(t) X(t), A X(t)\rangle_{H}\right.$ as

$$
\begin{aligned}
\mathbb{E}\left(\langle P(t) X(t), A X(t)\rangle_{H} \leq\right. & \left|\mathbb{E}\left(\left\langle P(t) X(t), Y_{1}(t)\right\rangle_{H}\right)\right|+\left|\mathbb{E}\left(\langle P(t) X(t), I(t)\rangle_{H}\right)\right| \\
& +\left|\mathbb{E}\left(\langle P(t) X(t), B u(t)\rangle_{H}\right)\right| \\
\leq & \|P(t)\|_{\mathcal{L}(H)} \mathbb{E}\left(\|X(t)\|_{H}\right) \mathbb{E}\left(\left\|Y_{1}(t)\right\|_{H}\right) \\
& +\|P(t)\|_{\mathcal{L}(H)} \mathbb{E}\left(\|X(t)\|_{H}\right) \mathbb{E}\left(\|I(t)\|_{H}\right) \\
& +\left\|B^{\star} P(t)\right\|_{\mathcal{L}(H, U)} \mathbb{E}\left(\|u(t)\|_{U}\right) \mathbb{E}\left(\|X(t)\|_{H}\right) \\
\leq & c Q\left(\|A C\|_{\mathcal{L}(H)},\|A D\|_{\mathcal{L}(H)},\|P(t)\|_{\mathcal{L}(H)},\left\|B^{\star} P(t)\right\|_{\mathcal{L}(H, U)},\right. \\
& \left.\|u\|_{L^{2}\left(\Omega ; H_{0}^{1}([s, T] ; U)\right)},\|A x\|_{H}\right),
\end{aligned}
$$

where we used the continuous embedding $H_{0}^{1}([s, T] ; U) \subset C([s, T] ; U)$ in the last step and where $Q$ is a polynomial in the indicated norms. The right-hand side is finite which yields the desired result.
6. Dynamic programming: The Riccati equation and the optimal control. In the following lemma, we relate the optimization problem to the solution of the differential Riccati equation via a dynamic programing argument. This technique is paramount to a completion of squares technique which furnishes an expression for the cost functional in which the minimizer and minimum value of the cost functional can be immediately deduced. However, the use of Itô's formula in this argument requires $C^{2}$ trajectories, which means that the argument has to be performed on an approximate regularized version of the abstract stochastic differential equation, before passing through the limit.

Lemma 6.1. The quadratic cost functional (1.2) has the form

$$
\begin{gathered}
J(t, x, u)=\mathbb{E}\left(\int_{t}^{T} \|\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u(\tau)+\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)\right.\right. \\
\left.\left.+D^{\star} P(\tau) C\right) y(\tau) \|_{U}^{2} d \tau\right)+\langle P(t) x, x\rangle
\end{gathered}
$$

for $s \leq t \leq T$ and $s=T_{\text {max }}$, where $P(t)$ is a solution to the differential Riccati equation (5.1) and $y$ is the solution to (1.1) corresponding to $u \in M_{w}^{2}([s, T] ; U)$.

Proof. In order to apply Itô's formula, we must use an appropriate approximate problem satisfied by a sufficiently regular random variable, and, in particular, a strong solution of a stochastic differential equation. We follow $[\mathrm{H}]$ closely and consider the following stochastic differential equation

$$
d y_{n}=\left(A y_{n}+B u\right) d t+\left(C_{n} y_{n}+D_{n} u\right) d W_{t},
$$

where $R(n, A)=(n I-A)^{-1}$ is the resolvent of $A$, and $C_{n}$ is defined by $C_{n} \equiv$ $n R(n, A) C$, while $D_{n} \equiv n R(n, A) D$. Taking $u \in L^{2}\left(\Omega ; H_{0}^{1}([s, T] ; U)\right)$, we set

$$
v_{n}=y_{n}+A^{-1} B u .
$$

Now, let $P(t) \in C([s, T] ; \mathcal{L}(H))$ be a self-adjoint positive operator satisfying the differential Riccati equation (5.1) such that $B^{\star} P(\cdot) \in C_{\gamma}([s, T] ; \mathcal{L}(H, U))$. We rewrite $\left\langle P(t) y_{n}(t), y_{n}(t)\right\rangle$ in terms of $v_{n}$ as
$\psi\left(t, v_{n}, u\right)=\left\langle P(t) v_{n}(t), v_{n}(t)\right\rangle-2\left\langle P(t) v_{n}(t), A^{-1} B u(t)\right\rangle+\left\langle P(t) A^{-1} B u(t), A^{-1} B u(t)\right\rangle$.
We next observe that $v_{n}$ is a strong solution of the equation

$$
\begin{equation*}
d v_{n}=\left(A v_{n}+A^{-1} B u^{\prime}\right) d t+\left(C_{n} y_{n}+D_{n} u\right) d W_{t} \tag{6.3}
\end{equation*}
$$

where $u^{\prime}$ denotes $\frac{d}{d t} u$. In particular, taking $y(s)=x \in \mathcal{D}(A)$, and by the variation of parameters formula we get

$$
\begin{aligned}
y_{n}(t)= & e^{A(t-s)} x+\int_{s}^{t} e^{A(t-\tau)} B u(\tau) d \tau+\int_{s}^{t} e^{A(t-\tau)} C_{n} y_{n}(\tau) d W_{\tau} \\
& +\int_{s}^{t} e^{A(t-\tau)} D_{n} u(\tau) d W_{\tau}
\end{aligned}
$$

Integrating by parts in time in the first integral, we get

$$
\begin{aligned}
y_{n}(t)= & e^{A(t-s)} x-A^{-1} B u(t)+e^{A(t-s)} A^{-1} B u(s)+\int_{s}^{t} e^{A(t-\tau)} A^{-1} B u^{\prime}(\tau) d \tau \\
& +\int_{s}^{t} e^{A(t-\tau)}\left(C_{n} y_{n}+D_{n} u\right) d W_{\tau}
\end{aligned}
$$

Adding $A^{-1} B u(t)$ to both sides, we have

$$
\begin{aligned}
v_{n}(t)= & e^{A(t-s)}\left(x+A^{-1} B u(s)\right)+\int_{s}^{t} e^{A(t-\tau)} A^{-1} B u^{\prime}(\tau) d \tau+\int_{s}^{t} e^{A(t-\tau)} C_{n} y_{n} d W_{\tau} \\
& +\int_{s}^{t} e^{A(t-\tau)} D_{n} u d W_{\tau}
\end{aligned}
$$

which shows that $v_{n}$ is a solution to (6.3). Now, we can verify that $v_{n}(t) \in \mathcal{D}(A)$. Indeed, applying $A$ to the right-hand side, we have

$$
\begin{aligned}
& e^{A(t-s)}(A x+B u(s))+\int_{s}^{t} e^{A(t-\tau)} B u^{\prime}(\tau) d \tau+\int_{s}^{t} e^{A(t-\tau)} A C_{n} y_{n} d W_{\tau} \\
& \quad+\int_{s}^{t} e^{A(t-\tau)} A D_{n} u d W_{\tau}
\end{aligned}
$$

and $x \in \mathcal{D}(A)$ while

$$
\begin{aligned}
\mathbb{E}\left(\left\|\int_{s}^{t} e^{A(t-\tau)} B u^{\prime}(\tau) d \tau\right\|_{H}^{2}\right) & \leq \mathbb{E}\left(\int_{s}^{t} \frac{c}{(t-\tau)^{\gamma}}\left\|u^{\prime}(\tau)\right\|_{U} d \tau\right)^{2} \\
& \leq \widetilde{c} T^{1-2 \gamma} \mathbb{E}\left(\|u\|_{H^{1}([s, T] ; U)}^{2}\right)
\end{aligned}
$$

(note $\gamma<1 / 2$ ), where we used the singular estimate condition and Hölder's inequality in the last step. Moreover, we have by the boundedness of $A C_{n}$ and using Itô's isometry that

$$
\mathbb{E}\left(\left\|\int_{s}^{t} e^{A(t-\tau)} A C_{n} y_{n}(\tau) d W_{\tau}\right\|_{H}^{2}\right) \leq c_{1} \int_{s}^{t} \mathbb{E}\left(\left\|y_{n}(\tau)\right\|_{H}^{2}\right) d \tau<\infty
$$

where we used Theorem 3.1 in the last step. Moreover,

$$
\mathbb{E}\left(\left\|\int_{s}^{t} e^{A(t-\tau)} A D_{n} u d W_{\tau}\right\|_{H}^{2}\right) \leq c_{1} \int_{s}^{t} \mathbb{E}\left(\|u(\tau)\|_{U}^{2}\right) d \tau \leq c_{1}\|u\|_{M_{\omega}^{2}([s, T] ; U)}^{2}
$$

Hence, $v_{n} \in \mathcal{D}(A)$ which means it is a strong solution of (6.3).
We now can differentiate the expression for $\psi\left(t, v_{n}(t), u(t)\right)$ in (6.2) using Itô's formula [DZ1] to obtain

$$
\begin{aligned}
& d \psi\left(\tau, v_{n}(\tau), u(\tau)\right)=\left\langle P^{\prime}(\tau) v_{n}(\tau), v_{n}(\tau)\right\rangle d \tau+2\left\langle P(\tau) v_{n}(\tau), A v_{n}(\tau)+A^{-1} B u^{\prime}(\tau)\right\rangle d \tau \\
& \quad+2\left\langle P(\tau) v_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau} \\
& \quad+\left\langle P(\tau)\left(C_{n} y_{n}(\tau)+D_{n} u(\tau)\right), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d \tau \\
& \quad-2\left\langle P^{\prime}(\tau) v_{n}(\tau), A^{-1} B u(\tau)\right\rangle d \tau-2\left\langle P(\tau)\left(A v_{n}(\tau)+A^{-1} B u^{\prime}(\tau)\right), A^{-1} B u(\tau)\right\rangle d \tau \\
& \quad-2\left\langle P(\tau)\left(C_{n} y_{n}(\tau)+D_{n} u(\tau)\right), A^{-1} B u(\tau)\right\rangle d W_{\tau}-2\left\langle P(\tau) v_{n}(\tau), A^{-1} B u^{\prime}(\tau)\right\rangle d \tau \\
& \quad+\left\langle P^{\prime}(\tau) A^{-1} B u(\tau), A^{-1} B u(\tau)\right\rangle d \tau+2\left\langle P(\tau) A^{-1} B u^{\prime}(\tau), A^{-1} B u(\tau)\right\rangle d \tau .
\end{aligned}
$$

Substituting $y_{n}(\tau)$ back to eliminate $v_{n}(\tau)$ using self-adjointness of $P^{\prime}(\tau)$, we obtain

$$
\begin{aligned}
d\left\langle P(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle= & \left\langle P^{\prime}(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau+2\left\langle P(\tau) y_{n}(\tau), A y_{n}(\tau)+B u(\tau)\right\rangle d \tau \\
& +2\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau} \\
& +\left\langle P(\tau)\left(C_{n} y_{n}(\tau)+D_{n} u(\tau)\right), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d \tau
\end{aligned}
$$

We now recall that $P(\tau)$ solves the differential Riccati equation and, hence, we have

$$
\begin{aligned}
& d\left\langle P(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle=-\left\langle A^{\star} P(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau-\left\langle P(\tau) A y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau \\
& \quad-\left\langle R^{\star} R y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau-\left\langle C^{\star} P(\tau) C y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau \\
& \quad+\left\langle\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau),\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)\right\rangle d \tau \\
& \quad+2\left\langle P(\tau) y_{n}(\tau), A y_{n}(\tau)\right\rangle d \tau+2\left\langle P(\tau) y_{n}(\tau), B u(\tau)\right\rangle d \tau \\
& \quad+2\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau}+\left\langle P(\tau)\left(C_{n} y_{n}+D_{n} u\right), C_{n} y_{n}+D_{n} u\right\rangle d \tau
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
d\langle & \left.P(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle=-\left\|R y_{n}(\tau)\right\|_{Z}^{2} d \tau-\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau \\
& +\left\|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)\right\|_{U}^{2} d \tau \\
& +2\left\langle B^{\star} P(\tau) y_{n}(\tau), u(\tau)\right\rangle d \tau+2\left\langle D_{n}^{\star} P(\tau) C_{n} y_{n}(\tau), u(\tau)\right\rangle d \tau \\
& +\left\langle D_{n}^{\star} P(\tau) D_{n} u(\tau), u(\tau)\right\rangle d \tau+2\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau},
\end{aligned}
$$

where $\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}$ is well defined since $I+D^{\star} P(\tau) D$ is a positive operator. Adding $\|u(\tau)\|_{U}^{2} d \tau$ to both sides and adding and subtracting the term

$$
2\left\langle D^{\star} P(\tau) D u(\tau), u(\tau)\right\rangle d \tau+2\left\langle D^{\star} P(\tau) C y_{n}(\tau), u(\tau)\right\rangle d \tau
$$

to the right-hand side, we get

$$
\begin{aligned}
& \|u(\tau)\|_{U}^{2} d \tau+d\left\langle P(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle \\
& = \\
& \quad-\left\|R y_{n}(\tau)\right\|_{Z}^{2} d \tau-\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau \\
& \quad+\left\|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)\right\|_{U}^{2} d \tau \\
& \quad+2\left\langle\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau+2\left\langle\left(D_{n}^{\star} P(\tau) C_{n}-D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau \\
& \quad+\left\langle\left(I+D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau+\left\langle\left(I+D_{n}^{\star} P(\tau) D_{n}-D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau \\
& \quad+2\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau} .
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
& \|u(\tau)\|_{U}^{2} d s+d\left\langle P(\tau) y_{n}(\tau), y_{n}(\tau)\right\rangle \\
& =-\left\|R y_{n}(\tau)\right\|_{Z}^{2} d \tau-\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau \\
& \quad+\left\|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)-\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u\right\|_{U}^{2} d \tau \\
& \quad+2\left\langle\left(D_{n}^{\star} P(\tau) C_{n}-D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau \\
& \quad+\left\langle\left(I+D_{n}^{\star} P(\tau) D_{n}-D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau \\
& \quad+2\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau} .
\end{aligned}
$$

Integrating from $t$ to $T$ and using the condition $P(T)=G^{\star} G$ and $y_{n}(t)=x$, we have

$$
\begin{aligned}
& \int_{t}^{T}\|u(\tau)\|_{U}^{2} d \tau+\int_{t}^{T}\left\|R y_{n}(\tau)\right\|_{W}^{2} d \tau+\left\|G y_{n}(T)\right\|_{Z}^{2}=\langle P(t) x, x\rangle \\
& \quad-\int_{t}^{T}\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau \\
& \quad+\int_{t}^{T}\left\|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)-\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u\right\|_{U}^{2} d \tau \\
& \quad+2 \int_{t}^{T}\left\langle\left(D_{n}^{\star} P(\tau) C_{n}-D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau \\
& \quad+\int_{t}^{T}\left\langle\left(I+D_{n}^{\star} P(\tau) D_{n}-D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau \\
& \quad+2 \int_{t}^{T}\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle d W_{\tau}
\end{aligned}
$$

Since $\left\langle P(\tau) y_{n}(\tau), C_{n} y_{n}(\tau)+D_{n} u(\tau)\right\rangle$ is not $L^{2}\left(\Omega ; L^{2}([0, T], \mathbb{R})\right)$, we cannot simply apply the expected value to the equation above. However, we appeal to Proposition 7.10 in [D2], from which it suffices that all the integrands are $L^{1}\left(\Omega ; L^{1}([0, T], \mathbb{R})\right)$ to conclude that $\left\langle P(T) y_{n}(T), y_{n}(T)\right\rangle$ or $\left\|G y_{n}(T)\right\|_{Z}^{2}$ is $L^{1}(\Omega ; \mathbb{R})$ which means $\mathbb{E}\left(\left\|G y_{n}(T)\right\|_{Z}^{2}\right)$ $<\infty$ and that the expected value is

$$
\begin{aligned}
& \mathbb{E}\left(\left\|G y_{n}(T)\right\|_{Z}^{2}\right)=\langle P(t) x, x\rangle-\mathbb{E}\left(\int_{t}^{T}\|u(\tau)\|_{U}^{2} d \tau\right) \\
&- \mathbb{E}\left(\int_{t}^{T}\left\|R y_{n}(\tau)\right\|_{W}^{2} d \tau-\int_{t}^{T}\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau\right) \\
&+ \mathbb{E}\left(\int_{t}^{T} \|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)\right. \\
&\left.\quad-\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u \|_{U}^{2} d \tau\right) \\
&+ 2 \mathbb{E}\left(\int_{t}^{T}\left\langle\left(D_{n}^{\star} P(\tau) C_{n}-D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau\right) \\
&+ \mathbb{E}\left(\int_{t}^{T}\left\langle\left(I+D_{n}^{\star} P(\tau) D_{n}-D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau\right)
\end{aligned}
$$

Rearranging, we have

$$
\begin{align*}
J_{n} \equiv & J(t, x, u)=\langle P(t) x, x\rangle-\mathbb{E}\left(\int_{t}^{T}\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau\right)  \tag{6.4}\\
+ & \mathbb{E}\left(\int_{t}^{T} \|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau)\right. \\
& \left.\quad-\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u \|_{U}^{2} d \tau\right) \\
+ & 2 \mathbb{E}\left(\int_{t}^{T}\left\langle\left(D_{n}^{\star} P(\tau) C_{n}-D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau\right) \\
+ & \mathbb{E}\left(\int_{t}^{T}\left\langle\left(D_{n}^{\star} P(\tau) D_{n}-D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau\right) .
\end{align*}
$$

We next must show that $y_{n} \rightarrow y \in M_{\omega}^{2}([s, T] ; H)$ while the second and the last two terms in (6.4) go to zero as $n \rightarrow \infty$.

Estimating the norm of the difference $\mathbb{E}\left(\left\|y_{n}-y\right\|_{H}^{2}\right)$ we have

$$
\begin{aligned}
\mathbb{E}\left(\left\|y_{n}(t)-y(t)\right\|_{H}^{2}\right) \leq & c \mathbb{E}\left(\int_{s}^{t}\left\|e^{A(t-\tau)}\left(C_{n} y_{n}-C y\right)\right\|_{H} d W_{\tau}\right)^{2} \\
& +\mathbb{E}\left(\int_{s}^{t}\left\|e^{A(t-\tau)}\left(D_{n} u-D u\right)\right\|_{H} d W_{\tau}\right)^{2} \\
\leq & c \int_{s}^{t}\left\|C_{n}-C\right\|_{\mathcal{L}(H)}^{2} \mathbb{E}\left(\|y\|_{H}^{2}\right) d \tau+c \int_{s}^{t}\left\|C_{n}\right\|_{\mathcal{L}(H)}^{2} \mathbb{E}\left(\left\|y_{n}-y\right\|_{H}^{2}\right) d \tau \\
& +c \int_{s}^{t}\left\|D_{n}-D\right\|_{\mathcal{L}(U ; H)}^{2} \mathbb{E}\left(\|u\|_{U}^{2}\right) d \tau .
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$
\begin{aligned}
& \mathbb{E}\left(\left\|y_{n}(t)-y(t)\right\|_{H}^{2}\right) \leq c\left(\left\|C_{n}-C\right\|_{\mathcal{L}(H)}^{2}\|y\|_{M_{w}^{2}([s, T] ; H)}^{2}+\| D_{n}\right. \\
&\left.-D\left\|_{\mathcal{L}(H)}^{2}\right\| u \|_{M_{w}^{2}([s, T] ; U)}^{2}\right)\left\|C_{n}\right\|_{\mathcal{L}(H)}^{2} t .
\end{aligned}
$$

Integrating in time and noting that the sequence $C_{n}$ is uniformly bounded by a constant $M$ in the norm (since $C_{n} \rightarrow C$ ), then choosing $n$ sufficiently large, we finally get

$$
\int_{s}^{T} \mathbb{E}\left(\left\|y_{n}(t)-y(t)\right\|_{H}^{2}\right) d t \leq\left(c \epsilon\|y\|_{M_{w}^{2}([s, T] ; H)}^{2}+\epsilon\|u\|_{M_{w}^{2}([s, T] ; U)}^{2}\right) M \frac{T^{2}}{2} .
$$

This shows that $y_{n} \rightarrow y$ in $M_{w}^{2}([s, T] ; H)$.
Using standard arguments we can easily show that

$$
\mathbb{E}\left(\int_{t}^{T}\left\langle\left(C^{\star} P(\tau) C-C_{n}^{\star} P(\tau) C_{n}\right) y_{n}(\tau), y_{n}(\tau)\right\rangle d \tau\right) \rightarrow 0 \text { as } n \rightarrow \infty
$$

Similarly,
and

$$
2 \mathbb{E}\left(\int_{t}^{T}\left\langle\left(D_{n}^{\star} P(\tau) C_{n}-D^{\star} P(\tau) C\right) y_{n}(\tau), u(\tau)\right\rangle d \tau\right) \rightarrow 0
$$

$$
\mathbb{E}\left(\int_{t}^{T}\left\langle\left(D_{n}^{\star} P(\tau) D_{n}-D^{\star} P(\tau) D\right) u(\tau), u(\tau)\right\rangle d \tau\right) \rightarrow 0
$$

as $n \rightarrow \infty$.
As for the second term in (6.4), we have

$$
\begin{aligned}
& \mathbb{E}\left(\int_{t}^{T}\right. \|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y_{n}(\tau) \\
&\left.-\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u \|_{U}^{2} d \tau\right) \\
& \rightarrow \mathbb{E}\left(\int_{t}^{T} \|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y(\tau)\right. \\
&\left.\quad\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u \|_{U}^{2} d \tau\right)
\end{aligned}
$$

Therefore, the functional $J_{n}$ given in (6.4) converges to

$$
\begin{aligned}
J(t, x, u)=\langle P(t) x, x\rangle+\mathbb{E}\left(\int_{t}^{T}\right. & \|\left(I+D^{\star} P(\tau) D\right)^{-1 / 2}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) y(\tau) \\
& \left.-\left(I+D^{\star} P(\tau) D\right)^{1 / 2} u \|_{U}^{2} d \tau\right)
\end{aligned}
$$

We finally extend (6.1) for all $u \in M_{w}^{2}([s, T] ; U)$. By density of $L^{2}\left(\Omega ; H^{1}([s, T] ; U)\right) \subset$ $M_{w}^{2}([s, T] ; U)$, we then approximate $u \in M_{w}^{2}([s, T] ; U)$ by a sequence $u_{n} \in L^{2}\left(\Omega ; H^{1}\right.$ $([s, T] ; U)$ ), and pass through the limit. It is easy to show show that $y\left(u_{n}\right) \rightarrow y(u)$ in $M_{w}^{2}([s, T] ; H)$ (continuous dependence of $y$ on the control $u$ ). Hence, passing through the limit in $u_{n} \rightarrow u$, we have $y_{n} \rightarrow y(u)$ and (6.1) is valid for $u \in M_{w}^{2}([s, T] ; U)$. Since the argument in passing through the limit in $J$ is similar, it will not be repeated.
7. A global-in-time solution to the differential Riccati equation. We now extend the solution of the Riccati equation from $\left[T_{\max }, T\right.$ ] to any time interval $[s, T]$. We establish a global bound on $P(t)$ since

$$
\begin{aligned}
\langle P(t) x, x\rangle & \leq J(t, x ; u=0)=\mathbb{E}\left(\int_{t}^{T}\|R y(\tau)\|^{2} d \tau+\|G y(T)\|_{Z}^{2}\right) \\
& \leq c M^{2} T e^{2 \alpha T}\|x\|_{H}^{2}+c M^{2} e^{2 \alpha T}\|x\|_{H}^{2}=C_{T}\|x\|_{H}^{2}
\end{aligned}
$$

for all $t \in\left[T_{\max }, T\right]$ and thus $\|P(t)\|_{\mathcal{L}(H)} \leq\left\|P^{1 / 2}(t)\right\|_{\mathcal{L}(H)}^{2} \leq C_{T}$. This bound can be used to reiterate the proofs of Lemma 4.4 and Theorem 4.1 on a new interval [ $T_{1}, T_{\max }$ ] with $G=P^{1 / 2}\left(T_{\max }\right)$. The bound insures that the choice of the constant $c$ (which depends on $G$ ) in (4.8) is global and all the estimates are uniform and that $r$ and the time step $T_{\max }-T_{1}$ are the same. Hence, the results can be extended by repeated iteration on equal time steps to any initial time $s \geq 0$.

## 8. Uniqueness of solution to the differential Riccati equation.

TheOrem 8.1. The solution to the differential Riccati equation is unique in the class of self-adjoint operators in $C([0, T] ; \mathcal{L}(H))$ satisfying $B^{\star} P \in C_{\gamma}([s, T] ; \mathcal{L}(H, U))$.

Proof. Assume there is another solution $\widetilde{P}(t)$ to the Riccati equation in this class, then the same dynamic programming argument from the previous section leads to

$$
\min J(t, x, u)=\langle P(t) x, x\rangle=\langle\widetilde{P}(t) x, x\rangle
$$

for all $x \in H$. Hence, we have for any $x, y \in H$ that

$$
\begin{aligned}
0 & =\langle(P(t)-\widetilde{P}(t))(x+y),(x+y)\rangle \\
& =\langle(P(t)-\widetilde{P}(t)) x, x\rangle+\langle(P(t)-\widetilde{P}(t)) x, y\rangle+\langle(P(t)-\widetilde{P}(t)) y, x\rangle+\langle(P(t)-\widetilde{P}(t)) y, y\rangle \\
& =2\langle(P(t)-\widetilde{P}(t)) x, y\rangle
\end{aligned}
$$

by self-adjointness of $P$ and $\widetilde{P}$. Thus, $P(t)=\widetilde{P}(t)$.
9. Proof of main Theorems 2.1 and 2.2. We finally obtain our main results in this paper stated in Theorems 2.1 and 2.2. We start with Theorem 2.1.

Proof.
(i) From (6.1) in Lemma 6.1, the functional $J$ satisfies

$$
\inf _{u \in M_{\omega}([s, T] ; U)} J(s, x ; u)=\langle P(s) x, x\rangle,
$$

where $P(t)$ is the solution to the differential Riccati equation.
(ii) The existence of a solution to the differential Riccati equation in $C([s, T] ; \mathcal{L}(H))$ follows from Theorem 5.1, and the uniqueness was established in section 8.
(iii), (iv) The regularity properties of $P(t)$ and $B^{*} P(t)$ were established in Theorem 4.1.
Finally, we prove Theorem 2.2.
Proof.
(i), (iii) To show that the minimum of $J$ is realized in (6.1), we can establish the existence of a unique solution $u^{0} \in M_{w}^{2}([s, T] ; U)$ to the equation

$$
u^{0}(t, s ; x)=-\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) y\left(t, s, u^{0} ; x\right)
$$

via a fixed point argument on $M_{w}^{2}([s, T] ; U)$. Thus,

$$
u^{0}(s, t ; x)=-\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) y^{0}(t, s ; x)
$$

so that $J\left(s, x ; u^{0}\right)=\langle P(s) x, x\rangle$.
(ii) If follows from Theorem 3.1 that the corresponding optimal state $y^{0} \in L^{2}$ $(\Omega ; C([s, T] ; H))$.
(iv) It then follows by regularity properties of $B^{\star} P$ in (4.3) that

$$
\left\|u^{0}(t, s ; x)\right\|_{L^{2}(\Omega ; U)} \leq \frac{c}{(T-t)^{\gamma}}\|x\|_{H}
$$

10. Applications to control of PDEs. This section is devoted to an application of the theory to concrete PDE systems with unbounded control actions.
10.1. Theremoelastic plates with boundary control. We consider a stochastic model for a hinged thermoelastic plate with Neumann thermal boundary control. Let $W_{t}$ be a one dimensional Wiener process on a complete probability space $(\Sigma, \mathcal{F}, \mathcal{P})$. The system consists of a heat equation and a plate equation

$$
\left.\begin{array}{rl}
{[I-\rho \Delta] d w_{t}+\Delta^{2} w d t+\Delta \theta d t=\left(\nabla w+b w_{t}\right) d W_{t},} & \Omega \times[0, T],  \tag{10.1}\\
d \theta-\Delta \theta d t-\Delta w_{t} d t=\left(C_{31} \Delta w+C_{32} \nabla w_{t}+C_{33} \theta\right) d W_{t}, & \Omega \times[0, T],
\end{array}\right\}
$$

where $w(\omega, x, t)$ is the transversal displacement and $\theta(\omega, x, t)$ is the temperature of the plate which occupies the open domain $\Omega$ in $\mathbb{R}^{2}$ or $\mathbb{R}^{3}$, subject to the hinged boundary conditions

$$
\begin{equation*}
w=\Delta w=0, \quad \partial \Omega \times[0, T] \tag{10.2}
\end{equation*}
$$

and thermal control $u$ on the boundary

$$
\begin{equation*}
\frac{\partial \theta}{\partial \nu}+b \theta=u(x, t)+u(x, t) \dot{W}(t), \quad \partial \Omega \times[0, T] \tag{10.3}
\end{equation*}
$$

The functions $y(\omega, x, t) \equiv\left(w(\omega, x, t), w_{t}(\omega, x, t), \theta(\omega, x, t)\right)$ are random variables which take values in the finite energy space $\mathcal{H}$ defined by $\mathcal{H} \equiv H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \times$ $L^{2}(\Omega)$.

We are particularly interested in a Bolza-type optimal control of this system with the objective of minimizing an energy functional

$$
\begin{align*}
J\left(u, w, w_{t}, \theta\right)=\mathbb{E}( & \int_{0}^{T}\|u(\cdot, t)\|_{L^{2}(\partial \Omega)}^{2}+\|w(\cdot, t)\|_{H^{2}(\Omega)}^{2}+\left\|w_{t}(\cdot, t)\right\|_{H^{1}(\Omega)}^{2} \\
& \left.+\|\theta(\cdot, t)\|_{L^{2}(\Omega)}^{2} d t+\|w(\cdot, T)\|_{H^{2}(\Omega)}^{2}+\left\|w_{t}(\cdot, T)\right\|_{H^{1}(\Omega)}^{2}\right) \tag{10.4}
\end{align*}
$$

over all boundary controls $u \in M_{w}^{2}\left([0, T] ; L^{2}(\partial \Omega)\right)$, given initial data in the $\left(w_{0}, w_{1}, \theta_{0}\right)$ $\in \mathcal{H}$ finite energy space. This problem can be adapted to the abstract setting of the stochastic linear quadratic regulator, since the deterministic uncontrolled system is driven by a $C_{0}$-semigroup $e^{A t}$ while a control operator $B$ from the boundary to interior satisfies the singular estimate [BL].

Following [LT1, BL], we introduce the self-adjoint operator $\mathcal{A}$ on $L^{2}(\Omega)$ defined by

$$
\mathcal{A} h=\Delta^{2} h
$$

with domain

$$
\mathcal{D}(\mathcal{A})=\left\{h \in H^{4}(\Omega):\left.h\right|_{\partial \Omega}=\left.\frac{\partial}{\partial \nu} h\right|_{\partial \Omega}=0\right\}
$$

The fractional power $\mathcal{A}^{1 / 2}$ of this operator has a domain which can be identified with the space $H^{2}(\Omega) \times H_{0}^{1}(\Omega)$. We also introduce the self-adjoint operator $A_{N}$ on $L^{2}(\Omega)$

$$
A_{N} h=-\Delta h
$$

with domain

$$
\mathcal{D}\left(A_{N}\right)=\left\{h \in H^{2}(\Omega): \frac{\partial}{\partial \nu} h+h=0 \text { on } \partial \Omega\right\} .
$$

The operator $-A_{N}$ is well known to generate an analytic semigroup $e^{-A_{N} t}$ on the space $L^{2}(\Omega)$.

We also follow [BL] in introducing the operator $\mathcal{M}$ on $L^{2}(\Omega)$ given by

$$
\mathcal{M}=\left(I+\rho A_{N}\right)
$$

with the well-defined bounded inverse $\mathcal{M}^{-1}$. Additionally, we also introduce the Neumann map $N: L^{2}(\partial \Omega) \rightarrow L^{2}(\Omega)$ defined by

$$
\begin{array}{r}
N g=h \quad \Longleftrightarrow \quad \Delta h=0 \text { in } \Omega \\
\\
\frac{\partial h}{\partial \nu}+h=g \text { on } \partial \Omega
\end{array}
$$

It is well known that $A^{3 / 4-\epsilon} N$ is bounded $L^{2}(\partial \Omega) \rightarrow L^{2}(\Omega)$. The system can then be expressed in abstract form as

$$
d y(t)=(A y+B u) d t+(C y+D u) d W_{t},
$$

where

$$
y(t)=\left(\begin{array}{c}
w \\
w_{t} \\
\theta
\end{array}\right)
$$

and

$$
A=\left(\begin{array}{ccc}
0 & I & 0 \\
-\mathcal{M}^{-1} \mathcal{A} & 0 & \mathcal{M}^{-1} A_{N} \\
0 & -A_{N} & -A_{N}
\end{array}\right)
$$

and with domain

$$
\mathcal{D}(A)=\mathcal{D}\left(\mathcal{A}^{3 / 4}\right) \times \mathcal{D}\left(\mathcal{A}^{1 / 2}\right) \times \mathcal{D}\left(A_{N}\right)
$$

Moreover, the control operators $B, D$ are

$$
B=D=\left(\begin{array}{c}
0 \\
0 \\
A_{N} N
\end{array}\right)
$$

and the noise operator $C$ is

$$
C=\left(\begin{array}{ccc}
0 & 0 & 0 \\
\nabla & b & 0 \\
C_{31} \Delta & C_{32} \nabla & C_{33}
\end{array}\right)
$$

for real parameters $C_{31}, C_{32}, C_{33}, b$. Note here that the adjoint $B^{\star}: \mathcal{D}\left(A^{\star}\right) \rightarrow L^{2}(\partial \Omega)$ is defined by

$$
B^{\star}\left[x_{1}, x_{2}, x_{3}\right]=N^{\star} A_{N} x_{3}=\left.x_{3}\right|_{\partial \Omega}
$$

which is the restriction to the boundary $\partial \Omega$. As for the observation operators in (10.4), we take $R=I$ and $G=[I, I, 0]$ on the state space $\mathcal{H}$.

It was shown in [BL] that the set of assumptions in Assumption 1.1 are indeed satisfied. In particular, the critical singular estimate does hold with any $\gamma>1 / 4$ :

$$
\left\|e^{A t} B u\right\|_{\mathcal{H}} \leq \frac{C}{t^{1 / 4+\epsilon}}\|u\|_{L^{2}(\partial \Omega)}
$$

for every $u \in L^{2}(\partial \Omega)$, and $A^{-1} B$ is bounded from $L^{2}(\partial \Omega)$ to $\mathcal{H}$. Thus, we are in a position to apply the conclusions of Theorems 2.1 and 2.2. Thus we have the following theorem.

Theorem 10.1. Given initial data $\left(\theta_{0}, w_{0}, w_{1}\right) \in \mathcal{H}$, there exists a unique optimal control $u^{0} \in M_{w}^{2}\left([s, T] ; L^{2}(\partial \Omega)\right)$ to the stochastic thermoelastic plate system (10.1) with hinged boundary conditions (10.2) and Neumann thermal boundary control (10.3), which minimizes the cost functional (10.4). Moreover,

1. the optimal control $u^{0} \in C\left([s, T) ; L^{2}(\Sigma, \partial \Omega)\right)$ and

$$
\mathbb{E}\left(\left\|u^{0}(t)\right\|_{L^{2}(\partial \Omega)}^{2}\right) \leq\left(\frac{c}{t^{1 / 4+\epsilon}}\left(\left\|w_{0}\right\|_{H^{2}(\Omega)}+\left\|w_{1}\right\|_{H^{1}(\Omega)}+\left\|\theta_{0}\right\|_{L^{2}(\Omega)}\right)\right)^{2}
$$

2. the corresponding optimal state $\left(\theta^{0}(t), w^{0}(t), w_{t}^{0}(t)\right) \in C\left([s, T] ; L^{2}(\Sigma, \mathcal{H})\right)$ and

$$
\begin{aligned}
& \mathbb{E}\left(\left\|w^{0}(t)\right\|_{H^{2}(\Omega)}^{2}\right)+\mathbb{E}\left(\left\|w_{t}^{0}(t)\right\|_{H^{1}(\Omega)}^{2}\right)+\mathbb{E}\left(\left\|\theta^{0}(t)\right\|_{L^{2}(\Omega)}^{2}\right) \\
& \quad \leq c\left(\left\|w_{0}\right\|_{H^{2}(\Omega)}^{2}+\left\|w_{1}\right\|_{H^{1}(\Omega)}^{2}+\left\|\theta_{0}\right\|_{L^{2}(\Omega)}^{2}\right)
\end{aligned}
$$

3. the optimal control is given in feedback form

$$
u^{0}(t)=-\left(I+D^{\star} P(t) D\right)^{-1}\left(D^{\star} P(t) C+B^{\star} P(t)\right)\left[w^{0}(t), w_{t}^{0}(t), \theta^{0}(t)\right]^{T}
$$

for $B, D$ and $C$ defined above and where $P(t)$ is a self-adjoint positive operator on $\mathcal{H}$ satisfying the differential Riccati equation

$$
\begin{aligned}
& \left\langle\mathcal{A}^{1 / 2} p_{1 t}, \mathcal{A}^{1 / 2} y_{1}\right\rangle+\left\langle\mathcal{M}^{1 / 2} p_{2 t}, \mathcal{M}^{1 / 2} y_{2}\right\rangle+\left\langle p_{3 t}, y_{3}\right\rangle=-\left\langle\mathcal{A}^{1 / 2} p_{1}, \mathcal{A}^{1 / 2} y_{2}\right\rangle \\
& \quad+\left\langle p_{2}, \mathcal{A} y_{1}\right\rangle-\left\langle p_{2}, A_{N} y_{3}\right\rangle+\left\langle A_{N} p_{3}, y_{2}\right\rangle+\left\langle A_{N} p_{3}, y_{3}\right\rangle-\left\langle\mathcal{A}^{1 / 2} x_{2}, \mathcal{A}^{1 / 2} \hat{p}_{1}\right\rangle \\
& \quad+\left\langle\mathcal{A} x_{1}, \hat{p}_{2}\right\rangle-\left\langle A_{N} x_{3}, \hat{p}_{2}\right\rangle+\left\langle A_{N} x_{2}, \hat{p}_{3}\right\rangle+\left\langle A_{N} x_{3}, \hat{p}_{3}\right\rangle-\langle P(t) C x, C y\rangle \\
& \quad+\left\langle\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P C\right) x,\left(B^{\star} P(t)\right.\right. \\
& \left.\left.\quad+D^{\star} P C\right) y\right\rangle_{\partial \Omega},\left[p_{1}(T), p_{2}(T), p_{3}(T)\right] \\
& =\left[x_{1}, x_{2}, 0\right]
\end{aligned}
$$

for all $x=\left(x_{1}, x_{2}, x_{3}\right)$ and $y=\left(y_{1}, y_{2}, y_{3}\right)$ in $\mathcal{D}(A)$, where we denote $P(t) x=$ $\left[p_{1}(t), p_{2}(t), p_{3}(t)\right]$ and $P(t) y=\left[\hat{p}_{1}(t), \hat{p}_{2}(t), \hat{p}_{3}(t)\right]$, and by $\langle\cdot, \cdot\rangle$ the $L^{2}$ inner product on $\Omega$.
10.2. Fluid-structure interaction. Here we shall revisit the motivating example introduced in section 1.1. In particular, the system (1.4)-(1.6) with boundary conditions (1.7)-(1.9) can be expressed in the abstract form

$$
d Y=\mathcal{A}_{F S} Y d t+B g d t+C Y d W_{t}+D g d W_{t}
$$

with

$$
\mathcal{A}_{F S}=\left(\begin{array}{ccc}
A_{N} & A_{N} N \sigma & 0 \\
0 & 0 & I \\
0 & \operatorname{div}(\sigma) & 0
\end{array}\right)
$$

where $A_{N}: V \rightarrow V^{\prime}$ is defined by $\left\langle A_{N} \phi, v^{\prime}\right\rangle=-\langle\epsilon(\phi), \epsilon(v)\rangle$ and $V$ is the space

$$
V \equiv\left\{v \in H^{1}\left(\Omega_{f}\right): \operatorname{div} v=0,\left.v\right|_{\Gamma_{f}}=0\right\}
$$

while $N: H^{-1 / 2}\left(\Gamma_{s}\right) \rightarrow V$ is the map defined by

$$
\begin{gathered}
N g=h \Longleftrightarrow\left\langle A_{N} h, v\right\rangle=\langle g, v\rangle_{\Gamma_{s}}, \\
\left.h\right|_{\Gamma_{f}}=0,
\end{gathered}
$$

for every $v \in V$ which is well defined by the Lax-Milgram theorem [LTu3]. Denoting the finite energy space $H \times H^{1}\left(\Omega_{s}\right) \times L^{2}\left(\Omega_{s}\right)$ by $\mathcal{H}$, the operator $\mathcal{A}_{F S}$ generates a $C_{0}$-semigroup on the space $\mathcal{H}$. The control operators $B$ and $D$ are defined by

$$
B=D=\left(\begin{array}{c}
A_{N} N \\
0 \\
0
\end{array}\right)
$$

and $B: L^{2}\left(\Gamma_{s}\right) \rightarrow\left[\mathcal{D}\left(\mathcal{A}_{F S}^{\star}\right)\right]^{\prime}$ is the control operator [LTu3] which satisfies an incrementally weaker form of the singular estimate [LTu3]

$$
\left\|e^{\mathcal{A}_{F S} t} B f\right\|_{\mathcal{H}^{-\alpha}} \leq \frac{c}{t^{1 / 4+\epsilon}}\|f\|_{L^{2}\left(\Gamma_{s}\right)}
$$

for $\alpha>0$, where $\mathcal{H}^{\alpha}$ is the lower topology space $\mathcal{H}^{\alpha}=H \times H^{1-\alpha}\left(\Omega_{s}\right) \times H^{-\alpha}\left(\Omega_{s}\right)$.
However, this estimate is sufficient in order to address the control functional (1.10) with $\alpha=1$; cf. [LTu3]. In particular, we take our operator $R=[I, 0,0]$ and $G=[I, I, 0]$ and take the observation space $W \equiv \mathcal{H}$ and $Z \equiv \mathcal{H}^{-1}$. Moreover, we determine the noise operator $C$ as

$$
C=\left(\begin{array}{ccc}
c_{1} & 0 & 0  \tag{10.5}\\
0 & 0 & 0 \\
0 & c_{2} & 0
\end{array}\right)
$$

which is a bounded operator on the state space $\mathcal{H}$. Note here that the adjoint $B^{\star}$ : $\mathcal{D}\left(A^{\star}\right) \rightarrow L^{2}\left(\Gamma_{s}\right)$ is defined by

$$
B^{\star}\left[x_{1}, x_{2}, x_{3}\right]=N^{\star} A_{N} x_{1}=\left.x_{1}\right|_{\Gamma_{s}} .
$$

Now that the assumptions of Assumption 1.1 are all satisfied by the system, we can specialize Theorems 2.1 and 2.2 to this system to obtain the following optimal control result.

Theorem 10.2. Given initial data $\left(u_{0}, w_{0}, w_{1}\right) \in \mathcal{H}$, there exists a unique optimal control $g^{0} \in M_{w}^{2}\left([s, T] ; L^{2}\left(\Gamma_{s}\right)\right)$ to the stochastic fluid-structure interaction system (1.4)-(1.6) with boundary conditions (1.7)-(1.9), which minimizes the cost functional (1.10). Moreover,

1. the optimal control $g^{0} \in C\left([s, T) ; L^{2}\left(\Sigma, \Gamma_{s}\right)\right)$ and

$$
\mathbb{E}\left(\left\|g^{0}(t)\right\|_{L^{2}\left(\Gamma_{s}\right)}^{2}\right) \leq\left(\frac{c}{t^{1 / 4+\epsilon}}\left(\left\|u_{0}\right\|_{L^{2}\left(\Omega_{f}\right)}+\left\|w_{0}\right\|_{L^{2}\left(\Omega_{s}\right)}+\left\|w_{1}\right\|_{L^{2}\left(\Omega_{s}\right)}\right)\right)^{2}
$$

2. the corresponding optimal state $\left(\theta^{0}(t), w^{0}(t), w_{t}^{0}(t)\right) \in L^{2}\left([s, T] ; L^{2}(\Sigma, \mathcal{H})\right) \cap$ $C\left([s, T] ; L^{2}\left(\Sigma, \mathcal{H}_{-1}\right)\right)$ and

$$
\begin{aligned}
& \mathbb{E}\left(\left\|u^{0}(t)\right\|_{L^{2}\left(\Omega_{f}\right)}^{2}\right)+\mathbb{E}\left(\left\|w^{0}(t)\right\|_{L^{2}\left(\Omega_{s}\right)}^{2}\right)+\mathbb{E}\left(\left\|w_{t}^{0}(t)\right\|_{H^{-1}\left(\Omega_{s}\right)}^{2}\right) \\
& \quad \leq c\left(\left\|u_{0}\right\|_{L^{2}\left(\Omega_{f}\right)}^{2}+\left\|w_{0}\right\|_{H^{1}\left(\Omega_{s}\right)}^{2}+\left\|w_{1}\right\|_{L^{2}\left(\Omega_{s}\right)}^{2}\right)
\end{aligned}
$$

3. The optimal control is given in feedback form

$$
g^{0}(t)=-\left(I+D^{\star} P(t) D\right)^{-1}\left(D^{\star} P(t) C+B^{\star} P(t)\right)\left[u^{0}(t), w^{0}(t), w_{t}^{0}(t)\right]^{T}
$$

for $B, D$, and $C$ defined above and where $P(t)$ is a self-adjoint positive operator on $\mathcal{H}$ satisfying the differential Riccati equation
$\left\langle p_{1 t}, y_{1}\right\rangle_{f}+\left\langle\nabla p_{2 t}, \nabla y_{2}\right\rangle_{s}+\left\langle p_{3 t}, y_{3}\right\rangle_{s}=-\left\langle A_{N} x_{1}, \hat{p}_{1}\right\rangle_{f}-\left\langle A_{N} N \sigma\left(x_{2}\right), \hat{p}_{1}\right\rangle_{f}$
$-\left\langle\nabla x_{3}, \nabla \hat{p}_{2}\right\rangle_{s}-\left\langle\operatorname{div} \sigma\left(x_{2}\right), \hat{p}_{3}\right\rangle_{s}$
$-\left\langle p_{1}, A_{N} y_{1}\right\rangle_{f}-\left\langle p_{1}, A_{N} N \sigma\left(y_{2}\right)\right\rangle_{f}-\left\langle\nabla p_{2}, \nabla y_{3}\right\rangle_{s}$
$-\left\langle p_{3}, \operatorname{div} \sigma\left(y_{2}\right)\right\rangle_{s}-\left\langle c_{1} p_{1}, c_{1} y_{1}\right\rangle_{f}-\left\langle c_{2} p_{3}, c_{2} y_{2}\right\rangle_{s}-\left\langle x_{1}, y_{1}\right\rangle_{f}$
$+\left\langle\left.\left(I+D^{\star} P(t) D\right)^{-1}\left(1+c_{1}\right) p_{1}\right|_{\Gamma_{s}},\left.\left(1+c_{1}\right) \hat{p}_{1}\right|_{\Gamma_{s}}\right\rangle_{\Gamma_{s}}$,
$\left[p_{1}(T), p_{2}(T), p_{3}(T)\right]=\left[x_{1}, x_{2}, 0\right]$,
for every $x=\left(x_{1}, x_{2}, x_{3}\right)$ and $y=\left(y_{1}, y_{2}, y_{3}\right) \in \mathcal{D}\left(\mathcal{A}_{F S}\right)$, where $P(t) x=$
$\left[p_{1}(t), p_{2}(t), p_{3}(t)\right]$ and $P(t) y=\left[\hat{p}_{1}(t), \hat{p}_{2}(t), \hat{p}_{3}(t)\right]$, while $\langle\cdot, \cdot\rangle_{f}$ and $\langle\cdot, \cdot\rangle_{s}$ de-
note the $L^{2}$ inner product on $\Omega_{f}$ and $\Omega_{s}$, respectively.

Remark 10.3. The proof of this theorem requires extending the results of Theorems 2.1 and 2.2 to a generalized singular estimate condition on the observation spaces $\left\|R e^{A t} B f\right\|_{W} \leq \frac{c}{t^{\gamma}}\|f\|_{U}$ and $\left\|G e^{A t} B f\right\|_{Z} \leq \frac{c}{t^{\gamma}}\|f\|_{U} \forall f \in U$ for some $\gamma \in(0,1 / 2)$; cf. [Tu]. This leads to the continuity-in-time property to be satisfied by the observed optimal state space $R y^{0}$ only on the observation space $W$ as stated in part 2 of the above theorem.

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# A Numerical Approximation Framework for the Stochastic Linear Quadratic Regulator on Hilbert Spaces 

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Published online: 12 March 2016
© Springer Science+Business Media New York 2016


#### Abstract

We present an approximation framework for computing the solution of the stochastic linear quadratic control problem on Hilbert spaces. We focus on the finite horizon case and the related differential Riccati equations (DREs). Our approximation framework is concerned with the so-called "singular estimate control systems" (Lasiecka in Optimal control problems and Riccati equations for systems with unbounded controls and partially analytic generators: applications to boundary and point control problems, 2004) which model certain coupled systems of parabolic/hyperbolic mixed partial differential equations with boundary or point control. We prove that the solutions of the approximate finite-dimensional DREs converge to the solution of the infinite-dimensional DRE. In addition, we prove that the optimal state and control of the approximate finite-dimensional problem converge to the optimal state and control of the corresponding infinite-dimensional problem.


Keywords Stochastic linear quadratic regulator problems • Feedback control • Approximation schemes • Riccati equations

[^8]
## 1 Introduction

The deterministic linear quadratic control problem for infinite-dimensional systems has been extensively studied in the literature [7,8,30,31]. An approximation scheme for Riccati equations in infinite-dimensional spaces have been first proposed by Gibson [16], who developed an approximation framework in order to reduce inherently infinite-dimensional problems to finite-dimensional ones using Riccati integral equations. In [16] the deterministic problems with bounded control and observation operators were considered. The result proposed by Gibson requires the approximating problems to be defined on the entire original state space which leads to some technical difficulties. Assuming that the dynamics are driven by an analytic semigroup, Banks and Kunisch [3] propose an alternative framework which is more amenable to numerical implementations. In the same setting, convergence results for DREs can be found in [5], while results on convergence rates can be found in [26]. A complete Riccati theory and convergence analysis for infinite dimensional systems driven by analytic semigroups and a special class of unbounded control operators was developed by Lasiecka and Triggiani in [30]. However, up to our knowledge, convergence results for the stochastic linear quadratic control problem have not been studied in the literature. One of the reasons could be the fact that the computational cost of solving the stochastic linear quadratic regulator (LQR) problem is much higher compared to the cost in the deterministic case. In this paper, we extend the ideas presented in $[3,5,35]$ to the stochastic linear quadratic control problem.

The stochastic linear quadratic regulator problem in finite dimensions has been first studied by Kushner [27] and Wonham [39,40]. On the other hand, the control problem with stochastic coefficients and the corresponding backward stochastic Riccati equations have been treated in the finite-horizon and finite-dimensional case by many authors [9,10,22-25]. In [14] the authors provide the study of stochastic LQR problems subject to both multiplicative white noise and Markovian jumps in finite dimensions. In [41], one can find a comprehensive treatment of the linear quadratic optimal control problem in finite dimensions along with a feedback characterization of the optimal control via a matrix Riccati equation. We note here that the Riccati equation associated with the stochastic LQR problem is a deterministic differential equation, and thus the feedback relation between the optimal control and the optimal state, is deterministic, even though they are both random.

The infinite dimensional analog of the stochastic linear quadratic problem was solved in [21] using a dynamic programming approach. Da Prato [12] and Flandoli [15] later considered the stochastic LQR for systems driven by analytic semigroups with Dirichlet or Neumann boundary controls, but with disturbance in the state only. The infinite dimensional LQR with random coefficients have also been investigated recently in $[17,18]$ along with the associated backward stochastic Riccati equation.

In [36], a novel approach for solving the stochastic LQR based on the chaos expansion method in the framework of white noise analysis to the state equation was proposed. This numerical framework relies on solving standard deterministic Riccati equations. Efficient solvers for Riccati equations have been proposed in recent years [1,4-6,28].

For a class of control systems known as singular estimate control systems, where the dynamics are driven by strongly continuous semigroups and the kernel of the control-to-state map satisfies a singular estimate, the stochastic analog of the linear quadratic problem has been first treated by Hafizoglu [19]. These control systems capture certain systems of coupled parabolic/hyperbolic PDEs, with boundary or point control actions [29,32], and include systems with analytic dynamics as a special case [30]. Examples of such control systems appear in structure-acoustics, thermoelastic-plates and fluidstructure interactions [2,11,33]. Recently, a theoretical framework for the stochastic LQR has been laid out for singular estimate control systems in the presence of noise in the control and in the case of finite time penalization in the performance index [20]. In this paper, we consider the general setting described in [20] and propose an approximation scheme for solving the control problem and the associated Riccati equation. In particular, our task is two fold: First of all, we provide an approximation framework for the singular estimate control systems at the deterministic level which generalizes some of the results on approximation of analytic dynamics in [30], and second of all, we extend the results from the deterministic case $[3,5,16,30]$ to the stochastic case.

The paper is organized in the following manner: in Sect. 2 we state the stochastic LQR problem, give basic notions and derive the Riccati integral equation in terms of a semigroup. In Sect. 3, we develop a general convergence framework for the stochastic LQR problem. In Sect. 4, we present and prove our main results in Theorem 4.1 and Theorem 4.2. Finally, some applications in stochastic control are considered in Sect. 5.

## 2 The Stochastic LQR Problem

Let $\mathcal{U}$ and $\mathcal{H}$ be separable Hilbert spaces of controls and states respectively with norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{H}}$, generated by the corresponding scalar products. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\left(W_{t}\right)_{t \geq 0}$ a one-dimensional real valued standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ be the complete right continuous $\sigma$-algebra generated by $\left(W_{t}\right)_{t>0}$. We assume that all function spaces are adapted to the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$, i.e. we consider only $\mathcal{F}_{t}$-predictable processes. Let $L^{2}(\Omega)=$ $L^{2}(\Omega, \mathcal{F}, \mathbb{P})$ be a Hilbert space of square integrable real valued random variables endowed with the norm $\|F\|_{L^{2}(\Omega)}^{2}=\mathbb{E}_{\mathbb{P}}\left(F^{2}\right)$, for $F \in L^{2}(\Omega)$, induced by the scalar product $(F, G)_{L^{2}(\Omega)}=\mathbb{E}_{\mathbb{P}}(F G)$, for $F, G \in L^{2}(\Omega)$, and $\mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to the measure $\mathbb{P}$. Further on, we will write $\mathbb{E}$ for the expectation omitting $\mathbb{P}$. We denote by $L^{2}(\Omega, \mathcal{U})$ a Hilbert space of $\mathcal{U}$-valued square integrable random values and by $L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)$, the Hilbert space of square integrable $\mathcal{F}_{T}$-predictable $\mathcal{U}$-valued stochastic processes $u$ endowed with the norm

$$
\|u\|_{L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)}^{2}=\int_{0}^{T} \mathbb{E}\left(\|u(t)\|_{\mathcal{U}}^{2}\right) d t
$$

Let $C\left([0, T], L^{2}(\Omega, \mathcal{H})\right)$ be the Hilbert space of $\mathcal{F}_{T}$-predictable continuous $\mathcal{H}$-valued stochastic processes $y$ endowed with the norm

$$
\|y\|_{C\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)}^{2}=\sup _{t \in[0, T]} \mathbb{E}\left(\|y(t)\|_{\mathcal{H}}^{2}\right)
$$

The infinite dimensional stochastic linear quadratic regulator (SLQR) optimal control problem on Hilbert spaces is given by an Itô stochastic differential equation

$$
\begin{align*}
d y(t) & =(A y(t)+B u(t)) d t+(C y(t)+D u(t)) d W_{t}, \quad t \in[0, T]  \tag{1}\\
y(0) & =y_{0}
\end{align*}
$$

and the quadratic cost functional

$$
\begin{equation*}
J(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|Q y\|_{\mathcal{W}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\|G y(T)\|_{\mathcal{Z}}^{2}\right] \tag{2}
\end{equation*}
$$

The dynamics of the problem, the operator $A$, is deterministic and represents an infinitesimal generator of a (strongly continuous) $C_{0}$-semigroup $e^{A t}$ on the state space $\mathcal{H}$. Operators $A$ and $C$ are operators on $\mathcal{H}$, while operators $B$ and $D$ are operators acting from the control space $\mathcal{U}$ to the state space $\mathcal{H}$. The observation spaces $\mathcal{W}$ and $\mathcal{Z}$ are also Hilbert spaces. We denote by $\mathcal{D}(S)$ the domain of a certain operator $S$, and by $S^{\star}$ the adjoint operator of $S$.

It is clear that $y$ in (1) and (2) is a stochastic process $y(t, \omega)$, i.e. for fixed $t$ it represents a random variable and for fixed $\omega$ we obtain a realization of the process. For simplicity we write $y(t)$.

We consider the stochastic LQR problem under the following conditions:
Assumption 2.1 (a) The linear operator $A$ is an infinitesimal generator of a $C_{0}{ }^{-}$ semigroup $e^{A t}$ on the space $\mathcal{H}$.
(b) The linear control operator $B$ acts from $\mathcal{U} \rightarrow\left[\mathcal{D}\left(A^{\star}\right)\right]^{\prime}$ or equivalently $A^{-1} B$ is bounded from $\mathcal{U} \rightarrow \mathcal{H}$.
(c) There exists a number $\gamma \in(0,1 / 2)$ such that the control to state map kernel $e^{A t} B$ satisfies the singular estimate

$$
\left\|e^{A t} B u\right\|_{\mathcal{H}} \leq \frac{c}{t^{\gamma}}\|u\|_{\mathcal{U}}
$$

for every $u \in \mathcal{U}$ and $0<t<1$.
(d) The operators $Q: \mathcal{H} \rightarrow \mathcal{W}, G: \mathcal{H} \rightarrow \mathcal{Z}$ and $C: \mathcal{H} \rightarrow \mathcal{H}$ are all bounded linear operators.

Clearly, we consider the problem for unbounded control operator $B$, particularly singular estimate control systems under the Assumptions 2.1.

The aim of the stochastic LQR problem is to minimize the cost functional $J(u)$ over the set of square integrable controls $u \in L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)$, which are adapted in the filtration. We denote the optimal control by $u_{*}$ and the optimal state by $y_{*}$ so

$$
J\left(u_{*}\right)=\min _{u} J(u)
$$

### 2.1 Strong and Mild Solutions

Let $u \in L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)$. An $\mathcal{H}$-valued adapted process $y=y(t, \omega)$ is a strong solution of the state equation (1) over [0, T] if:
(a) $y(t)$ takes values in $\mathcal{D}(A) \cap \mathcal{D}(C)$ for almost all $t$ and $\omega$;
(b) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}}+\|A y(s)\|_{\mathcal{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|C y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=$ $1 ;$
(c) for arbitrary $t \in[0, T]$ and $\mathbb{P}$-almost surely, it satisfies the integral equation

$$
y(t)=y_{0}+\int_{0}^{t} A y(s) d s+\int_{0}^{t} B u(s) d s+\int_{0}^{t} C y(s) d W_{s}+\int_{0}^{t} D u(s) d W_{s} .
$$

An $\mathcal{H}$-valued adapted process $y(t, \omega)$ is a mild solution of the state equation (1) over $[0, T]$ if
(a) $y$ takes values in $\mathcal{D}(C)$;
(b) $P\left(\int_{0}^{T}\|y(s)\|_{\mathcal{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|C y(s)\|_{\mathcal{H}}^{2} d s<\infty\right)=1 ;$
(c) for arbitrary $t \in[0, T]$ and $\mathbb{P}$-almost surely, it satisfies the integral equation

$$
\begin{aligned}
y(t)=e^{A t} y_{0}+\int_{0}^{t} e^{A(t-s)} B u(s) d s & +\int_{0}^{t} e^{A(t-s)} C y(s) d W_{s} \\
& +\int_{0}^{t} e^{A(t-s)} D u(s) d W_{s}
\end{aligned}
$$

(d) $B u$ and $D u$ are $\mathcal{F}_{t}$ measurable Bochner integrable $\mathcal{H}$-valued functions.

Mild solutions are the limits of strong solutions. In the case of a deterministic state equation, i.e. when $C=D=0$, a mild solution $y \in L^{2}([0, T] ; \mathcal{H})$ can be written in the form

$$
y(t)=e^{A t} y_{0}+\int_{0}^{t} e^{A(t-s)} B u(s) d s, \quad t \in[0, T]
$$

It is well known that when $B$ is bounded, and given $u \in L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{U})\right)$ (Bochner integrable) and the initial data $y_{0} \in \mathcal{H}$, there exists a unique mild solution $y \in$ $L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)$ to the controlled state equation (1), see [13]. Using a standard argument, the existence is extended to the case of $B$ unbounded under Assumptions 2.1 (a) and (b), see [19].

### 2.2 Optimal Control

It is well known, cf. [21], that if all the operators $B, C, D$ appearing in (1) and (2) are bounded, then the optimal control is given by

$$
\begin{equation*}
u_{*}(t)=-\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) y_{*}(t) \tag{3}
\end{equation*}
$$

where $P(t)$ is a positive self-adjoint operator solving the Riccati equation for every $v, w \in \mathcal{D}(A)$

$$
\begin{align*}
& \langle\dot{P} v, w\rangle+\langle P A v, w\rangle+\left\langle A^{\star} P v, w\right\rangle+\left\langle C^{\star} P C v, w\right\rangle+\left\langle Q^{\star} Q v, w\right\rangle \\
& -\left\langle\left(B^{\star} P+D^{\star} P C\right)^{\star}\left(I+D^{\star} P D\right)^{-1}\left(B^{\star} P+D^{\star} P C\right) v, w\right\rangle=0,  \tag{4}\\
& P(T) v=G^{\star} G v . \tag{5}
\end{align*}
$$

Note that since $\left(I+D^{\star} P(t) D\right)$ is a positive operator, the inverse operator $(I+$ $\left.D^{\star} P(t) D\right)^{-1}$ is well defined and bounded on the control space $\mathcal{U}$. Moreover, if the initial condition $y_{0}$ is deterministic, the optimal cost is

$$
\min _{u \in L^{2}\left([0, T], L^{2}(\Omega, \mathcal{U})\right.} J(u)=J\left(u_{*}\right)=J\left(u_{*}, y_{*}\left(y_{0}, u_{*}\right)\right)=\left\langle P(0) y_{0}, y_{0}\right\rangle_{\mathcal{H}} .
$$

The proof of this feedback characterization in the more general framework of singular estimates (Assumptions 2.1) can be found in [19] when $D=0, G=0$ but $0 \leq \gamma<1$, and in [20] for bounded $D$ and $G$ under the condition $0 \leq \gamma<1 / 2$. In particular, we include the following theorems from [20].

Theorem 2.1 Under Assumptions 2.1, there exists a positive self-adjoint operator $P(t) \in C([0, T] ; \mathcal{L}(\mathcal{H}))$ satisfying the Riccati equation (4) and (5). Moreover, the following statements hold:
(i) The solution $P(t)$ is unique in the class of positive self adjoint operators in $C([0, T] ; \mathcal{L}(\mathcal{H}))$.
(ii) The solution $P(t)$ satisfies the estimate

$$
\begin{equation*}
\|P(t) x\|_{\mathcal{H}} \leq c\|x\|_{\mathcal{H}}, \quad \forall t \in[0, T], x \in \mathcal{H} . \tag{6}
\end{equation*}
$$

(iii) The operator $B^{\star} P(t)$ satisfies the estimate

$$
\begin{equation*}
\left\|B^{\star} P(t) x\right\|_{\mathcal{H}} \leq \frac{c}{(T-t)^{\gamma}}\|x\|_{\mathcal{H}}, \quad \forall t \in[0, T), x \in \mathcal{H} . \tag{7}
\end{equation*}
$$

Theorem 2.2 Under Assumptions 2.1, the optimal control problem of minimizing (2) subject to the differential equation (1) with initial condition $y_{0} \in L^{2}(\Omega, \mathcal{H})$ has a unique solution $u_{*}\left(\cdot ; y_{0}\right) \in C\left([0, T) ; L^{2}(\Omega, \mathcal{U})\right)$ and a corresponding optimal state $y_{*}\left(\cdot ; y_{0}\right) \in C\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)$. Moreover,
(i) The optimal control $u_{*}$ satisfies the estimate

$$
\mathbb{E}\left(\left\|u_{*}\left(t ; y_{0}\right)\right\|_{\mathcal{U}}^{2}\right) \leq \frac{c}{(T-t)^{2 \gamma}} \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right) \quad \forall t \in[0, T) .
$$

(ii) The optimal state $y_{*}$ satisfies the estimate

$$
\mathbb{E}\left(\left\|y_{*}\left(t ; y_{0}\right)\right\|_{\mathcal{H}}^{2}\right) \leq c \cdot \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right) \quad \forall t \in[0, T] .
$$

(iii) The optimal control $u_{*}$ has the feedback characterization in terms of the optimal state

$$
u_{*}\left(t ; y_{0}\right)=-\left(I+D^{\star} P D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) y_{*}(t)
$$

where $P(t)$ is the unique solution to the DRE (4)-(5).
(iv) The minimum of the functional (2) is given by

$$
J\left(u_{*}\right)=\min _{u \in L^{2}\left([0, T], L^{2}(\Omega, \mathcal{U})\right)} J(u)=\mathbb{E}\left\langle P(0) y_{0}, y_{0}\right\rangle_{\mathcal{H}}
$$

Remark 2.1 The deterministic analogue of these theorems, i.e the case $C=0$ and $D=0$, can be found in $[29,32]$. The framework of Assumptions 2.1 generalizes the class of systems driven by analytic semigroups with unbounded control operators $B$ satisfying $A^{-\gamma} B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. A complete analysis of this case, along with numerical approximations can be found in [30].

Remark 2.2 The theorems above also hold for $\gamma$ in $[1 / 2,1$ ) when $G=0$. In this case there is no singularity at the final time $T$ for the operator $B^{\star} P$ or the optimal control, [19]. In particular, the optimal control is continuous on [0, T], i.e. $u_{*} \in$ $L^{2}(\Omega ; C([0, T] ; \mathcal{U})$.

Remark 2.3 For nonzero $G$, the case of $\gamma \in[1 / 2,1)$ is more challenging even in the deterministic case where additional assumptions on $G$, are required for the existence of a minimum for the cost functional (2), see [30].

Remark 2.4 The theorems above can also be applied under an unbounded noise control operator $D$ satisfying the exact same conditions on $B$ stated in Assumptions 2.1 (b) and (c). This is particularly interesting because it allows the inclusion of noise in boundary control of certain classes of coupled PDEs.

In the next section, we show that the solution of the approximate finite-dimensional DREs converges to the solution of the DRE (4), under the general singular estimates framework, which extends the case when $B$ is a bounded operator, the case $\gamma=0$.

### 2.3 Preliminary Results

Since operator $A$ is an infinitesimal generator of a $C_{0}$-semigroup $e^{A t}$ and is defined on a dense subset of a Hilbert space $\mathcal{H}$, we have the uniform bound

$$
\left\|e^{A t}\right\|_{\mathcal{L}(\mathcal{H})} \leq M e^{\alpha t}, \quad \text { for } t \in[0, T]
$$

for some positive constants $\alpha, M$.
Using the feedback expression for the optimal control (3), the state equation (1) can be expressed as

$$
\begin{equation*}
d y(t)=A y(t) d t+\tilde{B}(t) y(t) d t+\tilde{C}(t) y(t) d W_{t}, \quad y(0)=y_{0} \tag{8}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{B}(t)=-B\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right),  \tag{9}\\
& \tilde{C}(t)=C-D\left(I+D^{\star} P(t) D\right)^{-1}\left(B^{\star} P(t)+D^{\star} P(t) C\right) . \tag{10}
\end{align*}
$$

We first state a lemma on the existence of a unique mild solution $y$ to (8) which is continuous, i.e. $y \in \mathcal{C} \equiv C\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)$. This is a standard result for uniformly bounded operators $\tilde{B}(t)$ and $\tilde{C}(t)$ on $\mathcal{H}$, cf. [13]. Via standard estimates, the existence result can be extended to the general case of $\tilde{B}(t)$ and $\tilde{C}(t)$ satisfying the estimates

$$
\begin{equation*}
\left\|e^{A t} \tilde{B}(t) x\right\|_{\mathcal{H}} \leq \frac{c}{(T-t)^{\gamma}(t-\tau)^{\gamma}}\|x\|_{\mathcal{H}} \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|e^{A t} \tilde{C}(t) x\right\|_{\mathcal{H}} \leq \frac{c}{(T-t)^{\gamma}(t-\tau)^{\gamma}}\|x\|_{\mathcal{H}} \tag{12}
\end{equation*}
$$

for some $\gamma<1 / 2$ for all $x \in \mathcal{H}$ and $t \in[0, T)$. Note that if $\tilde{B}$ and $\tilde{C}$ are defined by (9) and (10), then these estimates follow from Theorem 2.1. If $\gamma \geq 1 / 2$, then the mild solution is only $L^{2}\left([0, T] ; L^{2}(\Omega, \mathcal{H})\right)$ unless $G=0$, cf. [19, 20].

Lemma 2.1 Under Assumptions 2.1 and operators $\tilde{B}$ and $\tilde{C}$ satisfying (11) and (12) respectively, there exists a mild solution $y(t) \in \mathcal{C}$ to the equation

$$
d y(t)=A y(t) d t+\tilde{B} y(t) d t+\tilde{C} y(t) d W_{t}, \quad y(0)=y_{0} \in \mathcal{H}
$$

satisfying

$$
\begin{equation*}
y(t)=e^{A t} y_{0}+\int_{0}^{t} e^{A(t-s)} \tilde{B} y(s) d s+\int_{0}^{t} e^{A(t-s)} \tilde{C} y(s) d W_{s} \tag{13}
\end{equation*}
$$

In Lemma 2.1 we assumed the deterministic initial condition, i.e. $y_{0} \in \mathcal{H}$. The same statement will also hold if the initial condition is assumed to be a square integrable $\mathcal{H}$-valued random variable, i.e. satisfying $\mathbb{E}\left\|y_{0}\right\|_{\mathcal{H}}^{2}<\infty$.

In the following lemma, we prove an integral formulation of the differential Riccati equation (4), in terms of the semigroup $e^{A t}$ only, which will be used in the convergence scheme. The analogous formulation for the deterministic case $(C=0, D=0$ and $B$ bounded) can be found in [16].

Lemma 2.2 Under the assumptions stated in Sect. 2.1, the Riccati integral equation corresponding to the differential Riccati equation (4) can be expressed in terms of the semigroup $e^{A t}$ as

$$
\begin{align*}
P(t) \varphi= & \int_{t}^{T} e^{A^{\star}(\tau-t)} Q^{\star} Q e^{A(\tau-t)} \varphi d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)}\left(P(\tau) B+C^{\star} P(\tau) D\right)\left(I+D^{\star} P(\tau) D\right)^{-1} . \\
& \cdot\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) e^{A(\tau-t)} \varphi d \tau \\
& +e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)} \varphi+\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P(\tau) C e^{A(\tau-t)} \varphi d \tau . \tag{14}
\end{align*}
$$

Proof We consider the integral form of (4), which is given by

$$
\begin{align*}
P(t) \varphi= & \int_{t}^{T} e^{A^{\star}(\tau-t)} Q^{\star} Q \Phi(\tau, t) \varphi d \tau+\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P(\tau) C \Phi(\tau, t) \varphi d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P^{\star}(\tau) D\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(\tau)\right. \\
& \left.+D^{\star} P(\tau) C\right) \Phi(\tau, t) \varphi d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)} P^{\star}(\tau) B\left(I+D^{\star} P(\tau) D\right)^{-1} D^{\star} P(\tau) C \Phi(\tau, t) \varphi d \tau \\
& +e^{A^{\star}(T-t)} G^{\star} G \Phi(T, t) \varphi, \tag{15}
\end{align*}
$$

where $\Phi(t, s)$ is the solution to the equation

$$
\begin{equation*}
\Phi(\tau, t) x=e^{A(\tau-t)} x-\int_{t}^{\tau} e^{A(\tau-\eta)} B\left(I+D^{\star} P^{\star}(\eta) D\right)^{-1} B^{\star} P(\eta) \Phi(\eta, t) x d \eta \tag{16}
\end{equation*}
$$

see [20]. Note that $\Phi(\cdot, s) \in \mathcal{L}(\mathcal{H}, C([s, T], \mathcal{H})$ satisfies the differential equation

$$
\left\langle\frac{d}{d t} \Phi(t, s) v, w\right\rangle=\left\langle\left(A-B B^{*} P(t)\right) \Phi(t, s) v, w\right\rangle, \quad \Phi(s, s)=I
$$

$v \in \mathcal{H}, w \in \mathcal{D}\left(A^{*}\right)$. The positive operator $P(t)$ is self-adjoint and the operator $\Phi(t, s)$ is an evolution operator on $C([s, T] ; \mathcal{H})$, see [20].

Using formula (2.7) from [16, p. 540], the evolution can be expressed as

$$
\begin{equation*}
\Phi(\tau, t) \varphi=e^{A(\tau-t)} \varphi-\int_{t}^{\tau} \Phi(\tau, \eta) B\left(I+D^{\star} P^{\star}(\eta) D\right)^{-1} B^{\star} P(\eta) e^{A(\eta-t)} \varphi d \eta . \tag{17}
\end{equation*}
$$

Therefore, we can use this formula to rewrite the Riccati equation (15) in terms of the semigroup $e^{A(--.)}$ only. Thus, the integral Riccati equation can be expressed as

$$
P(t) \varphi=\int_{t}^{\tau} e^{A^{\star}(\tau-t)} \mathcal{M}(\tau) \Phi(\tau, t) \varphi d \tau+e^{A^{\star}(\tau-t)} G^{\star} G \Phi(\tau, t) \varphi
$$

for $s<t<\tau<T$, where

$$
\begin{aligned}
\mathcal{M}(\tau)= & Q^{\star} Q+C^{\star} P(\tau) C-C^{\star} P(\tau) D\left(I+D^{\star} P(\tau) D\right)^{-1}\left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) \\
& -P(\tau) B\left(I+D^{\star} P(\tau) D\right)^{-1} D^{\star} P(\tau) C
\end{aligned}
$$

Substituting the expression (17) for the evolution into (15) we have

$$
\begin{aligned}
P(t) \varphi= & \int_{t}^{T} e^{A^{\star}(\tau-t)} \mathcal{M}(\tau) e^{A(\tau-t)} \varphi d \tau \\
& +\int_{t}^{T} e^{A^{\star}(\tau-t)} \mathcal{M}(\tau) \int_{t}^{\tau} \Phi(\tau, \eta) \mathcal{K}(\eta) e^{A(\eta-t)} \varphi d \eta d \tau \\
& +e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)} \varphi \\
& +e^{A^{\star}(T-t)} G^{\star} G \int_{t}^{T} \Phi(T, \eta) \mathcal{K}(\eta) e^{A(\eta-t)} \varphi d \eta
\end{aligned}
$$

where $\mathcal{K}(\eta)=-B\left(I+D^{\star} P(\eta) D\right)^{-1} B^{\star} P(\eta)$.
Changing the order of integration in the second term and using evolution properties of the semigroup we get

$$
\begin{aligned}
P(t) \varphi= & \int_{t}^{T} e^{A^{\star}(\tau-t)} \mathcal{M}(\tau) e^{A(\tau-t)} \varphi d \tau \\
& +\int_{t}^{\tau} e^{A^{\star}(\eta-t)} \int_{\eta}^{T} e^{A^{\star}(\tau-\eta)} \mathcal{M}(\tau) \Phi(\tau, \eta) \mathcal{K}(\eta) e^{A(\eta-t)} \varphi d \tau d \eta \\
& +e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)} \varphi \\
& +\int_{t}^{T} e^{A^{\star}(\eta-t)} e^{A^{\star}(T-\eta)} G^{\star} G \Phi(T, \eta) \mathcal{K}(\eta) e^{A(\eta-t)} \varphi d \eta \\
= & \int_{t}^{T} e^{A^{\star}(\tau-t)} \mathcal{M}(\tau) e^{A(\tau-t)} \varphi d \tau \\
& +\int_{t}^{T} e^{A^{\star}(\eta-t)} P(\eta) \mathcal{K}(\eta) e^{A(\eta-t)} \varphi d \eta \\
& +e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)} \varphi
\end{aligned}
$$

Hence, we obtain the integral Riccati equation in terms of the semigroup

$$
\begin{aligned}
P(t) \varphi= & \int_{t}^{T} e^{A^{\star}(\tau-t)} Q^{\star} Q e^{A(\tau-t)} \varphi d \tau \\
& -\int_{t}^{T} e^{A^{\star}(\tau-t)}\left(P(\tau) B+C^{\star} P(\tau) D\right)\left(I+D^{\star} P(\tau) D\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& \left(B^{\star} P(\tau)+D^{\star} P(\tau) C\right) e^{A(\tau-t)} \varphi d \tau \\
+ & e^{A^{\star}(T-t)} G^{\star} G e^{A(T-t)} \varphi+\int_{t}^{T} e^{A^{\star}(\tau-t)} C^{\star} P(\tau) C e^{A(\tau-t)} \varphi d \tau
\end{aligned}
$$

as claimed in (14).
We will use the form (14) of the Riccati equation to prove the convergence results in Sect. 4.

## 3 Approximation Scheme

In this section, we develop a general convergence framework which can be used in computational techniques for solving the stochastic $L Q R$ problem. The results given here generalize the deterministic results proposed in $[3,5,16,30]$ to the stochastic case. In particular, the last reference [30] addresses the case of analytic semigroups $e^{A t}$ and unbounded operators $B: \mathcal{U} \rightarrow\left[\mathcal{D}\left(A^{\star}\right)\right]^{\prime}$ satisfying $A^{-\gamma} B: \mathcal{U} \rightarrow \mathcal{H}$, which was generalized by the singular estimate framework [32].

Let $\left(\mathcal{V}^{N}\right)_{N \in \mathbb{N}}$, be a sequence of finite-dimensional linear subspaces of $\mathcal{H} \cap \mathcal{D}\left(B^{\star}\right)$ and let

$$
\Pi^{N}: \mathcal{H} \rightarrow \mathcal{V}^{N}, \quad N \in \mathbb{N},
$$

be the canonical orthogonal projections. Assume that for every $N \in \mathbb{N}$ the operator $A^{N} \in \mathcal{L}\left(\mathcal{V}^{N}\right)$ is an infinitesimal generator of a $C_{0}$-semigroup $e^{A^{N} t}$ on $\mathcal{V}^{N}$ and thus $\left(e^{A^{N} t}\right)_{N \in \mathbb{N}}$ is a sequence of strongly continuous semigroups on $\mathcal{V}^{N}$. Given operators $B^{N} \in \mathcal{L}\left(\mathcal{U}, \mathcal{V}^{N}\right), G^{N}, Q^{N}, C^{N} \in \mathcal{L}\left(\mathcal{V}^{N}\right)$, we consider the family of finite dimensional stochastic LQR problems on $\mathcal{V}^{N}$ :

$$
\begin{align*}
d y^{N}(t) & =\left(A^{N} y^{N}(t)+B^{N} u(t)\right) d t+\left(C^{N} y^{N}(t)+D^{N} u(t)\right) d W_{t}, t \in[0, T] \\
y^{N}(0) & =y_{0}^{N} \tag{18}
\end{align*}
$$

and the cost functional

$$
\begin{equation*}
J^{N}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\left\|Q^{N} y^{N}\right\|_{\mathcal{H}}^{2}+\|u\|_{\mathcal{U}}^{2}\right) d t+\left\|G^{N} y^{N}(T)\right\|_{\mathcal{H}}^{2}\right] . \tag{19}
\end{equation*}
$$

For simplicity we consider the observation spaces $\mathcal{W}=\mathcal{Z}=\mathcal{H}$.
The optimal control is given in feedback form by

$$
\begin{equation*}
u_{*}^{N}(t)=-\left(I+D^{N^{\star}} P^{N}(t) D^{N}\right)^{-1}\left(B^{N^{\star}} P^{N}(t)+D^{N^{\star}} P^{N}(t) C^{N}\right) y_{*}^{N}(t) \tag{20}
\end{equation*}
$$

where $P^{N}(t) \in \mathcal{L}\left(\mathcal{V}^{N}\right)$ is the unique positive self-adjoint solution of the differential Riccati equation:

$$
\begin{align*}
& \dot{P}^{N}+P^{N} A^{N}+A^{N^{\star}} P^{N}+C^{N^{\star}} P^{N} C^{N}+Q^{N^{\star}} Q^{N} \\
& -\left(B^{N^{\star}} P^{N}+D^{N^{\star}} P^{N} C^{N}\right)^{\star}\left(I+D^{N^{\star}} P^{N} D^{N}\right)^{-1}\left(B^{N^{\star}} P^{N}+D^{N^{\star}} P^{N} C^{N}\right)=0 \\
& \quad\left(I+D^{N^{\star}} P^{N} D^{N}\right)>0 \\
& \quad P^{N}(T)=G^{N^{\star}} G^{N} \tag{21}
\end{align*}
$$

and $y_{*}^{N}(t)$ is the optimal state. We refer the reader to [41] for an extensive treatment of the finite dimensional stochastic LQR.

We impose the following assumptions on the approximation operators:
Assumption 3.1 (i) For all $x \in \mathcal{H}$, the semigroup $e^{A^{N} t} \Pi^{N} x$ converges in $\mathcal{H}$ to $e^{A t} x$ uniformly on $[0, T]$ as $N \rightarrow \infty$ and in particular there exists $N_{0} \in \mathbb{N}$ such that for $N \geq N_{0}$, we have

$$
\left\|\left(e^{A^{N}} t \Pi^{N}-e^{A t}\right) x\right\|_{\mathcal{H}} \leq \frac{c}{N}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H}, \quad \forall t \in[0, T]
$$

(ii) For all $x \in \mathcal{H}$, the semigroups $e^{A^{N \star} t} \Pi^{N} x$ converge in $\mathcal{H}$ to $e^{A^{\star} t} x$ uniformly on $[0, T]$ and in particular for $N \geq N_{0}$

$$
\left\|\left(e^{A^{N \star} t} \Pi^{N}-e^{A^{\star} t}\right) x\right\|_{\mathcal{H}} \leq \frac{c}{N}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H}, \quad \forall t \in[0, T] .
$$

(iii) For all $x \in \mathcal{V}^{N}$ we have for $N \geq N_{0}$

$$
\left\|B^{N \star} \Pi^{N} x\right\|_{\mathcal{U}} \leq c N^{\gamma}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H}
$$

(iv) The projections $\Pi^{N}$ satisfy the convergence estimate for $N \geq N_{0}$

$$
\left\|B^{\star}\left(\Pi^{N}-I\right) x\right\|_{\mathcal{U}} \leq \frac{c}{N}\|x\|_{\mathcal{D}\left(B^{\star}\right)}, \quad \forall x \in \mathcal{D}\left(B^{\star}\right)
$$

(v) For all $x \in \mathcal{D}\left(B^{\star}\right), B^{N \star} \Pi^{N} x$ converges to $B^{\star} x$ in $\mathcal{U}$ and for $N \geq N_{0}$

$$
\left\|\left(B^{\star}-B^{N \star} \Pi^{N}\right) x\right\|_{\mathcal{U}} \leq \frac{c}{N}\|x\|_{\mathcal{D}\left(B^{\star}\right)}, \quad \forall x \in \mathcal{D}\left(B^{\star}\right)
$$

(iv) The approximations $B^{N \star}$ satisfy the uniform singular estimate

$$
\begin{equation*}
\left\|B^{N \star} e^{A^{N \star} t} \Pi^{N} x\right\|_{\mathcal{U}} \leq \frac{c}{t^{\gamma}}\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H}, \quad \forall t \in[0, T) \tag{22}
\end{equation*}
$$

for $N \geq N_{0}$. The parameter $\gamma \in(0,1 / 2)$ is the exponent of singularity in Assumption 2.1 (c) satisfied by $e^{A t} B$.
(vii) For all $v \in \mathcal{U}, D^{N} v \rightarrow D v$ in $\mathcal{H}$ and for all $\varphi \in \mathcal{H}$, we have $D^{N \star} \Pi^{N} \varphi \rightarrow D^{\star} \varphi$ in $\mathcal{U}$ such that for $N \geq N_{0}$

$$
\left\|\left(D^{N}-D\right) v\right\|_{\mathcal{H}} \leq \frac{c}{N}\|v\|_{\mathcal{U}}, \quad \forall v \in \mathcal{U}
$$

and

$$
\left\|\left(D^{N \star} \Pi^{N}-D^{\star}\right) \varphi\right\|_{\mathcal{U}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

(viii) For all $\varphi \in \mathcal{H}$, we have $C^{N} \Pi^{N} \varphi \rightarrow C \varphi$ and $C^{N \star} \Pi^{N} \varphi \rightarrow C^{\star} \varphi$ in $\mathcal{H}$ such that for $N \geq N_{0}$

$$
\left\|\left(C^{N} \Pi^{N}-C\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H}
$$

and

$$
\left\|\left(C^{N \star} \Pi^{N}-C^{\star}\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

(ix) For all $\varphi \in \mathcal{H}$, we have $Q^{N} \Pi^{N} \varphi \rightarrow Q \varphi$ and $Q^{N \star} \Pi^{N} \varphi \rightarrow Q^{\star} \varphi$ in $\mathcal{H}$ such that for $N \geq N_{0}$

$$
\left\|\left(Q^{N} \Pi^{N}-Q\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H},
$$

and

$$
\left\|\left(Q^{N \star} \Pi^{N}-Q^{\star}\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

(x) For all $\varphi \in \mathcal{H}$, we have $G^{N} \Pi^{N} \varphi \rightarrow G \varphi$ and $G^{N \star} \Pi^{N} \varphi \rightarrow G^{\star} \varphi$ in $\mathcal{H}$ such that for $N \geq N_{0}$

$$
\left\|\left(G^{N} \Pi^{N}-G\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H},
$$

and

$$
\left\|\left(G^{N \star} \Pi^{N}-G^{\star}\right) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N}\|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H} .
$$

Assumption (i) implies that $\Pi^{N} \varphi \rightarrow \varphi$ for all $\varphi \in \mathcal{H}$, which indicates the sense in which the subspaces $\mathcal{V}^{N}$ approximate $\mathcal{H}$.

Remark 3.1 If $e^{A t}$ is an analytic semigroup, and $A^{-\gamma} B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, then one can alternatively assume the condition of uniform analyticity on the approximations [30]

$$
\begin{equation*}
\left\|\left(A^{N}\right)^{\theta} e^{A^{N} t}\right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c}{t^{\theta}} \tag{23}
\end{equation*}
$$

This assumption along with the uniform bound on $B^{N}$

$$
\begin{equation*}
\left\|B^{N} x\right\|_{\mathcal{H}} \leq\left\|A^{\gamma} x\right\|_{\mathcal{H}} \tag{24}
\end{equation*}
$$

can replace the uniform singular estimate condition (vi) in Assumptions 3.1. We refer the reader to [30] for a detailed treatment of this case.

## 4 Convergence

We next state the main convergence results, showing convergence of the solution of the approximate differential Riccati equation (21) to the solution of the original Riccati equation (4).

Theorem 4.1 Under Assumptions 2.1 and Assumptions 3.1, for $\varphi \in \mathcal{H}, P^{N}(t) \Pi^{N}$ $\varphi \rightarrow P(t) \varphi$ uniformly on $[0, T]$ in $\mathcal{H}$ as $N \rightarrow \infty$, and in particular

$$
\begin{equation*}
\left\|P^{N}(t) \Pi^{N} \varphi-P(t) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N^{1-\gamma}}\|\varphi\|_{\mathcal{H}} \tag{25}
\end{equation*}
$$

for $N \geq N_{0}$ and for all $t \in[0, T]$. Moreover, for all $t \in[0, T)$, we have

$$
\begin{equation*}
\left\|B^{N \star} P^{N}(t) \Pi^{N} \varphi-B^{\star} P(t) \varphi\right\|_{\mathcal{H}} \leq \frac{c}{N^{1-\gamma}(T-t)^{\gamma}}\|\varphi\|_{\mathcal{H}} \tag{26}
\end{equation*}
$$

The second theorem below establishes convergence of the optimal pair $u_{*}^{N}$ and $y_{*}^{N}$ of the $N$ problem (18) and (19) to the optimal pair $u_{*}$ and $y_{*}$ of (1) and (2).

Theorem 4.2 Under Assumptions 2.1 and Assumptions 3.1 and given the condition $\mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right)<\infty$, we have

$$
\begin{equation*}
y_{*}^{N} \rightarrow y_{*} \text { uniformly as } N \rightarrow \infty \text { on }[0, T] \text { in } L^{2}(\Omega, \mathcal{H}) \tag{27}
\end{equation*}
$$

and in particular

$$
\begin{equation*}
\mathbb{E}\left(\left\|y_{*}^{N}\left(t ; y_{0}^{N}\right)-y_{*}\left(t ; y_{0}\right)\right\|_{\mathcal{H}}^{2}\right) \leq \frac{c}{N^{2(1-\gamma)}} \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right), \quad \forall t \in[0, T] \tag{28}
\end{equation*}
$$

while

$$
\begin{equation*}
u_{*}^{N} \rightarrow u_{*} \text { uniformly as } N \rightarrow \infty \text { on }[0, T-\epsilon] \text { in } L^{2}(\Omega, \mathcal{U}), \quad \epsilon>0 \tag{29}
\end{equation*}
$$

and in particular

$$
\begin{equation*}
\mathbb{E}\left(\left\|u_{*}^{N}\left(t ; y_{0}^{N}\right)-u_{*}\left(t ; y_{0}\right)\right\|_{\mathcal{U}}^{2}\right) \leq \frac{c}{N^{2(1-\gamma)}(T-t)^{2 \gamma}} \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right), \quad \forall t \in[0, T) \tag{30}
\end{equation*}
$$

### 4.1 Preliminary Results

Lemma 4.1 The following convergence estimate

$$
\begin{equation*}
\left\|\left(B^{\star} e^{A^{\star} t}-B^{N \star} e^{A^{N \star} t} \Pi^{N}\right) x\right\|_{\mathcal{U}} \leq \frac{c}{N^{1-\gamma} t^{\gamma}}\|x\|_{\mathcal{H}} \tag{31}
\end{equation*}
$$

follows from Assumptions 3.1 (i)-(vi) for every $x \in \mathcal{H}, t \in[0, T)$, and $N \geq N_{0}$.
Proof We estimate the term as

$$
\begin{aligned}
\left\|\left(B^{\star} e^{A^{\star} t}-B^{N \star} e^{A^{N \star t}} \Pi^{N}\right) x\right\|_{\mathcal{U}} \leq & \left\|\left(B^{\star}-B^{N \star} \Pi^{N}\right) e^{A^{\star} t} x\right\|_{\mathcal{U}}+ \\
& +\left\|B^{N \star} \Pi^{N}\left(e^{A^{\star} t}-e^{A^{N \star} t} \Pi^{N}\right) x\right\|_{\mathcal{U}} \\
\leq & \frac{c}{N}\left\|e^{A^{\star} t} x\right\|_{\mathcal{D}\left(B^{\star}\right)}+c N^{\gamma}\left\|\left(e^{A^{N \star} t} \Pi^{N}-e^{A^{\star} t}\right) x\right\|_{\mathcal{H}}
\end{aligned}
$$

where we used Assumptions 3.1 (v) and (iii), respectively. Hence, by the singular estimate condition and Assumption 3.1 (ii), we have

$$
\begin{aligned}
\left\|\left(B^{\star} e^{A^{\star t}}-B^{N \star} e^{A^{N \star} t} \Pi^{N}\right) x\right\|_{\mathcal{U}} & \leq \frac{c}{N t^{\gamma}}\|x\|_{\mathcal{H}}+\frac{c}{N^{1-\gamma}}\|x\|_{\mathcal{H}} \\
& \leq \frac{c_{T}}{N^{1-\gamma} t^{\gamma}}\|x\|_{\mathcal{H}},
\end{aligned}
$$

where $c_{T}=c+c T^{\gamma}$ and $\gamma \in\left(0, \frac{1}{2}\right)$.
Lemma 4.2 The solution $P^{N}$ to (21) satisfies the uniform estimates:

$$
\begin{array}{r}
\left\|P^{N}(t) \Pi^{N} x\right\|_{\mathcal{H}} \leq c\|x\|_{\mathcal{H}}, \\
\left\|B^{N \star} P^{N}(t) \Pi^{N} x\right\|_{\mathcal{U}} \leq \frac{c}{(T-t)^{\gamma}}\|x\|_{\mathcal{H}}, \tag{33}
\end{array}
$$

for all $t \in[0, T), x \in \mathcal{H}$ and $N \geq N_{0}$.
Proof Assumptions 3.1 guarantee uniform bounds independent of $N$ on all operators $e^{A^{N} t}, Q^{N}, G^{N}, C^{N}, D^{N}$ and their adjoints. Moreover, we have a uniform singular estimate assumption 3.1 (vi) satisfied by $e^{A^{N} t} B^{N}$. Hence, (32) and (33) follow from (6) and (7) of Theorem 2.1 applied to the N problem in (18) and (19).

### 4.2 Proof of Theorem 4.1

Proof It suffices to prove convergence of solutions of the equivalent integral formulation (14). Let $P$ be the solution of the Riccati integral equation (14) and let $P^{N}$ be the solution of the N approximate integral equation

$$
\begin{align*}
P^{N}(t) x= & \int_{t}^{T} e^{A^{N^{\star}}(\tau-t)} Q^{N^{\star}} Q^{N} e^{A^{N}(\tau-t)} x d \tau \\
& -\int_{t}^{T} e^{A^{N^{\star}}(\tau-t)}\left(P^{N}(\tau) B^{N}+C^{N^{\star}} P^{N}(\tau) D^{N}\right)\left(I+D^{N^{\star}} P^{N}(\tau) D^{N}\right)^{-1} \\
& \cdot\left(B^{N^{\star}} P^{N}(\tau)+D^{N^{\star}} P^{N}(\tau) C^{N}\right) e^{A^{N}(\tau-t)} x d \tau \\
& +e^{A^{N^{\star}}(T-t)} G^{N^{\star}} G^{N} e^{A^{N}(T-t)} x \\
& +\int_{t}^{T} e^{A^{N^{\star}(\tau-t)} C^{N^{\star}} P^{N}(\tau) C^{N} e^{A^{N}(\tau-t)} x d \tau .} \$ \text { (34) } \tag{34}
\end{align*}
$$

Estimating the difference $\left\|P^{N}(t) \Pi^{N}-P(t)\right\|_{\mathcal{L}(\mathcal{H})}$ via (32) and (33) using standard arguments we obtain

$$
\begin{align*}
& \| P^{N}(t) \Pi^{N}-P(t) \|_{\mathcal{L}(\mathcal{H})} \\
& \leq \int_{t}^{T} c k_{N}(\tau) d \tau+c g_{N}(t)+c \int_{t}^{T} \frac{1}{(T-\tau)^{\gamma}}\left\|B^{N \star} P^{N}(\tau) \Pi^{N}-B^{\star} P(\tau)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} d \tau \\
& \quad+\int_{t}^{T} \frac{c}{(T-\tau)^{2 \gamma}} d_{N}(\tau) d \tau+\int_{t}^{T} \frac{c}{(T-\tau)^{\gamma}}\left\|P^{N}(\tau) \Pi^{N}-P(\tau)\right\|_{\mathcal{L}(\mathcal{H})} d \tau \tag{35}
\end{align*}
$$

where

$$
\begin{align*}
k_{N}(\tau) \equiv & \left\|Q^{N \star} Q^{N} \Pi^{N}-Q^{\star} Q\right\|_{\mathcal{L}(\mathcal{H})}+\left\|e^{A^{N \star} \tau} \Pi^{N}-e^{A^{\star} \tau}\right\|_{\mathcal{L}(\mathcal{H})} \\
& +\left\|e^{A^{N} \tau} \Pi^{N}-e^{A \tau}\right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c}{N} \tag{36}
\end{align*}
$$

by Assumptions 3.1 (i), (ii) and (ix) for all $\tau \leq T$ while

$$
\begin{align*}
g_{N}(t) \equiv & \left\|e^{A^{N \star}(T-t)} \Pi^{N}-e^{A^{\star}(T-t)}\right\|_{\mathcal{L}(\mathcal{H})}+\left\|e^{A^{N}(T-t)} \Pi^{N}-e^{A(T-t)}\right\|_{\mathcal{L}(\mathcal{H})} \\
& +\left\|G^{N \star} \Pi^{N}-G^{\star}\right\|_{\mathcal{L}(\mathcal{H})}+\left\|G^{N} \Pi^{N}-G\right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c}{N} \tag{37}
\end{align*}
$$

for all $\tau \leq T$ by Assumptions 3.1 (i), (ii) and (x). Similarly,

$$
\begin{align*}
d_{N}(\tau) & \equiv\left\|e^{A^{N \star} \tau} \Pi^{N}-e^{A^{\star} \tau}\right\|_{\mathcal{L}(\mathcal{H})}+\left\|e^{A^{N} \tau} \Pi^{N}-e^{A \tau}\right\|_{\mathcal{L}(\mathcal{H})} \\
& +\left\|C^{N} \Pi^{N}-C\right\|_{\mathcal{L}(\mathcal{H})}+\left\|D^{N}-D\right\|_{\mathcal{L}(\mathcal{U}, \mathcal{H})} \\
& +\left\|C^{N \star} \Pi^{N}-C^{\star}\right\|_{\mathcal{L}(\mathcal{H})}+\left\|D^{N \star} \Pi^{N}-D^{\star}\right\|_{\mathcal{L}(\mathcal{U}, \mathcal{H})} \leq \frac{c}{N} \tag{38}
\end{align*}
$$

for all $\tau \leq T$ by Assumptions 3.1 (i), (ii), (vii) and (viii). Therefore, (35) becomes

$$
\begin{aligned}
& \left\|P^{N}(t) \Pi^{N}-P(t)\right\|_{\mathcal{L}(\mathcal{H})} \\
& \quad \leq \frac{c}{N}\left(1+T+T^{1-2 \gamma}\right)+\int_{t}^{T} \frac{c}{(T-\tau)^{\gamma}} \| B^{N \star} P^{N}(\tau) \Pi^{N} \\
& \quad-B^{\star} P(\tau) \|_{\mathcal{L}(\mathcal{H}, \mathcal{U})} d \tau
\end{aligned}
$$

$$
\begin{equation*}
+\int_{t}^{T} \frac{c}{(T-\tau)^{\gamma}}\left\|P^{N}(\tau) \Pi^{N}-P(\tau)\right\|_{\mathcal{L}(\mathcal{H})} d \tau \tag{39}
\end{equation*}
$$

Similarly, applying $B^{\star}$ and $B^{N \star}$ to (14) and (34) respectively, we can estimate $(T-t)^{\gamma}\left\|B^{N \star} P^{N}(t) \Pi^{N}-B^{\star} P(t)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})}$, using the uniform singular estimate (22) in Assumption 3.1 (iv) in addition to the uniform estimates (32) and (33), so that we obtain

$$
\begin{align*}
& (T-t)^{\gamma}\left\|B^{N \star} P^{N}(t) \Pi^{N}-B^{\star} P(t)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})} \\
& \leq(T-t)^{\gamma} \int_{t}^{T} c \tilde{k}_{N}(\tau) d \tau+(T-t)^{\gamma} \tilde{g}_{N}(t) \\
& +c(T-t)^{\gamma} \int_{t}^{T} \frac{(T-\tau)^{\gamma}}{(\tau-t)^{\gamma}(T-\tau)^{2 \gamma}}\left\|B^{N \star} P^{N}(\tau) \Pi^{N}-B^{\star} P(\tau)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})} d \tau \\
& +(T-t)^{\gamma} \int_{t}^{T} \frac{c}{(T-\tau)^{2 \gamma}} \tilde{d}_{N}(\tau) d \tau \\
& +(T-t)^{\gamma} \int_{t}^{T} \frac{c}{(\tau-t)^{\gamma}(T-\tau)^{\gamma}}\left\|P^{N}(\tau) \Pi^{N}-P(\tau)\right\|_{\mathcal{L}(\mathcal{H})} d \tau \tag{40}
\end{align*}
$$

where

$$
\tilde{k}_{N}(\tau) \equiv \frac{k_{N}(\tau)}{(\tau-t)^{\gamma}}+\left\|B^{N \star} e^{A^{N \star}(\tau-t)} \Pi^{N}-B^{\star} e^{A^{\star}(\tau-t)}\right\|_{\mathcal{L}(\mathcal{H})}
$$

while

$$
\tilde{g}_{N}(t) \equiv \frac{g_{N}(t)}{(T-t)^{\gamma}}+\left\|B^{N \star} e^{A^{N \star}(T-t)} \Pi^{N}-B^{\star} e^{A^{\star}(T-t)}\right\|_{\mathcal{L}(\mathcal{H})}
$$

and

$$
\tilde{d}_{N}(\tau) \equiv \frac{d_{N}(\tau)}{(\tau-t)^{\gamma}}+\left\|B^{N \star} e^{A^{N \star}(\tau-t)} \Pi^{N}-B^{\star} e^{A^{\star}(\tau-t)}\right\|_{\mathcal{L}(\mathcal{H})}
$$

Again, using (31) in Lemma 4.1 and (36)-(38), note that

$$
\begin{aligned}
\tilde{k}_{N}(\tau) & \leq \frac{c}{N^{1-\gamma}(\tau-t)^{\gamma}} \\
\tilde{g}_{N}(t) & \leq \frac{c}{N^{1-\gamma}(T-t)^{\gamma}} \\
\tilde{d}_{N}(\tau) & \leq \frac{c}{N^{1-\gamma}(\tau-t)^{\gamma}} .
\end{aligned}
$$

Thus, (40) becomes

$$
\begin{align*}
(T & -t)^{\gamma}\left\|B^{N \star} P^{N}(t) \Pi^{N}-B^{\star} P(t)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})} \\
& \leq \frac{c}{N^{1-\gamma}}\left(1+T+T^{1-2 \gamma}\right) \\
& +c(T-t)^{\gamma} \int_{t}^{T} \frac{(T-\tau)^{\gamma}}{(\tau-t)^{\gamma}(T-\tau)^{2 \gamma}}\left\|B^{N \star} P^{N}(\tau) \Pi^{N}-B^{\star} P(\tau)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})} d \tau \\
& +(T-t)^{\gamma} \int_{t}^{T} \frac{c}{(\tau-t)^{\gamma}(T-\tau)^{\gamma}}\left\|P^{N}(\tau) \Pi^{N}-P(\tau)\right\|_{\mathcal{L}(\mathcal{H})} d \tau \tag{41}
\end{align*}
$$

Adding the two estimates (39) and (41) and noting that the right hand side is finite, we apply Grönwall's inequality to obtain

$$
\begin{gathered}
(T-t)^{\gamma}\left\|B^{N \star} P^{N}(t) \Pi^{N}-B^{\star} P(t)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})}+\left\|P^{N}(t) \Pi^{N}-P(t)\right\|_{\mathcal{L}(\mathcal{H})} \\
\quad \leq \frac{c\left(1+T+T^{1-2 \gamma}\right)}{N^{1-\gamma}} \exp \left((T-t)^{\gamma} \int_{t}^{T} \frac{c}{(\tau-t)^{\gamma}(T-\tau)^{2 \gamma}} d \tau\right)
\end{gathered}
$$

Now, since $\gamma<1 / 2$ the time integral can be estimated as

$$
\begin{aligned}
&(T-t)^{\gamma} \int_{t}^{T} \frac{c}{(\tau-t)^{\gamma}(T-\tau)^{2 \gamma}} d \tau \\
& \leq c(T-t)^{\gamma}\left(\int_{t}^{\frac{T+t}{2}} \frac{d \tau}{(\tau-t)^{\gamma}(T-t)^{2 \gamma}}+\int_{\frac{T+t}{2}}^{T} \frac{d \tau}{(T-t)^{\gamma}(T-\tau)^{2 \gamma}}\right) \\
& \quad \leq c\left(\int_{t}^{\frac{T+t}{2}} \frac{d \tau}{(\tau-t)^{2 \gamma}}+\int_{\frac{T+t}{2}}^{T} \frac{d \tau}{(T-\tau)^{2 \gamma}}\right) \\
& \quad \leq \frac{2^{2 \gamma} c}{1-2 \gamma}(T-t)^{1-2 \gamma} \leq \frac{2^{2 \gamma} c}{1-2 \gamma} T^{1-2 \gamma}
\end{aligned}
$$

Hence,

$$
(T-t)^{\gamma}\left\|B^{N \star} P^{N}(t) \Pi^{N}-B^{\star} P(t)\right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})}+\left\|P^{N}(t) \Pi^{N}-P(t)\right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c_{T}}{N^{1-\gamma}}
$$

and the result is established.

### 4.3 Proof of Theorem 4.2

Proof Substituting (20) into (18), the optimal states $y_{*}^{N}$ are then the solutions of the stochastic differential equations

$$
\begin{equation*}
d y^{N}(t)=A^{N} y^{N}(t) d t+\tilde{B}^{N} y^{N}(t) d t+\tilde{C}^{N} y^{N}(t) d W_{t}, \quad y^{N}(0)=y_{0}^{N} \tag{42}
\end{equation*}
$$

for $t \in[0, T]$ where

$$
\begin{aligned}
& \tilde{B}^{N}(t)=-B^{N}\left(I+D^{N^{\star}} P^{N}(t) D^{N}\right)^{-1}\left(B^{N^{\star}} P^{N}(t)+D^{N^{\star}} P^{N}(t) C^{N}\right) \\
& \tilde{C}^{N}(t)=C^{N}-D^{N}\left(I+D^{N^{\star}} P^{N}(t) D^{N}\right)^{-1}\left(B^{N^{\star}} P^{N}(t)+D^{N^{\star}} P^{N}(t) C^{N}\right)
\end{aligned}
$$

From Assumptions 3.1 (i)-(x) and the convergence estimates (25) and (26) of Theorem 4.1, we have

$$
\begin{array}{r}
\left\|\left(e^{A^{N}(t-\tau)} \tilde{B}^{N}(\tau)-e^{A(t-\tau)} \tilde{B}(\tau)\right) x\right\|_{\mathcal{H}} \leq \frac{c}{N^{1-\gamma}(t-\tau)^{\gamma}(T-\tau)^{\gamma}}\|x\|_{\mathcal{H}} \\
\left\|e^{A^{N}(t-\tau)} \tilde{B}^{N}(\tau) x\right\|_{\mathcal{H}} \leq \frac{c}{(t-\tau)^{\gamma}(T-\tau)^{\gamma}}\|x\|_{\mathcal{H}} \\
\left\|\left(e^{A^{N}(t-\tau)} \tilde{C}^{N}(t)-e^{A(t-\tau)} \tilde{C}(t)\right) x\right\|_{\mathcal{H}} \leq \frac{c}{N^{1-\gamma}(T-\tau)^{\gamma}}\|x\|_{\mathcal{H}} \\
\left\|e^{A^{N}(t-\tau)} \tilde{C}^{N}(\tau) x\right\|_{\mathcal{H}} \leq \frac{c}{(T-\tau)^{\gamma}}\|x\|_{\mathcal{H}} \tag{46}
\end{array}
$$

for all $x \in \mathcal{H}$.
We apply Lemma 2.1 and represent the solutions $y_{*}(t)$ and $y_{*}^{N}(t)$ of (8) and (42) respectively in the form (13), i.e.

$$
\begin{gathered}
y_{*}(t)=e^{A t} y_{0}-\int_{0}^{t} e^{A(t-\tau)} \tilde{B} y_{*}(\tau) d \tau+\int_{0}^{t} e^{A(t-\tau)} \tilde{C} y_{*}(\tau) d W_{\tau} \quad \text { and } \\
y_{*}^{N}(t)=e^{A^{N} t} y_{0}^{N}-\int_{0}^{t} e^{A^{N}(t-\tau)} \tilde{B}^{N} y_{*}^{N}(\tau) d \tau+\int_{0}^{t} e^{A^{N}(t-\tau)} \tilde{C}^{N} y_{*}^{N}(\tau) d W_{\tau},
\end{gathered}
$$

with $y_{0}^{N}=\Pi^{N} y_{0}$. Taking the difference of $y_{*}(t)$ and $y_{*}^{N}(t)$ and factorizing corresponding terms, we obtain

$$
\begin{align*}
y_{*}(t)-y_{*}^{N}(t)= & \left(e^{A t}-e^{A^{N} t} \Pi^{N}\right) y_{0}-\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{B} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{B}^{N} y_{*}^{N}(\tau)\right) d \tau \\
& +\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{C} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{C}^{N} y_{*}^{N}(\tau)\right) d W_{\tau} \tag{47}
\end{align*}
$$

for $t \in[0, T]$. The second term on the right hand side of (47) can be expressed in the form

$$
\begin{align*}
& \int_{0}^{t}\left(e^{A(t-\tau)} \tilde{B} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{B}^{N} y_{*}^{N}(\tau)\right) d \tau \\
& \left.\quad=\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{B}-e^{A^{N}(t-\tau)} \tilde{B}^{N}\right) y_{*}(\tau)+e^{A^{N}(t-\tau)} \tilde{B}^{N}\left(y_{*}(\tau)-y_{*}^{N}(\tau)\right)\right) d \tau \tag{48}
\end{align*}
$$

and similarly the third term on the right hand side of (47) can be written as follows

$$
\begin{align*}
& \int_{0}^{t}\left(e^{A(t-\tau)} \tilde{C} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{C}^{N} y_{*}^{N}(\tau)\right) d W_{\tau} \\
& \left.\quad=\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{C}-e^{A^{N}(t-\tau)} \tilde{C}^{N}\right) y_{*}(\tau)+e^{A^{N}(t-\tau)} \tilde{C}^{N}\left(y_{*}(\tau)-y_{*}^{N}(\tau)\right)\right) d W_{\tau} \tag{49}
\end{align*}
$$

We want to show that $\left\|y_{*}-y_{*}^{N}\right\|_{\mathcal{C}}^{2} \rightarrow 0$, when $N \rightarrow \infty$. Therefore, we estimate

$$
\begin{align*}
& \mathbb{E}\left(\left\|y_{*}(t)-y_{*}^{N}(t)\right\|_{\mathcal{H}}^{2}\right) \\
& \leq 3 \cdot \mathbb{E}\left(\left\|\left(e^{A t}-e^{A^{N} t} \Pi^{N}\right) y_{0}\right\|_{\mathcal{H}}^{2}\right) \\
& \quad+3 \cdot \mathbb{E}\left(\left\|\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{B} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{B}^{N} y_{*}^{N}(\tau)\right) d \tau\right\|_{\mathcal{H}}^{2}\right) \\
& \quad+3 \cdot \mathbb{E}\left(\left\|\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{C} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{C}^{N} y_{*}^{N}(\tau)\right) d W_{\tau}\right\|_{\mathcal{H}}^{2}\right) \tag{50}
\end{align*}
$$

We will estimate the right hand side by estimating each of the three terms separately. First,

$$
\begin{equation*}
\mathbb{E}\left(\left\|\left(e^{A t}-e^{A^{N} t} \Pi^{N}\right) y_{0}\right\|_{\mathcal{H}}^{2}\right) \leq \frac{c}{N^{2}} \cdot \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right) \tag{51}
\end{equation*}
$$

which holds because Assumption 3.1 (i). Next, using (48), the second term in (50) can be estimated in the following way

$$
\begin{align*}
& \mathbb{E}\left(\left\|\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{B} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{B}^{N} y_{*}^{N}(\tau)\right) d \tau\right\|_{\mathcal{H}}^{2}\right) \\
& \leq\left\|y_{*}\right\|_{\mathcal{C}}^{2}\left(\int_{0}^{t} \frac{c}{N^{1-\gamma}(t-\tau)^{2 \gamma}} d \tau\right)^{2}+\mathbb{E}\left(\int_{0}^{t} \frac{c}{(t-\tau)^{2 \gamma}}\left\|y_{*}(\tau)-y_{*}^{N}(\tau)\right\|_{\mathcal{H}} d \tau\right)^{2} \\
& \leq \frac{c T^{2(1-2 \gamma)}}{N^{2(1-\gamma)}}\left\|y_{*}\right\|_{\mathcal{C}}^{2}+T^{(1-2 \gamma)} \int_{0}^{t} \frac{c}{(t-\tau)^{2 \gamma}} \mathbb{E}\left(\left\|y_{*}(\tau)-y_{*}^{N}(\tau)\right\|_{\mathcal{H}}^{2}\right) d \tau \tag{52}
\end{align*}
$$

where we used (43) and the uniform estimate (44). At last, we estimate the term $\mathbb{E}\left(\left\|\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{C} y_{*}(\tau)-e^{A^{N}(t-\tau)} \tilde{C}^{N} y_{*}^{N}(\tau)\right) d W_{\tau}\right\|_{\mathcal{H}}^{2}\right)$. Applying the Itô isometry and using (49), we obtain

$$
\begin{aligned}
& \mathbb{E}\left(\left\|\int_{0}^{t}\left(e^{A(t-\tau)} \tilde{C} y_{*}(\tau)-e^{A^{N}(t-s)} \tilde{C}^{N} y_{*}^{N}(\tau)\right) d W_{\tau}\right\|_{\mathcal{H}}^{2}\right) \\
& \quad \leq 2 \cdot \mathbb{E}\left(\int_{0}^{t}\left\|\left(e^{A(t-\tau)} \tilde{C}-e^{A^{N}(t-\tau)} \tilde{C}^{N}\right) y_{*}(\tau)\right\|_{\mathcal{H}}^{2} d \tau\right)
\end{aligned}
$$

$$
\begin{align*}
& +2 \cdot \mathbb{E}\left(\int_{0}^{t}\left\|e^{A^{N}(t-\tau)} \tilde{C}^{N}\left(y_{*}(\tau)-y_{*}^{N}(\tau)\right)\right\|_{\mathcal{H}}^{2} d \tau\right) \\
& \leq \frac{c T^{2(1-2 \gamma)}}{N^{2(1-\gamma)}}\left\|y_{*}\right\|_{\mathcal{C}}^{2}+\int_{0}^{t} \frac{c}{(t-\tau)^{2 \gamma}} \mathbb{E}\left(\left\|y_{*}(\tau)-y_{*}^{N}(\tau)\right\|_{\mathcal{H}}^{2}\right) d \tau \tag{53}
\end{align*}
$$

where again we used (45) and the uniform estimate (46).
We finally combine estimates (51), (52) and (53) together in (50) and get

$$
\begin{aligned}
\mathbb{E}\left(\left\|y_{*}(t)-y_{*}^{N}(t)\right\|_{\mathcal{H}}^{2}\right) \quad & \leq \frac{c_{T}}{N^{2(1-\gamma)}} \cdot\left\|y_{*}\right\|_{\mathcal{C}}^{2}+\frac{c_{T}}{N^{2}} \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right) \\
+ & c_{T} \int_{0}^{t} \frac{c}{(t-\tau)^{2 \gamma}} \mathbb{E}\left(\left\|y_{*}(\tau)-y_{*}^{N}(\tau)\right\|_{\mathcal{H}}^{2}\right) d \tau
\end{aligned}
$$

Noting again that $\gamma<1 / 2$ and the right hand side is finite, we apply the Grönwall's inequality to obtain

$$
\begin{equation*}
\mathbb{E}\left(\left\|y_{*}(t)-y_{*}^{N}(t)\right\|_{\mathcal{H}}^{2}\right) \leq\left(\frac{c_{T} c}{N^{2(1-\gamma)}}+\frac{c_{T}}{N^{2}}\right) \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right) e^{c_{T} T^{1-2 \gamma}} \tag{54}
\end{equation*}
$$

In the last step we used the bound $\left\|y_{*}\right\|_{\mathcal{C}}^{2} \leq c \cdot \mathbb{E}\left(\left\|y_{0}\right\|_{\mathcal{H}}^{2}\right)$ from Theorem 2.2 (ii). Therefore, the right hand side of (54) tends to 0 as $N \rightarrow \infty$ and (29) holds, which means the optimal trajectories $y_{*}^{N}$ of the finite dimensional problems converge to the corresponding optimal trajectory $y_{*}$ in $\mathcal{C}$. Consequently, the convergence of the optimal controls $u_{*}^{N}$ to $u_{*}$ and the estimate (30) follows from the feedback relation (3) and the convergence estimate (26) on $B^{N \star} P^{N}$ in Theorem 4.1.

Remark 4.1 As in the deterministic case, see [3,5], it is possible to prove analogous theorems to Theorems 4.1 and 4.2 without the requirement $\mathcal{V}^{N} \subseteq \mathcal{H}$. If we assume that $(\mathcal{H},\|\cdot\|),\left(\mathcal{V}^{N},\|\cdot\|_{N}\right)$ are Hilbert spaces (in general $\mathcal{V}^{N} \nsubseteq \mathcal{H}$ ), with $e^{A t}, e^{A^{N} t}$ strongly continuous semigroups on $\mathcal{H}$ and $\mathcal{V}^{N}$ respectively, with a slight modification of the set of Assumptions 3.1.

Remark 4.2 Our results can be extended to the non-autonomous case, i.e. the case in which stochastic partial differential equations (PDEs) of the form (1) have timevarying coefficients. Approximation results for the deterministic non-autonomous case can be found in $[5,16]$.

Remark 4.3 The approximation scheme, proposed in this paper, could be extended to optimal control problems with state equations given in more general form, when stochastic perturbations are of Wick type within white noise framework, see [34].

## 5 Applications in SPDE Control

The approximation framework described in this paper can be used for a large class of parabolic systems. In particular, our assumptions are fulfilled in the case of parabolic systems with disturbance in the state and in the control. Let us consider the deterministic system

$$
\begin{equation*}
\frac{\partial y}{\partial t}=\sum_{i, j=1}^{n} \frac{\partial}{\partial x_{i}}\left(a_{i j}(x) \frac{\partial y}{\partial x_{j}}\right)+\sum_{i=1}^{n} b_{i}(x) \frac{\partial y}{\partial x_{i}}+c y+B u(t) \tag{55}
\end{equation*}
$$

for $t>0, x \in \mathcal{G} \subset \mathbb{R}^{n}$ with Dirichlet boundary conditions $\left.y\right|_{\partial \mathcal{G}}=0$ and known initial data $\left.y\right|_{t=0}=\phi$. This model appears in connection with insect dispersal investigations and was studied by Banks and Kunisch in [3]. There, the authors prove that the operators related to (55) fulfill the assumptions needed for the application of the approximation framework to the deterministic linear quadratic regulator problem. For the operators involving the deterministic part of (1), we assume the same Assumptions 3.1 as in the deterministic linear quadratic regulator problem, see hypotheses [3, (H2') p 688]. In addition, if we assume operators $C$ and $D$ to be deterministic and to satisfy Assumptions 3.1 (vii) and (viii), then the results are extended to the stochastic analogue of this system.

### 5.1 Specific Example

We consider an example coming from an important industrial task of cooling a rail in a rolling mill $[37,38]$. This problem arises in a rolling mill when different stages in the production process require different temperatures of the raw material. An infinitely long steel profile is assumed so that a 2-dimensional heat diffusion process is considered. Exploiting the symmetry of the workpiece, an artificial boundary $\Gamma_{0}$ is introduced on the symmetry axis, and $\Gamma=\bigcup \Gamma_{i}$ for $i=1, \ldots 7$. A linearized version of the model has the form

$$
\begin{aligned}
c \varrho x_{t}(\xi, t) & =\lambda \Delta x(\xi, t) \text { in } \Omega \times(0, T) \\
\lambda \partial_{\nu} x(\xi, t)+\kappa x(\xi, t) & =\kappa u_{i} \text { on } \Gamma_{i} \text { where } i=0, \ldots, 7, \\
x(\xi, 0) & =x_{0}(\xi) \text { in } \Omega
\end{aligned}
$$

for $\lambda, \kappa>0$, where $x(\xi, t)$ represents the temperature at time $t$ at point $\xi$, and $u_{i}$ represents the temperature in the profile surface $i=1, \ldots . .7$ to be controlled We define the problem on the state space $\mathcal{H} \equiv L^{2}(\Omega)$ and introduce the operator $A$ defined as $A=\lambda \Delta$, with the domain

$$
\mathcal{D}(A)=\left\{x \in H^{2}(\Omega):\left.\left(\lambda \partial_{\nu} x+\kappa x\right)\right|_{\Gamma_{i}}=0\right\}
$$

so that $-A$ is a positive self-adjoint operator and in fact $A$ generates an analic semigroup on $\mathcal{H}$. We also introduce the map $N: L^{2}(\Gamma) \rightarrow L^{2}(\Omega)$ which is defined by

$$
N \phi=h \Longleftrightarrow \Delta h=0,\left.\left(\lambda \partial_{\nu} h+\kappa h\right)\right|_{\Gamma}=\phi
$$

The map $N$ is well defined since the above elliptic problem has a unique solution and is bounded $L^{2}\left(\Gamma_{i}\right) \rightarrow L^{2}(\Omega)$. Moreover, we have that $A^{3 / 4-\epsilon} N: L^{2}\left(\Gamma_{i}\right) \rightarrow L^{2}(\Omega)$.

We rewrite the system above in the abstract formulation

$$
\begin{array}{rlrl}
c \varrho x_{t}(t) & =A(x(t)-\kappa N u) \\
x(0) & = & x_{0} & \text { in } \mathcal{H},
\end{array}
$$

and extending the action of the operator $A$ onto $L^{2}(\Omega) \rightarrow[\mathcal{D}(A)]^{\prime}$ we have

$$
\begin{aligned}
c \varrho x_{t}(t) & =A x(t)+B u \text { on }[\mathcal{D}(A)]^{\prime}, \\
x(0) & =x_{0} \quad \text { in } \mathcal{H},
\end{aligned}
$$

with the control operator $B: L^{2}\left(\Gamma_{i}\right) \rightarrow[\mathcal{D}(A)]^{\prime}$ defined by

$$
B \equiv-\kappa A N
$$

Note that the operator $B$ is unbounded when considered as acting on $L^{2}(\Gamma)$, and the control $\left.u\right|_{\Gamma_{i}}=u_{i}$.

We next incorporate noise or disturbance into the system state and control, with $W(t)$ representing a one dimensional real valued Brownian motion on a complete probability space $(\Sigma, \mathcal{F}, \mathbb{P})$. In particular, we have the system

$$
\begin{aligned}
c \varrho d x(\xi, t) & =\lambda \Delta x(\xi, t) d t+c x(\xi, t) d W_{t} \quad \text { in } \Omega \times(0, T), \\
\lambda \partial_{\nu} x(\xi, t)+\kappa x(\xi, t) & =\kappa u_{i}+r u_{i} \dot{W}_{t} \quad \text { on } \Gamma_{i}, i=0, \ldots, 7 \\
x(\xi, 0) & =x_{0}(\xi) \text { in } \Omega
\end{aligned}
$$

where the control noise operator $D$ is similarly captured by $D=-r A N$ where $r>0$. The state noise operator here is simply $C=c I$ where $c>0$.

The objective is to minimize the cost functional

$$
\begin{equation*}
J(x, u)=E\left(\int_{0}^{T}\|x(t)\|_{L^{2}(\Omega)}^{2}+\|u(t)\|_{L^{2}(\Gamma)}^{2} d t+\|x(T)\|_{L^{2}(\Omega)}^{2}\right) \tag{56}
\end{equation*}
$$

over all random variables $u \in L^{2}\left([0, T] ; L^{2}\left(\Sigma, L^{2}(\Gamma)\right)\right)$. The operator $A$ here generates an analytic semigroup while the control operator $B=A N$ satisfies $A^{-1 / 4+\epsilon} B$ : $L^{2}(\Gamma) \rightarrow L^{2}(\Omega)$. The singular estimate assumption $2.1(\mathrm{c})$ is automatically satisfied with $\gamma=1 / 4-\epsilon$ since

$$
\left\|e^{A t} B u\right\|_{L^{2}(\Omega)}=\left\|e^{A t} A^{1 / 4-\epsilon} A^{-1 / 4+\epsilon} B u\right\|_{L^{2}(\Omega)} \leq \frac{C}{t^{1 / 4-\epsilon}}\|u\|_{L^{2}(\Gamma)}
$$

by analyticity of the semigroup. Hence, the abstract approximation framework developed in this paper can be applied to this control problem.

## 6 Conclusions

We present an approximation framework for the computation of the finite time stochastic linear quadratic control problem on Hilbert spaces. We proved that the solutions of the approximate finite-dimensional DREs converge to the solutions of the infinitedimensional DREs. In addition, we prove that the sequence of solutions to the approximate finite dimensional optimal control problems, converges to the optimal solutions of the original infinite dimensional problem. Our approximation framework holds for a large class of parabolic systems and mixed parabolic hyperbolic couple systems with boundary or point control (also known as Singular Estimate Control Systems). Moreover, our results can be extended to the non-autonomous case, i.e. the case in which stochastic PDEs considered here have time-varying coefficients. In the same setting, approximation results for the infinite horizon as well as convergence rates can be developed. This is work in progress and will be reported somewhere else.

Acknowledgements The paper was partially supported by the project Solution of large-scale Lyapunov Differential Equations (P 27926) founded by the Austrian Science Foundation.

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## Noname manuscript No.

(will be inserted by the editor)

# A splitting/polynomial chaos expansion approach for stochastic evolution equations 

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Received: date / Accepted: date


#### Abstract

In this paper we combine deterministic splitting methods with a polynomial chaos expansion method for solving stochastic parabolic evolution problems. The stochastic differential equation is reduced to a system of deterministic equations that we solve explicitly by splitting methods. The method can be applied to a wide class of problems where the related stochastic processes are given uniquely in terms of stochastic polynomials. A comprehensive convergence analysis is provided and numerical experiments validate our approach.


Keywords Splitting methods • Polynomial chaos expansion.
Mathematics Subject Classification (2010) 60H15 • 65J10 • 60H40. 60H35 - 11B83.

[^9]
## 1 Introduction

Splitting methods are numerical methods for solving differential equations, both ordinary and partial differential equations (PDEs), involving operators that are decomposable into a sum of (differential) operators. These methods are used to improve the speed of calculations for problems involving decomposable operators and to solve multidimensional PDEs by reducing them to a sum of one-dimensional problems [10]. Splitting methods have been successfully applied to many types of PDEs, e.g. [14,16]. Exponential splitting methods are applied in cases when the explicit solution of a splitted equation can be computed. Such computations often rely on applying fast Fourier techniques, see for instant [38]. Resolvent splitting is used in cases when the splitted equation cannot be solved explicitly [17,34]; here we consider this type of methods.

There are also many results in the literature about the approximation of solutions of SPDEs using splitting methods, see e.g. $[2,3,4,5,9,12,15]$ and references therein. In [12] a splitting method for nonlinear stochastic equations of Schrödinger type is proposed. There the authors approximate the solution of the problem by a sequence of solutions of two types of equations: one without stochastic term and other containing only the stochastic term. They prove that an appropriate combination of the solutions of these equations converges strongly to the solution of the original problem. Exponential integrators for nonlinear Schrödinger equations with white noise dispersion were proposed in [5]. For a stochastic incompressible time-dependent Stokes equation different time-splitting methods were studied in [4]. In [2] the convergence of a DouglasRachford type splitting algorithm is presented for general SPDEs driven by linear multiplicative noise. In this work a splitting/polynomial chaos expansion is considered for stochastic evolution equations. Our approach has not been considered in the literature for solving these types of SPDEs so far.

We consider stochastic evolution equations of the form

$$
\begin{align*}
d u(t) & =((A+B) u(t)+f(t)) d t+(C u(t)+g(t)) d B(t)  \tag{1}\\
u(0) & =u^{0}
\end{align*}
$$

where $A, B$ and $C$ are differential operators acting on Hilbert space valued stochastic processes, $\left\{B_{t}\right\}_{t \geq 0}$ is a cylindrical Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f$ and $g$ are deterministic functions. In [30] equation (1) involving Gaussian noise terms was solved in an appropriate weighted Wiener chaos space. The deterministic problem that corresponds to (1), i.e., the case where $C=0$ and $g=0$, for particular $A u=\partial_{x}\left(a \partial_{x} u\right), B u=\partial_{y}\left(b \partial_{y} u\right)$ and $f$ was studied in [10]. We consider equation (1) involving a non-Gaussian noise term. Namely, we consider inhomogeneous parabolic evolution equations involving the operators that can be split in $A+B$ and uniformly distributed random inputs. These equations, can be also written in the form

$$
\begin{align*}
u_{t}(t, x, \omega) & =(A+B) u(t, x, \omega)+G(t, x, \omega) \\
u(0, x, \omega) & =u^{0}(x, \omega) \tag{2}
\end{align*}
$$

where $G$ represents the noise term, see e.g. $[20,25,28,29,30]$. The existence of a random parameter $\omega$ is due to uncertainties coming from initial conditions and/or a random force term. Therefore, the solution is considered to be a stochastic process.

Stochastic processes with finite second moments on white noise spaces can be represented in series expansion form in terms of a family of orthogonal stochastic polynomials. The classes of orthogonal polynomials are chosen depending on the underlying probability measure [19,20]. Namely, the Askey scheme of hypergeometric orthogonal polynomials and the Sheffer system [36, 37] can be used to define several discrete and continuous distribution types [39]. For example, in the case of the Gaussian measure, the orthogonal basis of the space of random variables with finite second moments is constructed by the use of the Hermite polynomials. We consider problems with non-Gaussian random inputs. The noise term is considered to be uniformly distributed. It is known that in order to obtain a square integrable solution of (1) with deterministic initial condition, it is enough to assume that the operator $A-\frac{1}{2} C C^{*}$ is elliptic and that the stochastic part (the noise term) is sufficiently regular, see e.g. [8]. In this work, the assumptions on the input data for problem (2) will be set such that the existence of a square integrable solution is always established. We do not consider solutions which are generalized stochastic processes as in $[28,30]$, since our focus is on numerical treatment.

Our approach is general enough to be applied to problems with additive noise, problems involving multiplicative noise and problems with convolutiontype noise [28]. For instance, with this approach the heat equation with random potential, the heat equation in random (inhomogeneous and anisotropic) media and the Langevin equation can be solved. If (1) does have a sufficiently regular solution, this solution can be projected on an orthonormal basis in some Hilbert space, resulting in a system of equations for the corresponding Fourier coefficients. Thus, we use the so-called polynomial chaos method or the chaos expansion method and define the solution of (1) as a formal Fourier series with the coefficients computed by solving the corresponding system of deterministic PDEs [30]. With this method, the deterministic part of a solution is separated from its random part. Particularly, in the case of Gaussian noise, the orthonormal basis of stochastic polynomials involves the Hermite polynomials and in the case when the noise term is uniformly distributed, the orthonormal basis involves the Legendre polynomials [36]. By construction, the solution is strong in the probabilistic sense. It is uniquely determined by the coefficients, free terms, initial condition and the noise term. The coefficients in the Fourier series are uniquely determined by equation (1) and are computed by solving (numerically) the corresponding lower-triangular system of deterministic parabolic equations. The polynomial chaos method has been successfully applied for solving general classes of SPDEs. The list of references is long, here we mention just a few $[20,28,32,33]$. In $[25,26,27]$ this approach has been recently applied to the stochastic optimal regulator control problem [13].

Practical application of the Wiener polynomial chaos involves two truncations, truncation with respect to the number of the random variables and
truncation with respect to the order of the orthogonal Askey polynomials used (in the particular case considered, the Legendre polynomials), see e.g. [21].

The paper is organized as follows. In Section 2 we introduce the notation and basic concepts used in the following sections. In Section 3 we present the splitting/polynomial chaos expansion approach and provide a complete convergence analysis. Finally, in Section 4 we validate our approach with a numerical experiment.

## 2 Preliminaries

In this section we briefly recall polynomial chaos representations of random variables and stochastic processes. Particular emphasis is given to Legendre polynomials and the corresponding Wiener-Legendre expansion, and to the Karhunen-Loève expansion.

### 2.1 Polynomial chaos representation

Let $\mathcal{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ be the set of sequences of non-negative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right), \alpha_{i} \in \mathbb{N}_{0}$, $i=1,2, \ldots, m, m \in \mathbb{N}$. Particularly, $(0,0, \ldots)$ is the zero vector. We denote by $\varepsilon^{(k)}=(0, \cdots, 0,1,0, \cdots), k \in \mathbb{N}$ the $k$ th unit vector. The length of $\alpha \in \mathcal{I}$ is the sum of its components $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$.

First, we briefly recall the main results from the Wiener-Itô chaos expansion. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with the Gaussian probability measure $\mu$ and let $(L)^{2}=L^{2}(\Omega, \mathcal{F}, \mu)$ denote the space of random variables with finite second moments on the probability space $(\Omega, \mathcal{F}, \mu)$. The space $(L)^{2}$ is a Hilbert space. The scalar product of two random variables $F, G \in(L)^{2}$ is given by

$$
(F(\omega), G(\omega))_{(L)^{2}}=\mathbb{E}(F(\omega) G(\omega))
$$

where $\mathbb{E}$ denotes the expectation with respect to the measure $\mu$.
Let $\left\{h_{n}\right\}_{n \in \mathbb{N}_{0}}$ be the Hermite polynomials given through the recursion

$$
\begin{aligned}
h_{0}(x) & =1 \\
h_{1}(x) & =x \\
h_{n+1}(x) & =x h_{n}(x)+n h_{n-1}(x) \quad \text { for } n \geq 2, x \in \mathbb{R}
\end{aligned}
$$

Define the $\alpha$ th Fourier-Hermite polynomial as the product

$$
H_{\alpha}(\boldsymbol{\xi}(\omega))=H_{\left(\alpha_{1}, \alpha_{2}, \ldots\right)}\left(\left(\xi_{1}(\omega), \xi_{2}(\omega), \ldots\right)\right)=\prod_{i \in \mathbb{N}} h_{\alpha_{i}}\left(\xi_{i}(\omega)\right),
$$

represented in terms of the Hermite polynomials evaluated at appropriate components of the sequence $\boldsymbol{\xi}=\left(\xi_{1}, \xi_{2}, \ldots\right)$ of independent Gaussian variables
with zero mean and unit variance. Especially,

$$
\begin{aligned}
H_{(0,0, \ldots)}(\boldsymbol{\xi}(\omega)) & =\prod_{i \in \mathbb{N}} h_{0}\left(\xi_{i}(\omega)\right)=1 \quad \text { and } \\
H_{\varepsilon^{(k)}}(\boldsymbol{\xi}(\omega)) & =h_{1}\left(\xi_{k}(\omega)\right) \prod_{i \neq k, i \in \mathbb{N}} h_{0}\left(\xi_{i}(\omega)\right)=\xi_{k}(\omega), \quad k \in \mathbb{N} .
\end{aligned}
$$

Theorem 1 (Wiener-Itô chaos expansion theorem, [20]) Each square integrable random variable $F \in(L)^{2}$ can be uniquely represented in the form

$$
\begin{equation*}
F(\boldsymbol{\xi}(\omega))=\sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\boldsymbol{\xi}(\omega)), \tag{3}
\end{equation*}
$$

where $f_{\alpha} \in \mathbb{R}$ for $\alpha \in \mathcal{I}$. Moreover, it holds

$$
\|F\|_{(L)^{2}}^{2}=\sum_{\alpha \in \mathcal{I}} f_{\alpha}^{2}\left\|H_{\alpha}\right\|_{(L)^{2}}^{2}<\infty .
$$

The family of stochastic polynomials $\left\{H_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ forms an orthogonal basis of $(L)^{2}$ such that

$$
\begin{equation*}
\mathbb{E}\left(H_{\alpha} H_{\beta}\right)=\alpha!\delta_{\alpha \beta}, \tag{4}
\end{equation*}
$$

for all $\alpha, \beta \in \mathcal{I}$, see [20]. Here $\delta_{\alpha \beta}$ denotes the Kronecker delta. Thus, the sequence of the coefficients in (3), which is a sequence of real numbers, is obtained from $f_{\alpha}=\frac{1}{\alpha!} \mathbb{E}\left(F H_{\alpha}\right), \alpha \in \mathcal{I}$. Also, we have

$$
\mathbb{E}\left(H_{(0,0, \ldots)}\right)=1 \quad \text { and } \quad \mathbb{E}\left(H_{\alpha}\right)=0 \text { for }|\alpha|>0 .
$$

Property (4) is a consequence of the orthogonality of the Hermite polynomials

$$
\int_{\mathbb{R}} h_{n}(x) h_{m}(x) d \mu(x)=\frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} h_{n}(x) h_{m}(x) e^{-\frac{x^{2}}{2}} d x=n!\delta_{m, n}
$$

for all $m, n \in \mathbb{N}$.
In [39] it was shown that the initial construction of the Wiener chaos which corresponds to the Gaussian measure and Hermite polynomials can be extended also to other types of measures, where instead of the Hermite polynomials other classes of orthogonal polynomials from the Askey scheme [36] are used. For example, the Gamma distribution corresponds to the Laguerre polynomials and thus to the Wiener-Laguerre chaos, while the Beta distribution is related to the Jacobi polynomials and thus to the Wiener-Jacobi chaos etc. Moreover, in [36] it was proven that the optimal exponential convergence rate for each Wiener-Askey chaos can be realized.

In this paper, we deal with stochastic evolution problems with non-Gaussian random inputs which are uniformly distributed. From the Askey scheme of orthogonal polynomials it follows that the uniform distribution, as a special case of the Beta distribution, corresponds to the special class of the Jacobi polynomials, the Legendre polynomials. Therefore, we are going to work with the Wiener-Legendre polynomial chaos.
2.2 Wiener-Legendre chaos representation

Denote by $\left\{p_{n}(x)\right\}_{n \in \mathbb{N}_{0}}$ the Legendre polynomials on $[-1,1]$. These polynomials are defined by the recursion

$$
\begin{align*}
p_{0}(x) & =1 \\
p_{1}(x) & =x  \tag{5}\\
(n+1) p_{n+1}(x) & =(2 n+1) x p_{n}(x)-n p_{n-1}(x) \quad \text { for } n \geq 1 .
\end{align*}
$$

They can be also obtained from Rodrigues' formula [36]

$$
p_{n}(x)=\frac{1}{2^{n} n!} \frac{d^{n}}{d x^{n}}\left(x^{2}-1\right)^{n}
$$

The Legendre polynomials satisfy the second order differential equation $\left(1-x^{2}\right) p_{n}^{\prime \prime}(x)-2 x p_{n}^{\prime}(x)+n(n+1) p_{n}(x)=0$, which appears in physics when solving the Laplace equation in spherical coordinates [36]. These polynomials are orthogonal and it holds

$$
\begin{equation*}
\int_{-1}^{1} p_{m}(x) p_{n}(x) d x=\frac{2}{2 n+1} \delta_{m, n}, \quad m, n \in \mathbb{N}_{0} \tag{6}
\end{equation*}
$$

The previous property (6) is equivalent to the orthogonality relation with respect to the uniform measure, i.e., the measure with the constant weighting function $w(x)=\frac{1}{2}$.

We consider square integrable random variables and stochastic processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the measure $\mathbb{P}$ generated by the uniform distribution. Let $(L)^{2}=L^{2}(\Omega, \mathcal{F}, \mathbb{P})$ be the Hilbert space of square integrable random variables with respect to the measure $\mathbb{P}$.
We define the $\alpha$ th Fourier-Legendre polynomial as the product

$$
\begin{equation*}
L_{\alpha}(\boldsymbol{\xi}(\omega))=\prod_{i \in \mathbb{N}} p_{\alpha_{i}}\left(\xi_{i}(\omega)\right), \quad \alpha=\left(\alpha_{1}, \alpha_{2}, \ldots\right) \in \mathcal{I}, \tag{7}
\end{equation*}
$$

where $\left\{p_{n}\right\}_{n \in \mathbb{N}_{0}}$ are the Legendre polynomials and $\boldsymbol{\xi}=\left(\xi_{1}, \xi_{2}, \ldots\right)$ is a sequence of independent uniformly distributed random variables with zero mean and unit variance. Note that the product in (7) is finite since each $\alpha \in \mathcal{I}$ has only finitely many nonzero components. Particularly,

$$
\begin{aligned}
L_{(0,0, \ldots)}(\boldsymbol{\xi}(\omega)) & =1 \quad \text { and } \\
L_{\varepsilon^{(k)}}(\boldsymbol{\xi}(\omega)) & =\xi_{k}(\omega) \quad \text { for } k \in \mathbb{N} .
\end{aligned}
$$

We also have

$$
\begin{equation*}
\mathbb{E}\left(L_{(0,0, \ldots)}\right)=1 \quad \text { and } \quad \mathbb{E}\left(L_{\alpha}(\boldsymbol{\xi}(\omega))\right)=0 \text { for }|\alpha|>0, \tag{8}
\end{equation*}
$$

since $\boldsymbol{\xi}(\omega)$ has zero mean. Moreover, from the orthogonality (6) of the Legendre polynomials we obtain that the family of the Fourier-Legendre polynomials $\left\{L_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ is also orthogonal and

$$
\begin{equation*}
\mathbb{E}\left(L_{\alpha} L_{\beta}\right)=\mathbb{E} L_{\alpha}^{2} \delta_{\alpha, \beta}=\frac{1}{\prod_{k \in \mathbb{N}}\left(2 \alpha_{k}+1\right)} \delta_{\alpha \beta} \tag{9}
\end{equation*}
$$

for all $\alpha, \beta \in \mathcal{I}$.
Now we formulate the representation of a random variable in an analoguous way to Theorem 1.
Theorem 2 (Wiener-Legendre chaos expansion theorem) Each random variable $F \in(L)^{2}$ can be uniquely represented in the form

$$
\begin{equation*}
F(\boldsymbol{\xi}(\omega))=\sum_{\alpha \in \mathcal{I}} f_{\alpha} L_{\alpha}(\boldsymbol{\xi}(\omega)) \tag{10}
\end{equation*}
$$

where

$$
f_{\alpha}=\frac{1}{\mathbb{E}\left(L_{\alpha}^{2}\right)} \mathbb{E}\left(F L_{\alpha}\right), \quad \alpha \in \mathcal{I}
$$

is the corresponding sequence of real coefficients. Moreover, it holds

$$
\|F\|_{(L)^{2}}^{2}=\sum_{\alpha \in \mathcal{I}} f_{\alpha}^{2} \mathbb{E} L_{\alpha}^{2}=\sum_{\alpha \in \mathcal{I}} \frac{f_{\alpha}^{2}}{\prod_{k \in \mathbb{N}}\left(2 \alpha_{k}+1\right)}<\infty .
$$

Remark 1 We note here that the chaos representation (10) of a random variable with finite second moment with respect to the underlying probability measure $\mathbb{P}$ can be extended also to square integrable stochastic processes, where a family of real numbers $f_{\alpha}$ is replaced by an appropriate family of functions with values in a certain Banach space $X$. Particularly, an $X$-valued square integrable process $u=u(t, x, \omega)$ can be represented as

$$
\begin{equation*}
u(t, x, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t, x) L_{\alpha}(\boldsymbol{\xi}(\omega)) \tag{11}
\end{equation*}
$$

In this context, the notation $u \in C([0, T], X) \otimes(L)^{2}$ means that the coefficients of the process $u$ given in the form (11) satisfy $u_{\alpha} \in C([0, T], X)$ for all $\alpha \in \mathcal{I}$. Additionally, the estimate

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{C([0, T], X)}^{2} \mathbb{E} L_{\alpha}^{2}=\sum_{\alpha \in \mathcal{I}} \sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
$$

holds, where the expectation $\mathbb{E} L_{\alpha}^{2}$ is given by (9). Similarly, a process $u \in$ $C^{1}([0, T], X) \otimes(L)^{2}$ can be represented in the form (11), where its coefficients $u_{\alpha} \in C^{1}([0, T], X)$ for all $\alpha \in \mathcal{I}$. Moreover, it holds

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{C^{1}([0, T], X)}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
$$

### 2.3 Karhunen-Loève expansion

The Karhunen-Loève expansion gives a way to represent a stochastic process as an infinite linear combination of orthogonal functions on a bounded interval. It is used to represent spatially varying random inputs in stochastic models. Various applications of the Karhunen-Loève expansion can be found in uncertainty propagation through dynamical systems with random parameter functions $[7,11,24]$.

Theorem 3 (Karhunen-Loève expansion theorem, [11]) Let $v(x, \omega)$ be a spatially varying square integrable random field defined over the spatial domain D and a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with mean $\bar{v}(x)$ and continuous covariance function $C_{v}\left(x_{1}, x_{2}\right)$. Then, $v(x, \omega)$ can be represented in the form

$$
\begin{equation*}
v(x, \omega)=\bar{v}(x)+\sum_{k \in \mathbb{N}} \sqrt{\lambda_{k}} e_{k}(x) Z_{k}(\omega), \tag{12}
\end{equation*}
$$

where $\lambda_{k}$ and $e_{k}, k \in \mathbb{N}$ are the eigenvalues and eigenfunctions of the covariance function, i.e., they solve the integral equation

$$
\begin{equation*}
\int_{\mathrm{D}} C_{v}\left(x_{1}, x_{2}\right) e_{k}\left(x_{2}\right) d x_{2}=\lambda_{k} e_{k}\left(x_{1}\right), \quad x_{1} \in \mathrm{D}, k \in \mathbb{N} \tag{13}
\end{equation*}
$$

and $Z_{k}$ are uncorrelated zero mean random variables that have unit variance.
For some particular covariance functions $C_{v}$, the eigenpairs $\left(\lambda_{k}, e_{k}\right)_{k \in \mathbb{N}}$ are known a priory, and the eigenvalues $\lambda_{k}$ decay as $k$ increases. In general, the eigenvalues and eigenvectors of the covariance function have to be calculated numerically, i.e., by solving the discrete version of (13). This constitutes the bottleneck of the method as it requires a large number of calculations.

In practical applications, the series are truncated, i.e., the random field is approximated by

$$
\begin{equation*}
\tilde{v}(x, \omega)=\bar{v}(x)+\sum_{k=1}^{n} \sqrt{\lambda_{k}} e_{k}(x) Z_{k}(\omega), \tag{14}
\end{equation*}
$$

which is the finite representation with the minimal mean square error over all such finite representations.

Remark 2 Comparing the representation (12) with the form (10) we conclude that the random field $v$ is represented in terms of the Wiener-Askey polynomial chaos of orders zero and one, i.e., it is equivalent to the representation

$$
\begin{equation*}
v(x, \omega)=\bar{v}(x)+\sum_{k \in \mathbb{N}} v_{\varepsilon^{(k)}}(x) L_{\varepsilon^{(k)}}(\mathbf{Z}(\omega)), \tag{15}
\end{equation*}
$$

since $Z_{k}(\omega)=L_{\varepsilon^{(k)}}(\mathbf{Z}(\omega)), k \in \mathbb{N}$ with $\mathbf{Z}(\omega)=\left(Z_{1}(\omega), Z_{2}(\omega), \ldots\right)$ being a sequence of uncorrelated uniformly distributed zero mean random variables that have unit variance. The truncated version of the representation (15) is given by

$$
\begin{equation*}
\tilde{v}(x, \omega)=\bar{v}(x)+\sum_{k=1}^{n} v_{\varepsilon^{(k)}}(x) L_{\varepsilon^{(k)}}(\mathbf{Z}(\omega)) . \tag{16}
\end{equation*}
$$

There, $n$ corresponds to the finite number of random variables of the sequence $\mathbf{Z}=\left(Z_{1}, Z_{2}, \ldots Z_{n}\right)$ that are applied in the approximation. This is used in Section 4.

More details on methods based on stochastic polynomial representations can be found, for example, in $[1,6,11,24,39]$.

## 3 Splitting methods for SPDEs

In this section, we introduce a new numerical method which combines the Wiener-Askey polynomial chaos expansion [39] with deterministic splitting methods [10]. The method is then applied to problem (1) with non-Gaussian random inputs. First, we are going to state a theorem on the existence and uniqueness of the solution of (2). Then, we recall some convergence results of splitting methods in the deterministic setting. Finally, we provide a convergence analysis of our approach which is the main result of this section. Thorough this section we denote $\mathcal{L}=A+B$.

### 3.1 Existence and uniqueness of the solution

Recall that a solution of the considered stochastic evolution problem (2) belongs to the space of square integrable stochastic processes whose coefficients are continuously differentiable deterministic functions with values in $X$.

Definition 1 A process $u$ is a (classical) solution of (2) if $u \in C([0, T], X) \otimes$ $(L)^{2} \cap C^{1}((0, T], X) \otimes(L)^{2}$ and if $u$ satisfies (2) pointwise.

Let the following assumptions hold:
(A1) Let $\mathcal{L}$ be a coordinatewise operator defined on some domain $\mathcal{D}(\mathcal{L})$ dense in $X$, i.e.,

$$
\mathcal{L} u=\sum_{\alpha \in \mathcal{I}} \mathcal{L}\left(u_{\alpha}\right) L_{\alpha}
$$

for $u$ of the form (11). Moreover, let $\mathcal{L}$ be the infinitesimal generator of a $C_{0}$ semigroup $\left(S_{t}\right)_{t \geq 0}$ of type $(M, w)$, i.e.,

$$
\left\|S_{t}\right\|_{L(X)} \leq M e^{w t}, \quad t \geq 0
$$

for some $M>0$ and $w \in \mathbb{R}$.
(A2) Let $u^{0} \in X \otimes(L)^{2}$ and $\mathcal{L} u^{0} \in X \otimes(L)^{2}$, i.e.,

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2}<\infty \quad \text { and } \quad \sum_{\alpha \in \mathcal{I}}\left\|\mathcal{L} u_{\alpha}^{0}\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
$$

(A3) The noise process is given in the form $G(t, x, \omega)=\sum_{\alpha \in \mathcal{I}} g_{\alpha}(t, x) L_{\alpha} \in$ $C^{1}([0, T], X) \otimes(L)^{2}$, i.e., it holds

$$
\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{C^{1}([0, T], X)}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
$$

We note here that the derivative is a coordinatewise operator, i.e., for a process $u \in C^{1}([0, T], X) \otimes(L)^{2}$ it holds

$$
\frac{d}{d t} u(t, \omega)=\frac{d}{d t}\left(\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) L_{\alpha}(\boldsymbol{\xi}(\omega))\right)=\sum_{\alpha \in \mathcal{I}}\left(\frac{d}{d t} u_{\alpha}(t)\right) L_{\alpha}(\boldsymbol{\xi}(\omega))
$$

Theorem 4 (Existence and uniqueness of the solution) If the assumptions (A1)-(A3) hold, then the stochastic Cauchy problem

$$
\begin{equation*}
u_{t}(t, \omega)=\mathcal{L} u(t, \omega)+G(t, \omega), \quad u(0, \omega)=u^{0}(\omega) \tag{17}
\end{equation*}
$$

has a unique solution

$$
\begin{equation*}
u(t, \omega)=\sum_{\alpha \in \mathcal{I}}\left(S_{t} u_{\alpha}^{0}+\int_{0}^{t} S_{t-s} g_{\alpha}(s) d s\right) L_{\alpha}(\omega) \tag{18}
\end{equation*}
$$

in $C^{1}([0, T], X) \otimes(L)^{2}$.
Proof We present the main steps of the proof. We are looking for a solution in chaos representation form

$$
u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) L_{\alpha}(\omega)
$$

Then, by applying the chaos expansion method, the stochastic equation (17) is transformed to the infinite system of deterministic problems

$$
\begin{align*}
\frac{d}{d t} u_{\alpha}(t) & =\mathcal{L} u_{\alpha}(t)+g_{\alpha}(t)  \tag{19}\\
u_{\alpha}(0) & =u_{\alpha}^{0}
\end{align*}
$$

for all $\alpha \in \mathcal{I}$ that can be solved in parallel. Since $g_{\alpha} \in C^{1}([0, T], X)$ the inhomogeneous initial value problem (19) has a solution $u_{\alpha}(t) \in C^{1}((0, T], X)$ for all $\alpha \in \mathcal{I}$. Moreover, the solution $u_{\alpha}$ is given by

$$
u_{\alpha}(t)=S_{t} u_{\alpha}^{0}+\int_{0}^{t} S_{t-s} g_{\alpha}(s) d s, \quad t \in[0, T]
$$

see [35]. Thus, for all fixed $\alpha \in \mathcal{I}$ the solution $u_{\alpha}(t)$ exists for all $t \in[0, T]$, and it is a unique classical solution on the whole interval $[0, T]$. Also,

$$
\frac{d}{d t} u_{\alpha}(t)=S_{t} \mathcal{L} u_{\alpha}^{0}+\int_{0}^{t} S_{t-s} \frac{d}{d s} g_{\alpha}(s) d s+S_{t} g_{\alpha}(0), \quad \alpha \in \mathcal{I}, t \in[0, T]
$$

Moreover, the series $\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) L_{\alpha}$ converges in $C^{1}([0, T], X) \otimes(L)^{2}$. Namely, from the assumptions ( $A 1$ )-(A3) we obtain

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{C^{1}([0, T], X)}^{2} \mathbb{E} L_{\alpha}^{2}=\sum_{\alpha \in \mathcal{I}}\left(\sup _{t \in[0, T]}\left\|u_{\alpha}(t)\right\|_{X}^{2}+\sup _{t \in[0, T]}\left\|\frac{d}{d t} u_{\alpha}(t)\right\|_{X}^{2}\right) \mathbb{E} L_{\alpha}^{2} \\
& \leq c \sum_{\alpha \in \mathcal{I}}\left(\left\|u_{\alpha}^{0}\right\|_{X}^{2}+\left\|\mathcal{L} u_{\alpha}^{0}\right\|_{X}^{2}+\left\|g_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}\right) \mathbb{E} L_{\alpha}^{2}<\infty
\end{aligned}
$$

where $c=c(M, w, T)$ is a constant depending on $M, w$ and $T$.
Remark 3 If an operator $A$ is the infinitesimal generator of a $C_{0}$ semigroup and $B$ is a bounded operator then the operator $\mathcal{L}=A+B$ is also the infinitesimal generator of a $C_{0}$ semigroup and Theorem 4 holds. In particular, Theorem 4 also holds for analytic semigroups.
3.2 Splitting methods for deterministic problems

We briefly recall the convergence of two operator resolvent splitting methods: resolvent Lie splitting (a first-order method) and trapezoidal resolvent splitting (a second-order method). Resolvent splitting methods for the time integration of abstract evolution equations were studied in [17]. The convergence properties of splitting methods for inhomogeneous evolution equations were analyzed in [34]. Other splitting methods were also considered in the literature. For example, exponential splitting methods for homogeneous problems with unbounded operators were presented in [14,16]. The inhomogeneous case was studied in [10]. Error bounds for exponential operator splittings were further discussed in [23].

### 3.2.1 Analytic setting

Let $X$ be an arbitrary Hilbert space with norm denoted by $\|\cdot\|$. Let $X^{*}$ be the dual space of $X$. For $t \in[0, T]$ we consider the inhomogeneous evolution equation

$$
\begin{align*}
\frac{d}{d t} u(t) & =\mathcal{L} u(t)+g(t)  \tag{20}\\
& =A u(t)+B u(t)+g(t), \quad u(0)=u^{0}
\end{align*}
$$

where $(\mathcal{D}(\mathcal{L}), \mathcal{L}),(\mathcal{D}(A), A)$ and $(\mathcal{D}(B), B)$ are linear unbounded operators in $X$ such that $\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ and $g:[0, T] \rightarrow X$. We recall the main results from [17] and [34].

Let the following assumptions hold:
(a1) The operators $(\mathcal{D}(\mathcal{L}), \mathcal{L}),(\mathcal{D}(A), A)$ and $(\mathcal{D}(B), B)$ are maximal dissipative and densely defined in $X$.
(a2) $\mathcal{D}\left(\mathcal{L}^{2}\right) \subseteq \mathcal{D}(A B)$
(a3) Let $0 \in \rho(\mathcal{L})$, let $\mathcal{L}^{-1} g(t) \in \mathcal{D}(A B)$ for all $t \in[0, T]$ and

$$
\max _{0 \leq t \leq T}\left\|A B \mathcal{L}^{-1} g(t)\right\| \leq c
$$

with a moderate constant $c$.
Recall that an operator $(\mathcal{D}(G), G)$ is maximal dissipative in $X$ if the following conditions hold:
(i) for every $x \in \mathcal{D}(G)$ there exists an element $f \in F(x)=\left\{h \in X^{*}: h(x)=\right.$ $\left.\|x\|^{2}=\|h\|^{2}\right\} \subseteq X^{*}$ such that $\operatorname{Re} f(G x) \leq 0$ and
(ii) range $(I-G)=X$.

Since we assumed that $X$ is a Hilbert space, every maximal dissipative operator in $X$ is densely defined. The assumption (a1) is equivalent to claiming
that the operators generate $C_{0}$ semigroups of contractions on $X$, see [35]. Additionally, from (a1) the following estimates hold

$$
\left\|(I-h A)^{-1}\right\| \leq 1 \quad \text { and } \quad\left\|(I-h B)^{-1}\right\| \leq 1 \quad \text { for all } h \geq 0
$$

We recall briefly the results from regularity theory for analytic semigroups needed in the following sections.

Theorem 5 ([31]) Let $\mathcal{L}$ be the generator of an analytic semigroup and let the data of problem (20) satisfy

$$
u^{0} \in \mathcal{D}(\mathcal{L}), \quad g \in C^{\theta}([0, T], X)
$$

for some $\theta>0$. Then, the exact solution of problem (20) is given by the variation of constants formula

$$
\begin{equation*}
u(t)=e^{t \mathcal{L}} u^{0}+\int_{0}^{t} e^{(t-\tau) \mathcal{L}} g(\tau) d \tau, \quad 0 \leq t \leq T \tag{21}
\end{equation*}
$$

It possesses the regularity

$$
u \in C^{1}([0, T], X) \cap C([0, T], D(\mathcal{L}))
$$

The same regularity is obtained if $g$ is only continuous but has a slightly improved spatial regularity, see [31, Corollary 4.3.9].

Theorem 6 ([34]) Let $\mathcal{L}$ be the generator of an analytic semigroup. Under the further assumptions

$$
\begin{equation*}
u^{0} \in D(\mathcal{L}), \quad \mathcal{L} u^{0}+g(0) \in D(\mathcal{L}), \quad g \in C^{1+\theta}([0, T], X) \tag{22}
\end{equation*}
$$

for some $\theta>0$, the solution (21) of the evolution equation (20) possesses the improved regularity

$$
\begin{equation*}
u \in C^{2}([0, T], X) \cap C^{1}([0, T], D(\mathcal{L})) \tag{23}
\end{equation*}
$$

In the following we present two deterministic resolvent splitting methods [22], the resolvent Lie splitting and the resolvent trapezoidal splitting, that were both applied to inhomogeneous evolution equations (20) in [34].

### 3.2.2 Resolvent Lie splitting

The exact solution of the evolution equation (20) is given by the variation of constants formula (21). Then, at time $t_{n+1}=t_{n}+h$, with a positive step size $h$, the solution can be written as

$$
u\left(t_{n+1}\right)=e^{h \mathcal{L}} u\left(t_{n}\right)+\int_{0}^{h} e^{(h-s) \mathcal{L}} g\left(t_{n}+s\right) d s
$$

After expanding $g\left(t_{n}+s\right)$ in Taylor form we obtain
$u\left(t_{n+1}\right)=e^{h \mathcal{L}} u\left(t_{n}\right)+\int_{0}^{h} e^{(h-s) \mathcal{L}}\left(g\left(t_{n}\right)+s g^{\prime}\left(t_{n}\right)+\int_{t_{n}}^{t_{n}+s}\left(t_{n}+s-\tau\right) g^{\prime \prime}(\tau) d \tau\right) d s$,
see [34]. For resolvent Lie splitting, the numerical solution of (20) at time $t_{n+1}$ is denoted by $u^{n+1}$ and it is given by

$$
\begin{equation*}
u^{n+1}=(I-h B)^{-1}(I-h A)^{-1}\left(u^{n}+h g\left(t_{n}\right)\right) \tag{24}
\end{equation*}
$$

Theorem 7 (Resolvent Lie splitting, [34]) Let the assumptions (a1), (a2) and (a3) be fulfilled and let the solution satisfy (23). Then the resolvent Lie splitting (24) is first-order convergent, i.e., the global error satisfies the bound

$$
\begin{equation*}
\left\|u\left(t_{n}\right)-u^{n}\right\| \leq C h, \quad 0 \leq t_{n} \leq T \tag{25}
\end{equation*}
$$

with a constant $C$ that can be chosen uniformly on $[0, T]$ and, in particular, independently of $n$ and $h$.

Remark 4 The constant $C$ in (25) depends on derivatives of the solution $u$ and on $A B \mathcal{L}^{-1} g(t)$, which are uniformly bounded on $[0, T]$ due to the asumptions of Theorem 7. A detailed proof is given in [34].

In particular, for a homogeneous evolution problem $(g=0)$ the global error (25) can be estimated as

$$
\left\|u\left(t_{n}\right)-u^{n}\right\| \leq \operatorname{ch}\left(\left\|u^{0}\right\|+\left\|\mathcal{L} u^{0}\right\|+\left\|\mathcal{L}^{2} u^{0}\right\|\right)
$$

where the positive constant $c$ is independent on $n$ and $h$, see [17].
We note that the full-order convergence of Lie resolvent splitting only requires additional smoothness in space of the inhomogeneity $g$.

### 3.2.3 The trapezoidal splitting

For a trapezoidal splitting method, the numerical solution of (20) at time $t_{n+1}=t_{n}+h$ with a positive time step size $h$ is given by
$u^{n+1}=\left(I-\frac{h}{2} B\right)^{-1}\left(I-\frac{h}{2} A\right)^{-1}\left(\left(I+\frac{h}{2} A\right)\left(I+\frac{h}{2} B\right) u^{n}+\frac{h}{2}\left(g\left(t_{n}\right)+g\left(t_{n+1}\right)\right)\right)$
with $u^{0}=u(0)$.
As we are considering a second-order method, we need more regularity of the solution. For analytic semigroups, this requirement can be expressed in terms of the data. The following modification of the assumption (a3) is needed:
(a4) Let $0 \in \rho(\mathcal{L})$, let $\mathcal{L}^{-1} g^{\prime}(t) \in \mathcal{D}(A B)$ for all $t \in[0, T]$ and

$$
\max _{0 \leq t \leq T}\left\|A B \mathcal{L}^{-1} g^{\prime}(t)\right\| \leq c
$$

with a moderate constant $c$.
Since we assumed $X$ to be a Hilbert space, it follows from assumption (a1) that the estimates

$$
\left\|(I+h A)(I-h A)^{-1}\right\| \leq 1 \quad \text { and } \quad\left\|(I+h B)(I-h B)^{-1}\right\| \leq 1
$$

hold for all $h>0$.

Theorem 8 ([34]) Let $\mathcal{L}$ be the generator of an analytic semigroup. If

$$
\begin{align*}
g & \in C^{2+\theta}([0, T], X), \\
u^{0} & \in \mathcal{D}(\mathcal{L}), \quad \mathcal{L} u^{0}+g(0) \in \mathcal{D}(\mathcal{L}), \quad \mathcal{L}^{2} u^{0}+\mathcal{L} g(0)+g^{\prime}(0) \in \mathcal{D}(\mathcal{L}) \tag{27}
\end{align*}
$$

for some $\theta>0$, then the exact solution (21) of the inhomogeneous evolution equation (20) satisfies

$$
\begin{equation*}
u \in C^{3}([0, T], X) \cap C^{2}([0, T], D(\mathcal{L})) \tag{28}
\end{equation*}
$$

Theorem 9 (The trapezoidal splitting method, [34]) Let the assumptions (a1), (a2) and (a4) be fulfilled and let the solution satisfy (28). Then the trapezoidal splitting method (26) is second-order convergent, i.e., the global error satisfies the bound

$$
\begin{equation*}
\left\|u\left(t_{n}\right)-u^{n}\right\| \leq C h^{2}, \quad 0 \leq t_{n} \leq T \tag{29}
\end{equation*}
$$

with a constant $C$ that can be chosen uniformly on $[0, T]$ and, in particular, independently of $n$ and $h$.

Remark 5 The constant $C$ in (29) depends on derivatives of the solution $u$ and on $A B \mathcal{L}^{-1} g^{\prime}(t)$, which are uniformly bounded on $[0, T]$ due to the asumptions of Theorem 9. More details are given in [34].

### 3.3 Convergence analysis

In order to solve problem (17) numerically, we approximate the solution $u$ by the truncated chaos representation form

$$
\begin{equation*}
\tilde{u}=\sum_{\alpha \in \mathcal{I}_{m, K}} u_{\alpha} L_{\alpha}, \tag{30}
\end{equation*}
$$

where $\mathcal{I}_{m, K}=\left\{\alpha \in \mathcal{I}: \alpha=\left(\alpha_{1}, \ldots, \alpha_{m}, 0,0, \ldots\right),|\alpha| \leq K\right\}$. Here, $K \in \mathbb{N}$ is the highest degree of Legendre polynomials and $m \in \mathbb{N}$ is the number of random variables we want to use in the approximation (30). The $m$-dimensional random vector $\boldsymbol{\xi}=\left(\xi_{1}, \ldots, \xi_{m}\right)$ has independent and identically distributed components $\xi_{i} \sim \mathcal{U}([-1,1])$ for $i=1, \ldots, m$. The choice of $m$ and $K$ influences the accuracy of the approximation. They can be chosen so that the norm of the approximation remainder $u-\tilde{u}$ is smaller than a given tolerance. The sum in (30) has

$$
\begin{equation*}
P=\frac{(m+K)!}{m!K!} \tag{31}
\end{equation*}
$$

terms, which means that $P$ coefficients of the solution will be computed. Thus, only the first $P$ equations of the system (19) are solved and in this way the approximation of the solution of the system is obtained. The global error of the proposed numerical scheme depends on the error generated by the truncation
of the chaos expansion and the error of the discretisation method. Also, the statistics $\mathbb{E} \tilde{u}$ and $\operatorname{Var} \tilde{u}$ of the approximated solution can be calculated in terms of the obtained discretized coefficients. For more details on the truncation (30) see for instance [39]. In the following, we consider the two numerical resolvent splitting methods, Lie splitting and trapezoidal splitting, and provide error analysis for both of them.

Theorem 10 (Error generated by the truncation of the WienerLegendre chaos expansion) Let $\tilde{u}$ denote the truncated chaos representation of the solution $u$ of the stochastic evolution problem (17) given in the form (30). Let the assumptions (A1)-(A3) hold. Then, $\tilde{u}$ approximates the solution $u$ and the approximation error satisfies the a priori bound

$$
\begin{align*}
& \|u-\tilde{u}\|_{C^{1}([0, T], X) \otimes(L)^{2}}^{2} \\
& \quad \leq c \sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, K}}\left(\left\|u_{\alpha}^{0}\right\|_{X}^{2}+\left\|\mathcal{L} u_{\alpha}^{0}\right\|_{X}^{2}+\left\|g_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}\right) \mathbb{E} L_{\alpha}^{2}<\infty . \tag{32}
\end{align*}
$$

Proof The approximation error due to the elimination of the higher order components of the Wiener-Legendre chaos expansion and the truncation of the noise term is obtained by

$$
\begin{aligned}
& \|u-\tilde{u}\|_{C^{1}([0, T], X) \otimes(L)^{2}}^{2}=\left\|\sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, K}} u_{\alpha} L_{\alpha}\right\|_{C^{1}([0, T], X) \otimes(L)^{2}}^{2} \\
& \quad=\sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, K}}\left\|u_{\alpha}\right\|_{C^{1}([0, T], X)}^{2} \mathbb{E} L_{\alpha}^{2} \\
& \quad \leq c \sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, K}}\left(\left\|u_{\alpha}^{0}\right\|_{X}^{2}+\left\|\mathcal{L} u_{\alpha}^{0}\right\|_{X}^{2}+\left\|g_{\alpha}\right\|_{C^{1}([0, T], X)}^{2}\right) \mathbb{E} L_{\alpha}^{2}
\end{aligned}
$$

which is finite by the assumptions (A1)-(A3). In the last estimate, we employed the bound derived in the proof of Theorem 4.

Theorem 11 (Discretization error) Let $\tilde{u}$ denote the truncated chaos representation of the solution $u$ of the stochastic evolution problem (17) given in the form (30). Let a square integrable process $\tilde{u}_{\text {dis }}^{n}$ be given in the form

$$
\tilde{u}_{d i s}^{n}=\sum_{\alpha \in \mathcal{I}_{m, K}} u_{\alpha, \text { dis }}^{n} L_{\alpha}
$$

where its coefficients $u_{\alpha, \text { dis }}^{n}, \alpha \in \mathcal{I}_{m, K}$ are numerical approximations of $u_{\alpha}$ for $\alpha \in \mathcal{I}_{m, K}$ at time $t_{n}=n h$ with a positive step size $h$. Assume that the coefficients $u_{\alpha}$ are sufficiently regular and the approximation

$$
\begin{equation*}
\left\|u_{\alpha}\left(t_{n}\right)-u_{\alpha, d i s}^{n}\right\|_{X} \leq e_{\alpha}, \quad \alpha \in \mathcal{I}_{m, K} \tag{33}
\end{equation*}
$$

holds for the particular numerical method applied. Then, the difference between $\tilde{u}$ evaluated at $t_{n}$ and $\tilde{u}_{\text {dis }}^{n}$ can be estimated by the a priori bound

$$
\begin{aligned}
\left\|\tilde{u}\left(t_{n}\right)-\tilde{u}_{d i s}^{n}\right\|_{X \otimes(L)^{2}}^{2} & \leq \sum_{\alpha \in \mathcal{I}_{m, K}}\left\|u_{\alpha}\left(t_{n}\right)-u_{\alpha, d i s}^{n}\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2} \\
& \leq \sum_{\alpha \in \mathcal{I}_{m, K}} e_{\alpha}^{2} \mathbb{E} L_{\alpha}^{2}<\infty .
\end{aligned}
$$

Proof From Parseval's identity and the orthogonality of the polynomial basis $\left\{L_{\alpha}\right\}$, and using that the error (33) for a concrete numerical method, we obtain

$$
\begin{aligned}
\left\|\tilde{u}\left(t_{n}\right)-\tilde{u}_{d i s}^{n}\right\|_{X \otimes(L)^{2}}^{2} & =\left\|\sum_{\alpha \in \mathcal{I}_{m, K}} u_{\alpha}\left(t_{n}\right) L_{\alpha}-\sum_{\alpha \in \mathcal{I}_{m, K}} u_{\alpha, d i s}^{n} L_{\alpha}\right\|_{X \otimes(L)^{2}}^{2} \\
& =\sum_{\alpha \in \mathcal{I}_{m, K}}\left\|u_{\alpha}\left(t_{n}\right)-u_{\alpha, d i s}^{n}\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2} \\
& \leq \sum_{\alpha \in \mathcal{I}_{m, K}} e_{\alpha}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
\end{aligned}
$$

which completes the proof.
In order to apply the splitting methods in the setting of [34], we are going to consider the analytic case and adapt Theorem 11. We replace the assumption (A1) with the assumption:
(B1) Let $(A, \mathcal{D}(A)),(B, \mathcal{D}(B))$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ be coordinatewise operators that generate analytic semigroups of contractions on $X$. Let $\mathcal{D}\left(\mathcal{L}^{2}\right) \subseteq \mathcal{D}(A B)$.
Further, for the case of the resolvent Lie splitting we replace the assumptions ( $A 2$ ) and ( $A 3$ ) by:
(B2) The noise process given by

$$
\begin{equation*}
G=\sum_{\alpha \in \mathcal{I}} g_{\alpha} L_{\alpha} \tag{34}
\end{equation*}
$$

belongs to $C^{1+\theta}([0, T], X) \otimes(L)^{2}$ for some $\theta>0$, i.e.,

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{C^{1+\theta}([0, T], X)}^{2} \mathbb{E} L_{\alpha}^{2}<\infty \tag{35}
\end{equation*}
$$

holds.
(B3) Let $u^{0} \in \mathcal{D}(\mathcal{L}) \otimes(L)^{2}$ and $\mathcal{L} u^{0}+G(0) \in \mathcal{D}(\mathcal{L}) \otimes(L)^{2}$, i.e.,

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{D(\mathcal{L})}^{2} \mathbb{E} L_{\alpha}^{2}<\infty \quad \text { and } \quad \sum_{\alpha \in \mathcal{I}}\left\|\mathcal{L} u_{\alpha}^{0}+g_{\alpha}(0)\right\|_{D(\mathcal{L})}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
$$

(B4) Let $0 \in \rho(\mathcal{L})$, let $\mathcal{L}^{-1} G(t) \in \mathcal{D}(A B) \otimes(L)^{2}$ for all $t \in[0, T]$ and let the coefficients $g_{\alpha}$ of $G$ given by (34), satisfy the estimate

$$
\max _{0 \leq t \leq T}\left\|A B \mathcal{L}^{-1} g_{\alpha}(t)\right\| \leq c_{\alpha}, \quad 0 \leq t \leq T
$$

with a moderate constant $c_{\alpha}$ for each $\alpha \in \mathcal{I}$.

Note that, under these assumptions, the existence theorem, Theorem 4, still holds. Particularly, for the resolvent Lie splitting it reads:

Theorem 12 Let $\mathcal{L}$ be the generator of an analytic semigroup. Under the assumptions (B2) and (B3), the solution (18) of the stochastic evolution problem (2) posseses the improved regularity

$$
\begin{equation*}
u \in C^{2}([0, T], X) \otimes(L)^{2} \cap C^{1}([0, T], \mathcal{D}(\mathcal{L})) \otimes(L)^{2} \tag{36}
\end{equation*}
$$

Proof By the method of chaos expansion, the stochastic evolution problem (2) transforms to the system of deterministic problems (19). From (B2) and (B3) it follows that $u_{\alpha}^{0}$ and $g_{\alpha}$ for each $\alpha \in \mathcal{I}$ satisfy the assumptions (22). After applying Theorem 6 we obtain the improved regularity $u_{\alpha} \in C^{2}([0, T], X) \cap$ $C^{1}([0, T], D(\mathcal{L})), \alpha \in \mathcal{I}$.

Theorem 13 (Discretization error, the resolvent Lie splitting) Let the assumptions (B1)-(B4) be fulfilled. Then, for the resolvent Lie splitting, Theorem 11 holds with

$$
e_{\alpha} \leq c_{\alpha} h, \quad \alpha \in \mathcal{I}_{m, K}
$$

The constants $c_{\alpha}$ can be chosen uniformly on $[0, T]$ and, in particular, independently of $n$ and $h$.

Proof The coefficients $u_{\alpha}$, for each $\alpha \in \mathcal{I}_{m, K}$ are the exact solutions of the deterministic initial value problems (19) and $u_{\alpha, \text { dis }}^{n}$ are their numerical approximations obtained by the resolvent Lie splitting (24). Moreover, $u_{\alpha}$ satisfy the assumptions (22) for all $\alpha \in \mathcal{I}$. Thus, we can apply Theorem 7 to each initial value problem (19) and obtain the global estimate (25) for each $\alpha \in \mathcal{I}_{m, K}$, i.e. $e_{\alpha} \leq c_{\alpha} h$, for $\alpha \in \mathcal{I}_{m, K}$. This leads to the desired result.

In the case of the trapezoidal resolvent splitting, we need the following additional assumptions:
(B5) The noise process $G$ given by (34) belongs to $C^{2+\theta}([0, T], X) \otimes(L)^{2}$ for some $\theta>0$.
(B6) Let $\mathcal{L}^{2} u^{0}+\mathcal{L} G(0)+G^{\prime}(0) \in \mathcal{D}(\mathcal{L}) \otimes(L)^{2}$, i.e.,

$$
\sum_{\alpha \in \mathcal{I}}\left\|\mathcal{L}^{2} u_{\alpha}^{0}+\mathcal{L} g_{\alpha}(0)+g_{\alpha}^{\prime}(0)\right\|_{D(\mathcal{L})}^{2} \mathbb{E} L_{\alpha}^{2}<\infty
$$

(B7) Let $0 \in \rho(\mathcal{L})$, let $\mathcal{L}^{-1} G^{\prime}(t) \in \mathcal{D}(A B) \otimes(L)^{2}$ for all $t \in[0, T]$ and let the coefficients $g_{\alpha}$ of $G$ given by (34), satisfy the estimate

$$
\max _{0 \leq t \leq T}\left\|A B \mathcal{L}^{-1} g_{\alpha}^{\prime}(t)\right\| \leq c_{\alpha}
$$

with a moderate constant $c_{\alpha}$ for each $\alpha \in \mathcal{I}$.

Theorem 14 Let $\mathcal{L}$ be the generator of an analytic semigroup. Under the assumptions (B3), (B5) and (B6), the solution (18) of the stochastic evolution problem (2) posseses the improved regularity

$$
u \in C^{3}([0, T], X) \otimes(L)^{2} \cap C^{2}([0, T], \mathcal{D}(\mathcal{L})) \otimes(L)^{2} .
$$

Proof The method of chaos expansion transforms the stochastic evolution problem (2) to the system of deterministic problems (19). From (B3), (B5) and (B6) it follows that $u_{\alpha}^{0}$ and $g_{\alpha}$ for each $\alpha \in \mathcal{I}$ satisfy the assumptions (27). Then, the improved regularity $u_{\alpha} \in C^{3}([0, T], X) \cap C^{2}([0, T], D(\mathcal{L}))$, for $\alpha \in \mathcal{I}$ follows from Theorem 8 .

Theorem 15 (Discretization error, the trapezoidal resolvent splitting) Let the assumptions (B1), (B3) and (B5)-(B7) be fulfilled. Then, for the trapezoidal resolvent splitting, Theorem 11 holds with

$$
e_{\alpha} \leq c_{\alpha} h^{2}, \quad \alpha \in \mathcal{I}_{m, K}
$$

The constants $c_{\alpha}$ can be chosen uniformly on $[0, T]$ and, in particular, independently of $n$ and $h$.

Proof From the assumptions it follows that the coefficients $u_{\alpha}^{0}$ and $g_{\alpha}$ satisfy (27) for each $\alpha \in \mathcal{I}_{m, K}$. We apply the trapezoidal resolvent splitting (26) in order to obtain the approximation $u_{\alpha, \text { dis }}^{n}$ of the exact solution $u_{\alpha}\left(t_{n}\right)$ evaluated at $t_{n}$ of the initial value problem (19) for each $\alpha \in \mathcal{I}_{m, K}$. Thus, by Theorem 9 we obtain the global error estimate (29), i.e. $e_{\alpha} \leq c h^{2}$ for each $\alpha \in \mathcal{I}_{m, K}$.

Denote by $\frac{1}{2} \Delta$ the constant on the right hand side of the estimate (32) obtained in Theorem 10. The full error estimates of the Wiener-Legendre chaos expansion combined with the two splitting methods are given in the following theorem.

## Theorem 16 (Full error estimate)

(1) Let the assumptions of Theorem 13 hold. Then, the full error estimate of the Wiener-Legendre chaos expansion combined with the resolvent Lie splitting satisfies the following bound

$$
\begin{equation*}
\left\|u\left(t_{n}\right)-\tilde{u}_{d i s}^{n}\right\|_{X \otimes(L)^{2}}^{2} \leq \Delta+c h^{2} \tag{37}
\end{equation*}
$$

(2) Let the assumptions of Theorem 15 hold. Then, the full error estimate of the Wiener-Legendre chaos expansion combined with the trapezoidal resolvent splitting satisfies the bound

$$
\begin{equation*}
\left\|u\left(t_{n}\right)-\tilde{u}_{d i s}^{n}\right\|_{X \otimes(L)^{2}}^{2} \leq \Delta+c h^{4} \tag{38}
\end{equation*}
$$

Proof The full error estimate reads

$$
\begin{aligned}
\| u\left(t_{n}\right) & -\tilde{u}_{d i s}^{n}\left\|_{X \otimes(L)^{2}}^{2}=\right\| \sum_{\alpha \in \mathcal{I}} u_{\alpha}\left(t_{n}\right) L_{\alpha}-\sum_{\alpha \in \mathcal{I}_{m, K}} u_{\alpha, d i s}^{n} L_{\alpha} \|_{X \otimes(L)^{2}}^{2} \\
& =\left\|\sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, K}} u_{\alpha}\left(t_{n}\right) L_{\alpha}+\sum_{\alpha \in \mathcal{I}_{n, K}}\left(u_{\alpha}\left(t_{n}\right)-u_{\alpha, d i s}^{n}\right) L_{\alpha}\right\|_{X \otimes(L)^{2}}^{2} \\
& \leq 2 \sum_{\alpha \in \mathcal{I} \backslash \mathcal{I}_{m, K}}\left\|u_{\alpha}\left(t_{n}\right)\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2}+2 \sum_{\alpha \in \mathcal{I}_{m, K}}\left\|u_{\alpha}\left(t_{n}\right)-u_{\alpha, d i s}^{n}\right\|_{X}^{2} \mathbb{E} L_{\alpha}^{2} \\
& \leq \Delta+2 \sum_{\alpha \in \mathcal{I}_{m, K}} e_{\alpha}^{2} \mathbb{E} L_{\alpha}^{2}
\end{aligned}
$$

by the triangle inequality and the orthogonality property (9). We apply Theorem 10 to the first term. In the case of the resolvent Lie splitting, the estimate (37) follows after applying Theorem 13, while in case of the trapezoidal resolvent splitting, Theorem 15 leads to the desired estimate (38).

## 4 Numerical Results

In this section, we validate the proposed method and the convergence analysis presented in the previous section. For this purpose, we consider the twodimensional problem

$$
\begin{equation*}
u_{t}=\mathcal{L} u+v+1, \quad u(0)=0,\left.\quad u\right|_{\partial \mathrm{D}}=0 \tag{39}
\end{equation*}
$$

where the operator $\mathcal{L}$ is defined by $\mathcal{L} u=(A+B) u=\left(a u_{x}\right)_{x}+\left(b u_{y}\right)_{y}$ over the spatial domain $\mathrm{D}=[-1,1]^{2}$ with state variables $x$ and $y$, spatial non-Gaussian noise $v$ given in the form (12) and $t \in[0, T]$ for some $T>0$. This problem is an example of the problem class (2) with zero initial and boundary conditions. The solution $u$ of the considered problem (39) is given in its polynomial chaos representation (11) and approximated by a truncated expansion (30) in terms of Fourier-Legendre polynomials. The truncation procedure is explained in detail in Section 3.3.

Consider the set of multiindices $\mathcal{I}_{m, K} \subset \mathcal{I}$, i.e.,

$$
\mathcal{I}_{m, K}=\left\{\alpha \in \mathcal{I}: \alpha=\left(\alpha_{1}, \ldots, \alpha_{m}, 0,0, \ldots\right),|\alpha| \leq K\right\} .
$$

In this section, elements $\alpha \in \mathcal{I}_{m, K}$ will be denoted as $m$-tuples $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right)$, omitting the components $\alpha_{j}=0, j \geq m+1$. Moreover, we set

$$
\varepsilon^{(k)}=\left(\varepsilon_{1}^{(k)}, \ldots, \varepsilon_{m}^{(k)}\right), \quad \varepsilon_{j}^{(k)}=\delta_{k j}
$$

For fixed $m \in \mathbb{N}$ we consider an index function

$$
K_{m}: \mathcal{I}_{m, K} \rightarrow\{0,1, \ldots, P-1\}
$$

which enumerates multi-indices $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right) \in \mathcal{I}_{m, K}$. The function $K_{m}$ is a bijection and each $\alpha \in \mathcal{I}_{m, K}$ corresponds to a unique $K_{m}(\alpha)=p \in$ $\{0,1, \ldots P-1\}$.

For our purpose, we define the function $K_{m}$ by
$K_{m}(0,0, \ldots, 0,0)=0$,
$K_{m}\left(\varepsilon^{(k)}\right)=k \quad$ for $1 \leq k \leq m$,
$K_{m}\left(\varepsilon^{(k)}+\varepsilon^{(\ell)}\right)=m+(m-1)+\ldots+(m-k+1)+\ell \quad$ for $1 \leq k \leq \ell \leq m$,
...
$K_{m}(0,0, \ldots, 0, K)=P-1$.
We use the index function $K_{m}$ to enumerate the Fourier-Legendre polynomials $L_{\alpha}$ for each $\alpha \in \mathcal{I}_{m, K}$. Thus, we denote by $\left(\Phi_{p}\right)_{p \in\{0,1, \ldots, P-1\}}$ the ordered Fourier-Legendre polynomials

$$
\Phi_{p}(\boldsymbol{\xi}(\omega))=\Phi_{K_{m}(\alpha)}(\boldsymbol{\xi}(\omega))=L_{\alpha}(\boldsymbol{\xi}(\omega))
$$

for $p=K_{m}(\alpha), \alpha \in \mathcal{I}_{m, K}$, where we use the definition (7) of the FourierLegendre polynomials. For example, following the just introduced notation, we have $\Phi_{0}(\boldsymbol{\xi}(\omega))=L_{(0,0, \ldots, 0)}(\boldsymbol{\xi}(\omega))=1$ and

$$
\Phi_{k}(\boldsymbol{\xi}(\omega))=L_{\varepsilon^{(k)}}(\boldsymbol{\xi}(\omega))=\xi_{k}(\omega) \text { for } 1 \leq k \leq m .
$$

Also, by applying the definition of the Legendre polynomials (5) we have

$$
\Phi_{m+1}(\boldsymbol{\xi}(\omega))=L_{(2,0, \ldots, 0)}(\boldsymbol{\xi}(\omega))=p_{2}\left(\xi_{1}(\omega)\right)=\frac{3}{2} \xi_{1}^{2}(\omega)-\frac{1}{2}
$$

as well as

$$
\Phi_{m+2}(\boldsymbol{\xi}(\omega))=L_{(1,1,0, \ldots, 0)}(\boldsymbol{\xi}(\omega))=p_{1}\left(\xi_{1}(\omega)\right) p_{1}\left(\xi_{2}(\omega)\right)=\xi_{1}(\omega) \xi_{2}(\omega)
$$

Moreover, it holds

$$
\Phi_{P-1}(\boldsymbol{\xi}(\omega))=L_{(0,0, \ldots, 0, K)}(\boldsymbol{\xi}(\omega))=p_{K}\left(\xi_{n}(\omega)\right)
$$

In the next step, we represent the solution $u$ of problem (39) by its truncated polynomial chaos expansion (30) and the noise term by its representation (16). Inserting the representations in (39) gives

$$
\sum_{\alpha \in \mathcal{I}_{m, K}}\left(u_{\alpha}\right)_{t} L_{\alpha}=\sum_{\alpha \in \mathcal{I}_{m, K}} \mathcal{L} u_{\alpha} L_{\alpha}+\bar{v}+1+\sum_{j=1}^{m} \sqrt{\lambda_{j}} e_{j} Z_{j} .
$$

By performing a Galerkin projection we obtain

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{I}_{m, K}}\left(u_{\alpha}\right)_{t} \mathbb{E}\left(L_{\alpha} L_{\beta}\right)= \\
& \quad=\sum_{\alpha \in \mathcal{I}_{m, K}} \mathcal{L} u_{\alpha} \mathbb{E}\left(L_{\alpha} L_{\beta}\right)+(\bar{v}+1) \mathbb{E} L_{\beta}+\sum_{j=1}^{m} \sqrt{\lambda_{j}} e_{j} \mathbb{E}\left(Z_{j} L_{\beta}\right)
\end{aligned}
$$

for $\beta \in \mathcal{I}_{m, K}$. Then, by applying the properties of the Fourier-Legendre polynomials (8) and (9), we obtain a system of deterministic equations (19). Particularly,
(i) for $|\alpha|=0$ :

$$
\begin{equation*}
\left(u_{(0,0, \ldots, 0)}\right)_{t}=\mathcal{L} u_{(0,0, \ldots, 0)}+\bar{v}+1, \quad u_{(0,0, \ldots, 0)}(0)=0,\left.\quad u_{(0,0, \ldots, 0)}\right|_{\partial \mathrm{D}}=0 \tag{40}
\end{equation*}
$$

(ii) for $|\alpha|=1$, i.e., $\alpha=\varepsilon^{(k)}, 1 \leq k \leq m$ :

$$
\begin{equation*}
\left(u_{\varepsilon^{(k)}}\right)_{t}=\mathcal{L} u_{\varepsilon^{(k)}}+\sqrt{\lambda_{k}} e_{k}, \quad u_{\varepsilon^{(k)}}(0)=0,\left.\quad u_{\varepsilon^{(k)}}\right|_{\partial \mathrm{D}}=0 \tag{41}
\end{equation*}
$$

(iii) for $|\alpha|>1$ :

$$
\begin{equation*}
\left(u_{\alpha}\right)_{t}=\mathcal{L} u_{\alpha}, \quad u_{\alpha}(0)=0,\left.\quad u_{\alpha}\right|_{\partial D}=0 \tag{42}
\end{equation*}
$$

From (42) we clearly deduce that $u_{\alpha} \equiv 0$ for $|\alpha|>1$. In the calculations we also used $\mathbb{E}\left(L_{(0,0, \ldots, 0)} Z_{j}\right)=\mathbb{E} Z_{j}=0$ for $j \geq 1$ and

$$
\mathbb{E}\left(Z_{j} L_{\beta}\right)=\mathbb{E}\left(p_{1}\left(Z_{j}\right) L_{\beta}\right)=\mathbb{E}\left(L_{\varepsilon^{(j)}} L_{\beta}\right)=\delta_{\beta, \varepsilon^{(j)}} \mathbb{E} L_{\varepsilon^{(j)}}^{2}=\delta_{\beta, \varepsilon^{(j)}} \cdot \frac{1}{3}
$$

This particularly implies

$$
\sum_{j=1}^{m} \sqrt{\lambda_{j}} e_{j} \mathbb{E}\left(Z_{j} L_{\varepsilon^{(k)}}\right)=\sqrt{\lambda_{k}} e_{k} \quad \text { for } \quad 1 \leq k \leq m
$$

which was used in equation (41).
The obtained system (40), (41) and (42) can be represented in terms of the index function $K_{m}$, i.e., in the form

$$
\begin{equation*}
\left(u_{p}\right)_{t}=\mathcal{L} u_{p}+g_{p}, \quad u_{p}(0)=0,\left.\quad u_{p}\right|_{\partial D}=0 \tag{43}
\end{equation*}
$$

for $0 \leq p \leq P-1$, where each $p$ corresponds to an $\alpha \in \mathcal{I}_{m, K}$ Each equation in (43) has the form of an inhomogeneous deterministic initial value problem, where the inhomogeneities $g_{p}$ are given by: $g_{0}=\bar{v}+1$ and $g_{p}=\sqrt{\overline{\lambda_{p}}} e_{p}$ for $1 \leq p \leq m$ and $g_{p}=0$ for $m<p \leq P-1$.

One way to approximate numerically a problem of the form

$$
u_{t}=(A+B) u+g, \quad u(0)=u^{0},\left.\quad u\right|_{\partial \mathcal{D}}=0
$$

with $\mathrm{D}=[-1,1]^{2}$ is to define a grid consisting of $N \times N$ equidistant computational points and define the discrete operators $A_{s}$ and $B_{s}$ by

$$
\begin{aligned}
& \left(A_{s} u^{\mathrm{dis}}\right)_{i, j}=\frac{1}{2 s}\left(\frac{\mathrm{~d}}{\mathrm{~d} x} a_{i, j}\left(u_{i+1, j}^{\mathrm{dis}}-u_{i-1, j}^{\mathrm{dis}}\right)\right)+\frac{1}{s^{2}}\left(a_{i, j}\left(u_{i+1, j}^{\mathrm{dis}}-2 u_{i, j}^{\mathrm{dis}}+u_{i-1, j}^{\mathrm{dis}}\right)\right), \\
& \left(B_{s} u^{\mathrm{dis}}\right)_{i, j}=\frac{1}{2 s}\left(\frac{\mathrm{~d}}{\mathrm{~d} y} b_{i, j}\left(u_{i, j+1}^{\mathrm{dis}}-u_{i, j-1}^{\mathrm{dis}}\right)\right)+\frac{1}{s^{2}}\left(b_{i, j}\left(u_{i, j+1}^{\mathrm{dis}}-2 u_{i, j}^{\mathrm{dis}}+u_{i, j-1}^{\mathrm{dis}}\right)\right),
\end{aligned}
$$

where

$$
\begin{gathered}
\frac{\mathrm{d}}{\mathrm{~d} x} a_{i, j}=\frac{\mathrm{d}}{\mathrm{~d} x} a(i s, j s), \quad \text { and } a_{i, j}=a(i s, j s), \\
\frac{\mathrm{d}}{\mathrm{~d} y} b_{i, j}=\frac{\mathrm{d}}{\mathrm{~d} y} b(i s, j s), \quad \text { and } b_{i, j}=b(i s, j s)
\end{gathered}
$$

for $i, j=1, \ldots, N$ and $s=2 /(N+1)$. Due to the homogeneous Dirichlet boundary conditions we have:

$$
u_{0, j}^{\mathrm{dis}}=u_{N+1, j}^{\mathrm{dis}}=u_{i, 0}^{\mathrm{dis}}=u_{i, N+1}^{\mathrm{dis}}=0
$$

for all $i, j=0, \ldots, N+1$. By setting $\mathcal{L}_{s}=A_{s}+B_{s}$ we obtain the discretized problem

$$
\frac{\mathrm{d}}{\mathrm{~d} t} u^{\mathrm{dis}}=\mathcal{L}_{s} u^{\mathrm{dis}}+g_{s}(t), \quad u^{\mathrm{dis}}(0)=0
$$

where $g_{s}$ denotes the discretization of the inhomogeneity $g$.
Note that the number $P$ of partial differential equations one has to solve in (43) increases fast due to the factorials occurring in (31). Since $g_{p}=0$ for all $m<p \leq P-1, u_{p}=0$ is consequently the solution of the $p$ th partial differential equation of (43). Therefore, we only have to solve the first $m+1$ partial differential equations instead of all $P$. Further, we see that the solution does not depend on the highest degree $K$ of the $m$-dimensional Legendre polynomials.

Let $u_{p}^{n}$ denote the numerical solution $u_{p}$ at time $t_{n}=h n$ and $g_{p}^{n}$ the function $g_{p}$ evaluated at time $t_{n}$. By setting

$$
\begin{equation*}
u_{p}^{n+1}=\left(I-h A_{s}\right)^{-1}\left(I-h B_{s}\right)^{-1}\left(u_{p}^{n}+h g_{p}^{n}\right) \tag{44}
\end{equation*}
$$

the Lie resolvent splitting method is defined, see (24).
The trapezoidal splitting method is given by
$u_{p}^{n+1}=\left(I-\frac{h}{2} B_{s}\right)^{-1}\left(I-\frac{h}{2} A_{s}\right)^{-1}\left[\left(I+\frac{h}{2} A_{s}\right)\left(I+\frac{h}{2} B_{s}\right) u_{p}^{n}+\frac{h}{2}\left(g_{p}^{n}+g_{p}^{n+1}\right)\right]$,
see (26).
In our numerical experiment, we consider (39) with constant coefficients $a(x, y)=b(x, y)=1$ for all $(x, y) \in \mathrm{D}=[-1,1]^{2}$ and set $T=1$. Note that for some $p \in\{0, \ldots, m\}$ the inhomogeneities $g_{p}$ might be incompatible with the boundary conditions at the corners of the spatial domain D. Such an incompatibility results in order reduction, see [18]. This in particular leads to large errors near the corners of D . To overcome this problem, we apply the modified Lie resolvent splitting [18] in this situation.

For $p \in\{0, \ldots, m\}$, let $u_{p}$ be the solution of the partial differential equation (43). Let $I=\{1,2,3,4\}$ be the set of indices of the corners of the spatial domain D. They are enumerated from 1 to 4 counter-clockwise starting from the corner with coordinates $(-1,-1)$. Suppose that the inhomogeneity $g_{p}$ does not vanish at the corners $I_{p} \subset I$. Let $g_{p, i}(t)$ denote the value of the function $g_{p}$ at corner $i \in I_{p}$ and time $t \geq 0$. For $g_{p, i}(0) \neq 0$ we set

$$
f_{i}=\frac{P_{i} g_{p}(0)}{g_{p, i}(0)}
$$

where the polynomials $P_{i}$ are given by

$$
\begin{aligned}
P_{1} & =\frac{1}{4}(x-1)(y-1), & P_{2} & =-\frac{1}{4}(x+1)(y-1), \\
P_{3} & =\frac{1}{4}(x+1)(y+1), & P_{4} & =-\frac{1}{4}(x-1)(y+1) .
\end{aligned}
$$

These four polynomials form a partition of unity.
Let $v_{i}$ be the solution of the stationary problem

$$
\mathcal{L} v_{i}=f_{i} \text { in } \mathrm{D},\left.\quad v_{i}\right|_{\partial \mathrm{D}}=0,
$$

for $i \in I_{p}$. Note that $v_{i}$ can be computed once and for all. Then, let

$$
\tilde{g}_{p}(t)=g_{p}(t)+\sum_{i \in I_{p}} g_{p, i}^{\prime}(t) v_{i}-g_{p, i}(t) f_{i}, \quad \tilde{u}_{p, 0}=u_{p}(0)+\sum_{i \in I_{p}} g_{p, i}(0) v_{i}
$$

and apply the resolvent Lie splitting to the problem

$$
\left(\tilde{u}_{p}\right)_{t}=\mathcal{L} \tilde{u}_{p}(t)+\tilde{g}_{p}(t), \quad \tilde{u}_{p}(0)=\tilde{u}_{p, 0},\left.\quad \tilde{u}_{p}\right|_{\partial \mathrm{D}}=0
$$

By setting

$$
\begin{equation*}
u_{p}^{n, \bmod }=\tilde{u}_{p}^{n}-\sum_{i \in I_{p}} g_{p, i}(n h) v_{i} \quad \text { for } \quad n \in \mathbb{N} \tag{45}
\end{equation*}
$$

we obtain the modified splitting scheme. Note that in our case $g_{p, i}^{\prime}(t)=0$ for all $i \in I_{p}$ and for all $p=0, \ldots, m$ since none of the inhomogeneities $g_{p}$ is time dependent.

In the implementation, the set $I_{p}$ for $p=0, \ldots, m$ is constructed by checking the values of the inhomogeneities $g_{p}$ at the corners, i.e.,

$$
I_{p}=\left\{i \in\{1,2,3,4\}| | g_{p, i}(0) \mid \geq \mathrm{TOL}\right\}
$$

for a user chosen tolerance TOL. If $I_{p}=\emptyset$, the standard Lie resolvent splitting given in (44) is applied.

In the following, we consider problem (39) with $v$ given by (12) with covariance function

$$
C_{v}(\mathbf{x}, \mathbf{y})=\exp \left\{-\|\mathbf{x}-\mathbf{y}\|^{2}\right\}
$$

The reference solution $u_{p}^{\text {ref }}$ at time $t$ is calculated according to

$$
u_{p}^{\mathrm{ref}}(t)=\exp (t \mathcal{L}) u_{p}(0)+t \varphi_{1}(t \mathcal{L}) g_{p}
$$

where $\varphi_{1}(z)=\frac{\exp (z)-1}{z}$ and $\exp (\cdot)$ denotes the matrix exponential. In all the examples shown we fix the highest degree of ordered Fourier-Legendre polynomials to $K=3$ and use a maximal number of $m=120$ uncorrelated zero-mean random variables $Z_{j}$ used in the truncated Karhunen-Loève expansion (14). If not stated explicitly, we fix the number of computational points to $N \times N=40 \times 40$.

Figure 1 illustrates the impact of the modification of the Lie resolvent splitting method. The figure shows the pointwise error of the numerical solution


Fig. 1 Pointwise error of $u_{0}$ over the domain $\mathcal{D}=[-1,1]^{2}$ for the Lie splitting (left) and the modified Lie splitting (right).
at time $T=1$, i.e., $\left|u_{0}(T)-u_{0}^{\text {ref }}(T)\right|$ over the spatial domain $\mathrm{D}=[-1,1]^{2}$ when calculated with the Lie splitting and the modified Lie splitting given in (44) and (45), respectively. The pointwise error of the solution $u_{0}$ is not only reduced at all the four corners of the domain D but also approximately decreases by an order of magnitude.

Figure 2 shows the discrete $L^{2}$ error of $u_{p}, p=0, \ldots, 7$ calculated with different time step sizes $h$. The time step sizes are set to $h_{q}=2^{q}$ for $q=$ $-13, \ldots,-4$. The blue line denotes the error of the modified Lie splitting scheme of order 1 . The red line and the green line illustrate the error of the Crank-Nicolson scheme and the trapezoidal splitting method, both of order two. The black dashed lines have slope 1 and 2, respectively. We see that for each $m$, the order plots confirm the respective orders of the methods which can be derived from theory.

The empirical variance $\operatorname{Var}(u)$ of $u$ is given by

$$
\operatorname{Var}(u)=\mathbb{E}[u-\mathbb{E}(u)]=\sum_{p=1}^{P} u_{p}^{2} \mathbb{E}\left(\Phi_{p}^{2}\right),
$$

where we used the linearity of $\mathbb{E}$ and the orthogonality of the Fourier-Legendre polynomials. Furthermore, since $u_{\alpha} \equiv 0$ for $|\alpha|>1$, i.e., $u_{p} \equiv 0$ for $p>m$, the number of non-zero summands in the sum is $m$ and since $\mathbb{E}\left(\Phi_{p}^{2}\right)=\frac{1}{3}$ for $1<p \leq m, \operatorname{Var}(u)$ reduces to

$$
\operatorname{Var}(u)=\frac{1}{3} \sum_{p=1}^{m} u_{p}^{2} .
$$










| ー ー ー $h$ー・ー $h^{2}$$\square$ Modified Lie Splitting$\square$ Crank－Nicolson$\square$ Trapezoidal Splitting |
| :---: |
|  |  |
|  |  |
|  |  |
|  |  |

Fig． 2 Order plots for the first eight different solutions $u_{p}, p=0, \ldots, 7$ computed with the correspondent methods．

Figure 3 shows the discrete $L^{2}$ error of the empirical variance of $u$ at time $T=1$ where the summation is truncated at different $n$ ．The time step $h$ used for the calculations is $h=2^{-10}$ ．Here，we clearly see the superiority of


Fig． 3 Discrete $L^{2}$ error of $\operatorname{Var}(u)$ for different number of variables $m$ used in the Karhunen－ Loève expansion．The employed methods are：Crank－Nicolson（CN），modified Lie splitting （MLSPL），and trapezoidal splitting（TSPL）．
the methods of order two compared to the modified Lie splitting for which the numerical approximation error prevails over the error induced by the truncation of the sum.

Finally, we report the computational work which is needed to solve the system of partial differential equations given in (40) - (42). Table 1 summarizes the computational time needed to obtain one solution of the system of partial differential equations as a function of the number of spatial grid points $N \times N=$ $2^{k} \times 2^{k}$ for $k=2,3, \ldots 7$. The highest number of grid points we are able to use (16 384) is quite low due to the fact that the calculation of the eigenvalues and eigenfunctions of the integral equation given in (13) requires the storage of a dense matrix of the size $N^{2} \times N^{2}$. We clearly see that the Crank-Nicolson method is by far the slowest. Both splitting methods perform approximately the for smaller $N$, while for $N=2^{7}$, Lie splitting starts to clearly outperform trapezoidal splitting in terms of computational time.

Table 1 Average computational time (in seconds) for the calculation of one solution $u_{m}$ for different degrees of freedom $N$, i.e., the number of computational points used in the discretization of D and the operator $\mathcal{L}$. The employed methods are: Crank-Nicolson (CN), modified Lie splitting (MLSPL) and trapezoidal splitting (TSPL).

| $N \times N$ | CN $[\mathrm{s}]$ | MLSPL $[\mathrm{s}]$ | TSPL $[\mathrm{s}]$ |
| :---: | :---: | :---: | :---: |
| $4 \times 4$ | 0.0133 | 0.0373 | 0.0530 |
| $8 \times 8$ | 0.0214 | 0.0241 | 0.0379 |
| $16 \times 16$ | 0.1005 | 0.1008 | 0.0948 |
| $32 \times 32$ | 0.4098 | 0.3652 | 0.4047 |
| $64 \times 64$ | 2.7620 | 1.8051 | 1.8237 |
| $128 \times 128$ | 41.1284 | 9.3921 | 13.5091 |

Acknowledgements This work was partially supported by a Research grant for Austrian graduates granted by the Office of the Vice Rector for Research of University of Innsbruck. The computational results presented have been partially achieved using the HPC infrastructure LEO of the University of Innsbruck. A. Kofler was supported by the program Nachwuchsförderung 2014 at University of Innsbruck. H. Mena was supported by the Austrian Science Fund - project id: P27926.

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# Operator differential-algebraic equations with noise arising in fluid dynamics 

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Received: 9 December 2015 / Accepted: 12 May 2016 / Published online: 24 May 2016
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#### Abstract

We study linear semi-explicit stochastic operator differential algebraic equations (DAEs) for which the constraint equation is given in an explicit form. In particular, this includes the Stokes equations arising in fluid dynamics. We combine a white noise polynomial chaos expansion approach to include stochastic perturbations with deterministic regularization techniques. With this, we are able to include Gaussian noise and stochastic convolution terms as perturbations in the differential as well as in the constraint equation. By the application of the polynomial chaos expansion method, we reduce the stochastic operator DAE to an infinite system of deterministic operator DAEs for the stochastic coefficients. Since the obtained system is very sensitive to perturbations in the constraint equation, we analyze a regularized version of the system. This then allows to prove the existence and uniqueness of the solution of the initial stochastic operator DAE in a certain weighted space of stochastic processes.


[^10]Keywords Operator DAE • Noise disturbances • Chaos expansion • Itô-Skorokhod integral • Stochastic convolution • Regularization

Mathematics Subject Classification 65J10 • 60H40 60H30 • 35R60

## 1 Introduction

The governing equations of an incompressible flow of a Newtonian fluid are described by the Navier-Stokes equations [43]. Therein, one searches for the evolution of a velocity field $u$ and the pressure $p$ to given initial data, a volume force, and boundary conditions. For results on the existence of a (unique) solution, we refer to [20], [42, Ch. 25], and [43, Ch.III].

In this paper, we consider the linear case but allow a more general constraint, namely that the divergence of the velocity does not vanish. Note that this changes the analysis and numerics since the state-of-the-art methods are often tailored for the particular case of a vanishing divergence. An application with non-vanishing divergence is given by the optimal control problem constrained by the Navier-Stokes equations where the cost functional includes the pressure [23].

The Navier-Stokes equations, as well as the corresponding linearized equations, can be formulated as differential-algebraic equations (DAEs) in an abstract setting [3,4]. These so-called operator DAEs correspond to the weak formulation in the framework of partial differential equations (PDEs). As generalization of finite-dimensional DAEs, see [19,25,26] for an introduction, also here considered constrained PDEs suffer from instabilities and ill-posedness. This is the reason why the stable approximation of the pressure (which is nothing else than a Lagrange multiplier to enforce the incompressibility) is a great challenge.

One solution strategy is to perform a regularization which corresponds to an index reduction in the finite-dimensional setting. With this, the issue of instabilities with respect to perturbations is removed. In the case of fluid dynamics, this has been shown in [4].

In this paper, we study the stochastic version of operator DAEs considered in the framework of white noise analysis and chaos expansions of generalized stochastic processes $[18,21,39]$. More precisely, we consider semi-explicit operator DAEs with perturbations of stochastic type. We combine the polynomial chaos expansion approach from the white noise theory with the deterministic theory of operator DAEs. Particularly, in the fluid flow case, we deal with the stochastic equations of the form

$$
\begin{aligned}
\dot{u}(t)-\Delta u(t)+\nabla p(t) & =\mathcal{F}(t)+\text { "noise", } \\
& =\mathcal{G}(t)+\text { "noise" }
\end{aligned}
$$

with an initial value for $u(0)$. In order to preserve the mean dynamics, we deal with stochastic perturbations of zero mean. This implies that the expected value of the stochastic solution equals the solution of the corresponding deterministic operator DAE. For the "noise" processes we consider either a general Gaussian white noise process or perturbations which can be expressed in the form of a stochastic convolution.

Within this paper, we consider the Gaussian white noise space $(\Omega, \mathcal{F}, \mu)$ with the Gaussian probability measure $\mu$ to be the underlying probability space. Instead, the same analysis can be provided also on Poissonian white noise space $(\Omega, \mathcal{F}, \nu)$, with the Poissonian probability measure $v$, on fractional Gaussian white noise space $\left(\Omega, \mathcal{F}, \mu_{H}\right)$, or on fractional Poissonian white noise space $\left(\Omega, \mathcal{F}, \nu_{H}\right)$, for $H \in(0,1)$. This follows from the existence of unitary mappings between Gaussian and Poissonian white noise spaces, and between Gaussian and fractional Gaussian white noise spaces [27].

With the application of the polynomial chaos expansion method, also known as the propagator method, the problem of solving the initial stochastic equations is reduced to the problem of solving an infinite triangular systems of deterministic operator DAEs, which can be solved recursively. Summing up all coefficients of the expansion and proving convergence in an appropriate space of stochastic processes, one obtains the stochastic solution of the initial problem.

The chaos expansion methodology is a very useful technique for solving many types of stochastic differential equations, linear and nonlinear, see e.g. [6,18,29,30,32$34,40,46]$. The main statistical properties of the solution, its mean, variance, and higher moments, can be calculated from the formulas involving only the coefficients of the chaos expansion representation $[16,36]$.

The proposed method allows to apply regularization techniques from the theory of deterministic operator DAEs to the related stochastic system. Applications arise in fluid dynamics, but are not only restricted to this case. The same procedure can be used to regularize other classes of equations that fulfill our setting. A specific example with the operators of the Malliavin calculus is described in Sect. 5. For this reason, in the present paper, we develop a general abstract setting based on white noise analysis and chaos expansions. Numerical experiments with truncated chaos expansions, i.e., stochastic Galerkin methods, are not included in this paper. However, once we regularize each system, it becomes numerically well-posed [3] and then the stochastic equation is well-posed as well.

The paper is organized as follows. In Sect. 2 we introduce the concept of (deterministic) operator DAEs with special emphasis on applications in fluid dynamics. Considering perturbation results for such systems, we detect the necessity of a regularization in order to allow stochastic perturbations. The stochastic setting for the chaos expansion is then given in Sect. 3. Furthermore, we discuss stochastic noise terms in the differential as well as in the constraint equation and the systems which result from the chaos expansions, Theorems 6,8 and 9 . The extension to more general cases is then subject of Sect. 4. Therein, we consider more general operators and stochastic convolution terms. We also provide proofs of the convergence of the obtained solutions in appropriate spaces of generalized stochastic processes, Theorem 11. In Sect. 5 we consider shortly a specific example of DEAs that involve stochastic operators arising in Malliavin calculus. The proof of existence of a unique solution in a space of generalized stochastic processes is given in Theorem 13. Finally, we discuss extensions of our results to specific types of nonlinear equations.

## 2 Operator DAEs

In this section we introduce the concept of operator DAEs, analyze the influence of perturbations, provide regularization of operator DAEs, and state stability results.

### 2.1 Abstract setting

First we consider operator DAEs (also called PDAEs) which equal constrained PDEs in the weak setting or DAEs in an abstract framework [3,15]. Thus, we work with generalized derivatives in time and space. In particular, we consider semi-explicit operator DAEs for which the constraint equation is explicitly stated.

We consider real, separable, and reflexive Banach spaces $\mathcal{V}$ and $\mathcal{Q}$ and a real Hilbert space $\mathcal{H}$. Furthermore, we assume that we have a Gel'fand triple of the form

$$
\mathcal{V} \subseteq \mathcal{H} \subseteq \mathcal{V}^{*}
$$

which means that $\mathcal{V}$ is continuously and densely embedded in $\mathcal{H}$ [47, Ch. 23]. As a consequence, well-known embedding theorems yield the continuous embedding

$$
\left\{v \in L^{2}(T ; \mathcal{V}): \dot{v} \in L^{2}\left(T ; \mathcal{V}^{*}\right)\right\} \hookrightarrow C(T ; \mathcal{H})
$$

Note that $L^{2}(T ; \mathcal{V})$ denotes the Bochner space of abstract functions on a time interval $T$ with values in $\mathcal{V}$, see [14, Ch. 7.1] for an introduction. The corresponding norm of $L^{2}(T ; \mathcal{V})$, which we denote by $\|\cdot\|_{L^{2}(\mathcal{V})}$, is given by

$$
\|u\|_{L^{2}(\mathcal{V})}^{2}:=\|u\|_{L^{2}(T ; \mathcal{V})}^{2}:=\int_{T}\|u(t)\|_{\mathcal{V}}^{2} \mathrm{~d} t
$$

The (deterministic) problem of interest has the form

$$
\begin{align*}
\dot{u}(t)+K u(t)+B^{*} \lambda(t) & =F(t) \text { in } \mathcal{V}^{*},  \tag{1a}\\
B u(t) & =G(t) \text { in } \mathcal{Q}^{*}, \tag{1b}
\end{align*}
$$

with (consistent) initial condition $u(0)=u^{0} \in \mathcal{H}$. The need of consistent initial values is one characteristic of DAEs in the finite dimensional setting [10,25]. The condition in the infinite-dimensional case is discussed in Remark 1 below.

Furthermore, we need the operators and right-hand sides of (1) to satisfy the following assumptions.

Assumption 1 1. The right-hand sides of (1) satisfy

$$
F \in L^{2}\left(T ; \mathcal{V}^{*}\right) \text { and } G \in H^{1}\left(T ; \mathcal{Q}^{*}\right) \hookrightarrow C\left(T ; \mathcal{Q}^{*}\right)
$$

2. The constraint operator $B: \mathcal{V} \rightarrow \mathcal{Q}^{*}$ is linear and there exists a right-inverse which is denoted by $B^{-}$.
3. Operator $K: \mathcal{V} \rightarrow \mathcal{V}^{*}$ is linear, positive on the kernel of $B$, and continuous.

Note that the involved operators $B: \mathcal{V} \rightarrow \mathcal{Q}^{*}$ and $K: \mathcal{V} \rightarrow \mathcal{V}^{*}$ can be extended to Nemytskii mappings of the form $B: L^{2}(T ; \mathcal{V}) \rightarrow L^{2}\left(T ; \mathcal{Q}^{*}\right)$ and $K: L^{2}(T ; \mathcal{V}) \rightarrow$ $L^{2}\left(T ; \mathcal{V}^{*}\right)$, see [41, Ch. 1.3]. From here onwards, we restrict ourselves to the linear case.

As search space for the solution $(u, \lambda)$ we consider

$$
u \in L^{2}(T ; \mathcal{V}) \text { with } \dot{u} \in L^{2}\left(T ; \mathcal{V}^{*}\right) \text { and } \lambda \in L^{2}(T ; \mathcal{Q})
$$

Note that the actual meaning of equation (1a) is that for all test functions $v \in \mathcal{V}$ and $\Phi \in C^{\infty}(T)$ it holds that

$$
\int_{T}\left\langle\dot{u}(t)+K u(t)+B^{*} \lambda(t), v\right\rangle \Phi(t) \mathrm{d} t=\int_{T}\langle F(t), v\rangle \Phi(t) \mathrm{d} t
$$

Remark 1 (Consistent initial values) DAEs require consistent initial data because of the given constraints which also apply to the initial condition. This remains valid for the operator case. However, since we allow $u^{0} \in \mathcal{H}$, the constraint operator $B$ is not applicable to $u^{0}$. In this case, the condition has the form

$$
u^{0}=u_{B}^{0}+B^{-} G(0)
$$

where $u_{B}^{0}$ is an arbitrary element from the closure of the kernel of $B$ in $\mathcal{H}$ [4,15]. If $u^{0} \in \mathcal{V}$ is given, then we get the same decomposition but with $u_{B}^{0} \in \operatorname{Ker} B$.

In the following, we write $a \lesssim b$ meaning that there exists a positive constant $c$ such that $a \leq c b$. We show that the solution is bounded in terms of the initial data, the right-hand sides, and their derivative, cf. [3, Sect. 6.1.3].

Theorem 1 (Stability estimate) Given Assumption 1 and consistent initial data $u^{0}=$ $u_{B}^{0}+B^{-} G(0) \in \mathcal{H}$, the solution of the operator DAE (1) satisfies the estimate

$$
\begin{equation*}
\|u\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{B}^{0}\right\|_{\mathcal{H}}^{2}+\|F\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\|G\|_{H^{1}\left(\mathcal{Q}^{*}\right)^{2}}^{2} \tag{2}
\end{equation*}
$$

Proof We consider a splitting of the space $\mathcal{V}=\mathcal{V}_{B} \oplus \mathcal{V}^{\mathrm{c}}$ which we will also use later within the regularization in Sect. 2.3. Therein, $\mathcal{V}_{B}$ denotes the kernel of the operator $B$ and $\mathcal{V}^{\mathrm{c}}$ is any complementary space. This gives a unique decomposition $u=u_{1}+u_{2}$ where $u_{1}, u_{2}$ take values in $\mathcal{V}_{B}, \mathcal{V}^{c}$, respectively. Thus, we have $B u=B u_{2}=G$ and therefore $u_{2}=B^{-} G$. The assumption on $G$ implies $u_{2} \in H^{1}(T ; \mathcal{V})$ and

$$
\left\|u_{2}\right\|_{L^{2}(\mathcal{V})} \lesssim\|G\|_{L^{2}\left(\mathcal{Q}^{*}\right)}, \quad\left\|\dot{u}_{2}\right\|_{L^{2}(\mathcal{V})} \lesssim\|\dot{G}\|_{L^{2}\left(\mathcal{Q}^{*}\right)}
$$

It remains to find a bound of $u_{1}$. For this, we insert $u_{1}$ in (1a) as test function which leads to

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|u_{1}\right\|_{\mathcal{H}}^{2}+\left\|u_{1}\right\|_{\mathcal{V}}^{2} & \lesssim\left\langle\dot{u}_{1}, u_{1}\right\rangle+\left\langle\mathcal{K} u_{1}, u_{1}\right\rangle \\
& =\left\langle F, u_{1}\right\rangle-\left\langle\dot{u}_{2}, u_{1}\right\rangle-\left\langle\mathcal{K} u_{2}, u_{1}\right\rangle \\
& \lesssim\|F\|_{\mathcal{V}^{*}}\left\|u_{1}\right\| \mathcal{V}+\|\dot{G}\|_{\mathcal{Q}^{*}}\left\|u_{1}\right\| \mathcal{V}+\|G\|_{\mathcal{Q}^{*}}\left\|u_{1}\right\| \mathcal{V} .
\end{aligned}
$$

Note that the Lagrange multiplier $\lambda$ vanishes, since the test function is element of $\mathcal{V}_{B}$. Thus, by the Young's inequality we obtain

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left\|u_{1}\right\|_{\mathcal{H}}^{2}+\left\|u_{1}\right\|_{\mathcal{V}}^{2} \lesssim\|F\|_{\mathcal{V}^{*}}^{2}+\|G\|_{\mathcal{Q}}^{2}+\|\dot{G}\|_{\mathcal{Q}^{*}}^{2}
$$

An integration of this estimate over the given time interval $T=\left[0, t_{\text {end }}\right]$ finally leads to

$$
\left\|u_{1}\left(t_{\text {end }}\right)\right\|_{\mathcal{H}}^{2}+\left\|u_{1}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{1}(0)\right\|_{\mathcal{H}}^{2}+\|F\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\|G\|_{L^{2}(\mathcal{Q})}^{2}+\|\dot{G}\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2} .
$$

This completes the proof, since $u_{1}(0)=u_{B}^{0}$.
Remark 2 Throughout the paper, we concentrate on results for the variable $u$ which corresponds to the velocity in terms of fluid flow applications. Similar results for the Lagrange multiplier $\lambda$ (respectively the pressure) are valid but require stronger regularity assumptions on $F$ and $u^{0}$. For a detailed stability analysis of the Lagrange multiplier, we refer to [50, Ch. 3.1.2]. Note that Assumption 1 is not sufficient to prove $\lambda \in L^{2}(T ; \mathcal{Q})$.

Since this paper focuses on fluid flows, we show that the linear Stokes equations fit into the given framework. Note that also the Navier-Stokes equations may be considered in the given setting if we allow the operator $K$ in (1) to be nonlinear. However, we exclude the nonlinear case in this paper.

Example 1 (Stokes equations) The linear Stokes equations provide a leading-order simplification of the Navier-Stokes equations and describe the incompressible flow of a Newtonian fluid in a bounded domain $D$, cf. [43]. We consider homogeneous Dirichlet boundary conditions and set

$$
\mathcal{V}=\left[H_{0}^{1}(D)\right]^{d}, \quad \mathcal{H}=\left[L^{2}(D)\right]^{d}, \quad \mathcal{Q}=L^{2}(D) / \mathbb{R}
$$

Furthermore, we define $G \equiv 0, B=$ div with dual operator $B^{*}=-\nabla$, and $K$ which equals the weak form of the Laplace operator, i.e.,

$$
\langle K u, v\rangle:=\int_{D} \nabla u \cdot \nabla v \mathrm{~d} x .
$$

The solution $u$ describes the velocity of the fluid whereas $\lambda$ measures the pressure. The operator equations (1) then equal the weak formulation of the Stokes equations

$$
\dot{u}-\Delta u+\nabla \lambda=f, \quad \nabla \cdot u=0, \quad u(0)=u^{0}
$$

For the Stokes equations with stochastic noise, we refer to Example 7 below.
Example 2 (Linearized Navier-Stokes equations) With a simple modification of the operator $K$, the framework given in Example 1 includes any linearization of the NavierStokes equations such as the Oseen equations.

Given the characteristic velocity $u_{\infty}$, the Oseen equations include the operator

$$
\langle K u, v\rangle:=\int_{D}\left(u_{\infty} \cdot \nabla\right) u v+v \nabla u \cdot \nabla v \mathrm{~d} x
$$

or even

$$
\langle K u, v\rangle:=\int_{D}\left(u_{\infty} \cdot \nabla\right) u v+(u \cdot \nabla) u_{\infty} v+v \nabla u \cdot \nabla v \mathrm{~d} x .
$$

Note that $u$ describes the 'disturbance velocity', i.e., the variation around $u_{\infty}$.
Although we focus here on applications in fluid dynamics, we emphasize that the given framework is not restricted to this class. Further examples are given by PDEs with boundary control [11] (with $B$ being the trace operator) as well as applications in elastodynamics which leads to second-order systems of similar structure [2].

### 2.2 Influence of perturbations

DAEs are known for its high sensitivity to perturbations. The reason for this is that derivatives of the right-hand sides appear in the solution. In particular, this implies that a certain smoothness of the right-hand sides is necessary for the existence of solutions. Furthermore, the numerical approximation is much harder than for ODEs, since small perturbations - such as round-off errors or errors within iterative methods - may have a large influence [38].

The resulting level of difficulty in the numerical approximation of DAEs is measured by the so-called index. There exist several index concepts [35] and we use here the differentiation index, see [10, Def. 2.2.2] for a precise definition. A comparable index concept for operator DAEs which may be used to classify systems of the form (1) does not exist. Thus, in order to obtain information about stability issues it is advisable to analyse the influence of perturbations. Furthermore, a spatial discretization of system (1) by finite elements (under some basic assumptions) leads to a DAE of index 2. Note that the understanding of the index is not crucial for the further reading of this paper. However, we comment on the index from time to time for additional insight.

We consider system (1) with additional perturbations $\delta \in L^{2}\left(T ; \mathcal{V}^{*}\right)$ and $\theta \in$ $H^{1}\left(T ; \mathcal{Q}^{*}\right)$. The perturbed solution $(\hat{u}, \hat{\lambda})$ then satisfies the system

$$
\begin{aligned}
\dot{\hat{u}}+K \hat{u}+B^{*} \hat{\lambda} & =F+\delta \text { in } \mathcal{V}^{*} \\
B \hat{u} & =G+\theta \text { in } \mathcal{Q}^{*} .
\end{aligned}
$$

Let $e_{1}$ denote the difference of $u$ and $\hat{u}$ projected to the kernel of the constraint operator $B$. Accordingly, we denote the projected initial error by $e_{1,0}$. In [5] it is shown that with the given assumptions on the operators $K$ and $B$ of Assumption 1, we have

$$
\begin{equation*}
\left\|e_{1}\right\|_{C(T ; \mathcal{H})}^{2}+\left\|e_{1}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|e_{1,0}\right\|_{\mathcal{H}}^{2}+\|\delta\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\|\theta\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}+\|\dot{\theta}\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2} . \tag{3}
\end{equation*}
$$

This estimate shows that the error depends on the derivative of the perturbation $\theta$. Note that this is crucial if we consider stochastic perturbations in Sect. 3 where we apply the chaos expansion method to reduce the given problem to an infinite number of deterministic systems. Similar to index reduction procedures for DAEs, cf. [10,25], the operator DAE can be regularized in view of an improved behaviour with respect to perturbations.

### 2.3 Regularization of operator DAEs

In this subsection, we introduce an operator DAE which is equivalent to (1), but where the solution of the perturbed system does not depend on derivatives of the perturbations. Furthermore, a semi-discretization in space of the regularized system directly leads to a DAE of index 1 and thus, is better suited for numerical integration [25].

In the case of the Stokes equations, the right-hand side $G$ vanishes since we search for divergence-free velocities. In this case, the constrained system is often reduced to the kernel of the constraint operator $B$ which leads to an operator ODE, i.e., a timedependent PDE. However, with the stochastic noise term in the constraint, we cannot ignore the inhomogeneity anymore. In addition, the inclusion of $G$ enlarges the class of possible applications. Thus, we propose to apply a regularization of the operator DAE.

For the regularization we follow the procedure introduced first in [2] for secondorder systems. The idea is to add the derivative of the constraint, the so-called hidden constraint, to the system. In order to balance the number of equations and variables, we add a so-called dummy variable $v_{2}$ to the system. The assumptions are as before, but we split the space $\mathcal{V}$ into $\mathcal{V}=\mathcal{V}_{B} \oplus \mathcal{V}^{\mathrm{c}}$ were

$$
\mathcal{V}_{B}:=\operatorname{Ker} B
$$

and $\mathcal{V}^{\mathrm{c}}$ is any complementary space on which $B$ is invertible, i.e., there exists a rightinverse of $B$, namely $B^{-}: \mathcal{Q}^{*} \rightarrow \mathcal{V}^{\mathcal{C}}$ with $B B^{-} q=q$ for all $q \in \mathcal{Q}^{*}$. In the example of the Stokes equations, cf. Example $1, \mathcal{V}_{B}$ is the space of divergence-free functions which build a proper subspace of $\mathcal{V}$ and $\mathcal{V}^{\mathcal{C}}$ equals its orthogonal complement in $\mathcal{V}$. We then search for a solution $\left(u_{1}, u_{2}, v_{2}, \lambda\right)$ where $u_{1}$ takes values in $\mathcal{V}_{B}$ and $u_{2}, v_{2}$ in the complement $\mathcal{V}^{\text {c }}$. The extended (but equivalent) system then reads

$$
\begin{align*}
\dot{u}_{1}(t)+v_{2}(t)+K\left(u_{1}(t)+u_{2}(t)\right)+B^{*} \lambda(t) & =F(t) \text { in } \mathcal{V}^{*},  \tag{4a}\\
B u_{2}(t) & =G(t) \text { in } \mathcal{Q}^{*},  \tag{4b}\\
B v_{2}(t) & =\dot{G}(t) \text { in } \mathcal{Q}^{*} \tag{4c}
\end{align*}
$$

with initial condition

$$
\begin{equation*}
u_{1}(0)=u_{B}^{0}-B^{-} G(0) \in \mathcal{H} \tag{4d}
\end{equation*}
$$

Recall that $u_{B}^{0}$ is an element of the closure of $\mathcal{V}_{B}$ in $\mathcal{H}$, cf. Remark 1. The connection of system (1) and (4) is given by $u=u_{1}+u_{2}$ and $v_{2}=\dot{u}_{2}$. Note, however, that in system (4) $u_{2}$ is not differentiated anymore and corresponds to an algebraic variable in the finite-dimensional case. For the regularized formulation (4) we obtain the following stability result.
Theorem 2 (Influence of perturbations) Let Assumption 1 be satisfied and consider perturbations $\delta \in L^{2}\left(T ; \mathcal{V}^{*}\right)$ and $\theta, \xi \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$ of the right-hand sides of (4) with the corresponding perturbed solution $\left(\hat{u}_{1}, \hat{u}_{2}, \hat{v}_{2}, \hat{\lambda}\right)$. Then, the error in $u_{1}$, namely $e_{1}=\hat{u}_{1}-u_{1}$, satisfies the estimate

$$
\begin{equation*}
\left\|e_{1}\right\|_{C(T ; \mathcal{H})}^{2}+\left\|e_{1}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|e_{1,0}\right\|_{\mathcal{H}}^{2}+\|\delta\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\|\theta\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}+\|\xi\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2} \tag{5}
\end{equation*}
$$

Proof We introduce the remaining errors $e_{2}:=\hat{u}_{2}-u_{2}, e_{v}:=\hat{v}_{2}-v_{2}$, and $e_{\lambda}:=\hat{\lambda}-\lambda$. The difference of the original and the perturbed problem then yields an operator DAE for $e_{1}, e_{2}, e_{v}$, and $e_{\lambda}$ of the form (4), namely

$$
\begin{aligned}
& \dot{e}_{1}(t)+e_{v}(t)+K\left(e_{1}(t)+e_{2}(t)\right)+B^{*} e_{\lambda}(t)=\delta(t) \\
& B e_{2}(t)=\theta(t) \\
& \text { in } \mathcal{V}^{*} \\
& B e_{v}(t)=\xi(t) \\
& \text { in } \mathcal{Q}^{*}
\end{aligned}
$$

with initial condition $e_{1}(0)=e_{1,0}$. From this point on, we follow the arguments of the proof of Theorem 1, using

$$
\left\|e_{2}\right\|_{L^{2}(\mathcal{V})} \lesssim\|\theta\|_{L^{2}\left(\mathcal{Q}^{*}\right)}, \quad\left\|e_{v}\right\|_{L^{2}(\mathcal{V})} \lesssim\|\xi\|_{L^{2}\left(\mathcal{Q}^{*}\right)}
$$

instead of the estimates of $u_{2}$ and $\dot{u}_{2}$ therein. Thus, we obtain the estimate

$$
\left\|e_{1}(t)\right\|_{\mathcal{H}}^{2}+\left\|e_{1}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|e_{1}(0)\right\|_{\mathcal{H}}^{2}+\|\delta\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\|\theta\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}+\|\xi\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}
$$

for all $t \in T$. Thus, maximizing over $t$ and using the initial condition, we obtain the stated assertion.
Note that, in contrast to the original formulation, estimate (5) does not depend on derivatives of the perturbations. This is crucial when we consider stochastic perturbations.

## 3 Inclusion of stochastic perturbations

In this section, we consider the operator DAE (1) with additional stochastic perturbation terms, also called noise terms. Clearly, we perturb the deterministic system with zero mean disturbances. First, we consider the noise, only in the differential equation, i.e., we study

$$
\begin{align*}
\dot{u}(t)+\mathcal{K} u(t)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+" \text { noise" }  \tag{6a}\\
\mathcal{B} u(t) & =\mathcal{G}(t) \tag{6b}
\end{align*}
$$

Afterwards, we also add a noise term in the constraint equation,

$$
\begin{align*}
\dot{u}(t)+\mathcal{K} u(t)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+\text { "noise" },  \tag{7a}\\
\mathcal{B} u(t) & =\mathcal{G}(t)+\text { "noise". } \tag{7b}
\end{align*}
$$

As discussed in Sect. 2.2, perturbations in the second equation, i.e., in the constraint equation, lead to instabilities. Thus, we also consider the regularized operator equations (4) with stochastic perturbations. In any case, we assume a consistent initial condition of the form $u(0)=u^{0}$. Note that with the inclusion of stochastic perturbations, we also allow the initial data $u^{0}$ to be random.

From the modeling point of view, noise may enter the physical system either as temporal fluctuations of internal degrees of freedom or as random variations of some external control parameters; internal randomness often reflects itself in additive noise terms, while external fluctuations gives rise to multiplicative noise terms. Moreover, the additive noises may appear in various forms, ranging from the space time white noise to colored noises generated by some infinite dimensional Brownian motion with a prescribed covariance operator [13].

### 3.1 Preliminaries

In this section, we recall some basic facts and notions of the white noise theory, random variables, stochastic processes, and operators. Then we apply the chaos expansion method in order to solve stated problems.

### 3.1.1 White noise space

We consider stochastic DAEs in the white noise framework. For this, the spaces of stochastic test and generalized functions are built by use of series decompositions via orthogonal functions as a basis with certain weight sequences. The classical Hida approach [21] suggests to start with a Gel'fand triple

$$
\mathcal{E} \subseteq L^{2}(\mathbb{R}) \subseteq \mathcal{E}^{\prime}
$$

with continuous inclusions, formed by a nuclear space $\mathcal{E}$ and its dual $\mathcal{E}^{\prime}$. As basic probability space we set $\Omega=\mathcal{E}^{\prime}$ endowed with the Borel sigma algebra of the weak topology and an appropriate probability measure, see [21,22]. Without loss of generality, in this paper we assume that the underlying probability space is the Gaussian white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathcal{B}, \mu\right)$. Therefore, we take $\mathcal{E}$ and $\mathcal{E}^{\prime}$ to be the Schwartz spaces of rapidly decreasing test functions $S(\mathbb{R})$ and tempered distributions $S^{\prime}(\mathbb{R})$, respectively, and $\mathcal{B}$ the Borel sigma algebra generated by the weak topology on $S^{\prime}(\mathbb{R})$. By the Bochner-Minlos theorem, there exists a unique measure $\mu$ on $\left(S^{\prime}(\mathbb{R}), \mathcal{B}\right)$ such that for each $\phi \in S(\mathbb{R})$ the relation

$$
\int_{S^{\prime}(\mathbb{R})} e^{\langle\omega, \phi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\phi\|_{L^{2}(\mathbb{R})}^{2}}
$$

holds, where $\langle\omega, \phi\rangle$ denotes the action of a tempered distribution $\omega \in S^{\prime}(\mathbb{R})$ on a test function $\phi \in S(\mathbb{R})$. We denote by $L^{2}(\Omega, \mu)$, or in short $L^{2}(\Omega)$, the space of square integrable random variables $L^{2}(\Omega)=L^{2}(\Omega, \mathcal{B}, \mu)$. It is the Hilbert space of random variables which have finite second moments. Here, the scalar product is $(F, G)_{L^{2}(\Omega)}=\mathbb{E}_{\mu}(F \cdot G)$, where $\mathbb{E}_{\mu}$ denotes the expectation with respect to the measure $\mu$. In the sequel, we omit $\mu$ and simply write $\mathbb{E}$.

In the case of a Gaussian measure, the orthogonal polynomial basis of $L^{2}(\Omega)$ can be represented as a family of orthogonal Fourier-Hermite polynomials defined by use of the Hermite functions and the Hermite polynomials. We denote by $\left\{h_{n}(x)\right\}_{n \in \mathbb{N}_{0}}$ the family of Hermite polynomials and $\left\{\xi_{n}(x)\right\}_{n \in \mathbb{N}}$ the family of Hermite functions, where

$$
\begin{array}{ll}
h_{n}(x)=(-1)^{n} e^{\frac{x^{2}}{2}} \frac{d^{n}}{d x^{n}}\left(e^{-\frac{x^{2}}{2}}\right), & n \in \mathbb{N}_{0}, \\
\xi_{n}(x)=\frac{1}{\sqrt[4]{\pi} \sqrt{(n-1)!}} e^{-\frac{x^{2}}{2}} h_{n-1}(\sqrt{2} x), & n \in \mathbb{N},
\end{array}
$$

for $x \in \mathbb{R}$. The family of Hermite polynomials forms an orthogonal basis of the space $L^{2}(\mathbb{R})$ with respect to the Gaussian measure $d \mu=\frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}} d x$, while the family of Hermite functions forms a complete orthonormal system in $L^{2}(\mathbb{R})$ with respect to the Lebesque measure. We follow the characterization of the Schwartz spaces in terms of the Hermite basis [17]. Clearly, the Schwartz space of rapidly decreasing functions can be constructed as the projective limit of the family of spaces

$$
S_{l}(\mathbb{R})=\left\{f(t)=\sum_{k \in \mathbb{N}} a_{k} \xi_{k}(t) \in L^{2}(\mathbb{R}):\|f\|_{l}^{2}=\sum_{k \in \mathbb{N}} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{N}_{0}
$$

The Schwartz space of tempered distributions is isomorphic to the inductive limit of the family of spaces

$$
S_{-l}(\mathbb{R})=\left\{F(t)=\sum_{k \in \mathbb{N}} b_{k} \xi_{k}(t):\|F\|_{-l}^{2}=\sum_{k \in \mathbb{N}} b_{k}^{2}(2 k)^{-l}<\infty\right\}, l \in \mathbb{N}_{0}
$$

It holds that $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$ and $S^{\prime}(\mathbb{R})=\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$. The action of a generalized function $F=\sum_{k \in \mathbb{N}} b_{k} \xi_{k} \in S^{\prime}(\mathbb{R})$ on a test function $f=\sum_{k \in \mathbb{N}} a_{k} \xi_{k} \in$ $S(\mathbb{R})$ is given by $\langle F, f\rangle=\sum_{k \in \mathbb{N}} a_{k} b_{k}$.

### 3.1.2 Spaces of random variables

Let $\mathcal{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ be the set of sequences of non-negative integers which have only finitely many nonzero components $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right), \alpha_{i} \in \mathbb{N}_{0}, i=1,2, \ldots, m$, $m \in \mathbb{N}$. The $k$-th unit vector $\varepsilon^{(k)}=(0, \ldots, 0,1,0, \ldots), k \in \mathbb{N}$, is the sequence of zeros with the entry 1 as the $k$-th component and $\mathbf{0}$ is the multi-index with only zero components. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. We
say $\alpha \geq \beta$ if $\alpha_{k} \geq \beta_{k}$ for all $k \in \mathbb{N}$ and thus $\alpha-\beta=\left(\alpha_{1}-\beta_{1}, \alpha_{2}-\beta_{2}, \ldots\right)$. For $\alpha<\beta$ the difference $\alpha-\beta$ is not defined. Particularly, for $\alpha_{k}>0$ we have $\alpha-\varepsilon^{(k)}=\left(\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0, \ldots\right), k \in \mathbb{N}$. We denote $(2 \mathbb{N})^{\alpha}=$ $\prod_{k=1}^{\infty}(2 k)^{\alpha_{k}}$.

Theorem 3 ([49]) It holds that $\sum_{\alpha \in \mathcal{I}}(2 \mathbb{N})^{-p \alpha}<\infty$ if and only if $p>1$.
The proof can be foud in the paper of Zhang [49], also in [22, Prop. 2.3.3].
We define by

$$
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in \mathcal{I},
$$

the Fourier-Hermite orthogonal polynomial basis of $L^{2}(\Omega)$ such that $\left\|H_{\alpha}\right\|_{L^{2}(\Omega)}^{2}=$ $\mathbb{E}\left(H_{\alpha}\right)^{2}=\alpha!$. In particular, $H_{0}(\omega)=H_{(0,0, \ldots)}(\omega)=1$, and for the $k$-th unit vector $H_{\varepsilon^{(k)}}(\omega)=h_{1}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}$.

Theorem 4 ([22]) (Wiener-Itô chaos expansion theorem) Each random variable $f \in$ $L^{2}(\Omega)$ has a unique representation of the form

$$
f(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega), \quad a_{\alpha} \in \mathbb{R}, \omega \in \Omega
$$

such that it holds

$$
\|f\|_{L^{2}(\Omega)}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2} \alpha!<\infty .
$$

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions. They have no point value for $\omega \in \Omega$ but an average value with respect to a test random variable. Following the idea of the construction of $S^{\prime}(\mathbb{R})$ as an inductive limit space over $L^{2}(\Omega)$ with appropriate weights [48], one can define stochastic generalized random variable spaces over $L^{2}(\Omega)$ by adding certain weights in the convergence condition of the series expansion. Several spaces of this type, weighted by a sequence $q=\left(q_{\alpha}\right)_{\alpha \in \mathcal{I}}$, denoted by $(Q)_{-\rho}$, for $\rho \in[0,1]$ were described in [27]. Thus a Gel'fand triple

$$
(Q)_{\rho} \subseteq L^{2}(\Omega) \subseteq(Q)_{-\rho}
$$

is obtained, where the inclusions are again continuous. The most common weights and spaces appearing in applications are $q_{\alpha}=(2 \mathbb{N})^{\alpha}$ which correspond to the Kondratiev spaces of stochastic test functions $(S)_{\rho}$ and stochastic generalized functions $(S)_{-\rho}$, for $\rho \in[0,1]$. Exponential weights $q_{\alpha}=e^{(2 \mathbb{N})^{\alpha}}$ are linked with the exponential growth spaces of stochastic test functions $\exp (S)_{\rho}$ and stochastic generalized functions $\exp (S)_{-\rho}[21,22,27,39,40]$. In this paper, we consider the largest Kondratiev space of
stochastic distributions, i.e., $\rho=1$. For definition of the Kondratiev spaces we follow [22].

The space of the Kondratiev test random variables $(S)_{1}$ can be constructed as the projective limit of the family of spaces

$$
(S)_{1, p}=\left\{f(\omega)=\sum_{\alpha \in \mathcal{I}} a_{\alpha} H_{\alpha}(\omega) \in L^{2}(\Omega):\|f\|_{1, p}^{2}=\sum_{\alpha \in \mathcal{I}} a_{\alpha}^{2}(\alpha!)^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}
$$

$p \in \mathbb{N}_{0}$. The space of the Kondratiev generalized random variables $(S)_{-1}$ can be constructed as the inductive limit of the family of spaces
$(S)_{-1,-p}=\left\{F(\omega)=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}(\omega):\|f\|_{-1,-p}^{2}=\sum_{\alpha \in \mathcal{I}} b_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\}, p \in \mathbb{N}_{0}$.
It holds that $(S)_{1}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{1, p}$ and $(S)_{-1}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-1, p}$. The action of a generalized random variable $F=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}(\omega) \in(S)_{-1}$ on a test random variable $f=\sum_{\alpha \in \mathcal{I}} b_{\alpha} H_{\alpha}(\omega) \in(S)_{1}$ is given by $\langle F, f\rangle=\sum_{\alpha \in \mathcal{I}} \alpha!a_{\alpha} b_{\alpha}$. It holds that $(S)_{1}$ is a nuclear space with the Gel'fand triple structure

$$
(S)_{1} \subseteq L^{2}(\Omega) \subseteq(S)_{-1}
$$

with continuous inclusions. Moreover, for $0 \leq p \leq q$ it holds $(S)_{1, q} \subseteq(S)_{1, p} \subseteq$ $(S)_{1,0} \subseteq L^{2}(\Omega) \subseteq(S)_{-1,0} \subseteq(S)_{-1,-p} \subseteq(S)_{-1,-q}$. The proof of nuclearity of $(S)_{1}$ can be found in [21] and in [22, Lemma 2.8.2].

The problem of pointwise multiplications of generalized stochastic functions in the white noise analysis is overcome by introducing the Wick product, which represents the stochastic convolution. The fundamental theorem of stochastic calculus states the relation of the Wick multiplication to the Itô-Skorokhod integration [22].

Let $L$ and $S$ be random variables given in their chaos expansion representations $L=\sum_{\alpha \in \mathcal{I}} \ell_{\alpha} H_{\alpha}$ and $S=\sum_{\alpha \in \mathcal{I}} s_{\alpha} H_{\alpha}, \ell_{\alpha}, s_{\alpha} \in \mathbb{R}$ for all $\alpha \in \mathcal{I}$. Then, the Wick product $L \diamond S$ is defined by

$$
\begin{equation*}
L \diamond S=\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} \ell_{\alpha} s_{\beta}\right) H_{\gamma}(\omega) \tag{8}
\end{equation*}
$$

Note here that the space $L^{2}(\Omega)$ is not closed under the Wick multiplication.
Example 3 Consider the random variable $F=\sum_{k \in \mathbb{N}} \frac{1}{k} H_{\varepsilon^{(k)}}$ and its Wick square $F^{\diamond 2}=F \diamond F=\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n(n-k)} H_{\varepsilon^{(n)}}$. Then $F \in L^{2}(\Omega)$, since $\|F\|_{L^{2}(\Omega)}^{2}=$ $\sum_{k \in \mathbb{N}} \frac{1}{k^{2}}<\infty$. In contrast, its Wick square $F^{\diamond 2}$ is not an element of $L^{2}(\Omega)$, since it holds that

$$
\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} \frac{1}{n(n-k)}\right)^{2} \geq \sum_{k=1}^{\infty} \frac{1}{k(k+1)}=\sum_{k=1}^{\infty}\left(1-\frac{1}{k+1}\right)=+\infty
$$

Kondratiev spaces $(S)_{1}$ and $(S)_{-1}$ are closed under the Wick multiplication. For the proof we refer to [22, Lemma 2.4.4].

### 3.1.3 Stochastic processes

Classical stochastic process can be defined as a family of functions $v: T \times \Omega \rightarrow$ $\mathbb{R}$ such that for each fixed $t \in T, v(t, \cdot)$ is an $\mathbb{R}$-valued random variable and for each fixed $\omega \in \Omega, v(\cdot, \omega)$ is an $\mathbb{R}$-valued deterministic function, called trajectory. Here, following [39], we generalize the definition of a classical stochastic process and define generalized stochastic processes. By replacing the space of trajectories with some space of deterministic generalized functions, or by replacing the space of random variables with some space of generalized random variables, different types of generalized stochastic processes can be obtained. In this manner, we obtain processes generalized with respect to the $t$ argument, the $\omega$ argument, or even with respect to both arguments [22,39].

A very general concept of generalized stochastic processes, based on chaos expansions was introduced in [39] and further developed in [27,28]. In [22] generalized stochastic processes are defined as measurable mappings $T \rightarrow(S)_{-1}$. Thus, they are defined pointwise with respect to the parameter $t \in T$ and generalized with respect to $\omega \in \Omega$. We define such processes by their chaos expansion representations in terms of an orthogonal polynomial basis.

Let $\tilde{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\tilde{X}}$ and let $\tilde{X}^{\prime}$ denote its dual space. If, for example, $\tilde{X}$ is a space of functions on $\mathbb{R}$ such as $\tilde{X}=C^{k}(T)$ or $\tilde{X}=L^{2}(\mathbb{R})$, we obtain stochastic processes. The definition of processes where $\tilde{X}$ is not a normed space, but a nuclear space topologized by a family of seminorms, e.g. $\tilde{X}=S(\mathbb{R})$ is given in [39].

Let $u$ have the formal expansion $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$, where $f_{\alpha} \in X$ and $\alpha \in \mathcal{I}$. We define the spaces

$$
\left.\begin{array}{rl}
X \otimes(S)_{1, p} & =\left\{f:\|f\|_{X \otimes(S)_{1, p}}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right. \\
X \otimes(S)_{-1,-p} & =\left\{f:\|f\|_{X \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right.
\end{array}\right\},
$$

where $X$ denotes an arbitrary Banach space (both possibilities $X=\tilde{X}$ and $X=\tilde{X}^{\prime}$ are allowed).

Definition 1 Generalized stochastic processes and test stochastic processes in Kondratiev sense are elements of the spaces respectively

$$
X \otimes(S)_{-1}=\bigcup_{p \in \mathbb{N}} X \otimes(S)_{-1,-p} \quad \text { and } \quad X \otimes(S)_{1}=\bigcap_{p \in \mathbb{N}} X \otimes(S)_{1, p} .
$$

In this case the symbol $\otimes$ denotes the projective tensor product of two spaces, i.e., $\tilde{X}^{\prime} \otimes(S)_{-1}$ is the completion of the tensor product with respect to the $\pi$-topology.

Remark 3 From the nuclearity of the Kondratiev space $(S)_{1}$ it follows that $(\tilde{X} \otimes$ $\left.(S)_{1}\right)^{\prime} \cong \tilde{X}^{\prime} \otimes(S)_{-1}$. Moreover, $\tilde{X}^{\prime} \otimes(S)_{-1}$ is isomorphic to the space of linear bounded mappings $\tilde{X} \rightarrow\left(\tilde{X}_{-1}\right.$, and it is also isomporphic to the space of linear bounded mappings $(S)_{1} \rightarrow \tilde{X}^{\prime}$. More details can be found in [28,32,39].

Throughout the paper we consider generalized stochastic processes $u$ which belong to $X \otimes(S)_{-1}$ and are given by the chaos expansion form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}=u_{\mathbf{0}}(t)+\sum_{k \in \mathbb{N}} u_{\varepsilon^{(k)}} \otimes H_{\varepsilon^{(k)}}+\sum_{|\alpha|>1} u_{\alpha} \otimes H_{\alpha} \tag{9}
\end{equation*}
$$

Therein, the coefficients $u_{\alpha} \in X$ satisfy for some $p \in \mathbb{N}_{0}$ the convergence condition

$$
\|u\|_{X \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

The value $p$ corresponds to the level of singularity of the process $u$. Note that the deterministic part of $u$ in (9) is the coefficient $u_{\mathbf{0}}$, which represents the generalized expectation of $u$. In the applications of fluid flows, the space $X$ equals one of the Sobolev-Bochner spaces $L^{2}(T ; \mathcal{V})$ or $L^{2}(T ; \mathcal{Q})$.

Example 4 If $X=L^{2}(\mathbb{R})$, then $u \in L^{2}(\mathbb{R}) \otimes L^{2}(\Omega)$ is given in the chaos expansion form $u(t, \omega)=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(t) H_{\alpha}(\omega), t \in \mathbb{R}, \omega \in \Omega$ such that

$$
\|u\|_{L^{2}(\mathbb{R}) \otimes L^{2}(\Omega)}^{2}=\sum_{\alpha \in \mathcal{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}(\mathbb{R})}^{2}=\sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}} \alpha!\left|u_{\alpha}(t)\right|^{2} \mathrm{~d} t<\infty
$$

Stochastic processes which are elements of the space $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}=$ $\bigcup_{p, l \in \mathbb{N}} X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}$ are defined similarly, cf. [27-29,31]. More precisely, $F \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ has a chaos expansion representation

$$
\begin{equation*}
F=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathcal{I}} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k} \tag{10}
\end{equation*}
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} a_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathcal{I}} a_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-1}$, and $a_{\alpha, k} \in X$. Thus, for some $p, l \in \mathbb{N}_{0}$, it holds that

$$
\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left\|a_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty
$$

The generalized expectation of $F$ is the zero-th coefficient in the expansion representation (10), i.e., it is given by $\sum_{k \in \mathbb{N}} a_{\mathbf{0}, k} \otimes \xi_{k}=b_{\mathbf{0}}$.

Space of processes with finite second moments and square integrable trajectories $X \otimes L^{2}(\mathbb{R}) \otimes(L)^{2}$. It is isomporphic to $X \otimes L^{2}(\mathbb{R} \times \Omega)$ and if $X$ is a separable Hilbert space, then it is also isomorphic to $L^{2}(\mathbb{R} \times \Omega, X)$.

Example 5 Consider $X=C^{k}(T), k \in \mathbb{N}$, where $T$ denotes a time interval. From the nuclearity of $(S)_{1}$ and the arguments provided in Remark 3 it follows that $C^{k}\left(T ;(S)_{-1}\right)=C^{k}(T) \otimes(S)_{-1}$, i.e., differentiation of a stochastic process can be carried out componentwise in the chaos expansion, cf. [28,32]. This means that a stochastic process $u(t, \omega)$ is $k$ times continuously differentiable if and only if all of its coefficients $u_{\alpha}, \alpha \in \mathcal{I}$ are in $C^{k}(T)$. The same holds for Banach space valued stochastic processes, i.e., for elements of $C^{k}(T ; X) \otimes(S)_{-1}$, where $X$ is an arbitrary Banach space. These processes can be regarded as elements of the tensor product space

$$
C^{k}\left(T ; X \otimes(S)_{-1}\right)=C^{k}(T ; X) \otimes(S)_{-1}=\bigcup_{p=0}^{\infty} C^{k}(T ; X) \otimes(S)_{-1,-p}
$$

Since we consider weak solutions, i.e., solutions in Sobolev-Bochner spaces such as $L^{2}(T ; X)$, it also holds $L^{2}\left(T ; X \otimes(S)_{-1}\right)=L^{2}(T ; X) \otimes(S)_{-1}$, as well as $H^{1}(T ; X \otimes$ $\left.(S)_{-1}\right)=H^{1}(T ; X) \otimes(S)_{-1}$.

In this way, by representing stochastic processes in their polynomial chaos expansion form, we are able to separate the deterministic component from the randomness of the process.
Example 6 Brownian motion $B_{t}(\omega):=\left\langle\omega, \chi_{[0, t]}\right\rangle, \omega \in S^{\prime}(\mathbb{R}), t \geq 0$ is defined by passing though the limit in $L^{2}(\mathbb{R})$, where $\chi_{[0, t]}$ is the characteristic function on $[0, t]$. The chaos expansion representation has the form

$$
B_{t}(\omega)=\sum_{k \in \mathbb{N}} \int_{0}^{t} \xi_{k}(s) \mathrm{d} s H_{\varepsilon^{(k)}}(\omega)
$$

Note that for fixed $t, B_{t}$ is an element of $L^{2}(\Omega)$. Brownian motion is a Gaussian process with zero expectation and the covariance function $E\left(B_{t}(\omega) B_{s}(\omega)\right)=\min \{t, s\}$. Furthermore, almost all trajectories are continuous, but nowhere differentiable functions.

Singular white noise is defined by the formal chaos expansion

$$
\begin{equation*}
W_{t}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega) \tag{11}
\end{equation*}
$$

and is an element of the space $C^{\infty}(\mathbb{R}) \otimes(S)_{-1,-p}$ for $p>1$, cf. [22]. With weak derivatives in the $(S)_{-1}$ sense, it holds that $\frac{\mathrm{d}}{\mathrm{d} t} B_{t}=W_{t}$. Both, Brownian motion and singular white noise, are Gaussian processes and have chaos expansion representations via Fourier-Hermite polynomials with multi-indeces of length one, i.e., belong to the Wiener chaos space of order one.
More general, the chaos expansion of a Gaussian process $G_{t}$ in $S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$, which belongs to the Wiener chaos space of order one, is given by

$$
\begin{equation*}
G_{t}(\omega)=\sum_{k=1}^{\infty} m_{k}(t) H_{\varepsilon^{(k)}}(\omega)=\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} m_{k n} \xi_{n}(t) H_{\varepsilon^{(k)}}(\omega) \tag{12}
\end{equation*}
$$

with coefficients $m_{k}$ being deterministic generalized functions and $m_{k n} \in \mathbb{R}$ such that the condition

$$
\sum_{k=1}^{\infty}\left\|m_{k}\right\|_{-l}^{2}(2 k)^{-p}=\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} m_{k n}^{2}(2 n)^{-l}(2 k)^{-p}<\infty
$$

holds for some $l, p \in \mathbb{N}_{0}$. One can also consider a generalized Gaussian process $G \in X \otimes(S)_{-1}$ with a Banach space $X$ of the form

$$
G=\sum_{k=1}^{\infty} m_{k} H_{\varepsilon^{(k)}}
$$

with coefficients $m_{k} \in X$ that satisfy

$$
\begin{equation*}
\sum_{k=1}^{\infty}\left\|m_{k}\right\|_{X}^{2}(2 k)^{-p}<\infty \tag{13}
\end{equation*}
$$

For example, in Sect. 3.3 we deal with $X=L^{2}\left(T ; \mathcal{V}^{*}\right)$.
The Wick product of two stochastic processes is defined in an analogue way as it was defined for random variables in (8) and generalized random variables [30]. Let $F$ and $G$ be stochastic processes given in their chaos expansion forms $F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha}$ and $G=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha}, f_{\alpha}, g_{\alpha} \in X$ for all $\alpha \in \mathcal{I}$. Assuming that $f_{\alpha} g_{\beta} \in X$, for all $\alpha, \beta \in \mathcal{I}$, the Wick product $F \diamond G$ is defined by

$$
\begin{equation*}
F \diamond G=\sum_{\gamma \in \mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) \otimes H_{\gamma} \tag{14}
\end{equation*}
$$

The examples considered in this paper use either $X=L^{2}\left(T ; \mathcal{V}^{*}\right)$ or $X=C^{k}(T)$. The space of stochastic processes $X \otimes(S)_{-1}$ is closed under the Wick multiplication. This is stated in the following theorem. The proof can be found in [28].

Theorem 5 ([28]) Consider $F \in X \otimes(S)_{-1,-p_{1}}$ and $G \in X \otimes(S)_{-1,-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$. Then the Wick product $F \diamond G$ is a well-defined element in the space $X \otimes(S)_{-1,-q}$ for $q \geq p_{1}+p_{2}+2$.

### 3.1.4 Coordinatewise operators

We follow the classification of stochastic operators given in [32] and consider the following two classes. We say that an operator $\mathcal{A}$ defined on $X \otimes(S)_{-1}$ is a coordinatewise operator if it is composed of a family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathcal{I}}, A_{\alpha}: X \rightarrow X$, $\alpha \in \mathcal{I}$, such that for a process $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-1}, u_{\alpha} \in X, \alpha \in \mathcal{I}$ it holds that

$$
\mathcal{A} u=\sum_{\alpha \in \mathcal{I}} A_{\alpha} u_{\alpha} \otimes H_{\alpha}
$$

If $A_{\alpha}=A$ for all $\alpha \in \mathcal{I}$, then the operator $\mathcal{A}$ is called a simple coordinatewise operator.

### 3.2 Chaos expansion approach

We return to the stochastic operator DAEs (6) and (7) where the noise terms are generalized Gaussian stochastic processes as given in (12). Within the next two subsections, we consider the influence of these perturbations. Applying the chaos expansion method, we transform the stochastic systems into deterministic problems, which we solve by induction over the length of the multi-index $\alpha$. Clearly, we represent all the processes appearing in the stochastic equation by their chaos expansion forms and, since the representation in the Fourier-Hermite polynomial basis is unique, equalize the coefficients. In this section, we assume $\mathcal{K}$ and $\mathcal{B}$ to be simple coordinatewise operators, i.e., for $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$ we have

$$
\begin{equation*}
\mathcal{K} u=\sum_{\alpha \in \mathcal{I}} K u_{\alpha} \otimes H_{\alpha} \quad \text { and } \quad \mathcal{B} u=\sum_{\alpha \in \mathcal{I}} B u_{\alpha} \otimes H_{\alpha} . \tag{15}
\end{equation*}
$$

Note that this implies that $\mathcal{B}^{*}$ is a simple coordinatewise operator as well. A more general case of coordinatewise operators is considered in Sect. 4. In the following, we assume that $K$ and $B$ are linear and that they satisfy Assumption 1. For the right-hand side of the differential equation (6a), namely stochastic process $\mathcal{F}$, and the constraint (6b), namely stochastic process $\mathcal{G}$, we assume that they are given in the chaos expansion forms

$$
\begin{equation*}
\mathcal{F}=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes H_{\alpha} \quad \text { and } \quad \mathcal{G}=\sum_{\alpha \in \mathcal{I}} g_{\alpha} \otimes H_{\alpha} . \tag{16}
\end{equation*}
$$

Therein, corresponding to the deterministic setting of Sect. 2.1, the deterministic coefficients satisfy $f_{\alpha} \in L^{2}\left(T ; \mathcal{V}^{*}\right)$ and $g_{\alpha} \in H^{1}\left(T ; \mathcal{Q}^{*}\right)$. Furthermore, we assume that for some positive $p$ it holds that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{L}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \quad \text { and } \quad \sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}<\infty . \tag{17}
\end{equation*}
$$

Remark 4 Since the family of spaces $(S)_{-1,-p}$ is monotone, i.e., it holds that $(S)_{-1,-p_{1}} \subset(S)_{-1,-p}$ for $p_{1}<p$, we may assume in (17) that all the convergence conditions hold for the same level of singularity $p$. Clearly, for two different $p_{1}$ and $p_{2}$ we can take $p$ to be $p=\max \left\{p_{1}, p_{2}\right\}$ and thus, obtain that generalized stochastic processes satisfies (17) in the biggest space $\left(S_{-1,-p}\right.$. In that sense, we use in the sequel always the same level of singularity $p$.

We seek for solutions $u$ and $\lambda$ of stochastic operator DAEs (6) and (7), which are stochastic processes belonging to $L^{2}(\mathcal{V}) \otimes(S)_{-1}$ and $L^{2}(\mathcal{Q}) \otimes(S)_{-1}$, respectively. Their chaos expansions are given by

$$
\begin{equation*}
u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha} \quad \text { and } \quad \lambda=\sum_{\alpha \in \mathcal{I}} \lambda_{\alpha} \otimes H_{\alpha} . \tag{18}
\end{equation*}
$$

The aim is to calculate the unknown coefficients $u_{\alpha}$ and $\lambda_{\alpha}$ for all $\alpha \in \mathcal{I}$, which then give the overall solutions $u$ and $\lambda$. Furthermore, we are going to prove bounds on the solutions, provided that the stated assumptions on the given processes $\mathcal{F}$ and $\mathcal{G}$, the initial condition, and the noise terms are fulfilled.

Considering the stochastic operator DAE equations (6) and (7), we apply at first the chaos expansion method to the initial condition $u(0)=u^{0}$ and obtain

$$
u^{0}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}(0) H_{\alpha}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha}
$$

Thus, the initial condition reduces to the family of conditions $u_{\alpha}(0)=u_{\alpha}^{0} \in \mathcal{H}$ for every $\alpha \in \mathcal{I}$. In order to achieve consistency, the initial data has to be of the form

$$
\begin{equation*}
u_{\alpha}^{0}=u_{B, \alpha}^{0}+B^{-} g_{\alpha}(0), \quad \alpha \in \mathcal{I} \tag{19}
\end{equation*}
$$

with an arbitrary $u_{B, \alpha}^{0}$ from the closure of the kernel of $B$ in $\mathcal{H}$ and $B^{-}$denoting the right-inverse of the operator $B, \mathrm{cf}$. Remark 1 .

### 3.3 Noise in the differential equation

Consider the system (6) with a stochastic perturbation given in the form of a generalized Gaussian stochastic process in the Wiener chaos space of order one as in (12), i.e., we consider the initial value problem

$$
\begin{gather*}
\dot{u}(t)+\mathcal{K} u(t)+\mathcal{B}^{*} \lambda(t)=\mathcal{F}(t)+G_{t} \\
\mathcal{B} u(t)=\mathcal{G}(t), \quad u(0)=u^{0}=u_{\mathcal{B}}^{0}+\mathcal{B}^{-} \mathcal{G}(0) \tag{20}
\end{gather*}
$$

Example 7 (Randomly forced Stokes equation) We consider the randomly forced Stokes equation, i.e. Stokes equation with noise forcing term. In this case, the operator equation (20) is equal to the weak formulation of the stochastically perturbed Stokes equations

$$
\dot{u}-\Delta u+\nabla \lambda=\tilde{f}, \quad \nabla \cdot u=0, \quad u(0)=u^{0}
$$

where the flow $\tilde{f}=f+G_{t}, t \in T$, is subject to an external forcing. We refer the reader to [12] for a detailed explanation. Note that, in general, additive noise is interpreted in applications as a perturbation of the original model (in this case Example 4).

We summarize the needed requirements in the following assumption.
Assumption 2 1. Operators $\mathcal{K}$ and $\mathcal{B}$ are simple coordinatewise operators with corresponding deterministic operators $K: \mathcal{V} \rightarrow \mathcal{V}^{*}$ and $B: \mathcal{V} \rightarrow \mathcal{Q}^{*}$, which satisfy the assumptions stated in Assumption 1.
2. The stochastic processes $\mathcal{F}$ and $\mathcal{G}$ are given in their chaos expansion forms (16) such that the conditions in (17) hold.
3. The process $G_{t}$ is a Gaussian noise term represented in the form (12), with $m_{k} \in$ $L^{2}\left(T ; \mathcal{V}^{*}\right), k \in \mathbb{N}$, such that (13) holds.
4. The stochastic process $u^{0}$ has the chaos expansion form $u^{0}=\sum_{\alpha \in \mathcal{I}} u_{\alpha}^{0} H_{\alpha}$ such that for some $p \in \mathbb{N}_{0}$ it holds that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}^{0}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{21}
\end{equation*}
$$

Remark 5 If the initial data is consistent, then Assumption 2 and equation (19) imply that condition (21) can be replaced by

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}}\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{22}
\end{equation*}
$$

with $u_{B, \alpha}^{0}$ given in (19).
Theorem 6 Let Assumption 2 be satisfied. Then, for any consistent initial data there exists a unique solution $u \in L^{2}(T ; \mathcal{V}) \otimes(S)_{-1}$ of the stochastic DAE (20).

Proof We represent all the processes in (20) in their chaos expansion forms, apply (15) and thus, reduce it to an infinite triangular system of deterministic initial value problems, which can be solved recursively over the length of multi-index $\alpha$. We obtain the system

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}}\left(\dot{u}_{\alpha}(t)+K u_{\alpha}(t)+B^{*} \lambda_{\alpha}(t)\right) H_{\alpha}(\omega) & =\sum_{\alpha \in \mathcal{I}} f_{\alpha}(t) H_{\alpha}(\omega)+\sum_{k \in \mathbb{N}} m_{k}(t) H_{\alpha}(\omega) \\
\sum_{\alpha \in \mathcal{I}} B u_{\alpha}(t) H_{\alpha}(\omega) & =\sum_{\alpha \in \mathcal{I}} g_{\alpha}(t) H_{\alpha}(\omega)
\end{aligned}
$$

with $u(0)=u^{0}$, i.e., initial data with coefficients given in (19) that satisfy (21). Thus,

1. for $|\alpha|=0$, i.e., for $\alpha=\mathbf{0}=(0,0, \ldots)$, we have to solve

$$
\begin{equation*}
\dot{u}_{\mathbf{0}}(t)+K u_{\mathbf{0}}(t)+B^{*} \lambda_{\mathbf{0}}(t)=f_{\mathbf{0}}(t), B u_{\mathbf{0}}(t)=g_{\mathbf{0}}(t), \quad u_{\mathbf{0}}=u_{B, \mathbf{0}}^{0}+B^{-} g_{\mathbf{0}}(0) \tag{23}
\end{equation*}
$$

Note that system (23) is a deterministic problem of the form (1), where $F$ and $G$ from (1) are equal to $f_{0}$ and $g_{0}$, respectively. Moreover, the system (23) can be obtained by taking the expectation of the system (20). The assumptions on the operators and right-hand sides $f_{0} \in L^{2}\left(T ; \mathcal{V}^{*}\right), g_{0} \in H^{1}\left(T ; \mathcal{Q}^{*}\right)$ imply the existence of a solution $u_{\mathbf{0}}, \lambda_{\mathbf{0}}$.
2. for $|\alpha|=1$, i.e., for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$, we obtain the system

$$
\begin{equation*}
\dot{u}_{\varepsilon^{(k)}}(t)+K u_{\varepsilon^{(k)}}(t)+B^{*} \lambda_{\varepsilon^{(k)}}(t)=f_{\varepsilon^{(k)}}(t)+m_{k}(t), B u_{\varepsilon^{(k)}}(t)=g_{\varepsilon^{(k)}}(t) \tag{24}
\end{equation*}
$$

with initial condition $u_{\varepsilon^{(k)}}(0)=u_{B, \varepsilon^{(k)}}^{0}+B^{-} g_{\varepsilon^{(k)}}(0)$. For each $k \in \mathbb{N}$ system (24) is a deterministic initial value problem of the form (1), with the choice $F=f_{\varepsilon^{(k)}}+m_{k}$ and $G=g_{\varepsilon^{(k)}}$.
3. for $|\alpha|>1$, we finally solve

$$
\begin{equation*}
\dot{u}_{\alpha}(t)+K u_{\alpha}(t)+B^{*} \lambda_{\alpha}(t)=f_{\alpha}(t), B u_{\alpha}(t)=g_{\alpha}(t), \quad u_{\alpha}(0)=u_{B, \alpha}^{0}+B^{-} g_{\alpha}(0) . \tag{25}
\end{equation*}
$$

Again, system (25) is a deterministic operator DAE, which can be solved in the same manner as the system (23).

From (23) we obtain $u_{\boldsymbol{0}}$ and $\lambda_{\boldsymbol{0}}$. Further, from (24) we obtain the coefficients $u_{\alpha}$ and $\lambda_{\alpha}$ for $|\alpha|=1$ and from (25) the remaining coefficients. Note that all these systems may be solved in parallel.

As the last step of the analysis, we prove the convergence of the obtained solution in the space of Kondratiev generalized stochastic processes, i.e., we prove that $\|u\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1}}^{2}<\infty$, for $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}$. More precisely we show that

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

holds for some $p \in \mathbb{N}_{0}$. For this, we apply the estimate from Theorem 1 to the deterministic operator DAEs (23)-(25) for the coefficients $u_{\alpha}$. For $u_{0}$ we obtain by Theorem 1 the estimate

$$
\begin{equation*}
\left\|u_{\mathbf{0}}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{B, \mathbf{0}}^{0}\right\|_{\mathcal{H}^{2}}^{2}+\left\|f_{\mathbf{0}}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\mathbf{0}}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2} . \tag{26}
\end{equation*}
$$

Similarly, for $|\alpha|=1$ and $|\alpha|>1$, we obtain respectively the estimates

$$
\begin{aligned}
\left\|u_{\varepsilon^{(k)}}\right\|_{L^{2}(\mathcal{V})}^{2} & \lesssim\left\|u_{B, \varepsilon^{(k)}}^{0}\right\|_{\mathcal{H}^{2}}^{2}+\left\|f_{\varepsilon^{(k)}}+m_{k}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\varepsilon^{(k)}}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}, k \in \mathbb{N} \text { and } \\
\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2} & \lesssim\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}^{2}}^{2}+\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}, \quad|\alpha|>1
\end{aligned}
$$

Note that the involved constants are equal for all estimates, since we have assumed simple coordinatewise operators. Summarizing the results, we obtain

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2}(2 \mathbb{N})^{-p \alpha} \lesssim & \sum_{\alpha \in \mathcal{I}}\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}^{2}}^{2}(2 \mathbb{N})^{-p \alpha}+\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{k=1}^{\infty}\left\|m_{k}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 k)^{-p}+\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}<\infty,
\end{aligned}
$$

where we have used the linearity, the triangular inequality, and the relation $(2 \mathbb{N})^{\varepsilon^{(k)}}=$ $2 k, k \in \mathbb{N}$. The assumptions (13), (17), and (21) show that the right-hand side is bounded and thus, completes the proof.

Remark 6 If the process $\mathcal{F}$ in (20) is a deterministic function, then it can be represented by $\mathcal{F}=f_{\mathbf{0}}$, since the remaining coefficients satisfy $f_{\alpha}=0$ for all $|\alpha|>0$. Therefore, systems (24) and (25) further simplify.

As mentioned in Remark 2, a similar result can be formulated for the Lagrange multiplier if we assume stronger regularity assumptions. For completeness we state the following result for the Lagrange multiplier but leave out the proof.
Theorem 7 Let Assumption 2 be satisfied. Assume additionally $f_{\alpha} \in L^{2}\left(T ; \mathcal{H}^{*}\right)$ and $u_{B, \alpha}^{0} \in \mathcal{V}$ and let the operator $K$ be symmetric. Then, for any consistent initial data there exists a unique Lagrange multiplier $\lambda \in L^{2}(T ; \mathcal{Q}) \otimes(S)_{-1}$ of the stochastic operator DAE (20).

One may also consider stochastic operator DAEs (20) which include a more general form of the Gaussian noise $G_{t}$, i.e.,

$$
\begin{equation*}
G_{t}(\omega)=\sum_{|\alpha|>0} m_{\alpha}(t) H_{\alpha}(\omega), \tag{27}
\end{equation*}
$$

where $G_{t}$ has also non-zero coefficients of order greater than one. The solution for this case can be provided similarly to the presented case for Gaussian noise in the Wiener chaos space of order one.

Theorem 8 Let the assumptions 1, 2 and 4 from Assumption 2 hold and let the process $G_{t}$ be a Gaussian process of the form (27) such that for some $p \geq 0$ it holds that

$$
\sum_{|\alpha|>0}\left\|m_{\alpha}\right\|_{L^{2}\left(\nu^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

Then, for any consistent initial data that satisfies (21) the stochastic operator DAE (20) has a unique solution $u \in L^{2}(T ; \mathcal{V}) \otimes(S)_{-1}$.

Proof The system of deterministic DAEs obtained from (20) by applying the chaos expansion method contains (23) for $|\alpha|=0$ and

$$
\begin{align*}
\dot{u}_{\alpha}(t)+K u_{\alpha}(t)+B^{*} \lambda_{\alpha}(t) & =f_{\alpha}(t)+m_{\alpha}(t), \\
B u_{\alpha}(t) & =g_{\alpha}(t), \tag{28}
\end{align*}
$$

with the condition $u_{\alpha}(0)=u_{B, \alpha}^{0}+B^{-} g_{\alpha}(0)$, for $|\alpha|>0$. By solving the obtained systems, we obtain the unknown coefficients $u_{\alpha}, \alpha \in \mathcal{I}$. By applying Theorem 1 , we obtain the estimates (26) for $|\alpha|=0$ and

$$
\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}+\left\|f_{\alpha}+m_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}
$$

for $|\alpha|>0$. Hence, the solution $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} H_{\alpha}$ satisfies the estimate

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2}(2 \mathbb{N})^{-p \alpha} \lesssim & \sum_{\alpha \in \mathcal{I}}\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \alpha}+\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)^{*}(2 \mathbb{N})^{-p \alpha}}^{2} \\
& +\sum_{|\alpha|>0}^{\infty}\left\|m_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}+\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)^{*}}^{2}(2 \mathbb{N})^{-p \alpha}<\infty . \tag{29}
\end{align*}
$$

This shows that $u$ belongs to $L^{2}(T ; \mathcal{V}) \otimes(S)_{-1}$.

Remark 7 Let the assumptions of Theorem 8 hold. If we assume additionally that $f_{\alpha} \in L^{2}\left(T ; \mathcal{H}^{*}\right), u_{B, \alpha}^{0} \in \mathcal{V}$ and the operator $K$ is symmetric, then there exists a unique Lagrange multiplier $\lambda \in L^{2}(T ; \mathcal{Q}) \otimes(S)_{-1}$ of the stochastic operator DAE (20).

### 3.4 Noise in the constraint equation

Consider the stochastic operator DAE (7), where the noise terms are given in the form of two Gaussian white noise processes $G_{t}$ and $G_{t}^{(1)}$ belonging to the Wiener chaos space of order one. More precisely, we consider the initial value problem

$$
\begin{align*}
\dot{u}(t)+\mathcal{K} u(t)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+G_{t} \\
\mathcal{B} u(t) & =\mathcal{G}(t)+G_{t}^{(1)} \tag{30}
\end{align*}
$$

with the initial condition $u(0)=u^{0}$. Note that the initial data $u^{0}$ has to be consistent again. Here, the consistency condition includes the perturbation $G_{t}^{(1)}$ such that the consistent initial data of the unperturbed problem may not be consistent in this case. We assume

$$
\begin{equation*}
G_{t}^{(1)}(\omega)=\sum_{k=1}^{\infty} m_{k}^{(1)}(t) H_{\varepsilon^{(k)}}(\omega) \tag{31}
\end{equation*}
$$

where $m_{k}^{(1)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$. We still keep the Assumption 2 for the operators $\mathcal{K}$ and $\mathcal{B}$, processes $\mathcal{F}$ and $\mathcal{G}$ and Gaussian noise $G_{t}$. Note that $m_{k} \in L^{2}\left(T ; \mathcal{V}^{*}\right), k \in \mathbb{N}$. Then, system (30) reduces to the following deterministic systems:

1. for $|\alpha|=0$, i.e., for $\alpha=\mathbf{0}=(0,0, \ldots)$, we obtain

$$
\begin{equation*}
\dot{u}_{\mathbf{0}}(t)+K u_{\mathbf{0}}(t)+B^{*} \lambda_{\mathbf{0}}(t)=f_{\mathbf{0}}(t), B u_{\mathbf{0}}(t)=g_{\mathbf{0}}(t), \quad u_{0}=u_{\mathbf{0}}^{0} \tag{32}
\end{equation*}
$$

2. for $|\alpha|=1$, i.e., for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$, we have

$$
\begin{align*}
\dot{u}_{\varepsilon^{(k)}}(t)+K u_{\varepsilon^{(k)}}(t)+B^{*} \lambda_{\varepsilon^{(k)}}(t) & =f_{\varepsilon^{(k)}}(t)+m_{k}(t) \\
B u_{\varepsilon^{(k)}}(t) & =g_{\varepsilon^{(k)}}(t)+m_{k}^{(1)}(t), \\
u_{\varepsilon^{(k)}}(0) & =u_{\varepsilon^{(k)}}^{0} \tag{33}
\end{align*}
$$

3. for $|\alpha|>1$, we solve

$$
\begin{gather*}
\dot{u}_{\alpha}(t)+K u_{\alpha}(t)+B^{*} \lambda_{\alpha}(t)=f_{\alpha}(t) \\
B u_{\alpha}(t)=g_{\alpha}(t) \\
u_{\alpha}(0)=u_{\alpha}^{0} \tag{34}
\end{gather*}
$$

We emphasize that the operator DAEs (32)-(34) can be solved in parallel again. However, system (33) is deterministic with a perturbation in the constraint, cf. Sect. 2.2
with $\theta=m_{k}^{(1)}$. The estimate (3) shows that this results in instabilities such that the stochastic truncation cannot converge. To see this, note that a computation as in the proof of Theorem 6 includes terms of the form $\left\|\dot{m}_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}$. Thus, the assumed boundedness

$$
\sum_{k=1}^{\infty}\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}(2 k)^{-p}<\infty
$$

is not sufficient to bound the terms which involve the derivatives of $m_{k}^{(1)}$. Consequently, we have to consider the regularized formulation.

### 3.5 Regularization

We have seen that the solution is very sensitive to perturbations in the constraint equation. As for the deterministic case in Sect. 2.3, we need a regularization. The extended (but equivalent) system to (30) with stochastic noise terms, has the form

$$
\begin{align*}
\dot{u}_{1}(t)+v_{2}(t)+\mathcal{K}\left(u_{1}(t)+u_{2}(t)\right)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+G_{t}  \tag{35a}\\
\mathcal{B} u_{2}(t) & =\mathcal{G}(t)+G_{t}^{(1)}  \tag{35b}\\
\mathcal{B} v_{2}(t) & =\dot{\mathcal{G}}(t)+G_{t}^{(2)} \tag{35c}
\end{align*}
$$

Note that, because of the extension of the system, we consider another perturbation $G_{t}^{(2)}$ in (35), represented in the form

$$
\begin{equation*}
G_{t}^{(2)}(\omega)=\sum_{k=1}^{\infty} m_{k}^{(2)}(t) H_{\varepsilon^{(k)}}(\omega) \tag{36}
\end{equation*}
$$

The chaos expansion approach leads again to a system of deterministic operator DAEs. Since the perturbations have zero mean and are of order one only, we only consider the case with $\alpha=\varepsilon^{(k)}$, which leads to

$$
\begin{align*}
\dot{u}_{1, \varepsilon^{(k)}}(t)+v_{2, \varepsilon^{(k)}}(t)+K\left(u_{1, \varepsilon^{(k)}}+u_{2, \varepsilon^{(k)}}\right)(t)+B^{*} \lambda_{\varepsilon^{(k)}}(t) & =f_{\varepsilon^{(k)}}(t)+m_{k}(t) \\
B u_{2, \varepsilon^{(k)}}(t) & =g_{\varepsilon^{(k)}}(t)+m_{k}^{(1)}(t), \\
B v_{2, \varepsilon^{(k)}}(t) & =\dot{g}_{\varepsilon^{(k)}}(t)+m_{k}^{(2)}(t) \tag{37}
\end{align*}
$$

Note here that the obtained system (37) corresponds to the perturbed extended system with perturbations in the constraint equation, i.e., it corresponds to the system

$$
\begin{align*}
\dot{u}_{1, \varepsilon^{(k)}}(t)+v_{2, \varepsilon^{(k)}}(t)+K\left(u_{1, \varepsilon^{(k)}}+u_{2, \varepsilon^{(k)}}\right)(t)+B^{*} \lambda_{\varepsilon^{(k)}}(t) & =f_{\varepsilon^{(k)}}(t)+m_{k}(t) \\
B u_{2, \varepsilon^{(k)}}(t) & =g_{\varepsilon^{(k)}}(t) \\
B v_{2, \varepsilon^{(k)}}(t) & =\dot{g}_{\varepsilon^{(k)}}(t) \tag{38}
\end{align*}
$$

that is equivalent to (24). Therefore, the stochastic operator DAE (30) can be treated as perturbed stochastic operator DAE (20) with the perturbation appearing in its constraint equation.

Recall that the formulation (37) allows an estimate of the coefficients $u_{1, \varepsilon^{(k)}}$ without the derivatives of the perturbations, cf. estimate (5). This then leads to a uniform bound of the solution $u_{1}, u_{2}$, similarly as in Theorem 6 . Furthermore, the regularization solves the problem of finding consistent initial data. Here, the condition reads $u_{1, \varepsilon^{(k)}}(0)=$ $u_{1, \varepsilon^{(k)}}^{0}$ and thus, does not depend on the perturbations. Finally, Theorem 9 summarizes the discussion. Therein we use the following notation. We denote by $\left(u_{1}, u_{2}, v_{2}, \lambda_{2}\right)$ the solution of

$$
\begin{align*}
\dot{u}_{1}(t)+v_{2}(t)+\mathcal{K}\left(u_{1}(t)+u_{2}(t)\right)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+G_{t}  \tag{39a}\\
\mathcal{B} u_{2}(t) & =\mathcal{G}(t)  \tag{39b}\\
\mathcal{B} v_{2}(t) & =\dot{\mathcal{G}}(t) \tag{39c}
\end{align*}
$$

and by $\left(\hat{u}_{1}, \hat{u}_{2}, \hat{v}_{2}, \hat{\lambda}_{2}\right)$ the solution of its perturbed operator DAE (35), while by $\mathbf{e}_{1}$ we denote the error in $u_{1}$, i.e. $\mathbf{e}_{1}=\hat{u}_{1}-u_{1}$.

Theorem 9 Let the Assumption 2 hold. Consider the perturbations $G_{t}^{(1)}$ and $G_{t}^{(2)}$ of the right hand sides of the operator DAE (39) that are of the forms (31) and (36), with the coefficients $m_{k}^{(1)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$ and $m_{k}^{(2)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$ such that

$$
\begin{equation*}
\sum_{k=1}^{\infty}\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}(2 k)^{-p}<\infty \text { and } \sum_{k=1}^{\infty}\left\|m_{k}^{(2)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}(2 k)^{-p}<\infty \tag{40}
\end{equation*}
$$

for some $p \in \mathbb{N}_{0}$. Then, the error $\mathbf{e}_{1}$ satisfies the following estimate

$$
\begin{align*}
& \left\|\mathbf{e}_{1}\right\|_{C(T ; \mathcal{H}) \otimes(S)_{-1,-p}}^{2}+\left\|\mathbf{e}_{1}\right\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1,-p}}^{2} \\
& \lesssim \sum_{k \in \mathbb{N}}\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}(2 k)^{-p}+\sum_{k \in \mathbb{N}}\left\|m_{k}^{(2)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}(2 k)^{-p}<\infty . \tag{41}
\end{align*}
$$

Proof After applying the chaos expansion method to the extended operator DAE (39) we obtain the system of deterministic problems, i.e. for $|\alpha|=1$ we obtain (38), while for all $|\alpha| \neq 1$ we obtain

$$
\begin{align*}
\dot{u}_{1, \alpha}(t)+v_{2, \alpha}(t)+K\left(u_{1, \alpha}(t)+u_{2, \alpha}(t)\right)+B^{*} \lambda_{\alpha}(t) & =f_{\alpha}(t) \\
B u_{2, \alpha}(t) & =g_{\alpha}(t) \\
B v_{2, \alpha}(t) & =\dot{g}_{\alpha}(t) \tag{42}
\end{align*}
$$

On the other hand, by applying the chaos expansion method to the extended operator DAE (35) we obtain the deterministic systems, i.e. for $|\alpha|=1$ we obtain (37) and for $|\alpha| \neq 1$ the system (42).

The difference between the systems (38)-(42) of the original problem (39) and the system (37)-(42) of the perturbed problem (35) is seen only for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$. Thus, only nonzero coefficients of the error $\mathbf{e}_{1}$ are obtained for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$, i.e., $\mathbf{e}_{1, \varepsilon^{(k)}}=\hat{u}_{1, \varepsilon^{(k)}}-u_{1, \varepsilon^{(k)}}$ and $\mathbf{e}_{1, \alpha}=\hat{u}_{1, \alpha}-u_{1, \alpha}=0$, for $|\alpha| \neq 1$. Similarly, the remaining nonzero errors are $\mathbf{e}_{2, \varepsilon^{(k)}}=\hat{u}_{2, \varepsilon^{(k)}}-u_{2, \varepsilon^{(k)}}, \mathbf{e}_{v, \varepsilon^{(k)}}=\hat{v}_{\varepsilon^{(k)}}-v_{\varepsilon^{(k)}}$ and $\mathbf{e}_{\lambda, \varepsilon^{(k)}}=\hat{\lambda}_{\varepsilon^{(k)}}-\lambda_{\varepsilon^{(k)}}$. Moreover, the notation of Theorem 2, we have $\delta_{\varepsilon^{(k)}}(t)=0$, $\theta_{\varepsilon^{(k)}}=m_{k}^{(1)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$ and $\xi_{\varepsilon^{(k)}}(t)=m_{k}^{(2)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right), k \in \mathbb{N}$ and $\mathbf{e}_{1,0}=0$. Therefore, we apply Theorem 2 and obtain the estimates

$$
\begin{equation*}
\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{C(T ; \mathcal{H})}^{2}+\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{L^{2}(T ; \mathcal{V})}^{2} \lesssim\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2}+\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}^{2} \tag{43}
\end{equation*}
$$

for all $k \in \mathbb{N}$. Since it holds

$$
\begin{aligned}
\left\|\mathbf{e}_{1}\right\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1}}^{2} & =\sum_{k \in \mathbb{N}}\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{L^{2}(\mathcal{V})}^{2}(2 k)^{-p}+\sum_{|\alpha| \neq 1}\left\|\mathbf{e}_{1, \alpha}\right\|_{L^{2}(\mathcal{V})}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{k \in \mathbb{N}}\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{L^{2}(\mathcal{V})}^{2}(2 k)^{-p}
\end{aligned}
$$

and similarly

$$
\left\|\mathbf{e}_{1}\right\|_{C(T ; \mathcal{H}) \otimes(S)_{-1}}^{2}=\sum_{k \in \mathbb{N}}\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{C(T ; \mathcal{H})}^{2}(2 k)^{-p}
$$

also holds, then by (43) we obtain

$$
\begin{aligned}
& \left\|\mathbf{e}_{1}\right\|_{C(T ; \mathcal{H}) \otimes(S)_{-1,-p}}^{2}+\left\|\mathbf{e}_{1}\right\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1,-p}}^{2} \\
& \quad=\sum_{k=1}^{\infty}\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{C(T ; \mathcal{H})}^{2}(2 k)^{-p}+\sum_{k=1}^{\infty}\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{L^{2}(\mathcal{V})}^{2}(2 k)^{-p} \\
& \quad=\sum_{k=1}^{\infty}\left(\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{C(T ; \mathcal{H})}^{2}+\left\|\mathbf{e}_{1, \varepsilon^{(k)}}\right\|_{L^{2}(\mathcal{V})}^{2}\right)(2 k)^{-p} \\
& \quad \lesssim \sum_{k \in \mathbb{N}}\left(\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}+\left\|m_{k}^{(2)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}\right)(2 k)^{-p}<\infty
\end{aligned}
$$

and the estimate (41) follows.

### 3.6 Convergence of the truncated expansion

In practice, only the coefficients $u_{\alpha}, \lambda_{\alpha}$ for multi-indices of a maximal length $P$, i.e., up to a certain order $P$, can be computed. Thus, the infinite sum has to be truncated such
that a given tolerance is achieved. Clearly, denoting by $\tilde{u}$ the approximated (truncated) solution and $u_{r}$ the truncation error, i.e.,

$$
\tilde{u}=\sum_{|\alpha| \leq P} u_{\alpha} \otimes H_{\alpha} \quad \text { and } \quad u_{r}=\sum_{|\alpha|>P} u_{\alpha} \otimes H_{\alpha},
$$

we can represent the process as $u=\tilde{u}+u_{r}$. In applications, one computes $u_{\alpha}$ for $|\alpha|<P$ such that the desired bound $\left\|u_{r}\right\|_{\mathcal{V} \otimes L^{2}(\Omega)}=\|u-\tilde{u}\|_{\mathcal{V} \otimes L^{2}(\Omega)} \leq \epsilon$ is carried out. Convergence in $L^{2}$ is attained if the sum is truncated properly [24,33,46]. The truncation procedure relies on the regularity of the solution, the type of noise, and the discretization method for solving the deterministic equations involved, see e.g. [8] for finite element methods. Numerical treatment of elliptic PDEs perturbed by Gaussian noise with error estimate in appropriate weighted space of stochastic processes is presented in [45].

```
Algorithm 3.1 Main steps of the numerical approximation
    Find finite dimensional approximations of the infinite dimensional Gaussian processes.
    Choose a finite set of polynomials \(H_{\alpha}\) and truncate the random series.
    Regularize the operator DAEs if necessary.
    Compute/approximate the solutions of the resulting systems.
    Generate \(H_{\alpha}\) to compute the approximate solution.
    Compute the approximate statistics of the solution from the obtained coefficients.
```

Similar results for specific equations can be found, e.g., in [1,7,9]. A general truncation method is stated in [24]. The same ideas can be applied to our equations once we have performed the regularization to the deterministic system (such that operator DAE is well-posed in each level), the convergence of the truncated expansion is, in general, guaranteed by the stability result of Theorem 6 . The main steps of the numerical approach are sketched in Algorithm 3.1.

## 4 More general cases

This section is devoted to the discussion of two generalizations. First, we consider general coordinatewise operators instead of simple coordinatewise operators as in the previous section. Thus, following the definition from Sect.3.1.4, we allow the operators $\mathcal{K}$ and $\mathcal{B}$ to be composed out from families of deterministic operators $\left\{K_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ and $\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, respectively, which may not be the same for all multi-indices. Second, we replace the Gaussian noise term by a stochastic integral term. The mean dynamics will remain unchanged, while the perturbation in the differential equation will be given in the form of a stochastic convolution.

Throughout this section we keep the following assumptions.
Assumption 3 1. The operator $\mathcal{K}$ is a coordinatewise operator that corresponds to a family $\left\{K_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ of deterministic operators $K_{\alpha}: \mathcal{V} \rightarrow \mathcal{V}^{*}, \alpha \in \mathcal{I}$. The operators $K_{\alpha}, \alpha \in \mathcal{I}$, are linear, continuous, and positive on the kernel of $B$.
2. The constraint operator $\mathcal{B}$ is a coordinatewise operator that corresponds to $a$ family $\left\{B_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ of deterministic operators $B_{\alpha}: \mathcal{V} \rightarrow \mathcal{Q}^{*}, \alpha \in \mathcal{I}$. The opertors $B_{\alpha}$ are linear and for every $\alpha \in \mathcal{I}$ there exists a right-inverse which is denoted by $B_{\alpha}^{-}$.
3. The operators $K_{\alpha}$ and $B_{\alpha}$ are uniformly bounded.
4. The stochastic processes $\mathcal{F}$ and $\mathcal{G}$ are given in their chaos expansion forms (16) such the conditions (17) hold.

### 4.1 Coordinatewise operators

In the given application, we consider the coordinatewise operators $\mathcal{K}, \mathcal{B}$ with

$$
\mathcal{K} u=\sum_{\alpha \in \mathcal{I}} K_{\alpha} u_{\alpha} \otimes H_{\alpha} \quad \text { and } \quad \mathcal{B} u=\sum_{\alpha \in \mathcal{I}} B_{\alpha} u_{\alpha} \otimes H_{\alpha}
$$

such that Assumption 3 holds and the processes $G_{t}^{(1)}$ and $G_{t}^{(2)}$ are of the forms (31) and (36). This also implies that $\mathcal{B}^{*}$ is a coordinatewise operator, which corresponds to the family of operators $\left\{B_{\alpha}^{*}\right\}_{\alpha \in \mathcal{I}}$ such that for $\lambda=\sum_{\alpha \in \mathcal{I}} \lambda_{\alpha} H_{\alpha}$ it holds that

$$
\mathcal{B}^{*} \lambda=\sum_{\alpha \in \mathcal{I}} B_{\alpha}^{*} \lambda_{\alpha} \otimes H_{\alpha}
$$

The chaos expansion method applied to the system with the Gaussian noise in the constraint equation (30) then leads to the following deterministic systems:

1. for $|\alpha|=0$, i.e., for $\alpha=\mathbf{0}$,

$$
\begin{aligned}
\dot{u}_{\mathbf{0}}(t)+K_{\mathbf{0}} u_{\mathbf{0}}(t)+B_{\mathbf{0}}^{*} \lambda_{\mathbf{0}}(t) & =f_{\mathbf{0}}(t) \\
B_{\mathbf{0}} u_{\mathbf{0}}(t) & =g_{\mathbf{0}}(t), u_{\mathbf{0}}=u_{\mathbf{0}}^{0}
\end{aligned}
$$

2. for $|\alpha|=1$, i.e., for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$,

$$
\begin{aligned}
\dot{u}_{\varepsilon^{(k)}}(t)+K_{\varepsilon^{(k)}} u_{\varepsilon^{(k)}}(t)+B_{\varepsilon^{(k)}}^{*} \lambda_{\varepsilon^{(k)}}(t) & =f_{\varepsilon^{(k)}}(t)+m_{k}^{(1)}(t) \\
B_{\varepsilon^{(k)}} u_{\varepsilon^{(k)}}(t) & =g_{\varepsilon^{(k)}}(t)+m_{k}^{(2)}(t)
\end{aligned}
$$

with $u_{\varepsilon^{(k)}}(0)=u_{\varepsilon^{(k)}}^{0}$.
3. for the remaining $|\alpha|>1$,

$$
\begin{aligned}
\dot{u}_{\alpha}(t)+K_{\alpha} u_{\alpha}(t)+B_{\alpha}^{*} \lambda_{\alpha}(t) & =f_{\alpha}(t) \\
B_{\alpha} u_{\alpha}(t) & =g_{\alpha}(t), u_{\alpha}(0)=u_{\alpha}^{0}
\end{aligned}
$$

As before, these systems may be solved in parallel. Furthermore, since the constraint equation includes again a perturbation, a regularization as in Sect. 3.5 is necessary. We omit further details here.

### 4.2 Stochastic convolution

Consider the problem (7), where the stochastic disturbance is given in terms of a stochastic convolution term. More precisely, we are dealing with the problem of the form

$$
\begin{align*}
\dot{u}(t)+\mathcal{K} u(t)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+\delta(\mathcal{C} u), \\
\mathcal{B} u(t) & =\mathcal{G}(t)+G_{t}^{(1)} \tag{44}
\end{align*}
$$

with a consistent initial condition $u(0)=u^{0}$. We assume that Assumption 3 holds for operators $\mathcal{K}$ and $\mathcal{B}$ and processes $\mathcal{F}$ and $\mathcal{G}$. Additionally, we assume that $G_{t}^{(1)}$ is a Gaussian noise as in (12). The term $\delta(\mathcal{C u})$ stays for an Itô-Skorokhod stochastic integral. The Skorokhod integral is a generalization of the Itô integral for processes which are not necessarily adapted. The fundamental theorem of stochastic calculus connects the Itô-Skorokhod integral with the Wick product by

$$
\begin{equation*}
\delta(\mathcal{C} u)=\int_{\mathbb{R}} \mathcal{C} u \mathrm{~d} B_{t}=\int_{\mathbb{R}} \mathcal{C} u \diamond W_{t} \mathrm{~d} t, \tag{45}
\end{equation*}
$$

where the integral on the right-hand side of the relation is the Riemann integral and the derivative is taken in sense of distributions [22]. We assume that the operator $\mathcal{C}$ is a linear coordinatewise operator composed of a family of uniformly bounded operators $\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ such that $\mathcal{C} u$ is integrable in the Skorokhod sense [22]. The stochastic integral is the Itô-Skorokhod integral and it exists not only for processes adapted to the filtration but also for non-adapted ones. It is equal to the Riemann integral of a process $\mathcal{C} u$, stochastically convoluted with a singular white noise.

The operator $\delta$ is the adjoint operator of the Malliavin derivative $\mathbb{D}$. Their composition is known as the Ornstein-Uhlenbeck operator $\mathcal{R}$ which is a self-adjoint operator. These operators are the main operators of an infinite dimensional stochastic calculus of variations called the Malliavin calculus [37]. We consider these operators in Sect. 5.

For adapted processes $v$ the Itô integral and the Skorokhod integral coincide, i.e., $I(v)=\delta(v)$. Because of this fact, we refer to the stochastic integral as the ItôSkorokhod integral. Applying the definition of the Wick product (14) to the chaos expansion representation (9) of a process $v$ and the representation (11) of a singular white noise in the definition (45) of $\delta(v)$, we obtain a chaos expansion representation of the Skorokhod integral. Clearly, for $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha}(t) H_{\alpha}$ we have

$$
v \diamond W_{t}=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha}(t) \xi_{k}(t) H_{\alpha+\varepsilon^{(k)}}(\omega),
$$

and thus, it holds that

$$
\begin{equation*}
\delta(v)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k} H_{\alpha+\varepsilon^{(k)}}(\omega) . \tag{46}
\end{equation*}
$$

Therein, we have used that $v_{\alpha}(t)=\sum_{k \in \mathbb{N}} v_{\alpha, k} \xi_{k}(t) \in L^{2}(\mathbb{R})$ is the chaos expansion representation of $v_{\alpha}$ in the orthonormal Hermite functions basis with coefficients $v_{\alpha, k} \in \mathbb{R}$. Therefore, we are able to represent stochastic perturbations appearing in the stochastic equation (44) explicitly. Note that $\delta(v)$ belongs to the Wiener chaos space of higher order than $v$, see also $[22,28]$.

Definition 2 We say that a $L^{2}(\mathbb{R})$-valued stochastic process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha}$, with coefficients $v_{\alpha}(t)=\sum_{k \in \mathbb{N}} v_{\alpha, k} \xi_{k}(t), v_{\alpha, k} \in \mathbb{R}$, for all $\alpha \in \mathcal{I}$ is integrable in the Itô-Skorokhod sense if it holds that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}|\alpha| \alpha!<\infty \tag{47}
\end{equation*}
$$

Then, the Itô-Skorokhod integral of $v$ is of the form (46) and we write $v \in \operatorname{Dom}(\delta)$.
Theorem 10 The Skorokhod integral $\delta$ of an $L^{2}(\mathbb{R})$-valued stochastic process is a linear and continuous mapping

$$
\delta: \operatorname{Dom}(\delta) \rightarrow L^{2}(\Omega)
$$

Proof Let $v$ satisfy condition (47). Then we have

$$
\begin{aligned}
\|\delta(v)\|_{L^{2}(\Omega)}^{2} & =\left\|\sum_{|\beta|>0} \sum_{k \in \mathbb{N}} v_{\beta-\varepsilon^{(k)}, k} H_{\beta}\right\|_{L^{2}(\Omega)}^{2}=\sum_{|\beta|>0}\left(\sum_{k \in \mathbb{N}} v_{\beta-\varepsilon^{(k)}, k}\right)^{2} \beta! \\
& =\sum_{\alpha \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}} v_{\alpha, k} \sqrt{\alpha_{k}+1}\right)^{2} \alpha!\leq c \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha, k}^{2}|\alpha| \alpha!<\infty
\end{aligned}
$$

where we used $\beta!=\left(\alpha+\varepsilon^{(k)}\right)!=\left(\alpha_{k}+1\right) \alpha!$, for $\alpha \in \mathcal{I}, k \in \mathbb{N}$.
A detailed analysis of the domain and the range of operators of the Malliavin calculus in spaces of stochastic distributions can be found in [28,29,31].

First, we solve the stochastic operator DAE (44) with the stochastic perturbations given in terms of a stochastic convolution and without disturbance in the constraint equation. In order to prove the convergence of obtained solution in the Kondratiev space of generalized processes it is necessary to assume uniform boundness of the family of operators $C_{\alpha}, \alpha \in \mathcal{I}$. Then, we consider the stochastic operator DAE (44) with perturbation in the constraint equation that is given by a Gaussian noise term.

Theorem 11 Let Assumption 3 hold for the operators $\mathcal{K}$ and $\mathcal{B}$ and stochastic processes $\mathcal{F}$ and $\mathcal{G}$ and let $\mathcal{C}$ be a coordinatewise operator that corresponds to a family of deterministic operators $\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}}, C_{\alpha}: \mathcal{V} \rightarrow \mathcal{V}^{*}$ for $\alpha \in \mathcal{I}$ that satisfy

$$
\begin{equation*}
\left\|C_{\alpha}\right\| \leq d<1, \quad \text { for all } \alpha \in \mathcal{I} \tag{48}
\end{equation*}
$$

Then, for any consistent initial data that satisfies (21) there exists a unique solution $u \in L^{2}(T ; \mathcal{V}) \otimes(S)_{-1}$ of the stochastic operator DAE

$$
\begin{align*}
\dot{u}(t)+\mathcal{K} u(t)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+\delta(\mathcal{C} u)  \tag{49a}\\
\mathcal{B} u(t) & =\mathcal{G}(t) \tag{49b}
\end{align*}
$$

Proof We are looking for the solution in the chaos expansion form (9). For this, we apply the polynomial chaos expansion method to problem (49) and obtain the following systems of deterministic operator DAEs:
$1^{\circ}$ for $|\alpha|=0$, i.e., for $\alpha=\mathbf{0}$,

$$
\begin{align*}
\dot{u}_{\mathbf{0}}(t)+K_{\mathbf{0}} u_{\mathbf{0}}(t)+B_{\mathbf{0}}^{*} \lambda_{\mathbf{0}}(t) & =f_{\mathbf{0}}(t) \\
B_{\mathbf{0}} u_{\mathbf{0}}(t) & =g_{\mathbf{0}}(t) \tag{50}
\end{align*}
$$

$2^{\circ}$ for $|\alpha|=1$, i.e., for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$,

$$
\begin{align*}
\dot{u}_{\varepsilon^{(k)}}(t)+K_{\varepsilon^{(k)}} u_{\varepsilon^{(k)}}(t)+B_{\varepsilon^{(k)}}^{*} \lambda_{\varepsilon^{(k)}}(t) & =f_{\varepsilon^{(k)}}+\left(C_{\mathbf{0}} u_{\mathbf{0}}\right)_{k},  \tag{51}\\
B_{\varepsilon^{(k)}} u_{\varepsilon^{(k)}}(t) & =g_{\varepsilon^{(k)}}(t)
\end{align*}
$$

$3^{\circ}$ for $|\alpha|>1$,

$$
\begin{align*}
\dot{u}_{\alpha}(t)+K_{\alpha} u_{\alpha}(t)+B_{\alpha}^{*} \lambda_{\alpha}(t) & =f_{\alpha}(t)+\sum_{k \in \mathbb{N}}\left(C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}}\right)_{k} \\
B_{\alpha} u_{\alpha}(t) & =g_{\alpha}(t) \tag{52}
\end{align*}
$$

Note that the corresponding initial conditions are given as in systems (32)-(34). The term $\left(C_{0} u_{0}\right)_{k}$ appearing in (51) represents the $k$ th component of the action of the operator $C_{0}$ on the solution $u_{0}$ obtained in the previous step, i.e., on the solution of the system (50). Similarly, the term $\left(C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}}\right)_{k}$ from (52) represents the $k$ th coefficient obtained by the action of the operator $C_{\alpha-\varepsilon^{(k)}}$ on $u_{\alpha-\varepsilon^{(k)}}$ calculated in the previous steps. We use the convention that $C_{\alpha-\varepsilon^{(k)}}$ exists only for those $\alpha \in \mathcal{I}$ for which $\alpha_{k} \geq 1$. Therefore, the $\operatorname{sum} \sum_{k \in \mathbb{N}}\left(C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}}\right)_{k}$ has as many summands as the multi-index $\alpha$ has non-zero components. For example, for $\alpha=(2,0,1,0,0, \ldots)$ with two non-zero components $\alpha_{1}=2$ and $\alpha_{3}=1$, the sum has two terms $\left(C_{(1,0,1,0,0, \ldots)} u_{(1,0,1,0,0, \ldots)}\right)_{1}$ and $\left(C_{(2,0,0,0,0, \ldots)} u_{(2,0,0,0,0, \ldots)}\right)_{3}$.

We point out that, in contrast to the previous cases, the unknown coefficients are obtained by recursion. Thus, in order to calculate $u_{\alpha}$, we need the solutions $u_{\beta}$ for $\beta<\alpha$ from the previous steps. Also this case can be found in applications, see for example [24, 29, 31, 33].

We apply the estimate (2) from Theorem 1 to the deterministic operator DAEs (50)(52) for the coefficients $u_{\alpha}$ in each step recursively and then prove the convergence of $u$ in $L^{2}(T ; \mathcal{V}) \otimes(S)_{-1}$. Particularly, we have to show that for some $p \in \mathbb{N}$ it holds

$$
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

For $|\alpha|=0$, from the system (50) and by (2) we estimate the coefficient $u_{\mathbf{0}}$, i.e.,

$$
\left\|u_{\mathbf{0}}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{B, \mathbf{0}}^{0}\right\|_{\mathcal{H}}^{2}+\left\|f_{\mathbf{0}}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\mathbf{0}}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2} .
$$

For $|\alpha|=1$, i.e., for $\alpha=\varepsilon^{(k)}, k \in \mathbb{N}$, by the system (51) we obtain the estimate

$$
\left\|u_{\varepsilon^{(k)}}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{B, \varepsilon^{(k)}}^{0}\right\|_{\mathcal{H}}^{2}+\left\|f_{\varepsilon^{(k)}}+\left(C_{\mathbf{0}} u_{\mathbf{0}}\right)_{k}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\varepsilon^{(k)}}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}, \quad k \in \mathbb{N},
$$

while for $|\alpha|>1$ from (52) we obtain

$$
\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2} \lesssim\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}+\left\|f_{\alpha}+\sum_{k \in \mathbb{N}}\left(C_{\alpha-\varepsilon^{(k)}} u_{\left.\alpha-\varepsilon^{(k)}\right)}\right)_{k}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}+\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}
$$

We sum up all the coefficients and apply the obtained estimates. Thus, we get

$$
\begin{align*}
\sum_{\alpha \in \mathcal{I}}\left\|u_{\alpha}\right\|_{L^{2}(\mathcal{V})}^{2}(2 \mathbb{N})^{-p \alpha} \lesssim & \sum_{\alpha \in \mathcal{I}}\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \alpha}+\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{\alpha \in \mathcal{I},|\alpha|>0}\left(\sum_{k \in \mathbb{N}}\left(C_{\alpha-\varepsilon^{(k)}} u_{\left.\alpha-\varepsilon^{(k)}\right)_{k}}\right)^{2}(2 \mathbb{N})^{-p \alpha}\right. \tag{53}
\end{align*}
$$

From the assumptions (17) and (22) it follows that the first three summands on the right hand side of (53) are finite. The last term can be estimated in the following way

$$
\begin{aligned}
& \sum_{|\alpha|>0}\left(\sum _ { k \in \mathbb { N } } \left(C_{\alpha-\varepsilon(k)} u_{\left.\left.\alpha-\varepsilon^{(k)}\right)_{k}\right)^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\beta \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left(C_{\beta} u_{\beta}\right)_{k}(2 k)^{-\frac{p}{2}}\right)^{2}(2 \mathbb{N})^{-p \beta}} \quad \leq \sum_{\beta \in \mathcal{I}}\left(\sum_{k \in \mathbb{N}}\left(C_{\beta} u_{\beta}\right)_{k}^{2} \sum_{k \in \mathbb{N}}(2 k)^{-p}\right)(2 \mathbb{N})^{-p \beta} \leq M \sum_{\beta \in \mathcal{I}}\left\|C_{\beta} u_{\beta}\right\|^{2}(2 \mathbb{N})^{-p \beta}\right.\right. \\
& \quad \leq M d \sum_{\beta \in \mathcal{I}}\left\|u_{\beta}\right\|^{2}(2 \mathbb{N})^{-p \beta}=M d\|u\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1,-p}^{2}}
\end{aligned}
$$

Therein, we have first used the substitution $\alpha=\beta+\varepsilon^{(k)}$ and the property

$$
(2 \mathbb{N})^{\beta+\varepsilon^{(k)}}=(2 \mathbb{N})^{\beta} \cdot(2 \mathbb{N})^{\varepsilon^{(k)}}=(2 \mathbb{N})^{\beta} \cdot(2 k),
$$

then the Cauchy-Schwartz inequality, the uniformly boundness of the family $\left\{C_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ from (48), and at last the sum $M=\sum_{k \in \mathbb{N}}(2 k)^{-p}<\infty$ for $p>1$. Finally, putting
everything together in (53), we obtain

$$
\begin{aligned}
\|u\|_{L^{2}(\mathcal{V}) \otimes(S)-1}^{2} \leq & c\left(\sum_{\alpha \in \mathcal{I}}\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \alpha}+\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}\right. \\
& \left.+\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha}\right)+M d\|u\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1}}^{2}
\end{aligned}
$$

We group the two summands with the term $\|u\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1}}^{2}$ on the left hand side of the inequality and obtain

$$
\begin{aligned}
\|u\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1}}^{2}(1-M d) \lesssim & \sum_{\alpha \in \mathcal{I}}\left\|u_{B, \alpha}^{0}\right\|_{\mathcal{H}}^{2}(2 \mathbb{N})^{-p \alpha}+\sum_{\alpha \in \mathcal{I}}\left\|f_{\alpha}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha} \\
& +\sum_{k=1}^{\infty}\left\|m_{k}\right\|_{L^{2}\left(\mathcal{V}^{*}\right)}^{2}(2 k)^{-p}+\sum_{\alpha \in \mathcal{I}}\left\|g_{\alpha}\right\|_{H^{1}\left(\mathcal{Q}^{*}\right)}^{2}(2 \mathbb{N})^{-p \alpha} .
\end{aligned}
$$

Since (48) holds, one can choose $p$ large enough so that $1-M d>0$. With this, we have proven that the solution $u$ of (49) the norm $\|u\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1}}^{2}$ is finite and thus, complete the proof of theorem norm.

Let us now consider briefly the stochastic operator DAE (44). This problem corresponds to the stochastic operator DAE (49) with additional disturbance in the constraint equation. Similar to Theorem 9, the regularization is needed and will be provided only for the coefficients $u_{\alpha}$, when $|\alpha|=1$. Thus one can obtain the error estimate of the solution of the initial problem (49) and the perturbed one (44), i.e. of the solutions of their corresponding problems in extended forms. Here we state the theorem, but omit the proof.

Theorem 12 Let the assumptions of Theorem 11 hold. Let $\left(u_{1}, u_{2}, v_{2}, \lambda_{2}\right)$ be the solution of operator DAE

$$
\begin{aligned}
\dot{u}_{1}(t)+v_{2}(t)+\mathcal{K}\left(u_{1}(t)+u_{2}(t)\right)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+\delta(C u) \\
\mathcal{B} u_{2}(t) & =\mathcal{G}(t) \\
\mathcal{B} v_{2}(t) & =\dot{\mathcal{G}}(t)
\end{aligned}
$$

and ( $\hat{u}_{1}, \hat{u}_{2}, \hat{v}_{2}, \hat{\lambda}_{2}$ ) the solution of the corresponding perturbed operator DAE

$$
\begin{aligned}
\dot{u}_{1}(t)+v_{2}(t)+\mathcal{K}\left(u_{1}(t)+u_{2}(t)\right)+\mathcal{B}^{*} \lambda(t) & =\mathcal{F}(t)+\delta(C u) \\
\mathcal{B} u_{2}(t) & =\mathcal{G}(t)+G_{t}^{(1)} \\
\mathcal{B} v_{2}(t) & =\dot{\mathcal{G}}(t)+G_{t}^{(1)},
\end{aligned}
$$

where the perturbations $G_{t}^{(1)}$ and $G_{t}^{(2)}$ are considered to be of the forms (31) and (36), with the coefficients $m_{k}^{(1)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$ and $m_{k}^{(2)} \in L^{2}\left(T ; \mathcal{Q}^{*}\right)$ such that (40) holds
for some $p \in \mathbb{N}_{0}$. . Then, the error $\mathbf{e}_{1}=\hat{u}_{1}-u_{1}$ satisfies the following estimate

$$
\begin{aligned}
& \left\|\mathbf{e}_{1}\right\|_{C(T ; \mathcal{H}) \otimes(S)_{-1,-p}}^{2}+\left\|\mathbf{e}_{1}\right\|_{L^{2}(\mathcal{V}) \otimes(S)_{-1,-p}}^{2} \\
& \lesssim \sum_{k \in \mathbb{N}}\left\|m_{k}^{(1)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}(2 k)^{-p}+\sum_{k \in \mathbb{N}}\left\|m_{k}^{(2)}\right\|_{L^{2}\left(\mathcal{Q}^{*}\right)}(2 k)^{-p}<\infty \text {. }
\end{aligned}
$$

With this result, we close this section and consider a further generalization, namely the fully stochastic case.

## 5 An example involving operators of Malliavin calculus

We present an example involving operators of Malliavin calculus which has the same structure as the deterministic operator DAE (1). Although this example does not arise in fluid dynamics it is related with the extension of our results to nonlinear equations in particular Navier-Stokes equation. Thus, we consider a semi-explicit systems including the stochastic operators from the Malliavin calculus and use their duality relations. Denote by $\mathbb{D}$ and $\delta$ the Malliavin derivative operator and the Itô-Skorokhod integral, respectively. As mentioned above, the Itô-Skorokhod integral is the adjoint operator of the Malliavin derivative, i.e., the duality relationship

$$
\mathbb{E}(F \cdot \delta(u))=\mathbb{E}(\langle\mathbb{D} F, u\rangle),
$$

holds for stochastic functions $u$ and $F$ belonging to appropriate spaces [37].
Assume that the stochastic operator $\mathcal{K}$ is a coordinatewise operator such that the corresponding deterministic operators $\left\{K_{\alpha}\right\}_{\alpha \in \mathcal{I}}$ are densely defined on a given Banach space $X$. Taking in (1) the operators $\mathcal{B}=\mathbb{D}$ and thus $\mathcal{B}^{*}=\delta$, we can consider the stochastic operator DAE of the form

$$
\begin{gather*}
\dot{u}+\mathcal{K} u+\delta \lambda=v \\
\mathbb{D} u=y \tag{54}
\end{gather*}
$$

such that the initial condition $u(0)=u^{0}$ holds and given stochastic processes $v$ and $y$.

The results concerning the generalized Malliavin calculus and the equations involving these operators can be found in [28,29,31,32]. The chaos expansion method combined with the regularization techniques presented in the previous sections can be applied also in this case. Here we present the direct chaos expansion approach and prove the convergence of the obtained solution.

In the generalized $S^{\prime}(\mathbb{R})$ setting, the operators of the Malliavin calculus are defined as follows:

1. The Malliavin derivative, namely $\mathbb{D}$, as a stochastic gradient in the direction of white noise, is a linear and continuous mapping $\mathbb{D}: \operatorname{Dom}(\mathbb{D}) \subseteq X \otimes(S)_{-1} \rightarrow$
$X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ given by

$$
\begin{equation*}
\mathbb{D} u=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon_{k}} \tag{55}
\end{equation*}
$$

for $u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}, u_{\alpha} \in X, \alpha \in \mathcal{I}$. We say that a process $u$ is differentiable in Malliavin sence, i.e., it belongs to the domain $\operatorname{Dom}(\mathbb{D})$ if and only if for some $p \in \mathbb{N}_{0}$ it holds that

$$
\sum_{\alpha \in \mathcal{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

The operator $\mathbb{D}$ reduces the order of the Wiener chaos space and it holds that the kernel $\operatorname{Ker}(\mathbb{D})$ consists of constant random variables, i.e., random variables having the chaos expansion in the Wiener chaos space of order zero. In terms of quantum theory, this operator corresponds to the annihilation operator.
2. The Itô-Skorokhod integral, namely $\delta$, is a linear and continuous mapping $\delta: X \otimes$ $S^{\prime}(\mathbb{R}) \otimes(S)_{-1} \rightarrow X \otimes(S)_{-1}$ given by

$$
\delta(F)=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_{\alpha} \otimes v_{\alpha, k} \otimes H_{\alpha+\varepsilon_{k}}, \text { for } F=\sum_{\alpha \in \mathcal{I}} f_{\alpha} \otimes\left(\sum_{k \in \mathbb{N}} v_{\alpha, k} \xi_{k}\right) \otimes H_{\alpha}
$$

Note that the domain $\operatorname{Dom}(\delta)=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. The operator $\delta$ is the adjoint operator of the Malliavin derivative. It increases the order of the Wiener chaos space and in terms of quantum theory $\delta$ corresponds to the creation operator.
3. The Ornstein-Uhlenbeck operator, namely $\mathcal{R}$, as the composition $\delta \circ \mathbb{D}$, is the stochastic analogue of the Laplacian. It is a linear and continuous mapping $\mathcal{R}: X \otimes$ $(S)_{-1} \rightarrow X \otimes(S)_{-1}$ given by

$$
\mathcal{R}(u)=\sum_{\alpha \in \mathcal{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} \quad \text { for } u=\sum_{\alpha \in \mathcal{I}} u_{\alpha} \otimes H_{\alpha}
$$

Clearly, $\mathcal{R}$ is a coordinatewise operator and its domain $\operatorname{Dom}(\mathcal{R})$ coincides with the domain $\operatorname{Dom}(\mathbb{D})$. In terms of quantum theory, the operator $\mathcal{R}$ corresponds to the number operator. It is a self-adjoint operator with eigenvectors equal to the basis elements $H_{\alpha}, \alpha \in \mathcal{I}$, i.e., $\mathcal{R}\left(H_{\alpha}\right)=|\alpha| H_{\alpha}, \alpha \in \mathcal{I}$. Therefore, Gaussian processes from the Wiener chaos space of order one with zero expectation are the only fixed points for the Ornstein-Uhlenbeck operator [28,31].

In this section we present the direct method of solving system (54), which relies on the results obtained in $[27,29,31]$. First, we solve the second equation with the initial condition in (54) and obtain the solution $u$ in the space of stochastic processes $X \otimes(S)_{-1}$. Then by subtracting the obtained solution $u$ in the first equation of (54) we solve an integral equation and obtain the explicit form of $\lambda$ in the space of generalized $S^{\prime}(\mathbb{R})$-stochastic processes.

Theorem 13 Let the operator $\mathcal{K}$ satisfy the assumptions $1^{\circ}$ and $3^{\circ}$ of Assumption 3. Let a process $y \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$ have a chaos representation $y=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} y_{\alpha, k} \otimes$ $\xi_{k} \otimes H_{\alpha}$ and a process $v \in X \otimes(S)_{-1}$ have a chaos representation $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} \otimes H_{\alpha}$ such that $\mathbb{E} v=K_{0} u^{0}$. Then the stochastic problem (54) with the initial condition $\mathbb{E} u=u^{0} \in X$ has a unique solution $u \in X \otimes(S)_{-1}$ and $\lambda \in X \otimes S(\mathbb{R}) \otimes(S)_{-1}$ given respectively by

$$
\begin{equation*}
u=u^{0}+\sum_{\alpha \in \mathcal{I},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} y_{\alpha-\varepsilon}(k), k \in H_{\alpha} \tag{56}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{v_{\alpha+\varepsilon^{(k)}}^{(1)}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha}, \tag{57}
\end{equation*}
$$

where $v^{(1)}=v-\dot{u}-\mathcal{K} u$.
Proof We search for the solution represented of the form (18). The initial value problem involving the Malliavin derivative operator

$$
\begin{equation*}
\mathbb{D} u=y, \quad \mathbb{E} u=u^{0} \in X \tag{58}
\end{equation*}
$$

can be solved by applying the integral operator on both sides of the equation. For a given process $y \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-q}, l \in \mathbb{N}_{0}, q>l+1$, represented in its chaos expansion form $y=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} y_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$, the equation (58) has a unique solution in $\operatorname{Dom}(\mathbb{D})$ given by (56), [27,29]. Clearly, it holds that

$$
\|u\|_{X \otimes(S)_{-1,-q}}^{2} \leq\left\|u^{0}\right\|_{X}^{2}+c\|y\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-1,-q}^{2}}^{2}<\infty .
$$

The operator $\mathcal{K}$ is a coordinatewise operator and corresponds to a uniformly bounded family of operators $\left\{K_{\alpha}\right\}_{\alpha \in \mathcal{I}}$, i.e., it holds that $\left\|K_{\alpha}\right\| \leq M, \alpha \in \mathcal{I}$. For $u \in X \otimes(S)_{-1} \bigcap \operatorname{Dom}(\mathbb{D})$ it holds that

$$
\|\mathcal{K} u\|_{X \otimes(S)_{-1,-q}}^{2}=\sum_{\alpha \in \mathcal{I}}\left\|K_{\alpha} u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \leq M\|u\|_{X \otimes(S)_{-1,-q}}^{2}<\infty
$$

and thus we conclude that $\mathcal{K} u \in X \otimes(S)_{-1,-q}$. Since $y_{\alpha} \in X \otimes S_{-l}(\mathbb{R})$, we can apply the formula for derivatives of the Hermite functions [22]. Thus,

$$
\dot{y}_{\alpha}=\sum_{k \in \mathbb{N}} y_{\alpha, k} \otimes \frac{d}{d t} \xi_{k}=\sum_{k \in \mathbb{N}} y_{\alpha, k} \otimes\left(\sqrt{\frac{k}{2}} \xi_{k-1}-\sqrt{\frac{k+1}{2}} \xi_{k+1}\right)
$$

and it holds that $\dot{y}_{\alpha} \in X \otimes S_{-l-1}(\mathbb{R})$. We note that the problem $\mathbb{D} \dot{u}=\dot{y}$ with the initial condition $\mathbb{E} \dot{u}=u^{1} \in X$ also holds and it can be solved as equation (58). Hence, the following estimate holds

$$
\|\dot{u}\|_{X \otimes(S)_{-1,-q}}^{2} \leq\left\|u^{1}\right\|_{X}^{2}+c\|\dot{y}\|_{X \otimes S_{-l-1}(\mathbb{R}) \otimes(S)_{-1,-q}}^{2}<\infty .
$$

Let $v \in X \otimes(S)_{-1,-q}$ and denote by $v^{(1)}=v-\dot{u}-\mathcal{K} u$. From the given assumptions it follows $v^{(1)} \in X \otimes(S)_{-1,-q}$ and it has zero expectation. Let

$$
v_{1}=\sum_{\alpha \in \mathcal{I},|\alpha| \geq 1} v_{\alpha}^{(1)} \otimes H_{\alpha}, \quad v_{\alpha}^{(1)} \in X
$$

Then the integral equation

$$
\delta \lambda=v_{1}
$$

has a unique solution $\lambda$ in $X \otimes S_{-l-1}(\mathbb{R}) \otimes(S)_{-1,-q}$, for $l>q$, given in the form (57), see $[28,31]$. The estimate

$$
\|v\|_{X \otimes(S)_{-1,-q}}^{2} \leq c\left(\|u\|_{X \otimes(S)_{-1,-q}}^{2}+\|v\|_{X \otimes(S)_{-1,-q}}^{2}+\|\dot{u}\|_{X \otimes(S)_{-1,-q}}^{2}\right)
$$

also holds.
Theorem 14 Let $y=\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} y_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-1}$. The initial value problem (58) is equivalent to the system of two initial values problems

$$
\begin{equation*}
\mathbb{D} u_{1}=0, \quad \mathbb{E} u_{1}=u^{0} \in X \quad \text { and } \quad \mathbb{D} u_{2}=y, \quad \mathbb{E} u_{2}=0 \tag{59}
\end{equation*}
$$

where $u=u_{1}+u_{2}$.
Proof Let $u_{1}$ and $u_{2}$ be the solutions of the system (59). From the linearity of the operator $\mathbb{D}$ and the linearity of $\mathbb{E}$ it follows $\mathbb{D} u=\mathbb{D}\left(u_{1}+u_{2}\right)=\mathbb{D} u_{1}+\mathbb{D} u_{2}=y$ and $\mathbb{E} u=\mathbb{E}\left(u_{1}+u_{2}\right)=\mathbb{E} u_{1}+\mathbb{E} u_{2}=u^{0}$. Thus the superposition of $u_{1}$ and $u_{2}$ solves (58).

Let now $u$ be the solution of (58). By Theorem 13 it has chaos expansion representation form (56). The kernel of $\mathbb{D}$, i.e., $\operatorname{Ker}(\mathbb{D})$ is equal to $\mathcal{H}_{0}$ and therefore $u$ can be expressed in the form $u=u_{1}+u_{2}$, where $u_{1} \in \operatorname{Ker}(\mathbb{D})$ and $u_{2} \in \operatorname{Im}(\mathbb{D})$. Thus, by (56) we conclude that $\mathbb{D} u_{1}=0$ and $\mathbb{E} u_{1}=u^{0}$, while $\mathbb{D} u_{2}=y$ and $\mathbb{E} u_{2}=0$.

### 5.1 Extension to nonlinear equations

In [36] the authors show that a random polynomial nonlinearity can be expanded in a Taylor alike series involving Wick products and Malliavin derivatives. This result has been applied to the nonlinear advection term in the Navier-Stokes equations [44]. There a detailed study of the accuracy and computational efficiency of these Wicktype approximations is shown. We point out that following the same approach we can extend the ideas presented in this paper to Navier-Stokes equations. Specifically, by the product formula, of two square-integrable stochastic processes $u$ and $v$,

$$
u v=\sum_{i=0}^{P} \frac{\mathbb{D}^{(i)} u \diamond \mathbb{D}^{(i)} v}{i!}
$$

where $\diamond$ denotes the Wick product and $\mathbb{D}^{(i)}$ is the $i$ th order of the Malliavin derivative operator, one can construct approximations of finite stochastic order. Particularly, the nonlinear advection term in the Navier-Stokes equations can be approximated by

$$
\begin{equation*}
(u \cdot \nabla) u \simeq \sum_{i=0}^{Q} \frac{\left(\mathbb{D}^{(i)} \diamond \nabla\right) \mathbb{D}^{(i)} u}{i!} \tag{60}
\end{equation*}
$$

where $Q$ denotes the highest stochastic order in the Wick-Malliavin expansion. The zero-order approximation $(u \cdot \nabla) u \simeq(u \diamond \nabla) u$ is known as the Wick approximation, while $(u \cdot \nabla) u \simeq(u \diamond \nabla) u+(\mathbb{D} u \diamond \nabla) \mathbb{D} u$ is the first-order Wick-Malliavin approximation [44]. As the Malliavin derivate has an explicit chaos expansion representation form (55), the formula (60) allows us to express the nonlinear advection term in terms of chaos expansions. Therefore, the ideas presented in this paper for the linear semiexplicit stochastic operator DAEs can be extended to Navier-Stokes equations and in general to equations with nonlinearities of the type (60). Moreover, the multiplication formula

$$
v G=v \diamond G+\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) v_{\alpha+\varepsilon^{(k)}} g_{k} H_{\alpha}
$$

holds for a Gaussian process $G=g_{0}+\sum_{k \in \mathbb{N}} g_{k} H_{\varepsilon^{(k)}} \in X \otimes(S)_{-1}$ and a process $v=\sum_{\alpha \in \mathcal{I}} v_{\alpha} H_{\alpha} \in X \otimes(S)_{-1}[28,31]$. The equations involving higher orders of the Malliavin derivarive operator were solved in [31]. Thus, the results proved in this paper and the ones in $[36,44]$ can be generalized for this type of general processes (not necessary square integrable). We intent to investigate this in a future work.

## 6 Conclusion

We have analyzed the influence of stochastic perturbations to linear operator DAEs of semi-explicit structure. With the application of the polynomial chaos expansion, we could reduce the problem to a system of deterministic operator DAEs. Since the obtained system is very sensitive to perturbations in the constraint equation, we analyze a regularized version of the system. With this, we have proven the existence and uniqueness of a solution of the stochastic operator DAE in a weighted space of generalized stochastic processes. Examples analyzed in this paper are the Stokes equations and the linearized Navier-Stokes equations. Moreover, the results of this paper can be extended to a certain type of nonlinear equations including Navier-Stokes.

Acknowledgements Open access funding provided by University of Innsbruck and Medical University of Innsbruck. R. Altmann was supported by the ERC Advanced Grant "Modeling, Simulation and Control of Multi-Physics Systems" MODSIMCONMP. H. Mena was supported by the project Solution of large-scale Lyapunov Differential Equations (P 27926) founded by the Austrian Science Foundation. Moreover, the authors would like to thank the referees for their valuable suggestions which helped to improve this paper.

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# Equations Involving Malliavin Calculus Operators 

Applications and Numerical Approximation

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ISSN 2191-8198
SpringerBriefs in Mathematics
ISBN 978-3-319-65677-9
DOI 10.1007/978-3-319-65678-6

Library of Congress Control Number: 2017949124
Mathematics Subject Classification (2010): 60G20, 60G22, $65 J 10,60 \mathrm{H} 07,60 \mathrm{H} 40,60 \mathrm{H} 10,60 \mathrm{H} 15$, $60 \mathrm{H} 20,60 \mathrm{H} 30,60 \mathrm{H} 35,35 \mathrm{R} 60,34 \mathrm{H} 05,93 \mathrm{E} 20$, 47D06, 46N30, 49N10, 93C20
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# To Branko and Zvezdana <br> Tijana Levajković <br> To Marco and Juanita 

Hermann Mena

## Preface

This book provides a comprehensive and unified introduction to stochastic differential equations involving Malliavin calculus operators. Particularly, it contains results on generalized stochastic processes, linear quadratic optimization and specific applications that appear in the literature as a series of papers ranging from classical Hilbert spaces to numerical approximations in a white noise analysis setting. The intended audience are researchers and graduate students interested in stochastic partial differential equations and related fields. This book is self-contained for readers familiar with white noise analysis and Malliavin calculus. A major contribution of this book is the development of generalized Malliavin calculus in the framework of white noise analysis, based on chaos expansion representation of stochastic processes and its application for solving several classes of stochastic differential equations with singular data, i.e., singular coefficients and singular initial conditions, involving the main operators of Malliavin calculus.

This book is divided into four chapters. The first, entitled White Noise Analysis and Chaos Expansions, includes notation and provides the reader with the theoretical background needed to understand the subsequent chapters. In particular, we introduce spaces of random variables and stochastic processes, and consider processes that have finite variance on classical and fractional Gaussian white noise probability spaces. We also present processes with infinite variance, particularly Kondratiev stochastic distributions. We introduce the Wick and ordinary multiplication of the processes and state where these operations are well defined.

In Chap. 2, Generalized Operators of Malliavin Calculus, the Malliavin derivative operator $\mathbb{D}$, the Skorokhod integral $\delta$ and the Ornstein-Uhlenbeck operator $\mathscr{R}$ are introduced in terms of chaos expansions. The main properties of the operators, which are known in the literature for the square integrable processes, are proven using the chaos expansion approach and extended for generalized and test stochastic processes. Moreover, we discuss fractional versions of these operators. Chapter 3, Equations Involving Malliavin Calculus Operators, is devoted to the study of several types of stochastic differential equations that involve the operators of Malliavin calculus, introduced in the previous chapter. In particular, we describe the range of the operators $\mathbb{D}, \delta$ and $\mathscr{R}$.

In Chap. 4, we present applications of the chaos expansion method in optimal control and stochastic partial differential equations. In particular, we consider the stochastic linear quadratic optimal control problem where the state equation is given by a stochastic differential equation of the Ito-Skorokhod type with different forms of noise disturbances, operator differential algebraic equations arising in fluid dynamics, stationary equations and fractional versions of the studied equations. Moreover, we provide a numerical framework based on chaos expansions and perform numerical simulations.

We would like to express our gratitude to our institutions, University of Innsbruck (Austria) and Yachay Tech (Ecuador) for giving us the opportunity to work in great environments. H. Mena thanks the support of the Austrian Science Foundation (FWF) - project id: P27926. Special thanks go to all our co-authors and colleagues, who contributed greatly in making this project enjoyable and successful. Last but not least, we would like to thank our families for their love and support.

Innsbruck, Austria
August 2017

Tijana Levajković
Hermann Mena

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# Chapter 1 <br> White Noise Analysis and Chaos Expansions 


#### Abstract

In the framework of white noise analysis, random variables and stochastic processes can be represented in terms of Fourier series in a Hilbert space orthogonal basis, namely in their chaos expansion forms. We briefly summarize basic concepts and notations of white noise analysis, characterize different classes of stochastic processes (test, square integrable and generalized stochastic processes) in terms of their chaos expansion representations and review the main properties of the Wick calculus and stochastic integration.


### 1.1 Introduction

White noise analysis, introduced by Hida in [9] and further developed by many authors [11, 26], as a discipline of infinite dimensional analysis, has found applications in solving stochastic differential equations (SDEs) and thus in the modeling of stochastic dynamical phenomena arising in physics, economy, biology [6, 27, 29, 34, 37]. In this context, white noise analysis was proposed as an infinite dimensional analogue of the Schwartz theory of deterministic generalized functions.

Stochastic processes with infinite variance, e.g. the white noise process, appear in many cases as solutions of SDEs. The Hida spaces and the Kondratiev spaces $[9,11]$ have been introduced as the stochastic analogues of the Schwartz space of tempered distributions in order to provide a strict theoretical meaning for this kind of processes. The spaces of test processes contain highly regular processes which allow one to detect the action of generalized processes. The chaos expansion of a stochastic process provides a series decomposition of a square integrable process in a Hilbert space orthogonal basis built upon a class of special functions, Hermite polynomials and functions, in the framework of white noise analysis. In order to build spaces of stochastic test and generalized functions, one has to use series decompositions via orthogonal functions as a basis, with certain weight sequences. Here, we follow the classical Hida approach [9], which suggests to start with a nuclear space $\mathscr{G}$ and its dual $\mathscr{G}^{\prime}$, such that $\mathscr{G} \subseteq L^{2}(\mathbb{R}) \subseteq \mathscr{G}^{\prime}$, and then take the basic probability space to be $\Omega=\mathscr{G}^{\prime}$ endowed with the Borel sigma algebra of the weak topology and an appropriate probability measure $P$.

We consider $P$ to be either the Gaussian white noise probability measure $\mu$ or the fractional Gaussian white noise probability measure $\mu_{H}$. In these cases the orthogonal basis of $L^{2}(P)$ can be constructed from any orthogonal basis of $L^{2}(\mathbb{R})$ that belongs to $\mathscr{G}$ and from the Hermite polynomials [11]. Note that, in the case of a Poissonian measure the orthogonal basis of $L^{2}(P)$ is constructed using the Charlier polynomials together with the orthogonal basis of $L^{2}(\mathbb{R})$. We will focus on the case where $\mathscr{G}$ and $\mathscr{G}^{\prime}$ are the Schwartz spaces of rapidly decreasing test functions $S(\mathbb{R})$ and tempered distributions $S^{\prime}(\mathbb{R})$. In this case, the orthogonal family of $L^{2}(\mathbb{R})$ can be represented in terms of the Hermite functions.

Fractional Brownian motion $b_{t}^{(H)}$ is one-parameter extension of a standard Brownian motion $b_{t}$ and the main properties of such a Gaussian process depend on values of the Hurst parameter $H \in(0,1)$. Fractional Brownian motion, as a process with independent increments which have a long-range dependence and self-similarity properties, found many applications modeling wide range of problems in hydrology, telecommunications, queuing theory and mathematical finance [3, 6]. A specific construction of stochastic integrals with respect to fractional Brownian motion defined for all possible values $H \in(0,1)$, was introduced by Elliot and van der Hoek in [7]. Several different definitions of stochastic integration for fractional Brownian motion appear in literature [3, 7, 33, 36]. We follow [7] and use the definition of the fractional white noise spaces by use of the fractional transform mapping for all values of $H \in(0,1)$ and the extension of the action of the fractional transform operator to a class of generalized stochastic processes. The main properties of the fractional transform operator and the connection of a fractional Brownian motion with a classical Brownian motion on the classical white noise space were presented in $[3,19]$.

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions. They have no point value for $\omega \in \Omega$, only an average value with respect to a test random variable $[9,11,16]$. We introduce the Kondratiev spaces of stochastic distributions $(S)_{-\rho}^{P}, \rho \in[0,1]$, with respect to the probability measure $P$, and thus obtain a Gel'fand triplet $(S)_{\rho}^{P} \subseteq L^{2}(P) \subseteq(S)_{-\rho}^{P}$.

We consider generalized stochastic processes to be measurable mappings from $\mathbb{R}$ into $(S)_{-\rho}^{P}$ or more general, elements of a tensor product $X \otimes(S)_{-\rho}^{P},[23,37]$. Altogether we work with three different types of processes: test, square integrable and generalized stochastic processes. We will characterize them in terms of chaos expansions and review the main properties of the Wick calculus and stochastic integration. Finally, we will introduce coordinatewise operators and convolution type of operators acting on all considered sets of stochastic processes.

### 1.2 Deterministic Background

The Schwartz spaces of rapidly decreasing functions $S(\mathbb{R})$ and tempered distributions $S^{\prime}(\mathbb{R})$ can be characterized in terms of Fourier series representations in the Hermite
functions orthonormal basis. This gives a motivation to build analogous spaces of stochastic elements which allow the chaos decompositions in terms of an orthogonal basis of the Fourier-Hermite polynomials.

The Hermite polynomial of order $n$ for $n \in \mathbb{N}_{0}$ is defined by

$$
h_{n}(x)=(-1)^{n} e^{\frac{x^{2}}{2}} \frac{d^{n}}{d x^{n}}\left(e^{-\frac{x^{2}}{2}}\right), \quad x \in \mathbb{R}
$$

and the Hermite function of order $n+1, n \in \mathbb{N}_{0}$, is defined by

$$
\begin{equation*}
\xi_{n+1}(x)=\frac{1}{\sqrt[4]{\pi} \sqrt{n!}} e^{-\frac{x^{2}}{2}} h_{n}(\sqrt{2} x), \quad x \in \mathbb{R} \tag{1.1}
\end{equation*}
$$

The following relations for the derivatives of the Hermite polynomials and the Hermite functions hold $h_{n}^{\prime}(x)=n h_{n-1}(x), n \in \mathbb{N}$ and

$$
\begin{equation*}
\xi_{n}^{\prime}(x)=\sqrt{\frac{n}{2}} \xi_{n-1}-\sqrt{\frac{n+1}{2}} \xi_{n+1}, \quad n \in \mathbb{N} \tag{1.2}
\end{equation*}
$$

Moreover, $\left|\xi_{n}\right| \leq c n^{-\frac{1}{12}}$ for $|x| \leq 2 \sqrt{n}$ and $\left|\xi_{n}\right| \leq c e^{-\gamma x^{2}}$ for $|x|>2 \sqrt{n}$ for constants $c$ and $\gamma$ independent of $n$, [11]. Recall, the family $\left\{\frac{1}{\sqrt{n!}} h_{n}\right\}_{n \in \mathbb{N}_{0}}$ forms an orthonormal basis of the space $L^{2}(\mathbb{R})$ with respect to the Gaussian measure $d \mu=\frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}} d x$. The family of Hermite functions $\left\{\xi_{n+1}\right\}_{n \in \mathbb{N}_{0}}$ constitutes a complete orthonormal system of $L^{2}(\mathbb{R})$ with respect to the Lebesque measure. Namely, every deterministic function $g \in L^{2}(\mathbb{R})$ have Fourier series representation

$$
\begin{equation*}
g(x)=\sum_{k \in \mathbb{N}} a_{k} \xi_{k}(x) \tag{1.3}
\end{equation*}
$$

with coefficients $a_{k}=\left(g, \xi_{k}\right)_{L^{2}(\mathbb{R})} \in \mathbb{R}$ satisfying $\sum_{k \in \mathbb{N}} a_{k}^{2}<\infty$.
The Schwartz space of rapidly decreasing functions is defined as

$$
S(\mathbb{R})=\left\{f \in C^{\infty}(\mathbb{R}): \forall \alpha, \beta \in \mathbb{N}_{0},\|f\|_{\alpha, \beta}=\sup _{x \in \mathbb{R}}\left|x^{\alpha} D^{\beta} f(x)\right|<\infty\right\}
$$

and the topology on $S(\mathbb{R})$ is given by the family of seminorms $\|f\|_{\alpha, \beta}$. The space $S(\mathbb{R})$ is a nuclear countable Hilbert space and the family of Hermite functions $\left\{\xi_{n}\right\}_{n \in \mathbb{N}}$ forms an orthonormal basis of $S(\mathbb{R})$, [12]. The Schwartz space of rapidly decreasing functions can be constructed as the projective limit $S(\mathbb{R})=\bigcap_{l \in \mathbb{N}_{0}} S_{l}(\mathbb{R})$ of the family of spaces

$$
\begin{equation*}
S_{l}(\mathbb{R})=\left\{\varphi=\sum_{k=1}^{\infty} a_{k} \xi_{k} \in L^{2}(\mathbb{R}):\|\varphi\|_{l}^{2}=\sum_{k=1}^{\infty} a_{k}^{2}(2 k)^{l}<\infty\right\}, l \in \mathbb{N}_{0} \tag{1.4}
\end{equation*}
$$

Note that $S_{l}(\mathbb{R}), l \in \mathbb{Z}$ is a Hilbert spaces endowed with the scalar product $\langle\cdot, \cdot\rangle_{l}$, $l \in \mathbb{Z}$ given by $\left\langle\xi_{k}, \xi_{j}\right\rangle_{l}=0$ for $k \neq j$ and $\left\langle\xi_{k}, \xi_{j}\right\rangle_{l}=\left\|\xi_{k}\right\|_{l}^{2}=(2 k)^{l}$ for $k=j$. Moreover, $S_{l_{1}} \subseteq S_{l_{2}}$, for $l_{1} \geq l_{2}$. The Schwartz space of tempered distributions $S^{\prime}(\mathbb{R})$ is the dual space of the space $S(\mathbb{R})$ equipped with the strong topology, which is equivalent to the inductive topology. Its elements are called generalized functions or distributions [12]. The Schwartz space of tempered distributions is isomorphic to the inductive limit $S^{\prime}(\mathbb{R})=\bigcup_{l \in \mathbb{N}_{0}} S_{-l}(\mathbb{R})$ of the family of spaces

$$
S_{-l}(\mathbb{R})=\left\{f=\sum_{k=1}^{\infty} b_{k} \xi_{k}:\|f\|_{-l}^{2}=\sum_{k=1}^{\infty} b_{k}^{2}(2 k)^{-l}<\infty\right\}, l \in \mathbb{N}_{0}
$$

The action of a generalized function $f=\sum_{k \in \mathbb{N}} b_{k} \xi_{k} \in S^{\prime}(\mathbb{R})$ onto a test function $\varphi=\sum_{k \in \mathbb{N}} a_{k} \xi_{k} \in S(\mathbb{R})$ is given by $\langle f, \varphi\rangle=\sum_{k \in \mathbb{N}} a_{k} b_{k}$. Also, $S(\mathbb{R}) \subseteq L^{2}(\mathbb{R}) \subseteq$ $S^{\prime}(\mathbb{R})$ is a Gel'fand triple with continuous inclusions.

### 1.3 Spaces of Random Variables

Throughout this manuscript we work on two Gaussian white noise probability spaces, namely the classical $\left(S^{\prime}(\mathbb{R}), \mathscr{B}, \mu\right)$ and the fractional $\left(S^{\prime}(\mathbb{R}), \mathscr{B}, \mu_{H}\right)$ which respectively correspond to the Gaussian probability measures $\mu$ and $\mu_{H}$, Sects.1.3.1 and 1.3.6. In this section, we recall notions of random variables, not only square integrable but also those which are elements of the Hida-Kondratiev spaces. We state the famous Wiener-Itô chaos expansion theorem and characterize random variables through their chaos expansion forms. We follow the ideas from [7, 9, 11, 15].

### 1.3.1 Gaussian White Noise Space

Consider the Schwartz space of tempered distributions $S^{\prime}(\mathbb{R})$, the Borel sigmaalgebra $\mathscr{B}$ generated by the weak topology on $S^{\prime}(\mathbb{R})$ and a given characteristic function $C$. Recall, a mapping $C: S(\mathbb{R}) \rightarrow \mathbb{C}$ given on a nuclear space $S(\mathbb{R})$ is called a characteristic function if it is continuous, positive definite, i.e.,

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} z_{i} \bar{z}_{j} C\left(\varphi_{i}-\varphi_{j}\right) \geq 0
$$

for all $\varphi_{1}, \ldots, \varphi_{n} \in S(\mathbb{R})$ and $z_{1}, \ldots, z_{n} \in \mathbb{C}$, and if it satisfies $C(0)=1$. Then, by the Bochner-Minlos theorem, there exists a unique probability measure $P$ on $\left(S^{\prime}(\mathbb{R}), \mathscr{B}\right)$ such that for all $\varphi \in S(\mathbb{R})$ the relation $\mathbb{E}_{P}\left(e^{i\langle\omega, \varphi\rangle}\right)=C(\varphi)$ holds [9, 11]. Here $\mathbb{E}_{P}$ denotes the expectation with respect to the measure $P$ and $\langle\omega, \varphi\rangle$ denotes the dual
pairing between a tempered distribution $\omega \in S^{\prime}(\mathbb{R})$ and a rapidly decreasing function $\varphi \in S(\mathbb{R})$. Further on, we will omit writing the measure $P$. Thus,

$$
\begin{equation*}
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \varphi\rangle} d P(\omega)=C(\varphi), \quad \varphi \in S(\mathbb{R}) \tag{1.5}
\end{equation*}
$$

The triplet $\left(S^{\prime}(\mathbb{R}), \mathscr{B}, P\right)$ is called the white noise probability space and the measure $P$ is called the white noise probability measure.

If we choose in (1.5) the characteristic function of a Gaussian random variable

$$
\begin{equation*}
C(\varphi)=\exp \left[-\frac{1}{2}\|\varphi\|_{L^{2}(\mathbb{R})}^{2}\right], \quad \varphi \in S(\mathbb{R}) \tag{1.6}
\end{equation*}
$$

then the corresponding unique measure $P$ from the Bochner-Minlos theorem is called the Gaussian white noise measure and is denoted by $\mu$. The triplet $\left(S^{\prime}(\mathbb{R}), \mathscr{B}, \mu\right)$ is called the Gaussian white noise probability space.

The space $L^{2}(\mu)=L^{2}\left(S^{\prime}(\mathbb{R}), \mathscr{B}, \mu\right)$ is the Hilbert space of square integrable random variables on $S^{\prime}(\mathbb{R})$ with respect to the Gaussian measure $\mu$. Thus, from (1.5) and (1.6) it follows

$$
\begin{equation*}
\int_{S^{\prime}(\mathbb{R})} e^{i\langle\omega, \varphi\rangle} d \mu(\omega)=e^{-\frac{1}{2}\|\varphi\|_{L^{2}(\mathbb{R})}^{2}}, \quad \varphi \in S(\mathbb{R}) \tag{1.7}
\end{equation*}
$$

From (1.7) we conclude that the random element $\langle\omega, \varphi\rangle, \varphi \in S(\mathbb{R}), \omega \in S^{\prime}(\mathbb{R})$ is a centered Gaussian square integrable random variable with the variance

$$
\begin{equation*}
\operatorname{Var}(\langle\omega, \varphi\rangle)=\mathbb{E}\left(\langle\omega, \varphi\rangle^{2}\right)=\|\varphi\|_{L^{2}(\mathbb{R})}^{2} \tag{1.8}
\end{equation*}
$$

The element $\langle\omega, \varphi\rangle$ is called smoothed white noise and the mapping $J_{1}: \varphi \rightarrow\langle\omega, \varphi\rangle$, $\varphi \in S(\mathbb{R})$ can be extended to an isometry from $L^{2}(\mathbb{R})$ to $L^{2}(\mu)$.

Example 1.1 Brownian motion. By extending the action of a distribution $\omega \in S^{\prime}(\mathbb{R})$ not only onto test functions from $S(\mathbb{R})$ but also onto elements of $L^{2}(\mathbb{R})$ we obtain Brownian motion with respect to the measure $\mu$ in the form

$$
b_{t}(\omega)=\langle\omega, \chi[0, t]\rangle, \quad \omega \in S^{\prime}(\mathbb{R})
$$

where $\chi[0, t]$ represents the characteristic function of interval $[0, t], t \in \mathbb{R}$. To be precise, $\langle\omega, \chi[0, t]\rangle$ is a well defined element of $L^{2}(\mu)$ for all $t$, defined by $\lim _{n \rightarrow \infty}\left\langle\omega, \varphi_{n}\right\rangle$, where $\varphi_{n} \rightarrow \chi[0, t], n \rightarrow \infty$ in $L^{2}(\mathbb{R})$. It has a zero expectation and its covariance function equals

$$
\mathbb{E}(\langle\omega, \chi[0, t]\rangle\langle\omega, \chi[0, s]\rangle)=\mathbb{E}\left(b_{t}(\omega) b_{s}(\omega)\right)=\min \{t, s\}, \quad t, s>0
$$

Recall, Brownian motion is a Gaussian process whose almost all trajectories are continuous but nowhere differentiable functions [14, 39].

Example 1.2 The Itô integral. For a deterministic function $f \in L^{2}(\mathbb{R})$, the smoothed white noise $\langle\omega, f\rangle$ can be represented in the form of a stochastic integral with respect to Brownian motion

$$
\langle\omega, f\rangle=\int_{\mathbb{R}} f(t) d b_{t}(\omega)
$$

Clearly, $\langle\omega, f\rangle$ equals to the (one-fold) Itô integral $I_{1}(f)$. Thus, $\mathbb{E}\left(I_{1}(f)\right)=0$ and also the Itô isometry $\left\|I_{1}(f)\right\|_{L^{2}(\mu)}^{2}=\|f\|_{L^{2}(\mathbb{R})}^{2}$ holds.
Remark 1.1 For different choices of positive definite functionals $C(\varphi)$ in (1.5) one can obtain different white noise probabilistic measures, which then correspond to such functionals. In particular, $C(\varphi)$ is the characteristic function of the compound Poisson random variable then the corresponding white noise measure is the Poissonian white noise measure. In [34] the authors replaced the characteristic function $C(\varphi)$ by a completely monotonic function defined by the Mittag-Leffler function of order $0<\beta \leq 1$ and obtained the gray noise measure, which is more general then the white noise measure. With a similar construction, one can also obtain the Lévy white noise measure [6, 42].

### 1.3.2 Wiener-Itô Chaos Expansion of Random Variables

Denote by $\mathscr{I}=\left(\mathbb{N}_{0}^{\mathbb{N}}\right)_{c}$ the set of sequences of non-negative integers which have finitely many nonzero components. Its elements are multi-indices $\alpha \in \mathscr{I}$ of the form $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0 \ldots\right), \alpha_{i} \in \mathbb{N}_{0}, i=1,2, \ldots, m, m \in \mathbb{N}$, where $\operatorname{Index}(\alpha)=\max \left\{k \in \mathbb{N}: \alpha_{k} \neq 0\right\}=m$. Particularly, $\mathbf{0}=(0,0, \ldots)$ denotes the zeroth vector and $\varepsilon^{(k)}=(0, \cdots, 0,1,0, \cdots), k \in \mathbb{N}$ is the $k$ th unit vector. The length of $\alpha \in \mathscr{I}$ is defined by $|\alpha|=\sum_{k=1}^{\infty} \alpha_{k}$. Operations with multi-indices are carried out componentwise, e.g. $\alpha!=\prod_{k=1}^{\infty} \alpha_{k}!,\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$ and $(2 \mathbb{N})^{\alpha}=$ $\prod_{i \in \mathbb{N}}(2 i)^{\alpha_{i}}$. Note that $\alpha>\mathbf{0}$ if it has at least one nonzero component, i.e., $\alpha_{k}>0$ for some $k \in \mathbb{N}$. We say $\alpha \geq \beta$ if it holds $\alpha_{k} \geq \beta_{k}$ for all $k \in \mathbb{N}$ and in that case $\alpha-\beta=\left(\alpha_{1}-\beta_{1}, \alpha_{2}-\beta_{2}, \ldots\right)$. Particularly, for $\alpha_{k}>0$ we have $\alpha-\varepsilon^{(k)}=$ $\left(\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0, \ldots\right), k \in \mathbb{N}$. For $\alpha<\beta$ the difference $\alpha-\beta$ is not defined.

Lemma 1.1 ([20]) The following estimates hold:

$$
\begin{array}{ll}
1^{\circ} & \binom{\alpha}{\beta} \leq 2^{|\alpha|} \leq(2 \mathbb{N})^{\alpha}, \quad \alpha \in \mathscr{I}, \quad \beta \leq \alpha, \\
2^{\circ} & (\theta+\beta)!\leq \theta!\beta!(2 \mathbb{N})^{\theta+\beta}, \quad \theta, \beta \in \mathscr{I} .
\end{array}
$$

Proof $1^{\circ}$ Since $\binom{n}{k} \leq 2^{n}$, for all $n \in \mathbb{N}_{0}$ and $0 \leq k \leq n$, it follows that

$$
\binom{\alpha}{\beta}=\prod_{i \in \mathbb{N}}\binom{\alpha_{i}}{\beta_{i}} \leq \prod_{i \in \mathbb{N}} 2^{\alpha_{i}}=2^{|\alpha|} \leq \prod_{i \in \mathbb{N}}(2 i)^{\alpha_{i}}=(2 \mathbb{N})^{\alpha}
$$

for all $\alpha \in \mathscr{I}$ and $\mathbf{0} \leq \beta \leq \alpha$.
$2^{\circ}$ Let $\alpha \geq \beta$. From part $1^{\circ}$ and $\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$ we obtain the following inequality $\alpha!\leq \beta!(\alpha-\beta)!(2 \mathbb{N})^{\alpha}$, which after substituting $\theta=\alpha-\beta$ leads to desired estimate $(\theta+\beta)!\leq \theta!\beta!(2 \mathbb{N})^{\theta+\beta}$ for all $\theta, \beta \in \mathscr{I}$.

As a consequence of Lemma 1.1 we have that the estimates $|\alpha|!\geq \alpha!$ and $(2 \alpha)!\leq$ $(2 \alpha)!!^{2}=\left(2^{|\alpha|} \alpha!\right)^{2}$ hold for $\alpha \in \mathscr{I}$.

Theorem 1.1 ([45]) It holds that

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{-p \alpha}<\infty \text { if and only if } p>1 \tag{1.9}
\end{equation*}
$$

The proof can be found in $[11,45]$.
Remark 1.2 Consider a sequence of real numbers $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$. We denote by $a^{\alpha}=\prod_{k=1}^{\infty} a_{k}^{\alpha_{k}}, \frac{a^{\alpha}}{\alpha!}=\prod_{k=1}^{\infty} \frac{a_{k}^{\alpha_{k}}}{\alpha_{k}!}$ and $(2 \mathbb{N} a)^{\alpha}=\prod_{k=1}^{\infty}\left(2 k a_{k}\right)^{\alpha_{k}}$. The result (1.9) is used to verify the statement

$$
\sum_{\alpha \in \mathscr{I}}(2 \mathbb{N} a)^{-p \alpha}<\infty \quad \text { if and only if } \quad p>1
$$

Definition 1.1 For a given $\alpha \in \mathscr{I}$ the $\alpha$-th Fourier-Hermite polynomial is defined by

$$
\begin{equation*}
H_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right), \quad \alpha \in \mathscr{I} . \tag{1.10}
\end{equation*}
$$

For each $\alpha \in \mathscr{I}$ the product (1.10) has finitely many terms, since each $\alpha$ has finitely many nonzero components and it holds $h_{0}(x)=1$. Particularly, for $\alpha=\mathbf{0}$ the zeroth Fourier-Hermite polynomial is $H_{0}(\omega)=1$, for the $k$ th unit vector $\varepsilon^{(k)}$ the Fourier-Hermite polynomial is

$$
H_{\varepsilon^{(k)}}(\omega)=h_{1}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=\left\langle\omega, \xi_{k}\right\rangle=\int_{\mathbb{R}} \xi_{k}(t) d B_{t}(\omega)=I_{1}\left(\xi_{k}\right), \quad k \in \mathbb{N}
$$

and for $\alpha=(2,0,1,0, \ldots)$ we have
$H_{(2,0,1,0, \ldots)}(\omega)=h_{2}\left(\left\langle\omega, \xi_{1}\right\rangle\right) h_{1}\left(\left\langle\omega, \xi_{3}\right\rangle\right)=\left(\left\langle\omega, \xi_{1}\right\rangle^{2}-1\right)\left\langle\omega, \xi_{3}\right\rangle=\left(I_{1}\left(\xi_{1}\right)^{2}-1\right) I_{1}\left(\xi_{3}\right)$.
Theorem 1.2 ([11]) The family of Fourier-Hermite polynomials $\left\{H_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ forms an orthogonal basis of the space $L^{2}(\mu)$, where $\left\|H_{\alpha}\right\|_{L^{2}(\mu)}^{2}=\mathbb{E}\left(H_{\alpha}^{2}\right)=\alpha$ !.

Theorem 1.3 ([11, 44]) (Wiener-Itô chaos expansion theorem) Each square integrable random variable $F \in L^{2}(\mu)$ has a unique representation of the form

$$
\begin{equation*}
F(\omega)=\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}(\omega), \quad c_{\alpha} \in \mathbb{R}, \omega \in \Omega, \tag{1.11}
\end{equation*}
$$

such that it holds

$$
\begin{equation*}
\|F\|_{L^{2}(\mu)}^{2}=\sum_{\alpha \in \mathscr{I}} c_{\alpha}^{2} \alpha!<\infty \tag{1.12}
\end{equation*}
$$

The coefficients are unique and are obtained from $c_{\alpha}=\frac{1}{\alpha!} \mathbb{E}\left(F H_{\alpha}\right), \alpha \in \mathscr{I}$.
Example 1.3 A smoothed white noise $\langle\omega, \varphi\rangle$, where $\varphi \in S(\mathbb{R})$ and $\omega \in S^{\prime}(\mathbb{R})$ is a zero-mean Gaussian random variable with the variance (1.8). By (1.3) and (1.4) we represent $\varphi=\sum_{k=1}^{\infty}\left(\varphi, \xi_{k}\right)_{L^{2}(\mathbb{R})} \xi_{k} \in S(\mathbb{R})$. Then, the chaos expansion representation of $\langle\omega, \varphi\rangle$ is given by

$$
\langle\omega, \varphi\rangle=\sum_{k=1}^{\infty}\left(\varphi, \xi_{k}\right)_{L^{2}(\mathbb{R})}\left\langle\omega, \xi_{k}\right\rangle=\sum_{k=1}^{\infty}\left(\varphi, \xi_{k}\right)_{L^{2}(\mathbb{R})} H_{\varepsilon^{(k)}}(\omega) .
$$

Definition 1.2 The spaces $\mathscr{H}_{k}$ that are obtained by closing the linear span of the $k$ th order Hermite polynomials in $L^{2}(\mu)$ are called the Wiener chaos spaces of order $k$

$$
\mathscr{H}_{k}=\left\{F \in L^{2}(\Omega): F=\sum_{\alpha \in \mathscr{\mathscr { F }},|\alpha|=k} c_{\alpha} H_{\alpha}\right\}, \quad k \in \mathbb{N}_{0} .
$$

Particularly, $\mathscr{H}_{0}$ is the set of constant random variables, $\mathscr{H}_{1}$ is a set of Gaussian random variables, $\mathscr{H}_{2}$ is a space of quadratic Gaussian random variables and so on. Since each $\mathscr{H}_{k}, k \in \mathbb{N}_{0}$ is a closed subspace of $L^{2}(\mu)$ the Wiener-Itô chaos expansion theorem can be stated in the form $L^{2}(\mu)=\bigoplus_{k=0}^{\infty} \mathscr{H}_{k}$. Therefore, every $F \in L^{2}(\mu)$ can be uniquely represented in the form

$$
F(\omega)=\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}(\omega)=\sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k} c_{\alpha} H_{\alpha}(\omega)\right)=\sum_{k=0}^{\infty} F_{k}(\omega),
$$

where $F_{k}(\omega)=\sum_{|\alpha|=k} c_{\alpha} H_{\alpha}(\omega) \in \mathscr{H}_{k}, k \in \mathbb{N}_{0}, \omega \in S^{\prime}(\mathbb{R})$.
Remark 1.3 The Wiener-Itô chaos expansion theorem, Theorem 1.3, can be formulated also in terms of iterated Itô integrals. Although this formulation will not play central role in our presentation, for completeness we include it here. Clearly, the second formulation of the Wiener-Itô chaos expansion theorem states that each $F \in L^{2}(\mu)$ is determined by a unique family of symmetric deterministic functions $f_{n}$, such that

$$
\begin{equation*}
F(\omega)=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right), \tag{1.13}
\end{equation*}
$$

where $I_{n}$ denotes $n$-fold iterated Itô integral. The connection between two formulations (1.11) and (1.13) was provided by Itô in [13]. For $\alpha=\left(\alpha_{1}, \cdots, \alpha_{m}, 0, \cdots\right) \in$ $\mathscr{I}$ of the length $n$ the symmetrized tensor product with factors $\xi_{1}, \ldots \xi_{m}$ is defined by $\xi^{\widehat{\otimes} \alpha}=\xi_{1}^{\otimes \alpha_{1}} \widehat{\otimes} \ldots \widehat{\otimes} \xi_{m}^{\otimes \alpha_{m}}$, where each $\xi_{i}$ is taken $\alpha_{i}$ times. Then, it holds

$$
H_{\alpha}(\omega)=\int_{\mathbb{R}^{n}} \xi^{\widehat{\otimes} \alpha}(t) d b_{t}^{\otimes n}(\omega)
$$

The connection between two chaos expansion forms is given by $f_{n}=\sum_{|\alpha|=n} c_{\alpha} \xi_{n}^{\hat{\otimes} \alpha}$. Further on we will focus only on the formulation (1.11).

Theorem 1.4 ([23]) All random variables which belong to the space $\mathscr{H}_{1}$ are Gaussian random variables.

Proof Random variables that belong to the space $\mathscr{H}_{1}$ are linear combinations of elements $\left\langle\omega, \xi_{k}\right\rangle, k \in \mathbb{N}, \omega \in S^{\prime}(\mathbb{R})$. From the definition of the Gaussian measure (1.7) we obtain $\mathbb{E}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=0$ and $\operatorname{Var}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=\mathbb{E}\left(\left\langle\omega, \xi_{k}\right\rangle^{2}\right)=$ $\left\|\xi_{k}\right\|_{L^{2}(\mathbb{R})}^{2}=1$. Due to the form of the characteristic function we conclude that $\left\langle\omega, \xi_{k}\right\rangle: \mathscr{N}(0,1), k \in \mathbb{N}$. Thus, every finite linear combination of Gaussian random variables $\sum_{k=1}^{n} a_{k}\left\langle\omega, \xi_{k}\right\rangle$ is a Gaussian random variable and the limit of Gaussian random variables $\sum_{k=1}^{\infty} a_{k}\left\langle\omega, \xi_{k}\right\rangle=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} a_{k}\left\langle\omega, \xi_{k}\right\rangle$ is also Gaussian.

We note that $\mathscr{H}_{1}$ is the closed Gaussian space generated by the random variables $b_{t}(\omega), t \geq 0$, see Example 1.7 and also [39].

Remark 1.4 Although the space $L^{2}(\mu)$ is constructed with respect to the Gaussian measure $\mu$, it contains all square integrable random variables, not just those with Gaussian distribution but also all absolutely continuous, singularly continuous, discrete and mixed type distributions. All elements in $\mathscr{H}_{0} \oplus \mathscr{H}_{1}$ are Gaussian (those with zero expectation are strictly in $\mathscr{H}_{1}$ ), but the converse is not true. There exist Gaussian random variables with higher order chaos expansions. Representative elements of $\mathscr{H}_{0} \oplus \mathscr{H}_{1} \oplus \mathscr{H}_{2}$ are for example, quadratic Gaussian random variables and the Chi-square distribution as a finite sum of independent quadratic Gaussian variables. Discrete random variables with finite variance belong to $\bigoplus_{k=0}^{\infty} \mathscr{H}_{k}$, i.e., their chaos expansion forms consist of multi-indices of all lengths. All finite sums, i.e., partial sums of a chaos expansion correspond to absolutely continuous distributions or almost surely constant distributions. There is no possibility to obtain discrete random variables by using finite sums in the Wiener-Itô expansion, see [23].

The multiplication formula for the ordinary product of Fourier-Hermite polynomials is proven in [11] and is given by

$$
\begin{equation*}
H_{\alpha} \cdot H_{\beta}=\sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \tag{1.14}
\end{equation*}
$$

for all $\alpha, \beta \in \mathscr{I}$. Particularly, for $k, j \in \mathbb{N}$ it holds $H_{\varepsilon^{(k)}} \cdot H_{\varepsilon^{(j)}}=\left\{\begin{array}{cc}H_{2 \varepsilon^{(k)}}+1, k=j \\ H_{\varepsilon^{(k)}+\varepsilon^{(j)}}, k \neq j\end{array}\right.$. In next section we introduce suitable spaces, called Kondratiev spaces, that will contain random variables which do not satisfy (1.12), i.e., random variables with infinite variances.

### 1.3.3 Kondratiev Spaces

Following the ideas introduced in $[9,11,26,27]$ we define weighted spaces of stochastic test functions and stochastic generalized functions, which represent the stochastic analogue of the Schwartz spaces. The choice of weights depends on a concrete problem studied. Here we will work with polynomial weights and HidaKondratiev spaces of test and generalized random variables. We characterize these spaces in terms of chaos expansions [2, 11]. In [8, 15] the authors used $\mathscr{S}$-transform. Another type of weighted spaces, spaces of exponential growth, was investigated and characterized in [19].

Definition 1.3 Let $\rho \in[0,1]$.
$1^{\circ}$ The space of Kondratiev test random variables $(S)_{\rho}$ consists of elements $f=\sum_{\alpha \in \mathscr{I}} b_{\alpha} H_{\alpha} \in L^{2}(\mu), b_{\alpha} \in \mathbb{R}, \alpha \in \mathscr{I}$, such that

$$
\begin{equation*}
\|f\|_{\rho, p}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho} b_{\alpha}^{2}(2 \mathbb{N})^{p \alpha}<\infty \text { for all } p \in \mathbb{N}_{0} \tag{1.15}
\end{equation*}
$$

$2^{\circ}$ The space of Kondratiev generalized random variables $(S)_{-\rho}$ consists of formal expansions of the form $F=\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}, c_{\alpha} \in \mathbb{R}, \alpha \in \mathscr{I}$, such that

$$
\begin{equation*}
\|F\|_{-\rho,-p}^{2}=\sum_{\alpha \in \mathscr{\mathscr { F }}} \alpha!^{1-\rho} c_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \quad \text { for some } \quad p \in \mathbb{N}_{0} . \tag{1.16}
\end{equation*}
$$

The generalized expectation of $F$ is defined as $\mathbb{E}(F)=c_{0}$, i.e., it is the zeroth coefficient in the chaos expansion of $F$. For $F \in L^{2}(\mu)$ it coincides with the expectation.

The space $(S)_{\rho}$ can be constructed as the projective limit $(S)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{\rho, p}$ of the family $(S)_{\rho, p}=\left\{f=\sum_{\alpha \in \mathscr{I}} b_{\alpha} H_{\alpha} \in L^{2}(\mu):\|f\|_{\rho, p}^{2}<\infty\right\}, p \in \mathbb{N}_{0}$. The Kondratiev space $(S)_{-\rho}$ can be constructed as the inductive limit $(S)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-\rho,-p}$ of the family $(S)_{-\rho,-p}=\left\{F=\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}:\|F\|_{-\rho,-p}^{2}<\infty\right\}, p \in \mathbb{N}_{0}$. It also holds $(S)_{\rho} \subseteq L^{2}(\mu) \subseteq(S)_{-\rho}$, with continuous inclusions. The largest space of the Kondratiev generalized random variables is $(S)_{-1}$ and is obtained for $\rho=1$, while the smallest one is obtained for $\rho=0$. The spaces $(S)_{0}=(S)$ and $(S)_{-0}=(S)^{*}$, obtained for $\rho=0$, are called the Hida spaces of test and generalized random variables. Hence, for $\rho \in[0,1]$ we obtain a sequence of spaces such that

$$
\begin{equation*}
(S)_{1, p} \subseteq(S)_{\rho, p} \subseteq(S)_{0, p} \subseteq L^{2}(\mu) \subseteq(S)_{-0,-p} \subseteq(S)_{-\rho,-p} \subseteq(S)_{-1,-p} \tag{1.17}
\end{equation*}
$$

For all $p \geq q \geq 0$ it holds $(S)_{\rho, p} \subseteq(S)_{\rho, q} \subseteq L^{2}(\mu) \subseteq(S)_{-\rho,-q} \subseteq(S)_{-\rho,-p}$ and the inclusions denote continuous embeddings with $(S)_{0,0}=L^{2}(\mu)$. In [9, 11] it was proven that the spaces $(S)_{1}$ and $(S)_{0}$ are nuclear. We denote by $\ll \cdot, \cdot>_{\rho}$ the dual pairing between $(S)_{-\rho}$ and $(S)_{\rho}$. Its action is given by $\ll F, f>_{\rho}=$ $\ll \sum_{\alpha \in \mathscr{\mathscr { G }}} c_{\alpha} H_{\alpha}, \sum_{\alpha \in \mathscr{\mathscr { I }}} b_{\alpha} H_{\alpha}>_{\rho}=\sum_{\alpha \in \mathscr{\mathscr { I }}} \alpha!c_{\alpha} b_{\alpha}$. Especially, for $\rho=0$ and any fixed $p \in \mathbb{Z}$ the space $(S)_{0, p}$ is a Hilbert space endowed with the scalar product $\ll H_{\alpha}, H_{\beta}>_{0, p}=0$ for $\alpha \neq \beta$ and $\ll H_{\alpha}, H_{\beta}>_{0, p}=\alpha!(2 \mathbb{N})^{p \alpha}$ for $\alpha=\beta$, extended by linearity and continuity to $\ll F, f \gg_{0, p}=\sum_{\alpha \in \mathscr{I}} \alpha!c_{\alpha} b_{\alpha}(2 \mathbb{N})^{p \alpha}$. In case of random variables with finite variances we have $\ll F, f>_{0,0}=(F, f)_{L^{2}(\mu)}=$ $\mathbb{E}(F f)$.

Remark 1.5 Kondratiev spaces modified by a sequence. Definition 1.3 can be generalized for polynomial weights which are modified by a given sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}$, $a_{k} \geq 1$. The obtained spaces are introduced in [22] and are called the Kondratiev spaces modified by the sequence $a$. Let $\rho \in[0,1]$ and let $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$.
$1^{\circ}$ The space of Kondratiev test random variables modified by the sequence $a$, denoted by $(S a)_{\rho}$, consists of elements $f=\sum_{\alpha \in \mathscr{\mathscr { I }}} b_{\alpha} H_{\alpha} \in L^{2}(\mu), b_{\alpha} \in \mathbb{R}$, $\alpha \in \mathscr{I}$, such that

$$
\begin{equation*}
\|f\|_{\rho, p}^{2}=\sum_{\alpha \in \mathscr{\mathscr { I }}} \alpha!^{1+\rho} b_{\alpha}^{2}(2 \mathbb{N} a)^{p \alpha}<\infty \text { for all } p \in \mathbb{N}_{0} . \tag{1.18}
\end{equation*}
$$

$2^{\circ}$ The space of Kondratiev generalized random variables modified by the sequence $a$, denoted by $(S a)_{-\rho}$, consists of formal expansions of the form $F=\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}, c_{\alpha} \in \mathbb{R}, \alpha \in \mathscr{I}$ such that

$$
\begin{equation*}
\|F\|_{-\rho,-p}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho} c_{\alpha}^{2}(2 \mathbb{N} a)^{-p \alpha}<\infty \quad \text { for some } \quad p \in \mathbb{N}_{0} \tag{1.19}
\end{equation*}
$$

It is clear that for $a_{k}=1, k \in \mathbb{N}$ these spaces reduce to the Kondratiev spaces $(S)_{\rho}$ and $(S)_{-\rho}$. For all $\rho \in[0,1]$ we have $(S a)_{\rho} \subseteq L^{2}(\mu) \subseteq(S a)_{-\rho}$.

### 1.3.4 Hilbert Space Valued Kondratiev Type Random Variables

Let $\mathscr{H}$ be a separable Hilbert space with the orthonormal basis $\left\{s_{j}\right\}_{j \in \mathbb{N}}$ and the inner product $(\cdot, \cdot)_{\mathscr{H}}$. We denote by $L^{2}(\Omega, \mathscr{H})$ the space of random variables on $\Omega$ with values in $\mathscr{H}$, which are square integrable with respect to the white noise measure $\mu$. It is a Hilbert space equipped with the inner product $\ll F, G \gg_{L^{2}(\Omega, \mathscr{H})}=$ $\mathbb{E}\left((F, G)_{\mathscr{H}}\right)$, for all $F, G \in L^{2}(\Omega, \mathscr{H})$. The family of functions $\left\{\frac{1}{\sqrt{\alpha!}} H_{\alpha} s_{j}\right\}_{j \in \mathbb{N}, \alpha \in \mathscr{\mathscr { I }}}$ forms an orthonormal basis of $L^{2}(\Omega, \mathscr{H})$. Hence, each $F \in L^{2}(\Omega, \mathscr{H})$ can be represented in the chaos expansion form

$$
\begin{equation*}
F(\omega)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} s_{k} H_{\alpha}(\omega) \tag{1.20}
\end{equation*}
$$

where $a_{\alpha, k} \in \mathbb{R}, \omega \in \Omega$ such that

$$
\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!a_{\alpha, k}^{2}<\infty
$$

In the following we define $\mathscr{H}$-valued Kondratiev test and generalized random variables over $L^{2}(\Omega, \mathscr{H})$.

Definition 1.4 ([41]) Let $\rho \in[0,1]$.
$1^{\circ}$ The space of $\mathscr{H}$-valued Kondratiev test random variables $S(\mathscr{H})_{\rho}$ consists of functions $f \in L^{2}(\Omega, \mathscr{H})$ given in the chaos expansion form (1.20) such that for all $p \in \mathbb{N}_{0}$ it holds

$$
\|f\|_{S(\mathscr{H})_{\rho, p}}^{2}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!^{1+\rho} a_{\alpha, k}^{2}(2 \mathbb{N})^{p \alpha}=\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho} a_{\alpha, k}^{2}(2 \mathbb{N})^{p \alpha}<\infty
$$

$2^{\circ}$ The corresponding $\mathscr{H}$-valued Kondratiev space of generalized random variables $S(\mathscr{H})_{-\rho}$ consists of formal expansions of the form

$$
F(\omega)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} b_{\alpha, k} s_{k} H_{\alpha}(\omega), \quad b_{\alpha, k} \in \mathbb{R}
$$

such that for some $p \in \mathbb{N}_{0}$ it holds

$$
\|F\|_{S(\mathscr{H})_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!^{1-\rho} b_{\alpha, k}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho} b_{\alpha, k}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

Note here that $f \in S(\mathscr{H})_{\rho}$ can be expressed in several ways

$$
f(\omega)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} a_{\alpha, k} s_{k} H_{\alpha}(\omega)=\sum_{\alpha \in \mathscr{I}} a_{\alpha} H_{\alpha}(\omega)=\sum_{k \in \mathbb{N}} a_{k}(\omega) s_{k}
$$

$a_{\alpha}=\left(f, H_{\alpha}\right)_{L^{2}(\mu)}=\sum_{k \in \mathbb{N}} a_{\alpha, k} s_{k} \in \mathscr{H}$ and $a_{k}(\omega)=\left(f, s_{k}\right)_{\mathscr{H}}=\sum_{\alpha \in \mathscr{I}} a_{\alpha, k}$ $H_{\alpha}(\omega) \in(S)_{\rho}$, with $a_{\alpha, k}=\ll f, s_{k} H_{\alpha} \gg_{L^{2}(\Omega, \mathscr{H})} \in \mathbb{R}$ for $k \in \mathbb{N}, \alpha \in \mathscr{I}$. Similarly, a generalized random variable $F \in S(\mathscr{H})_{-\rho}$ can be expressed as

$$
F(\omega)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} b_{\alpha, k} s_{k} H_{\alpha}(\omega)=\sum_{\alpha \in \mathscr{I}} b_{\alpha} H_{\alpha}(\omega)=\sum_{k \in \mathbb{N}} b_{k}(\omega) s_{k}
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} b_{\alpha, k} s_{k} \in \mathscr{H}$ and $b_{k}(\omega)=\sum_{\alpha \in \mathscr{I}} b_{\alpha, k} H_{\alpha}(\omega) \in(S)_{-\rho}, b_{\alpha, k} \in$ $\mathbb{R}$ for $k \in \mathbb{N}$ and $\alpha \in \mathscr{I}$. The action of $F$ onto $f$ is given by $\ll F, f \gg=$ $\sum_{\alpha \in \mathscr{I}} \alpha!\left(b_{\alpha}, a_{\alpha}\right)_{\mathscr{H}}$. Since $(S)_{\rho}$ is nuclear, the following important results is valid

$$
\begin{equation*}
S(\mathscr{H})_{-\rho} \cong(S)_{-\rho} \otimes \mathscr{H} . \tag{1.21}
\end{equation*}
$$

The isomorphism (1.21) with tensor product spaces is investigated in [41]. The space $S(\mathscr{H})_{\rho}$ is a countably Hilbert space and $S(\mathscr{H})_{\rho} \subseteq L^{2}(\Omega, \mathscr{H}) \subseteq S(\mathscr{H})_{-\rho}$. An important example arises when the separable Hilbert space $\mathscr{H}$ is the space $L^{2}(\mathbb{R})$ with the Hermite functions orthonormal basis $\left\{\xi_{i}\right\}_{i \in \mathbb{N}}$.
Remark 1.6 Hilbert space valued Kondratiev type random variables modified by a sequence that belong to the space $\mathrm{Sa}(\mathscr{H})_{\rho}$ of test variables and the space $\mathrm{Sa}(\mathscr{H})_{-\rho}$ of distributions are defined in a similar way. In the convergence conditions for $\|f\|_{S a(\mathscr{H})_{\rho, p}}^{2}$ and $\|F\|_{S a(\mathscr{H})_{-\rho,-p}},(2 \mathbb{N})^{\alpha}$ is replaced by $(2 \mathbb{N} a)^{\alpha}$.

### 1.3.5 Wick Product

The problem of pointwise multiplication of generalized functions, in the framework of white noise analysis, is overcome by introducing the Wick product. Historically, the Wick product first arose in quantum physics as a renormalization operation. It is closely connected to the $\mathscr{S}$-transform [8, 15]. The most important property of the Wick multiplication is its relation to the Itô-Skorokhod integration. For more details we refer to [11, 26, 28].

The Wick product is well defined in the Hida and the Kondratiev spaces of test and generalized stochastic functions [9, 11, 16]. In [37] it is defined for stochastic test functions and distributions of exponential growth.
Definition 1.5 Let $\rho \in[0,1]$. Let $F$ and $G$ be random variables given in the forms $F(\omega)=\sum_{\alpha \in \mathscr{I}} f_{\alpha} H_{\alpha}(\omega)$ and $G(\omega)=\sum_{\beta \in \mathscr{I}} g_{\beta} H_{\beta}(\omega)$, for $f_{\alpha}, g_{\beta} \in \mathbb{R}, \alpha \in \mathscr{I}$. Their Wick product $F \diamond G$ is a random variable defined by

$$
\begin{equation*}
F \diamond G(\omega)=\sum_{\gamma \in \mathscr{\mathscr { I }}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) H_{\gamma}(\omega) . \tag{1.22}
\end{equation*}
$$

From (1.22) we obtain $H_{\alpha} \diamond H_{\beta}=H_{\alpha+\beta}$ for $\alpha, \beta \in \mathscr{I}$ and particularly for $j, k \in \mathbb{N}$ it holds $H_{\varepsilon^{(k)}} \diamond H_{\varepsilon^{(j)}}=H_{\varepsilon^{(k)}+\varepsilon^{(j)}}$. The Wick product can be interpreted as a stochastic convolution. It also represents a renormalization of the ordinary product and the highest order stochastic approximation of the ordinary product [32]. Note here that the space $L^{2}(\mu)$ is not closed under the Wick multiplication.
Example 1.4 The random variable $F(\omega)=\sum_{n=1}^{\infty} \frac{1}{n} H_{\varepsilon^{(n)}}(\omega)$ belongs to $L^{2}(\mu)$ since $\|F\|_{L^{2}(\Omega)}^{2}=\sum_{n=1}^{\infty} \frac{1}{n^{2}}<\infty$. The Wick product $F \diamond F$, also called the Wick square $F^{\diamond 2}$, is not an element of $L^{2}(\mu)$ since

$$
\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} \frac{1}{n(n-k)}\right)^{2} \geq \sum_{k=1}^{\infty} \frac{1}{k(k+1)}=\sum_{k=1}^{\infty}\left(1-\frac{1}{k+1}\right)=+\infty .
$$

In [11] it was proven that the spaces $(S)_{0},(S)_{-0},(S)_{1}$ and $(S)_{-1}$ are closed under the Wick multiplication. In the following theorem we prove that this also holds for the spaces $(S)_{\rho}$ and $(S)_{-\rho}$ for any $\rho \in[0,1]$.

Theorem 1.5 Let $\rho \in[0,1]$. The Kondratiev spaces $(S)_{\rho}$ and $(S)_{-\rho}$ are closed under the Wick multiplication.
Proof Let $F, G \in(S)_{-\rho}$. Then, $F(\omega)=\sum_{\alpha \in \mathscr{G}} f_{\alpha} H_{\alpha}(\omega)$ and $G(\omega)=\sum_{\beta \in \mathscr{\mathscr { F }}} g_{\beta}$ $H_{\beta}(\omega)$ for $f_{\alpha}, g_{\beta} \in \mathbb{R}, \alpha, \beta \in \mathscr{I}$ and from (1.16) there exist $p_{1}, p_{2} \geq 0$ such that

$$
\|F\|_{-\rho,-p_{1}}^{2}=\sum_{\alpha \in \mathscr{\mathscr { I }}} \alpha!^{1-\rho} f_{\alpha}^{2}(2 \mathbb{N})^{-p_{1} \alpha}<\infty,\|G\|_{-\rho,-p_{2}}^{2}=\sum_{\beta \in \mathscr{\mathscr { I }}} \beta!^{1-\rho} g_{\beta}^{2}(2 \mathbb{N})^{-p \alpha}<\infty .
$$

The Wick product is given by $F \diamond G(\omega)=\sum_{\gamma \in \mathscr{I}} c_{\gamma} H_{\gamma}(\omega)$ with $c_{\gamma}=\sum_{\alpha+\beta=\gamma}$ $f_{\alpha} g_{\beta}$. Then, for $q \geq p_{1}+p_{2}+3-\rho$ we obtain

$$
\begin{aligned}
& \|F \diamond G\|_{-\rho,-q}^{2}=\sum_{\gamma \in \mathscr{I}} \gamma!^{1-\rho} c_{\gamma}^{2}(2 \mathbb{N})^{-q \gamma} \leq \sum_{\gamma \in \mathscr{I}} r^{1-\rho}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right)^{2}(2 \mathbb{N})^{-\left(p_{1}+p_{2}+3-\rho\right) \gamma} \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} \gamma!^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{1}}{2} \alpha}(2 \mathbb{N})^{-\frac{p_{2}}{2} \beta}(2 \mathbb{N})^{-\frac{(1-\rho)}{2} \gamma}\right)^{2} \\
& \leq \sum_{\gamma \in \mathscr{\mathscr { I }}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\left(\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}\right)^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{1}}{2} \alpha}(2 \mathbb{N})^{-\frac{p_{2}}{2} \beta}(2 \mathbb{N})^{\left.-\frac{(1-\rho)}{2}(\alpha+\beta)\right)^{2}}\right. \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma} \alpha!^{1-\rho} f_{\alpha}^{2}(2 \mathbb{N})^{-p_{1} \alpha}\right)\left(\sum_{\alpha+\beta=\gamma} \beta!^{1-\rho} g_{\beta}^{2}(2 \mathbb{N})^{-p_{2} \alpha}\right) \\
& \leq m \cdot\|F\|_{-\rho,-p_{1}}^{2} \cdot\|G\|_{-\rho,-p_{2}}^{2}<\infty,
\end{aligned}
$$

where $m=\sum_{\gamma \in \mathscr{\mathscr { H }}}(2 \mathbb{N})^{-2 \gamma}<\infty$ by (1.9). First we used $\alpha \leq \gamma, \beta \leq \gamma$, then applied the inequality $\gamma!=(\alpha+\beta)!\leq \alpha!\beta!(2 \mathbb{N})^{\gamma}$ proved in Lemma 1.1 part $2^{\circ}$ and at last used the Cauchy-Schwarz inequality.

Assume now $F, G \in(S)_{\rho}$. By (1.15) for $p_{1}, p_{2} \geq 0$ the estimates hold

$$
\|F\|_{\rho, p_{1}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho} f_{\alpha}^{2}(2 \mathbb{N})^{p_{1} \alpha}<\infty, \quad\|G\|_{\rho, p_{2}}^{2}=\sum_{\beta \in \mathscr{\mathscr { I }}} \beta!^{1+\rho} g_{\beta}^{2}(2 \mathbb{N})^{p_{2} \alpha}<\infty .
$$

Then, $\|F \diamond G\|_{\rho, q}^{2}=\sum_{\gamma \in \mathscr{\mathscr { I }}} \gamma!^{1+\rho}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right)^{2}(2 \mathbb{N})^{q \gamma}$ is finite since

$$
\begin{aligned}
& \| F \diamond G \|_{\rho, q}^{2}=\sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma} \gamma!^{\frac{1+\rho}{2}} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2} \gamma}\right)^{2} \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma}\left(\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}\right)^{\frac{1+\rho}{2}} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2}(\alpha+\beta)}\right)^{2} \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma} \alpha!^{1+\rho} f_{\alpha}^{2}(2 \mathbb{N})^{(q+3+\rho) \alpha}\right)\left(\sum_{\alpha+\beta=\gamma} \beta!^{1+\rho} g_{\beta}^{2}(2 \mathbb{N})^{(q+3+\rho) \beta}\right) \\
& \quad \leq m \cdot\|F\|_{\rho, p_{1}}^{2} \cdot\|G\|_{\rho, p_{2}}^{2}<\infty,
\end{aligned}
$$

for $q+3+\rho \leq \min \left\{p_{1}, p_{2}\right\}$, where $m=\sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}<\infty$.
If $F$ is a deterministic function then the Wick product $F \diamond G$ reduces to the ordinary product $F \cdot G$. This follows from the property $\left.H_{0}\right\rangle H_{\beta}=H_{0} \cdot H_{\beta}$, for $\beta \in \mathscr{I}$. The Wick product is a commutative, associative operation, distributive with respect to addition. Moreover, whenever $F, G$ and $F \diamond G$ are $\mu$-integrable it holds

$$
\begin{equation*}
\mathbb{E}(F \diamond G)=\mathbb{E} F \cdot \mathbb{E} G, \tag{1.23}
\end{equation*}
$$

where the independence of $F$ and $G$ is not required [11].
Definition 1.6 Let $\rho \in[0,1]$. The Wick powers of $F \in(S)_{-\rho}$ are defined inductively

$$
F^{\diamond 0}=1, \quad F^{\diamond k}=F \diamond F^{\diamond(k-1)}, \quad k \in \mathbb{N} .
$$

Definition 1.7 Let $P_{m}(x)=\sum_{k=0}^{m} p_{k} x^{k}, p_{m} \in \mathbb{R}, p_{m} \neq 0, x \in \mathbb{R}$ be a polynomial of degree $m$ with real coefficients. The Wick version $P_{m}^{\diamond}:(S)_{-\rho} \rightarrow(S)_{-\rho}$ of the polynomial $P_{m}$ is defined by

$$
P_{m}^{\diamond}(F)=\sum_{k=0}^{m} p_{k} F^{\diamond k}, \quad \text { for } F \in(S)_{-\rho}
$$

For $F, G \in(S)_{-\rho}$ and $n \in \mathbb{N}$ the element $(F+G)^{\diamond n}$ also belongs to $(S)_{-\rho}$ and the binomial formula holds

$$
(F+G)^{\diamond n}=\sum_{k=0}^{n}\binom{n}{k} F^{\diamond k} \diamond G^{\diamond(n-k)} .
$$

Particularly, if $n=2$ we obtain $(F+G)^{\diamond 2}=F^{\diamond 2}+2 F \diamond G+G^{\diamond 2}$. Note that all analytic functions have their Wick versions. For example, the Wick exponential of $F \in(S)_{-\rho}$ is defined as a formal sum

$$
\begin{equation*}
\exp ^{\diamond} F=\sum_{n=0}^{\infty} \frac{F^{\diamond n}}{n!}, \tag{1.24}
\end{equation*}
$$

while the Wick versions of trigonometric functions are defined as formal sums

$$
\begin{equation*}
\sin ^{\diamond} F=\sum_{n=1}^{\infty}(-1)^{k-1} \frac{F^{\diamond(2 k-1)}}{(2 k-1)!}, \quad \cos ^{\diamond} F=\sum_{n=0}^{\infty}(-1)^{k} \frac{F^{\diamond 2 k}}{(2 k)!} . \tag{1.25}
\end{equation*}
$$

For $F, G \in(S)_{-\rho}$ by (1.23), (1.24) and (1.25) we obtain $\mathbb{E}\left(\exp ^{\diamond} F\right)=\exp (\mathbb{E} F)$, $\mathbb{E}\left(\sin ^{\diamond} F\right)=\sin (\mathbb{E} F)$ and $\mathbb{E}\left(\cos ^{\diamond} F\right)=\cos (\mathbb{E} F)$. Moreover, the element $\exp ^{\diamond}(F+G) \in(S)_{-\rho}$ and $\exp ^{\diamond}(F+G)=\exp ^{\diamond} F \diamond \exp ^{\diamond} G$, see [11].

The most important property of the Wick multiplication is its relation to the ItôSkorokhod integration, also known as the fundamental theorem of calculus [9, 11], which will be discussed in Chap. 2.

### 1.3.6 Fractional Gaussian White Noise Space

In [7] the authors developed fractional white noise theory for Hurst parameter $H \in(0,1)$. They introduced the fractional transform operator $M^{(H)}$ which connects fractional Brownian motion $b_{t}^{(H)}$ and standard Brownian motion $b_{t}$ on the white noise probability space $\left(S^{\prime}(\mathbb{R}), \mathscr{B}, \mu\right)$. We extend these results for $\mathscr{H}$-valued Brownian motion $B_{t}$ and $\mathscr{H}$-valued white noise $W_{t}$ and their corresponding fractional versions $B_{t}^{(H)}$ and $W_{t}^{(H)}$.

Definition 1.8 ([7]) Let $H \in(0,1)$. The fractional transform operator $M^{(H)}$ : $S(\mathbb{R}) \rightarrow L^{2}(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ is defined by

$$
\begin{equation*}
\widehat{M^{(H)} f}(y)=|y|^{\frac{1}{2}-H} \widehat{f}(y), \quad y \in \mathbb{R}, \quad f \in S(\mathbb{R}) \tag{1.26}
\end{equation*}
$$

where $\widehat{f}(y)=\int_{\mathbb{R}} e^{-i x y} f(x) d x$ denotes the Fourier transform of $f$.
Equivalently, the operator $M^{(H)}$ for all $H \in(0,1)$ can be defined as a constant multiple of

$$
\begin{equation*}
-\frac{d}{d x} \int_{\mathbb{R}}(t-x)|t-x|^{H-\frac{3}{2}} f(t) d t \tag{1.27}
\end{equation*}
$$

where the constant is chosen so that (1.26) holds. The operator $M^{(H)}$ has the structure of a convolution operator. Particularly, from (1.27) it follows that for $H \in\left(0, \frac{1}{2}\right)$ the fractional operator is of the form $M^{(H)} f(x)=C_{H} \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}} d t$, then for $H \in\left(\frac{1}{2}, 1\right)$ it is of the form $M^{(H)} f(x)=C_{H} \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}} d t$ and for $H=\frac{1}{2}$ it reduces to the identity operator, i.e., $M^{\left(\frac{1}{2}\right)} f(x)=f(x)$. The normalizing constant is $C_{H}=\left(2 \Gamma\left(H-\frac{1}{2}\right) \cos \left(\frac{\pi}{2}\left(H-\frac{1}{2}\right)\right)\right)^{-1}$ and $\Gamma$ is the Gamma function.

From (1.26) we have that the inverse fractional transform operator of $M^{(H)}$ is the operator $M^{(1-H)}$ defined by $\widehat{M^{(1-H)}} f(y)=|y|^{H-\frac{1}{2}} \widehat{f}(y)$ for $y \in \mathbb{R}, f \in S(\mathbb{R})$.

Denote by

$$
L_{H}^{2}(\mathbb{R})=\left\{f: \mathbb{R} \rightarrow \mathbb{R} ; M^{(H)} f(x) \in L^{2}(\mathbb{R})\right\}
$$

the closure of $S(\mathbb{R})$ with respect to the norm $\|f\|_{L_{H}^{2}(\mathbb{R})}=\left\|M^{(H)} f\right\|_{L^{2}(\mathbb{R})}$ for $f \in S(\mathbb{R})$ induced by the inner product

$$
(f, g)_{L_{H}^{2}(\mathbb{R})}=\left(M^{(H)} f, M^{(H)} g\right)_{L^{2}(\mathbb{R})} .
$$

The operator $M^{(H)}$ is a self-adjoint operator and for $f, g \in L^{2}(\mathbb{R}) \cap L_{H}^{2}(\mathbb{R})$ we have $\left(f, M^{(H)} g\right)_{L_{H}^{2}(\mathbb{R})}=\left(\widehat{f}, \widehat{M^{(H)} g}\right)_{L^{2}(\mathbb{R})}=\int_{\mathbb{R}}|y|^{\frac{1}{2}-H} \widehat{f}(y) \widehat{g}(y) d y=\left(M^{(H)} f, g\right)_{L_{H}^{2}(\mathbb{R})}$.

Remark 1.7 For fixed $H \in\left(\frac{1}{2}, 1\right)$ we denote by $\phi(s, t)=H(2 H-1)|s-t|^{2 H-2}$, $s, t \in \mathbb{R}$. Then,

$$
\begin{equation*}
\int_{\mathbb{R}}\left(M^{(H)} f(x)\right)^{2} d x=c_{H} \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) d s d t \tag{1.28}
\end{equation*}
$$

with constant $c_{H}$. The property (1.28) was used in $[7,10,18,31]$ in order to adapt the classical white noise calculus to the fractional one.

Theorem 1.6 ( $[4,7])$ Let $M^{(H)}: L_{H}^{2}(\mathbb{R}) \rightarrow L^{2}(\mathbb{R})$ defined by $(1.26)$ be the extension of the operator $M^{(H)}$ from Definition 1.8. Then, $M^{(H)}$ is an isometry between the two Hilbert spaces $L^{2}(\mathbb{R})$ and $L_{H}^{2}(\mathbb{R})$. The functions

$$
\begin{equation*}
e_{n}^{(H)}(x)=M^{(1-H)} \xi_{n}(x), \quad n \in \mathbb{N}, \tag{1.29}
\end{equation*}
$$

belong to $S(\mathbb{R})$ and form an orthonormal basis in $L_{H}^{2}(\mathbb{R})$.
From (1.29) it also follows that $e_{n}^{(1-H)}(x)=M^{(H)} \xi_{n}(x), n \in \mathbb{N}$, where we used the fact that $M^{(1-H)}$ is the inverse operator of the operator $M^{(H)}$. Following [4, 7] we extend $M^{(H)}$ onto $S^{\prime}(\mathbb{R})$ and define the fractional operator $M^{(H)}: S^{\prime}(\mathbb{R}) \rightarrow S^{\prime}(\mathbb{R})$ by $\left\langle M^{(H)} \omega, f\right\rangle=\left\langle\omega, M^{(H)} f\right\rangle$ for $f \in S(\mathbb{R}), \omega \in S^{\prime}(\mathbb{R})$.

We denote by

$$
L^{2}\left(\mu_{H}\right)=L^{2}\left(\mu \circ M^{(1-H)}\right)=\left\{G: \Omega \rightarrow \mathbb{R} ; G \circ M^{(H)} \in L^{2}(\mu)\right\}, \quad H \in(0,1)
$$

the stochastic analogue of $L_{H}^{2}(\mathbb{R})$, see [3]. It is the space of square integrable functions on $S^{\prime}(\mathbb{R})$ with respect to fractional Gaussian white noise measure $\mu_{H}$. Thus, the triple $\left(S^{\prime}(\mathbb{R}), \mathscr{B}, \mu_{H}\right)$ denotes the fractional Gaussian white noise space.

Since $G \in L^{2}\left(\mu_{H}\right)$ if and only if $G \circ M^{(H)} \in L^{2}(\mu)$, it follows that $G$ has the chaos expansion of the form

$$
\begin{aligned}
G\left(M^{(H)} \omega\right) & =\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}(\omega)=\sum_{\alpha \in \mathscr{I}} c_{\alpha} \prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle\omega, \xi_{i}\right\rangle\right) \\
& =\sum_{\alpha \in \mathscr{I}} c_{\alpha} \prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle\omega, M^{(H)} e_{i}^{(H)}\right\rangle\right)=\sum_{\alpha \in \mathscr{I}} c_{\alpha} \prod_{i=1}^{\infty} h_{\alpha_{i}}\left(\left\langle M^{(H)} \omega, e_{i}^{(H)}\right\rangle\right) .
\end{aligned}
$$

Definition 1.9 For a given $\alpha \in \mathscr{I}$ the $\alpha$-th fractional Fourier-Hermite polynomial is defined by

$$
\begin{equation*}
\widetilde{H}_{\alpha}(\omega)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, e_{k}^{(H)}\right\rangle\right), \quad \alpha \in \mathscr{I} . \tag{1.30}
\end{equation*}
$$

The family $\left\{\widetilde{H}_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ forms an orthogonal basis of $L^{2}\left(\mu_{H}\right)$ and for all $\alpha \in \mathscr{I}$ it holds $\left\|\widetilde{H}_{\alpha}\right\|_{L^{2}\left(\mu_{H}\right)}^{2}=\alpha$ !. Therefore, Theorem 1.3 can be formulated for fractional square integrable random variables.

Theorem 1.7 Each $G \in L^{2}\left(\mu_{H}\right)$ can be uniquely represented in the form

$$
G(\omega)=\sum_{\alpha \in \mathscr{I}} c_{\alpha} \widetilde{H}_{\alpha}(\omega), \quad c_{\alpha} \in \mathbb{R}, \alpha \in \mathscr{I}
$$

such that $\|G\|_{L^{2}\left(\mu_{H}\right)}^{2}=\sum_{\alpha \in \mathscr{I}} c_{\alpha}^{2} \alpha!$ is finite and $\|G\|_{L^{2}\left(\mu_{H}\right)}=\left\|G \circ M^{(H)}\right\|_{L^{2}(\mu)}$ holds.
Let $\rho \in[0,1]$. The fractional Kondratiev spaces $(S)_{\rho}^{(H)}$ and $(S)_{-\rho}^{(H)}$ are defined in an analogous way as the spaces $(S)_{\rho}$ and $(S)_{-\rho}$ in Sect. 1.3.3.

Definition 1.10 Let $\rho \in[0,1]$ and let $F=\sum_{\alpha \in \mathscr{I}} b_{\alpha} \widetilde{H}_{\alpha}, f_{\alpha} \in \mathbb{R}, \alpha \in \mathscr{I}$. The Kondratiev space of fractional random variables $(S)_{\rho}^{(H)}$ is defined as the projective limit $(S)_{\rho}^{(H)}=\bigcap_{p \in \mathbb{N}_{0}}(S)_{\rho, p}$ of the family $(S)_{\rho, p}^{(H)}=\left\{F \in L^{2}\left(\mu_{H}\right):\|F\|_{\rho, p}^{2}<\infty\right\}$, $p \in \mathbb{N}_{0}$. The Kondratiev space of fractional generalized random variables $(S)_{-\rho}^{(H)}$ is constructed as the inductive limit $(S)_{-\rho}^{(H)}=\bigcup_{p \in \mathbb{N}_{0}}(S)_{-\rho,-p}^{(H)}$ of the family $(S)_{-\rho,-p}^{(H)}=$ $\left\{F:\|F\|_{-\rho,-p}^{2}<\infty\right\}, p \in \mathbb{N}_{0}$.

Similarly as in Sect. 1.3.4 one can define spaces of $\mathscr{H}$-valued fractional test and generalized random variables. Moreover the Wick product in $(S)_{\rho}^{(H)}$ and $(S)_{-\rho}^{(H)}$ is defined as in Definition 1.5, where $H_{\alpha}$ is replaced by $\widetilde{H}_{\alpha}$.

### 1.4 Stochastic Processes

In this section we characterize different classes of stochastic processes in terms of chaos expansion representation forms. Particularly, beside classical processes (those with finite variances) we deal with test and generalized stochastic processes.

Recall, a real valued (classical) stochastic process can be defined as a family of functions $v: T \times \Omega \rightarrow \mathbb{R}$ such that for each fixed $t \in T, v(t, \cdot)$ is an $\mathbb{R}$-valued random variable and for each fixed $\omega \in \Omega, v(\cdot, \omega)$ is an $\mathbb{R}$-valued deterministic function, called trajectory. Here, following [37], we generalize the definition of classical stochastic processes and define test and generalized stochastic processes. By replacing the space of trajectories with a space of deterministic generalized functions, or by replacing the space of random variables with a space of generalized random variables, different types of generalized stochastic processes can be obtained. In this manner, processes generalized with respect to the $t$ argument, the $\omega$ argument, or even with respect to both arguments can be obtained [11, 37]. A very general concept of generalized stochastic processes, based on chaos expansions was introduced in [37] and further developed in [19, 20, 23]. In [11] generalized stochastic processes are defined as measurable mappings $I \rightarrow(S)_{-1}$ (one can consider also other spaces of generalized random variables instead). Thus, they are defined pointwise with respect to the parameter $t \in I$ and generalized with respect to $\omega \in \Omega$. Detailed survey on generalization of classical stochastic processes is given in [37], where several classes of generalized stochastic processes were distinguished and represented in appropriate chaos expansions. Here we will consider a class of generalized stochastic process to be wider than the one in [11]. We follow [20, 23, 37, 41] to define such processes and characterize them in terms of chaos expansion representations in orthogonal polynomial basis.

In [37] the authors considered generalized stochastic processes to be linear and continuous mappings from a certain space of deterministic functions $\mathscr{T}$ into the Kondratiev space $(S)_{-1}$, i.e., elements of $\mathscr{L}\left(\mathscr{T},(S)_{-1}\right)$ and proved expansion theorems. Particularly, this class of generalized processes contains stochastic processes with coefficients in the Schwartz space of tempered distributions $S^{\prime}(\mathbb{R})$. Following these ideas, we consider generalized stochastic processes to be elements of a tensor product space of the form $X \otimes(S)_{-\rho}, \rho \in[0,1]$, where $X$ is a Banach space.

### 1.4.1 Chaos Expansion Representation of Stochastic Processes

Let $\tilde{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\tilde{X}}$ and let $\tilde{X}^{\prime}$ denote its dual space. In this section we describe $\tilde{X}$-valued random variables. Most notably, if $\tilde{X}$ is a space of functions on $\mathbb{R}$, e.g. $\tilde{X}=C^{k}([0, T]), T>0$ or $\tilde{X}=L^{2}(\mathbb{R})$, we obtain the notion of a stochastic process. We will also define processes where $\tilde{X}$ is not a normed space, but a nuclear space topologized by a family of seminorms, e.g. $\tilde{X}=S(\mathbb{R})$, see [23, 37].

Definition 1.11 Let a process $u$ has the formal expansion

$$
\begin{equation*}
u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}, \quad \text { where } u_{\alpha} \in X, \alpha \in \mathscr{I} \tag{1.31}
\end{equation*}
$$

Let $\rho \in[0,1]$. We define the following spaces:

$$
\begin{gather*}
X \otimes(S)_{\rho, p}=\left\{u:\|u\|_{X \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\},  \tag{1.32}\\
X \otimes(S)_{-\rho,-p}=\left\{u:\|u\|_{X \otimes(S)_{-\rho,-p}^{2}}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\},
\end{gather*}
$$

where $X$ denotes an arbitrary Banach space (allowing both possibilities $X=\tilde{X}$, $\left.X=\tilde{X}^{\prime}\right)$. Especially, for $\rho=0$ and $p=0$ we denote by $X \otimes(S)_{0,0}$ the space

$$
X \otimes L^{2}(\mu)=\left\{u:\|u\|_{X \otimes L^{2}(\mu)}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\} .
$$

Definition 1.12 Test stochastic processes and generalized stochastic processes in Kondratiev sense are respectively elements of the spaces

$$
X \otimes(S)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}} X \otimes(S)_{\rho, p}, \quad X \otimes(S)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}} X \otimes(S)_{-\rho,-p}, \quad \rho \in[0,1] .
$$

Remark 1.8 The symbol $\otimes$ denotes the projective tensor product of two spaces, i.e., $\tilde{X}^{\prime} \otimes(S)_{-\rho}$ is the completion of the tensor product with respect to the $\pi$-topology. The Kondratiev space $(S)_{\rho}$ is nuclear and thus $\left(\tilde{X} \otimes(S)_{\rho}\right)^{\prime} \cong \tilde{X}^{\prime} \otimes(S)_{-\rho}$. Note that $\tilde{X}^{\prime} \otimes(S)_{-\rho}$ is isomorphic to the space of linear bounded mappings $\tilde{X} \rightarrow(S)_{-\rho}$, and it is also isomporphic to the space of linear bounded mappings $(S)_{\rho} \rightarrow \tilde{X}^{\prime}$. More details can be found in [23, 37, 43].

The action of a generalized stochastic process $u$, represented in the form (1.31), onto a test function $\varphi \in X$ gives a generalized random variable from the space $(S)_{-\rho}$

$$
\ll u, \varphi \gg=\sum_{\alpha \in \mathscr{I}}\left\langle u_{\alpha}, \varphi\right\rangle H_{\alpha} \in(S)_{-\rho}
$$

and the action of such process $u$ onto a test random variable from the Kondratiev space $\theta \in(S)_{\rho}$ gives a generalized deterministic function in $\tilde{X}^{\prime}$

$$
\langle u, \theta\rangle=\sum_{\alpha \in \mathscr{I}} \ll H_{\alpha}, \theta \gg u_{\alpha} \in \tilde{X}^{\prime}
$$

The most common examples used in applications are Schwartz spaces $X=S(\mathbb{R})$ and $X^{\prime}=S^{\prime}(\mathbb{R})$, distributions with compact support $X=\mathscr{E}(\mathbb{R})$ and $X^{\prime}=\mathscr{E}^{\prime}(\mathbb{R})$, the Sobolev spaces $X=W_{0}^{1,2}(\mathbb{R})$ and $X^{\prime}=W^{-1,2}(\mathbb{R})$. In applications of fluid flows, the space $X$ is one of the Sobolev-Bochner spaces $L^{2}([0, T], \mathscr{V})$ and $L^{2}\left([0, T], \mathscr{V}^{*}\right)$, where $\mathscr{V}$ is a real, separable, and reflexive Banach space such that we have a Gel'fand triple of the form $\mathscr{V} \subseteq \mathscr{H} \subseteq \mathscr{V}^{*}$ and $\mathscr{H}$ is a real Hilbert space [1].

Example 1.5 If $X=S(\mathbb{R})$ then for $\theta=\sum_{\beta \in \mathscr{I}} \theta_{\beta} H_{\beta} \in(S)_{\rho}$ the action of process $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} H_{\alpha}$, for $u_{\alpha}=\sum_{k \in \mathbb{N}} u_{\alpha, k} \xi_{k}(t) \in S^{\prime}(\mathbb{R})$ on $\theta=\sum_{\beta \in \mathscr{I}} \theta_{\beta} H_{\beta}$ is given by

$$
\begin{aligned}
\langle u, \theta\rangle & =\sum_{\alpha \in \mathscr{I}} u_{\alpha} \ll H_{\alpha}, \sum_{\beta \in \mathscr{I}} \theta_{\beta} H_{\beta} \gg=\sum_{\alpha \in \mathscr{I}} \theta_{\alpha} u_{\alpha} \alpha! \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!\theta_{\alpha} u_{\alpha, k} \xi_{k}(t)=\sum_{k \in \mathbb{N}}\left(\sum_{\alpha \in \mathscr{I}} \alpha!\theta_{\alpha} u_{\alpha, k}\right) \xi_{k}(t)
\end{aligned}
$$

A generalized stochastic processes $u$ which belong to $X \otimes(S)_{-\rho}$ given by (1.31) can be written in the form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}=u_{\mathbf{0}}+\sum_{k \in \mathbb{N}} u_{\varepsilon^{(k)}} \otimes H_{\varepsilon^{(k)}}+\sum_{|\alpha|>1} u_{\alpha} \otimes H_{\alpha} \tag{1.33}
\end{equation*}
$$

such that the coefficients $u_{\alpha} \in X$ for some $p \in \mathbb{N}_{0}$ satisfy

$$
\begin{equation*}
\|u\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{1.34}
\end{equation*}
$$

The value $p$ corresponds to the level of singularity of the process $u$. Note that the deterministic part of $u$ in (1.33) is the coefficient $u_{\mathbf{0}}$, which represents the generalized expectation of $u$. In this way, by representing stochastic processes in their polynomial chaos expansion forms, we are able to separate the deterministic component from the randomness of the process.

Example 1.6 If $X=L^{2}(\mathbb{R})$, then $u \in L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$ is given in the chaos expansion form $u(t, \omega)=\sum_{\alpha \in \mathscr{I}} u_{\alpha}(t) H_{\alpha}(\omega), t \in \mathbb{R}, \omega \in \Omega$ such that

$$
\|u\|_{L^{2}(\mathbb{R}) \otimes L^{2}(\mu)}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}(\mathbb{R})}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!\int_{\mathbb{R}}\left|u_{\alpha}(t)\right|^{2} d t<\infty
$$

Example 1.7 The chaos expansion representation of a Brownian motion $b_{t}(\omega)$, for $\omega \in S^{\prime}(\mathbb{R}), t \geq 0$, considered in Example 1.1, is given by

$$
\begin{equation*}
b_{t}(\omega)=\sum_{k \in \mathbb{N}} \int_{0}^{t} \xi_{k}(s) \mathrm{d} s H_{\mathcal{E}^{(k)}}(\omega) \tag{1.35}
\end{equation*}
$$

such that for all $k \in \mathbb{N}$ the coefficients $\int_{0}^{t} \xi_{k}(s) d s$ are in $C^{\infty}(\mathbb{R})$. Moreover, for fixed $t, b_{t}$ is an element of $L^{2}(\mu)$.

Example 1.8 Singular white noise is defined by the formal chaos expansion

$$
\begin{equation*}
w_{t}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega) \tag{1.36}
\end{equation*}
$$

Since $\sum_{k=1}^{\infty}\left|\xi_{k}(t)\right|^{2}>\sum_{k=1}^{\infty} \frac{1}{k}=\infty$ and $\sum_{k=1}^{\infty}\left|\xi_{k}(t)\right|^{2}(2 k)^{-p}<\infty$ holds for $p>1$ it follows that it is an element of the space $C^{\infty}(\mathbb{R}) \otimes(S)_{0,-p}$ for $p>1$. Moreover, since the inclusions (1.17) hold, it can be considered to be an element of $C^{\infty}(\mathbb{R}) \otimes(S)_{-\rho,-p}$ for all $\rho \in[0,1]$ and $p>1$. With weak derivatives in the $(S)_{-\rho}$ sense, it holds that $\frac{\mathrm{d}}{\mathrm{d} t} b_{t}=w_{t}$. Both, Brownian motion and singular white noise are Gaussian processes and have chaos expansion representations given in terms of the Fourier-Hermite polynomials with multi-indices of length one, i.e., they both belong to the Wiener chaos space of order one.

Example 1.9 Let $X$ be a Banach space. A generalized Gaussian process in $X \otimes(S)_{-\rho}$, which belongs to the Wiener chaos space of order one, is given in the form

$$
\begin{equation*}
G=\sum_{k=1}^{\infty} m_{k} H_{\varepsilon^{(k)}} \tag{1.37}
\end{equation*}
$$

with the coefficients $m_{k} \in X$ such that for some $p \in \mathbb{N}_{0}$

$$
\begin{equation*}
\sum_{k=1}^{\infty}\left\|m_{k}\right\|_{X}^{2}(2 k)^{-p}<\infty \tag{1.38}
\end{equation*}
$$

Particularly, the chaos expansion form (1.37) for $G_{t} \in S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ in the Wiener chaos space of order one transforms to

$$
G_{t}(\omega)=\sum_{k=1}^{\infty} m_{k}(t) H_{\varepsilon^{(k)}}(\omega)=\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} m_{k n} \xi_{n}(t) H_{\varepsilon^{(k)}}(\omega)
$$

and the condition (1.38) modifies to

$$
\sum_{k=1}^{\infty}\left\|m_{k}\right\|_{-l}^{2}(2 k)^{-p}=\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} m_{k n}^{2}(2 n)^{-l}(2 k)^{-p}<\infty
$$

for some $l, p \in \mathbb{N}_{0}$. Here, the coefficients $m_{k}$ are deterministic generalized functions represented as formal sums $m_{k}(t)=\sum_{n=1}^{\infty} m_{k n} \xi_{n}(t), k \in \mathbb{N}$, with $m_{k n} \in \mathbb{R}$.
Example 1.10 A real valued fractional Brownian motion $b_{t}^{(H)}(\omega), H \in(0,1)$ is given by

$$
\begin{equation*}
b_{t}^{(H)}(\omega)=\sum_{k=1}^{\infty}\left(\int_{0}^{t} \xi_{k}(s) d s\right) \widetilde{H}_{\varepsilon^{(k)}}(\omega) . \tag{1.39}
\end{equation*}
$$

For $t$ fixed $b_{t}^{(H)}$ is an element of $L^{2}\left(\mu_{H}\right)$, see [19]. Recall, fractional Brownian motion is a Gaussian process with zero expectation and the covariance function

$$
\begin{equation*}
\mathbb{E}\left(b_{s}^{(H)} b_{t}^{(H)}\right)=\frac{1}{2}\left(|t|^{2 H}+|s|^{2 H}-|t-s|^{2 H}\right), \quad s, t \in \mathbb{R} \tag{1.40}
\end{equation*}
$$

It is one-parameter extension of a Brownian motion and it depends on the Hurst index $H$. Fractional Brownian motion is a centered Gaussian process with dependent increments which have long-range dependence, modified by the Hurst parameter, and self-similarity properties. For $H=\frac{1}{2}$ the covariance function can be written as $\mathbb{E}\left(b_{t}^{\left(\frac{1}{2}\right)} b_{s}^{\left(\frac{1}{2}\right)}\right)=\min \{s, t\}$ and the process $b_{t}^{\left(\frac{1}{2}\right)}$ reduces to $b_{t}$, which has independent increments. Moreover, for $H \neq \frac{1}{2}$ fractional Brownian motion is neither a semimartingale nor a Markov process. In addition, it holds

$$
\mathbb{E}\left(b_{t}^{(H)}-b_{s}^{(H)}\right)^{2}=|t-s|^{2 H} .
$$

According to the Kolmogorov continuity criterion fractional Brownian motion $b^{(H)}$ has a continuous modification. The parameter $H$ controls the regularity of trajectories. The covariance function (1.40) is homogeneous of order $2 H$, thus fractional Brownian motion is an $H$ self-similar process, i.e., $b_{k t}^{(H)}=k^{H} b_{t}^{(H)}, k>0$. For any $n \in \mathbb{Z}, n \neq 0$ the autocovariance function is given by

$$
\begin{aligned}
r(n)=\mathbb{E}\left[b_{1}^{(H)}\left(b_{n+1}^{(H)}-b_{n}^{(H)}\right)\right] & =H(2 H-1) \int_{0}^{1} \int_{n}^{n+1}(u-v)^{2 H-2} d u d v \\
& \sim H(2 H-1)|n|^{2 H-1}, \quad \text { when }|n| \rightarrow \infty .
\end{aligned}
$$

Therefore, the increments are positively correlated for $H \in\left(\frac{1}{2}, 1\right)$ and negatively correlated for $H \in\left(0, \frac{1}{2}\right)$. More precisely, for $H \in\left(\frac{1}{2}, 1\right)$ fractional Brownian motion has the long-range dependence property $\sum_{n=1}^{\infty} r(n)=\infty$ and for $H \in\left(0, \frac{1}{2}\right)$ the short-range property $\sum_{n=1}^{\infty}|r(n)|<\infty$. For applications we refer to [3, 10, 31, 33, 35, 40].

Remark 1.9 Generalized stochastic processes as elements of $X \otimes(S a)_{-\rho}$. Similarly to Definition 1.12, one can define stochastic processes which are elements of $X \otimes$ $(S a)_{-\rho}$ and $X \otimes(S a)_{\rho}, \rho \in[0,1]$. We denote by

$$
\begin{aligned}
X \otimes(S a)_{\rho, p} & =\left\{u:\|u\|_{X \otimes(S a)_{\rho, p}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 a \mathbb{N})^{p \alpha}<\infty\right\}, \\
X \otimes(S a)_{-\rho,-p} & =\left\{u:\|u\|_{X \otimes(S a)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 a \mathbb{N})^{-p \alpha}<\infty\right\} .
\end{aligned}
$$

Then, test and generalized stochastic processes of the Kondratiev type modified by a sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1, k \in \mathbb{N}$ are elements of the spaces respectively

$$
X \otimes(S a)_{\rho}=\bigcap_{p \in \mathbb{N}_{0}} X \otimes(S a)_{\rho, p} \quad \text { and } \quad X \otimes(S a)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}} X \otimes(S a)_{-\rho,-p}
$$

Therefore, these processes have the chaos expansion form (1.31) such that for some $p \in \mathbb{N}_{0}$ it holds

$$
\|u\|_{X \otimes(S a)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}}\left\|u_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N} a)^{-p \alpha}<\infty .
$$

Definition 1.13 Let $\widetilde{v}$ have the formal expansion

$$
\begin{equation*}
\tilde{v}=\sum_{\alpha \in \mathscr{I}} \tilde{v}_{\alpha} \otimes \widetilde{H}_{\alpha}, \quad \text { where } \tilde{v}_{\alpha} \in X, \alpha \in \mathscr{I} \tag{1.41}
\end{equation*}
$$

Let $\rho \in[0,1]$ and let $X$ be an arbitrary Banach space. We define the spaces:

$$
\begin{gathered}
X \otimes(S)_{\rho, p}^{(H)}=\left\{\widetilde{v}:\|\widetilde{v}\|_{X \otimes(S)_{p, p}^{(H)}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|\widetilde{v}_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}, \\
X \otimes(S)_{-\rho,-p}^{(H)}=\left\{\widetilde{v}:\|\widetilde{v}\|_{X \otimes(S)_{-\rho,-p}^{(H)}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho}\left\|\widetilde{v}_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\} .
\end{gathered}
$$

Especially, for $\rho=0$ and $p=0$, we denote the space $X \otimes(S)_{0,0}^{(H)}$ by

$$
X \otimes L^{2}\left(\mu_{H}\right)=\left\{\widetilde{v}:\|\widetilde{v}\|_{X \otimes L^{2}\left(\mu_{H}\right)}^{2}=\sum_{\alpha \in \mathscr{\mathscr { I }}} \alpha!\left\|\widetilde{v}_{\alpha}\right\|_{X}^{2}<\infty\right\} .
$$

Definition 1.14 Test and generalized fractional stochastic processes in Kondratiev sense are respectively elements of the spaces

$$
X \otimes(S)_{\rho}^{(H)}=\bigcap_{p \in \mathbb{N}_{0}} X \otimes(S)_{\rho, p}^{(H)} \text { and } X \otimes(S)_{-\rho}^{(H)}=\bigcup_{p \in \mathbb{N}_{0}} X \otimes(S)_{-\rho,-p}^{(H)} .
$$

### 1.4.2 Schwartz Spaces Valued Stochastic Processes

A general setting of Schwartz spaces valued generalized stochastic process was provided in [41] and further developed in [19, 22-24].
Definition 1.15 Let $F$ has a formal expansion

$$
F=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}, \quad \text { where } f_{\alpha, k} \in X, \alpha \in \mathscr{I}, K \in \mathbb{N} .
$$

Let $\rho \in[0,1]$. Define the following spaces:

$$
\begin{aligned}
X \otimes & S_{l}(\mathbb{R}) \otimes(S)_{\rho, p} \\
& =\left\{F:\|F\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!^{1+\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{l}(2 \mathbb{N})^{p \alpha}<\infty\right\}
\end{aligned}
$$

$X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$

$$
=\left\{F:\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!^{1-\rho}\left\|f_{\alpha, k}\right\|_{X}^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<\infty\right\}
$$

Definition 1.16 The Schwartz space of generalized stochastic processes in Kondratiev sense are elements of the space

$$
X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}=\bigcup_{p, l \in \mathbb{N}_{0}} X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}
$$

while the Schwartz space of test stochastic processes in Kondratiev sense are elements of the space

$$
X \otimes S(\mathbb{R}) \otimes(S)_{\rho}=\bigcap_{p, l \in \mathbb{N}_{0}} X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}
$$

Therefore, $S^{\prime}(\mathbb{R})$-valued generalized stochastic processes as elements of $X \otimes S^{\prime}(\mathbb{R}) \otimes$ $(S)_{-\rho}$ are given in the chaos expansion form

$$
\begin{equation*}
F=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{I}} b_{\alpha} \otimes H_{\alpha}=\sum_{k \in \mathbb{N}} c_{k} \otimes \xi_{k} \tag{1.42}
\end{equation*}
$$

where $b_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathscr{I}} f_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$ and $f_{\alpha, k} \in X$. Its generalized expectation is the zeroth coefficient in the expansion representation (1.42), i.e., it is given by $\mathbb{E}(F)=\sum_{k \in \mathbb{N}} f_{\mathbf{0}, k} \otimes \xi_{k}=b_{\mathbf{0}}$. Moreover,

$$
\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}}\left\|b_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-p \alpha}=\sum_{k \in \mathbb{N}}\left\|c_{k}\right\|_{X \otimes(S)_{-\rho,-p}}^{2}(2 k)^{-l}
$$

On the other side, $S(\mathbb{R})$-valued test processes as elements of $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$, which are given by chaos expansions of the form (1.42), where $b_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes$ $\xi_{k} \in X \otimes S(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathscr{I}} f_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S)_{\rho}$ and $f_{\alpha, k} \in X$.

The Hida spaces are obtained for $\rho=0$. The space $X \otimes L^{2}(\mathbb{R}) \otimes(S)_{0,0}$ is denoted by

$$
X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)=\left\{F:\|F\|_{X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)}^{2}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha!\left\|f_{\alpha, k}\right\|_{X}^{2}<\infty\right\}
$$

and represents the space of processes with finite second moments and square integrable trajectories. It is isomporphic to $X \otimes L^{2}(\mathbb{R} \times \Omega)$ and if $X$ is a separable Hilbert space, then it is also isomorphic to $L^{2}(\mathbb{R} \times \Omega ; X)$, see [43].

Example 1.11 Let $T>0$ and let $X=C^{k}([0, T]), k \in \mathbb{N}$. From the nuclearity of $(S)_{\rho}$ and the arguments provided in Remark 1.8 it follows that $C^{k}\left([0, T],(S)_{-\rho}\right)$ is isomorphic to $C^{k}([0, T]) \otimes(S)_{-\rho}$, i.e., differentiation of a stochastic process can be carried out componentwise in the chaos expansion [23,25]. This means that a stochastic process $u(t, \omega)$ is $k$ times continuously differentiable if and only if all its coefficients $u_{\alpha}, \alpha \in \mathscr{I}$ are in $C^{k}([0, T])$. The same holds for Banach space valued stochastic processes, i.e., for elements of $C^{k}([0, T], X) \otimes(S)_{-\rho}$, where $X$ is an arbitrary Banach space. These processes can be regarded as elements of the space $C^{k}\left([0, T], X \otimes(S)_{-\rho}\right)$ which is isomorphic to

$$
C^{k}([0, T], X) \otimes(S)_{-\rho}=\bigcup_{p \in \mathbb{N}_{0}} C^{k}([0, T], X) \otimes(S)_{-\rho,-p}
$$

In applications in fluid mechanics [1] we considered weak solutions of SDEs, i.e., we dealt with stochastic processes whose coefficients are elements in SobolevBochner spaces such as $L^{2}([0, T], X)$. Moreover, due to the nuclearity of the Kondratiev spaces we have $L^{2}\left([0, T], X \otimes(S)_{-\rho}\right) \cong L^{2}([0, T], X) \otimes(S)_{-\rho}$ and $H^{1}\left([0, T], X \otimes(S)_{-\rho}\right) \cong H^{1}([0, T], X) \otimes(S)_{-\rho}$.

Remark 1.10 Let $\rho \in[0,1]$. The Schwartz spaces valued generalized and test stochastic processes modified by a sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1, k \in \mathbb{N}$ are respectively elements of $X \otimes S^{\prime}(\mathbb{R}) \otimes(S a)_{-\rho}$ and $X \otimes S(\mathbb{R}) \otimes(S a)_{\rho}$. Then, $F \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S a)_{-\rho}$ can be represented in the chaos expansion form (1.42), where $b_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \in X \otimes S^{\prime}(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathscr{\mathscr { I }}} f_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S a)_{-\rho}$ and $f_{\alpha, k} \in X$, such that for some $p, l \in \mathbb{N}_{0}$ it holds

$$
\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S a)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left\|f_{\alpha, k}\right\|_{X}^{2} \alpha!^{1-\rho}(2 k)^{-l}(2 \mathbb{N} a)^{-p \alpha}<\infty .
$$

On the other hand, $F \in X \otimes S(\mathbb{R}) \otimes(S a)_{\rho}$ can be represented in the form (1.42), where $b_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \in X \otimes S(\mathbb{R}), c_{k}=\sum_{\alpha \in \mathscr{I}} f_{\alpha, k} \otimes H_{\alpha} \in X \otimes(S a)_{\rho}$ and $f_{\alpha, k} \in X$, such that for all $p, l \in \mathbb{N}_{0}$ it holds

$$
\|F\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S a)_{\rho, p}}^{2}=\sum_{\alpha \in \mathscr{J}} \sum_{k \in \mathbb{N}}\left\|f_{\alpha, k}\right\|_{X}^{2} \alpha!^{1-\rho}(2 k)^{l}(2 \mathbb{N} a)^{p \alpha}<\infty .
$$

Example 1.12 Let $\mathscr{H}$ be a separable Hilbert space and let $X=L^{2}([0, T], \mathscr{H})$. A square integrable $\mathscr{H}$-valued stochastic processes $v$ is an element of $L^{2}([0, T] \times$ $\Omega, \mathscr{H}) \cong L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ and is of the form

$$
\begin{align*}
v(t, \omega) & =\sum_{\alpha \in \mathscr{I}} v_{\alpha}(t) H_{\alpha}(\omega) \\
& =v_{\mathbf{0}}(t)+\sum_{k \in \mathbb{N}} v_{\varepsilon^{(k)}}(t) H_{\varepsilon^{(k)}}(\omega)+\sum_{|\alpha|>1} v_{\alpha}(t) H_{\alpha}(\omega), \quad t \in[0, T], \tag{1.43}
\end{align*}
$$

where $v_{\alpha} \in L^{2}([0, T], \mathscr{H})$ such that it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathscr{H})}^{2} \alpha!<\infty . \tag{1.44}
\end{equation*}
$$

A process $v$ with the chaos expansion representation (1.43) that instead of (1.44) satisfies the condition

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathscr{H})}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}<\infty \tag{1.45}
\end{equation*}
$$

belongs to $L^{2}([0, T], \mathscr{H}) \otimes(S)_{-\rho}$ and is considered to be a generalized stochastic process. The coefficient $v_{\mathbf{0}}(t)$ is the deterministic part of $v$ in (1.43) and is the generalized expectation of the process $v$.

Denote by $\left\{\mathbf{e}_{n}(t)\right\}_{n \in \mathbb{N}}$ the orthonormal basis of $L^{2}([0, T], \mathscr{H})$, i.e., the basis obtained by diagonalizing the orthonormal basis $\left\{b_{i}(t) s_{j}\right\}_{i, j \in \mathbb{N}}$, where $\left\{b_{i}(t)\right\}_{i \in \mathbb{N}}$ is the orthonormal basis of $L^{2}([0, T])$ and $\left\{s_{j}\right\}_{j \in \mathbb{N}}$ is the orthonormal basis of $\mathscr{H}$. The coefficients $v_{\alpha}(t) \in L^{2}([0, T], \mathscr{H}), \alpha \in \mathscr{I}$ can be represented in the form

$$
v_{\alpha}(t)=\sum_{j \in \mathbb{N}} v_{\alpha, j}(t) s_{j}=\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} v_{\alpha, j, i} b_{i}(t) s_{j}, \quad \alpha \in \mathscr{I}
$$

with $v_{\alpha, j} \in L^{2}([0, T])$ and $v_{\alpha, j, i} \in \mathbb{R}$. Then, (1.43) can be rewritten in the form

$$
v(t, \omega)=\sum_{\alpha \in \mathscr{I}} v_{\alpha}(t) H_{\alpha}(\omega)=\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} v_{\alpha, j, i} s_{j} b_{i}(t) H_{\alpha}(\omega) .
$$

After a diagonalization of $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N},(i, j) \mapsto n=n(i, j)$, it can be rearranged to

$$
v(t, \omega)=\sum_{\alpha \in \mathscr{\mathscr { O }}} \sum_{n \in \mathbb{N}} v_{\alpha, n} \mathbf{e}_{n}(t) H_{\alpha}(\omega), \quad v_{\alpha, n} \in \mathbb{R}, \omega \in \Omega, t \in[0, T] .
$$

Example 1.13 An $\mathscr{H}$-valued Brownian motion $\left\{B_{t}\right\}_{t \geq 0}$ on $(\Omega, \mathscr{F}, P)$ is a family of mappings $B_{t}: \mathscr{H} \rightarrow L^{2}(P)$ such that $\left\{B_{t} h\right\}_{h \in \mathscr{H}, t \geq 0}$ is a Gaussian centered process with the covariance $\mathbb{E}_{P}\left(B_{t} h B_{s} h_{1}\right)=\left(h, h_{1}\right)_{\mathscr{H}} \min \{t, s\}$, for $t, s \geq 0, h, h_{1} \in \mathscr{H}$. Moreover, for every $h \in \mathscr{H}$ the process $\left\{B_{t} h\right\}$ is a real Brownian motion [5]. Let $\left\{b_{t}^{(i)}\right\}_{i \in \mathbb{N}}$ be a sequence of independent one dimensional Brownian motions of the form (1.35). An $\mathscr{H}$-valued Brownian motion is then given by

$$
B_{t}(\omega)=\sum_{i=1}^{\infty} b_{t}^{(i)} s_{i}=\sum_{k=1}^{\infty} \theta_{k}(t) H_{\varepsilon^{(k)}}(\omega),
$$

where $\theta_{k}(t)=\delta_{n(i, j), k}\left(\int_{0}^{t} \xi_{j}(s) d s\right) s_{i}$, and $\delta_{n(i, j), k}$ is the Kronecker delta function, see [41]. Particularly, $B_{t}$ with values in $\mathbb{R}$ reduces to the standard Brownian motion $b_{t}$.

Example 1.14 An $\mathscr{H}$-valued white noise process is given by the formal sum

$$
\begin{equation*}
W_{t}(\omega)=\sum_{k=1}^{\infty} \mathbf{e}_{k}(t) H_{\varepsilon^{(k)}}(\omega) . \tag{1.46}
\end{equation*}
$$

Following [5], an $\mathscr{H}$-valued white noise can be also defined as $\sum_{n \in \mathbb{N}} w_{t}^{n}(\omega) s_{n}$, where $w_{t}^{(n)}(\omega)$ are independent copies of one dimensional white noise (1.36) and $\left\{s_{n}\right\}_{n \in \mathbb{N}}$ is the orthonormal basis of $\mathscr{H}$. This definition can be reduced to (1.46) since

$$
\sum_{n \in \mathbb{N}} w_{t}^{(n)}(\omega) s_{n}=\sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \xi_{k}(t) H_{\varepsilon^{(k)}}(\omega) s_{n}=\sum_{i \in \mathbb{N}} \xi_{i}(t) H_{\varepsilon^{(i)}}(\omega) s_{n}=\sum_{j=1}^{\infty} \mathbf{e}_{j}(t) H_{\varepsilon^{(j)}}(\omega),
$$

where $\left\{\mathbf{e}_{j}\right\}_{j \in \mathbb{N}}$ is the orthogonal basis of $L^{2}(\mathbb{R}, \mathscr{H})$ obtained by diagonalizing the basis $\left\{\xi_{k}(t) s_{n}\right\}_{k, n \in \mathbb{N}}$ and $\xi_{n}$ are the Hermite functions (1.1). The process $W_{t}$ is an element in $S(\mathscr{H})_{-\rho}$, for all $\rho \in[0,1]$.

More generally, chaos expansion representation of an $\mathscr{H}$-valued Gaussian process that belongs to the Wiener chaos space of order one is given by

$$
\begin{equation*}
G_{t}(\omega)=\sum_{k \in \mathbb{N}} g_{k}(t) H_{\varepsilon^{(k)}}(\omega)=\sum_{k \in \mathbb{N}}\left(\sum_{i \in \mathbb{N}} g_{k i} \mathbf{e}_{i}(t)\right) H_{\varepsilon^{(k)}}(\omega), \tag{1.47}
\end{equation*}
$$

with $g_{k} \in L^{2}([0, T], \mathscr{H})$ and $g_{k i}=\left(g_{k}, \mathbf{e}_{i}\right)_{L^{2}([0, T], \mathscr{H})}$ is a real constant. If the condition

$$
\begin{equation*}
\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{L^{2}([0, T], \mathscr{H})}^{2}<\infty \tag{1.48}
\end{equation*}
$$

is fulfilled, then $G_{t}$ belongs to the space $L^{2}([0, T] \times \Omega, \mathscr{H}) \cong L^{2}([0, T], \mathscr{H}) \otimes$ $L^{2}(\mu)$. If the sum in (1.48) is infinite then the representation (1.47) is formal, and if additionally $\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{L^{2}([0, T], \mathscr{H})}^{2}(2 \mathbb{N})^{-p \varepsilon^{(k)}}=\sum_{k \in \mathbb{N}}\left\|g_{k}\right\|_{L^{2}([0, T], \mathscr{H})}^{2}(2 k)^{-p}<\infty$ holds for some $p \in \mathbb{N}_{0}$, then for each $t$ the process $G_{t}$ belongs to the Kondratiev space of stochastic distributions $(S)_{-\rho}$, see [19, 26, 37].

From the representation (1.47) we conclude that a Gaussian noise can be interpreted as a colored noise with the representation operator $N$ and the correlation function $\mathscr{C}=N N^{\star}$ such that

$$
\sum_{k \in \mathbb{N}} N^{\star} f_{k}(t) H_{\varepsilon^{(k)}}(\omega)=\sum_{k \in \mathbb{N}} N^{\star}\left(\sum_{i \in \mathbb{N}} f_{k i} \mathbf{e}_{i}(t)\right) H_{\varepsilon^{(k)}}(\omega)=\sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{N}} \lambda_{i} f_{k i} \mathbf{e}_{i}(t) H_{\varepsilon^{(k)}}(\omega),
$$

with $N^{\star} \mathbf{e}_{i}(t)=\lambda_{i} \mathbf{e}_{i}(t), i \in \mathbb{N}$, [30]. Hence, in the following we will assume the color noise to be a Gaussian process of the form

$$
\begin{equation*}
L_{t}(\omega)=\sum_{k \in \mathbb{N}} l_{k} \mathbf{e}_{k}(t) H_{\varepsilon^{(k)}}(\omega) \tag{1.49}
\end{equation*}
$$

with a sequence of real coefficients $\left\{l_{k}\right\}_{k \in \mathbb{N}}$ such that for some $p \in \mathbb{N}$ it holds

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} l_{k}^{2}(2 k)^{-p}<\infty \tag{1.50}
\end{equation*}
$$

Example 1.15 For $X=L^{2}([0, T], \mathscr{H})$ we obtain the space of $\mathscr{H}$-valued fractional stochastic processes. In prticular, $\widetilde{v} \in L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)$ is uniquely defined by

$$
\begin{equation*}
\widetilde{v}_{t}(\omega)=\sum_{\alpha \in \mathscr{I}} \tilde{v}_{\alpha}(t) \tilde{H}_{\alpha}(\omega) \tag{1.51}
\end{equation*}
$$

where $\tilde{v}_{\alpha} \in L^{2}([0, T], \mathscr{H}), \alpha \in \mathscr{I}$ such that $\sum_{\alpha \in \mathscr{I}}\left\|\widetilde{v}_{\alpha}\right\|_{X}^{2} \alpha!<\infty$. Moreover, (1.51) can be written in the form $\widetilde{v}_{t}(\omega)=\sum_{\alpha \in \mathscr{I}} \sum_{n \in \mathbb{N}} v_{\alpha, n} \mathbf{e}_{n}(t) \widetilde{H}_{\alpha}(\omega), v_{\alpha, n} \in \mathbb{R}$, $\omega \in \Omega, t \in[0, T]$. Fractional generalized processes $\widetilde{v}$ from $L^{2}([0, T], \mathscr{H}) \otimes(S)_{-\rho}^{(H)}$ has a chaos expansion representation of the form (1.51) such that (1.45) holds.

### 1.4.3 Fractional Operator $\mathscr{M}$

In this section we introduce an isometry $\mathscr{M}$ between the space of square integrable fractional random variables $L^{2}\left(\mu_{H}\right)$ and the space of integrable random variables $L^{2}(\mu)$, and then extend this mapping to an isometry between the spaces stochastic processes $X \otimes(S)_{-\rho}^{H}$ and $X \otimes(S)_{-\rho}$. Since the operator $M^{(H)}$, defined in Sect. 1.3.6, is self-adjoint, we can for each $\alpha \in \mathscr{I}$ connect the Fourier-Hermite polynomials (1.10) and (1.30) in the following way

$$
\begin{aligned}
H_{\alpha}(\omega) & =\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, \xi_{k}\right\rangle\right)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle\omega, M^{(H)} e_{k}^{(H)}\right\rangle\right)=\prod_{k=1}^{\infty} h_{\alpha_{k}}\left(\left\langle M^{(H)} \omega, e_{k}^{(H)}\right\rangle\right) \\
& =\widetilde{H}_{\alpha}\left(M^{(H)} \omega\right)
\end{aligned}
$$

and similarly $\widetilde{H}_{\alpha}(\omega)=H_{\alpha}\left(M^{(1-H)} \omega\right)$. Hence, we define a fractional operator $\mathscr{M}$ as follows.

Definition 1.17 ([19]) Let $\mathscr{M}: L^{2}\left(\mu_{H}\right) \rightarrow L^{2}(\mu)$ be defined by

$$
\mathscr{M}\left(\tilde{H}_{\alpha}(\omega)\right)=H_{\alpha}(\omega), \quad \alpha \in \mathscr{I}, \omega \in \Omega
$$

The action of $\mathscr{M}$ can be seen as a transformation of the corresponding elements of the orthogonal basis $\left\{\widetilde{H}_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ into $\left\{H_{\alpha}\right\}_{\alpha \in \mathscr{I}}$. The fractional operators $\mathscr{M}$ and $M^{(1-H)}$ correspond to each other. For $G=\sum_{\alpha \in \mathscr{\mathscr { I }}} c_{\alpha} \widetilde{H}_{\alpha}(\omega) \in L^{2}\left(\mu_{H}\right)$ by linearity and continuity we extend $\mathscr{M}$ to

$$
\begin{equation*}
\mathscr{M}(G)=\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} c_{\alpha} \widetilde{H}_{\alpha}(\omega)\right)=\sum_{\alpha \in \mathscr{I}} c_{\alpha} H_{\alpha}(\omega) . \tag{1.52}
\end{equation*}
$$

Theorem 1.8 ([19]) Operator $\mathscr{M}$ is an isometry between spaces of classical Gaussian and fractional Gaussian random variables.

Proof The operator $\mathscr{M}$ is an isometry between $L^{2}\left(\mu_{H}\right)$ and $L^{2}(\mu)$ because it holds $\left\|\mathscr{M}\left(\widetilde{H}_{\alpha}\right)\right\|_{L^{2}(\mu)}^{2}=\left\|H_{\alpha}\right\|_{L^{2}(\mu)}^{2}=\alpha!=\left\|\widetilde{H}_{\alpha}\right\|_{L^{2}\left(\mu_{H}\right)}^{2}$.

Further on, each pair of elements $F$ and $\widetilde{F}$, that are connected via $\mathscr{M}$, will be called the associated pairs. The coefficients of the chaos expansion representations of associated elements $F$ and $\widetilde{F}$ coincide.

Lemma 1.2 Let $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} H_{\alpha} \in L^{2}(\mu)$ and $\widetilde{F}=\sum_{\alpha \in \mathscr{I}} \widetilde{f}_{\alpha} \widetilde{H}_{\alpha} \in L^{2}\left(\mu_{H}\right)$. Then, $F$ and $\widetilde{F}$ are associated if and only if $\widetilde{f}_{\alpha}=f_{\alpha}$ for all $\alpha \in \mathscr{I}$.

Proof Let $F$ and $\widetilde{F}$ be associated. Then,

$$
\sum_{\alpha \in \mathscr{I}} f_{\alpha} H_{\alpha}=F=\mathscr{M}(\widetilde{F})=\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} \widetilde{f}_{\alpha} \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathscr{I}} \widetilde{f}_{\alpha} H_{\alpha} .
$$

Due to the uniqueness of the chaos expansion representation in $\left\{H_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ we obtain $f_{\alpha}=\widetilde{f_{\alpha}}$ for all $\alpha \in \mathscr{I}$.

The action of the operator $\mathscr{M}$ can be extended to Kondratiev space of stochastic distributions $\mathscr{M}:(S)_{-\rho}^{(H)} \rightarrow(S)_{-\rho}$ by (1.52) for $G \in(S)_{-\rho}^{(H)}$. The extension, also denoted by $\mathscr{M}$, is well defined since there exists $p \in \mathbb{N}$ such that it holds $\sum_{\alpha \in \mathscr{J}} \alpha!^{1-\rho} f_{\alpha}^{2}(2 \mathbb{N})^{-p \alpha}<\infty$. In an analogous way, the action of the operator $\mathscr{M}$ can be extended to all classes of stochastic processes (test, square integrable and generalized) and $\mathscr{H}$-valued stochastic processes.

Example 1.16 The connection between a Brownian motion $b_{t}$ and a fractional Brownian motion $b_{t}^{(H)}$, that are respectively represented by (1.35) and (1.39), in terms of the operator $\mathscr{M}$ is given by $\mathscr{M}^{-1}\left(b_{t}\right)=b_{t}^{(H)}$.

Example 1.17 One dimensional real valued fractional singular white noise $w_{t}^{(H)}$ for fixed $t$ is an element of fractional Kondratiev space $(S)_{-\rho}^{(1-H)}$. It is defined by the formal expansion

$$
\begin{equation*}
w_{t}^{(H)}(\omega)=\sum_{k=1}^{\infty} \xi_{k}(t) \widetilde{H}_{\varepsilon^{(k)}}(\omega) . \tag{1.53}
\end{equation*}
$$

It is integrable and the relation $\frac{d}{d t} b_{t}^{(H)}=w_{t}^{(H)}$ holds in the sense of distributions. Moreover, by combining (1.36) and (1.53) we obtain

$$
\mathscr{M}^{-1}\left(w_{t}\right)=\mathscr{M}^{-1}\left(\sum_{k=1}^{\infty} \xi_{k} H_{\varepsilon^{(k)}}\right)=\sum_{k=1}^{\infty} \xi_{k} \widetilde{H}_{\varepsilon^{(k)}}(\omega)=w_{t}^{(H)} .
$$

Example 1.18 An $\mathscr{H}$-valued fractional white noise in the fractional space is given by

$$
\begin{equation*}
W_{t}^{(H)}(\omega)=\sum_{k=1}^{\infty} \mathbf{e}_{k}(t) \widetilde{H}_{\varepsilon^{(k)}}(\omega), \tag{1.54}
\end{equation*}
$$

where $\left\{\mathbf{e}_{k}\right\}_{k \in \mathbb{N}}$ is an orthonormal basis in $L^{2}([0, T], \mathscr{H})$. By (1.46) and (1.54), the relations $\mathscr{M}\left(W_{t}^{(H)}\right)=W_{t}$ and $\mathscr{M}^{-1}\left(W_{t}\right)=W_{t}^{(H)}$ follow.

From here onwards we will keep the following notation: all processes denoted with tilde in superscript will be considered as elements of a certain fractional space. Therefore, due to Lemma 1.2, each process $v=\sum_{\alpha \in \mathscr{I}} v_{\alpha} H_{\alpha}$ from an $\mathscr{H}$-valued classical space (particularly $L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ or $\left.L^{2}([0, T], \mathscr{H}) \otimes(S)_{-\rho}\right)$ will be associated to a process $\widetilde{v}=\sum_{\alpha \in \mathscr{I}} v_{\alpha} \widetilde{H}_{\alpha}$ from the corresponding $\mathscr{H}$-valued fractional space (particularly $L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)$ or $\left.L^{2}([0, T], \mathscr{H}) \otimes(S)_{-\rho}^{(H)}\right)$ via the fractional mapping $\mathscr{M}$, i.e., $\mathscr{M}(\widetilde{v})=v$. Since the coefficients of processes $\widetilde{v}$ and $v$ are equal it also follows

$$
\begin{equation*}
\|\widetilde{v}\|_{L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!\left\|v_{\alpha}\right\|_{L^{2}([0, T], \mathscr{H})}^{2}=\|v\|_{L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)}^{2} . \tag{1.55}
\end{equation*}
$$

### 1.4.4 Multiplication of Stochastic Processes

In this section we deal with two types of products of stochastic processes. First, we generalize the definition of the Wick product of random variables to the set of generalized stochastic processes as it was done in [11, 21, 24]. For this purpose we will assume that $X$ is closed under multiplication, i.e., that $x \cdot y \in X$, for all $x, y \in X$. Then, we consider the ordinary (usual) product of stochastic processes.

Definition 1.18 For stochastic processes $F$ and $G$ given in chaos expansion forms (1.31), their Wick product $F \diamond G$ is a stochastic process defined by

$$
\begin{equation*}
F \diamond G=\sum_{\gamma \in \mathscr{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right) \otimes H_{\gamma} . \tag{1.56}
\end{equation*}
$$

In [23] it was proven that the spaces of stochastic processes $X \otimes(S)_{-1}$ and $X \otimes(S)_{1}$ are closed under the Wick multiplication, providing the conditions only in terms
of the level of singularity $p$. Here we prove the closability of the Wick product in $X \otimes(S)_{-\rho}$ and $X \otimes(S)_{\rho}$ for $\rho \in[0,1]$ by stating the conditions involving both the level of singularity $p$ and $\rho$.

Theorem 1.9 Let $\rho \in[0,1]$ and let $F$ and $G$ be given in their chaos expansion forms $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha}$ and $G=\sum_{\alpha \in \mathscr{I}} g_{\alpha} \otimes H_{\alpha}, f_{\alpha}, g_{\alpha} \in X, \alpha \in \mathscr{I}$.
$1^{\circ}$ If $F \in X \otimes(S)_{-\rho,-p_{1}}$ and $G \in X \otimes(S)_{-\rho,-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{-\rho,-q}$, for $q \geq p_{1}+p_{2}+3-\rho$.
$2^{\circ}$ If $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{\rho, p_{2}}$ for $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{\rho, q}$, for $q+3+\rho \leq \min \left\{p_{1}, p_{2}\right\}$.

Proof $1^{\circ}$ Let $F \in X \otimes(S)_{-\rho,-p_{1}}$ and $G \in X \otimes(S)_{-\rho,-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$. Then, from (1.56) by the Cauchy-Schwarz inequality and Lemma 1.1, part $2^{\circ}$, the following holds

$$
\begin{aligned}
& \|F \diamond G\|_{X \otimes(S)_{-\rho,-q}^{2}}^{2}=\sum_{\gamma \in \mathscr{I}} \gamma!^{1-\rho}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-q \gamma} \\
& \leq \sum_{\gamma \in \mathscr{I}} \gamma!^{1-\rho}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-\left(p_{1}+p_{2}+3-\rho\right) \gamma} \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\left(\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}\right)^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{1}++p_{2}+1-\rho}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left(\sum_{\alpha+\beta=\gamma}\left\|f_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p_{1} \alpha}\right)\left(\sum_{\alpha+\beta=\gamma}\left\|g_{\beta}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p_{2} \beta}\right) \\
& \leq m \cdot\|F\|_{X \otimes(S)_{-\rho,-p 1}}^{2} \cdot\|G\|_{X \otimes(S)_{-\rho,-p 2}}^{2}<\infty,
\end{aligned}
$$

since $m=\sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}<\infty$.
$2^{\circ}$ Let now $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{\rho, p_{2}}$ for all $p_{1}, p_{2} \in \mathbb{N}_{0}$. Then, the chaos expansion form of $F \diamond G$ is given by (1.56) and

$$
\begin{aligned}
& \|F \diamond G\|_{X \otimes(S)_{\rho, q}}^{2}=\sum_{\gamma \in \mathscr{\mathscr { I }}} \gamma!^{1+\rho}\left\|\sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{q \gamma} \\
& =\sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} \gamma!^{\frac{1+\rho}{2}} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2} \gamma}\right\|_{X}^{2} \\
& \leq \sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}\left\|\sum_{\alpha+\beta=\gamma} \alpha!^{\frac{1+\rho}{2}} \beta!^{\frac{1+\rho}{2}}(2 \mathbb{N})^{\frac{1+\rho}{2}(\alpha+\beta)} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\frac{q+2}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \leq m\left(\sum_{\alpha+\beta=\gamma} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{(q+3+\rho) \alpha}\right)\left(\sum_{\alpha+\beta=\gamma} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{(q+3+\rho) \beta}\right) \\
& \leq m \cdot\|F\|_{X \otimes(S)_{\rho, p_{1}}^{2}}^{2} \cdot\|G\|_{X \otimes(S)_{\rho, p_{2}}^{2}}^{2}<\infty
\end{aligned}
$$

for $q+3+\rho \leq \min \left\{p_{1}, p_{2}\right\}$, where $m=\sum_{\gamma \in \mathscr{I}}(2 \mathbb{N})^{-2 \gamma}<\infty$.

Remark 1.11 A test stochastic process $u \in X \otimes(S)_{\rho, p}, p \geq 0$ can be considered also as an element in $X \otimes(S)_{-\rho,-q}, q \geq 0$ since it holds $\|u\|_{X \otimes(S)_{-\rho,-q}}^{2} \leq\|u\|_{X \otimes(S)_{\rho, p}}^{2}$.

Therefore, if $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{-\rho,-p_{2}}$ for some $p_{1}, p_{2} \in \mathbb{N}_{0}$, then $F \diamond G$ is a well defined element in $X \otimes(S)_{-\rho,-q}$, for $q \geq p_{2}+3+\rho$. This follows from Theorem 1.9 part $1^{\circ}$ by letting $p_{1}=0$.

Applying the formula for the product of Fourier-Hermite polynomials (1.14), the ordinary product $F \cdot G$ of two stochastic processes $F$ and $G$ can be defined. Thus, formally we obtain

$$
\begin{align*}
F \cdot G & =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \otimes H_{\alpha} \cdot H_{\beta} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{\mathscr { I }}} f_{\alpha} g_{\beta} \otimes \sum_{0 \leq \gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}  \tag{1.57}\\
& =F \diamond G+\sum_{\alpha \in \mathscr{\mathscr { C }}} \sum_{\beta \in \mathscr{\mathscr { I }}} f_{\alpha} g_{\beta} \otimes \sum_{0<\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} .
\end{align*}
$$

After the change of variables $\delta=\alpha-\gamma, \theta=\beta-\gamma$ and $\tau=\delta+\theta$ we obtain

Therefore, we can rearrange the sums for $F \cdot G$ and obtain

$$
\begin{align*}
& F \cdot G=F \diamond G+\sum_{\tau \in \mathscr{\mathscr { C }}} \sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\substack{\nu=0 . \delta \leq \tau \\
\gamma+\tau \tau \delta \beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_{\tau}  \tag{1.58}\\
& =\sum_{\tau \in \mathscr{\mathscr { I }}} \sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} a_{\alpha, \beta, \tau} H_{\tau},
\end{align*}
$$

where

$$
\begin{equation*}
a_{\alpha, \beta, \tau}=\sum_{\substack{\gamma \in, \in, s \leq \tau \\ \gamma+\tau-\delta=\beta, \gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} . \tag{1.59}
\end{equation*}
$$

Note that for each $\alpha, \beta, \tau \in \mathscr{I}$ fixed, there exists a unique pair of multi-indices $\gamma, \delta \in \mathscr{I}$ such that $\delta \leq \tau$ and $\gamma+\tau-\delta=\beta, \gamma+\delta=\alpha$. Moreover, both $\alpha+\beta$ and $|\alpha-\beta|$ are odd (respectively even) if and only if $\tau$ is odd (respectively even). Also, $\alpha+\beta \geq \tau \geq|\alpha-\beta|$. Thus,

$$
a_{\alpha, \beta, \tau}=\frac{\alpha!\beta!}{\left(\frac{\alpha+\beta-\tau}{2}\right)!\left(\frac{\alpha-\beta+\tau}{2}\right)!\left(\frac{\beta-\alpha+\tau}{2}\right)!} .
$$

For example, if $\tau=(2,0,0,0, \ldots)$, then the coefficient next to $H_{\tau}$ in (1.58) is $f_{(0,0,0, \ldots)} g_{(2,0,0, \ldots)}+f_{(1,0,0, \ldots)} g_{(1,0,0 \ldots)}+f_{(2,0,0, \ldots)} g_{(0,0,0, \ldots)}+3 f_{(1,0,0, \ldots)} g_{(3,0,0, \ldots)}+$ $4 f_{(2,0,0, \ldots)} g_{(2,0,0, \ldots)}+3 f_{(3,0,0, \ldots)} g_{(1,0,0, \ldots)}+18 f_{(3,0,0, \ldots)} g_{(3,0,0, \ldots)}+\cdots$.

Lemma 1.3 ([20]) Let $\alpha, \beta, \tau \in \mathscr{I}$ and $a_{\alpha, \beta, \tau}$ be defined as in (1.59). Then,

$$
a_{\alpha, \beta, \tau} \leq(2 \mathbb{N})^{\alpha+\beta} .
$$

The proof is rather technical and it is omitted here. We refer the reader to [20].
Theorem 1.10 The following hold:
$1^{\circ}$ If $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{\rho, p_{2}}$, for $p_{1}, p_{2} \in \mathbb{N}_{0}$, then the ordinary product $F \cdot G$ is a well defined element in $X \otimes(S)_{\rho, q}$ for $q+7+\rho \leq \min \left\{p_{1}, p_{2}\right\}$.
$2^{\circ}$ If $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{-\rho,-p_{2}}$, for $p_{1}-p_{2}>8$, then their ordinary product $F \cdot G$ is well defined and belongs to $X \otimes(S)_{-\rho,-q}$ for $q \geq p_{2}+7-\rho$.

Proof $1^{\circ}$ Let $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{\rho, p_{2}}$, for $p_{1}, p_{2} \in \mathbb{N}_{0}$. By Lemma 1.3, Lemma 1.1 and the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
& \left.\|F \cdot G\|_{X \otimes(S)}^{2}\right)_{\rho, \mathscr{q}}=\sum_{\tau \in \mathscr{I}} \tau!^{1+\rho}\left\|\sum_{\alpha, \beta \in \mathscr{I}} f_{\alpha} g_{\beta} a_{\alpha, \beta, \tau}\right\|_{X}^{2}(2 \mathbb{N})^{q \tau} \\
& \leq \sum_{\tau \in \mathscr{I}} \tau!^{1+\rho}\left\|\sum_{\tau \leq \alpha+\beta} \tau!^{!+\rho} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\alpha+\beta}(2 \mathbb{N})^{\frac{q}{\tau} \tau}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\tau \leq \alpha+\beta}(2 \mathbb{N})^{2(\alpha+\beta)} f_{\alpha} g_{\beta}\left(\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}\right)^{\frac{1+\rho}{2}}(2 \mathbb{N})^{\frac{q}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\alpha, \beta \in \mathscr{I}}(2 \mathbb{N})^{-\beta} \alpha!^{\frac{1+\rho}{2}} f_{\alpha}(2 \mathbb{N})^{-\alpha} \beta!^{\frac{1+\rho}{2}} g_{\beta}(2 \mathbb{N})^{\frac{1}{2}(\alpha+\beta)}\right\|_{X}^{2} \\
& \left.\leq \sum_{\tau \in \mathscr{I}} 2 \mathbb{N}\right)^{-2 \tau}\left(\sum_{\alpha, \beta \in \mathscr{I}} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{r \alpha}(2 \mathbb{N})^{-2 \beta} \sum_{\alpha, \beta \in \mathscr{I}} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{r \beta}(2 \mathbb{N})^{-2 \alpha}\right) \\
& \left.\leq m\left(\sum_{\beta \in \mathscr{I}}(2 \mathbb{N})^{-2 \beta} \sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p_{1} \alpha}\right)\left(\sum_{\alpha \in \mathscr{I}} 2 \mathbb{N}\right)^{-2 \alpha} \sum_{\beta \in \mathscr{I}} \beta!^{1+\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p_{2} \beta}\right) \\
& \leq m\left(c_{1} \sum_{\alpha \in \mathscr{\mathscr { I }}} \alpha!^{2}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p_{1} \alpha}\right)\left(c_{2} \sum_{\beta \in \mathscr{I}} \alpha!^{2}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p_{2} \beta}\right) \\
& =m c_{1} c_{2}\|F\|_{X \otimes(S) \rho_{\rho, p_{1}}^{2}}\|G\|_{X \otimes(S)_{\rho, p_{2}}^{2}<\infty,}^{<\infty}
\end{aligned}
$$

for $r=q+7+\rho \leq \min \left\{p_{1}, p_{2}\right\}$, where $m=\sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}, c_{1}=\sum_{\beta \in \mathscr{I}}(2 \mathbb{N})^{-2 \beta}$ and $c_{2}=\sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{-2 \alpha}$ are finite.
$2^{\circ}$ Let $F \in X \otimes(S)_{\rho, p_{1}}$ and $G \in X \otimes(S)_{-\rho,-p_{2}}$, for $p_{1}, p_{2} \in \mathbb{N}_{0}$ such that $p_{1}>p_{2}+8$, and assume $q \geq p_{2}+7-\rho$. Then,

$$
\|F \cdot G\|_{X \otimes(S)_{-\rho,-q}}^{2}=\sum_{\tau \in \mathscr{\mathscr { F }}} \tau!^{1-\rho}\left\|\sum_{\alpha, \beta \in \mathscr{\mathscr { F }}} f_{\alpha} g_{\beta} a_{\alpha, \beta, \tau}\right\|_{X}^{2}(2 \mathbb{N})^{-q \tau}
$$

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$$
\begin{aligned}
& \leq \sum_{\tau \in \mathscr{I}}\left\|\sum_{\tau \leq \alpha+\beta} \tau!^{\frac{1-\rho}{2}} f_{\alpha} g_{\beta}(2 \mathbb{N})^{\alpha+\beta}(2 \mathbb{N})^{-\frac{p_{2}+7-\rho}{2} \tau}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\tau \leq \alpha+\beta}(2 \mathbb{N})^{2(\alpha+\beta)} f_{\alpha} g_{\beta}\left(\alpha!\beta!(2 \mathbb{N})^{\alpha+\beta}\right)^{\frac{1-\rho}{2}}(2 \mathbb{N})^{-\frac{p_{2}+7-\rho}{2}}(\alpha+\beta)\right\|_{X}^{2} \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\alpha, \beta \in \mathscr{I}}(2 \mathbb{N})^{-\beta} \alpha!^{\frac{1-\rho}{2}} f_{\alpha}(2 \mathbb{N})^{-\frac{p_{2}^{2}}{2} \alpha}(2 \mathbb{N})^{-\alpha} \beta!^{\frac{1-\rho}{2}} g_{\beta}(2 \mathbb{N})^{-\frac{p_{2}}{2} \beta}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left\|\sum_{\alpha, \beta \in \mathscr{I}}(2 \mathbb{N})^{-\beta} \alpha!^{\frac{1+\rho}{2}} f_{\alpha}(2 \mathbb{N})^{\frac{p_{1} \alpha}{2}}(2 \mathbb{N})^{-\alpha} \beta!^{\frac{1-\rho}{2}} g_{\beta}(2 \mathbb{N})^{-\frac{p_{2}}{2} \beta}\right\|_{X}^{2} \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left(\sum_{\alpha, \beta \in \mathscr{I}} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p_{2} \alpha-2 \beta} \sum_{\alpha, \beta \in \mathscr{I}} \beta!^{1-\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p_{2} \beta-2 \alpha}\right) \\
& \leq \sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}\left(c_{1} \sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p_{1} \alpha}\right)\left(c_{2} \sum_{\beta \in \mathscr{I}} \beta!^{1-\rho}\left\|g_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p_{2} \beta}\right) \\
& \leq m c_{1} c_{2}\|F\|_{X \otimes(S)_{\rho, p 1}}^{2}\|G\|_{X \otimes(S)_{-\rho, p_{2}}^{2}<\infty,}<\infty,
\end{aligned}
$$

where $m=\sum_{\tau \in \mathscr{I}}(2 \mathbb{N})^{-2 \tau}, c_{1}=\sum_{\beta \in \mathscr{I}}(2 \mathbb{N})^{-2 \beta}, c_{2}=\sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{-2 \alpha}$ are finite.
In [20] the authors proved similar theorem to Theorem 1.10, where the conditions were given only in terms of level of singularities.

Remark 1.12 For $F, G \in X \otimes L^{2}(\mu)$ the ordinary product $F \cdot G$ will not necessarily belong to $X \otimes L^{2}(\mu)$, but due to the Hölder inequality it will belong to $X \otimes L^{1}(\mu)$. This also follows from the fact that $F \diamond G$ for random variables $F$ and $G$ does not necessarily belong to $L^{2}(\mu)$, see Example 1.4.

### 1.5 Operators

We consider two classes of operators defined on sets of stochastic processes, coordinatewise operators and convolution type operators. We follow the classification of stochastic operators given in [17, 25].

Definition 1.19 We say that an operator A defined on $X \otimes(S)_{-\rho}$ is:
$1^{\circ}$ a coordinatewise operator if it is composed of a family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}}$, $A_{\alpha}: X \rightarrow X, \alpha \in \mathscr{I}$, such that for a process $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$, $u_{\alpha} \in X, \alpha \in \mathscr{I}$ it holds that

$$
\begin{equation*}
\mathbf{A} u=\sum_{\alpha \in \mathscr{I}} A_{\alpha} u_{\alpha} \otimes H_{\alpha} \tag{1.60}
\end{equation*}
$$

$2^{\circ}$ a simple coordinatewise operator if $A_{\alpha}=A$ for all $\alpha \in \mathscr{I}$, i.e., if it holds that

$$
\mathbf{A} u=\sum_{\alpha \in \mathscr{I}} A\left(u_{\alpha}\right) \otimes H_{\alpha}=A\left(u_{0}\right)+\sum_{|\alpha|>0} A\left(u_{\alpha}\right) \otimes H_{\alpha}
$$

Definition 1.20 A fractional operator $\widetilde{\mathbf{A}}: X \otimes(S)_{-\rho}^{(H)} \rightarrow X \otimes(S)_{-\rho}^{(H)}$ is a coordinatewise operator if it is composed of a family of operators $\widetilde{A}_{\alpha}: X \rightarrow X, \alpha \in \mathscr{I}$, such that $\widetilde{u}=\sum_{\alpha \in \mathscr{I}} \widetilde{u}_{\alpha} \otimes \widetilde{H}_{\alpha} \in X \otimes(S)_{-\rho}^{(H)}, \widetilde{u}_{\alpha} \in X, \alpha \in \mathscr{I}$ it holds

$$
\begin{equation*}
\widetilde{\mathbf{A}} u=\sum_{\alpha \in \mathscr{I}} \widetilde{A}_{\alpha} \widetilde{u}_{\alpha} \otimes \widetilde{H}_{\alpha} \tag{1.61}
\end{equation*}
$$

If $\widetilde{A}_{\alpha}=\widetilde{A}$ for all $\alpha \in \mathscr{I}$ then $\mathbf{A}$ is a simple coordinatewise operator.
Remark 1.13 Definitions 1.19 and 1.20 can be modified for the operators acting on the spaces of square integrable processes $X \otimes L^{2}(\mu)$ and $X \otimes L^{2}\left(\mu_{H}\right)$ and spaces of test processes $X \otimes(S)_{\rho}$ and $X \otimes(S)_{\rho}^{(H)}$.

Example 1.19 The time differentiation can be carried out componentwise in the chaos expansion, Example 1.11, and thus the differentiation operator is a simple coordinatewise operator. In the following chapters we will work with generalized operators of the Malliavin calculus. Particularly, the Ornstein-Uhlenbeck operator, defined by (2.14), is an example of a coordinatewise operator, while the Malliavin derivative, defined by (2.2), is not a coordinatewise operator. In [25] it was proven that the Skorokhod integral, defined by (2.9), can be represented in the form of a convolution type operator.

Lemma 1.4 Let $\mathbf{A}$ be a coordinatewise operator that corresponds to a family of deterministic operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}}$. If the operators $A_{\alpha}: X \rightarrow X, \alpha \in \mathscr{I}$ are uniformly bounded by $c>0$ then $\mathbf{A}$ is also bounded.

Proof $1^{\circ}$ Let first $\mathbf{A}: X \otimes L^{2}(\mu) \rightarrow X \otimes L^{2}(\mu)$ and $\left\|A_{\alpha}\right\|_{L(X)} \leq c$ for all $\alpha \in \mathscr{I}$. Let $v=\sum_{\alpha \in \mathscr{I}} v_{\alpha} H_{\alpha}$. Then, $\|\mathbf{A} v\|_{X \otimes L^{2}(\mu)}^{2} \leq c^{2} \sum_{\alpha \in \mathscr{I}}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!=$ $c^{2}\|v\|_{X \otimes L^{2}(\mu)}^{2}$ and the operator $\mathbf{A}$ is bounded with $\|\mathbf{A}\|_{L(X) \otimes L^{2}(\mu)} \leq c$.
$2^{\circ}$ For $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$, such that $\left\|A_{\alpha}\right\|_{L(X)} \leq c$ for all $\alpha \in \mathscr{I}$ we have $\|\mathbf{A} v\|_{X \otimes(S)_{-\rho,-p}}^{2} \leq c^{2} \sum_{\alpha \in \mathscr{I}}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}=c^{2}\|v\|_{X \otimes(S)_{-\rho,-p}}^{2}$ and $\mathbf{A}$ is bounded.
$3^{\circ}$ Similarly, for a coordinatewise operator $\mathbf{A}: X \otimes(S)_{\rho} \rightarrow X \otimes(S)_{\rho}$, such that $\left\|A_{\alpha}\right\|_{L(X)} \leq c$ we have $\|\mathbf{A} v\|_{X \otimes(S)_{\rho, p}}^{2} \leq c^{2}\|v\|_{X \otimes(S)_{\rho, p}}^{2}$ and $\|\mathbf{A}\|_{L(X) \otimes(S)_{\rho}} \leq c$.

Lemma 1.5 ([25]) Let $\mathbf{A}$ be a coordinatewise operator for which all $A_{\alpha}, \alpha \in \mathscr{I}$, are polynomially bounded, i.e., $\left\|A_{\alpha}\right\|_{L(X)} \leq R(2 \mathbb{N})^{r \alpha}$ for some $r, R>0$. Then, $\mathbf{A}$ is a bounded operator:

$$
\begin{aligned}
& 1^{\circ} \mathbf{A}: X \otimes(S)_{-\rho,-p} \rightarrow X \otimes(S)_{-\rho,-q} \text { for } q \geq p+2 r, \text { and } \\
& 2^{\circ} \mathbf{A}: X \otimes(S)_{\rho, p} \rightarrow X \otimes(S)_{\rho, q} \text { for } q+2 r \leq p .
\end{aligned}
$$

Proof $1^{\circ}$ For $q \geq p+2 r$ we obtain $\|\mathbf{A}\|_{L(X) \otimes(S)_{-\rho}} \leq R$. Clearly, from (1.60) by (1.34) we obtain the estimate

$$
\begin{aligned}
\|\mathbf{A}(v)\|_{X \otimes(S)_{-\rho,-q}}^{2} & \leq R^{2} \sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{2 r \alpha}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-q \alpha} \\
& \leq R^{2} \sum_{\alpha \in \mathscr{I}}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}=R^{2}\|v\|_{X \otimes(S)_{-\rho,-p}}^{2}<\infty .
\end{aligned}
$$

$2^{\circ}$ For $q \leq p-2 r$ we obtain the estimate

$$
\begin{aligned}
\|\mathbf{A}(v)\|_{X \otimes(S)_{\rho, q}}^{2} & \leq R^{2} \sum_{\alpha \in \mathscr{\mathscr { I }}}(2 \mathbb{N})^{2 r \alpha}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{q \alpha} \\
& \leq R^{2} \sum_{\alpha \in \mathscr{I}}\left\|v_{\alpha}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha}=R^{2}\|v\|_{X \otimes\left(S S_{\rho, p}\right.}^{2}<\infty
\end{aligned}
$$

and thus $\|\mathbf{A}\|_{L(X) \otimes(S)_{\rho, p}} \leq R$.
Remark 1.14 The condition stating that the deterministic operators $A_{\alpha}, \alpha \in \mathscr{I}$ are polynomially bounded can be formulated as $\sum_{\alpha \in \mathscr{I}}\left\|A_{\alpha}\right\|_{L(X)}^{2}(2 \mathbb{N})^{-r \alpha}<\infty$ for some $r>0$, see [25].

Definition 1.21 The Wick convolution type operator $\mathbf{T} \diamond$ for $y=\sum_{\alpha \in \mathscr{I}} y_{\alpha} H_{\alpha}$ is defined by

$$
\begin{equation*}
\mathbf{T} \diamond(y)=\sum_{\alpha \in \mathscr{\mathscr { A }}} \sum_{\beta \leq \alpha} T_{\beta}\left(y_{\alpha-\beta}\right) H_{\alpha}=\sum_{\gamma \in \mathscr{\mathscr { I }}} \sum_{\alpha+\beta=\gamma} T_{\alpha}\left(y_{\beta}\right) H_{\gamma} . \tag{1.62}
\end{equation*}
$$

If the operators $T_{\alpha}, \alpha \in \mathscr{I}$ are assumed to be polynomially bounded and linear on $X$, then $\mathbf{T} \diamond$ is well-defined operator on $X \otimes(S)_{-\rho}$ and also on $X \otimes(S)_{\rho}$, see [25].
Lemma 1.6 ([25]) If $T_{\alpha}, \alpha \in \mathscr{I}$, satisfy $\sum_{\alpha \in \mathscr{I}}\left\|T_{\alpha}\right\|_{L(X)}(2 \mathbb{N})^{-\frac{p}{2} \alpha}<\infty$ for some $p>0$, then $\mathbf{T} \diamond$ is well-defined as a mapping $\mathbf{T} \diamond: X \otimes(S)_{-\rho,-p} \rightarrow X \otimes(S)_{-\rho,-p}$.

Proof For $v \in X \otimes(S)_{-\rho,-p}$ by the generalized Minkowski inequality we obtain

$$
\begin{aligned}
& \|\mathbf{T} \diamond(y)\|_{X \otimes(S)_{-\rho,-p}}^{2} \leq \sum_{\gamma \in \mathscr{I}}\left(\sum_{\alpha+\beta=\gamma}\left\|T_{\alpha}\right\|_{L(X)}\left\|y_{\beta}\right\|_{X}\right)^{2} \gamma!^{1-\rho}(2 \mathbb{N})^{-p \gamma} \\
& \quad \leq \sum_{\gamma \in \mathscr{I}}\left(\sum_{\alpha+\beta=\gamma}\left\|T_{\alpha}\right\|_{L(X)}(2 \mathbb{N})^{-\frac{p}{2} \alpha}\left\|y_{\beta}\right\|_{X}(2 \mathbb{N})^{-\frac{p}{2} \beta} \gamma!^{\frac{1-\rho}{2}}\right)^{2} \\
& \quad \leq\left(\sum_{\alpha \in \mathscr{I}}\left\|T_{\alpha}\right\|_{L(X)}(2 \mathbb{N})^{-\frac{p}{2} \alpha}\right)^{2} \sum_{\gamma \in \mathscr{I}}\left\|y_{\gamma}\right\|_{X}^{2} \gamma!^{1-\rho}(2 \mathbb{N})^{-p \gamma}<\infty .
\end{aligned}
$$

For more details about $\mathbf{T} \triangleleft$ we refer to [21, 25, 38].

In Chaps. 3 and 4 we deal with operators on $X \otimes L^{2}\left(\mu_{H}\right)$ and $X \otimes L^{2}(\mu)$ for a special choice $X=L^{2}([0, T], \mathscr{H})$. Therefore, here we state some properties of the fractional mapping $\mathscr{M}$ that connects these particular spaces.
Theorem 1.11 ([17]) The fractional mapping $\mathscr{M}$ satisfies the following properties:
(1) Let the operators $\widetilde{\mathbf{O}}: L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right) \rightarrow L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)$ and $\mathbf{O}: L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu) \rightarrow L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ be coordinatewise operators that correspond to the same family of operators $O_{\alpha}$ : $L^{2}([0, T], \mathscr{H}) \rightarrow L^{2}([0, T], \mathscr{H}), \alpha \in \mathscr{I}$. Then it holds $\mathscr{M}(\widetilde{\mathbf{O}} \widetilde{v})=\mathbf{O}(\mathscr{M} \widetilde{v})$,
(2) $\mathscr{M}$ is linear and also $\mathscr{M}(\widetilde{u} \diamond \widetilde{y})=\mathscr{M}(\widetilde{u}) \diamond \mathscr{M}(\widetilde{y})$,
(3) $\mathscr{M}\left(\mathbb{E}_{\mu_{H}} \widetilde{v}\right)=\mathbb{E}_{\mu}(\mathscr{M} \widetilde{v})$,
for $\tilde{v} \in L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)$ and $\widetilde{u}, \tilde{y} \in L^{2}([0, T], \mathscr{H}) \otimes(S)_{-\rho}^{(H)}$.
Proof Since the action of $\mathscr{M}$ on a stochastic process given in the form (1.41) is reflected as its action on the orthogonal basis of $L^{2}\left(\mu_{H}\right)$,the following are valid: $1^{\circ}$ Let $\tilde{v} \in L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)$. From (1.60), (1.61) and (1.52) we obtain

$$
\mathscr{M}(\widetilde{\mathbf{O}} \widetilde{v})=\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} O_{\alpha} v_{\alpha} \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathscr{I}} O_{\alpha} v_{\alpha} H_{\alpha}=\mathbf{O}\left(\sum_{\alpha \in \mathscr{I}} v_{\alpha} H_{\alpha}\right)=\mathbf{O}(\mathscr{M} \widetilde{v}) .
$$

$2^{\circ}$ By definition, the fractional operator $\mathscr{M}$ is linear. It also holds

$$
\begin{aligned}
\mathscr{M}(\widetilde{u} \diamond \tilde{y}) & =\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} u_{\alpha} y_{\beta} \widetilde{H}_{\alpha+\beta}\right)=\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} u_{\alpha} y_{\beta} H_{\alpha+\beta} \\
& =\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} u_{\alpha} \widetilde{H}_{\alpha}\right) \diamond \mathscr{M}\left(\sum_{\beta \in \mathscr{I}} y_{\beta} \widetilde{H}_{\beta}\right)=\mathscr{M}(\widetilde{u}) \diamond \mathscr{M}(\widetilde{y}) .
\end{aligned}
$$

$3^{\circ}$ For $\widetilde{v} \in L^{2}([0, T], \mathscr{H}) \otimes L^{2}\left(\mu_{H}\right)$ the element $\mathbb{E}_{\mu_{H}} \widetilde{v}$ is the zeroth coefficient of fractional expansion of $\widetilde{v}$, i.e., $\mathbb{E}_{\mu_{H}} \widetilde{v}=v_{0}$. Thus, $\mathscr{M}\left(\mathbb{E}_{\mu_{H}} \widetilde{v}\right)=v_{0}$. On the other hand, $\mathbb{E}_{\mu}(\mathscr{M} \widetilde{v})$ is the zeroth coefficient of the expansion of $\mathscr{M} \widetilde{v}$, which is also equal to $v_{0}$. Thus, $\mathscr{M}\left(\mathbb{E}_{\mu_{H}} \widetilde{v}\right)=\mathbb{E}_{\mu}(\mathscr{M} \widetilde{v})$.
Theorem 1.12 ([17]) For a differentiable $\mathscr{H}$-valued process $\tilde{z}$ from a fractional space it holds

$$
\mathscr{M}\left(\frac{d}{d t} \widetilde{z}\right)=\frac{d}{d t}(\mathscr{M} \widetilde{z}) .
$$

Proof Differentiation of a stochastic process is a simple coordinatewise operator, i.e., a process is considered to be differentiable if and only if its coordinates are differentiable deterministic functions. The assertion follows by applying $\mathscr{M}$ to $\frac{d}{d t} \widetilde{z}=\sum_{\alpha \in \mathscr{I}} \frac{d}{d t} z_{\alpha}(t) \widetilde{H}_{\alpha}(\omega)=\sum_{\alpha \in \mathscr{I}} z_{\alpha}^{\prime}(t) \widetilde{H}_{\alpha}(\omega)$. We obtain

$$
\mathscr{M}\left(\frac{d}{d t} \widetilde{z}\right)=\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} z_{\alpha}^{\prime}(t) \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathscr{I}} z_{\alpha}^{\prime}(t) H_{\alpha}=\frac{d}{d t}\left(\sum_{\alpha \in \mathscr{I}} z_{\alpha}(t) H_{\alpha}\right)=\frac{d}{d t}(\mathscr{M} \bar{z}) .
$$

Following [19] we consider the extension of the fractional operator $M^{(H)}$ from $S^{\prime}(\mathbb{R}) \rightarrow S^{\prime}(\mathbb{R})$ onto spaces of generalized stochastic processes.
Definition 1.22 Let $\mathbf{M}=M^{(H)} \otimes I d: S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho} \rightarrow S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}, \rho \in[0,1]$ be given by

$$
\begin{equation*}
\mathbf{M}\left(\sum_{\alpha \in \mathscr{I}} a_{\alpha}(t) \otimes H_{\alpha}(\omega)\right)=\sum_{\alpha \in \mathscr{I}} M^{(H)} a_{\alpha}(t) \otimes H_{\alpha}(\omega) \tag{1.63}
\end{equation*}
$$

The restriction of $\mathbf{M}$ onto $L_{H}^{2}(\mathbb{R}) \otimes L^{2}(\mu)$ is an isometry mapping $L_{H}^{2}(\mathbb{R}) \otimes$ $L^{2}(\mu) \rightarrow L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$.

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# Chapter 2 <br> Generalized Operators of Malliavin Calculus 


#### Abstract

In this chapter we extend Malliavin calculus from the classical finite variance setting to generalized processes with infinite variance and their corresponding test processes. The domain and range of the main operators of Malliavin calculuss are characterized on spaces of test and generalized processes. Some properties, such as integration by parts formula, the product rules with respect to ordinary and Wick multiplication and the chain rule are proved.


### 2.1 Introduction

The Malliavin derivative $\mathbb{D}$, the Skorokhod integral $\delta$ and the Ornstein-Uhlenbeck operator $\mathscr{R}$ play a crucial role in the stochastic calculus of variations, an infinitedimensional differential calculus on white noise spaces, also called the Malliavin calculus $[2,4,18,19,21,23]$. In stochastic analysis, the Malliavin derivative characterizes densities of distributions, the Skorokhod integral is an extension of the Itô integral to non-adapted processes, and the Ornstein-Uhlenbeck operator plays the role of the stochastic Laplacian. Additionally, the Malliavin derivative appears as the adjoint operator of the Skorokhod integral, while their composition, the OrnsteinUhlenbeck operator, is linear, unbounded and self-adjoint operator. These operators are interpreted in quantum theory respectively as the annihilation, the creation and the number operators.

Originally, the Malliavin derivative was introduced by Paul Malliavin in order to provide a probabilistic proof of Hörmander's sum of squares theorem for hypoelliptic operators and to study the existence and regularity of a density for the solution of stochastic differential equations [17]. Nowadays, besides applications concerning the existence and smoothness of a density for the probability law of random variables, it has found significant applications in stochastic control and mathematical finance, particularly in option pricing and computing the Greeks (the Greeks measure the stability of the option price under variations of the parameters) via the Clark-Ocone formula [3, 18, 22]. Recently in [20] a novel connection between the Malliavin calculus and Stein's method was discovered, which can be used to estimate the
distance of a random variable from Gaussian variables. In [14] this relationship was reviewed using the chaos expansion method.

In the classical setting [4, 16, 19], the domain of these operators is a strict subset of the set of processes with finite second moments leading to Sobolev type normed spaces. We recall these classical results and denote the corresponding domains with a"zero"in order to retain a nice symmetry between test and generalized processes. A more general characterization of the domain of these operators in Kondratiev generalized function spaces has been derived in [10, 12, 13]. Surjectivity of the operators for generalized processes for $\rho=1$ has been developed in [14, 15], while a setting for the domains of these operators for $\rho \in[0,1]$ and for test processes was developed in [8, 11]. We summarize these recent results, construct the domain of the operators and prove that they are linear and bounded within the corresponding spaces.

We adopt the notation from $[11,14,15]$ and denote the domains of all the operators in the Kondratiev space of distributions by a"minus"sign to reflect the fact that they correspond to generalized processes and the domains for test processes denote by a"plus" sign.

The Malliavin derivative of generalized stochastic processes has first been considered in [1] using the $\mathscr{S}$-transform of stochastic exponentials and chaos expansions with $n$-fold Itô integrals with some vague notion of the Itô integral of a generalized function. Our approach is different, it relies on chaos expansions via Hermite polynomials and it provides more precise results. A fine gradation of generalized and test functions is followed where each level has a Hilbert structure and consequently each level of singularity has its own domain, range, set of multipliers etc. We develop the calculus including the integration by parts formula, product rules, the chain rule, using the interplay of generalized processes with their test processes and different types of dual pairings. We apply the chaos expansion method to illustrate several known results in Malliavin calculus and thus provide a comprehensive insight into its capabilities. For example, we prove some well-known classical results, such as the commutator relationship between $\mathbb{D}$ and $\delta$ and the relation between Itô integration and Riemann integration. A further analysis including examples and applications can be found in [14].

### 2.2 The Malliavin derivative

In this section, we define the Malliavin derivative operator $\mathbb{D}$ on spaces of generalized stochastic processes, test stochastic processes and classical stochastic processes. We describe the domains in terms of chaos expansion representations.

Definition 2.1 Let $\rho \in[0,1]$ and let $u \in X \otimes(S)_{-\rho}$ be a generalized stochastic process given in the chaos expansion form $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}, u_{\alpha} \in X, \alpha \in \mathscr{I}$. We say that $u$ belongs to $\operatorname{Dom}_{-\rho,-p}(\mathbb{D})$ if there exists $p \in \mathbb{N}_{0}$ such that

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty \tag{2.1}
\end{equation*}
$$

and its Malliavin derivative is defined by

$$
\begin{equation*}
\mathbb{D} u=\sum_{|\alpha|>0} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha} \tag{2.2}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$, i.e., for a multi-index $\alpha=$ $\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathscr{I}$ we have $H_{\alpha-\varepsilon^{(k)}}=0$ if $\alpha_{k}=0$ and $H_{\alpha-\varepsilon^{(k)}}=H_{\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k-1}, \alpha_{k}-1, \alpha_{k+1}, \ldots, \alpha_{m}, 0,0, \ldots\right)}$ if $\alpha_{k} \geq 1$.

Thus, the domain of the Malliavin derivative in $X \otimes(S)_{-\rho}$ is given by

$$
\begin{align*}
\operatorname{Dom}_{-\rho}(\mathbb{D}) & =\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-\rho,-p}(\mathbb{D}) \\
& =\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{-\rho}: \sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\} \tag{2.3}
\end{align*}
$$

All processes that belong to $\operatorname{Dom}_{-\rho}(\mathbb{D})$ are called Malliavin differentiable. The operator $\mathbb{D}$ is also called the stochastic gradient.

The following theorem characterizes the range of the Malliavin derivative operator.

Theorem 2.1 ([8]) The Malliavin derivative of a stochastic process $u \in X \otimes(S)_{-\rho}$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{-\rho,-p}(\mathbb{D}) \rightarrow X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}
$$

for $l>p+1$ and $p \in \mathbb{N}_{0}$.
Proof Let $u, v \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ such that $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}, v=\sum_{\alpha \in \mathscr{I}} v_{\alpha} \otimes H_{\alpha}$. Then, $a u+b v \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ for all $a, b \in \mathbb{R}$. Clearly, from Definition 2.1 and

$$
\begin{align*}
\mathbb{D}(a u+b v) & =\mathbb{D}\left(\sum_{\alpha \in \mathscr{I}}\left(a u_{\alpha}+b v_{\alpha}\right) \otimes H_{\alpha}\right)=\sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}} \alpha_{k}\left(a u_{\alpha}+b v_{\alpha}\right) \otimes H_{\alpha-\varepsilon^{(k)}} \\
& =a \sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes H_{\alpha-\varepsilon^{(k)}}+b \sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}} \alpha_{k} v_{\alpha} \otimes H_{\alpha-\varepsilon^{(k)}}  \tag{2.4}\\
& =a \mathbb{D}(u)+b \mathbb{D}(v)
\end{align*}
$$

we conclude that $\mathbb{D}$ is a linear operator.
Note that the following $(2 \mathbb{N})^{\varepsilon^{(k)}}=(2 k)$, and $\left\|\xi_{k}\right\|_{-l}^{2}=(2 k)^{-l}, k \in \mathbb{N}$ hold. Assume that $u$ satisfies (2.1) for some $p \geq 0$. Thus, we have

$$
\begin{aligned}
\|\mathbb{D} u\|_{\left.X \otimes S_{-l}(\mathbb{R}) \otimes(S)\right)_{-\rho,-p}}^{2} & =\left\|\sum_{\alpha \in \mathscr{\mathscr { A }}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2} \\
& =\sum_{\alpha \in \mathscr{\mathscr { C }}}\left\|\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2}\left\|u_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}(2 k)^{-l}\right) \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{2}\left\|u_{\beta}\right\|_{X}^{2}(2 k)^{-l}\left(\frac{\beta!}{\beta_{k}}\right)^{1-\rho}(2 k)^{p}\right)(2 \mathbb{N})^{-p \beta} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{1+\rho}(2 k)^{-(l-p)}\right)\left\|u_{\beta}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq \sum_{\beta \in \mathscr{\mathscr { I }}}\left(\sum_{k=1}^{\infty} \beta_{k}\right)^{1+\rho}\left(\sum_{k=1}^{\infty}(2 k)^{-(l-p)}\right)\left\|u_{\beta}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& =c \sum_{\beta \in \mathscr{\mathscr { I }}}|\beta|^{1+\rho} \beta!^{1-\rho}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p \beta}=c\|u\|_{D_{D o m_{-\rho,-p}(\mathbb{D})}^{2}<\infty,}
\end{aligned}
$$

where $c=\sum_{k \in \mathbb{N}}(2 k)^{-(l-p)}<\infty$ for $l>p+1$. We used the substitution $\beta=\alpha+\varepsilon^{(k)}$, i.e., $\beta_{k}=\alpha_{k}+1$ and $\beta!=\beta_{k}\left(\beta-\varepsilon^{(k)}\right)!, k \in \mathbb{N}$, the Cauchy-Schwarz inequality and the estimate $\sum_{k \in \mathbb{N}} \beta_{k}^{1+\rho} \leq\left(\sum_{k \in \mathbb{N}} \beta_{k}\right)^{1+\rho}=|\beta|^{1+\rho}$.

For $\rho=1$ the result of the previous theorem was proven in [13] and the case $\rho=0$ was studied in [11, 14].

## Lemma 2.1 The following properties hold:

$1^{\circ}$ For $p \leq q$ it holds $\operatorname{Dom}(\mathbb{D})_{-\rho,-p} \subseteq \operatorname{Dom}(\mathbb{D})_{-\rho,-q}$.
$2^{\circ}$ The smallest domain is $\operatorname{Dom}_{-0}(\mathbb{D})$ and the largest domain is Dom $_{-1}(\mathbb{D})$, i.e., for $\rho \in(0,1), p \geq 0$ it holds $\operatorname{Dom}_{-0,-p}(\mathbb{D}) \subset \operatorname{Dom}_{-\rho,-(p+\rho)}(\mathbb{D}) \subset$ $\operatorname{Dom}_{-1,-(p+1)}(\mathbb{D})$.

Proof $1^{\circ}$ The statement follows from the fact that for $p \leq q$ it holds

$$
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \leq \sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} .
$$

$2^{\circ}$ Since $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ for $\alpha \in \mathscr{I}$ we obtain

$$
\begin{aligned}
\|u\|_{\text {Dom }_{-1,-(p+1)}(\mathbb{D})} & =\sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(p+1) \alpha} \\
& \leq \sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-(p+\rho) \alpha} \\
& \leq \sum_{\alpha \in \mathscr{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\|u\|_{D o m_{-0,-p}(\mathbb{D})}^{2}
\end{aligned}
$$

Remark 2.1 Let $u \in \operatorname{Dom}_{-\rho}(\mathbb{D})$. Then, $u \in \operatorname{Dom}_{-\rho}(\mathbb{D} a)$, for a given sequence $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$, for all $k \in \mathbb{N}$. Indeed, there exists $p \geq 0$ such that

$$
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha} \leq c \sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty .
$$

Now we characterize the domain of the Malliavin derivative operator on a set of test stochastic processes $X \otimes(S)_{\rho}$.

Definition 2.2 Let $\rho \in[0,1]$ and let $v \in X \otimes(S)_{\rho}$ be given in the form $v=$ $\sum_{\alpha \in \mathscr{I}} v_{\alpha} \otimes H_{\alpha}, v_{\alpha} \in X, \alpha \in \mathscr{I}$. We say that $u$ belongs to $\operatorname{Dom}_{\rho, p}(\mathbb{D})$ if

$$
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1-\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty, \quad \text { for all } \quad p \in \mathbb{N}_{0}
$$

Thus, the domain of the Malliavin derivative operator in $X \otimes(S)_{\rho}$ is the projective limit of the spaces $\operatorname{Dom}_{\rho, p}(\mathbb{D})$, i.e.,

$$
\begin{align*}
\operatorname{Dom}_{\rho}(\mathbb{D}) & =\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{\rho, p}(\mathbb{D}) \\
& =\bigcap_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{\rho, p}: \sum_{\alpha \in \mathscr{I}}|\alpha|^{1-\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} . \tag{2.5}
\end{align*}
$$

Theorem 2.2 ([11]) The Malliavin derivative of a test stochastic process $v \in X \otimes$ $(S)_{\rho}$ is a linear and continuous mapping

$$
\mathbb{D}: \quad \operatorname{Dom}_{\rho, p}(\mathbb{D}) \rightarrow X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}, \quad \text { for } \quad p>l+1
$$

Proof Let $v=\sum_{\alpha \in \mathscr{I}} v_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{\rho, p}(\mathbb{D})$. The assertion follows from (2.2) and

$$
\begin{aligned}
& \|\mathbb{D} v\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}^{2}=\left\|\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) v_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha}\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}} \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}}\left\|\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) v_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2}\left\|v_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}(2 k)^{l}\right) \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{2}\left\|v_{\beta}\right\|_{X}^{2}(2 k)^{l}\left(\frac{\beta!}{\beta_{k}}\right)^{1+\rho}(2 k)^{-p}\right)(2 \mathbb{N})^{p \beta} \\
& \quad=\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{1-\rho}(2 k)^{-(p-l)}\right)\left\|v_{\beta}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \quad \leq c^{1-\rho} \sum_{\beta \in \mathscr{\mathscr { I }}}|\beta|^{1-\rho} \beta!^{1+\rho}\left\|v_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta}=c^{1-\rho}\|v\|_{D o m}^{2},(\mathbb{D})<\infty,
\end{aligned}
$$

where $\sum_{k \in \mathbb{N}} \beta_{k}^{1-\rho}(2 k)^{l-\rho} \leq\left(\sum_{k \in \mathbb{N}} \beta_{k}\right)^{1-\rho}\left(\sum_{k \in \mathbb{N}}(2 k)^{\frac{l-p}{1-\rho}}\right)^{1-\rho} \leq|\beta|^{1-\rho} \cdot c^{1-\rho}$, and $c=\sum_{k \in \mathbb{N}}(2 k)^{\frac{l-p}{1-\rho}} \leq \sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$, for $p>l+1$. In the previous estimates $\beta=\alpha+\varepsilon^{(k)}, \alpha!=\left(\beta-\varepsilon^{(k)}\right)!=\frac{\beta!}{\beta_{k},}, k \in \mathbb{N}$ and $(2 \mathbb{N})^{-p \varepsilon^{(k)}}=(2 k)^{-p}, k \in \mathbb{N}$. The linearity property of $\mathbb{D}$ on $\operatorname{Dom}_{\rho, p}(\mathbb{D})$ follows from (2.4).

Lemma 2.2 The following properties hold:

```
\(1^{\circ} \operatorname{Dom}_{\rho, q}(\mathbb{D}) \subseteq \operatorname{Dom}_{\rho, p}(\mathbb{D})\), for \(p \leq q\).
\(2^{\circ}\) The smallest domain is \(\operatorname{Dom}_{1}(\mathbb{D})\) and the largest is \(\operatorname{Dom}_{0}(\mathbb{D})\), i.e., for \(\rho \in(0,1)\)
\(\operatorname{Dom}_{1, p+1}(\mathbb{D}) \subset \operatorname{Dom}_{\rho, p+\rho}(\mathbb{D}) \subset \operatorname{Dom}_{0, p}(\mathbb{D})\).
```

Proof The first statement follows from $(2 \mathbb{N})^{p} \leq(2 \mathbb{N})^{q}$, which holds for all $p \leq q$, while the second is a consequence of $|\alpha| \leq(2 \mathbb{N})^{\alpha}$, for all $\alpha \in \mathscr{I}$.

Definition 2.3 For a square integrable stochastic process $u \in X \otimes L^{2}(\mu)$ the domain of $\mathbb{D}$ is given by

$$
\begin{equation*}
\operatorname{Dom}_{0}(\mathbb{D})=\left\{u \in X \otimes L^{2}(\mu): \sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\} . \tag{2.6}
\end{equation*}
$$

Theorem 2.3 The Malliavin derivative of a process $u \in \operatorname{Dom}_{0}(\mathbb{D})$ is a linear and continuous mapping

$$
\mathbb{D}: \operatorname{Dom}_{0}(\mathbb{D}) \rightarrow X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)
$$

Proof Let $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\mathbb{D})$. Then,

$$
\begin{aligned}
\|\mathbb{D} u\|_{X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)}^{2} & =\left\|\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha}\right\|_{X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)}^{2} \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}}\left\|\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2} \alpha! \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2}\left\|u_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}\right) \alpha! \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}^{2}\left\|u_{\beta}\right\|_{X}^{2} \frac{1}{\beta_{k}}\right) \beta! \\
& =\sum_{|\beta| \geq 1}\left(\sum_{k \in \mathbb{N}} \beta_{k}\right)\left\|u_{\beta}\right\|_{X}^{2} \beta!=\sum_{|\beta| \geq 1}|\beta| \beta!\left\|u_{\beta}\right\|_{X}^{2}<\infty .
\end{aligned}
$$

The linearity property of the operator $\mathbb{D}$ on $\operatorname{Dom}_{0}(\mathbb{D})$ follows from (2.4).
For $\rho \in[0,1]$ and all $p \in \mathbb{N}$ we obtained $\operatorname{Dom}_{\rho, p}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq$ $\operatorname{Dom}_{-\rho,-p}(\mathbb{D})$, and therefore $\operatorname{Dom}_{\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-\rho}(\mathbb{D})$. Moreover, using the estimate $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ it follows that

$$
\begin{align*}
X \otimes(S)_{-\rho,-(p-2)} & \subseteq \operatorname{Dom}_{-\rho,-p}(\mathbb{D}) \subseteq X \otimes(S)_{-\rho,-p}, \quad p>3, \quad \text { and } \\
X \otimes(S)_{\rho, p+2} & \subseteq \operatorname{Dom}_{\rho, p}(\mathbb{D}) \subseteq X \otimes(S)_{\rho, p}, \quad p>0 \tag{2.7}
\end{align*}
$$

Remark 2.2 Let $v \in \operatorname{Dom}_{\rho}(\mathbb{D} a)$. Then, $u \in \operatorname{Dom}_{\rho}(\mathbb{D})$, for a given sequence $a=$ $\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$, for all $k \in \mathbb{N}$. Indeed, for all $p \in \mathbb{N}_{0}$ we obtain

$$
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{p \alpha}<\infty .
$$

Hence, $\operatorname{Dom}_{\rho}(\mathbb{D} a) \subseteq \operatorname{Dom}_{\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{0}(\mathbb{D}) \subseteq \operatorname{Dom}_{-\rho}(\mathbb{D}) \subseteq \operatorname{Dom}_{-\rho}(\mathbb{D} a)$.
Remark 2.3 For $u \in \operatorname{Dom}_{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\mathbb{D})$ it is usual to write

$$
\mathbb{D}_{t} u=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k}(t) \otimes H_{\alpha-\varepsilon^{(k)}}
$$

in order to emphasize that the Malliavin derivative takes a random variable into a process, i.e., that $\mathbb{D} u$ is a function of $t$. Moreover, the formula

$$
\mathbb{D}_{t} F(\omega)=\lim _{h \rightarrow 0} \frac{1}{h}\left(F\left(\omega+h \cdot \chi_{[t, \infty)}\right)-F(\omega)\right), \quad \omega \in S^{\prime}(\mathbb{R})
$$

justifies the name stochastic derivative for the Malliavin operator. Since generalized functions do not have point values, this notation would be somewhat misleading for $u \in \operatorname{Dom}_{-\rho}(\mathbb{D})$. Therefore we omit the index $t$ in $\mathbb{D}_{t}$ that usually appears in the literature and write $\mathbb{D}$.

Remark 2.4 Higher orders of the Malliavin derivative operator are defined recursively, i.e., $\mathbb{D}^{(k)}=\mathbb{D} \circ \mathbb{D}^{(k-1)}, k \geq 1$ and $\mathbb{D}^{0}=I d$. For higher order derivatives to be well-defined, it is necessary that each result of the application of the operator $\mathbb{D}$ remains in its domain. For this purpose we note that if $u \in X \otimes(S)_{-\rho,-q}$ for some $q \geq 0$, then for $p \geq q+2$ it holds $u \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$, see (2.7). Thus, $\mathbb{D}^{(2)}: \operatorname{Dom}_{-\rho,-p}(\mathbb{D}) \xrightarrow{\mathbb{D}} X \otimes S_{-l_{1}} \otimes(S)_{-\rho,-p} \subseteq S_{-l_{1}} \otimes \operatorname{Dom}_{-\rho,-(p+2)}(\mathbb{D})$ $\xrightarrow{\mathbb{D}} S_{-l_{1}} \otimes S_{-l_{2}} \otimes X \otimes(S)_{-\rho,-(p+2)}$, where $l_{1}>p+1$ and $l_{2}>p+3$. Similarly, for any $k \in \mathbb{N}$ the operator $\mathbb{D}^{(k)}$ maps $X \otimes(S)_{-\rho,-(p-2)} \subset \operatorname{Dom}_{-\rho,-p}(\mathbb{D}) \rightarrow$ $X \otimes S_{-l_{1}} \otimes S_{-l_{2}} \otimes \cdots \otimes S_{-l_{k}} \otimes(S)_{-\rho,-(p+2 k)}$ where $l_{j}>p+1+2(j-1), 1 \leq j \leq k$. More details can be found in [15].

### 2.3 The Skorokhod Integral

The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $L^{2}(\mu)$-sense. In [13] the authors extended the definition of the Skorokhod integral from Hilbert space valued processes to the class of $S^{\prime}$-valued generalized processes. Further development in this direction was proposed in $[8,11,13,14]$. Here we summarize these results.

Definition 2.4 Let $\rho \in[0,1]$. Let $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ such that $f_{\alpha} \in X \otimes S^{\prime}(\mathbb{R})$ is given by $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k}, f_{\alpha, k} \in X$. Then, $F$ belongs to $\operatorname{Dom}_{-\rho,-l,-p}(\delta)$ if it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1-\rho} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-p \alpha}<\infty . \tag{2.8}
\end{equation*}
$$

Thus, the chaos expansion of its Skorokhod integral is given by

$$
\begin{equation*}
\delta(F)=\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}=\sum_{\alpha>\mathbf{0}} \sum_{k \in \mathbb{N}} f_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} . \tag{2.9}
\end{equation*}
$$

The domain of the Skorokhod integral operator for generalized stochastic processes in $\mathscr{X}=X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ is denoted by $\operatorname{Dom}_{-\rho}(\delta)$ and is given as the inductive limit of the spaces $\operatorname{Dom}_{-\rho,-l,-p}(\delta), l, p \in \mathbb{N}_{0}$, i.e.,

$$
\operatorname{Dom}_{-\rho}(\delta)=\bigcup_{p>l+1} \operatorname{Dom}_{-\rho,-l,-p}(\delta)=\bigcup_{p>l+1}\left\{F \in \mathscr{X}:\|F\|_{\text {Dom }_{-\rho,-l,-p}}^{2}<\infty\right\},
$$

where $\|F\|_{D_{o m-\rho,-l .-p}}^{2}$ is given by (2.8). Each stochastic process $F \in \operatorname{Dom}_{-\rho}(\delta)$ is called integrable in the Skorokhod sense.

Theorem 2.4 Let $\rho \in[0,1]$. The Skorokhod integral $\delta$ is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{-\rho,-l,-p}(\delta) \rightarrow X \otimes(S)_{-\rho,-p}, \quad p>l+1
$$

Proof A linear combination of two Skorokhod integrable processes $F, G$ is again a Skorokhod integrable process $a F+b G, a, b \in \mathbb{R}$. Namely, we have

$$
\begin{align*}
\delta(a F+b G) & =\delta\left(\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(a f_{\alpha, k}+b g_{\alpha, k}\right) \otimes \xi_{k} \otimes H_{\alpha}\right) \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(a f_{\alpha, k}+b g_{\alpha, k}\right) \otimes H_{\alpha+\varepsilon^{(k)}}  \tag{2.10}\\
& =a \sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}+b \sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}} \\
& =a \delta(F)+b \delta(G)
\end{align*}
$$

For the second part of the statement we use the Cauchy-Schwarz inequality and the estimates $\beta_{k}+1 \leq\left|\beta+\varepsilon^{(k)}\right|=|\beta|+1 \leq 2|\beta|$, when $\beta>\mathbf{0}, k \in \mathbb{N}$. Clearly,

$$
\begin{aligned}
&\|\delta(F)\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha>\mathbf{0}}\left\|\sum_{k \in \mathbb{N}} f_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
&= \sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} f_{\beta, k}\left(\beta_{k}+1\right)^{\frac{1-\rho}{2}}(2 k)^{-\frac{p}{2}}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
&= \sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} f_{\beta, k}(2 k)^{-\frac{l}{2}}\left(\beta_{k}+1\right)^{\frac{1-\rho}{2}}(2 k)^{-\frac{p-l}{2}}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq \sum_{k \in \mathbb{N}}\left\|f_{\mathbf{0}, k}\right\|_{X}^{2}(2 k)^{-l} \sum_{k \in \mathbb{N}}(2 k)^{p-l} \\
& \quad+2^{1-\rho} \sum_{\beta>\mathbf{0}}\left(\sum_{k \in \mathbb{N}}\left\|f_{\beta, k}\right\|_{X}^{2}(2 k)^{-l} \sum_{k \in \mathbb{N}}(2 k)^{l-p}\right)|\beta|^{1-\rho} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq m\left\|f_{\mathbf{0}}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}+2 m \sum_{\beta>\mathbf{0}}\left\|f_{\beta}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}|\beta|^{1-\rho} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta}<\infty
\end{aligned}
$$

where $f_{\alpha} \in X \otimes S_{-l}(\mathbb{R})$ for $\alpha \in \mathscr{I}$ and $m=\sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$ for $p>l+1$.
Remark 2.5 In the previous theorem the range of the operator $\delta$ is the space $X \otimes$ $(S)_{-\rho,-p}$ with the same level of singularity $p$ as in the domain of the operator $\delta$, i.e., $\operatorname{Dom}_{-\rho,-l,-p}(\delta)$, where $p>l+1$. On the other hand, the range of $\delta$ can be seen as the space $X \otimes(S)_{-\rho,-q}$, with lower level of singularity $q$ that depends on $\rho$. Clearly, by applying the estimate $|\beta| \leq(2 \mathbb{N})^{\beta}, \beta \in \mathscr{I}$ we obtain

$$
\begin{aligned}
\|\delta(F)\|_{X \otimes(S)_{-\rho,-p}}^{2} & \leq m\left\|f_{\mathbf{0}}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}+2 m \sum_{\beta>\mathbf{0}}\left\|f_{\beta}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}|\beta|^{1-\rho} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq 2 m \sum_{\beta \in \mathscr{I}}\left\|f_{\beta}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-(p-1+\rho) \beta}
\end{aligned}
$$

$$
\begin{aligned}
& \leq 2 m \sum_{\beta \in \mathscr{I}}\left\|f_{\beta}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-q \beta} \\
& =2 m\|F\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}}^{2}<\infty
\end{aligned}
$$

for $q \leq p-1+\rho$ and $p>l+1$. Since $|\alpha|^{1-\rho} \geq 1$ for $\alpha>\mathbf{0}$, then it also holds

$$
\begin{aligned}
& \|u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}}\left\|u_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& \quad \leq \sum_{\alpha \in \mathscr{I}}|\alpha|^{1-\rho}\left\|u_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}=\|u\|_{D_{o m_{-\rho,-l,-p}(\delta)}^{2}}
\end{aligned}
$$

Thus, for $p \in \mathbb{N}$ we have

$$
X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-(p-1+\rho)} \subseteq \operatorname{Dom}_{-\rho,-l,-p}(\delta) \subseteq X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}
$$

Particularly, the domain $\operatorname{Dom}_{-1}(\delta)$ was characterized in [13, 15].
Next, we characterize the domains $\operatorname{Dom}_{\rho}(\delta)$ and $\operatorname{Dom}_{0}(\delta)$ of the Skorokhod integral operator for test processes from $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ and square integrable processes from $X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$, as minor modifications of those presented in [11, 14].
Definition 2.5 Let $\rho \in[0,1]$. Let $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ be a test $S(\mathbb{R})$-valued stochastic process and let $f_{\alpha} \in X \otimes S(\mathbb{R})$ be given by the expansion $f_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k}, f_{\alpha, k} \in X$. We say that the process $F$ belongs to $\operatorname{Dom}_{\rho, l, p}(\delta)$ if

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}(2 \mathbb{N})^{p \alpha}<\infty \tag{2.11}
\end{equation*}
$$

Then, the chaos expansion form of the Skorokhod integral of $F$ is given by (2.9).
The domain of the Skorokhod integral for test stochastic processes in $X \otimes S(\mathbb{R}) \otimes$ $(S)_{\rho}$ is denoted by $\operatorname{Dom}_{\rho}(\delta)$ and is given as the projective limit of the spaces $\operatorname{Dom}_{\rho, l, p}(\delta), l, p \in \mathbb{N}_{0}$, i.e.,

$$
\begin{aligned}
\operatorname{Dom}_{\rho}(\delta) & =\bigcap_{l>p+1} \operatorname{Dom}_{\rho, l, p}(\delta) \\
& =\bigcap_{l>p+1}\left\{F \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}:\|F\|_{\operatorname{Dom}_{\rho, l, p}(\delta)}^{2}<\infty\right\}
\end{aligned}
$$

where $\|F\|_{D o m_{\rho, l, p}(\delta)}^{2}$ is defined by (2.11). All test processes $F$ that belong to Dom $_{\rho}(\delta)$ are called Skorokhod integrable.
Theorem 2.5 The Skorokhod integral $\delta$ of a $S_{l}(\mathbb{R})$-valued stochastic test process is a linear and continuous mapping

$$
\delta: \operatorname{Dom}_{\rho, l, p}(\delta) \rightarrow X \otimes(S)_{\rho, p}, \quad l>p+1, \quad p \in \mathbb{N}
$$

Proof Let $U=\sum_{\alpha \in \mathscr{\mathscr { F }}} u_{\alpha} \otimes H_{\alpha} \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}, u_{\alpha}=\sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \in$ $X \otimes S_{l}(\mathbb{R}), u_{\alpha, k} \in X$, for $p, l \geq 1$. Then, from (2.9) by the substitution $\alpha=\beta+\varepsilon^{(k)}$, $\alpha!=\left(\beta_{k}+1\right) \beta!$ and the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
& \|\delta(U)\|_{X \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha>0}\left\|\sum_{k \in \mathbb{N}} u_{\alpha-\varepsilon^{(k), k}}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} u_{\beta, k}\left(\beta_{k}+1\right)^{\frac{1+\rho}{2}}(2 k)^{\frac{p}{2}}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& =\sum_{\beta \in \mathscr{\mathscr { I }}}\left\|\sum_{k \in \mathbb{N}} u_{\beta, k}(2 k)^{\frac{l}{2}}\left(\beta_{k}+1\right)^{\frac{1+\rho}{2}}(2 k)^{\frac{p-l}{2}}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq \sum_{k \in \mathbb{N}}\left\|u_{\mathbf{0}, k}\right\|_{X}^{2}(2 k)^{l} \sum_{k \in \mathbb{N}}(2 k)^{p-l} \\
& \quad+4 \sum_{\beta>\mathbf{0}}\left(\sum_{k \in \mathbb{N}}\left\|u_{\beta, k}\right\|_{X}^{2}(2 k)^{l} \sum_{k \in \mathbb{N}}(2 k)^{p-l}\right)|\beta|^{1+\rho} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \quad \leq m\left\|u_{\mathbf{0}}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}+4 m \sum_{\beta>\mathbf{0}}\left\|u_{\beta}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}|\beta|^{1+\rho} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta}<\infty,
\end{aligned}
$$

where $m=\sum_{k \in \mathbb{N}}(2 k)^{p-l}<\infty$ for $l>p+1$. Moreover, the linearity property of $\delta$ on the set of test processes follows from (2.10).

Remark 2.6 With the same arguments as in Remark 2.5 and by $|\alpha| \leq(2 \mathbb{N})^{\alpha}, \alpha \in \mathscr{I}$ we obtain

$$
\begin{aligned}
\|\delta(U)\|_{X \otimes(S)_{\rho, p}}^{2} & \leq m\left\|u_{0}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}+4 m \sum_{\beta>0}\left\|u_{\beta}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}|\beta|^{1+\rho} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq m\left\|u_{\mathbf{0}}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}+4 m \sum_{\beta \in \mathscr{I}}\left\|u_{\beta}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{(p+1+\rho) \beta} \\
& \leq m\left\|u_{\mathbf{0}}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2}+4 m\|U\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, q}}^{2}<\infty,
\end{aligned}
$$

for all $q \geq p+1+\rho$. Since it holds $|\alpha|^{1+\rho} \geq 1$ for $\alpha>\mathbf{0}$, and from

$$
\|u\|_{X \otimes S_{l}(\mathbb{R}) \otimes\left(S S_{\rho, p}\right.}^{2} \leq \sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha|^{1+\rho}\left\|u_{\alpha}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha}=\|u\|_{D_{o m_{\rho, l p}(\delta)}^{2}}^{2},
$$

we obtain for $l>p+1$ the inclusions

$$
\begin{equation*}
X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p+1+\rho} \subseteq \operatorname{Dom}_{\rho, l, p}(\delta) \subseteq X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p} \tag{2.12}
\end{equation*}
$$

Definition 2.6 Let a square integrable stochastic processes $F \in X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$ be of the form $F=\sum_{\alpha \in \mathscr{\mathscr { V }}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}, f_{\alpha, k} \in X$. The process $F$ is Skorokhod integrable if it belongs to the space $\operatorname{Dom}_{0}(\delta)$, i.e., if it holds

$$
\begin{equation*}
\operatorname{Dom}_{0}(\delta)=\left\{F \in X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu): \sum_{\alpha \in \mathscr{I}}|\alpha| \alpha!\left\|f_{\alpha}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2}<\infty\right\} \tag{2.13}
\end{equation*}
$$

Theorem 2.6 The Skorokhod integral $\delta$ is a linear and continuous mapping

$$
\delta: \quad \operatorname{Dom}_{0}(\delta) \rightarrow X \otimes L^{2}(\mu) .
$$

Proof Let $F=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\delta)$. Then,

$$
\begin{aligned}
& \|\delta(F)\|_{X \otimes L^{2}(\mu)}^{2}=\sum_{\alpha>\mathbf{0}}\left\|\sum_{k \in \mathbb{N}} f_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \alpha!=\sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} f_{\beta, k} \sqrt{\beta_{k}+1}\right\|_{X}^{2} \beta! \\
& \leq \sum_{\beta \in \mathscr{I}}\left(\sum_{k \in \mathbb{N}}\left\|f_{\beta, k}\right\|_{X} \sqrt{\beta_{k}+1}\right)^{2} \beta!\leq \sum_{k \in \mathbb{N}}\left\|f_{\mathbf{0}, k}\right\|_{X}^{2}+2 \sum_{\beta>\mathbf{0}} \sum_{k \in \mathbb{N}}\left\|f_{\beta, k}\right\|_{X}^{2}|\beta| \beta! \\
& =\left\|f_{\mathbf{0}}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2}+2 \sum_{\beta>\mathbf{0}}\left\|f_{\beta}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2}|\beta| \beta!<\infty .
\end{aligned}
$$

Remark 2.7 Higher orders of the Skorokhod integral are considered in [15]. Let $\delta^{0}=I d$ and $k$ th order of the operator $\delta$ is defined recursively by $\delta^{(k)}=\delta \circ \delta^{(k-1)}$, $k \in \mathbb{N}$. We proved in Theorem 2.4 that $\delta: X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p} \rightarrow X \otimes(S)_{-\rho,-p}$, for $p>l+1$. Thus, for any $k \in \mathbb{N}$, the opertaor $\delta^{(k)}$ maps $X \otimes S_{-l_{1}} \otimes S_{-l_{2}} \otimes \cdots \otimes$ $S_{-l_{k}} \otimes(S)_{-\rho,-p} \rightarrow X \otimes(S)_{-\rho,-p}$ for $p>\max \left\{l_{1}, l_{2}, \ldots, l_{k}\right\}+1$.

### 2.4 The Ornstein-Uhlenbeck Operator

The third main operator of the Malliavin calculus is the Ornstein-Uhlenbeck operator. We describe the domain and the range of the Ornstain-Uhlenbeck operator for different classes of stochastic processes [8, 11, 14, 15].

Definition 2.7 The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathscr{R}=\delta \circ \mathbb{D}$ and is called the Ornstein-Uhlenbeck operator.

Since the estimate $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ holds for all $\alpha \in \mathscr{I}$, the image of the Malliavin derivative is included in the domain of the Skorokhod integral and thus we can define their composition. For example, for $v \in \operatorname{Dom}_{-\rho,-l,-p}(\delta)$ and $q+1-\rho \leq p$ we obtain

$$
\begin{aligned}
\|v\|_{D_{o m}^{-\rho,-l,-p}(\delta)}^{2} & =\sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha|^{1-\rho} \alpha!^{1-\rho}\left\|v_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq \sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho}\left\|v_{\alpha}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2}(2 \mathbb{N})^{-q \alpha}=\|v\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}^{2}},
\end{aligned}
$$

i.e., $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q} \subseteq \operatorname{Dom}_{-\rho,-l,-p}(\mathbb{D})$ for $q+1-\rho \leq p$. From Theorem 2.1 and Theorem 2.4 we obtain additional conditions $l>q+1$ and $p>l+1$ and thus for $p>q+2$ the operator $\mathscr{R}$ is well defined in $X \otimes(S)_{-\rho}$.

Theorem 2.7 For a Malliavin differentiable stochastic process u that is represented in the form $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}$, the Ornstein-Uhlenbeck operator is given by

$$
\begin{equation*}
\mathscr{R}(u)=\sum_{\alpha \in \mathscr{\mathscr { A }}}|\alpha| u_{\alpha} \otimes H_{\alpha} . \tag{2.14}
\end{equation*}
$$

Proof Since the image of the Malliavin derivative of a process $u$ is included in the domain of the Skorokhod integral, the Ornstein-Uhlenbeck operator is well defined. We combine (2.2) and (2.9) and obtain

$$
\begin{aligned}
\mathscr{R}(u)=\delta(\mathbb{D} u) & =\delta\left(\sum_{\alpha>0} \sum_{k \in \mathbb{N}} \alpha_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}\right) \\
& =\sum_{\alpha \in \mathscr{I}}\left(\sum_{k \in \mathbb{N}} \alpha_{k}\right) u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{I}}|\alpha| u_{\alpha} \otimes H_{\alpha} .
\end{aligned}
$$

Remark 2.8 For a special choice of $u=u_{\alpha} \otimes H_{\alpha}, \alpha \in \mathscr{I}$ we obtain that the FourierHermite polynomials are eigenfunctions of $\mathscr{R}$ and the corresponding eigenvalues are $|\alpha|, \alpha \in \mathscr{I}$, i.e.,

$$
\begin{equation*}
\mathscr{R}\left(u_{\alpha} \otimes H_{\alpha}\right)=|\alpha| u_{\alpha} \otimes H_{\alpha} . \tag{2.15}
\end{equation*}
$$

The domain of the Ornstein-Uhlenbeck operator in $X \otimes(S)_{-\rho}$ is given as the inductive limit $\operatorname{Dom}_{-\rho}(\mathscr{R})=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-\rho,-p}(\mathscr{R})$ of the spaces

$$
\begin{equation*}
\operatorname{Dom}_{-\rho,-p}(\mathscr{R})=\left\{u \in X \otimes(S)_{-\rho,-p}: \sum_{\alpha \in \mathscr{I}}|\alpha|^{2} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty\right\} . \tag{2.16}
\end{equation*}
$$

Theorem 2.8 The operator $\mathscr{R}$ is a linear and continuous mapping

$$
\mathscr{R}: \operatorname{Dom}_{-\rho,-p}(\mathscr{R}) \rightarrow X \otimes(S)_{-\rho,-p}, \quad p \in \mathbb{N}_{0} .
$$

Moreover, $\operatorname{Dom}_{-\rho}(\mathscr{R}) \subseteq$ Dom $_{-\rho}(\mathbb{D})$, while for $\rho=1$ they coincide.
Proof The Ornstein-Uhlenbeck operator is linear, i.e., by (2.14) for processes $u, v \in$ $\operatorname{Dom}_{-\rho,-p}(\mathscr{R})$ it holds that $a u+b v \in \operatorname{Dom}_{-\rho,-p}(\mathscr{R}), a, b \in \mathbb{R}$ and

$$
\begin{align*}
\mathscr{R}(a u+b v) & =\mathscr{R}\left(\sum_{\alpha \in \mathscr{I}}\left(a u_{\alpha}+b v_{\alpha}\right) \otimes H_{\alpha}\right)=\sum_{\alpha \in \mathscr{I}}|\alpha|\left(a u_{\alpha}+b v_{\alpha}\right) \otimes H_{\alpha} \\
& =a \sum_{\alpha \in \mathscr{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}+b \sum_{\alpha \in \mathscr{I}}|\alpha| v_{\alpha} \otimes H_{\alpha}=a \mathscr{R}(u)+b \mathscr{R}(v) . \tag{2.17}
\end{align*}
$$

For $u=\sum_{\alpha \in \mathscr{\mathscr { G }}} u_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{-\rho,-p}(\mathscr{R}), p \in \mathbb{N}$ we obtain

$$
\begin{gathered}
\|\mathscr{R} u\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{\mathscr { I }}}\left\|u_{\alpha}\right\|_{X}^{2}|\alpha|^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}=\|u\|_{D o m_{-\rho,-p}(\mathscr{R})}^{2}<\infty \text { and } \\
\sum_{\alpha \in \mathscr{I}}\left\|u_{\alpha}\right\|_{X}^{2}|\alpha|^{1+\rho} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathscr{I}}\left\|u_{\alpha}\right\|_{X}^{2}|\alpha|^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} .
\end{gathered}
$$

Particularly, for $\rho=1$ we have

$$
\|\mathscr{R} u\|_{X \otimes(S)_{-1,-p}}^{2}=\sum_{\alpha \in \mathscr{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}=\|u\|_{D o m_{-1,-p}(\mathbb{D})}^{2}<\infty, \quad p \in \mathbb{N}_{0},
$$

and thus $\operatorname{Dom}_{-1}(\mathscr{R})=\operatorname{Dom}_{-1}(\mathbb{D})$.
Particular case $\rho=1$ was considered in [10].
In next chapter we will show that Gaussian processes with zero expectation are the only fixed points of the Ornstein-Uhlenbeck operator, see Remark 3.2.

The domain of the Ornstein-Uhlenbeck operator in the space $X \otimes(S)_{\rho}$ is defined as the projective limit $\operatorname{Dom}_{\rho}(\mathscr{R})=\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{\rho, p}(\mathscr{R})$ of the spaces

$$
\begin{equation*}
\operatorname{Dom}_{\rho, p}(\mathscr{R})=\left\{v \in X \otimes(S)_{\rho, p}: \sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}|\alpha|^{2}\left\|v_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\} . \tag{2.18}
\end{equation*}
$$

Theorem 2.9 ( $[8,11])$ The operator $\mathscr{R}$ is a linear and continuous mapping

$$
\mathscr{R}: \quad \operatorname{Dom}_{\rho, p}(\mathscr{R}) \rightarrow X \otimes(S)_{\rho, p}, \quad p \in \mathbb{N}_{0} .
$$

Moreover, it holds $\operatorname{Dom}_{\rho}(\mathbb{D}) \supsetneq \operatorname{Dom}_{\rho}(\mathscr{R})$.
Proof Let $v=\sum_{\alpha \in \mathscr{J}} v_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{\rho, p}(\mathscr{R}), p \geq 0$. Then,

$$
\|\mathscr{R} v\|_{X \otimes(S)_{\rho, p}}^{2}=\sum_{\alpha \in \mathscr{\mathscr { A }}}\left\|v_{\alpha}\right\|_{X}^{2}|\alpha|^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha}=\|v\|_{\text {Dom }_{p}(\mathscr{R})}^{2}<\infty
$$

and the statement follows. The operator $\mathscr{R}$ is linear as the composition of two linear operators, thus (2.17) holds.

From the following inequalities

$$
\sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha} \leq \sum_{\alpha \in \mathscr{I}} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{(p+2) \alpha}
$$

we conclude the inclusions

$$
\begin{aligned}
X \otimes(S)_{\rho, p+2} & \subseteq \operatorname{Dom}_{\rho, p}(\mathscr{R}) \subseteq X \otimes(S)_{\rho, p}, \quad p \in \mathbb{N} \quad \text { and thus } \\
X \otimes(S)_{-\rho,-(p-2)} & \subseteq \operatorname{Dom}_{-\rho,-p}(\mathscr{R}) \subseteq X \otimes(S)_{-\rho,-p}
\end{aligned}
$$

The definition of the domain of the Ornstein-Uhlenbeck operator in the space of square integrable processes corresponds to the classical definition. Denote by

$$
\begin{equation*}
\operatorname{Dom}_{0}(\mathscr{R})=\left\{u \in X \otimes L^{2}(\mu): \sum_{\alpha \in \mathscr{I}} \alpha!|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2}<\infty\right\} \tag{2.19}
\end{equation*}
$$

Theorem 2.10 The operator $\mathscr{R}$ is a linear and continuous mapping

$$
\mathscr{R}: \quad \operatorname{Dom}_{0}(\mathscr{R}) \rightarrow X \otimes L^{2}(\mu)
$$

Moreover, it holds $\operatorname{Dom}_{0}(\mathbb{D}) \supsetneq \operatorname{Dom}_{0}(\mathscr{R})$.
Proof Let $u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{0}(\mathscr{R})$. Then $\mathscr{R}(u)=\sum_{\alpha \in \mathscr{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}$ and

$$
\|\mathscr{R}(u)\|_{X \otimes L^{2}(\mu)}^{2}=\sum_{\alpha \in \mathscr{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2} \alpha!=\|u\|_{\text {Dom }_{0}(\mathscr{R})}^{2}<\infty
$$

From $|\alpha| \leq|\alpha|^{2}, \alpha \in \mathscr{I}$ it follows that

$$
\|u\|_{D o m_{0}(\mathbb{D})}^{2}=\sum_{\alpha \in \mathscr{I}}|\alpha|\left\|u_{\alpha}\right\|_{X}^{2} \alpha!\leq \sum_{\alpha \in \mathscr{I}}|\alpha|^{2}\left\|u_{\alpha}\right\|_{X}^{2} \alpha!=\|u\|_{D_{0 m_{0}}(\mathscr{R})}^{2}
$$

and we conclude $\operatorname{Dom}_{0}(\mathbb{D}) \supset \operatorname{Dom}_{0}(\mathscr{R})$.
Characterization of the domain and range of the operator $\mathscr{R}$ and its properties on $X \otimes(S)_{1}$ and $X \otimes L^{2}(\mu)$ were discussed in [10, 14]. Moreover, for this particular cases the surjectivity of the mappings was proven in [11, 14, 15].

Remark 2.9 Note that $\mathbb{D}: \mathscr{H}_{k} \rightarrow \mathscr{H}_{k-1}$ reduces the Wiener chaos space order and therefore Malliavin differentiation corresponds to the annihilation operator, while $\delta: \mathscr{H}_{k} \rightarrow \mathscr{H}_{k+1}$ increases the chaos order and thus the Skorokhod integration corresponds to the creation operator. Clearly, $\mathscr{R}: \mathscr{H}_{k} \rightarrow \mathscr{H}_{k}$ and the OrnsteinUhlenbeck operator corresponds to the number operator in quantum theory.

Remark 2.10 The domain $\operatorname{Dom}_{-\rho}(\mathscr{R} a)$, where $a=\left(a_{k}\right)_{k \in \mathbb{N}}, a_{k} \geq 1$, is given by

$$
\sum_{\alpha \in \mathscr{I}}|\alpha|^{2} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha}<\infty
$$

Hence, for $p>1$ from

$$
\sum_{\alpha \in \mathscr{\mathscr { F }}}|\alpha|^{2} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} a)^{-p \alpha} \leq c \sum_{\alpha \in \mathscr{I}}|\alpha|^{2} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\infty
$$

where $c=\sum_{\alpha \in \mathscr{I}} a^{-p \alpha}<\infty$, it follows if $u \in \operatorname{Dom}_{-\rho}(\mathscr{R})$ then $u \in \operatorname{Dom}_{-\rho}(\mathscr{R} a)$.
Remark 2.11 Let $P_{m}(t)=\sum_{k=0}^{m} p_{k} t^{k}, t \in \mathbb{R}$ be a polynomial of degree $m$ with real coefficients and $p_{m} \neq 0$. Consider the operator $P(\mathscr{R})=\sum_{k=0}^{m} p_{k} \mathscr{R}^{k}$, where $\mathscr{R}^{0}=$ $I d$ denotes the identity operator and the higher orders of $\mathscr{R}$ are obtained recursively $\mathscr{R}^{k}=\mathscr{R} \circ \mathscr{R}^{k-1}, k>0$. The action of the operator $\mathscr{R}^{k}$ on $u=\sum_{\alpha \in \mathscr{\mathscr { I }}} u_{\alpha} \otimes H_{\alpha}$ is given by

$$
\mathscr{R}^{k} u=\sum_{\alpha \in \mathscr{I}}|\alpha|^{k} u_{\alpha} \otimes H_{\alpha} .
$$

The domain of $\mathscr{R}^{k}$ in $X \otimes(S)_{-\rho}$ is $\operatorname{Dom}_{-\rho}\left(\mathscr{R}^{k}\right)=\bigcup_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{-\rho,-p}\left(\mathscr{R}^{k}\right)$, where $\operatorname{Dom}_{-\rho,-p}\left(\mathscr{R}^{k}\right)=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{-\rho}: \sum_{\alpha \in \mathscr{\mathscr { F }}}|\alpha|^{2 k} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha}<\right.$ $\infty\}$. Moreover, using the estimate $|\alpha| \leq(2 \mathbb{N})^{\alpha}, \alpha \in \mathscr{I}$ we obtain for $q \leq p-2 k$

$$
\sum_{\alpha \in \mathscr{I}}|\alpha|^{2 k} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \leq \sum_{\alpha \in \mathscr{\mathscr { F }}} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha}
$$

Similarly, in the space of test processes $\operatorname{Dom}_{\rho}\left(\mathscr{R}^{k}\right)=\bigcap_{p \in \mathbb{N}_{0}} \operatorname{Dom}_{\rho, p}\left(\mathscr{R}^{k}\right)$, where $\operatorname{Dom}_{\rho, p}\left(\mathscr{R}^{k}\right)=\bigcup_{p \in \mathbb{N}_{0}}\left\{u \in X \otimes(S)_{\rho}: \sum_{\alpha \in \mathscr{I}}|\alpha|^{2 k} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}<\infty\right\}$, while in $X \otimes L^{2}(\mu)$ the domain $\operatorname{Dom}_{0}\left(\mathscr{R}^{k}\right)=\left\{u \in X \otimes L^{2}(\mu): \sum_{\alpha \in \mathscr{I}}|\alpha|^{2 k} \alpha!\left\|u_{\alpha}\right\|_{X}^{2}<\right.$ $\infty$ \}.

The action of the polynomial of the Ornstein-Uhlenbeck operator is given by

$$
\begin{equation*}
P_{m}(\mathscr{R}) u=\sum_{\alpha \in \mathscr{I}} P_{m}(|\alpha|) u_{\alpha} \otimes H_{\alpha}, \tag{2.20}
\end{equation*}
$$

for all $u$ in the domain of $\mathscr{R}^{m}$. From the estimate $\left|P_{m}(|\alpha|)\right| \leq c|\alpha|^{m},|\alpha|>0$, where $c=\max \left\{p_{0}, p_{1}, \ldots, p_{m}\right\}$ and

$$
\begin{aligned}
& \left\|P_{m}(\mathscr{R}) u\right\|_{X \otimes(S)_{-\rho,-p}}^{2}=\sum_{\alpha \in \mathscr{I}}\left\|u_{\alpha}\right\|_{X}^{2}\left|P_{m}(|\alpha|)\right|^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& \quad \leq\left|P_{m}(0)\right|^{2}\left\|u_{0}\right\|_{X}^{2}+c^{2} \sum_{|\alpha|>0}\left\|u_{\alpha}\right\|_{X}^{2}|\alpha|^{2 m} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& \quad=p_{0}^{2}\left\|u_{0}\right\|_{X}^{2}+c^{2}\|u\|_{\text {Dom }_{-\rho,-p}\left(\mathscr{R}^{m}\right)}^{2}<\infty,
\end{aligned}
$$

we conclude that $P_{m}(\mathscr{R})$ maps continuously $\operatorname{Dom}_{-\rho,-p}\left(\mathscr{R}^{m}\right) \rightarrow X \otimes(S)_{-\rho,-p}$. Similarly, on the spaces of test and square integrable processes, the operator $P_{m}(\mathscr{R})$ respectively maps continuously $\operatorname{Dom}_{\rho, p}\left(\mathscr{R}^{m}\right) \rightarrow X \otimes(S)_{\rho, p}$ and $\operatorname{Dom}_{0}\left(\mathscr{R}^{m}\right) \rightarrow$ $X \otimes L^{2}(\mu),[13]$.

### 2.5 Properties of the Operators of Malliavin Calculus

In this section we prove the main properties and relations between the operators of Malliavin calculus in terms of chaos expansions. We prove, for example the integration by parts formula, i.e., the duality relation between $\mathbb{D}$ and $\delta$, product rules for $\mathbb{D}$ and $\mathscr{R}$ and the chain rule.

In the classical $L^{2}$ setting it is known that the Skorokhod integral is the adjoint of the Malliavin derivative [19]. We extend this result in the next theorem and prove their duality by pairing a generalized process with a test process (the classical result is revisited in part $3^{\circ}$ of the following theorem).

Theorem 2.11 ([11, 14]) (Duality) Assume that either of the following hold:
$1^{\circ} F \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{\rho}(\delta)$
$2^{\circ} F \in \operatorname{Dom}_{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-\rho}(\delta)$
$3^{\circ} F \in \operatorname{Dom}_{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\delta)$

Then, the following duality relationship between the operators $\mathbb{D}$ and $\delta$ holds

$$
\begin{equation*}
\mathbb{E}(F \cdot \delta(u))=\mathbb{E}(\langle\mathbb{D} F, u\rangle) \tag{2.21}
\end{equation*}
$$

where (2.21) denotes the equality of the generalized expectations of two objects in $X \otimes(S)_{-\rho}$ and $\langle\cdot, \cdot\rangle$ denotes the dual paring of $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.

Proof First we show that the relationship (2.21) between $\mathbb{D}$ and $\delta$ holds formally. Let $u=\sum_{\beta \in \mathscr{I}} \sum_{j \in \mathbb{N}} u_{\beta, j} \otimes \xi_{j} \otimes H_{\beta}$ be a Skorokhod integrable process. Thus, by (2.9) it holds $\delta(u)=\sum_{\beta \in \mathscr{I}} \sum_{j \in \mathbb{N}} u_{\beta, j} \otimes H_{\beta+\varepsilon^{(j)}}$. Let $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha}$ be a Malliavin differentiable process. Then, by (2.2) it holds $\mathbb{D}(F)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+\right.$ 1) $f_{\alpha+\varepsilon^{(k)}} \otimes \xi_{k} \otimes H_{\alpha}$. Therefore, by (1.57) we obtain

$$
\begin{aligned}
F \cdot \delta(u) & =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\beta, j} \otimes H_{\alpha} \cdot H_{\beta+\varepsilon^{(j)}} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\beta, j} \otimes \sum_{\gamma \leq \min \left\{\alpha, \beta+\varepsilon^{(j)}\right\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta+\varepsilon^{(j)}}{\gamma} H_{\alpha+\beta+\varepsilon^{(j)}-2 \gamma} .
\end{aligned}
$$

The generalized expectation of $F \cdot \delta(u)$ is the zeroth coefficient in the previous sum, which is obtained when $\alpha+\beta+\varepsilon^{(j)}=2 \gamma$ and $\gamma \leq \min \left\{\alpha, \beta+\varepsilon^{(j)}\right\}$, i.e., only for the choice $\beta=\alpha-\varepsilon^{(j)}$ and $\gamma=\alpha, j \in \mathbb{N}$. Thus,

$$
\mathbb{E}(F \cdot \delta(u))=\sum_{\alpha \in \mathscr{I},|\alpha|>0} \sum_{j \in \mathbb{N}} f_{\alpha} u_{\alpha-\varepsilon^{(j)}, j} \cdot \alpha!=\sum_{\alpha \in \mathscr{I}} \sum_{j \in \mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot\left(\alpha+\varepsilon^{(j)}\right)!.
$$

On the other hand,

$$
\begin{aligned}
\langle\mathbb{D}(F), u\rangle & =\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{\beta \in \mathscr{\mathscr { O }}} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}}\left(\alpha_{k}+1\right) f_{\alpha+\varepsilon^{(k)}} u_{\beta, j}\left\langle\xi_{k}, \xi_{j}\right\rangle H_{\alpha} \cdot H_{\beta} \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{\beta \in \mathscr{\mathscr { I }}} \sum_{j \in \mathbb{N}}\left(\alpha_{j}+1\right) f_{\alpha+\varepsilon^{(j)}} u_{\beta, j} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}
\end{aligned}
$$

and its generalized expectation is obtained for $\alpha=\beta=\gamma$. Thus,

$$
\begin{aligned}
\mathbb{E}(\langle\mathbb{D}(F), u\rangle) & =\sum_{\alpha \in \mathscr{I}} \sum_{j \in \mathbb{N}}\left(\alpha_{j}+1\right) f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot \alpha! \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{j \in \mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha, j} \cdot\left(\alpha+\varepsilon^{(j)}\right)!=\mathbb{E}(F \cdot \delta(u)) .
\end{aligned}
$$

$1^{\circ}$ Let $\rho \in[0,1]$ be fixed. Let $F \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$ and $u \in \operatorname{Dom}_{\rho, r, s}(\delta)$, for some $p \in \mathbb{N}$ and all $r, s \in \mathbb{N}, r>s+1$. Then, $\mathbb{D} F \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$ for $l>p+1$. Since $r$ is arbitrary, we may assume that $r=l$ and denote by $\langle\cdot, \cdot\rangle$ the dual pairing between $S_{-l}(\mathbb{R})$ and $S_{l}(\mathbb{R})$. Moreover, $\langle\mathbb{D} F, u\rangle$ is well defined in $X \otimes(S)_{-\rho,-p}$. On the other hand, $\delta(u) \in X \otimes(S)_{\rho, s}$ and thus by Theorem 1.10, $F \cdot \delta(u)$ is also defined as an element in $X \otimes(S)_{-\rho,-k}$, for $k \geq p+7-\rho$. Since $s$ is arbitrary, one can take any $k \geq p+7-\rho$. This means that both objects, $F \cdot \delta(u)$ and $\langle\mathbb{D} F, u\rangle$ exist in $X \otimes(S)_{-\rho,-k}$, for $k \geq p+7-\rho$. Taking generalized expectations of $\langle\mathbb{D} F, u\rangle$ and $F \cdot \delta(u)$ we showed that the zeroth coefficients of the formal expansions are equal. Therefore, the duality formula (2.21) is valid.
$2^{\circ}$ Let $F \in \operatorname{Dom}_{\rho, p}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-\rho,-r,-s}(\delta)$, for some $r, s \in \mathbb{N}, s>r+1$ and all $p \in \mathbb{N}$. Then, $\mathbb{D} F \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}, l<p-1$ and $\langle\mathbb{D} F, u\rangle$ is a well defined object in $X \otimes(S)_{-\rho,-s}$. On the other hand, $\delta(u) \in X \otimes(S)_{-\rho,-s}$ and thus by Theorem 1.10, $F \cdot \delta(u)$ is also well defined and belongs to $X \otimes(S)_{-\rho,-k}$, for $k \geq s+7-\rho$. Thus, both $F \cdot \delta(u)$ and $\langle\mathbb{D} F, u\rangle$ belong to $X \otimes(S)_{-\rho,-k}$ for $k \geq s+7-\rho$.
$3^{\circ}$ For $F \in \operatorname{Dom}_{0}(\mathbb{D})$ and $u \in \operatorname{Dom}_{0}(\delta)$ the dual pairing $\langle\mathbb{D} F, u\rangle$ represents the inner product in $L^{2}(\mathbb{R})$ and the product $F \delta(u)$ is an element in $X \otimes L^{2}(\mu)$. Thus, the classical duality formula is valid.

The higher order duality formula, which connects the $k$ th order iterated Skorokhod integral and the Malliavin derivative operator of $k$ th order, $k \in \mathbb{N}$ is stated in the following theorem. Recall, in Remark 2.4 and Remark 2.7 we introduced the higher order operators $\mathbb{D}^{(k)}$ and $\delta^{(k)}$.

Theorem 2.12 ([14]) Let $f \in \operatorname{Dom}_{\rho}\left(\mathbb{D}^{(k)}\right)$ and $u \in \operatorname{Dom}_{-\rho}\left(\delta^{(k)}\right), k \in \mathbb{N}$. Then, the higher order duality formula

$$
\mathbb{E}\left(f \cdot \delta^{(k)}(u)\right)=\mathbb{E}\left(\left\langle\mathbb{D}^{(k)}(f), u\right\rangle\right)
$$

holds, where $\langle\cdot, \cdot\rangle$ denotes the duality pairing of $S^{\prime}(\mathbb{R})^{\otimes k}$ and $S(\mathbb{R})^{\otimes k}$.

Proof The assertion follows by induction and applying Theorem 2.11 successively $k$ times.

Remark 2.12 Note that Theorem 2.11 and Theorem 2.12 are special cases of a more general identity. It can be proven, under suitable assumptions that make all the products well defined, that the following holds

$$
\begin{equation*}
F \delta(u)=\delta(F u)+\langle\mathbb{D}(F), u\rangle, \tag{2.22}
\end{equation*}
$$

By taking the expectation in (2.22) and using the fact that $\mathbb{E}(\delta(F u))=0$, we obtain the duality relation (2.21).

A weaker type of duality then (2.21), which holds in Hida spaces of generalized processes was proven in [11]. Here we formulate the weak duality and omit its proof. A similar result is obtained in [14] for the Kondratiev type spaces when $\rho=1$.

Theorem 2.13 ([11]) (Weak duality) Consider $\rho=0$. Let $F \in \operatorname{Dom}_{-0,-p}(\mathbb{D})$ and $u \in \operatorname{Dom}_{-0,-q}(\mathbb{D})$, for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R}), n<q-1$, it holds that

$$
\ll\langle\mathbb{D} F, \varphi\rangle_{-r}, u>_{-r}=\ll F, \delta(\varphi u) \gg_{-r},
$$

for $r>\max \{q, p+1\}$.
The following theorem states that the Malliavin derivative indicates the speed of change in time between the ordinary product and the Wick product.

Theorem 2.14 ([15]) Let $h \in X \otimes(S)_{-\rho}$ and let $w_{t}$ denote white noise. Then,

$$
\begin{equation*}
h \cdot w_{t}-h \diamond w_{t}=\mathbb{D}(h) . \tag{2.23}
\end{equation*}
$$

Proof Let $h$ be of the form $h=\sum_{\alpha \in \mathscr{\mathscr { I }}} h_{\alpha} H_{\alpha}$ and $w_{t}=\sum_{n=1}^{\infty} \xi_{n}(t) H_{\varepsilon^{(n)}}$. Then,

$$
\begin{aligned}
& h \diamond w_{t}=\sum_{\gamma \in \mathscr{\mathscr { I }}} \sum_{\alpha+\varepsilon^{(n)}=\gamma} h_{\alpha} \xi_{n}(t) H_{\gamma}=\sum_{\gamma \in \mathscr{\mathscr { I }}} \sum_{n=1}^{\infty} h_{\gamma-\varepsilon^{(n)}} \xi_{n}(t) H_{\gamma} \text { and } \\
& h \cdot w_{t}=\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_{n}(t) H_{\alpha-\varepsilon^{(n)}} H_{\varepsilon^{(n)}} .
\end{aligned}
$$

Now by applying the formula (1.14) we obtain

$$
H_{\alpha-\varepsilon^{(n)}} \cdot H_{\varepsilon^{(n)}}=H_{\alpha}+\left(\alpha-\varepsilon^{(n)}\right)_{n} H_{\alpha-2 \varepsilon^{(n)}},
$$

where we used $\left(\underset{\varepsilon^{(k)}}{\alpha}\right)=\alpha_{k}, k \in \mathbb{N}$. Hence,

$$
h \cdot w_{t}=\sum_{\alpha \in \mathscr{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_{n}(t)\left(H_{\alpha}+\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}}\right),
$$

which implies

$$
\begin{aligned}
h \cdot w_{t}-h \diamond w_{t} & =\sum_{\alpha \in \mathscr{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_{n}(t)\left(\alpha_{n}-1\right) H_{\alpha-2 \varepsilon^{(n)}} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} \xi_{n}(t)\left(\alpha_{n}+1\right) H_{\alpha}=\mathbb{D}(h)
\end{aligned}
$$

The Malliavin derivative $\mathbb{D}$ is not the inverse operator of the Skorokhod integral $\delta$ and also they do not commute. However, the relation (2.24) holds.

Example 2.1 For $Z=\sum_{k \in \mathbb{N}} H_{2 \varepsilon^{(k)}}$ we have $\mathbb{D}\left(\frac{1}{2} Z\right)=w_{t}, \delta\left(w_{t}\right)=Z$ and $\mathscr{R}\left(\frac{1}{2} Z\right)=$ $\delta\left(\mathbb{D}\left(\frac{1}{2} Z\right)\right)=\delta\left(w_{t}\right)=Z$, where $w_{t}$ is singular white noise and $d_{t}$ the Dirac delta function. Moreover, $\mathbb{D}\left(\delta\left(w_{t}\right)\right)=\mathbb{D}(Z)=2 w_{t}$ while $\delta\left(\mathbb{D}\left(w_{t}\right)\right)=\delta\left(d_{t}\right)=w_{t}$ and thus $\mathbb{D}$ and $\delta$ do not commute.

Theorem 2.15 If $u \in \operatorname{Dom}_{-\rho}(\delta)$ then $\mathbb{D} u \in \operatorname{Dom}_{-\rho}(\delta)$ and it holds

$$
\begin{equation*}
\mathbb{D}(\delta u)=u+\delta(\mathbb{D} u) \tag{2.24}
\end{equation*}
$$

Proof Let $u$ be of the form $u=\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$. Then, $\delta(u)$ is of the form (2.9) and consequently

$$
\begin{aligned}
\mathbb{D}(\delta(u)) & =\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \sum_{i=1}^{\infty}\left(\alpha+\varepsilon^{(k)}\right)_{i} \xi_{i} \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes\left(\left(\alpha_{k}+1\right) \xi_{k} \otimes H_{\alpha}+\sum_{i \neq k} \alpha_{i} \xi_{i} \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}\right) \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}+\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{i} u_{\alpha, k} \otimes \xi_{i} \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \\
& =u+\delta(\mathbb{D}(u)) .
\end{aligned}
$$

The latter equality follows from
$\mathbb{D}(u)=\sum_{\alpha \in \mathscr{I}} \sum_{i=1}^{\infty} \alpha_{i}\left(\sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k}\right) \otimes \xi_{i} \otimes H_{\alpha-\varepsilon^{(i)}} \in X \otimes S^{\prime}(\mathbb{R}) \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$
which implies $\delta(\mathbb{D}(u))=\sum_{\alpha \in \mathscr{I}} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{i} u_{\alpha, k} \otimes \xi_{i} \otimes H_{\alpha-\varepsilon^{(i)}+\varepsilon^{(k)}}$.
Since $u \in \operatorname{Dom}_{-\rho,-l,-s}(\delta)$, from Theorem 2.4 it follows that $\delta u \in X \otimes(S)_{-\rho,-s}$, $s>l+1$, which by $(2.7)$ belongs to $\operatorname{Dom}_{-\rho,-s}(\mathbb{D})$. Then, $\mathbb{D}(\delta u)$ is a well defined element in $X \otimes S_{-l_{1}}(\mathbb{R}) \otimes(S)_{-\rho,-s}, l_{1}>s+1$, where $s$ is arbitrary. This means that
the left hand of $(2.24)$ is also an element in $X \otimes S_{-l_{1}}(\mathbb{R}) \otimes(S)_{-\rho,-s}$, thus $\mathbb{D} u$ must be in the domain of $\delta$.

The commutation relation (2.24) holds for processes $u \in \operatorname{Dom}_{\rho}(\delta)$ and also for $u \in \operatorname{Dom}_{0}(\delta)$. The proofs follow similarly.
Remark 2.13 Note that if $u \in X \otimes L^{2}(\mathbb{R}) \otimes(S)_{-\rho}$, then

$$
\delta(u)=\int_{\mathbb{R}} u \diamond w_{t} d t,
$$

where the right hand side is interpreted as the $X$-valued Bochner integral in the Riemann sense. This is in accordance with the known fact that Itô-Skorokhod integration with the rules of the Itô's calculus generates the same results as integration interpreted in the classical Riemann sense following the rules of ordinary calculus, if the integrand is interpreted as the Wick product with white noise [6]. For example,
$\int_{0}^{t_{0}} b_{t} d b_{t}=\delta\left(\chi_{\left[0, t_{0}\right]}(t) b_{t}\right)=\int_{0}^{t_{0}} b_{t} \diamond w_{t} d t=\int_{0}^{t_{0}} b_{t} \diamond b_{t}^{\prime} d t=\frac{1}{2} b_{t_{0}}^{\diamond 2}=\frac{1}{2}\left(b_{t_{0}}^{2}-t_{0}\right)$.
The general case follows easily from the definition of the Skorokhod integral. If $u=\sum_{\alpha \in \mathscr{\mathscr { I }}} u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$ is in $X \otimes L^{2}(\mathbb{R}) \otimes(S)_{-\rho}$ then $u_{\alpha, k}=\left(u_{\alpha}, \xi_{k}\right)_{L^{2}(\mathbb{R})}=\int_{\mathbb{R}} u_{\alpha}(t) \xi_{k}(t) d t$ for all $\alpha \in \mathscr{I}, k \in \mathbb{N}$. Thus,

$$
\begin{aligned}
\delta(u) & =\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha, k} \otimes H_{\alpha+\varepsilon^{(k)}}=\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} \int_{\mathbb{R}} u_{\alpha}(t) \xi_{k}(t) d t \otimes H_{\alpha+\varepsilon^{(k)}} \\
& =\int_{\mathbb{R}}\left(\sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} u_{\alpha}(t) \xi_{k}(t) \otimes H_{\alpha+\varepsilon^{(k)}}\right) d t \\
& =\int_{\mathbb{R}}\left(\sum_{\alpha \in \mathscr{I}} u_{\alpha}(t) \otimes H_{\alpha}\right) \diamond\left(\sum_{k=1}^{\infty} \xi_{k}(t) \otimes H_{\varepsilon^{(k)}}\right) d t=\int_{\mathbb{R}} u \diamond w_{t} d t .
\end{aligned}
$$

The following theorem states the product rule for the Ornstein-Uhlenbeck operator. Its special case for $F, G \in \operatorname{Dom}_{0}(\mathscr{R})$ states that $F \cdot G$ is also in $\operatorname{Dom}_{0}(\mathscr{R})$ and (2.25) holds. The proof can be found for example in [7].

Theorem 2.16 ([14]) (Product rule for $\mathscr{R}$ )
$1^{\circ}$ Let $F \in \operatorname{Dom}_{\rho}(\mathscr{R})$ and $G \in \operatorname{Dom}_{-\rho}(\mathscr{R})$. Then $F \cdot G \in \operatorname{Dom}_{-\rho}(\mathscr{R})$ and

$$
\begin{equation*}
\mathscr{R}(F \cdot G)=F \cdot \mathscr{R}(G)+G \cdot \mathscr{R}(F)-2 \cdot\langle\mathbb{D} F, \mathbb{D} G\rangle, \tag{2.25}
\end{equation*}
$$

holds, where $\langle\cdot, \cdot \cdot\rangle$ is the dual paring between $S^{\prime}(\mathbb{R})$ and $S(\mathbb{R})$.
$2^{\circ}$ Let $F, G \in \operatorname{Dom}_{-\rho}(\mathscr{R})$. Then $F \cdot G \in \operatorname{Dom}_{-\rho}(\mathscr{R})$ and

$$
\begin{equation*}
\mathscr{R}(F \diamond G)=F \diamond \mathscr{R}(G)+\mathscr{R}(F) \diamond G . \tag{2.26}
\end{equation*}
$$

Proof $1^{\circ}$ Assume $F \in \operatorname{Dom}_{\rho, q}(\mathscr{R})$ and $G \in \operatorname{Dom}_{-\rho,-p}(\mathscr{R})$. Then $\mathscr{R}(F) \in$ $X \otimes(S)_{\rho, q}$ and $\mathscr{R}(G) \in X \otimes(S)_{-\rho,-p}$. From Theorem 1.10 it follows that $F \mathscr{R}(G)$ and $G \cdot \mathscr{R}(F)$ are both well defined and belong to $X \otimes(S)_{-\rho,-s}$, for $s \geq p+7-\rho$. Similarly, $\langle\mathbb{D}(F), \mathbb{D}(G)\rangle$ belongs to $X \otimes(S)_{-\rho,-p}$, since $\mathbb{D}(F) \in X \otimes S_{l_{1}}(\mathbb{R}) \otimes$ $(S)_{\rho, q}$, where $l_{1}<q-1$ and $\mathbb{D}(G) \in X \otimes S_{-l_{2}}(\mathbb{R}) \otimes(S)_{-\rho,-p}$, where $l_{2}>p+1$ and the dual pairing is obtained for any $l \in\left[l_{1}, l_{2}\right]$. Thus, the right hand side of (2.25) is in $X \otimes(S)_{-\rho,-s}, s \geq p+7-\rho$. Hence, $F \cdot G \in \operatorname{Dom}_{-\rho,-s}(\mathscr{R})$.

Let $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}_{\rho}(\mathscr{R})$ and $G=\sum_{\beta \in \mathscr{I}} g_{\beta} \otimes H_{\beta} \in \operatorname{Dom}_{-\rho}(\mathscr{R})$. Then, $\mathscr{R}(F)=\sum_{\alpha \in \mathscr{I}}|\alpha| f_{\alpha} \otimes H_{\alpha}$ and $\mathscr{R}(G)=\sum_{\beta \in \mathscr{I}}|\beta| g_{\beta} \otimes H_{\beta}$.
The left hand side of (2.25) can be written in the form

$$
\begin{aligned}
\mathscr{R}(F \cdot G) & =\mathscr{R}\left(\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}\right) \\
& =\sum_{\alpha \in \mathscr{\mathscr { O }}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha+\beta-2 \gamma| H_{\alpha+\beta-2 \gamma} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{\mathscr { I }}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(|\alpha|+|\beta|-2|\gamma|) H_{\alpha+\beta-2 \gamma} .
\end{aligned}
$$

On the other hand, the first two terms on the right hand side of (2.25) are

$$
\begin{align*}
\mathscr{R}(F) \cdot G & =\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{\beta \in \mathscr{\mathscr { I }}} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha| H_{\alpha+\beta-2 \gamma} \text { and }  \tag{2.27}\\
F \cdot \mathscr{R}(G) & =\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{\beta \in \mathscr{\mathscr { I }}} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\beta| H_{\alpha+\beta-2 \gamma} . \tag{2.28}
\end{align*}
$$

Since $F \in \operatorname{Dom}_{\rho}(\mathscr{R}) \subset \operatorname{Dom}_{\rho}(\mathbb{D})$ and $G \in \operatorname{Dom}_{-\rho}(\mathscr{R}) \subseteq \operatorname{Dom}_{-\rho}(\mathbb{D})$ we have $\mathbb{D}(F)=\sum_{\alpha \in \mathscr{\mathscr { S }}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}$ and $\mathbb{D}(G)=\sum_{\beta \in \mathscr{\mathscr { S }}} \sum_{j \in \mathbb{N}} \beta_{j} g_{\beta} \otimes$ $\xi_{j} \otimes H_{\beta-\varepsilon^{(k)}}$. Thus, the third term on the right hand side of (2.25) is

$$
\begin{aligned}
& \langle\mathbb{D}(F), \mathbb{D}(G)\rangle=\left\langle\sum_{|\alpha|>0} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}, \sum_{|\beta|>0} \sum_{j \in \mathbb{N}} \beta_{j} g_{\beta} \otimes \xi_{j} \otimes H_{\left.\beta-\varepsilon^{(j)}\right\rangle}\right. \\
& =\sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} \alpha_{k} \beta_{j} f_{\alpha} g_{\beta}\left\langle\xi_{k}, \xi_{j}\right\rangle \otimes H_{\alpha-\varepsilon^{(k)}} \cdot H_{\beta-\varepsilon^{(j)}} \\
& =\sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \alpha_{k} \beta_{k} f_{\alpha} g_{\beta} \otimes \sum_{\gamma \leq \min \left\{\alpha-\varepsilon^{(k)}, \beta-\varepsilon^{(k)}\right\}} \gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} H_{\alpha+\beta-2 \varepsilon^{(k)}-2 \gamma},
\end{aligned}
$$

where we used the fact that $\left\langle\xi_{k}, \xi_{j}\right\rangle=0$ for $k \neq j$ and $\left\langle\xi_{k}, \xi_{j}\right\rangle=1$ for $k=j$.
Now we put $\theta=\gamma+\varepsilon^{(k)}$ and use the identities

$$
\alpha_{k} \cdot\binom{\alpha-\varepsilon^{(k)}}{\gamma}=\alpha_{k} \cdot\binom{\alpha-\varepsilon^{(k)}}{\theta-\varepsilon^{(k)}}=\theta_{k} \cdot\binom{\alpha}{\theta}, \quad k \in \mathbb{N},
$$

and $\theta_{k} \cdot\left(\theta-\varepsilon^{(k)}\right)!=\theta!$. Thus, we obtain

$$
\begin{aligned}
\langle\mathbb{D}(F), \mathbb{D}(G)\rangle & =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}} \theta_{k}^{2}\left(\theta-\varepsilon^{(k)}\right)!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}} \theta_{k} \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}}\left(\sum_{k \in \mathbb{N}} \theta_{k}\right) \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\theta \leq \min \{\alpha, \beta\}}|\theta| \theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2 \theta} .
\end{aligned}
$$

Combining all previous results, we obtain

$$
\begin{aligned}
\mathscr{R}(F \cdot G) & =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(|\alpha|+|\beta|-2|\gamma|) H_{\alpha+\beta-2 \gamma} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\alpha| H_{\alpha+\beta-2 \gamma} \\
& +\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}|\beta| H_{\alpha+\beta-2 \gamma} \\
& -2 \sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}}|\gamma| \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma} \\
& =\mathscr{R}(F) \cdot G+F \cdot \mathscr{R}(G)-2 \cdot\langle\mathbb{D}(F), \mathbb{D}(G)\rangle
\end{aligned}
$$

and thus (2.25) holds.
$2^{\circ}$ If $F \in \operatorname{Dom}_{-\rho,-p}(\mathscr{R})$ and $G \in \operatorname{Dom}_{-\rho,-q}(\mathscr{R})$, then $\mathscr{R}(F) \in X \otimes(S)_{-\rho,-p}$ and $\mathscr{R}(G) \in X \otimes(S)_{-\rho,-q}$. From Theorem 1.9 it follows that $\mathscr{R}(F) \diamond G$ and $\mathscr{R}(G) \diamond F$ belong to $X \otimes(S)_{-\rho,-(p+q+3-\rho)}$. Thus, the right hand side of (2.26) is in $X \otimes(S)_{-\rho,-(p+q+3-\rho)}$, i.e., $F \diamond G \in \operatorname{Dom}_{-\rho,-r}(\mathscr{R})$ for $r \geq p+q+3-\rho$.
From

$$
G \diamond \mathscr{R}(F)=\sum_{\gamma \in \mathscr{I}} \sum_{\alpha+\beta=\gamma}|\alpha| f_{\alpha} g_{\beta} H_{\gamma} \text { and } F \diamond \mathscr{R}(G)=\sum_{\gamma \in \mathscr{\mathscr { I }}} \sum_{\alpha+\beta=\gamma} f_{\alpha}|\beta| g_{\beta} H_{\gamma},
$$

it follows that

$$
G \diamond \mathscr{R}(F)+F \diamond \mathscr{R}(G)=\sum_{\gamma \in \mathscr{I}}|\gamma| \sum_{\alpha+\beta=\gamma} f_{\alpha} g_{\beta} H_{\gamma}=\mathscr{R}(F \diamond G)
$$

Corollary 2.1 Let $F \in \operatorname{Dom}_{\rho}(\mathscr{R})$ and $G \in \operatorname{Dom}_{-\rho}(\mathscr{R})$. Then,

$$
\begin{equation*}
\mathbb{E}(F \cdot \mathscr{R}(G))=\mathbb{E}(\langle\mathbb{D} F, \mathbb{D} G\rangle) \tag{2.29}
\end{equation*}
$$

The property (2.29) holds also for $F, G \in \operatorname{Dom}_{0}(\mathscr{R})$.
Proof From the chaos expansion form of $\mathscr{R}(F \cdot G)$ it follows that $\mathbb{E} \mathscr{R}(F \cdot G)=$ 0 and by taking the expectations on both sides of (2.27) and (2.28) the equality $\mathbb{E}(\mathscr{R}(F) \cdot G)=\mathbb{E}(F \cdot \mathscr{R}(G))$ follows. Then, from Theorem 2.16 we obtain that $0=2 \mathbb{E}(F \cdot \mathscr{R}(G))-2 \mathbb{E}(\langle\mathbb{D} F, \mathbb{D} G\rangle)$, which leads to the assertion (2.29).

In the classical literature $[18,19]$ it is proven that the Malliavin derivative satisfies the product rule (with respect to ordinary multiplication), i.e., if $F, G \in \operatorname{Dom}_{0}(\mathbb{D})$, then $F \cdot G \in \operatorname{Dom}_{0}(\mathbb{D})$ and (2.30) holds. The following theorem recapitulates this result and extends it for generalized and test processes, and also for the Wick multiplication [1, 14].

Theorem 2.17 (Product rule for $\mathbb{D}$ )
$1^{\circ}$ Let $F \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and $G \in \operatorname{Dom}_{\rho}(\mathbb{D})$. Then $F \cdot G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and it holds

$$
\begin{equation*}
\mathbb{D}(F \cdot G)=F \cdot \mathbb{D} G+\mathbb{D} F \cdot G \tag{2.30}
\end{equation*}
$$

$2^{\circ}$ Let $F, G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$. Then $F \diamond G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and

$$
\mathbb{D}(F \diamond G)=F \diamond \mathbb{D} G+\mathbb{D} F \diamond G
$$

Proof $\quad 1^{\circ}$ Assume that $F \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D}), G \in \operatorname{Dom}_{\rho, q}(\mathbb{D})$. Then $\mathbb{D}(F) \in$ $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}, l>p+1$, and $\mathbb{D}(G) \in X \otimes S_{k}(\mathbb{R}) \otimes(S)_{\rho, q}, k<q-1$. From Theorem 1.10 it follows that all products on the right hand side of (2.30) are well defined and $F \cdot \mathbb{D}(G) \in X \otimes S_{k}(\mathbb{R}) \otimes(S)_{-\rho,-r}, \mathbb{D}(F) \cdot G \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-r}$, for $r \geq p+7-\rho$. Thus, the right hand side of (2.30) can be embedded into $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-r}, r \geq p+7-\rho$. Then, $F \cdot G \in \operatorname{Dom}_{-\rho,-r}(\mathbb{D})$. Moreover,

$$
\begin{aligned}
& \mathbb{D}(F \cdot G)=\mathbb{D}\left(\sum_{\alpha \in \mathscr{I}} f_{\alpha} H_{\alpha} \cdot \sum_{\beta \in \mathscr{I}} g_{\beta} H_{\beta}\right)= \\
& \mathbb{D}\left(\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2 \gamma}\right)= \\
& \sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \{\alpha, \beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\left(\alpha_{k}+\beta_{k}-2 \gamma_{k}\right) \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& F \cdot \mathbb{D}(G)=\sum_{\alpha \in \mathscr{I}} f_{\alpha} H_{\alpha} \cdot \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}} \beta_{k} g_{\beta} \xi_{k} H_{\beta-\varepsilon^{(k)}}= \\
& \sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \left\{\alpha, \beta-\varepsilon^{(k)}\right\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} \beta_{k} \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
\end{aligned}
$$

and

$$
G \cdot \mathbb{D}(F)=\sum_{\alpha \in \mathscr{I}} \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}} f_{\alpha} g_{\beta} \sum_{\gamma \leq \min \left\{\alpha-\varepsilon^{(k)}, \beta\right\}} \gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta}{\gamma} \alpha_{k} \xi_{k} H_{\alpha+\beta-2 \gamma-\varepsilon^{(k)}}
$$

Summing up chaos expansions for $F \cdot \mathbb{D}(G)$ and $G \cdot \mathbb{D}(F)$ and applying the identities

$$
\alpha_{k}\binom{\alpha-\varepsilon^{(k)}}{\gamma}=\alpha_{k} \cdot \frac{\left(\alpha-\varepsilon^{(k)}\right)!}{\gamma!\left(\alpha-\varepsilon^{(k)}-\gamma\right)!}=\frac{\alpha!}{\gamma!(\alpha-\gamma)!} \cdot\left(\alpha_{k}-\gamma_{k}\right)=\binom{\alpha}{\gamma}\left(\alpha_{k}-\gamma_{k}\right)
$$

and

$$
\beta_{k}\binom{\beta-\varepsilon^{(k)}}{\gamma}=\binom{\beta}{\gamma}\left(\beta_{k}-\gamma_{k}\right)
$$

for all $\alpha, \beta \in \mathscr{I}, k \in \mathbb{N}$ and $\gamma \in \mathscr{I}$ such that $\gamma \leq \min \{\alpha, \beta\}$ and the expression $\left(\alpha_{k}-\gamma_{k}\right)+\left(\beta_{k}-\gamma_{k}\right)=\alpha_{k}+\beta_{k}-2 \gamma_{k}$ we obtain (2.30).
$2^{\circ}$ If $F \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$ and $G \in \operatorname{Dom}_{-\rho,-q}(\mathbb{D})$, then $\mathbb{D}(F) \in X \otimes S_{-l}(\mathbb{R}) \otimes$ $(S)_{-\rho,-p}, l>p+1$, and $\mathbb{D}(G) \in X \otimes S_{-k}(\mathbb{R}) \otimes(S)_{-\rho,-q}, k>q+1$. From Theorem 1.9 it follows that $\mathbb{D}(F) \diamond G$ and $F \diamond \mathbb{D}(G)$ both belong to $X \otimes S_{-m}(\mathbb{R}) \otimes$ $(S)_{-\rho,-(p+q+3-\rho)}, m=\max \{l, k\}$. Thus, $F \diamond G \in \operatorname{Dom}_{-\rho,-r}(\mathbb{D})$ for $r \geq p+q+$ $3-\rho$. It also holds

$$
\begin{aligned}
\mathbb{D}(F) \diamond G+F \diamond \mathbb{D}(G) & =\sum_{\gamma \in \mathscr{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \alpha_{k} f_{\alpha} g_{\beta} H_{\gamma}+\sum_{\gamma \in \mathscr{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \beta_{k} f_{\alpha} g_{\beta} H_{\gamma} \\
& =\sum_{\gamma \in \mathscr{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=\gamma} \gamma_{k} f_{\alpha} g_{\beta} H_{\gamma-\varepsilon^{(k)}}=\mathbb{D}(F \diamond G) .
\end{aligned}
$$

Theorem 2.18 Assume that either of the following hold:

$$
\begin{aligned}
& 1^{\circ} F \in \operatorname{Dom}_{-\rho}(\mathbb{D}), G \in \operatorname{Dom}_{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{\rho}(\delta), \\
& 2^{\circ} F, G \in \operatorname{Dom}_{\rho}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{-\rho}(\delta), \\
& 3^{\circ} F, G \in \operatorname{Dom}_{0}(\mathbb{D}) \text { and } u \in \operatorname{Dom}_{0}(\delta) .
\end{aligned}
$$

Then, the second integration by parts formula holds

$$
\begin{equation*}
\mathbb{E}(F\langle\mathbb{D} G, u\rangle)+\mathbb{E}(G\langle\mathbb{D} F, u\rangle)=\mathbb{E}(F G \delta(u)) . \tag{2.31}
\end{equation*}
$$

Proof The equation (2.31) follows directly from the duality formula (2.21) and the product rule (2.30). Assume the first case holds when $F \in \operatorname{Dom}_{-\rho}(\mathbb{D}), G \in$ $\operatorname{Dom}_{\rho}(\mathbb{D})$ and $u \in \operatorname{Dom}_{\rho}(\delta)$. Then $F \cdot G \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and we have

$$
\begin{aligned}
\mathbb{E}(F G \delta(u)) & =\mathbb{E}(\langle\mathbb{D}(F \cdot G), u\rangle)=\mathbb{E}(\langle F \cdot \mathbb{D}(G)+G \cdot \mathbb{D}(F), u\rangle) \\
& =\mathbb{E}(F\langle\mathbb{D}(G), u\rangle)+\mathbb{E}(G\langle\mathbb{D}(F), u\rangle)
\end{aligned}
$$

The second and third case can be proven in an analogous way.
A generalization of Theorem 2.17 for higher order derivatives, i.e., the Leibnitz formula is given in the next theorem.
Theorem 2.19 Let $F, G \in \operatorname{Dom}_{-\rho}\left(\mathbb{D}^{(k)}\right)$, $k \in \mathbb{N}$, then $F \diamond G \in \operatorname{Dom}_{-\rho}\left(\mathbb{D}^{(k)}\right)$ and the Leibnitz rule holds

$$
\mathbb{D}^{(k)}(F \diamond G)=\sum_{i=0}^{k}\binom{k}{i} \mathbb{D}^{(i)}(F) \diamond \mathbb{D}^{(k-i)}(G),
$$

such that $\mathbb{D}^{(0)}(F)=F$ and $\mathbb{D}^{(0)}(G)=G$.
Moreover, if $G \in \operatorname{Dom}_{\rho}\left(\mathbb{D}^{(k)}\right)$, then the ordinary product $F \cdot G \in \operatorname{Dom}_{-\rho}\left(\mathbb{D}^{(k)}\right)$ and

$$
\begin{equation*}
\mathbb{D}^{(k)}(F \cdot G)=\sum_{i=0}^{k}\binom{k}{i} \mathbb{D}^{(i)}(F) \cdot \mathbb{D}^{(k-i)}(G) . \tag{2.32}
\end{equation*}
$$

Proof The Leibnitz rule (2.32) follows by induction and applying Theorem 2.17. Clearly, (2.32) holds also if $F, G \in \operatorname{Dom}_{0}\left(\mathbb{D}^{(k)}\right)$ and $F \cdot G \in \operatorname{Dom}_{0}\left(\mathbb{D}^{(k)}\right)$.

The chain rule for the Malliavin derivative is stated in the following theorem. The case with square integrable processes has been known throughout the literature as a direct consequence of the definition of Malliavin derivatives as Fréchet derivatives [1]. Here we provide an alternative proof suited to the setting of chaos expansions.

Theorem 2.20 ([11, 14]) (The chain rule) Let $\phi$ be a twice continuously differentiable function with bounded derivatives.
$1^{\circ}$ If $F \in \operatorname{Dom}_{\rho}(\mathbb{D})\left(\right.$ or $F \in \operatorname{Dom}_{0}(\mathbb{D})$ ) then $\phi(F) \in \operatorname{Dom}_{\rho}(\mathbb{D})$ (respectively $\phi(F) \in \operatorname{Dom}_{0}(\mathbb{D})$ ) and the chain rule holds

$$
\begin{equation*}
\mathbb{D}(\phi(F))=\phi^{\prime}(F) \cdot \mathbb{D}(F) . \tag{2.33}
\end{equation*}
$$

$2^{\circ}$ If $F \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and $\phi$ is analytic then $\phi^{\diamond}(F) \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ and

$$
\begin{equation*}
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\phi^{\prime}(F) \diamond \mathbb{D}(F) \tag{2.34}
\end{equation*}
$$

Proof $\quad 1^{\circ}$ First we prove that (2.33) holds when $\phi$ is a polynomial of degree $n, n \in$ $\mathbb{N}$. Then, we use the Stone-Weierstrass theorem and approximate a continuously differentiable function $\phi$ by a polynomial $\widetilde{p}_{n}$ of degree $n$, and since we assumed that $\phi$ is regular enough, the derivative $\widetilde{p}_{n}^{\prime}$ will also approximate $\phi^{\prime}$.
Denote by $q_{n}(x)=x^{n}, n \in \mathbb{N}$ and let $p(x)=\sum_{k=0}^{n} a_{k} q_{k}(x)=\sum_{k=0}^{n} a_{k} x^{k}$ be a polynomial of degree $n$ with real coefficients $a_{0}, a_{1}, \ldots, a_{n}$, and $a_{n} \neq 0$. By induction on $n$, we prove the chain rule for $q_{n}$, i.e., we prove

$$
\begin{equation*}
\mathbb{D}\left(2_{n}(F)\right)=2_{n}^{\prime}(F) \cdot \mathbb{D}(F), \quad n \in \mathbb{N} \tag{2.35}
\end{equation*}
$$

For $n=1, q_{1}(x)=x$ and (2.35) holds since

$$
\mathbb{D}\left(q_{1}(F)\right)=\mathbb{D}(F)=1 \cdot \mathbb{D}(F)=q_{1}^{\prime}(F) \cdot \mathbb{D}(F)
$$

Assume (2.35) holds for $k \in \mathbb{N}$. Then, for $q_{k+1}=x^{k+1}$ by Theorem 2.17 we have

$$
\begin{aligned}
\mathbb{D}\left(q_{k+1}(F)\right) & =\mathbb{D}\left(F^{k+1}\right)=\mathbb{D}\left(F \cdot F^{k}\right)=\mathbb{D}(F) \cdot F^{k}+F \cdot \mathbb{D}\left(F^{k}\right) \\
& =\mathbb{D}(F) \cdot F^{k}+F \cdot k F^{k-1} \cdot \mathbb{D}(F)=(k+1) F^{k} \cdot \mathbb{D}(F)=q_{k+1}^{\prime}(F) \cdot \mathbb{D}(F)
\end{aligned}
$$

Thus, (2.35) holds for every $n \in \mathbb{N}$. Since $\mathbb{D}$ is a linear operator, (2.35) also holds for any polynomial $p_{n}$, i.e.,

$$
\mathbb{D}\left(p_{n}(F)\right)=\sum_{k=0}^{n} a_{k} \mathbb{D}\left(q_{k}(F)\right)=\sum_{k=0}^{n} a_{k} q_{k}^{\prime}(F) \cdot \mathbb{D}(F)=p_{n}^{\prime}(F) \cdot \mathbb{D}(F)
$$

Let $\phi \in C^{2}(\mathbb{R})$ and $F \in \operatorname{Dom}_{\rho, p}(\mathbb{D}), p \in \mathbb{N}$. Then, by the Stone-Weierstrass theorem, there exists a polynomial $\widetilde{p}_{n}$ such that

$$
\begin{aligned}
\left\|\phi(F)-\widetilde{p}_{n}(F)\right\|_{X \otimes(S)_{\rho, p}} & =\left\|\phi(F)-\sum_{k=0}^{n} a_{k} F^{k}\right\|_{X \otimes(S)_{\rho, p}} \rightarrow 0 \text { and } \\
\left\|\phi^{\prime}(F)-\widetilde{p}_{n}^{\prime}(F)\right\|_{X \otimes(S)_{\rho, p}} & =\left\|\phi^{\prime}(F)-\sum_{k=1}^{n} a_{k} k F^{k-1}\right\|_{X \otimes(S)_{\rho, p}} \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$. We denote by $\mathscr{X}_{l p}=X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}$. From (2.35) and the fact that $\mathbb{D}$ is a bounded operator, Theorem 2.1, we obtain for $l<p-1$

$$
\begin{aligned}
& \left\|\mathbb{D}(\phi(F))-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}=\left\|\mathbb{D}(\phi(F))-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\|_{\mathscr{X}_{l_{p}}} \\
& =\left\|\mathbb{D}(\phi(F))-\mathbb{D}\left(\widetilde{p_{n}}(F)\right)+\mathbb{D}\left(\widetilde{p_{n}}(F)\right)-\phi^{\prime}(F) \cdot \mathbb{D}(F)\right\| \mathscr{X}_{I_{p}} \\
& \leq\left\|\mathbb{D}(\phi(F))-\mathbb{D}\left(\widetilde{p_{n}}(F)\right)\right\|_{\mathscr{X}_{1 p}}+\left\|\mathbb{D}\left(\widetilde{p}_{n}(F)\right)-\phi^{\prime}(F) \mathbb{D}(F)\right\|_{\mathscr{X}_{1 p}} \\
& =\left\|\mathbb{D}\left(\phi(F)-\widetilde{p_{n}}(F)\right)\right\| \mathscr{X}_{1 p}+\left\|\widetilde{p}_{n}{ }^{\prime}(F) \mathbb{D}(F)-\phi^{\prime}(F) \mathbb{D}(F)\right\| \mathscr{X}_{1 p} \\
& \leq\|\mathbb{D}\| \cdot\left\|\left(\phi(F)-\widetilde{p_{n}}(F)\right)\right\|_{X \otimes(S)_{\rho, p}}+\left\|\widetilde{p}_{n}^{\prime}(F)-\phi^{\prime}(F)\right\| \cdot\|\mathbb{D}(F)\|_{X \otimes(S)_{\rho, p}} \rightarrow 0,
\end{aligned}
$$

as $n \rightarrow \infty$. From this also (2.33) follows together with the estimate

$$
\|\mathbb{D}(\phi(F))\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}} \leq\left\|\phi^{\prime}(F)\right\|_{X \otimes(S)_{\rho, p}} \cdot\|\mathbb{D}(F)\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}<\infty,
$$

and thus $\phi(F) \in \operatorname{Dom}_{\rho, p}(\mathbb{D})$.
$2^{\circ}$ The proof of (2.34) for the Wick version can be conducted in a similar manner. According to Theorem 2.17 we have $\mathbb{D}\left(F^{\diamond k}\right)=k F^{\diamond(k-1)} \diamond \mathbb{D}(F)$. If $\phi$ is an analytic function given by $\phi(x)=\sum_{k=0}^{\infty} a_{k} x^{k}$, then $\phi^{\prime}(x)=\sum_{k=1}^{\infty} a_{k} k x^{k-1}$ and consequently $\phi^{\diamond}(F)=\sum_{k=0}^{\infty} a_{k} F^{\diamond k}$ and $\phi^{\diamond}(F)=\sum_{k=1}^{\infty} a_{k} k F^{\diamond(k-1)}$. Thus, (2.34) follows from

$$
\mathbb{D}\left(\phi^{\diamond}(F)\right)=\sum_{k=0}^{\infty} a_{k} \mathbb{D}\left(F^{\diamond k}\right)=\sum_{k=0}^{\infty} a_{k} k F^{\diamond(k-1)} \diamond \mathbb{D}(F)=\phi^{\diamond}(F) \diamond \mathbb{D}(F) .
$$

Example 2.2 Let $b_{t}$ be Brownian motion, $w_{t}$ white noise and $d_{t_{0}}$ the Dirac delta function concentrated at $t_{0}$. Then, by the previous theorems we obtain $\mathbb{D}\left(b_{t_{0}}^{2}\right)=$ $2 b_{t_{0}} \cdot \mathbb{D}\left(b_{t_{0}}\right)=2 b_{t_{0}} \cdot \chi_{\left[0, t_{0}\right]}(t), \mathbb{D}\left(b_{t_{0}}^{\diamond 2}\right)=2 b_{t_{0}} \cdot \chi_{\left[0, t_{0}\right]}(t)$ and $\mathbb{D}\left(w_{t_{0}}^{\diamond 2}\right)=2 w_{t_{0}} \diamond \mathbb{D}\left(w_{t_{0}}\right)=$ $2 w_{t_{0}} \cdot d_{t_{0}}(t)$, since the Wick product reduces to the ordinary product if one of the multiplicands is deterministic. Also, $\mathbb{D}\left(\exp ^{\diamond}\left(w_{t_{0}}\right)\right)=\exp ^{\diamond}\left(w_{t_{0}}\right) \cdot d_{t_{0}}(t)$, or more general $\mathbb{D}\left(\exp ^{\diamond} \delta(h)\right)=\exp ^{\diamond} \delta(h) \cdot h$, for any $h \in S^{\prime}(\mathbb{R})$, which verifies that the stochastic exponentials are eigenvectors of the Malliavin derivative. More examples can be found in [14].

### 2.6 Fractional Operators of the Malliavin Calculus

Following [5, 10], in Sect. 1.3.6 we introduced the fractional transform $M^{(H)}$ and in Sect. 1.4.3 the isometry mapping $\mathscr{M}$ on spaces of random variables and stochastic processes. Now we define fractional operators of the Malliavin calculus.

Denote by $\mathbb{D}$ the Malliavin derivative and $\mathbb{D}^{(H)}$ the fractional Malliavin derivative on $X \otimes(S)_{-\rho}$ (respectively on $X \otimes(S)_{\rho}$ and $\left.X \otimes L^{2}(\mu)\right)$. We say that a process $F=\sum_{\alpha \in \mathscr{F}} f_{\alpha} \otimes H_{\alpha}, f_{\alpha} \in X$ is differentiable in Malliavin sense if its coefficients satisfy (2.3) (respectively (2.5) and (2.6)). Then, the chaos expansion form of its Malliavin derivative is given by (2.2), while the chaos expansion form of its fractional Malliavin derivative is given by

$$
\begin{equation*}
\mathbb{D}^{(H)} F=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes e_{k}^{(H)} \otimes H_{\alpha-\varepsilon^{(k)}} \tag{2.36}
\end{equation*}
$$

where $e_{k}^{(H)}=M^{(1-H)} \xi_{k}, k \in \mathbb{N}$. Denote by $\widetilde{\mathbb{D}}$ the Malliavin derivative and by $\widetilde{\mathbb{D}}^{(H)}$ the fractional Malliavin derivative on $X \otimes(S)_{-\rho}^{(H)}$ (respectively on $X \otimes(S)_{\rho}^{(H)}$ and $\left.X \otimes L^{2}\left(\mu_{H}\right)\right)$. If the coefficients of $\widetilde{F}=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes \widetilde{H}_{\alpha}, f_{\alpha} \in X, \alpha \in \mathscr{I}$ satisfy (2.3) (respectively (2.5) and (2.6)), then chaos expansion forms of these operators are

$$
\begin{align*}
\widetilde{\mathbb{D}} F & =\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes e_{k}^{(H)} \otimes \widetilde{H}_{\alpha-\varepsilon^{(k)}}  \tag{2.37}\\
\widetilde{\mathbb{D}}^{(H)} F & =\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes M^{(1-H)} e_{k}^{(H)} \otimes \widetilde{H}_{\alpha-\varepsilon^{(k)}}
\end{align*}
$$

Note that both $\operatorname{Dom}(\mathbb{D})=\operatorname{Dom}\left(\mathbb{D}^{(H)}\right)$ and $\operatorname{Dom}(\widetilde{\mathbb{D}})=\operatorname{Dom}\left(\widetilde{\mathbb{D}}^{(H)}\right)$ are determined by the condition (2.3) (respectively by (2.5) and (2.6)). The connection between $\mathbb{D}^{(H)}$ and $\mathbb{D}$ on a classical space and also between $\widetilde{\mathbb{D}}^{(H)}$ and $\widetilde{\mathbb{D}}$ on a fractional space is given through the mapping $\mathbf{M}=M^{(H)} \otimes I d$, defined by (1.63). In particular, let $\mathbb{D}^{(H)}: X \otimes(S)_{-\rho} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ and $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha} \in \operatorname{Dom}\left(\mathbb{D}^{(H)}\right)$. Then,

$$
\begin{equation*}
\mathbb{D}^{(H)} F=\mathbf{M}^{-1}\left(\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}}\right)=\mathbf{M}^{-1} \circ \mathbb{D} F . \tag{2.38}
\end{equation*}
$$

Similarly, $\widetilde{\mathbb{D}}^{(H)}: X \otimes(S)_{\widetilde{\sim}}^{(H)} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}^{(H)}$ and for $\widetilde{F} \in \operatorname{Dom}\left(\widetilde{\mathbb{D}}^{(H)}\right)$ it holds $\widetilde{\mathbb{D}}^{(H)} \widetilde{F}=\mathbf{M}^{-1} \circ \widetilde{\mathbb{D}} \widetilde{F}$.

Theorem 2.21 ([10]) For $F \in \operatorname{Dom}(\mathbb{D})$ it holds

$$
\begin{equation*}
\mathbb{D}^{(H)} F=\mathbf{M}^{-1} \circ \mathbb{D} F=\mathscr{M} \circ \widetilde{\mathbb{D}} \circ \mathscr{M}^{-1} F . \tag{2.39}
\end{equation*}
$$

Proof From (1.63), (2.2), (2.36) and (2.37) we obtain for all $F \in \operatorname{Dom}(\mathbb{D})$

$$
\begin{aligned}
& \mathscr{M} \circ \widetilde{\mathbb{D}} \circ \mathscr{M}^{-1}\left(\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha}\right)=\mathscr{M} \circ \widetilde{\mathbb{D}}\left(\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes \widetilde{H}_{\alpha}\right) \\
& =\mathscr{M}\left(\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes e_{k}^{(H)} \otimes \widetilde{H}_{\alpha-\varepsilon^{(k)}}\right)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} \alpha_{k} f_{\alpha} \otimes e_{k}^{(H)} \otimes H_{\alpha-\varepsilon^{(k)}},
\end{aligned}
$$

which by (2.38) equals $\mathbb{D}^{(H)} F$ and the assertion (2.39) follows.
Denote by $\delta^{(H)}$ the fractional Skorokhod integral on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ (respectively on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ and $\left.X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)\right)$ and by $\widetilde{\delta}$ the Skorokhod integral on the corresponding fractional space $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}^{(H)}$ (respectively on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{\rho}^{(H)}$ and $\left.X \otimes L^{2}(\mathbb{R}) \otimes L^{2}\left(\mu_{H}\right)\right)$. In particular, $u \in \operatorname{Dom}(\delta)$ if its co-
efficients satisfy (2.8) (respectively (2.11) and (2.13)) and the fractional Skorokhod integral is defined by

$$
\begin{equation*}
\delta^{(H)}(u)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} u_{\alpha, k}^{H} \otimes H_{\alpha+\varepsilon^{(k)}}, \tag{2.40}
\end{equation*}
$$

where $u_{\alpha, k}^{H}=\left(u_{\alpha}, e_{k}^{(H)}\right), \alpha \in \mathscr{I}$ and $k \in \mathbb{N}$. Let $\widetilde{u}=\sum_{\alpha \in \mathscr{\mathscr { I }}} u_{\alpha} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes$ $(S)_{-\rho}^{(H)}\left(\right.$ respectively on $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{\rho}^{(H)}$ and $\left.X \otimes L^{2}(\mathbb{R}) \otimes L^{2}\left(\mu_{H}\right)\right)$, such that the coefficients $u_{\alpha}=\sum_{k \in \mathbb{N}} u_{\alpha, k} \otimes \xi_{k}$ with $u_{\alpha, k} \in X$ satisfy (2.8) (respectively (2.11) and (2.13)). Then, the Skorokhod integral $\tilde{\delta}$ is of the form

$$
\begin{equation*}
\tilde{\delta}(\widetilde{u})=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} u_{\alpha, k} \otimes \widetilde{H}_{\alpha+\varepsilon^{(k)}} . \tag{2.41}
\end{equation*}
$$

Theorem 2.22 ([9]) For $\widetilde{u} \in \operatorname{Dom}_{0}(\widetilde{\delta})$ it holds $\mathscr{M}(\widetilde{\delta}(\widetilde{u}))=\delta(\mathscr{M}(\widetilde{u}))$.
Proof From (2.41) and the definition of $\mathscr{M}$ it holds $\mathscr{M}(\widetilde{\delta}(\widetilde{u}))=\delta(u)=\delta(\mathscr{M}(\widetilde{u}))$ for all associated pairs of processes $\widetilde{u}$ and $u=\mathscr{M} \widetilde{u}$. Since $\mathscr{M}$ is an isometry we have $\|\delta(u)\|_{X \otimes L^{2}(\mu)}^{2}=\|\mathscr{M}(\widetilde{\delta}(\widetilde{u}))\|_{X \otimes L^{2}(\mu)}^{2}=\|\widetilde{\delta}(\widetilde{u})\|_{X \otimes L^{2}\left(\mu_{H}\right)}^{2}$.

The fractional Ornstein-Uhlenbeck operator $\mathscr{R}^{(H)}$ on the classical space is defined as the composition $\mathscr{R}^{(H)}=\delta^{(H)} \circ \mathbb{D}^{(H)}$ and can be represented in the form

$$
\mathscr{R}^{(H)} u=\mathscr{R}^{(H)}\left(\sum_{\alpha \in \mathscr{\mathscr { }}} u_{\alpha} \otimes H_{\alpha}\right)=\sum_{\alpha \in \mathscr{\mathscr { A }}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\mathscr{R} u .
$$

Similarly, the Ornstein-Uhlenbeck operator $\widetilde{\mathscr{R}}=\widetilde{\delta} \circ \widetilde{\mathbb{D}}$ and the fractional OrnsteinUhlenbeck operators $\widetilde{\mathscr{R}}^{(H)}=\widetilde{\delta}^{(H)} \circ \widetilde{\mathbb{D}}^{(H)}$ in fractional spaces are also equal

$$
\widetilde{\mathscr{R}}^{(H)} \widetilde{u}=\widetilde{\mathscr{R}}^{(H)}\left(\sum_{\alpha \in \mathscr{\mathscr { F }}} \widetilde{u}_{\alpha} \otimes \widetilde{H}_{\alpha}\right)=\sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha| \widetilde{u}_{\alpha} \otimes \widetilde{H}_{\alpha}=\widetilde{\mathscr{R}} \widetilde{u} .
$$

The corresponding domains remain the same and, depending on a set of processes, are determined by (2.16), (2.18) or (2.19), see Sect. 2.4.

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## Chapter 3 <br> Equations Involving Mallivin Calculus Operators


#### Abstract

This chapter is devoted to the study of several classes of stochastic equations involving generalized operators of the Malliavin calculus. In particular, we prove the surjectivity of the main operators of the Malliavin calculus. We also consider equations involving the Malliavin derivative operator and the Wick product with a Gaussian process. Applying the chaos expansion method in white noise spaces, we solve these equations and obtain explicit forms of the solutions in appropriate spaces of stochastic processes.


### 3.1 Introduction

It is of great importance to solve explicitly stochastic differential equations (SDEs) involving operators of Malliavin calculus, since explicit expansions of solutions can be used in numerical simulations [2, 18, 24]. Particularly, we consider the following fundamental equations with the Ornstein-Uhlenbeck opertaor $\mathscr{R}$, the Malliavin operator $\mathbb{D}$ and the Skorokhod integral $\delta$

$$
\begin{equation*}
P_{m}(\mathscr{R}) u=g, \quad \mathbb{D} u=h, \quad \delta u=f \tag{3.1}
\end{equation*}
$$

where $P_{m}$ is a polynomial of order $m$ and $P_{m}(\mathscr{R})$ is of the form (2.20). We also consider Wick-type equations involving Malliavin derivative and a nonhomogeneous linear equation with $\mathbb{D}$, i.e.,

$$
\begin{equation*}
\mathbb{D} u=\mathbf{G} \diamond(\mathbf{A} u)+h, \quad \text { and } \quad \mathbb{D} u=c \otimes u+h, \tag{3.2}
\end{equation*}
$$

satisfying the initial condition $\mathbb{E} u=\widetilde{u}_{0}$, where $\mathbf{G}$ is a Gaussian process, A a coordinatewise operator, $c \in S^{\prime}(\mathbb{R})$ and $h$ is a Schwartz space valued generalized stochastic process. The three Equations. (3.1) have been considered in [12, 13]. They provide a full characterization of the range of all three operators. The study of the Wick-type equation in (3.2) was motivated by [13], where it was shown that Malliavin derivative indicates the rate of change in time between ordinary product and the Wick product, see Theorem 2.14 and (2.23). We also point out that the Wick product and
the Malliavin derivative play an important role in the analysis of nonlinear problems. For instance, in [23] the authors proved that in random fields, random polynomial nonlinearity can be expanded in a Taylor series involving Wick products and Malliavin derivatives, the so-called Wick-Malliavin series expansion. Since the Malliavin derivative represents a stochastic gradient in the direction of white noise, one can consider similar equations that include a stochastic gradient in the direction of more general stochastic process, like the ones studied in [17].

In order to solve these stochastic equations explicitly we apply the method of chaos expansions also called the propagator method. The initial SDE is thus reduced to an infinite triangular system of deterministic equations which can be solved by applying techniques from the deterministic theory of algebraic or (ordinary and partial) differential equations. Summing up all coefficients of the expansion, i.e., the solutions of the deterministic system, and proving its convergence in an appropriate space of stochastic processes, one obtains the solution of the initial equation.

Propagator method has been used for solving singular SDEs. It has been successfully applied to several classes of SPDEs. In $[11,21]$ the Dirichlet problem of elliptic stochastic equations was solved and in [14] parabolic equations with the Wick-type convolution operators were studied. Another type of equations have been investigated in $[5,15-17,19]$. Besides the fact that the chaos expansion method is easy to apply (since it uses orthogonal bases and series expansions), the advantage of the method is that it provides an explicit form of the solution. We avoid using the Hermite transform [4] or the $\mathscr{S}$-transform [3], since these methods depend on the ability to apply their inverse transforms. Our method requires only to find an appropriate weight factor to make the resulting series convergent. Moreover, polynomial chaos expansion approximations are known for being more efficient than Monte Carlo methods. Moreover, for non-Gaussian processes, convergence can be improved by changing the Hermite basis to another family of orthogonal polynomials (Charlier, Laguerre, Meixner, etc.) [25].

### 3.2 Equations with the Ornstein-Uhlenbeck Operator

We consider stochastic equations involving polynomials of the Ornstein-Uhlenbeck operator. We generalize results from $[9,10,12,13]$.

Theorem 3.1 Let $\rho \in[0,1]$ and let $P_{m}(t)=\sum_{k=0}^{m} p_{k} t^{k}, t \in \mathbb{R}$ be a polynomial of degree $m$ with real coefficients.
(a) If $P_{m}(k) \neq 0$, for $k \in \mathbb{N}_{0}$, then the equation $P_{m}(\mathscr{R}) u=g$ has a unique solution represented in the form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathscr{I}} \frac{g_{\alpha}}{P_{m}(|\alpha|)} \otimes H_{\alpha} . \tag{3.3}
\end{equation*}
$$

(b) If $P_{m}(k)=0$ for $k \in M$, where $M$ is a finite subset of $\mathbb{N}_{0}$ and $g_{\alpha}=0$ for $|\alpha|=$ $i \in M$ then the equation $P_{m}(\mathscr{R}) u=g$ with the conditions $u_{\alpha}=c_{i}$ for $|\alpha|=i \in M$ has a unique solution given by

$$
\begin{equation*}
u=\sum_{|\alpha| \notin M} \frac{g_{\alpha}}{P_{m}(|\alpha|)} \otimes H_{\alpha}+\sum_{|\alpha|=i \in M} c_{i} \otimes H_{\alpha} \tag{3.4}
\end{equation*}
$$

Moreover, the following hold:
$1^{\circ}$ If $g \in X \otimes(S)_{-\rho,-p}, p \in \mathbb{N}$ then $u \in \operatorname{Dom}_{-\rho,-p}\left(\mathscr{R}^{m}\right)$.
$2^{\circ}$ If $g \in X \otimes(S)_{\rho, p}, p \in \mathbb{N}$ then $u \in \operatorname{Dom}_{\rho, p}\left(\mathscr{R}^{m}\right)$.
$3^{\circ}$ If $g \in X \otimes L^{2}(\mu)$ then $u \in \operatorname{Dom}_{0}\left(\mathscr{R}^{m}\right)$.
Proof Let $g$ be of the form $g=\sum_{\alpha \in \mathscr{I}} g_{\alpha} \otimes H_{\alpha}$. We seek for a solution in the form

$$
\begin{equation*}
u=\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}, \quad u_{\alpha} \in X \tag{3.5}
\end{equation*}
$$

Applying (2.20), the equation $P_{m}(\mathscr{R}) u=g$ transforms to

$$
\sum_{\alpha \in \mathscr{I}} P_{m}(|\alpha|) u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{I}} g_{\alpha} \otimes H_{\alpha}
$$

which due to the uniqueness of the Wiener-Itô chaos expansion, reduces to the system of deterministic equations $P_{m}(|\alpha|) u_{\alpha}=g_{\alpha}$ for all $\alpha \in \mathscr{I}$. If $P_{m}(|\alpha|) \neq 0$ for all $\alpha \in \mathscr{I}$, then $u_{\alpha}=\frac{g_{\alpha}}{P_{m}(|\alpha|)}$ and the initial equation has a solution of the form (3.3). If $P_{m}(|\alpha|)=0$ for $|\alpha| \in M$ and $g_{\alpha}=0$ for $|\alpha| \in M$, then $u_{\alpha}=c_{i}$ for $|\alpha|=i \in M$ and $u_{\alpha}=\frac{g_{\alpha}}{P(|\alpha|)}$ for $|\alpha| \notin M$. Thus, in this case the solution is of the form (3.4).
$1^{\circ}$ Assume that $g \in X \otimes(S)_{-\rho,-p}$ such that $g=\sum_{|\alpha| \notin M} g_{\alpha} \otimes H_{\alpha}$, i.e., it satisfies the condition (1.34). We can also assume that $\sum_{|\alpha|=i \in M}\left\|c_{i}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-q \alpha}<\infty$ for $q \leq p-2 m$, because $M$ is a finite set. Then, $u \in \operatorname{Dom}_{-\rho,-p}\left(\mathscr{R}^{m}\right)$ since

$$
\begin{aligned}
\|u\|_{D o m_{-\rho,-p}\left(\mathscr{R}^{m}\right)}^{2} & =A+\sum_{|\alpha| \notin M}|\alpha|^{2 m} \frac{\left\|g_{\alpha}\right\|_{X}^{2}}{P_{m}(|\alpha|)^{2}} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& \leq A+\sum_{|\alpha| \notin M}\left\|g_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}=A+\|g\|_{X \otimes(S)_{-\rho,-p}}^{2}<\infty,
\end{aligned}
$$

where $A=\sum_{|\alpha|=i \in M}|\alpha|^{2 m}\left\|c_{i}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha}$.
$2^{\circ}$ Let $g \in X \otimes(S)_{\rho, p}$ such that $g=\sum_{|\alpha| \notin M} g_{\alpha} \otimes H_{\alpha}$ satisfying the condition (1.32). We additionally assume $\sum_{|\alpha|=i \in M}\left\|c_{i}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{q \alpha}<\infty$ for $q \geq p+$ $2 m$. Then, $u \in \operatorname{Dom}_{\rho, p}(\mathscr{R})$ because we have

$$
\begin{aligned}
\|u\|_{D_{o m_{\rho, p}\left(\mathscr{R}^{m}\right)}^{2}} & =B+\sum_{|\alpha| \notin M}|\alpha|^{2 m} \frac{\left\|u_{\alpha}\right\|_{X}^{2}}{P_{m}(|\alpha|)^{2}} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =B+\sum_{|\alpha| \notin M}\left\|g_{\alpha}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha}=B+\|g\|_{X \otimes(S)_{\rho, p}}^{2}<\infty
\end{aligned}
$$

where $B=\sum_{|\alpha|=i \in M}|\alpha|^{2 m}\left\|c_{i}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha}$.
$3^{\circ}$ If $g$ is square integrable and $\sum_{|\alpha|=i \in M}|\alpha|^{2 m}\left\|c_{i}\right\|_{X}^{2} \alpha!<\infty$, then the solution $u \in \operatorname{Dom}_{0}\left(\mathscr{R}^{m}\right)$ since

$$
\|u\|_{D_{0} m_{0}\left(\mathscr{R}^{m}\right)}^{2}=\sum_{|\alpha|=i \in M}|\alpha|^{2 m}\left\|c_{i}\right\|_{X}^{2} \alpha!+\sum_{|\alpha| \notin M}|\alpha|^{2 m} \alpha!\frac{\left\|g_{\alpha}\right\|_{X}^{2}}{P_{m}(|\alpha|)^{2}}<\infty
$$

Remark 3.1 For $P_{m}(t)=t^{m}, t \in \mathbb{R}$ the equation $P_{m}(\mathscr{R}) u=g$ reduces to

$$
\begin{equation*}
\mathscr{R}^{m} u=g, \quad \mathbb{E} u=\tilde{u}_{0} \in X \tag{3.6}
\end{equation*}
$$

This case was considered in [13]. Assuming that $g$ has zero generalized expectation, from Theorem 3.1 it follows that the Eq. (3.6) has a unique solution of the form

$$
u=\tilde{u}_{0}+\sum_{|\alpha|>0} \frac{g_{\alpha}}{|\alpha|^{m}} \otimes H_{\alpha}
$$

Remark 3.2 Note that $\mathscr{R} u=u$ if and only if $u \in \mathscr{H}_{1}$, i.e., Gaussian processes with zero expectation and first order chaos are the only fixed points for the OrnsteinUhlenbeck operator. For example, $\mathscr{R}\left(b_{t}\right)=b_{t}$ and $\mathscr{R}\left(w_{t}\right)=w_{t}$. Moreover, $\mathscr{H}_{m}$ is the eigenspace corresponding to the eigenvalue $m$ of the Ornstein-Uhlenbeck operator, for $m \in \mathbb{N}$. This is in compliance with (2.15).

Remark 3.3 If $\mathbb{E} u=0$ following [20], one can define the pseudo-inverse $\mathscr{R}^{-1}$. Particularly, the operator $\mathscr{R}^{-1}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ and for $u \in X \otimes(S)_{-\rho}$ such that $\mathbb{E} u=0$ is given in the form

$$
\mathscr{R}^{-1} u=\mathscr{R}^{-1}\left(\sum_{\alpha \in \mathscr{I},|\alpha|>0} u_{\alpha} \otimes H_{\alpha}\right)=\sum_{\alpha \in \mathscr{I},|\alpha|>0} \frac{u_{\alpha}}{|\alpha|} \otimes H_{\alpha}
$$

Thus,

$$
\mathscr{R} \mathscr{R}^{-1}(u)=u \quad \text { and } \quad \mathscr{R}^{-1} \mathscr{R}(u)=u
$$

In general, for $\mathbb{E} u \neq 0$, we have $\mathscr{R} \mathscr{R}^{-1}(u-\mathbb{E} u)=u$ and $\mathscr{R}^{-1} \mathscr{R}(u)=u$.

Corollary 3.1 Each stochastic process g can be represented as $g=\mathbb{E} g+\mathscr{R}(u)$,for some $u \in \operatorname{Dom}(\mathscr{R})$, where $\operatorname{Dom}(\mathscr{R})$ denotes the domain of $\mathscr{R}$ in one of the spaces $X \otimes(S)_{\rho}, X \otimes(S)_{-\rho}$ or $X \otimes L^{2}(\mu)$.

Proof If $\mathbb{E} g \neq 0$ then $g-\mathbb{E} g$ has zero expectation. Thus the assertion follows for $u=\mathscr{R}^{-1}(g-\mathbb{E} g)$.

Remark 3.4 If a stochastic process $f$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathscr{H}_{i}$ for some $m \in \mathbb{N}$, then the solution $u$ of the Eq. (3.6) belongs also to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathscr{H}_{i}$.

Another types of equations, for example equations with the exponential of the Ornstein-Uhlenbeck operator $e^{\mathscr{R}} u=g$, were solved in [7].

### 3.3 First Order Equation with the Malliavin Derivative Operator

We consider a first order equation involving the Malliavin derivative operator. The following result characterizes the family of stochastic processes that can be written as the Malliavin derivative of some stochastic process (for all cases square integrable, test and generalized stochastic processes). We generalize the results from [9, 12, 13].

Theorem 3.2 Let $\rho \in[0,1]$. Let a process $h$ be given in the chaos expansion representation form $h=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha}$ such that the coefficients $h_{\alpha, k}$ satisfy the condition

$$
\begin{equation*}
\frac{1}{\alpha_{k}} h_{\alpha-\varepsilon^{(k)}, k}=\frac{1}{\beta_{j}} h_{\beta-\varepsilon^{(j)}, j}, \tag{3.7}
\end{equation*}
$$

for all $\alpha+\varepsilon^{(k)}=\beta+\varepsilon^{(j)}$. Then, for each $\widetilde{u}_{0} \in X$ the equation

$$
\begin{equation*}
\mathbb{D} u=h, \quad \mathbb{E} u=\widetilde{u}_{0} \tag{3.8}
\end{equation*}
$$

has a unique solution и represented in the form

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha \in \mathscr{\mathscr { U }},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} . \tag{3.9}
\end{equation*}
$$

Moreover, the following holds:

```
\(1^{\circ}\) If \(h \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q>p+1\) then \(u \in \operatorname{Dom}_{-\rho,-q}(\mathbb{D})\).
\(2^{\circ}\) If \(h \in X \otimes S_{p}(\mathbb{R}) \otimes(S)_{\rho, q}, p>q+1\), then \(u \in \operatorname{Dom}_{\rho, q}(\mathbb{D})\).
\(3^{\circ}\) If \(h \in \operatorname{Dom}_{0}(\delta)\) then \(u \in \operatorname{Dom}_{0}(\mathbb{D})\).
```

Proof $1^{\circ}$ Applying the Skorokhod integral on both sides of (3.8) one obtains

$$
\mathscr{R} u=\delta(h),
$$

for a given $h \in \operatorname{Dom}_{-\rho,-l,-p}(\delta)$, for some $l, p \in \mathbb{N}$. From the initial condition it follows that the solution $u$ is given in the form $u=\widetilde{u}_{0}+\sum_{|\alpha|>0} u_{\alpha} \otimes H_{\alpha}$ and its coefficients are obtained from the system

$$
\begin{equation*}
|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}, \quad|\alpha|>0, \tag{3.10}
\end{equation*}
$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$. Hence, the solution $u$ is given in the form (3.9). Now, we prove that the solution $u$ belongs to the space $\operatorname{Dom}_{-\rho,-p}(\mathbb{D})$. Clearly,

$$
\begin{aligned}
& \left\|u-\tilde{u}_{0}\right\|_{D o m_{-\rho,-p}(\mathbb{D})}^{2}=\sum_{|\alpha|>0}|\alpha|^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{|\alpha|>0} \frac{|\alpha|^{1+\rho}}{|\alpha|^{2}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \alpha!^{1-\rho}(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} h_{\beta, k}\left(\frac{\beta_{k}+1}{\left|\beta+\varepsilon^{(k)}\right|}\right)^{\frac{1-\rho}{2}}(2 k)^{-\frac{p}{2}}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq \sum_{\beta \in \mathscr{I}}|\beta|^{1-\rho}\left\|\sum_{k \in \mathbb{N}} h_{\beta, k}(2 k)^{-\frac{1}{2}}(2 k)^{-\frac{p-l}{2}}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq \sum_{\beta \in \mathscr{I}}|\beta|^{1-\rho}\left(\sum_{k \in \mathbb{N}}\left\|h_{\beta, k}\right\|_{X}^{2}(2 k)^{-l} \sum_{k \in \mathbb{N}}(2 k)^{p-l}\right) \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq c \sum_{\beta \in \mathscr{\mathscr { I }}}|\beta|^{1-\rho}\left\|h_{\beta, k}\right\|_{X \otimes S_{-l}(\mathbb{R})}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& =c\|h\|_{D o m_{-\rho,-l,-p}(\delta)}^{2}<\infty,
\end{aligned}
$$

since $c=\sum_{k \in \mathbb{N}}(2 k)^{-(p-l)}<\infty$, for $p>l+1$. In the calculations we used the estimate $\frac{\beta_{k}+1}{\left|\beta_{k}+1\right|}=\frac{\beta_{k}+1}{|\beta|+1} \leq 1 \leq|\beta|$, for $\beta \in \mathscr{I}, k \in \mathbb{N}$ and the Cauchy-Schwarz inequality. Thus,
$2^{\circ}$ Let now $h \in X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$. The operator $\delta$ can again be applied onto $h$, since from (2.12) we have $h \in X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p} \subseteq \operatorname{Dom}_{\rho, l, p-1-\rho}(\delta)$. It remains to prove that the solution $u$ given in the form (3.9) belongs to $\operatorname{Dom}_{\rho, q}(\mathbb{D})$. Indeed,
3.3 First Order Equation with the Malliavin Derivative Operator

$$
\begin{aligned}
\left\|u-\tilde{u}_{0}\right\|_{D_{o m_{\rho, p}}^{2}(\mathbb{D})} & =\sum_{\alpha \in \mathscr{I}}|\alpha|^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{|\alpha|>0} \frac{|\alpha|^{1-\rho}}{|\alpha|^{2}}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \alpha!^{1+\rho}(2 \mathbb{N})^{p \alpha} \\
& =\sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} h_{\beta, k}\left(\frac{\beta_{k}+1}{\left|\beta+\varepsilon^{(k)}\right|}\right)^{\frac{1+\rho}{2}}(2 k)^{\frac{p}{2}}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq \sum_{\beta \in \mathscr{I}}\left(\sum_{k \in \mathbb{N}}\left\|h_{\beta, k}\right\|_{X}^{2}(2 k)^{l} \sum_{k \in \mathbb{N}}(2 k)^{p-l}\right) \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq c \sum_{\beta \in \mathscr{I}}\left(\sum_{k \in \mathbb{N}}\left\|h_{\beta, k}\right\|_{X}^{2}(2 k)^{l}\right) \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& =c \sum_{\beta \in \mathscr{I}}\left\|h_{\beta}\right\|_{X \otimes S_{l}(\mathbb{R})}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta}=c\|h\|_{X \otimes S_{l}(\mathbb{R}) \otimes(S)_{\rho, p}}^{2}<\infty
\end{aligned}
$$

since $c=\sum_{k \in \mathbb{N}}(2 k)^{p-l}<\infty$, for $l>p+1$. We used the estimate $\frac{\beta_{k}+1}{\left|\beta+\varepsilon^{(k)}\right|} \leq 1$, for $\beta \in \mathscr{I}, k \in \mathbb{N}$ and the Cauchy-Schwarz inequality. Thus,

$$
\|u\|_{D_{o m}, p}^{2}(\mathbb{D}) \leq 2\left(\left\|\tilde{u}_{0}\right\|_{X}^{2}+c\|h\|_{X \otimes S_{p}(\mathbb{R}) \otimes(S)_{\rho, q}}^{2}\right)<\infty .
$$

$3^{\circ}$ Let $h \in L^{2}(\mathbb{R}) \otimes L^{2}(\mu)$. Then,

$$
\begin{aligned}
\left\|u-\tilde{u}_{0}\right\|_{D o m_{0}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathscr{I}}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{|\alpha|>0} \frac{\alpha!}{|\alpha|}\left\|\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k}\right\|_{X}^{2} \\
& =\sum_{\beta \in \mathscr{I}}\left\|\sum_{k \in \mathbb{N}} h_{\beta, k}\left(\frac{\beta_{k}+1}{|\beta|+1}\right)^{\frac{1}{2}}\right\|_{X}^{2} \beta!\leq \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left\|h_{\beta, k}\right\|_{X}^{2} \beta! \\
& =\sum_{\alpha \in \mathscr{I}} \beta!\left\|h_{\beta}\right\|_{X \otimes L^{2}(\mathbb{R})}^{2}=\|h\|_{X \otimes L^{2}(\mathbb{R}) \otimes L^{2}(\mu)}^{2}<\infty
\end{aligned}
$$

In [10] for $\rho=1$ we provided another way for solving Eq. (3.8). Applying the chaos expansion method directly, we transformed Eq. (3.8) into a system of infinitely many equations of the form

$$
\begin{equation*}
u_{\alpha+\varepsilon^{(k)}}=\frac{1}{\alpha_{k}+1} h_{\alpha, k}, \quad \text { for all } \quad \alpha \in \mathscr{I}, k \in \mathbb{N} \tag{3.11}
\end{equation*}
$$

from which we calculated $u_{\alpha}$, by induction on the length of $\alpha$.

Denote by $r=r(\alpha)=\min \left\{k \in \mathbb{N}: \alpha_{k} \neq 0\right\}$, for a nonzero multi-index $\alpha \in \mathscr{I}$, i.e., let $r$ be the position of the first nonzero component of $\alpha$. Then, the first nonzero component of $\alpha$ is the $r$ th component $\alpha_{r}$, i.e., $\alpha=\left(0, \ldots, 0, \alpha_{r}, \ldots, \alpha_{m}, 0, \ldots\right)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of $\alpha$, except the $r$ th, which is $\alpha_{r}-1$. With the given notation we call $\alpha_{\varepsilon^{(r)}}$ the representative of $\alpha$ and write $\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}$. For $\alpha \in \mathscr{I},|\alpha|>0$ the set

$$
\mathscr{K}_{\alpha}=\left\{\beta \in \mathscr{I}: \alpha=\beta+\varepsilon^{(j)}, \text { for those } j \in \mathbb{N} \text { such that } \alpha_{j}>0\right\}
$$

is a nonempty set, because it contains at least the representative of $\alpha$, i.e., $\alpha_{\varepsilon^{(r)}} \in \mathscr{K}_{\alpha}$. Note that, if $\alpha=n \varepsilon^{(r)}, n \in \mathbb{N}$ then $\operatorname{Card}\left(\mathscr{K}_{\alpha}\right)=1$ and in all other cases $\operatorname{Card}\left(\mathscr{K}_{\alpha}\right)>$ 1. Further, for $|\alpha|>0, \mathscr{K}_{\alpha}$ is a finite set because $\alpha$ has finitely many nonzero components and $\operatorname{Card}\left(\mathscr{K}_{\alpha}\right)$ is equal to the number of nonzero components of $\alpha$. For example, the first nonzero component of $\alpha=(0,3,1,0,5,0,0, \ldots)$ is the second one and it has three nonzero components. It follows that $r=2, \alpha_{r}=3$, the representative of $\alpha$ is $\alpha_{\varepsilon^{(r)}}=\alpha-\varepsilon^{(2)}=(0,2,1,0,5,0,0, \ldots)$ and the set $\mathscr{K}_{\alpha}$ consists of three elements $\mathscr{K}_{\alpha}=\{(0,2,1,0,5,0, \ldots),(0,3,0,0,5,0, \ldots),(0,3,1,0,4,0, \ldots)\}$.

In [10] the coefficients $u_{\alpha}$ of the solution of (3.11) are obtained as functions of the representative $\alpha_{\varepsilon^{(r)}}$ of a nonzero multi-index $\alpha \in \mathscr{I}$ in the form

$$
u_{\alpha}=\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r)}, r, \quad \text { for }|\alpha| \neq 0, \alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)} .
$$

Theorem 3.3 ([10]) Let $\quad h=\sum_{\alpha \in \mathscr{\mathscr { L }}} \sum_{k \in \mathbb{N}} h_{\alpha, k} \otimes \xi_{k} \otimes H_{\alpha} \in X \otimes S_{-p}(\mathbb{R}) \otimes$ $(S)_{-\rho,-p}$, for some $p \in \mathbb{N}_{0}$ with $h_{\alpha, k} \in X$ such that

$$
\begin{equation*}
\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon^{(r)}, r}}=\frac{1}{\alpha_{j}} h_{\beta, j}, \tag{3.12}
\end{equation*}
$$

for the representative $\alpha_{\varepsilon^{(r)}}$ of $\alpha \in \mathscr{I},|\alpha|>0$ and all $\beta \in \mathscr{K}_{\alpha}$, such that $\alpha=\beta+\varepsilon^{(j)}$, for $j \geq r, r \in \mathbb{N}$. Then, (3.8) has a unique solution in $X \otimes(S)_{-\rho,-p}$ given in the chaos expansion form

$$
\begin{equation*}
u=\widetilde{u}_{0}+\sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)} \in \mathscr{I}} \frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon^{(r)}}, r} \otimes H_{\alpha} . \tag{3.13}
\end{equation*}
$$

Remark 3.5 Under the assumption (3.12), the obtained form of the solution (3.13) transforms to the form (3.9) obtained in [10]. This was provided in [7]. First we express all $h_{\beta, k}$ in condition (3.12) in terms of $h_{\alpha_{\varepsilon^{(r)}, r}}$, i.e., $h_{\beta, k}=\frac{\alpha_{j}}{\alpha_{r}} h_{\alpha_{\varepsilon^{(r)}, r}}$, where $\beta \in \mathscr{K}_{\alpha}$ correspond to the nonzero components of $\alpha$ in the following way: $\beta=\alpha-\varepsilon^{(k)}, k \in \mathbb{N}$, and $r \in \mathbb{N}$ is the first nonzero component of $\alpha$. Therefore, from the form of the coefficients (3.10) obtained in Theorem 3.2 we have

$$
\frac{1}{|\alpha|} \sum_{\beta \in \mathscr{K}_{\alpha}} h_{\beta, k}=\frac{1}{|\alpha|} \sum_{j \in \mathbb{N}, \alpha_{j} \neq 0} \frac{\alpha_{j}}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}=\frac{1}{|\alpha|} \frac{\sum_{j \in \mathbb{N}} \alpha_{j}}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}=\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}
$$

Thus, the forms (3.9) and (3.13) are equivalent.
Later we will present the direct approach for solving Wick-type equations with the Malliavin derivative operator.

Corollary 3.2 It holds that $\mathbb{D}(u)=0$ if and only if $u=\mathbb{E} u$.
In other words the kernel of the operator $\mathbb{D}$ is $\mathscr{H}_{0}$.
Theorem 3.4 ([1]) Let h be a given stochastic process satisfying (3.7) and let $\tilde{u}_{0} \in X$. The initial value problem (3.8) is equivalent to the system of two initial values problems

$$
\begin{equation*}
\mathbb{D} u_{1}=0, \quad \mathbb{E} u_{1}=\tilde{u}_{0} \quad \text { and } \quad \mathbb{D} u_{2}=h, \quad \mathbb{E} u_{2}=0 \tag{3.14}
\end{equation*}
$$

where $u=u_{1}+u_{2}$.
Proof Let $u_{1}$ and $u_{2}$ be the solutions of the system (3.14). From the linearity of the operator $\mathbb{D}$ and the linearity of $\mathbb{E}$ it follows $\mathbb{D} u=\mathbb{D}\left(u_{1}+u_{2}\right)=\mathbb{D} u_{1}+\mathbb{D} u_{2}=h$ and $\mathbb{E} u=\mathbb{E}\left(u_{1}+u_{2}\right)=\mathbb{E} u_{1}+\mathbb{E} u_{2}=\widetilde{u}_{0}$. Thus the superposition of $u_{1}$ and $u_{2}$ solves the Eq. (3.8).

Let now $u$ be the solution of (3.8). By Theorem 3.2 it has chaos expansion representation form (3.9). By Corollary 3.2 it follows $\operatorname{Ker}(\mathbb{D})=\mathscr{H}_{0}$ and therefore $u$ can be expressed in the form $u=u_{1}+u_{2}$, where $u_{1} \in \operatorname{Ker}(\mathbb{D})$ and $u_{2} \in \operatorname{Image}(\mathbb{D})$. Then, we conclude that $\mathbb{D} u_{1}=0$ and $\mathbb{E} u_{1}=u^{0}$ as well as $\mathbb{D} u_{2}=h$ and $\mathbb{E} u_{2}=0$.

Corollary 3.3 For every Skorokhod integrable process $h$ there exists a unique $u \in \operatorname{Dom}(\mathbb{D})$ such that $\mathbb{E} u=0$ and $h=\mathbb{D}(u)$ holds. (The statement holds for test, square integrable and generalized stochastic processes.)

Proof The assertion follows for $u=\mathscr{R}^{-1}(\delta(h))$.
Remark 3.6 If a stochastic process $h$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathscr{H}_{i}$ for some $m \in \mathbb{N}$, then the unique solution $u$ of the Eq. (3.8) belongs to the Wiener chaos space $\bigoplus_{i=0}^{m+1} \mathscr{H}_{i}$. Particularly, if the input function $h$ is a constant random variable i.e., an element of $\mathscr{H}_{0}$, then the solution $u$ of (3.8) is a Gaussian process.

### 3.4 Nonhomogeneous Equation with the Malliavin Derivative Operator

We consider now a nonhomogeneous linear Malliavin differential type equation

$$
\begin{equation*}
\mathbb{D} u=c \otimes u+h, \quad \mathbb{E} u=\widetilde{u}_{0} \tag{3.15}
\end{equation*}
$$

where $c \in S^{\prime}(\mathbb{R}), h$ is a Schwartz space valued generalized stochastic process and $\widetilde{u}_{0} \in X$. Especially, for $h=0$ the Eq. (3.15) reduces to the corresponding homogeneous equation $\mathbb{D} u=c \otimes u$ satisfying $\mathbb{E} u=\widetilde{u}_{0}$. In this case we obtain the generalized eigenvalue problem for the Malliavin derivative operator, which was solved in [10]. Moreover, it was proved that in a special case, obtained solution coincide with the stochastic exponential, see also Example 2.2. Additionally, putting $c=0$, the initial equation (3.15) transforms to (3.8).

Let $\alpha_{\varepsilon^{(r)}}$ be the representative of a nonzero multi-index $\alpha$, i.e., $\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}$, $\left|\alpha_{\varepsilon^{(r)}}\right|=|\alpha|-1$ and let $\operatorname{Card}\left(\mathscr{K}_{\alpha}\right)>1$. Then, we denote by $r_{1}$ the first nonzero component of $\alpha_{\varepsilon^{(r)}}$ and by $\alpha_{\varepsilon^{\left(r_{1}\right)}}$ its representative, i.e., $\alpha_{\varepsilon^{(r)}}=\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}}$ and $\left|\alpha_{\varepsilon^{(r 1)}}\right|=|\alpha|-2$. If $\operatorname{Card}\left(\mathscr{K}_{\alpha_{\varepsilon^{(r)}}}\right)>1$, then we denote by $r_{2}$ the first nonzero component of $\alpha_{\varepsilon^{\left(r_{1}\right)}}$ and with $\alpha_{\varepsilon^{\left(r_{2}\right)}}$ its representative, i.e., $\alpha_{\varepsilon^{\left(r_{1}\right)}}=\varepsilon^{\left(r_{2}\right)}+\alpha_{\varepsilon^{\left(r_{2}\right)}}$ and so on. With such a procedure we decompose $\alpha \in \mathscr{I}$ recursively by new representatives of the previous representatives and we obtain a sequence of $\mathscr{K}$-sets. Thus, for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}, 0,0, \ldots\right) \in \mathscr{I},|\alpha|=s+1$ there exists an increasing family of integers $1 \leq r \leq r_{1} \leq r_{2} \leq \ldots \leq r_{s} \leq m, s \in \mathbb{N}$ such that $\alpha_{\varepsilon^{(r s)}}=\mathbf{0}$ and every $\alpha$ is decomposed by the recurrent sum

$$
\begin{equation*}
\alpha=\varepsilon^{(r)}+\alpha_{\varepsilon^{(r)}}=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\alpha_{\varepsilon^{\left(r_{1}\right)}}=\ldots=\varepsilon^{(r)}+\varepsilon^{\left(r_{1}\right)}+\ldots+\varepsilon^{\left(r_{s}\right)}+\alpha_{\varepsilon^{(r)}} . \tag{3.16}
\end{equation*}
$$

Theorem 3.5 ([8]) Let $\rho \in[0,1]$. Let $c=\sum_{k=1}^{\infty} c_{k} \xi_{k} \in S^{\prime}(\mathbb{R})$ and let $h \in X \otimes$ $S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ with coefficients $h_{\alpha, k} \in X$ such that the following conditions $(C)$

$$
\begin{array}{rlrl}
\frac{1}{\alpha_{r}} h_{\alpha_{(r)}, r} & =\frac{1}{\beta_{k}} h_{\beta, k}, & & \beta \in \mathscr{K}_{\alpha},|\alpha|=1 \\
\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}} & =\frac{1}{\beta_{k} \beta_{k_{1}}} c_{k} h_{\beta_{1}, k_{1}}, & \beta \in \mathscr{K}_{\alpha}, \beta_{1} \in \mathscr{K}_{\alpha_{\varepsilon}^{(r)}},|\alpha|=2
\end{array}
$$

hold for all possible decompositions of $\alpha$ of the form (3.16). If $c_{k} \geq 2 k$ for all $k \in \mathbb{N}$, then (3.15) has a unique solution in $X \otimes(S c)_{-\rho}$ given by

$$
\begin{align*}
u= & u^{h o m}+u^{n h o m}=\sum_{\alpha \in \mathscr{I}} u_{\alpha}^{h o m} \otimes H_{\alpha}+\sum_{|\alpha|>0} u_{\alpha}^{n h o m} \otimes H_{\alpha} \\
=\widetilde{u}_{0} \otimes & \sum_{\alpha \in \mathscr{I}} \frac{c^{\alpha}}{\alpha!} H_{\alpha}+\sum_{|\alpha|>0}\left(\frac{1}{\alpha_{r}} h_{\alpha_{\varepsilon}(r), r}+\frac{1}{\alpha_{r} \alpha_{r_{1}}} c_{r} h_{\alpha_{\varepsilon}\left(r_{1}\right), r_{1}}\right.  \tag{3.17}\\
& \left.+\frac{1}{\alpha_{r} \alpha_{r_{1}} \alpha_{r_{2}}} c_{r} c_{r_{1}} h_{\alpha_{\varepsilon}\left(r_{2}\right), r_{2}}+\ldots+\frac{1}{\alpha!} c_{r} c_{r_{1}} \ldots c_{r_{s-1}} h_{0, r_{s}}\right) \otimes H_{\alpha}
\end{align*}
$$

where $u^{\text {hom }}$ is the solution of the corresponding homogeneous equation $\mathbb{D} u=c \otimes u$. The nonhomogeneous part $u^{\text {nhom }}$ of the solution $u$ is given by the the second sum in (3.17), which runs through nonzero $\alpha$ represented in the recursive form (3.16).

Here we omit the proof as it follows similarly to the one given in [8] for $\rho=1$. Note that the first subcondition in $(C)$ corresponds to (3.7) and equals (3.12).

### 3.5 Wick-Type Equations Involving the Malliavin Derivative

We consider a nonhomogeneous first order equation involving the Malliavin derivative operator and the Wick product with a Gaussian process $\mathbf{G}$

$$
\begin{equation*}
\mathbb{D} u=\mathbf{G} \diamond \mathbf{A} u+h, \quad \mathbb{E} u=\tilde{u}_{0}, \quad \tilde{u}_{0} \in X \tag{3.18}
\end{equation*}
$$

where $h$ is a $S^{\prime}(\mathbb{R})$-valued generalized stochastic process and $\mathbf{A}$ is a coordinatewise operator on the space $X \otimes(S)_{-\rho}$. We assume that a Gaussian process $\mathbf{G}$ belongs to $S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}$, for some $l, p>0$, i.e., it can be represented in the form

$$
\begin{equation*}
\mathbf{G}=\sum_{k \in \mathbb{N}} g_{k} \otimes H_{\varepsilon^{(k)}}=\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n} \xi_{n} \otimes H_{\varepsilon^{(k)}}, \quad g_{k n} \in \mathbb{R} \tag{3.19}
\end{equation*}
$$

such that $\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n}^{2}(2 n)^{-l}(2 k)^{-p}<\infty$. We also assume $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow$ $X \otimes(S)_{-\rho}$ to be a coordinatewise operator, i.e., a linear operator defined by $\mathbf{A}(f)=$ $\sum_{\alpha \in \mathscr{I}} A_{\alpha}\left(f_{\alpha}\right) \otimes H_{\alpha}$, for $f=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes(S)_{-\rho}$, where $A_{\alpha}: X \rightarrow X$, $\alpha \in \mathscr{I}$ are polynomially bounded for all $\alpha$, i.e., there exists $r>0$ such that $\sum_{\alpha \in \mathscr{I}}\left\|A_{\alpha}\right\|^{2}(2 \mathbb{N})^{-r \alpha}<\infty$. If $A_{\alpha}=A$ for all $\alpha \in \mathscr{I}$ then according to Definition 1.19, the operator $\mathbf{A}$ is a simple coordinatewise operator. Especially, for an operator A such that $A_{\alpha}=0$, the Eq. (3.18) reduces to the initial value problem (3.8).

As a case of study, here we will prove existence and uniqueness of a solution for a special form of the Eq. (3.18) and represent its solution in explicit form. Particularly, we assume $A_{\alpha}=I d, \alpha \in \mathscr{I}$ being the identity operator and a Gaussian process $G \in$
$S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}$ obtained from $\mathbf{G}$ by choosing $g_{k n}=g_{k}$, for $k=n$ and $g_{k n}=0$ for $k \neq n$. Clearly, we consider $G$ to be of the form

$$
\begin{equation*}
G=\sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes H_{\varepsilon^{(k)}}, \tag{3.20}
\end{equation*}
$$

which for some $l, q>0$ satisfies the condition

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} g_{k}^{2}(2 k)^{-q-l}<\infty \tag{3.21}
\end{equation*}
$$

For $p \geq l+q$ the condition (3.21) reduces to the condition (1.50).
First we solve the homogeneous version of (3.18), i.e., we find a Malliavin differentiable process whose derivative coincides with its Wick product with a certain Gaussian process $G$ of the form (3.20). We state a theorem and present a detailed proof. Then, we analyze the nonhomogeneous equation (3.18). Due to complicate notation we skip some technical details in solving (3.18), with $\mathbf{G}$ of the form (3.19).

Theorem 3.6 ([6]) Let $\rho \in[0,1]$ and let $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q, l>0$ be a Gaussian process of the form (3.20) whose coefficients $g_{k}, k \in \mathbb{N}$ satisfy the condition (3.21). If $g_{k} \geq 2 k$ for all $k \in \mathbb{N}$ then the initial value problem

$$
\begin{equation*}
\mathbb{D} u=G \diamond u, \quad \mathbb{E} u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X, \tag{3.22}
\end{equation*}
$$

has a unique solution in $\operatorname{Dom}(\mathbb{D} g)_{-\rho,-p}$ represented in the form

$$
\begin{equation*}
u=\widetilde{u}_{0} \otimes \sum_{\alpha=2 \beta \in \mathscr{I}} \frac{C_{\alpha}}{|\alpha|!!}\left(\prod_{k=1}^{\infty} g_{k}^{\beta_{k}}\right) H_{\alpha}=\widetilde{u}_{0} \otimes \sum_{2 \beta \in \mathscr{\mathscr { I }}} C_{2 \beta} \frac{g^{\beta}}{|2 \beta|!!} H_{2 \beta}, \tag{3.23}
\end{equation*}
$$

where $C_{\alpha}$ represents the number of all possible decomposition chains connecting multi-indices $\alpha$ and $\tilde{\alpha}$, such that $\tilde{\alpha}$ is the first successor of $\alpha$ having only one nonzero component that is obtained by the subtractions $\alpha-2 \varepsilon^{\left(p_{1}\right)}-\ldots-2 \varepsilon^{\left(p_{s}\right)}=\tilde{\alpha}$, for $p_{1}, \ldots, p_{s} \in \mathbb{N}, s \geq 0$.

Proof We are looking for a solution of (3.22) in the chaos expansion form (3.5), which is Malliavin differentiable and which admits the Wick multiplication with a Gaussian process of the form (3.20). This means that we are seeking for $u_{\alpha} \in X$ such that $\sum_{\alpha \in \mathscr{I}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} g)^{-p \alpha}<\infty$ for some $p>0$. Wick product of a process $u$ and a Gaussian process $G$, represented respectively in their chaos expansion forms (3.5) and (3.20), is a well defined element $G \diamond u$ given by

$$
G \diamond u=\sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes H_{\varepsilon^{(k)}} \diamond \sum_{\alpha \in \mathscr{\mathscr { I }}} u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{\mathscr { J }}} \sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} .
$$

Clearly, for $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}$ and $u \in X \otimes(S)_{-\rho,-p}$ the Wick product $G \diamond u$ belongs to $X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-s}, s \geq p+q+1-\rho$, because

$$
\begin{aligned}
&\|G \diamond u\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-s}}^{2}=\sum_{\alpha>0} \sum_{k \in \mathbb{N}} \alpha!^{1-\rho} g_{k}^{2}(2 k)^{-l}\left\|u_{\alpha-\varepsilon^{k}}\right\|_{X}^{2}(2 \mathbb{N})^{-s \alpha} \\
&=\sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(\beta_{k}+1\right)^{1-\rho} \beta!^{1-\rho} g_{k}^{2}(2 k)^{-l-s}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-s \beta} \\
& \leq \sum_{\beta \in \mathscr{I}} \sum_{k \in \mathbb{N}}(2 \mathbb{N})^{(1-\rho) \beta}(2 k)^{1-\rho} \beta!^{1-\rho} g_{k}^{2}(2 k)^{-l-s}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-s \beta} \\
& \leq \sum_{\beta \in \mathscr{I}}\left(\sum_{k \in \mathbb{N}} g_{k}^{2}(2 k)^{-l-q}\right) \beta!^{1-\rho}\left\|u_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
&=\|u\|_{X \otimes(S)_{-\rho,-p}}^{2} \cdot\|G\|_{S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}}^{2}<\infty
\end{aligned}
$$

where we used $\beta_{k}+1 \leq(2 \mathbb{N})^{\beta+\varepsilon^{(k)}}=(2 \mathbb{N})^{\beta}(2 k), \beta \in \mathscr{I}, k \in \mathbb{N}$. The estimates are also valid for processes in the Kondratiev space modified with a sequence $g=\left(g_{k}\right)_{k \in \mathbb{N}}$, for $g_{k} \geq 2 k, k \in \mathbb{N}$ since it holds $(2 \mathbb{N} g)^{-s \alpha} \leq(2 \mathbb{N})^{-2 s \alpha}$ for $s>0$.

Both, the Wick product $G \diamond u$ and the action of the Malliavin derivative on $u$, belong to the domain of the Skorokhod integral and therefore we can apply the operator $\delta$ on both sides of (3.22). Thus, we obtain the equation $\delta(\mathbb{D} u)=\delta(G \diamond u)$, which reduces to the equation written in terms of the Skorokhod integral $\delta$ and the Ornstein-Uhlenbeck operator $\mathscr{R}$

$$
\begin{equation*}
\mathscr{R} u=\delta(G \diamond u) . \tag{3.24}
\end{equation*}
$$

We replace all the processes in (3.24) with their chaos expansion expressions, apply operators $\mathscr{R}$ and $\delta$ and obtain unknown coefficients of a process $u$.

$$
\begin{gathered}
\mathscr{R}\left(\sum_{\alpha \in \mathscr{I}} u_{\alpha} \otimes H_{\alpha}\right)=\delta\left(\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{k} u_{\alpha} \otimes \xi_{k} \otimes H_{\alpha+\varepsilon^{(k)}}\right) \\
\sum_{\alpha \in \mathscr{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{k} u_{\alpha} \otimes H_{\alpha+2 \varepsilon^{(k)}}
\end{gathered}
$$

We select terms which correspond to multi-indices of length zero and one and obtain

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} u_{\varepsilon^{(k)}} \otimes H_{\varepsilon^{(k)}}+\sum_{|\alpha| \geq 2}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{|\alpha| \geq 2} \sum_{k \in \mathbb{N}} g_{k} u_{\alpha-2 \varepsilon^{(k)}} \otimes H_{\alpha} \tag{3.25}
\end{equation*}
$$

Due to the uniqueness of chaos expansion representations in the orthogonal Fourier-Hermite basis, we equalize corresponding coefficients on both sides of (3.25) and obtain the triangular system of deterministic equations

$$
\begin{gather*}
u_{\varepsilon^{(k)}}=0, \quad k \in \mathbb{N}  \tag{3.26}\\
|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} g_{k} u_{\alpha-2 \varepsilon^{(k)}}, \quad|\alpha| \geq 2 \tag{3.27}
\end{gather*}
$$

where by convention $\alpha-2 \varepsilon^{(k)}$ does not exist if $\alpha_{k}=0$ or $\alpha_{k}=1$, thus $u_{\alpha-2 \varepsilon^{(k)}}=0$ for $\alpha_{k} \leq 1$. We solve the system of Eqs. (3.26) and (3.27) by induction with respect to the length of multi-indices $\alpha$ and thus obtain the coefficients $u_{\alpha},|\alpha| \geq 1$.

First, from (3.27) it follows that $u_{\alpha}$ are represented in terms of $u_{\beta}$ such that $|\beta|=$ $|\alpha|-2$, where $u_{\beta}$ are obtained in the previous step of the induction procedure. From the initial condition $\mathbb{E} u=\widetilde{u}_{0}$ it follows that $u_{(0,0,0, \ldots)}=u_{0}=\widetilde{u}_{0}$ and from (3.26) we obtain coefficients $u_{\alpha}=0$ for all $|\alpha|=1$. For $|\alpha|=2$ there are two possibilities: $\alpha=2 \varepsilon^{(k)}, k \in \mathbb{N}$ and $\alpha=\varepsilon^{(k)}+\varepsilon^{(j)}, k \neq j, k, j \in \mathbb{N}$. From (3.27) it follows that

$$
u_{\alpha}= \begin{cases}\frac{1}{2} g_{k} \widetilde{u}_{0}, & \alpha=2 \varepsilon^{(k)} \\ 0, & \alpha=\varepsilon^{(k)}+\varepsilon^{(j)}, k \neq j\end{cases}
$$

Note that $\alpha=2 \varepsilon^{(k)}, k \in \mathbb{N}$ has only one nonzero component, so $\alpha=\alpha_{1}$, thus only one term appears in the sum (3.27) and $C_{\alpha}=1$.

For $|\alpha|=3$ the coefficients $u_{\alpha}=0$ because they are represented through the coefficients of the length one, which are zero. Moreover, for all $\alpha \in \mathscr{I}$ of odd length, i.e., for all $\alpha \in \mathscr{I}$ such that $|\alpha|=2 n+1, n \in \mathbb{N}$ the coefficients $u_{\alpha}=0$.

Our goal is to obtain a general form of the coefficients $u_{\alpha}$ for $\alpha \in \mathscr{I}$ of even length, i.e., for $|\alpha|=2 n, n \in \mathbb{N}$. Now, for $|\alpha|=4$ there are five different types of $\alpha$. Without loss of generality we consider
$\alpha \in\{(4,0,0, \ldots),(3,1,0,0 \ldots),(2,1,1,0, .),.(1,1,1,1,0,0, \ldots),(2,2,0,0, \ldots)\}$.
From (3.27) it follows $u_{(4,0,0, . .)}=\frac{1}{4} g_{1} u_{(2,0,0,0, \ldots)}$. Using the forms of $u_{\alpha}$ obtained in the previous steps we get $u_{(4,0,0, . .)}=\frac{1}{4} \frac{1}{2} g_{1}^{2} \widetilde{u}_{0}$. We also obtain $u_{(3,1,0, \ldots)}=$ $u_{(2,1,1,0, . .)}=u_{(1,1,1,1,0,0, \ldots)}=0 \quad$ and $\quad u_{(2,2,0,0 . .)}=\frac{1}{4}\left(g_{1} u_{(0,2,0, \ldots)}+g_{2} u_{(2,0,0, \ldots)}\right)=$ $\frac{1}{4} \frac{1}{2} g_{1} g_{2} \cdot \widetilde{u}_{0} \cdot 2$. It follows that only nonzero coefficients are obtained for multiindices of forms
$\alpha=4 \varepsilon^{(k)}, k \in \mathbb{N}$ and $\alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}, k \neq j, k, j \in \mathbb{N}$. Thus, for $|\alpha|=4$

$$
u_{\alpha}= \begin{cases}\frac{1}{4!!} g_{k}^{2} \widetilde{u}_{0}, & \alpha=4 \varepsilon^{(k)} \\ 2 \cdot \frac{1}{4!!} g_{k} g_{j} \widetilde{u}_{0}, & \alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}, k \neq j \\ 0, & \text { otherwise }\end{cases}
$$



Fig. $3.1 \alpha$ values

Since $\alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}$, for $k \neq j$ has two nonzero components, there are two terms in the sum (3.27) and $C_{\alpha}=2$. For example, $\alpha=(2,2,0,0, \ldots)$ can be decomposed in one of two following ways $\alpha=2 \varepsilon^{(1)}+(0,2,0,0, .$.$) or \alpha=2 \varepsilon^{(2)}+(2,0,0,0, .$.$) ,$ therefore $C_{(2,2,0,0, \ldots)}=2$.

For $|\alpha|=6$ we consider only multi-indices which have all their components even. For the rest $u_{\alpha}=0$. For example, from (3.27) and from the forms of the coefficients obtained in the previous steps it follows $u_{(6,0,0, \ldots)}=\frac{1}{6} g_{1} u_{(4,0,0 \ldots)}=\frac{1}{6} \frac{1}{4} \frac{1}{2} g_{1}^{3} \widetilde{u}_{0}$. Next, $u_{(4,2,0,0 \ldots)}=\frac{1}{6}\left(g_{1} u_{(2,2,0,0 \ldots)}+g_{2} u_{(4,0,0, \ldots)}\right)=3 \cdot \frac{1}{6} \frac{1}{4} \frac{1}{2} g_{1}^{2} g_{2} \widetilde{u}_{0}$. Finally, $u_{(2,2,2,0, \ldots)}=$ $g_{1} u_{(0,2,2,0, \ldots)}+g_{2} u_{(2,0,2,0, \ldots)}+g_{3} u_{(2,2,0,0, \ldots)}=6 \cdot \frac{1}{6} \frac{1}{4} \frac{1}{2} g_{1} g_{2} g_{3} \widetilde{u}_{0}$. The later coefficient is $C_{\alpha}=6$, meaning that there are six chain decompositions of $\alpha=$ $(2,2,2,0,0, \ldots)$ of the form $\alpha=2 \varepsilon^{\left(p_{1}\right)}+2 \varepsilon^{\left(p_{2}\right)}+\ldots+2 \varepsilon^{\left(p_{s}\right)}+\alpha_{1}$, with $\alpha_{1}$ having only one nonzero component. This case is illustrated in Fig.3.1b. For $\alpha=$ $(4,2,0,0, \ldots)$ we have $C_{\alpha}=3$, where all decomposing possibilities are described in Fig. 3.1a. Thus,

$$
u_{\alpha}= \begin{cases}\frac{1}{6!} g_{k}^{3} \widetilde{u}_{0}, & \alpha=6 \varepsilon^{(k)} \\ 3 \cdot \frac{1}{6!} g_{k}^{2} g_{j} \widetilde{u}_{0}, & \alpha=4 \varepsilon^{(k)}+2 \varepsilon^{(j)}, k \neq j \\ 6 \cdot \frac{1}{6!} g_{k} g_{j} g_{i} \widetilde{u}_{0}, & \alpha=2 \varepsilon^{(k)}+2 \varepsilon^{(j)}+2 \varepsilon^{(i)}, k \neq i, j, i \neq j \\ 0, & \text { otherwise }\end{cases}
$$

We proceed by the same procedure for all even multi-index lengths to obtain $u_{\alpha}$ in the form

$$
u_{\alpha}=\left\{\begin{array}{cl}
\frac{C_{\alpha}}{|\alpha|!!} \cdot g_{1}^{\beta_{1}} g_{2}^{\beta_{2}} \cdots g_{m}^{\beta_{m}} \widetilde{u}_{0}, & \alpha=2 \beta,|\alpha|=2 n, n \in \mathbb{N},  \tag{3.28}\\
0, & |\alpha|=2 n-1, n \in \mathbb{N}
\end{array}\right.
$$

where $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, 0,0, \ldots\right) \in \mathscr{I}, \beta_{1}, \ldots, \beta_{m} \in \mathbb{N}_{0}$ and $C_{\alpha}$ represents the number of decompositions of $\alpha$ in the way $\alpha=2 \varepsilon^{\left(p_{1}\right)}+\ldots+2 \varepsilon^{\left(p_{s}\right)}+\alpha_{1}$, for all possible $p_{1}, \ldots, p_{s}$, i.e., all branches paths that connect $\alpha$ and $\tilde{\alpha}=\left(0,0, \ldots, \widetilde{\alpha}_{i}, 0,0, ..\right)$,
for some $\widetilde{\alpha}_{i} \neq 0$. Note, for $\alpha=2 \beta=\left(2 \beta_{1}, 2 \beta_{2}, \ldots, 2 \beta_{m}, 0, ..\right) \in \mathscr{I}$ the coefficient $1 \leq C_{\alpha} \leq m$ !, i.e., $C_{\alpha}$ is maximal when all nonzero components of $\alpha$ are equal two.

Summing up all the coefficients in (3.28) we obtain the form of solution (3.23). It remains to prove the convergence of the solution $u$ in $\operatorname{Dom}(\mathbb{D} g)_{-\rho}$, i.e., to prove that the sum $\sum_{\alpha \in \mathscr{F}}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N} g)^{-p \alpha}$ is finite.

Since $g_{k} \geq 2 k$ for all $k \in \mathbb{N}$ it holds $(2 \mathbb{N} g)^{-p \alpha} \leq(2 \mathbb{N})^{-2 p \alpha}$, for $p>0$. Then, from the estimates $|\alpha| \leq(2 \mathbb{N})^{\alpha}$ and $C_{\alpha} \leq(2 \mathbb{N})^{\alpha}, \alpha \in \mathscr{I}$ by Theorem 1.1 for $p \geq 5+\rho$ it follows

$$
\begin{aligned}
&\|u\|_{\text {Dom }}^{-\rho,-p} \\
&(\mathbb{D} g) \\
&=\sum_{\alpha=2 \beta \in \mathscr{I}}|\alpha|^{1+\rho} C_{\alpha}^{2}\left\|\widetilde{u}_{0}\right\|_{X}^{2} \alpha!^{1-\rho} \frac{g^{\alpha}}{|\alpha|!!^{2}}(2 \mathbb{N} g)^{-p \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha=2 \beta \in \mathscr{I}} \frac{\alpha!^{1-\rho}}{|\alpha|!!^{2}}(2 \mathbb{N})^{(3+\rho) \alpha} g^{\alpha}(2 \mathbb{N} g)^{-p \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{(2+\rho) \alpha}(2 \mathbb{N} g)^{-(p-1) \alpha} \\
& \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{-(p-3-\rho) \alpha} \leq\left\|\widetilde{u}_{0}\right\|_{X}^{2} \sum_{\alpha \in \mathscr{I}}(2 \mathbb{N})^{-2 \alpha}<\infty .
\end{aligned}
$$

 because from Lemma 1.1 it holds $|\beta|!\geq \beta$ !.

Remark 3.7 The same procedure, described in the proof of Theorem 3.6 can be applied for solving equations with Gaussian processes given in a more general form (3.19). By applying the chaos expansion method, the problem of solving $\mathbb{D} u=\mathbf{G} \diamond u$, $\mathbb{E} u=\widetilde{u}_{0}$ reduces to the problem of solving the system of deterministic equations

$$
u_{\varepsilon^{(k)}}=0 \text {, for } k \in \mathbb{N} \text { and }|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n} u_{\alpha-\varepsilon^{(k)}-\varepsilon^{(n)}}, \quad|\alpha| \geq 2 \text {, }
$$

which corresponds to the system (3.26) and (3.27). Once we obtain the coefficients $u_{\alpha}, \alpha \in \mathscr{I}$ we have the chaos expansion representation of the solution in the form (3.5). Under the assumptions of Theorem 3.6 it can be proven that the obtained solution belongs to $\operatorname{Dom}(\mathbb{D} g)_{-\rho}$.

Theorem 3.7 Let $\rho \in[0,1]$ and let $G \in S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}, q, l>0$ be a Gaussian process of the form (3.20) whose coefficients $g_{k}, k \in \mathbb{N}$ satisfy (3.21). If $g_{k} \geq 2 k$ for all $k \in \mathbb{N}$ and if the coefficients of $h \in X \otimes S_{-l} \otimes(S)_{-\rho,-p}, l, p>0$ satisfy (C) for all possible decompositions of $\alpha$ of the form (3.16), then the nonhomogeneous equation

$$
\begin{equation*}
\mathbb{D} u=G \diamond u+h, \quad \mathbb{E} u=\widetilde{u}_{0}, \tag{3.29}
\end{equation*}
$$

for each $\widetilde{u}_{0} \in X$ has a unique solution in $\operatorname{Dom}(\mathbb{D} g)_{-\rho,-p}$ represented in the form $u=u^{\text {hom }}+u^{\text {nhom }}$, where $u^{\text {hom }}$ is the solution of the corresponding homogeneous equation (3.22) and is of the form (3.23) and $u^{\text {nhom }}$ is the nonhomogeneous part.

Since the proof is rather technical, we omit it here. We note that after applying the chaos expansion method to (3.29) we obtain the system

$$
u_{\varepsilon^{(k)}}=h_{0, k}, \text { for } k \in \mathbb{N} \text { and }|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} g_{k} u_{\alpha-\varepsilon^{(k)}-\varepsilon^{(n)}}+\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k},|\alpha| \geq 2
$$

Further analysis follows similarly as the analysis provided in the proof of Theorem 3.6, with additional difficulty arising from the fact that the coefficients $u_{\alpha}$, for $\alpha$ of odd lenght not necessarily vanishing.

Remark 3.8 If we consider the equation $\mathbb{D} u=\mathbf{G} \diamond u+h, \mathbb{E} u=\widetilde{u}_{0}$ with a Gaussian process $\mathbf{G}$ of the form (3.19) and $h \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ under the assumptions of Theorem 3.7, then the unknown coefficients $u_{\alpha}, \alpha \in \mathscr{I}$ of a solution $u$ are determined from the system of deterministic equations
$u_{\varepsilon^{(k)}}=h_{0, k}$, for $k \in \mathbb{N}$ and $|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} g_{k n} u_{\alpha-2 \varepsilon^{(k)}}+\sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)}, k},|\alpha| \geq 2$.
The solution $u$ belongs to the Kondratiev space of distributions modified by a sequence $g$ and it can be represented as a sum of the solution $u^{\text {hom }}$ that corresponds to the homogeneous part of equation and the nonhomogeneous part $u^{\text {nhom }}$ represented as a convolution of the coefficients $h_{\alpha}$ and $g_{k n}$.

Consider now a more general form of the nonhomogeneous problem

$$
\begin{equation*}
\mathbb{D} u=\mathscr{B}(G \diamond u)+h, \quad \mathbb{E} u=\widetilde{u}_{0}, \quad \widetilde{u}_{0} \in X, \tag{3.30}
\end{equation*}
$$

where $\mathscr{B}$ is a coordinatewise operator, i.e., $\mathscr{B}: X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho} \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes$ $(S)_{-\rho}$ is a linear operator defined by $\mathscr{B}(f)=\sum_{\alpha \in \mathscr{\mathscr { S }}} \mathbf{B}_{\alpha}\left(f_{\alpha}\right) \otimes H_{\alpha}$, for $f=\sum_{\alpha \in \mathscr{I}} f_{\alpha} \otimes H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$, where $\mathbf{B}_{\alpha}: X \otimes S^{\prime}(\mathbb{R}) \rightarrow X \otimes S^{\prime}(\mathbb{R})$, $\alpha \in \mathscr{I}$ are linear and of the form $\mathbf{B}_{\alpha}=\sum_{k \in \mathbb{N}} f_{\alpha, k} \otimes B_{\alpha, k}\left(\xi_{k}\right), \alpha \in \mathscr{I}$, such that $B_{\alpha, k}: S^{\prime}(\mathbb{R}) \rightarrow S^{\prime}(\mathbb{R}), k \in \mathbb{N}$. We also assume $\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left\|B_{\alpha, k}\right\|^{2}(2 k)^{-l}(2 \mathbb{N})^{-p \alpha}<$ $\infty$, for some $p, l>0$. Especially, if operator $\mathscr{B}$ is a simple coordinatewise operator of the form $B_{\alpha, k}=B=-\Delta+x^{2}+1, \alpha \in \mathscr{I}, k \in \mathbb{N}$ then, in order to solve (3.30) we can apply the same procedure explained in Theorem 3.6. Recall, the domain of $B$ contains $S^{\prime}(\mathbb{R})$ and the Hermite functions are eigenvectors of $B$ with $B \xi_{k}=2 k \xi_{k}$, $k \in \mathbb{N}$. We set $h=0$. Clearly,

$$
\begin{aligned}
\mathscr{B}(G \diamond u) & =\mathscr{B}\left(\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{k} \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}}\right)=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{k} B_{\alpha, k}\left(\xi_{k}\right) \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}} \\
& =\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{k} B \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}}=\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{k \in \mathbb{N}} g_{k} 2 k \xi_{k} \otimes u_{\alpha} \otimes H_{\alpha+\varepsilon^{(k)}}
\end{aligned}
$$

Therefore, after applying the operator $\delta$ we obtain

$$
\sum_{\alpha \in \mathscr{I}}|\alpha| u_{\alpha} \otimes H_{\alpha}=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} 2 k g_{k} u_{\alpha} \otimes H_{\alpha+2 \varepsilon^{(k)}} .
$$

The coefficients of the solution are obtained by induction from the system

$$
u_{\varepsilon^{(k)}}=0, \text { for all } k \in \mathbb{N}, \quad \text { and } \quad|\alpha| u_{\alpha}=\sum_{k \in \mathbb{N}} 2 k g_{k} u_{\alpha-2 \varepsilon^{(k)}},|\alpha| \geq 2 .
$$

Under the assumptions of Theorem 3.6 it can be proven that there exists a unique solution of equation in the space $\operatorname{Dom}(\mathbb{D} g)_{-\rho}$, for $p>5+\rho$ given in the form

$$
u=\widetilde{u}_{0} \otimes \sum_{2 \beta \in \mathscr{I}} \frac{C_{2 \beta}}{|2 \beta|!!}\left(\prod_{k=1}^{\infty}\left(2 k g_{k}\right)^{\beta_{k}}\right) H_{2 \beta} .
$$

### 3.6 Integral Equation

We consider an integral type equation involving the Skorokhod integral operator. In the following theorem we generalize results from [12, 13] for Schwartz valued test processes in $X \otimes S(\mathbb{R}) \otimes(S)_{\rho}$ and generalized processes from $X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$, $\rho \in[0,1]$.

Theorem 3.8 Let $\rho \in[0,1]$. Let $f$ be a stochastic process with zero expectation and chaos expansion representation form $f=\sum_{|\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}, f_{\alpha} \in X$. Then, the integral equation

$$
\begin{equation*}
\delta(u)=f, \tag{3.31}
\end{equation*}
$$

has a unique solution u given by

$$
\begin{equation*}
u=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} . \tag{3.32}
\end{equation*}
$$

Moreover, the following hold:

```
\(1^{\circ}\) If \(f \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D}), p \in \mathbb{N}\) then \(u \in \operatorname{Dom}_{-\rho,-l,-p}(\delta)\) for \(l>p+1\).
\(2^{\circ}\) If \(f \in \operatorname{Dom}_{\rho, p}(\mathbb{D}), p \in \mathbb{N}\) then \(u \in \operatorname{Dom}_{\rho, l, p}(\delta)\) for \(l<p-1\).
\(3^{\circ}\) If \(f \in \operatorname{Dom}_{0}(\mathbb{D})\), then \(u \in \operatorname{Dom}_{0}(\delta)\).
```

Proof $1^{\circ}$ We seek for the solution in Range $_{-\rho}(\mathbb{D})$. It is clear that $u \in$ Range $_{-\rho}(\mathbb{D})$ is equivalent to $u=\mathbb{D}(\widetilde{u})$, for some $\widetilde{u}$. This approach is general enough, since according to Theorem 3.2, for all $u \in X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-p}, l>p+1$ there exists $\widetilde{u} \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$ such that $u=\mathbb{D}(\widetilde{u})$ holds. Thus, the integral equation (3.31) is equivalent to the system of equations

$$
u=\mathbb{D}(\widetilde{u}), \quad \mathscr{R}(\widetilde{u})=f .
$$

The solution of $\mathscr{R}(\widetilde{u})=f$, by Theorem 3.1 for $m=1$, is given by

$$
\widetilde{u}=\widetilde{u}_{0}+\sum_{\alpha \in \mathscr{I},|\alpha| \geq 1} \frac{f_{\alpha}}{|\alpha|} \otimes H_{\alpha},
$$

where $\tilde{u}_{0}=\tilde{u}_{0}$ can be chosen arbitrarily. Finally, the solution of the initial equation (3.31) is obtained after applying the operator $\mathbb{D}$, i.e.,

$$
\begin{aligned}
u=\mathbb{D}(\widetilde{u}) & =\sum_{\alpha \in \mathscr{\mathscr { I }},|\alpha| \geq 1} \sum_{k \in \mathbb{N}} \alpha_{k} \frac{f_{\alpha}}{|\alpha|} \otimes \xi_{k} \otimes H_{\alpha-\varepsilon^{(k)}} \\
& =\sum_{\alpha \in \mathscr{\mathscr { I }}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{f_{\alpha+\varepsilon^{(k)}}}{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha} .
\end{aligned}
$$

It remains to prove the convergence of the solution (3.32) in $\operatorname{Dom}_{-\rho}(\delta)$.
We assume $f \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$, for some $p>0$. First we prove that $\widetilde{u} \in \operatorname{Dom}_{-\rho,-p}(\mathbb{D})$. Indeed,

$$
\begin{aligned}
\|\widetilde{u}\|_{D_{o m-\rho,-p}}^{2}(\mathbb{D}) & =\left\|\widetilde{u_{0}}\right\|_{X}^{2}+\sum_{|\alpha|>0}|\alpha|^{1+\rho} \alpha!^{1-\rho}\left\|\widetilde{u}_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& =\left\|\widetilde{u_{0}}\right\|_{X}^{2}+\sum_{|\alpha|>0} \frac{|\alpha|^{1+\rho}}{|\alpha|^{2}} \alpha!^{1-\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-p \alpha} \\
& \leq\left\|\widetilde{u_{0}}\right\|_{X}^{2}+\|f\|_{D o m_{-\rho,-p}^{2}(\mathbb{D})}^{2}<\infty .
\end{aligned}
$$

Finally, the solution $u \in \operatorname{Dom}_{-\rho,-l,-p}(\delta)$ for $l>p+1$ since it holds

$$
\begin{aligned}
\|u\|_{D_{o m-\rho,-l,-p}(\delta)}^{2} & =\sum_{\alpha \in \mathscr{\mathscr { I }}}|\alpha|^{1-\rho} \alpha!^{1-\rho}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2} \frac{\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}(2 k)^{-l}\right)(2 \mathbb{N})^{-p \alpha} \\
& =\sum_{|\beta|>0}\left(\sum_{k \in \mathbb{N}} \frac{\left|\beta-\varepsilon^{(k)}\right|^{1-\rho}}{|\beta|^{2}}\left\|f_{\beta}\right\|_{X}^{2}(2 k)^{-l+p} \frac{\beta_{k}^{2}}{\beta_{k}^{1-\rho}}\right) \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta} \\
& \leq c \sum_{|\beta|>0}|\beta|^{1+\rho}\left\|f_{\beta}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta}=c\|f\|_{D o m_{-\rho,-l,-p}(\mathbb{D})}^{2}<\infty .
\end{aligned}
$$

We used the substitution $\alpha=\beta-\varepsilon^{(k)},|\beta|>0$ and thus $\alpha!=\left(\beta-\varepsilon^{(k)}\right)!=\frac{\beta!}{\beta_{k}}$, $k \in \mathbb{N}$ and $(2 \mathbb{N})^{-p \alpha}=(2 \mathbb{N})^{-p \beta}(2 \mathbb{N})^{-p \varepsilon^{(k)}}=(2 \mathbb{N})^{-p \beta}(2 k)^{-p}$. We also applied the estimates $\frac{\left|\beta-\varepsilon^{(k)}\right|^{1-\rho}}{|\beta|^{2}}=\frac{(|\beta|-1)^{1-\rho}}{|\beta|^{2}} \leq 1$ and $\sum_{k \in \mathbb{N}} \beta_{k}^{1+\rho} \leq\left(\sum_{k \in \mathbb{N}} \beta_{k}\right)^{1+\rho}=|\beta|^{1+\rho}$, $|\beta|>0$ and the Cauchy-Schwarz inequality. Moreover, $c=\sum_{k \in \mathbb{N}}(2 k)^{p-l}<\infty$ for $l>p+1$. Note also that the coefficients $u_{\alpha}$ satisfy the conditions ( $C$ ).
$2^{\circ}$ The form of the solution (3.32) is obtained in a similar way as in the previous case. We prove that $\tilde{u} \in \operatorname{Dom}_{\rho, p}(\mathbb{D})$ and $u \in \operatorname{Dom}_{\rho, l, p}(\delta)$. We obtain

$$
\begin{aligned}
\|\widetilde{u}\|_{D^{2} m_{\rho, p}(\mathbb{D})}^{2} & =\sum_{\alpha \in \mathscr{I}}|\alpha|^{1-\rho} \alpha!^{1+\rho}\left\|u_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\sum_{|\alpha|>0}|\alpha|^{1-\rho} \alpha!^{1+\rho} \frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}}(2 \mathbb{N})^{p \alpha} \\
& \leq \sum_{|\alpha|>0}|\alpha|^{1+\rho} \alpha!^{1+\rho}\left\|f_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{p \alpha}=\|f\|_{D_{o m}, p}^{2}(\mathbb{D})<\infty
\end{aligned}
$$

and thus $\tilde{u} \in \operatorname{Dom}_{\rho, p}(\mathbb{D})$. Now, for $p>l+1$ it holds $c=\sum_{k \in \mathbb{N}}(2 k)^{l-p}<\infty$ and

$$
\begin{aligned}
\left\|u-\tilde{u}_{0}\right\|_{D_{o m}, l, p(\delta)}^{2} & =\sum_{|\alpha|>0}|\alpha|^{1+\rho} \alpha!^{1+\rho}\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2} \frac{\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}(2 k)^{-l}\right)(2 \mathbb{N})^{p \alpha} \\
& =\sum_{|\beta|>0}\left(\sum_{k \in \mathbb{N}} \frac{\left|\beta-\varepsilon^{(k)}\right|^{1+\rho}}{|\beta|^{2}} \cdot \frac{\beta_{k}^{2}}{\beta_{k}^{1+\rho}}(2 k)^{l-p}\right)\left\|f_{\beta}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq \sum_{|\beta|>0} \frac{(|\beta|-1)^{1+\rho}}{|\beta|^{2}} \cdot\left(\sum_{k \in \mathbb{N}} \beta_{k}^{1-\rho}(2 k)^{l-p}\right)\left\|f_{\beta}\right\|_{X}^{2} \beta!^{1+\rho}(2 \mathbb{N})^{p \beta} \\
& \leq c \sum_{\beta \in \mathscr{I}}|\beta|^{1-\rho} \beta!^{1+\rho}\left\|f_{\beta}\right\|_{X}^{2}(2 \mathbb{N})^{p \beta}=c\|f\|_{D o m_{\rho, l, p}(\mathbb{D})}^{2}<\infty .
\end{aligned}
$$

$3^{\circ}$ Let $f \in \operatorname{Dom}_{0}(\mathbb{D})$. In this case we have

$$
\|\widetilde{u}\|_{D o m_{0}(\mathbb{D})}^{2}-\left\|\tilde{u}_{0}\right\|_{X}^{2}=\sum_{|\alpha|>0}|\alpha| \alpha!\left\|u_{\alpha}\right\|_{X}^{2}=\sum_{|\alpha|>0}|\alpha| \alpha!\frac{\left\|f_{\alpha}\right\|_{X}^{2}}{|\alpha|^{2}} \leq\|f\|_{X \otimes L^{2}(\mu)}^{2}<\infty
$$

and thus $\tilde{u} \in \operatorname{Dom}_{0}(\mathbb{D})$. Also, by $\frac{|\beta|-1}{|\beta|^{2}} \leq 1, \beta \in \mathscr{I}$ we obtain

$$
\begin{aligned}
& \left\|u-\tilde{u}_{0}\right\|_{D_{\text {om }}^{0}(\delta)}^{2}=\sum_{|\alpha|>0}|\alpha| \alpha!\left(\sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right)^{2} \frac{\left\|f_{\alpha+\varepsilon^{(k)}}\right\|_{X}^{2}}{\left|\alpha+\varepsilon^{(k)}\right|^{2}}\right) \\
& \quad=\sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \beta_{k}^{2} \frac{\left\|f_{\beta}\right\|_{X}^{2}}{|\beta|^{2}} \cdot \frac{\beta!}{\beta_{k}}\left|\beta-\varepsilon^{(k)}\right|=\sum_{|\beta|>0}\left(\sum_{k \in \mathbb{N}} \beta_{k}\right) \frac{|\beta|-1}{|\beta|^{2}}\left\|f_{\beta}\right\|_{X}^{2} \beta! \\
& \quad \leq \sum_{\beta \in \mathscr{I}}|\beta| \beta!\left\|f_{\beta}\right\|_{X}^{2}=\|f\|_{D_{o m}(\mathbb{D})}^{2}<\infty .
\end{aligned}
$$

The obtained solution has symmetric kernel, i.e., it satisfies condition ( $C$ ).
Remark 3.9 If a stochastic process $f$ belongs to the Wiener chaos space $\bigoplus_{i=1}^{m} \mathscr{H}_{i}$ for some $m \in \mathbb{N}$, then the solution $u$ of the Eq. (3.31) belongs to the Wiener chaos space $\bigoplus_{i=0}^{m-1} \mathscr{H}_{i}$. Particularly, if $f$ is a quadratic Gaussian random process, i.e., an element of $\mathscr{H}_{2}$, then the solution $u$ to (3.31) is a Gaussian process.

Corollary 3.4 Each stochastic process $f$ can be represented as $f=\mathbb{E} f+\delta(u)$ for some Schwartz valued process u. The same holds for square integrable processes.

Proof The assertion follows for $u=\mathbb{D}\left(\mathscr{R}^{-1}(f-\mathbb{E} f)\right)$. This result reduces to the celebrated Itô representation theorem, i.e., the chaos expansion representation form (1.13), in case when $f$ is a square integrable adapted process [4, 22].

Remark 3.10 Applying the same techniques as in Theorems 3.2 and 3.8, by Remarks 2.4 and 2.7, one can solve the fundamental equations with higher order operators

$$
\mathbb{D}^{(k)} u=h, \quad \mathbb{E} u=\tilde{u}_{0}, \quad \mathbb{E}(\mathbb{D} u)=\tilde{u}_{1}, \ldots \mathbb{E}\left(\mathbb{D}^{(k-1)} u\right)=\tilde{u}_{k-1} \quad \text { and } \quad \delta^{(k)} u=f
$$

For more details we refer to [13].
Remark 3.11 All stochastic equations solved in this chapter can be interpreted, by the use of the isometric transformations $\mathscr{M}$ and $\mathbf{M}$ defined by (1.52) and (1.63) in Sects. 1.4.3 and 1.5 respectively, in fractional white noise space. Also, due to Theorems 2.21 and 2.22 the Malliavin derivative and the Skorokhod integral can be interpreted as their fractional counterparts in the corresponding fractional white noise space. For example, one can solve all versions of the initial value problem (3.8), i.e., to solve

$$
\widetilde{\mathbb{D}} \tilde{u}=f, \quad \mathbb{E}_{\mu_{H}} \tilde{u}=\tilde{u}_{0} \quad \mathbb{D}^{(H)} u=f, \quad \mathbb{E}_{\mu} u=u_{0} \quad \widetilde{\mathbb{D}}^{(H)} \tilde{u}=f, \quad \mathbb{E}_{\mu_{H}} \tilde{u}=\tilde{u}_{0}
$$

Moreover, the following fractional versions of the integral equation (3.31) can be solved

$$
\tilde{\delta} \tilde{u}=f \quad \text { and } \quad \delta^{(H)} u=f
$$

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# Chapter 4 <br> Applications and Numerical Approximation 


#### Abstract

In this chapter we present applications of the chaos expansion method in optimal control and stochastic partial differential equations. In particular, we consider the stochastic linear quadratic optimal control problem where the state equation is given by a stochastic differential equation of the Itô-Skorokhod type with different forms of noise disturbances, operator differential algebraic equations arising in fluid dynamics, stationary equations and fractional versions of the studied equations. Moreover, we provide a numerical framework based on chaos expansions and perform numerical simulations.


### 4.1 Introduction

Numerical methods for stochastic differential equations (SDEs) and uncertainty quantification based on the polynomial chaos approach became very popular in recent years [39, 44, 46]. They are highly efficient in practical computations providing fast convergence and high accuracy. We point out that, in order to apply the so-called stochastic Galerkin method, the derivation of explicit equations for the polynomial chaos coefficients is required. This is, as in the general chaos expansion, highly nontrivial and sometimes impossible. On the other hand, having an analytical representation of the solution all statistical information can be retrieved directly, e.g. mean, covariance function, variance and even sensitivity coefficients [39, 46]. The major challenge in stochastic simulations is the high dimensionality, which is even higher solving stochastic control problems, the same occur in the deterministic case [2]. In this chapter we provide an unified framework based on chaos expansions and deterministic theory of numerical analysis for solving stochastic optimal control problems and operator differential algebraic equations involving the operators of the Malliavin calculus. Moreover, we solve numerically the stationary form of nonhomogeneous equation corresponding to the Wick-type equations with the Laplace operator.

The linear quadratic Gaussian control problem for the control of finite-dimensional linear stochastic systems with Brownian motion is well understood [17]. The case with fractional Brownian motion [12-14] as well as the infinite dimensional case [15] have been studied recently. A more general problem arise if the noise depends on
the state variable, this is the so-called stochastic linear quadratic regulator (SLQR) problem. The SLQR problem in infinite dimensions was solved by Ichikawa in [23] using a dynamic programming approach. Da Prato [9] and Flandoli [16] later considered the SLQR for systems driven by analytic semigroups with Dirichlet or Neumann boundary controls and with disturbance in the state only. The infinite dimensional SLQR with random coefficients has been investigated in [19, 20] along with the associated backward stochastic Riccati equation. Recently, a theoretical framework for the SLQR has been laid for singular estimates control systems in the presence of noise in the control and in the case of finite time penalization in the performance index [21]. Considering the general setting described in [21, 28], an approximation scheme for solving the control problem and the associated Riccati equation has been proposed in [31]. In [30], a novel approach for solving the SLQR based on the concept of chaos expansion in the framework of white noise analysis was proposed. In [32] the results were extended to the SLQR problem with fractional Brownian motion. We consider the SLQR problem for state equations of the Itô-Skorokhod type, where the dynamics are driven by strongly continuous semigroups, and provide a numerical framework for solving the control problem using the chaos expansion approach. After applying the chaos expansion method to the state equation, we obtain a system of infinitely many deterministic partial differential equations in terms of the coefficients of the state and the control variables. For each equation we set up a control problem, which then results in a set of deterministic linear quadratic regulator problems. Solving these control problems, we find optimal coefficients for the state and the control. We prove the optimality of the solution expressed in terms of obtained coefficients compared to a direct approach. Moreover, we apply our result to a fully stochastic problem, in which the state, control and observation operators can be random. Finally, we consider problems involving state equations in more general form

$$
\begin{equation*}
\dot{y}=\mathbf{A} y+\mathbf{T} \Delta y+\mathbf{B} u, \quad y(0)=y^{0}, \tag{4.1}
\end{equation*}
$$

where $\mathbf{A}$ is an operator which generates a strongly continuous semigroup, and $\mathbf{T}$ is a linear bounded operator which combined with Wick product $\diamond$ introduces convolution-type perturbations into the equation. Equation (4.1) is related to Gaussian colored noise and it has been studied in [37], where the existence and uniqueness of its generalized solution was proven. Examples of this type of equations are the heat equation with random potential, the heat equation in random (inhomogeneous and anisotropic) media, the Langevin equation, etc. [37]. The related control problem for (4.1) leads to an optimal control defined in a space of generalized processes. A particular case of (4.1) together with an algebraic constraint arise in fluid dynamics, e.g. Stokes equations [1]. The resulting system is known as a semi-explicit operator differential algebraic equation (ODAE) and it has the form

$$
\dot{y}=\mathbf{A} y+\mathbf{B}^{\star} u+\mathbf{T} \Delta y+f, \quad \mathbf{B} y=g
$$

As an example, we present an ODAE involving generalized operators of Malliavin calculus. We particularly choose $\mathbf{B}$ to be the Skorohod integral $\delta$ and $\mathbf{B}^{\star}$ the Malliavin derivative $\mathbb{D}$ and then apply the results from Chap. 3. Finally, we solve fractional versions of the considered optimal control problems and OADEs. By using the fractional isometries $\mathscr{M}$ and $\mathbf{M}$, the fractional problems are transfered to the problems on classical space, i.e., the ones we have already solved.

### 4.2 A Stochastic Optimal Control Problem in Infinite Dimensions

We consider the infinite dimensional stochastic linear quadratic optimal control problem on finite horizon. It consists of the linear state equation

$$
\begin{equation*}
d y(t)=(\mathbf{A} y(t)+\mathbf{B} u(t)) d t+\mathbf{C} y(t) d B_{t}, \quad y(0)=y^{0}, \quad t \in[0, T] \tag{4.2}
\end{equation*}
$$

with respect to $\mathscr{H}$-valued Brownian motion $B_{t}$ in the classical Gaussian white noise space, and the quadratic cost functional. The operators $\mathbf{A}$ and $\mathbf{C}$ are operators on $\mathscr{H}$ and $\mathbf{B}$ acts from the control space $\mathscr{U}$ to the state space $\mathscr{H}$ and $y^{0}$ is a random variable. Spaces $\mathscr{H}$ and $\mathscr{U}$ are Hilbert spaces. The operators $\mathbf{B}$ and $\mathbf{C}$ are considered to be linear and bounded, while $\mathbf{A}$ could be unbounded. The objective is to minimize the quadratic functional

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathscr{W}}^{2}+\|u\|_{\mathscr{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathscr{Z}}^{2}\right] \tag{4.3}
\end{equation*}
$$

over all admissible controls $u$ and subject to the condition that $y$ satisfies the state Eq. (4.2). The operators $\mathbf{R}$ and $\mathbf{G}$ are bounded observation operators taking values in Hilbert spaces $\mathscr{W}$ and $\mathscr{Z}$ respectively, $\mathbb{E}$ denotes the expectation with respect to the Gaussian measure $\mu$ and $y_{T}=y(T)$. For the class of admissible controls we consider square integrable $\mathscr{U}$-valued adapted controls. The stochastic integration is taken with respect to $\mathscr{H}$-valued Brownian motion and the integral is considered as a Bochner-Pettis type integral [10, 45]. For $\mathbf{C}=0$ the Eq. (4.2) arises in the deterministic regulator problem and has been well understood in the literature [25, 26, 38]. A control process $u^{*}$ is called optimal if it minimizes the cost functional over all admissible control processes, i.e.,

$$
\min _{u} \mathbf{J}(u)=\mathbf{J}\left(u^{*}\right)
$$

The corresponding optimal trajectory is denoted by $y^{*}$. Thus, the pair $\left(y^{*}, u^{*}\right)$ is the optimal solution of the considered optimal control problem and is called the optimal pair. For simplicity, we take $\mathscr{W}=\mathscr{Z}=\mathscr{H}$.

Due to the fundamental theorem of stochastic calculus, for admissible square integrable processes, we consider an equivalent form of the state Eq. (4.2), i.e., its Wick version

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\mathbf{C} y(t) \diamond W_{t}, \quad y(0)=y^{0}, \quad t \in[0, T] \tag{4.4}
\end{equation*}
$$

We solve the optimal control problem (4.2)-(4.3) by combining the chaos expansion method with the deterministic optimal control theory. The following theorem gives the conditions for the existence of the optimal control in the feedback form using the associated Riccati equation. For more details on existence of mild solutions of (4.2) we refer the reader to [10] and for the optimal control and Riccati feedback synthesis we refer to [23].

Theorem 4.1 ([10, 23]) Let the following assumptions hold:
(a1) The linear operator $\mathbf{A}$ is an infinitesimal generator of a $C_{0}$-semigroup $\left(e^{\mathbf{A} t}\right)_{t \geq 0}$ on the space $\mathscr{H}$.
(a2) The linear control operator $\mathbf{B}$ is bounded $\mathscr{U} \rightarrow \mathscr{H}$.
(a3) The operators $\mathbf{R}, \mathbf{G}, \mathbf{C}$ are bounded linear operators.
Then, the optimal control $u^{*}$ of the linear quadratic problem (4.2)-(4.3) satisfies the feedback characterization in terms of the optimal state $y^{*}$

$$
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t)
$$

where $\mathbf{P}(t)$ is a positive self-adjoint operator solving the Riccati equation

$$
\begin{array}{r}
\dot{\mathbf{P}}(t)+\mathbf{P}(t) \mathbf{A}+\mathbf{A}^{\star} \mathbf{P}(t)+\mathbf{C}^{\star} \mathbf{P}(t) \mathbf{C}+\mathbf{R}^{\star} \mathbf{R}-\mathbf{P}(t) \mathbf{B B}^{\star} \mathbf{P}(t)=0,  \tag{4.5}\\
\mathbf{P}(T)=\mathbf{G}^{\star} \mathbf{G} .
\end{array}
$$

Here we also invoke the solution of the inhomogeneous deterministic control problem of minimizing the performance index

$$
\begin{equation*}
J(u)=\int_{0}^{T}\left(\|R x\|_{\mathscr{H}}^{2}+\|u\|_{\mathscr{U}}^{2}\right) d t+\|G x(T)\|_{\mathscr{H}}^{2} \tag{4.6}
\end{equation*}
$$

subject to the inhomogeneous differential equation

$$
\begin{equation*}
x^{\prime}(t)=A x(t)+B u(t)+f(t), \quad x(0)=x^{0} \tag{4.7}
\end{equation*}
$$

Besides the assumptions (al) and (a2), it is enough to assume that $f \in L^{2}((0, T), \mathscr{H})$, to obtain the optimal solution for the state and control $\left(x^{*}, u^{*}\right)$. The feedback form of the optimal control for the inhomogeneous problem (4.6)-(4.7) is given by

$$
\begin{equation*}
u^{*}(t)=-B^{\star} P_{d}(t) x^{*}(t)-B^{\star} k(t) \tag{4.8}
\end{equation*}
$$

where $P_{d}(t)$ solves the Riccati equation
4.2 A Stochastic Optimal Control Problem in Infinite Dimensions

$$
\begin{equation*}
\left\langle\left(\dot{P}_{d}+P_{d} A+A^{\star} P_{d}+R^{\star} R-P_{d} B B^{\star} P_{d}\right) v, w\right\rangle=0, \quad P_{d}(T) v=G^{\star} G v \tag{4.9}
\end{equation*}
$$

for all $v, w$ in $\mathscr{D}(A)$, while $k(t)$ is a solution of the auxiliary differential equation

$$
k^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k(t)+P_{d}(t) f(t)=0
$$

with the boundary conditions $P_{d}(T)=G^{\star} G$ and $k(T)=0$. For the homogeneous problem we refer to [25]. We also refer to [6, 7, 47] for better insight into the optimal control theory.

Let $g(t)$ be a $\mathscr{F}_{T}$-predictable Bochner integrable $\mathscr{H}$-valued function. An $\mathscr{H}$ valued adapted process $y(t)$ is a strong solution of the state Eq. (4.2) over [0, T] if $y(t)$ takes values in $D(\mathbf{A}) \cap D(\mathbf{C})$ for almost all $t$ and $\omega, P\left(\int_{0}^{T}\|y(s)\|_{\mathscr{H}}+\right.$ $\left.\|\mathbf{A} y(s)\|_{\mathscr{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathscr{H}}^{2} d s<\infty\right)=1$, and for arbitrary $t \in[0, T]$ and $P$-almost surely it satisfies the integral equation

$$
y(t)=y^{0}+\int_{0}^{t} \mathbf{A} y(s) d s+\int_{0}^{t} g(s) d s+\int_{0}^{t} \mathbf{C} y(s) d B_{s}
$$

An $\mathscr{H}$-valued adapted process $y(t)$ is a mild solution of the state equation

$$
d y(t)=(\mathbf{A} y(t)+g(t)) d t+\mathbf{C} y(t) d B_{t}, \quad y(0)=y^{0}
$$

over $[0, T]$ if the process $y(t)$ takes values in $D(\mathbf{C}), P\left(\int_{0}^{T}\|y(s)\|_{\mathscr{H}} d s<\infty\right)=1$ and $P\left(\int_{0}^{T}\|\mathbf{C} y(s)\|_{\mathscr{H}}^{2} d s<\infty\right)=1$ and for arbitrary $t \in[0, T]$ and $P$-almost surely it satisfies the integral equation

$$
y(t)=e^{\mathbf{A} t} y^{0}+\int_{0}^{t} e^{\mathbf{A}(t-s)} g(s) d s+\int_{0}^{t} e^{\mathbf{A}(t-s)} \mathbf{C} y(s) d B_{s} .
$$

Note that, under the assumptions of Theorem 4.1, and given a control process $u \in L^{2}([0, T], \mathscr{U}) \otimes L^{2}(\mu)$, i.e., $g(t)=\mathbf{B} u(t)$, and deterministic initial data, there exits a unique mild solution $y \in L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ of the controlled state Eq. (4.2), see [10].

Theorem 4.2 ([32]) Let the following assumptions hold:
(A1) The operator $\mathbf{A}: L^{2}([0, T], \mathscr{D}) \otimes L^{2}(\mu) \rightarrow L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ is a coordinatewise linear operator that corresponds to the family of deterministic operators $A_{\alpha}: L^{2}([0, T], \mathscr{D}) \rightarrow L^{2}([0, T], \mathscr{H}), \alpha \in \mathscr{I}$, where $A_{\alpha}$ are infinitesimal generators of strongly continuous semigroups $\left(e^{A_{\alpha} t}\right)_{\alpha \in \mathscr{I}}, t \geq 0$, defined on a common domain $\mathscr{D}$ that is dense in $\mathscr{H}$, such that for some $m, \theta>0$ and all $\alpha \in \mathscr{I}$

$$
\left\|\left(e^{A_{\alpha} t}\right)_{\alpha}\right\|_{L(\mathscr{H})} \leq m e^{\theta t}, \quad t \geq 0
$$

(A2) The operator $\mathbf{C}: L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu) \rightarrow L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ is a coordinatewise operator corresponding to a family of uniformly bounded deterministic operators $C_{\alpha}: L^{2}([0, T], \mathscr{H}) \rightarrow L^{2}([0, T], \mathscr{H}), \alpha \in \mathscr{I}$.
(A3) The control operator $\mathbf{B}$ is a simple coordinatewise operator $\mathbf{B}: L^{2}([0, T], \mathscr{U}) \otimes$ $L^{2}(\mu) \rightarrow L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)$ that is defined by a family of uniformly bounded deterministic operators $B_{\alpha}: L^{2}([0, T], \mathscr{U}) \rightarrow L^{2}([0, T], \mathscr{H})$, $\alpha \in \mathscr{I}$.
(A4) Opertors $\mathbf{R}$ and $\mathbf{G}$ are bounded coordinatewise operators corresponding to the families of deterministic operators $\left\{R_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\{G\}_{\alpha \in \mathscr{I}}$ respectively.
(A5) $\mathbb{E}\left\|y^{0}\right\|_{\mathscr{H}}^{2}<\infty$, such that $\mathbf{A} y^{0} \in \operatorname{Dom}(\mathbf{A})$.
Then, the optimal control problem (4.3)-(4.4) has a unique optimal control u* given in the chaos expansion form

$$
\begin{equation*}
u^{*}=-\sum_{\alpha \in \mathscr{I}} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha} \tag{4.10}
\end{equation*}
$$

where $P_{d, \alpha}(t)$ for every $\alpha \in \mathscr{I}$ solves the Riccati equation

$$
\begin{align*}
\dot{P}_{d, \alpha}(t)+P_{d, \alpha}(t) A_{\alpha}+A_{\alpha}^{\star} P_{d, \alpha}(t)+R_{\alpha} R_{\alpha}^{\star}-P_{d, \alpha}(t) B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t) & =0 \\
P_{d, \alpha}(T) & =G_{\alpha}^{\star} G_{\alpha} \tag{4.11}
\end{align*}
$$

and $k_{\alpha}(t)$ for each $\alpha \in \mathscr{I}$ solve the auxiliary differential equation

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+\left(A_{\alpha}^{\star}-P_{d, \alpha}(t) B_{\alpha} B_{\alpha}^{\star}\right) k_{\alpha}(t)+P_{d, \alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(t) \cdot \mathbf{e}_{i}(t)\right)=0 \tag{4.12}
\end{equation*}
$$

with the terminal condition $k_{\alpha}(T)=0$ and $y^{*}=\sum_{\alpha \in \mathscr{I}} y_{\alpha}^{*} H_{\alpha}$ is the optimal state.
Proof Since the operators $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are coordinatewise, by (1.60) their actions are given by $\mathbf{A} y(t, \omega)=\sum_{\alpha \in \mathscr{I}} A_{\alpha} y_{\alpha}(t) H_{\alpha}(\omega), \mathbf{B} u(t)=\sum_{\alpha \in \mathscr{I}} B_{\alpha} u_{\alpha}(t) H_{\alpha}(\omega)$ and $\mathbf{C} y(t, \omega)=\sum_{\alpha \in \mathscr{I}} C_{\alpha} y_{\alpha}(t) H_{\alpha}(\omega)$, for

$$
\begin{equation*}
y(t, \omega)=\sum_{\alpha \in \mathscr{I}} y_{\alpha}(t) H_{\alpha}(\omega), \quad u(t, \omega)=\sum_{\alpha \in \mathscr{I}} u_{\alpha}(t) H_{\alpha}(\omega) \tag{4.13}
\end{equation*}
$$

such that for all $\alpha \in \mathscr{I}$ the coefficients $y_{\alpha} \in L^{2}([0, T], \mathscr{H})$ and $u_{\alpha} \in L^{2}([0, T], \mathscr{U})$. From (A2) and (A3) we conclude that operators $\mathbf{C}$ and $\mathbf{B}$ are bounded. Namely, by Lemma 1.4 we obtain that $\|\mathbf{B} u\|_{L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)}^{2} \leq c^{2}\|u\|_{L^{2}([0, T], \mathscr{U}) \otimes L^{2}(\mu)}^{2}$ and $\|\mathbf{C} u\|_{L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)}^{2} \leq c_{1}^{2}\|y\|_{L^{2}([0, T], \mathscr{H}) \otimes L^{2}(\mu)}^{2}$, where $\left\|B_{\alpha}\right\| \leq c$ and $\left\|C_{\alpha}\right\| \leq c_{1}$ for all $\alpha \in \mathscr{I}$.

We divide the proof in several steps. First, we consider the Wick version (4.4) of the state Eq. (4.2), we apply the chaos expansion method and obtain a system of deterministic equations. By representing $y$ and $y^{0}$ in their chaos expansion forms, the initial condition $y(0)=y^{0}$, for a given $\mathscr{H}$-valued random variable $y^{0}$, is reduced
to a family of initial conditions for the coefficients of the state

$$
y_{\alpha}(0)=y_{\alpha}^{0}, \quad \text { for all } \alpha \in \mathscr{I}, \quad \text { where } y_{\alpha}^{0} \in \mathscr{H}, \alpha \in \mathscr{I} .
$$

With the chaos expansion method the state Eq.(4.4) transforms to the system of infinitely many deterministic initial value problems:

$$
1^{\circ} \text { for } \alpha=\mathbf{0} \text { : } \quad y_{\mathbf{0}}^{\prime}(t)=A_{\mathbf{0}} y_{\mathbf{0}}(t)+B_{\mathbf{0}} u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0},
$$

$2^{\circ}$ for $|\alpha|>0$ :

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=A_{\alpha} y_{\alpha}(t)+B_{\alpha} u_{\alpha}(t)+\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(t) \cdot \mathbf{e}_{i}(t), \quad y_{\alpha}(0)=y_{\alpha}^{0}, \tag{4.15}
\end{equation*}
$$

where the unknowns correspond to the coefficients of the control and the state variables. It describes how the stochastic state equation propagates chaos through different levels. Note that for $|\alpha|=0$, the Eq.(4.14) corresponds to the deterministic version of the problem (the state $y_{0}$ is the expected value of $y$ ). The terms $y_{\alpha-\varepsilon^{(i)}}(t)$ are obtained recursively with respect to the length of $\alpha$. The sum in (4.15) goes through all possible decompositions of $\alpha$, i.e., for all $i$ for which $\alpha-\varepsilon^{(i)}$ is defined. Therefore, the sum has as many terms as multi-index $\alpha$ has nonzero components. Hence, the solutions $y_{\alpha}$ to (4.15) are fully determined by the knowledge of $y_{\beta}$, $|\beta|<|\alpha|$. The existence and uniqueness of solutions of (4.14), (4.15) follow from the assumptions (A1), (A2) and (A3).

In the second step, we set up optimal control problems for each $\alpha$-level. We seek for the optimal control $u$ and the corresponding optimal state $y$ in the chaos expansion representation form (4.13), i.e., the goal is to obtain the unknown coefficients $u_{\alpha}$ and $y_{\alpha}$ for all $\alpha \in \mathscr{I}$. The problems are defined in the following way:
$1^{\circ}$ for $\alpha=\mathbf{0}$ the control problem (4.14) subject to

$$
\begin{equation*}
J\left(u_{\mathbf{0}}\right)=\int_{0}^{T}\left(\left\|R_{\mathbf{0}} y_{\mathbf{0}}(t)\right\|_{\mathscr{H}}^{2}+\left\|u_{\mathbf{0}}(t)\right\|_{\mathscr{U}}^{2}\right) d t+\left\|G_{\mathbf{0}} y_{\mathbf{0}}(T)\right\|_{\mathscr{H}}^{2}, \tag{4.16}
\end{equation*}
$$

$2^{\circ}$ for $|\alpha|>0$ the control problem (4.15) subject to

$$
\begin{equation*}
J\left(u_{\alpha}\right)=\int_{0}^{T}\left(\left\|R_{\alpha} y_{\alpha}(t)\right\|_{\mathscr{H}}^{2}+\left\|u_{\alpha}(t)\right\|_{\mathscr{U}}^{2}\right) d t+\left\|G_{\alpha} y_{\alpha}(T)\right\|_{\mathscr{H}}^{2}, \tag{4.17}
\end{equation*}
$$

and can be solved by the induction on the length of $\alpha \in \mathscr{I}$.
Next, we solve the family of deterministic control problems, i.e., we discuss the solution of the deterministic system of control problems (4.16) and (4.17):
$1^{\circ}$ For $\alpha=\mathbf{0}$ the state Eq.(4.14) is homogeneous, thus the optimal control for (4.14)-(4.16) is given in the feedback form

$$
\begin{equation*}
u_{\mathbf{0}}^{*}(t)=-B_{\mathbf{0}}^{\star} P_{d, \mathbf{0}}(t) y_{\mathbf{0}}^{*}(t) \tag{4.18}
\end{equation*}
$$

where $P_{d, \mathbf{0}}(t)$ solves the Riccati equation (4.9).
$2^{\circ}$ For each $|\alpha|>0$ the state Eq. (4.15) is inhomogeneous and the optimal control for (4.17) is given by

$$
\begin{equation*}
u_{\alpha}^{*}(t)=-B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t)-B_{\alpha}^{\star} k_{\alpha}(t) \tag{4.19}
\end{equation*}
$$

where $P_{d, \alpha}(t)$ solves the Riccati equation (4.11), while $k_{\alpha}(t)$ is a solution of the auxiliary differential equation (4.12) with the terminal condition $k_{\alpha}(T)=0$.

Summing up all the coefficients we obtain the optimal solution $\left(u^{*}, y^{*}\right)$ represented in terms of chaos expansions. Thus, the optimal state is given in the form $y^{*}=$ $\sum_{\alpha \in \mathscr{I}} y_{\alpha}^{*}(t) H_{\alpha}=y_{0}^{*}+\sum_{|\alpha|>0} y_{\alpha}^{*}(t) H_{\alpha}$ and the corresponding optimal control

$$
\begin{align*}
u^{*} & =\sum_{\alpha \in \mathscr{I}} u_{\alpha}^{*}(t) H_{\alpha}=u_{\mathbf{0}}^{*}+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha} \\
& =-B_{\mathbf{0}}^{\star} P_{d, \mathbf{0}}(t) y_{\mathbf{0}}^{*}-\sum_{|\alpha|>0} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{|\alpha|>0} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha}  \tag{4.20}\\
& =-\sum_{\alpha \in \mathscr{I}} B_{\alpha}^{\star} P_{d, \alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}-\sum_{\alpha \in \mathscr{I}} B_{\alpha}^{\star} k_{\alpha}(t) H_{\alpha} \\
& =-\mathbf{B}^{\star} \mathbf{P}_{d}(t) y^{*}(t)-\mathbf{B}^{\star} \mathscr{K}
\end{align*}
$$

where $\mathbf{P}_{d}(t)$ is a coordinatewise operator corresponding to the deterministic family of operators $\left\{P_{d, \alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\mathscr{K}$ is a stochastic process with coefficients $k_{\alpha}(t)$, i.e., a process of the form $\mathscr{K}=\sum_{\alpha \in \mathscr{I}} k_{\alpha}(t) H_{\alpha}$, with $k_{\mathbf{0}}=0$.

In the following step we prove the optimality of the obtained solution. Assuming (A1)-(A4) it follows that the assumptions of Theorem 4.1 are fulfilled and thus the optimal control problem (4.2)-(4.3) is given in feedback form by

$$
\begin{equation*}
u^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t) \tag{4.21}
\end{equation*}
$$

with a positive self-adjoint operator $\mathbf{P}(t)$ solving the stochastic Riccati equation (4.5). Since the state Eqs. (4.2) and (4.4) are equivalent, we are going to interpret the optimal solution (4.21), involving the Riccati operator $\mathbf{P}(t)$ in terms of chaos expansions. It holds $\mathbf{J}\left(u^{*}\right)=\min _{u} \mathbf{J}(u)$, for $u^{*}$ of the form (4.21).

On the other hand, the stochastic cost function $\mathbf{J}$ is related with the deterministic cost function $J$ by

$$
\begin{aligned}
\mathbf{J}(u) & =\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathscr{H}}^{2}+\|u\|_{\mathscr{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathscr{H}}^{2}\right] \\
& =\mathbb{E}\left(\int_{0}^{T}\|\mathbf{R} y\|_{\mathscr{H}}^{2} d t\right)+\mathbb{E}\left(\int_{0}^{T}\|u\|_{\mathscr{U}}^{2} d t\right) d t+\mathbb{E}\left(\left\|\mathbf{G} y_{T}\right\|_{\mathscr{H}}^{2}\right) \\
& =\sum_{\alpha \in \mathscr{I}} \alpha!\left\|R_{\alpha} y_{\alpha}\right\|_{L^{2}([0, T], \mathscr{H})}^{2}+\sum_{\alpha \in \mathscr{I}} \alpha!\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathscr{U})}^{2}+\sum_{\alpha \in \mathscr{I}} \alpha!\left\|G_{\alpha} y_{\alpha}(T)\right\|_{\mathscr{H}}^{2} \\
& =\sum_{\alpha \in \mathscr{I}} \alpha!\left(\left\|R_{\alpha} y_{\alpha}\right\|_{L^{2}([0, T], \mathscr{H})}^{2}+\left\|u_{\alpha}\right\|_{L^{2}([0, T], \mathscr{U})}^{2}+\left\|G_{\alpha} y_{\alpha}(T)\right\|_{\mathscr{H}}^{2}\right) \\
& =\sum_{\alpha \in \mathscr{I}} \alpha!J\left(u_{\alpha}\right) .
\end{aligned}
$$

Thus,

$$
\mathbf{J}\left(u^{*}\right)=\min _{u} \mathbf{J}(u)=\min _{u} \sum_{\alpha \in \mathscr{I}} \alpha!J\left(u_{\alpha}\right)=\sum_{\alpha \in \mathscr{I}} \alpha!\min _{u_{\alpha}} J\left(u_{\alpha}\right)=\sum_{\alpha \in \mathscr{I}} \alpha!J\left(u_{\alpha}^{*}\right)
$$

and therefore

$$
\begin{equation*}
u^{*}(t, \omega)=\sum_{\alpha \in \mathscr{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega) \tag{4.22}
\end{equation*}
$$

i.e., the optimal control obtained via direct Riccati approach $u^{*}$ coincides with the optimal control obtained via the chaos expansion approach $\sum_{\alpha \in \mathscr{I}} u_{\alpha}^{*}(t) H_{\alpha}(\omega)$. Moreover, the optimal states are the same and the existence and uniqueness of the solution of the optimal state equation via the chaos expansion approach follows from the direct Riccati approach.

Finally, we prove the convergence of the obtained chaos expansion of the optimal state. We include the feedback forms (4.18) and (4.19) of the optimal controls $u_{\alpha}^{*}$, $\alpha \in \mathscr{I}$ in the state Eqs. (4.14) and (4.15) and obtain the system

$$
\begin{align*}
& y_{\mathbf{0}}^{\prime}(t)=\left(A_{\mathbf{0}}-B_{\mathbf{0}} B_{\mathbf{0}}^{\star} P_{d, \mathbf{0}}(t)\right) y_{\mathbf{0}}(t), \quad \text { for }|\alpha|=0 \text { and } \\
& y_{\alpha}^{\prime}(t)=\left(A_{\alpha}-B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t)\right) y_{\alpha}(t)-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \mathbf{e}_{i}(t), \tag{4.23}
\end{align*}
$$

for $|\alpha| \geq 1$, with the initial conditions $y_{\alpha}(0)=y_{\alpha}^{0}, \alpha \in \mathscr{I}$.
From the assumption $(A 1)$ it follows that $A_{\alpha}, \alpha \in \mathscr{I}$ are infinitesimal generators of strongly continuous semigroups $\left(T_{t}\right)_{\alpha}=\left(e^{A_{\alpha} t}\right)_{\alpha}, t \geq 0$ which are uniformly bounded, i.e., $\left\|e^{A_{\alpha} t}\right\|_{\mathscr{L}(\mathscr{H})} \leq m e^{\theta t}, \alpha \in \mathscr{I}$ holds for some positive constants $m$ and $\theta$, where $\mathscr{L}(\mathscr{H})$ denotes the set of linear bounded mappings on $L^{2}([0, T], \mathscr{H})$. Moreover, the family $\left(T_{t}^{\star}\right)_{\alpha}=\left(e^{A_{\alpha}^{\star} t}\right)_{\alpha}, t \geq 0$ is a family of strongly continuous semigroups whose infinitesimal generators are $A_{\alpha}^{\star}, \alpha \in \mathscr{I}$, the adjoint operators of $A_{\alpha}, \alpha \in \mathscr{I}$. This follows from the fact that each Hilbert space is a reflexive Banach space, see [43]. We denote by $S_{\alpha}(t)=A_{\alpha}-B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t), \alpha \in \mathscr{I}$ and rewrite (4.23) in a simpler form

$$
\begin{array}{ll}
y_{\mathbf{0}}^{\prime}(t)=S_{\mathbf{0}}(t) y_{\mathbf{0}}(t) \\
y_{\alpha}^{\prime}(t)=S_{\alpha}(t) y_{\alpha}(t)+f_{\alpha}(t), & , \quad y_{\mathbf{0}}(0)=y_{0}^{0}  \tag{4.24}\\
y_{\alpha}(0)=y_{\alpha}^{0}, & |\alpha|>1
\end{array}
$$

where $f_{\alpha}(t)=-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}} C y_{\alpha-\varepsilon^{(i)}}(t) \mathbf{e}_{i}(t), \alpha \in \mathscr{I}$. The operators $S_{\alpha}(t)$, $\alpha \in \mathscr{I}$ can be understood as time dependent continuous perturbations of the operators $A_{\alpha}$. From Theorem 4.1 it follows that $P_{d, \alpha}(t), \alpha \in \mathscr{I}$ are self adjoint and uniformly bounded operators, i.e., $\left\|P_{d, \alpha}(t)\right\| \leq p, \alpha \in \mathscr{I}, t \in[0, T]$. The operators $B_{\alpha}$ and thus $B_{\alpha}^{\star}$ are uniformly bounded, i.e., for all $\alpha \in \mathscr{I}$ we have $\left\|B_{\alpha}\right\| \leq b$ and $\left\|B_{\alpha}^{*}\right\| \leq b, b>0$. Therefore, $B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}(t), \alpha \in \mathscr{I}$ are uniformly bounded. Hence, we can associate a family of evolution systems $U_{\alpha}(t, s), \alpha \in \mathscr{I}, 0 \leq s \leq t \leq T$ to the initial value problems (4.24) such that

$$
\left\|U_{\alpha}(t, s)\right\|_{L(\mathscr{H})} \leq e^{\theta_{1} t}, \quad \text { for all } 0 \leq s \leq t \leq T
$$

Recall, for all $\alpha \in \mathscr{I}$ we have that $U_{\alpha}(s, s)=I d, U_{\alpha}(t, s)=U_{\alpha}(t, r) U_{\alpha}(r, s)$ for $0 \leq t \leq r \leq s \leq T$ and $(t, s) \rightarrow U_{\alpha}(t, s)$ is continuous for $0 \leq s \leq t \leq T$. Moreover, $\frac{\partial}{\partial t} U_{\alpha}(t, s)=S_{\alpha}(t) U_{\alpha}(t, s)$ and $\frac{\partial}{\partial s} U_{\alpha}(t, s)=-U_{\alpha}(t, s) S_{\alpha}(s)$, for $0 \leq$ $t \leq s \leq T$.

The family of solution maps $U_{\alpha}(t, s) y_{\alpha}^{0}, \alpha \in \mathscr{I}$ to the non-autonomous system (4.24) is a family of evolutions which are in $C([0, T], \mathscr{H})$ since $B_{\alpha} B_{\alpha}^{\star} P_{d, \alpha}, \alpha \in \mathscr{I}$ are bounded for every $t$, and are for all $\alpha \in \mathscr{I}$ continuous in time, i.e., elements of $C([0, T], \mathscr{L}(\mathscr{H})),[43]$. The adjoint operators $\left(S_{\alpha}(t)\right)^{\star}=A_{\alpha}^{\star}+P_{d, \alpha}(t) B_{\alpha}^{\star} B_{\alpha}, \alpha \in$ $\mathscr{I}$ are associated to the corresponding adjoint evolution systems $U_{\alpha}^{\star}(t, s), \alpha \in \mathscr{I}$, $0 \leq s \leq t \leq T$, [43]. The operators $C_{\alpha}, \alpha \in \mathscr{I}$ are uniformly bounded and for all $\alpha \in \mathscr{I}$ it holds $\left\|C_{\alpha}\right\| \leq d, d>0$. For a fixed control $u$ it also holds $\mathbf{C} y \in \operatorname{Dom}_{0}(\delta)$, i.e., (2.13) holds for $\mathbf{C y}$.

Consider a small interval $\left[0, T_{0}\right]$, for fixed $T_{0} \in(0, T]$. Denote by $M_{1}(t)=e^{\theta_{1} t}$ and $M_{2}(t)=\frac{1}{2 \theta_{1}}\left(e^{2 \theta_{1} t}-1\right)^{2}$ for $t \in\left(0, T_{0}\right]$.

For every $y_{\alpha}^{0} \in \operatorname{Dom}\left(S_{\alpha}(t)\right)$ the mild solution of (4.24) is given in the form

$$
\begin{aligned}
& y_{\mathbf{0}}(t)=U_{\mathbf{0}}(t, 0) y_{\mathbf{0}}^{0} \text { and } \\
& y_{\alpha}(t)=U_{\alpha}(t, 0) y_{\alpha}^{0}+\int_{0}^{t} U_{\alpha}(t, s)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(s) \mathbf{e}_{i}(s)-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(s)\right) d s,
\end{aligned}
$$

for $|\alpha| \geq 1$ and $0 \leq s \leq t \leq T$ and $y_{\alpha}$ are continuous functions for all $\alpha \in \mathscr{I}$. The operators $C_{\alpha}, B_{\alpha}$ and $B_{\alpha}^{\star}, \alpha \in \mathscr{I}$ are uniformly bounded and therefore the inhomogeneity part of (4.23) belongs to the space $L^{2}\left(\left[0, T_{0}\right], \mathscr{H}\right)$, where functions $k_{\alpha}$, $\alpha \in \mathscr{I}$ are given in (4.12). Denote by $X_{0}=L^{2}\left(\left[0, T_{0}\right], \mathscr{H}\right), X_{1}=L^{2}\left(\left[T_{0}, T\right], \mathscr{H}\right)$, $\mathscr{X}=X_{0} \otimes L^{2}(\mu)$ and $\mathscr{X}_{1}=X_{1} \otimes L^{2}(\mu)$. Thus, it holds

$$
\begin{align*}
&\|y\|_{\mathscr{X}}^{2}= \sum_{\alpha \in \mathscr{\mathscr { I }}} \alpha!\left\|y_{\alpha}\right\|_{X_{0}}^{2}=\left\|y_{0}\right\|_{X_{0}}^{2}+\sum_{|\alpha| \geq 1} \alpha!\left\|y_{\alpha}\right\|_{X_{0}}^{2} \leq 2 \sum_{\alpha \in \mathscr{I}} \alpha!\left\|U_{\alpha}(t, 0) y_{\alpha}^{0}\right\|_{X_{0}}^{2} \\
&+2 \sum_{|\alpha| \geq 1} \alpha!\| \int_{0}^{t}\left(U_{\alpha}(t, s)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}(s) \mathbf{e}_{i}(s)-B_{\alpha} B_{\alpha}^{\star} k_{\alpha}(s)\right) d s \|_{X_{0}}^{2}\right. \\
& \leq 2 M_{1}^{2}\left(T_{0}\right) \sum_{\alpha \in \mathscr{I}} \alpha!\left\|y_{\alpha}^{0}\right\|_{X_{0}}^{2}  \tag{4.25}\\
&+8 M_{2}\left(T_{0}\right) d^{2} \sum_{|\alpha| \geq 1} \alpha!|\alpha|\left\|y_{\alpha}\right\|_{X_{0}}^{2}+4 M_{2}\left(T_{0}\right) b^{4} \sum_{|\alpha| \geq 1} \alpha!\left\|k_{\alpha}(s)\right\|_{X_{0}}^{2} \\
& \leq 2 M_{1}^{2}\left(T_{0}\right)\left\|y^{0}\right\|_{\mathscr{X}}^{2}+4 M_{2}\left(T_{0}\right) d^{2}\|y\|_{\text {Dom }}(\delta) \\
& 2 \\
& 4 M_{2}\left(T_{0}\right) b^{4}\|\mathscr{K}\|_{\mathscr{X}}^{2},
\end{align*}
$$

where $\|\mathscr{K}\|_{\mathscr{X}}^{2}=\sum_{\alpha \in \mathscr{I}}\left\|k_{\alpha}\right\|_{X_{0}}^{2} \alpha$ !. The coefficients $k_{\alpha}$ are the solutions of (4.12) and are expressed in terms of the adjoint evolution system $U_{\alpha}^{\star}(t, s), \alpha \in \mathscr{I}$. Clearly, the coefficients are of the form
$k_{\alpha}(t)=U_{\alpha}^{\star}(T, t) k_{\alpha}(T)+\int_{t}^{T} U_{\alpha}^{\star}(s, t) P_{d, \alpha}(s)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}} \mathbf{e}_{i}(s)\right) d s, t<T$
for $\alpha \in \mathscr{I}$. We denote by $\left\|U_{\alpha}^{\star}(T, t)\right\| \leq e^{\tilde{\theta} t}=M_{3}(t)$, for $\tilde{\theta}>0, \alpha \in \mathscr{I}$ and $M_{4}(t)=\frac{1}{2 \tilde{\theta}}\left(e^{2 \tilde{\theta}(T-t)}-1\right)^{2}$. Since $k_{\alpha}(T)=0$ we obtain

$$
\begin{aligned}
\|\mathscr{K}\|_{\mathscr{X}_{1}}^{2} & =\sum_{\alpha \in \mathscr{I}} \alpha!\left\|\int_{t}^{T} U_{\alpha}^{\star}(s, t) P_{d, \alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}} \mathbf{e}_{i}(s)\right) d s\right\|_{X_{1}}^{2} \\
& \leq 2 M_{4}\left(T_{0}\right) p^{2} d^{2} \sum_{\alpha \in \mathscr{I}} \alpha!|\alpha|\left\|y_{\alpha}\right\|_{X}^{2} \leq M_{4}\left(T_{0}\right) p^{2} d^{2}\|y\|_{D_{o o m_{0}(\delta)}^{2}}^{2}<\infty .
\end{aligned}
$$

Thus, $\|\mathscr{K}\|_{\mathscr{X}}^{2}<\infty$. With this bound we return to (4.25) and conclude that $\|y\|_{\mathscr{X}}^{2}<$ $\infty$. The interval ( $0, T$ ] can be covered by the intervals of the form $\left[k T_{0},(k+1) T_{0}\right]$ in finitely many steps. Thus, $y \in X \otimes L^{2}(\mu)$.

Theorem 4.2 is an extension of the one from [30], where the case with simple coordinatewise operators was considered. This convergence result plays a major role in the error analysis for truncating the chaos expansion in numerical simulations.

Remark 4.1 The previous results can be extended for optimal control problems with state equations of the form (4.1), in spaces of stochastic distributions. By replacing the uniformly boundedness conditions on the operators $B_{\alpha}$ and $C_{\alpha}, \alpha \in \mathscr{I}$ in (A2) and (A3) with the polynomial growth conditions of the type $\sum_{\alpha \in \mathscr{I}}\left\|C_{\alpha}\right\|^{2}(2 \mathbb{N})^{-s \alpha}<\infty$, for some $s>0$, it can be proven that for fixed admissible control, the state equation has a unique solution in the space $L^{2}([0, T], \mathscr{H}) \otimes(S)_{-\rho}, \rho \in[0,1]$. A similar theorem to Theorem 4.2 for the optimal control can be proven.

Remark 4.2 The SLQR problem (4.2)-(4.3) reduces to the one with the state equation considered as stochastic equation with respect to standard Brownian motion $b_{t}$ when $\mathscr{H}$-valued process $B_{t}$ is replaced by $\mathbb{R}$-valued process $B_{t}$, see Example 1.13.

The following theorem gives the characterization of the optimal solution (4.22) in terms of the solution of the stochastic Riccati equation (4.5).
Theorem 4.3 Let the conditions (A1)-(A5) from Theorem 4.2 hold and let $\mathbf{P}$ be a coordinatewise operator that corresponds to the family of operators $\left\{P_{\alpha}\right\}_{\alpha \in \mathscr{I}}$. Then, the solution of the optimal control problem (4.2)-(4.3) obtained via chaos expansion (4.20) is equal to the one obtained via Riccati approach (4.21) if and only if

$$
\begin{equation*}
C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha} y_{\alpha}^{*}(t)=P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot \mathbf{e}_{i}(t)\right), \quad|\alpha|>0, k \in \mathbb{N} \tag{4.26}
\end{equation*}
$$

hold for all $t \in[0, T]$.
Proof Let us assume first that (4.20) is equal to (4.21). Then,

$$
-\mathbf{B}^{\star} \mathbf{P}(t) y^{*}(t)=-\mathbf{B}^{\star} \mathbf{P}_{d}(t) y^{*}(t)-\mathbf{B}^{\star} \mathscr{K}
$$

and we obtain

$$
\left(\mathbf{P}(t)-\mathbf{P}_{d}(t)\right) y^{*}(t)=\mathscr{K} .
$$

The difference between $\mathbf{P}(t)$ and $\mathbf{P}_{d}(t)$ is expressed through the stochastic process $\mathscr{K}$, which comes from the influence of inhomogeneities. Assuming that $\mathbf{P}$ is a coordinatewise operator that corresponds to the family of operators $\{P\}_{\alpha \in \mathscr{I}}$, we will be able to see the action of stochastic operator $\mathbf{P}$ on the deterministic level, i.e., level of coefficients. Thus, for $y$ given in the chaos expansion form (4.13) and $\mathbf{P}(t) y^{*}=\sum_{\alpha \in \mathscr{I}} P_{\alpha}(t) y_{\alpha}^{*}(t) H_{\alpha}$ it holds

$$
\begin{equation*}
\sum_{\alpha \in \mathscr{I}}\left(P_{\alpha}(t)-P_{d, \alpha}(t)\right) y_{\alpha}^{*}(t) H_{\alpha}=\sum_{\alpha \in \mathscr{\mathscr { J }},|\alpha|>0} k_{\alpha}(t) H_{\alpha} . \tag{4.27}
\end{equation*}
$$

Since $k_{\mathbf{0}}(t)=0$ it follows $P_{\mathbf{0}}(t)=P_{d, \mathbf{0}}(t)$, for $t \in[0, T]$ and for $|\alpha|>0$

$$
\left(P_{\alpha}(t)-P_{d, \alpha}(t)\right) y_{\alpha}^{*}(t)=k_{\alpha}(t),
$$

such that (4.12) with the condition $k_{\alpha}(T)=0$ holds. We differentiate (4.27) and substitute (4.12), together with (4.5), (4.9) and (4.15). Thus, after all calculations we obtain for $|\alpha|=0$ the equation $\left(P_{\mathbf{0}}(t)-P_{d, \mathbf{0}}(t)\right) y_{\mathbf{0}}^{*}(t)=0$ and for $|\alpha|>0$

$$
C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha} y_{\alpha}^{*}(t)=P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}}^{*}(t) \cdot \mathbf{e}_{i}(t)\right), \quad k \in \mathbb{N} .
$$

Note that assuming (4.26) and $\mathbf{P}$ is a coordinatewise operator that corresponds to operators $P_{\alpha}, \alpha \in \mathscr{I}$ we go backwards in the analysis and prove that the optimal controls (4.21) and (4.20) are the same.
Remark 4.3 The condition (4.26) that characterizes the optimality represent the action of the stochastic Riccati operator in each level of the representation of the noise. Note that the stochastic Riccati equation (4.5) and the deterministic one (4.9) differ only in the term $C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha}$, i.e., the operator $C_{\alpha}^{\star} P_{\alpha}(t) C_{\alpha}, \alpha \in \mathscr{I}$ captures the stochasticity of the equation. Polynomial chaos projects the stochastic part in different levels of singularity, the way that Riccati operator acts in each level is given by (4.26).

Remark 4.4 Following our approach the numerical treatment of the SLQR problem relies on solving efficiently Riccati equations arising in the associated deterministic problems. In recent years, numerical methods for solving differential Riccati equations have been proposed [2-5, 27].

### 4.2.1 State Equation with a Delta-Noise

We apply the chaos expansion method to optimal control problems governing by state equations involving so-called delta noise. Particularly, we study the state equation of the form

$$
\begin{equation*}
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\delta(\mathbf{C} y(t)), \quad y(0)=y^{0}, \quad t \in[0, T], \tag{4.28}
\end{equation*}
$$

where $\delta$ denotes the Itô-Skorokhod integral, an integral of Bochner-Pettis type [45]. In the same setting we can also consider the state equation of the form

$$
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\delta_{t}(\mathbf{C} y(t)), \quad y(0)=y^{0}, \quad t \in[0, T],
$$

where $\delta_{t}(f)=\int_{0}^{t} f(s) d B_{s}, t \in[0, T]$ is the integral process. Note that it holds $\delta_{t}(f)=\delta\left(f \chi_{[0, t]}\right)$, and for $t \in[0, T]$ we have

$$
\delta_{t}(\mathbf{C} y)=\int_{0}^{t} \mathbf{C y}(s) d B_{s}=\int_{0}^{T} \mathbf{C y}(s) \chi_{[0, t]}(s) d B_{s}=\delta\left(\mathbf{C} y(s) \chi_{[0, t]}(s)\right) .
$$

The fact that $y$ appears in the stochastic integral implies that the noise contains a memory property [8]. The disturbance $\delta$ is a zero mean random variable for all $t \in[0, T]$, while $\delta_{t}$ is a zero mean stochastic process.

In [37] it was proven that for a coordinatewise operator on $X \otimes(S)_{-\rho}$ there exists a coordinatewise operator $\tilde{\mathbf{C}}$ such that there exists a one-to-one correspondence between $\tilde{\mathbf{C}} \diamond$ and $\delta \circ \mathbf{C}$, i.e., $\tilde{\mathbf{C}} \diamond y=\delta(\mathbf{C} y)$. Therefore, (4.28) can be written as

$$
\dot{y}(t)=\mathbf{A} y(t)+\mathbf{B} u(t)+\tilde{\mathbf{C}} \diamond y, \quad y(0)=y^{0} .
$$

Hence, there exists a correspondence between the Wick form perturbation and the stochastic integral representation.

In the following we apply the chaos expansion method for solving the SLQR problem related to (4.28). Since there is no explicit form of $\tilde{\mathbf{C}}$, the suggested polynomial chaos approach for solving the problem is quite promising. By applying the chaos expansion method to (4.28) and setting up optimal control problems for the coefficients $u_{\alpha}$ and $y_{\alpha}, \alpha \in \mathscr{I}$, we obtain the following system of deterministic optimal control problems:
$1^{\circ}$ for $\alpha=\mathbf{0}$ : the control problem (4.16) subject to

$$
\begin{equation*}
y_{\mathbf{0}}^{\prime}(t)=A y_{\mathbf{0}}(t)+B u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0} \tag{4.29}
\end{equation*}
$$

$2^{\circ}$ for $|\alpha|>\mathbf{0}$ the control problem (4.17) subject to

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=A y_{\alpha}(t)+B u_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}, \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{4.30}
\end{equation*}
$$

where $\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}$ denotes the $i$ th component of $C y_{\alpha-\varepsilon^{(i)}}$, i.e., a real number, obtained in the previous inductive step.

As in the previous section, for $|\alpha|=0$ the state Eq.(4.29) is homogeneous and the optimal control is given in the feedback form by the solution of the deterministic Riccati equation (4.9). On the other hand, for each $|\alpha|>0$ the state Eq.(4.30) is inhomogeneous with the inhomogeneity term $\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}\right)_{i}$, i.e., equation of the form (4.7). Thus, the optimal control is of the form (4.8) and is determined by the solutions of the auxiliary differential equations

$$
\begin{equation*}
k_{\alpha}^{\prime}(t)+\left(A^{\star}-P_{d}(t) B B^{\star}\right) k_{\alpha}(t)+P_{d}(t)\left(\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}\right)_{i}=0,\right. \tag{4.31}
\end{equation*}
$$

for $|\alpha|>0$ with the terminal condition $k_{\alpha}(T)=0$. Summing up all the coefficients obtained as optimal on each level $\alpha$, the optimal state is then given in the form $y^{*}=\sum_{\alpha \in \mathscr{I}} y_{\alpha}^{*}(t) H_{\alpha}=y_{0}^{*}+\sum_{|\alpha|>0} y_{\alpha}^{*}(t) H_{\alpha}$ and the corresponding optimal control $u^{*}=\sum_{\alpha \in \mathscr{I}}^{\alpha \in \mathscr{I}} u_{\alpha}^{*}(t) H_{\alpha}=u_{0}^{*}+\sum_{|\alpha|>0} u_{\alpha}^{*}(t) H_{\alpha}$. Thus, the optimal state for $|\alpha|=0$ is given by $y_{\mathbf{0}}^{\prime}(t)=\left(A-B B^{\star} P_{d}(t)\right) y_{\mathbf{0}}(t), y_{\mathbf{0}}(0)=y_{0}^{0}$ and for the levels $|\alpha|>0$ by $y_{\alpha}^{\prime}(t)=\left(A-B B^{\star} P_{d}(t)\right) y_{\alpha}(t)-B B^{\star} k_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C y_{\alpha-\varepsilon^{(i)}}(t)\right)_{i}, y_{\alpha}(0)=y_{\alpha}^{0}$, where $k_{\alpha}$ are solutions of (4.31). Finally, we point out that the convergence of the chaos expansions follows similarly to the one described in [30].

### 4.2.2 Random Coefficients

Let us consider a SLQR problem of the form

$$
\begin{equation*}
d y(t)=\left[\left(\overline{\mathbf{A}}+\mathbf{A}_{\sharp}\right) y(t)+\mathbf{B} u(t)\right] d t+\mathbf{C} y(t) d B_{t} \quad y(0)=y^{0} \tag{4.32}
\end{equation*}
$$

subject to the performance index

$$
\begin{equation*}
\mathbf{J}(u)=\mathbb{E}\left[\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathscr{H}}^{2}+\|u\|_{\mathscr{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathscr{Z}}^{2}\right] \tag{4.33}
\end{equation*}
$$

where $\overline{\mathbf{A}}$ is a deterministic operator and is the infinitesimal generator of a $C_{0}$ semigroup, while $\mathbf{A}_{\sharp}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ are allowed to be random. This problem was studied in $[19,20]$, where the authors proved that the optimal control is given in the feedback form in terms of $\mathbf{P}(t)$ that solves the backward stochastic Riccati equation

$$
\begin{aligned}
-d \mathbf{P}=\left(\mathbf{R}^{\star} \mathbf{R}+\overline{\mathbf{A}}^{\star} \mathbf{P}+\mathbf{P} \overline{\mathbf{A}}\right. & \left.-\mathbf{P B B}^{\star} \mathbf{P}+\mathbf{A}_{\sharp}^{\star} \mathbf{P}+\mathbf{P A}_{\sharp}\right) d t \\
& +\operatorname{Tr}\left(\mathbf{C}^{\star} \mathbf{P C}+\mathbf{C}^{\star} \mathbf{Q}+\mathbf{Q C}\right) d t+\mathbf{Q} d B_{t},
\end{aligned}
$$

with $\mathbf{P}_{\mathbf{0}}(T)=\mathbf{G}^{\star} \mathbf{G}$. The two operators $\mathbf{P}$ and $\mathbf{Q}$ are unknown and $\mathbf{Q}$ is sometimes referred as a martingale term [19, 20].

In case when the operators involved have chaos expansion representations, we can extend our approach also to solving the problem (4.32)-(4.33). Let $\overline{\mathbf{A}}$ be a coordinatewise operator composed of the family $\left\{\bar{A}_{\alpha}\right\}_{\alpha \in \mathscr{I}}$, where $\bar{A}_{\alpha}$ are infinitesimal generators of $C_{0}$-semigroups defined on a common domain that is dense in $\mathscr{H}$ and $\overline{\mathbf{A}}(F)=\sum_{\alpha \in \mathscr{I}} \bar{A}_{\alpha}\left(f_{\alpha}\right) H_{\alpha}$. The operators $\mathbf{A}_{\sharp}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ are also coordinatewise operators composed of the families of deterministic operators $\left\{A_{\alpha}^{\sharp}\right\}_{\alpha \in \mathscr{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathscr{I}}$, $\left\{C_{\alpha}\right\}_{\alpha \in \mathscr{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ respectively, and $\mathbf{A}_{\sharp}(F)=\sum_{\alpha \in \mathscr{I}} A_{\alpha}^{\sharp}\left(f_{\alpha}\right) H_{\alpha}$, $\mathbf{B}(U)=\sum_{\alpha \in \mathscr{I}} B_{\alpha}\left(u_{\alpha}\right) H_{\alpha}, \mathbf{C}(F)=\sum_{\alpha \in \mathscr{I}} C_{\alpha}\left(f_{\alpha}\right) H_{\alpha}, \mathbf{R}(F)=\sum_{\alpha \in \mathscr{I}} R_{\alpha}\left(f_{\alpha}\right) H_{\alpha}$ and $\mathbf{G}(F)=\sum_{\alpha \in \mathscr{I}} G_{\alpha}\left(f_{\alpha}\right) H_{\alpha}$ for $F=\sum_{\alpha \in \mathscr{I}} f_{\alpha} H_{\alpha}, f_{\alpha} \in \mathscr{H}$ and $U=$ $\sum_{\alpha \in \mathscr{I}} u_{\alpha} H_{\alpha}, u_{\alpha} \in \mathscr{U}$.

After applying the chaos expansion method to (4.32), we obtain the system
(a) for $|\alpha|=0$ :

$$
\begin{equation*}
y_{\mathbf{0}}^{\prime}(t)=\left(\bar{A}_{\mathbf{0}}+A_{\mathbf{0}}^{\sharp}\right) y_{\mathbf{0}}(t)+B_{\mathbf{0}} u_{\mathbf{0}}(t), \quad y_{\mathbf{0}}(0)=y_{\mathbf{0}}^{0}, \tag{4.34}
\end{equation*}
$$

(b) for $|\alpha|>0$ :

$$
\begin{equation*}
y_{\alpha}^{\prime}(t)=\left(\bar{A}_{\alpha}+A_{\alpha}^{\sharp}\right) y_{\alpha}(t)+B_{\alpha} u_{\alpha}(t)+\sum_{i \in \mathbb{N}}\left(C_{\alpha} y_{\alpha-\varepsilon^{(i)}}\right)_{i}, \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{4.35}
\end{equation*}
$$

Setting up control problems at each level, i.e., for (4.34)-(4.35), we obtain the optimal state in the form

$$
d y(t)=\left(\left(\overline{\mathbf{A}}+\mathbf{A}_{\sharp}-\mathbf{B B}^{\star} \overline{\mathbf{P}}\right) y(t)\right) d t+\mathbf{C} y(t) \diamond W_{t}-\mathbf{B B}^{\star} \bar{K}, \quad y(0)=y^{0},
$$

where $\overline{\mathbf{P}}$ is a coordinatewise operator composed by the family $\left\{P_{\alpha}\right\}_{\alpha \in \mathscr{I}}$. The operators $P_{\alpha}$ correspond to the solution of the Riccati equation for the coefficients $\bar{A}_{\alpha}, A_{\alpha}^{\sharp}$, $B_{\alpha}, C_{\alpha}, R_{\alpha}$ and $G_{\alpha}$. For $\alpha \in \mathscr{I}$ it holds

$$
\begin{equation*}
\dot{P}_{\alpha}+P_{\alpha}\left(A_{\alpha}+A_{\alpha}^{\sharp}\right)+\left(A_{\alpha}+A_{\alpha}^{\sharp}\right)^{\star} P_{\alpha}+R_{\alpha}^{\star} R_{\alpha}-\left(P_{\alpha} B_{\alpha} B_{\alpha}^{\star} P_{\alpha}\right)=0 \tag{4.36}
\end{equation*}
$$

with $P_{\alpha}(T)=G_{\alpha}^{\star} G_{\alpha}$. Note that (4.36) is a deterministic Riccati equation for each $\alpha$. Also $\overline{\mathscr{K}}=\sum_{\alpha \in \mathscr{I}} k_{\alpha} H_{\alpha}=k_{\varepsilon^{(i)}} H_{\varepsilon^{(i)}}+\sum_{|\alpha|>1} k_{\alpha} H_{\alpha}$ is a $\mathscr{H}$-valued stochastic process, where $k_{0}=0$ and $k_{\alpha}$, for $|\alpha| \geq 1$ are given by

$$
k_{\alpha}^{\prime}(t)+\left(A_{\alpha}^{\star}-P_{\alpha}(t) B_{\alpha} B_{\alpha}^{\star}\right) k_{\alpha}(t)+P_{\alpha}(t)\left(\sum_{i \in \mathbb{N}} C_{\alpha-\varepsilon^{(i)}} y_{\alpha-\varepsilon^{(i)}} \mathbf{e}_{i}\right)=0
$$

with final zero condition. Therefore, in order to control the system (4.32)-(4.33) we control each level through the chaos expansions. This implies solving deterministic control problems at each level. Although theoretically we have to solve infinitely many control problems, numerically, when approximating the solution by the $p$ th order chaos, we have to solve $\frac{(m+p)!}{m!p!}$ problems in order to achieve the convergence. The value of $p$ is in general equal to the number of uncorrelated random variables in the system and $m$ is typically chosen by some heuristic method [24, 46]. More details will be given in Sect.4.6.

### 4.2.3 Further Extensions

We consider now more general form of the state Eq. (4.1) for bounded coordinatewise operators $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{T} \diamond$, where the operator $\mathbf{T} \diamond$ is defined by (1.62). For more details about $\mathbf{T} \diamond$ we refer to [37]. We point out that in [37] the authors proved that (4.1), for fixed $u$, has a unique solution in a space of stochastic generalized processes. Here, we show that the optimal control problem (4.1)-(4.3) for a specific choice of operator $\mathbf{T}$ can be reduced to the problem (4.4)-(4.3), and thus its optimal control can be obtained by Theorem 4.2. Moreover, the corresponding fractional optimal control problem is considered in see Sect.4.5. This extension is connected to a Gaussian colored noise (1.49) with the condition (1.50).

We denote by $X=L^{2}([0, T], \mathscr{H})$.
Theorem 4.4 ([32]) Let $L_{t}$ be of the form (1.49) such that (1.50) holds. Let $\mathbf{N}$ be a coordinatewise operator which corresponds to a family of uniformly bounded (or polynomially bounded) operators $\left\{N_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ satisfy the assumptions
(A1)-(A4) of Theorem 4.2. Let $\mathbf{T}$ be a coordinatewise operator defined by a family of operators $\left\{T_{\alpha}\right\}_{\alpha \in \mathscr{I}}, T_{\alpha}: X \rightarrow X, \alpha \in \mathscr{I}$, such that for $|\beta| \leq|\alpha|$

$$
T_{\beta}\left(y_{\alpha-\beta}\right)=\left\{\begin{array}{ll}
N_{\alpha}\left(y_{\alpha}\right) & ,|\beta|=0  \tag{4.37}\\
l_{k} N_{\alpha-\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right) & ,|\beta|=1, \\
0 & ,|\beta|>1
\end{array} \text { i.e., } \beta=\varepsilon^{(k)}, k \in \mathbb{N}\right.
$$

for $y_{\alpha} \in X$. Then, the state Eq.(4.1) can be reduced to the state Eq. (4.4). Moreover, the optimal control problem (4.1)-(4.3) has a unique solution in $X \otimes(S)_{-\rho}$.

Proof By the definition (1.62) and the chaos expansion method, the state Eq. (4.1) reduces to the system

$$
\begin{align*}
& 1^{\circ} \text { for }|\alpha|=0 \\
& \qquad \dot{y}_{0}=\left(A_{0}+T_{\mathbf{0}}\right) y_{\mathbf{0}}+B_{0} u_{0}, \quad y_{0}(0)=y_{0}^{0}, \tag{4.38}
\end{align*}
$$

$2^{\circ}$ for $|\alpha| \geq 1$

$$
\begin{equation*}
\dot{y}_{\alpha}=\left(A_{\alpha}+T_{\mathbf{0}}\right) y_{\alpha}+B_{\alpha} u_{\alpha}+\sum_{\mathbf{0}<\beta \leq \alpha} T_{\beta}\left(y_{\alpha-\beta}\right), \quad y_{\alpha}(0)=y_{\alpha}^{0} . \tag{4.39}
\end{equation*}
$$

From (4.37) it follows that $T_{\mathbf{0}}\left(y_{\alpha}\right)=N_{\alpha}\left(y_{\alpha}\right), \alpha \in \mathscr{I}$ and also $T_{\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right)=$ $l_{k} N_{\alpha-\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right)$. We define $\hat{A}_{\alpha}=A_{\alpha}+N_{\alpha}, \alpha \in \mathscr{I}$. Since the family $\left\{N_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ is uniformly bounded and $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ are infinitesimal generators $C_{0}$-semigroups then the operators $\hat{A}_{\alpha}$ are also infinitesimal generators of $C_{0}$-semigroups and satisfy the condition (A1) of Theorem 4.2, see [43]. Thus, the system (4.38)-(4.39) transforms to
$1^{\circ}$ for $|\alpha|=0$

$$
\begin{equation*}
\dot{y}_{0}=\hat{A}_{0} y_{0}+B_{0} u_{0}, \quad y_{0}(0)=y_{0}^{0} \tag{4.40}
\end{equation*}
$$

$2^{\circ}$ for $|\alpha| \geq 1$

$$
\begin{equation*}
\dot{y}_{\alpha}=\hat{A}_{\alpha} y_{\alpha}+B_{\alpha} u_{\alpha}+\sum_{k \in \mathbb{N}} l_{k} N_{\alpha-\varepsilon^{(k)}}\left(y_{\alpha-\varepsilon^{(k)}}\right), \quad y_{\alpha}(0)=y_{\alpha}^{0} \tag{4.41}
\end{equation*}
$$

Define the operators $\hat{C}_{\mathbf{0}}=N_{\mathbf{0}}$ and $\hat{C}_{\alpha-\varepsilon^{(k)}}=l_{k} N_{\alpha-\varepsilon^{(k)}}$, for $|\alpha| \geq 1, k \in \mathbb{N}$. Therefore, the obtained system (4.40)-(4.41) corresponds to the state equation of the form

$$
\begin{equation*}
\dot{y}=\hat{\mathbf{A}} y+\mathbf{B} u+\hat{\mathbf{C}} \diamond W_{t} \tag{4.42}
\end{equation*}
$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ are coordinatewise operators corresponding to the families $\left\{\hat{A}_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\left\{\hat{C}_{\alpha}\right\}_{\alpha \in \mathscr{I}}$, respectively. Moreover, $\mathbf{B}, \hat{\mathbf{C}}$ satisfy the assumptions (A2)-(A4) of Theorem 4.2. Hence, it can be applied to the control problem (4.3)-(4.42).

### 4.3 Operator Differential Algebraic Equations

In this section we focus on semi-explicit operator differential algebraic equations (ODAEs). The abstract formulation of constraint partial differential equations (PDEs) of semi-explicit form has the structure of a ODAE, i.e., a differential equation subject to an algebraic constraint. These systems of equations are motivated by applications, for example by Stokes equations and linearized Navier-Stokes equations, and they are in most cases deterministic and finite-dimensional. However, recently ODAEs with additive noise have been studied in [1], where the chaos expansion method was combined with the deterministic regularization techniques. Here we study a stochastic version of this problem, i.e., a system of a linear semi-explicit stochastic equation subject to an algebraic constraint. We present an example of a stochastic system involving operators of Malliavin calculus which has the same structure as the deterministic ODAE

$$
\begin{equation*}
\dot{y}+\mathbf{K} y+\mathbf{B}^{*} u=f, \quad \mathbf{B} y=g \tag{4.43}
\end{equation*}
$$

Assume that the stochastic operator $\mathbf{K}$ is a coordinatewise operator such that the corresponding deterministic operators $\left\{K_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ are densely defined on a given Hilbert space $X$. Particularly, we are interested in the case when $\mathbf{B}=\mathbb{D}$ and $\mathbf{B}^{*}=\delta$. Thus the system (4.43) transforms to

$$
\dot{y}+\mathbf{K} y+\delta u=f, \quad \mathbb{D} y=g
$$

with the initial condition $\mathbb{E} y=y^{0}$ and given stochastic processes $f$ and $g$. Although this example does not arise in fluid dynamics it is related with the extension of our results to nonlinear equations, in particular Navier-Stokes equation. Thus, we consider a semi-explicit systems including the stochastic operators from the Malliavin calculus and use the duality (2.21). We also consider a more general case

$$
\begin{equation*}
\dot{y}=\mathbf{A} y+\mathbf{T} \diamond y+\delta u+f, \quad \mathbb{D} y=g \tag{4.44}
\end{equation*}
$$

that was studied for $\rho=1$ in [32]. The chaos expansion method combined with the regularization techniques can be applied also in this case. Here we present the direct chaos expansion approach and prove the convergence of the obtained solution.

Recall, the Malliavin derivative $\mathbb{D}$, as a stochastic gradient in the direction of white noise, is a linear and continuous mapping $\mathbb{D}: \operatorname{Dom}_{-\rho}(\mathbb{D}) \rightarrow X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ given by (2.2). It reduces the order of the Wiener chaos space and the kernel $\operatorname{Ker}(\mathbb{D})$ consists of constant random variables, Corollary 3.2. The Itô-Skorokhod integral $\delta$ is a linear and continuous mapping $\delta: X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ defined by (2.9). It is the adjoint operator of the Malliavin derivative, i.e., the duality (2.21) holds. It increases the order of the Wiener chaos space. The Ornstein-Uhlenbeck operator $\mathscr{R}$, defined as the composition $\delta \circ \mathbb{D}$, is a self adjoint linear and continuous
mapping $\mathscr{R}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$. It is a coordinatewise operator given by (2.14).

We reduce the system (4.44) to the following two problems: $\mathbb{D} y=g, \mathbb{E} y=y^{0}$ and $\delta(u)=v$ and then apply Theorems 3.2 and 3.8 , where we take $X$ to be the Hilbert space $L^{2}([0, T], \mathscr{H})$.

Theorem 4.5 Let $\rho \in[0,1]$. Let $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ be a coordinatewise operator corresponding to a uniformly bounded family of deterministic operators $A_{\alpha}: X \rightarrow X, \alpha \in \mathscr{I}$ and $\mathbf{T}$ be a coordinatewise operator that corresponds to a polynomially bounded family of operators $T_{\alpha}: X \rightarrow X, \alpha \in \mathscr{I}$. Let $g=$ $\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{\alpha, k} \xi_{k} H_{\alpha} \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ such that its coefficients $g_{\alpha k}$ satisfy the condition (3.7) and $f \in X \otimes(S)_{-\rho}$. Let $y^{0} \in X, y^{1} \in X$ be given and the actions $A_{0} y^{0}$ and $T_{0} y^{0}$ defined such that $\mathbb{E} f=A_{0} y^{0}+T_{0} y^{0}$. Then, the system (4.44) with the initial conditions $\mathbb{E} y=y^{0}$ and $\mathbb{E} \dot{y}=y^{1}$, has unique solution pair $y \in X \otimes(S)_{-\rho}$ and $u \in X \otimes S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}$ given respectively by

$$
\begin{align*}
& y=y^{0}+\sum_{|\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} g_{\alpha-\varepsilon^{(k)}, k} \otimes H_{\alpha} \quad \text { and }  \tag{4.45}\\
& u=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) \frac{v_{\alpha+\varepsilon^{(k)}}^{\left|\alpha+\varepsilon^{(k)}\right|} \otimes \xi_{k} \otimes H_{\alpha},}{|c|} \tag{4.46}
\end{align*}
$$

where $v=\dot{y}-\mathbf{A} y-\mathbf{T} \diamond y-f$.
Proof By Theorem 3.2, the initial value problem

$$
\begin{equation*}
\mathbb{D} y=g, \quad \mathbb{E} y=y^{0} \tag{4.47}
\end{equation*}
$$

for a process $g \in X \otimes S_{-p}(\mathbb{R}) \otimes(S)_{-\rho,-q}, p \in \mathbb{N}_{0}, q>p+1$, represented in its chaos expansion form $g=\sum_{\alpha \in \mathscr{I}} \sum_{k \in \mathbb{N}} g_{\alpha, k} \xi_{k} H_{\alpha}$, such that the condition (3.7) holds, has a unique solution $y \in \operatorname{Dom}_{-\rho}(\mathbb{D})$ represented in the form (4.45). Additionally, it holds

$$
\|y\|_{X \otimes(S)_{-\rho,-q}}^{2} \leq\left\|u^{0}\right\|_{X}^{2}+c\|g\|_{X \otimes S_{-l}(\mathbb{R}) \otimes(S)_{-\rho,-q}^{2}}^{2}<\infty .
$$

The operator $\mathbf{A}$ is a coordinatewise operator and it corresponds to an uniformly bounded family of operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}}$, i.e., it holds $\left\|A_{\alpha}\right\|_{L(X)} \leq M, \alpha \in \mathscr{I}$. Moreover, for $y \in \operatorname{Dom}_{-\rho,-q}(\mathbb{D})$ it holds

$$
\|\mathbf{A} y\|_{X \otimes(S)_{-\rho,-q}}^{2}=\sum_{\alpha \in \mathscr{I}} \alpha!^{1-\rho}\left\|A_{\alpha} y_{\alpha}\right\|_{X}^{2}(2 \mathbb{N})^{-q \alpha} \leq M^{2}\|y\|_{X \otimes(S)_{-\rho,-q}}^{2}<\infty
$$

and thus $\mathbf{A} y \in X \otimes(S)_{-\rho,-q}$. The operators $\left\{T_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ are polynomially bounded and by Lemma 1.6 it holds $\mathbf{T} \diamond: X \otimes(S)_{-\rho,-q} \rightarrow X \otimes(S)_{-\rho,-q}$. Since $g_{\alpha} \in X \otimes S_{-l}(\mathbb{R})$ we can use the formula (1.2) for derivatives of the Hermite functions and obtain

$$
\dot{g}_{\alpha}=\sum_{k \in \mathbb{N}} g_{\alpha, k} \otimes \frac{d}{d t} \xi_{k}=\sum_{k \in \mathbb{N}} g_{\alpha, k} \otimes\left(\sqrt{\frac{k}{2}} \xi_{k-1}-\sqrt{\frac{k+1}{2}} \xi_{k+1}\right)
$$

where $\dot{g}_{\alpha} \in X \otimes S_{-l-1}(\mathbb{R})$. We note that the problem $\mathbb{D} \dot{u}=\dot{y}$ with the initial condition $\mathbb{E} \dot{y}=y^{1} \in X$ can be solved as (4.47), again by applying Theorem 3.2. Moreover,

$$
\|\dot{y}\|_{X \otimes(S)_{-\rho,-q}}^{2} \leq\left\|y^{1}\right\|_{X}^{2}+c\|\dot{g}\|_{X \otimes S_{-l-1}(\mathbb{R}) \otimes(S)_{-\rho,-q}}^{2}<\infty .
$$

Let $f \in X \otimes(S)_{-\rho,-q}$ and denote by $v=\dot{y}-\mathbf{A} y-\mathbf{T} \diamond y-f$. From the given assumptions it follows $v \in X \otimes(S)_{-\rho,-q}$ and it has zero expectation. Then, by Theorem 3.8, for $v=\sum_{\alpha \in \mathscr{I},|\alpha|>1} v_{\alpha} \otimes H_{\alpha}$ the integral equation $\delta(u)=v$ has a unique solution $u$ in $X \otimes S_{-l-1}(\mathbb{R}) \otimes(S)_{-\rho,-q}$, for $l>q$, given in the form (4.46), Clearly, the estimate

$$
\|u\|_{X \otimes(S)_{-\rho,-q}}^{2} \leq c\left(\|y\|_{X \otimes(S)_{-\rho,-q}}^{2}+\|f\|_{X \otimes(S)_{-\rho,-q}}^{2}+\|\dot{y}\|_{X \otimes(S)_{-\rho,-q}}^{2}\right)
$$

also holds.
Remark 4.5 If the coefficients are not differentiable, then certain regularization techniques have to be applied [1].

### 4.3.1 Extension to Nonlinear Equations

In [42] the authors showed that a random polynomial nonlinearity can be expanded in a Taylor alike series involving Wick products and Malliavin derivatives. This result has been applied to the nonlinear advection term in the Navier-Stokes equations [44]. There a detailed study of the accuracy and computational efficiency of these Wicktype approximations has been shown. We point out that following the same approach the regularization techniques combined with chaos expansions [1] can be extended to Navier-Stokes equations. Specifically, by the product formula $u v=\sum_{i=0}^{P} \frac{\mathbb{D}^{(i)} u \diamond \mathbb{D}^{(i)} v}{i!}$, of two square-integrable stochastic processes $u$ and $v$, where $\mathbb{D}^{(i)}$ is the $i$ th order of the Malliavin derivative operator, one can construct approximations of finite stochastic order. Particularly, the nonlinear advection term in the Navier-Stokes equations can be approximated by

$$
\begin{equation*}
(u \cdot \nabla) u \simeq \sum_{i=0}^{Q} \frac{\left(\mathbb{D}^{(i)} \diamond \nabla\right) \mathbb{D}^{(i)} u}{i!} \tag{4.48}
\end{equation*}
$$

where $Q$ denotes the highest stochastic order in the Wick-Malliavin expansion. The zero-order approximation $(u \cdot \nabla) u \simeq(u \diamond \nabla) u$ is known as the Wick approximation, while $(u \cdot \nabla) u \simeq(u \diamond \nabla) u+(\mathbb{D} u \diamond \nabla) \mathbb{D} u$ is the first-order Wick-Malliavin approximation [44]. As the Malliavin derivate has an explicit chaos expansion representation
form (2.2), the formula (4.48) allows us to express the nonlinear advection term in terms of chaos expansions. Therefore, the ideas presented in [1] for the linear semiexplicit stochastic ODAEs (4.43) can be extended to Navier-Stokes equations and in general to equations with nonlinearities of the type (4.48). Moreover, by applying the multiplication formula (1.57) one can generalize (2.23) and obtain

$$
\begin{equation*}
v G=v \diamond G+\sum_{\alpha \in \mathscr{\mathscr { V }}} \sum_{k \in \mathbb{N}}\left(\alpha_{k}+1\right) v_{\alpha+\varepsilon^{(k)}} g_{k} H_{\alpha} \tag{4.49}
\end{equation*}
$$

for a Gaussian process of the form $G=g_{0}+\sum_{k \in \mathbb{N}} g_{k} H_{\varepsilon^{(k)}} \in X \otimes(S)_{-\rho}$ and a process $v=\sum_{\alpha \in \mathscr{I}} v_{\alpha} H_{\alpha} \in X \otimes(S)_{-\rho}$, [36]. Thus, the results proved in [1] and the ones in [42, 44] can be generalized for this type of processes (not necessary square integrable).

Remark 4.6 In [34] the authors defined a type of scalarized Wick product containing in itself an integral operator, i.e., the scalar product in $L^{2}(\mathbb{R})$ or the dual pairing $\langle\cdot, \cdot\rangle$ of a distribution in $S^{\prime}(\mathbb{R})$ and a test function in $S(\mathbb{R})$. Let $\rho \in[0,1]$. Then, for $a=\sum_{\alpha \in \mathscr{\mathscr { I }}} a_{\alpha} H_{\alpha} \in L^{2}(\mathbb{R}) \otimes(S)_{-\rho}$ and $b=\sum_{\beta \in \mathscr{I}} b_{\beta} H_{\beta} \in L^{2}(\mathbb{R}) \otimes(S)_{-\rho}$ the element $a \forall(S)_{-\rho}$ is defined by

$$
a \diamond b=\sum_{\gamma \in \mathscr{\mathscr { I }}} \sum_{\alpha+\beta=\gamma}\left\langle a_{\alpha}, b_{\beta}\right\rangle H_{\gamma} .
$$

Similarly, if $a \in S^{\prime}(\mathbb{R}) \otimes(S)_{-\rho}, b \in S(\mathbb{R}) \otimes(S)_{-\rho}$, the result will be $a \forall b \in(S)_{-\rho}$. The expression (4.49) can be rewritten as $v \cdot G=v \diamond G+\mathbb{D}(v) \diamond \mathbb{D}(G)$.

### 4.4 Stationary Equations

In this section we consider stationary equations of the form

$$
\begin{equation*}
\mathbf{A} y+\mathbf{T} \diamond y+f=0 \tag{4.50}
\end{equation*}
$$

where A : $X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}, \rho \in[0,1]$ and $\mathbf{T} \diamond: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ are the operators of the forms (1.60) and (1.62) respectively. We assume that $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\left\{T_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ are bounded operators such that $A_{\alpha}=\widetilde{A}_{\alpha}+C_{\alpha}, \alpha \in \mathscr{I}$. We also assume that $T_{0}$ and $\widetilde{A}_{\alpha}, \alpha \in \mathscr{I}$ are compact operators and $C_{\alpha}$ are self adjoint for all $\alpha \in \mathscr{I}$ such that $C_{\alpha}\left(H_{\alpha}\right)=r_{\alpha} H_{\alpha}, \alpha \in \mathscr{I}$. By combining the chaos expansion method with classical results of elliptic PDEs and the Fredholm alternative [18], we prove existence and uniqueness of the solution of (4.50).

Theorem 4.6 ([37]) Let $\rho \in[0,1]$. Let $\mathbf{A}: X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ and $\mathbf{T} \diamond:$ $X \otimes(S)_{-\rho} \rightarrow X \otimes(S)_{-\rho}$ be the operators, for which the following is satisfied:
$1^{\circ} \mathbf{A}$ is of the form $\mathbf{A}=\mathbf{B}+\mathbf{C}$, where $\mathbf{B} y=\sum_{\alpha \in \mathscr{I}} B_{\alpha} y_{\alpha} \otimes H_{\alpha}$ and $B_{\alpha}: X \rightarrow X$ are compact operators for all $\alpha \in \mathscr{I}, \mathbf{C} y=\sum_{\alpha \in \mathscr{I}} r_{\alpha} y_{\alpha} \otimes H_{\alpha}, r_{\alpha} \in \mathbb{R}, \alpha \in \mathscr{I}$, and $\mathbf{T}$ is of the form (1.62), where $T_{0}: X \rightarrow X$ is a compact operator. Assume that there exists $K>0$ such that for all $\alpha \in \mathscr{I}$

$$
-r_{\alpha}-\left\|B_{\alpha}\right\|-\left\|T_{0}\right\| \geq 0 \quad \text { and } \quad \sup _{\alpha \in \mathscr{I}}\left(\frac{1}{-r_{\alpha}-\left\|B_{\alpha}\right\|-\left\|T_{0}\right\|}\right)<K
$$

$2^{\circ} \mathbf{T}$ is of the form (1.62), where $T_{\beta}: X \rightarrow X, \beta>\mathbf{0}$ are bounded operators and there exists $p>0$ such that

$$
\begin{equation*}
K \sqrt{2} \sum_{\beta>\mathbf{0}}\left\|T_{\beta}\right\|(2 \mathbb{N})^{\frac{-p \beta}{2}}<1 \tag{4.51}
\end{equation*}
$$

$3^{\circ}$ For every $\alpha \in \mathscr{I}$

$$
\begin{equation*}
\operatorname{Ker}\left(B_{\alpha}+\left(1+r_{\alpha}\right) \operatorname{Id}+T_{\mathbf{0}}\right)=\{0\} . \tag{4.52}
\end{equation*}
$$

Then, for every $f \in X \otimes(S)_{-\rho,-p}$ there exists a unique solution $y \in X \otimes(S)_{-\rho,-p}$ of the Eq. (4.50).

Proof Equation (4.50) is equivalent to the equation $y-(\mathbf{B} y+\mathbf{C} y+y+\mathbf{T} \diamond y)=f$, which transforms to

$$
\sum_{\gamma \in \mathscr{I}}\left(y_{\gamma}-B_{\gamma} y_{\gamma}-\left(1+r_{\gamma}\right) y_{\gamma}-\sum_{\alpha+\beta=\gamma} T_{\alpha}\left(y_{\beta}\right)\right) \otimes H_{\gamma}=\sum_{\gamma \in \mathscr{I}} f_{\gamma} \otimes H_{\gamma}
$$

Due to uniqueness of the Wiener-Itô chaos expansion this is equivalent to

$$
\begin{equation*}
y_{\gamma}-\left(B_{\gamma}+\left(1+r_{\gamma}\right) I d+T_{\mathbf{0}}\right) y_{\gamma}=f_{\gamma}+\sum_{\mathbf{0}<\beta \leq \gamma} T_{\beta}\left(y_{\gamma-\beta}\right), \quad \gamma \in \mathscr{I} \tag{4.53}
\end{equation*}
$$

By the condition (4.52) it follows that for each $\gamma \in \mathscr{I}$ the homogeneous equation $y_{\gamma}-\left(B_{\gamma}+\left(1+r_{\gamma}\right) I d+T_{0}\right) y_{\gamma}=0$ has only trivial solution $y_{\gamma}=0$, see [18]. Since the operator $B_{\gamma}+\left(1+r_{\gamma}\right) I d+T_{0}$ is compact, the classical Fredholm alternative implies that for each $\gamma \in \mathscr{I}$ there exists a unique $y_{\gamma}$ that solves (4.53) and it is of the form

$$
y_{\gamma}=\left(I d-\left(\left(r_{\gamma}+1\right) I d+B_{\gamma}+T_{\mathbf{0}}\right)\right)^{-1}\left(f_{\gamma}+\sum_{\beta>\mathbf{0}} T_{\beta}\left(y_{\gamma-\beta}\right)\right), \quad \gamma \in \mathscr{I}
$$

so that

$$
\left\|y_{\gamma}\right\|_{X} \leq \frac{1}{-r_{\gamma}-\left\|B_{\gamma}\right\|-\left\|T_{\mathbf{0}}\right\|} \cdot\left(\left\|f_{\gamma}\right\|_{X}+\sum_{\beta>\mathbf{0}}\left\|T_{\beta}\right\|\left\|y_{\gamma-\beta}\right\|_{X}\right)
$$

for $\gamma \in \mathscr{I}$. It is left to prove that the obtained solution $y=\sum_{\gamma \in \mathscr{I}} y_{\gamma} \otimes H_{\gamma}$ converges in $X \otimes(S)_{-\rho}$. Indeed, for $p \geq 0$ we obtain the estimate

$$
\begin{aligned}
& \|y\|_{X \otimes(S)_{-\rho,-p}^{2}}=\sum_{\gamma \in \mathscr{I}} \gamma!^{1-\rho}\left\|y_{\gamma}\right\|_{X}^{2}(2 \mathbb{N})^{-p \gamma} \\
& \leq K^{2} \sum_{\gamma \in \mathscr{I}}\left(\left\|f_{\gamma}\right\|_{X}+\sum_{\gamma=\alpha+\beta, \alpha>\mathbf{0}}\left\|T_{\alpha}\right\|\left\|y_{\beta}\right\|_{X}\right)^{2} \gamma!^{1-\rho}(2 \mathbb{N})^{-p \gamma} \\
& \leq 2 K^{2}\left(\sum_{\gamma \in \mathscr{I}}\left\|f_{\gamma}\right\|_{X}^{2} \gamma!^{1-\rho}(2 \mathbb{N})^{-p \gamma}+\sum_{\gamma \in \mathscr{I}}\left(\sum_{\gamma=\alpha+\beta, \alpha>\mathbf{0}}\left\|T_{\alpha}\right\|\left\|y_{\beta}\right\|_{X}\right)^{2} \gamma!^{1-\rho}(2 \mathbb{N})^{-p \gamma}\right) \\
& \leq 2 K^{2}\left(\sum_{\gamma \in \mathscr{I}}\left\|f_{\gamma}\right\|_{X}^{2} \gamma!^{1-\rho}(2 \mathbb{N})^{-p \gamma}+\left(\sum_{\alpha>\mathbf{0}}\left\|T_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2} \sum_{\beta \in \mathscr{I}}\left\|y_{\beta}\right\|_{X}^{2} \beta!^{1-\rho}(2 \mathbb{N})^{-p \beta}\right) \\
& \leq 2 K^{2}\left(\|f\|_{X \otimes(S)_{-\rho,-p}^{2}}^{2}+\left(\sum_{\alpha>\mathbf{0}}\left\|T_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2}\|y\|_{\left.X \otimes(S)_{-\rho,-p}^{2}\right),}^{2}\right)
\end{aligned}
$$

which leads to

$$
\left(1-2 K^{2}\left(\sum_{\alpha>0}\left\|T_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2}\right) \cdot\|y\|_{X \otimes(S)_{-\rho,-p}}^{2} \leq 2 K^{2}\|f\|_{X \otimes(S)_{-\rho,-p}}^{2} .
$$

By the assumption (4.51) we have that $M=1-2 K^{2}\left(\sum_{\alpha>0}\left\|T_{\alpha}\right\|(2 \mathbb{N})^{-\frac{p \alpha}{2}}\right)^{2}>0$. This implies

$$
\|y\|_{X \otimes(S)_{-\rho,-p}^{2}}^{2} \leq \frac{2 K^{2}}{M}\|f\|_{X \otimes(S)_{-\rho,-p}}^{2}<\infty .
$$

Remark 4.7 Let $B_{\alpha}=0$ for all $\alpha \in \mathscr{I}$ and let $T_{\alpha}, \alpha \in \mathscr{I}$, be second order strictly elliptic partial differential operator. Let $\mathbf{C}=c P(\mathscr{R})$, for some $c \in \mathbb{R}$, where $\mathscr{R}$ is the Ornstein-Uhlenbeck operator and $P_{m}(t)=\sum_{k=0}^{m} p_{k} t^{k}, p_{k} \in \mathbb{R}, p_{m} \neq 0$. Then, the corresponding eigenvalues of $\mathbf{C}$ are $r_{\alpha}=c P(|\alpha|), \alpha \in \mathscr{I}$. Hence, (4.50) transforms to the elliptic equation with a perturbation term driven by the polynomial of the Ornstein-Uhlenbeck operator $\mathbf{T} \triangleleft y+c P(\mathscr{R}) y=f$ that was solved in [33]. Moreover, for $T_{\alpha}=0, \alpha \in \mathscr{I}$ the Eq. (4.50) reduces to $P(\mathscr{R}) y=g$ that we solved in Theorem 3.1. In addition, Theorem 4.6 can be applied to all elliptic problems of the form $\mathbf{A} u=h$ and $\mathbf{A} \diamond y=h$.

### 4.5 A Fractional Optimal Control Problem

We consider a fractional version of the stochastic optimal control problem (4.2)(4.3). The state equation is linear stochastic differential equation

$$
\begin{equation*}
d \widetilde{y}(t)=(\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)) d t+\widetilde{\mathbf{C}} \widetilde{y}(t) d B_{t}^{(H)} \quad \widetilde{y}(0)=\widetilde{y}^{0}, \quad t \in[0, T], \tag{4.54}
\end{equation*}
$$

with respect to a $\mathscr{H}$-valued fractional Brownian motion in the fractional Gaussian white noise space. The objective is to minimize the quadratic cost functional

$$
\begin{equation*}
\mathbf{J}^{(H)}(\widetilde{u})=\mathbb{E}_{\mu_{H}}\left[\int_{0}^{T}\left(\|\widetilde{\mathbf{R}} \widetilde{y}\|_{\mathscr{H}}^{2}+\|\widetilde{u}\|_{\mathscr{U}}^{2}\right) d t+\left\|\widetilde{\mathbf{G}} \widetilde{y}_{T}\right\|_{\mathscr{H}}^{2}\right] \tag{4.55}
\end{equation*}
$$

over all possible controls $\tilde{u}$ and subject to the condition that $\tilde{y}$ satisfies (4.54). A control process $\widetilde{u}^{*}$ is called optimal if $\min _{u} \mathbf{J}^{(H)}(\widetilde{u})=\mathbf{J}^{(H)}\left(\widetilde{u}^{*}\right)$. The corresponding trajectory is denoted by $\widetilde{y}^{*}$ and is called optimal. Thus, the pair $\left(\widetilde{y}^{*}, \widetilde{u}^{*}\right)$ is the optimal solution of the problem (4.54)-(4.55). The operators $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{C}}$ are defined on $\mathscr{H}$ and $\widetilde{\mathbf{B}}$ acts from the control space $\mathscr{U}$ to the state space $\mathscr{H}$ and $\widetilde{y}^{0}$ is a random variable. The operators $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ are considered to be linear and bounded, $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{G}}$ are bounded observation operators taking values in $\mathscr{H}$. Instead of the state Eq. (4.54), we consider its Wick version

$$
\begin{equation*}
\dot{\tilde{y}}(t)=\widetilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)+\widetilde{\mathbf{C}} \widetilde{y}(t) \diamond W_{t}^{(H)}, \quad \widetilde{y}(0)=y^{0}, \quad t \in[0, T] \tag{4.56}
\end{equation*}
$$

In Sect. 4.2 in Theorem 4.2 we stated conditions under which the stochastic control problem (4.2)-(4.3) has an optimal control given in the feedback form (4.10). In order to apply this result to the corresponding fractional control problem (4.54)-(4.55), we use the isometry mapping $\mathscr{M}$, defined in Sect. 1.4.3. We apply $\mathscr{M}$ to (4.55)(4.56) and transform it to (4.2)-(4.3). The solution of the fractional problem is thus obtained from the solution of the corresponding classical problem through the inverse fractional map.

Theorem 4.7 ([32]) Let the fractional operators $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}, \widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{G}}$ defined on fractional space be coorinatewise operators that correspond to the families $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}}$, $\left\{B_{\alpha}\right\}_{\alpha \in \mathscr{I}},\left\{C_{\alpha}\right\}_{\alpha \in \mathscr{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ respectively. Let the pair $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ be the optimal solution of the fractional stochastic optimal control problem (4.54)(4.55). Then, the pair $\left(\mathscr{M} \widetilde{u}^{*}, \mathscr{M} \widetilde{y}^{*}\right)$ is the optimal solution $\left(u^{*}, y^{*}\right)$ of the associated optimal control problem (4.2)-(4.3), where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ defined on classical space, are coorinatewise operators that correspond respectively to the same families of deterministic operators $\left\{A_{\alpha}\right\}_{\alpha \in \mathscr{I}},\left\{B_{\alpha}\right\}_{\alpha \in \mathscr{I}},\left\{C_{\alpha}\right\}_{\alpha \in \mathscr{I}},\left\{R_{\alpha}\right\}_{\alpha \in \mathscr{I}}$ and $\left\{G_{\alpha}\right\}_{\alpha \in \mathscr{I}}$.

Moreover, if $\left(u^{*}, y^{*}\right)$ is the optimal solution of the stochastic optimal control problem (4.2)-(4.3), then the pair $\left(\mathscr{M}^{-1} u^{*}, \mathscr{M}^{-1} y^{*}\right)$ is the optimal solution $\left(\widetilde{u}^{*}, \tilde{y}^{*}\right)$ of the corresponding fractional optimal control problem (4.54)-(4.55).

Proof Let $\left(\widetilde{u}^{*}, \widetilde{y}^{*}\right)$ be the optimal pair of the problem (4.54)-(4.55), i.e., its equivalent problem (4.55)-(4.56). Then $\min _{u} \mathbf{J}(u)=J\left(u^{*}\right)$, while $y^{*}$ solves (4.54) and also (4.56). After applying the chaos expansion method and the properties of the fractional operator $\mathscr{M}$ stated in Theorems 1.11 and 1.12 we transform (4.56) in fractional space to the corresponding state equation in the classical space, i.e.,

$$
\begin{aligned}
\dot{y}(t) & =\mathscr{M}\left(\tilde{\mathbf{A}} \widetilde{y}(t)+\widetilde{\mathbf{B}} \widetilde{u}(t)+\widetilde{\mathbf{C}} \widetilde{y}(t) \diamond W^{(H)}(t)\right) \\
& =\mathscr{M}(\widetilde{\mathbf{A}} \widetilde{y})+\mathscr{M}(\widetilde{\mathbf{B}} \widetilde{u})+\mathscr{M}(\widetilde{\mathbf{C}} \tilde{y}) \diamond \mathscr{M}\left(W_{t}^{(H)}\right)=\mathbf{A} y+\mathbf{B} u+\mathbf{C} y \diamond W_{t},
\end{aligned}
$$

where $y$ and $u$ are the associated processes to $\tilde{y}$ and $\tilde{u}$. Moreover, by Theorem 1.11 part $3^{\circ}$ and (1.55) the operator $\mathscr{M}$ transforms the cost functional $\mathbf{J}^{(H)}$ to

$$
\mathscr{M}\left(\mathbf{J}^{(H)}(\widetilde{u})\right)=\mathscr{M}\left(\mathbb{E}_{\mu_{H}}(\widetilde{v})\right)=\mathbb{E}_{\mu}(\mathscr{M} \widetilde{v})=\mathbb{E}_{\mu}(v)=\mathbf{J}(u)
$$

where $\widetilde{v}$ and $v$ are associated elements $\widetilde{v}=\int_{0}^{T}\left(\|\widetilde{\mathbf{R}} \widetilde{y}\|_{\mathscr{H}}^{2}+\|\widetilde{u}\|_{\mathscr{U}}^{2}\right) d t+\left\|\widetilde{\mathbf{G}}^{2} \tilde{y}_{T}\right\|_{\mathscr{H}}^{2}$ and $v=\int_{0}^{T}\left(\|\mathbf{R} y\|_{\mathscr{H}}^{2}+\|u\|_{\mathscr{U}}^{2}\right) d t+\left\|\mathbf{G} y_{T}\right\|_{\mathscr{H}}^{2}$.

Therefore, the fractional optimal control (4.54)-(4.55) has an optimal control represented in the feedback form. The optimal solution is obtained from Theorems 4.2 and 4.7 via the inverse fractional mapping $\mathscr{M}^{-1}$.

Remark 4.8 We note that in a similar way one can also solve fractional problem in the classical Gaussian white noise space by using the isometry mapping $\mathbf{M}$ from Definition 1.22.

### 4.6 Numerical Approximation

In this section we provide numerical approximations of two types of stochastic equations, elliptic and parabolic. Particularly, we study a stationary version of the Eq. (3.18) and a form of the initial value problem (4.1) with delta noise. By employing the method of chaos expansions the initial equation is transformed to a system of infinitely many deterministic equations, which is then truncated to a finite system of equations, solvable by standard numerical techniques. With this approach, the moments of the solution can be computed. Parts of this section appeared in [29].

### 4.6.1 Elliptic Equation

We consider the stationary equation with random coefficients

$$
\begin{equation*}
G \diamond \mathbf{A} u=h, \quad \mathbb{E} u=\tilde{u}_{0} \tag{4.57}
\end{equation*}
$$

Assume that $\mathbf{A}$ is a simple coordinatewise operator that corresponds to $A_{\alpha}=\Delta$, $\alpha \in \mathscr{I}$, the Laplace operator in two spatial dimensions and $G$ is a Gaussian random variable. Let $\mathscr{D}=\{(x, y):-1 \leq x \leq 1,-1 \leq y \leq 1\}$ be the spatial domain and let
$G=g_{0}+\sum_{k \in \mathbb{N}} g_{k} H_{\varepsilon^{(k)}}$ such that $\mathbb{E} G=g_{0}=10$ and $\operatorname{Var} G=\sum_{k \in \mathbb{N}} g_{k}^{2}-g_{0}^{2}=$ $3.3^{2}$.

We seek for $u(x, y, \omega)=\sum_{\alpha \in \mathscr{I}} u_{\alpha}(x, y) H_{\alpha}(\omega)$ which solves

$$
\begin{equation*}
G(\omega) \diamond \sum_{\alpha \in \mathscr{I}} \Delta u(x, y) H_{\alpha}(\omega)=h(x, y, \omega), \quad(x, y) \in \mathscr{D}, \omega \in \Omega \tag{4.58}
\end{equation*}
$$

In order to solve (4.58) numerically, we use finite dimensional approximations of $u$ in the Fourier-Hermite orthogonal polynomial basis. Similar techniques can be found in [22, 24, 46]. The main steps of the unified approach, based on chaos expansions and numerical scheme we are using, are sketched in Algorithm 4.6.1.

```
Algorithm 4.6.1 Main steps of the numerical scheme
    : Choose finite set of polynomials \(H_{\alpha}\) and truncate the random series to a finite random sum.
    Solve numerically the deterministic triangular system of equations by a suitable method.
    Compute the approximate statistics of the solutions from obtained coefficients.
    Generate \(H_{\alpha}\) and compute the approximate solution of the initial equation.
```

We denote by $\mathscr{I}_{m, p}$ the set of multi-indices of the form $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}, 0, \ldots\right) \in \mathscr{I}$ with $m=\max \left\{i \in \mathbb{N}: \alpha_{i} \neq 0\right\}$ such that $|\alpha| \leq p$. As first step, we represent the solution $u$ in its truncated chaos expansion form $u^{m, p}$, i.e., we approximate the solution by the chaos expansion in $\oplus_{k=0}^{p} \mathscr{H}_{k}$ with $m$ random variables

$$
\begin{equation*}
u^{m, p}(x, y, \omega)=\sum_{\alpha \in \mathscr{I}_{m, p}} u_{\alpha}^{m, p}(x, y) H_{\alpha}(\omega) \tag{4.59}
\end{equation*}
$$

The finite dimensional approximation (4.59) has exactly $P=\frac{(m+p)!}{m!p!}$ summands. The choice of $m$ and $p$ influences the accuracy of the approximation. They can be chosen so that the norm of the difference $u-u^{m, p}=\sum_{\alpha \in \mathscr{I} \backslash \mathscr{I}_{m, p}} u_{\alpha} H_{\alpha}$ is smaller then a prespecified error [24]. Knowing $m$ and $p$, the finite dimesional approximation of (4.58) leads to
$g_{0} \sum_{\alpha \in \mathscr{I}_{m, p}} \Delta u_{\alpha}^{m, p}(x, y) H_{\alpha}(\omega)+\sum_{\alpha \in \mathscr{I}_{m, p}} \sum_{k=1}^{m} g_{k} \Delta u_{\alpha}^{m, p}(x, y) H_{\alpha+\varepsilon^{(k)}}(\omega)=\sum_{\alpha \in \mathscr{I}_{m, p}} h_{\alpha} H_{\alpha}(\omega)$.
The unknown coefficients $u_{\alpha}^{m, p}, \alpha \in \mathscr{I}_{m, p}$ are obtained by the projection onto each element of the Fourier-Hermite basis $\left\{H_{\gamma}\right\}, \gamma \in \mathscr{I}_{m, p}$, i.e., by taking the expectations for all $\gamma \in \mathscr{I}_{m, p}$. Then, we obtain a triangular system of $P$ deterministic equations

$$
\begin{align*}
g_{0} \Delta u_{\mathbf{0}}^{m, p}=h_{\mathbf{0}}, & \text { for }|\alpha|=0  \tag{4.60}\\
g_{0} \Delta u_{\alpha}^{m, p}+\sum_{k=1}^{m} \Delta u_{\alpha-\varepsilon^{(k)}}^{m, p} g_{k}=h_{\alpha}, & 0<|\alpha| \leq p \tag{4.61}
\end{align*}
$$

that can be solved by induction on $|\alpha|$. The system (4.61) can be rewritten by recursive replacements of $\Delta u_{\alpha-\varepsilon^{(k)}}^{m, p}$ from lower order equations. Hence, we obtain

$$
\begin{aligned}
\Delta u_{\varepsilon^{(k)}}^{m, p} & =\frac{1}{g_{0}^{2}}\left(g_{0} h_{\varepsilon^{(k)}}-h_{0} g_{k}\right), \quad \text { for } \alpha=\varepsilon^{(k)} \\
\Delta u_{2 \varepsilon^{(k)}}^{m, p} & =\frac{1}{g_{0}^{3}}\left(g_{0}^{2} h_{2 \varepsilon^{(k)}}-g_{0} g_{k} h_{\varepsilon^{(k)}}+h_{0} g_{k}^{2}\right), \quad \text { for } \alpha=2 \varepsilon^{(k)} \\
\Delta u_{\varepsilon^{(k)}+\varepsilon^{(j)}}^{m, p} & =\frac{1}{g_{0}^{3}}\left(g_{0}^{2} h_{\varepsilon^{(k)}+\varepsilon^{(j)}}-g_{0} g_{k} h_{\varepsilon^{(j)}}-g_{0} g_{j} h_{\varepsilon^{(k)}}+h_{0} g_{k} g_{j}\right), \text { for } \alpha=2 \varepsilon^{(k)}
\end{aligned}
$$

Thus, all $P$ equations in the obtained system are Poisson equations

$$
\Delta u_{\alpha}^{m, p}=\frac{1}{g_{0}^{|\alpha|+1}} S(g, h), \quad \alpha \in \mathscr{I}_{m, p}
$$

where $S(g, h)$ is the convolution sum with terms of the form $g_{k_{1}}^{s_{1}} g_{k_{2}}^{s_{2}} \ldots g_{k_{n}}^{s_{n}} h_{\beta}$, for $\alpha=\beta+s_{1} \varepsilon^{\left(k_{1}\right)}+s_{2} \varepsilon^{\left(k_{2}\right)}+\cdots+s_{n} \varepsilon^{\left(k_{n}\right)}$. Next, we apply an appropriate numerical method to solve the obtained system (usually finite difference method, finite elements method or a combination of these two methods). As outcome we obtain the discretized approximation solution $u_{\alpha}^{m, p, d}, \alpha \in \mathscr{I}_{m, p}$ of the system and therefore, by recovering the random field, the calculated approximated solution

$$
u^{m, p, d}=\sum_{\alpha \in \mathscr{I}_{m, p}} u_{\alpha}^{m, p, d} H_{\alpha}
$$

of (4.58). The global error of the proposed numerical scheme depends on the error $\mathscr{E}_{1}$ generated by the truncation of the Wiener-Itô chaos expansion (4.59) and the error $\mathscr{E}_{2}$ influenced by the discretisation method, i.e.,

$$
u-u^{m, p, d}=\sum_{\alpha \in \mathscr{I} \backslash \mathscr{I}_{m, n}} u_{\alpha} H_{\alpha}+\sum_{\alpha \in \mathscr{I}_{m, n}}\left(u_{\alpha}^{m, p}-u_{\alpha}^{m, p, d}\right) H_{\alpha}
$$

We underline that the chaos expansion converges quite fast, i.e., even small values of $p$ may lead to very accurate approximation [24, 46]. The error $\mathscr{E}_{1}$ generated by the truncation of the Wiener-Itô chaos expansion, in $C^{2}(\mathscr{D}) \otimes L^{2}(\mu)$ is

$$
\mathscr{E}_{1}^{2}=\sum_{\alpha \in \mathscr{I} \backslash \mathscr{I}_{m, n}}\left\|u_{\alpha}^{m, p}(x, y)\right\|_{C^{2}(\mathscr{D})}^{2} \alpha!=\sum_{\alpha \in \mathscr{I} \backslash \mathscr{I}_{m, n}} \alpha!\iint_{\mathscr{D}} u_{\alpha}^{m, p}(x, y)^{2} d x d y
$$

while the error coming from discretization


Fig. 4.1 Expected value (a) and variance (b) of the solution

$$
\mathscr{E}_{2}^{2}=\sum_{\alpha \in \mathscr{I}_{m, n}}\left\|u_{\alpha}^{m, p}(x, y)-u_{\alpha}^{m, p, d}(x, y)\right\|_{C^{2}(\mathscr{D})}^{2} \alpha!
$$

depends on the numerical method performed.
For illustration, we take $m=15, p=3$ and then obtain $P=816$ deterministic equations in the system. We assume $h_{\alpha}=1$ for $|\alpha| \leq 3$ and $h_{\alpha}=0$ for $|\alpha|>3$. We use central differencing to discretize in the spatial dimensions and 170 grid cells in each spacial direction. Then, we solve numerically the resulting system (4.60)-(4.61). Once the coefficients of the expansion $u^{m, p, d}$ are obtained, we are able to compute all the moments of the random field. Particularly, the expectation $\mathbb{E} u=\mathbb{E} u^{m, p, d}=u_{0}$ and the variance $\operatorname{Var} u^{m, p, d}=\sum_{\alpha \in \mathscr{I}_{m, p}} \alpha!\left(u_{\alpha}^{m, p, d}\right)^{2}$ are plotted in Fig.4.1, on $z-$ axes over the domain $\mathscr{D}$. We can observe that the variance of the solution is relatively high. In general, this behavior is related to singularities.

### 4.6.1.1 3D Simulation

We perform a three dimensional simulation of (4.57), following the steps in Algorithm 4.6.1. The additional dimension increase the computational cost significantly, thus we perform the simulations on a coarser spatial grid. In Figs. 4.2 and 4.3 we plotted level sets (LS) at $t=0.808$ and $t=0.408$ of the expected value and variance of the solution by the polynomial chaos expansion approach (left) and Monte Carlo (MC) simulations (right). The error is plotted on the bottom.

Expected value of the solution LS


Variance of the solution LS


Error of the expepected value of the solution LS


Expected value of the solution LS MC


Variance of the solution LS MC


Error of the variance of the solution LS


Fig. 4.2 Level sets (LS) at $t=0.408$ of the expected value and variance of the solution by polynomial chaos (left) and Monte Carlo (MC) simulations (right). The error is plot it on the bottom

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Expected value of the solution LS


Variance of the solution LS


Error of the expepected value of the solution LS


Expected value of the solution LS MC


Variance of the solution LS MC


Error of the variance of the solution LS


Fig. 4.3 Level sets (LS) at $t=0.808$ of the expected value and variance of the solution by polynomial chaos (left) and Monte Carlo (MC) simulations (right). The error is plot it on the bottom
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Fig. 4.4 Expected values of the solution, at $t=0.088$ and $t=0.408$, of the equation with additive noise and Delta noise by polynomial chaos (left) and Monte Carlo (MC) simulations (right)

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Fig. 4.5 Variance of the solution, at $t=0.088$ and $t=0.408$, of the equation with additive noise and Delta noise by polynomial chaos (left) and Monte Carlo (MC) simulations (right)


Fig. 4.6 Error of expected values and variance of the solution at $t=0.408$ of the equation with additive noise and Delta noise with respect to Monte Carlo simulations

### 4.6.2 Parabolic Equation

Let us consider a general stochastic Cauchy problem

$$
\begin{align*}
\dot{u}(t, x, \omega) & =\mathbf{L} u(t, x, \omega)+\mathbf{T} \diamond u(t, x, \omega)+\delta(u(t, x, \omega)), \quad \text { in }(0, T]  \tag{4.62}\\
u(0, x, \omega) & =u_{0}(x, \omega),
\end{align*}
$$

where $\mathbf{L}$ generates a $C_{0}$-semigroup, $\mathbf{T}$ is a linear bounded operator defined by (1.62) and $\delta$ denotes the stochastic integral of Itô-Skorokhod type. In [37] the authors proved that (4.62) has a solution that can be represented in the chaos expansion form and stated its explicit form. Therefore, we can assume that the solution of (4.62) has a chaos expansion representation form and we perform the steps of Algorithm 4.6.1, as in Sect. 4.6.1.

In order to simplify the computations we consider $\mathbf{T}=0$, a two dimensional domain $\Omega=(-1,1)^{2}$ and $t \in(0,1)$, and $\mathbf{L}$ to be a simple coordinatewise operator that corresponds to $L\left(\partial_{x}, \partial_{y}\right)=\partial_{x}\left(a(x, y) \partial_{x}\right)+\partial_{y}\left(b(x, y) \partial_{y}\right)$, where $a(x, y)=$ $b(x, y)=1$. In addition, we compute Monte Carlo simulations to estimate the error. In the sake of comparison we also consider the case with Gaussian additive noise.

For all plots we consider the time instances $t=0.088$ and $t=0.408$. In Fig. 4.4, we plot expected values of the solution of the equation with additive noise and Delta noise by the polynomial chaos (left) and Monte Carlo (MC) simulations (right). In the same setting in Fig. 4.5, we plot Variance of the solution. In both cases, expected values and variances, we can observe that the results computed by the polynomial chaos and Monte Carlo simulations have the same behavior. Finally, using Monte Carlo simulations as true solution we compute the errors in Fig.4.6.

In this section we consider linear elliptic and parabolic stochastic partial differential equations. Recently, a new approach to approximate numerically certain types of these classes of equations have been proposed. This approach relies on a solving large-scale differential Lyapunov equation for the covariance [41]. As differential Lyapunov equations can be solved efficiently [2, 11, 40], this approach seems to have a great potential in applications, e.g. approximating the quasi-periodic climate pattern in the tropical Pacific Ocean known as El Niño phenomenon [41].

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[^0]:    2010 Mathematics Subject Classification. 60H40; 60H07; 60H10; 60G20
    Keywords. White noise space, series expansion, Hida space, Kondratiev space, Malliavin derivative, Skorokhod integral, OrnsteinUhlenbeck operator.

    Received: 30 May 2016; Accepted: 10 June 2016
    Communicated by Dragan S. Djordjević
    The paper was supported by the Ministry of Education, Science and Technological Development, Republic of Serbia.
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[^6]:    2010 Mathematics Subject Classification. Primary: 34H05, 49J55; Secondary: 15A24, 60H40, 60H30, 93E20.

    Key words and phrases. Stochastic linear quadratic optimal control problem, white noise analysis, polynomial chaos expansion method, Itô-Skorokhod integral, Riccati equations.
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[^7]:    *Received by the editors January 11, 2016; accepted for publication (in revised form) November 28, 2016; published electronically March 2, 2017.
    http://www.siam.org/journals/sicon/55-2/M105618.html
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[^10]:    Communicated by A. Constantin.
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