arXiv:1907.13420v2 [math.OC] 10 Feb 2020

A simplified derivation technique of topological derivatives for quasi-linear transmission problems

Peter Gangl* and Kevin Sturm†

February 11, 2020

Abstract

In this paper we perform the rigorous derivation of the topological derivative for optimization problems constrained by a class of quasi-linear elliptic transmission problems. In the case of quasi-linear constraints, techniques using fundamental solutions of the differential operators cannot be applied to show convergence of the variation of the states. Some authors succeeded showing this convergence with the help of technical computations under additional requirements on the problem. Our main objective is to simplify and extend these previous results by using a Lagrangian framework and a projection trick. Besides these generalisations the purpose of this manuscript is to present a systematic derivation approach for topological derivatives.

2010 Mathematics Subject Classification: Primary 49Q10; Secondary 49Qxx,90C46.
Keywords: topological derivative; quasi-linear problems; topology optimisation; asymptotic analysis; adjoint approach.

1 Introduction

The topological derivative of a shape functional $J = J(\Omega)$, where $\Omega \subset \mathbf{R}^d$, measures the sensitivity of the functional with respect to a topological perturbation of the shape Ω. The concept was first used in [11] in the context of linearized elasticity as a means to find optimal locations for introducing holes into an elastic structure. Later, the concept was introduced in a mathematically rigorous way in [20]. In the literature many research articles deal with the derivation of topological sensitivities of optimization problems which are constrained by linear partial differential equations (PDEs). We refer the reader to [2] as well as the monograph [17, pp. 3] and references therein. The topological derivative for a class of semilinear PDEs with the Laplace operator as the principal part was studied in [3, 14], and more recently in [21] using an averaged adjoint framework.

As it is mentioned in the recent book [18, Sec. 6.4, p.107],

*TU Graz, Steyrergasse 30/III, 8010 Graz, Austria, gangl(at)math.tugraz.at
†TU Wien, Wiedner Hauptstr. 8-10, 1040 Vienna, Austria, E-Mail: kevin.sturm(at)tuwien.ac.at

> "Extension to nonlinear problems in general can be considered the main challenge in the theoretical development of the topological derivative method. The difficulty arises when the nonlinearity comes from the main part of the operator, which at the same time suffers a topological perturbation."

This statement applies in particular to quasi-linear PDEs when the main part of the differential operator gets topologically perturbed. In this case, techniques based on fundamental solutions, as they are heavily used in the linear and semi-linear case, cannot be applied any more and other strategies have to be followed.

The first rigorous results of topological sensitivity analysis for shape functions constrained by quasi-linear PDEs were obtained in [4] where the authors consider a regularized version of the p-Poisson equation. Based on these results, the topological derivative for the quasi-linear equation of 2D magnetostatics was derived in [5] where also the numerical treatment of the obtained formula was addressed.

In this paper, we establish the topological derivative for a larger class of quasi-linear problems under more general assumptions. More precisely, given a fixed, open and bounded hold-all domain D and an open and measurable subset $\Omega \subset \mathsf{D}$, we study the topological sensitivity analysis of the tracking-type cost function

$$J(\Omega) = \int_{\mathsf{D}} |\nabla(u - u_d)|^2 \, dx \tag{1.1}$$

subject to the constraint that $u \in H_0^1(\mathsf{D})$ solves

$$\int_{\mathsf{D}} \mathscr{A}_\Omega(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\mathsf{D}} f \varphi \, dx \quad \text{for all } \varphi \in H_0^1(\mathsf{D}). \tag{1.2}$$

Here, $f \in L_2(\mathsf{D})$, $u_d \in H_0^1(\mathsf{D})$ and $\mathscr{A}_\Omega : \mathsf{D} \times \mathbf{R}^d \to \mathbf{R}^d$ is a piecewise nonlinear function defined by

$$\mathscr{A}_\Omega(x, y) := \begin{cases} a_1(y) & \text{for } x \in \Omega \\ a_2(y) & \text{for } x \in \mathsf{D} \setminus \Omega, \end{cases} \tag{1.3}$$

with $a_1, a_2 : \mathbf{R}^d \to \mathbf{R}^d$ being functions satisfying monotonicity and continuity assumptions.

The crucial ingredient for our result is the strong convergence (Theorem 4.3) of the variation of the direct states,

$$\nabla \left(\frac{(u_\varepsilon - u_0) \circ T_\varepsilon}{\varepsilon} \right) \to \nabla K \quad \text{strongly in } L_2(\mathbf{R}^d)^d, \tag{1.4}$$

where u_ε and u_0 correspond to the solutions to the perturbed and unperturbed state equation, respectively. As shown in [21], for semilinear problems only weak convergence in (1.4) is necessary to establish the topological derivative. For quasi-linear problems we need the strong convergence (1.4). The main contributions of this work are as follows:

- simplified analysis for derivation of topological derivative for quasi-linear equations
- generalisation of previous results
- relaxation of smoothness assumption on inclusion ω

The presented approach for deriving the topological derivative under a quasi-linear PDE constraint simplifies and also generalizes the approaches presented in [4] and [5] which is the subject of the following discussion.

The main difference between the presented approach and the results obtained in [4, 5] lies in the technique used to show (1.4), i.e. the strong convergence of the variation of the state on the rescaled bounded domain to the solution K of a transmission problem on the unbounded domain as $\varepsilon \to 0$. While, in our approach, this convergence is accomplished by the introduction of a projection $\hat{K}_\varepsilon$ of K into the space $H_0^1(\varepsilon^{-1}\mathsf{D})$, the main ingredient used in [4, 5] is a cut-off argument relying on explicit knowledge of the asymptotic behavior of K as $|x| \to \infty$. The authors successfully showed the necessary decay of K by a comparison principle. However, this was achieved using long, technical calculations which additionally required stronger assumptions on the data compared to what is presented here.

In particular, the authors of [4, 5] have to assume that $\omega = B_1(0)$ is the unit ball, whereas our approach remains valid for any open and bounded set $\omega \subset \mathbf{R}^d$ with $0 \in \omega$. Furthermore, the assumptions on the class of quasi-linear PDEs used here, i.e. Assumption A, are less restrictive than those used in [4, 5]. In particular, the proof technique used there requires the third derivative of the operator a_i to be bounded (see p.75 in [6] or Assumption 3.3. in [5]) and, in the case of [5], an additional nonphysical assumption on the materials which is not necessarily satisfied in practice (see Assumption 3.4 in [5]). We remark that the setting of [5] is fully covered in our analysis without this nonphysical assumption.

Moreover, in both [4] and [5], the case $z \in \Omega$ and $z \in \mathsf{D} \setminus \overline{\Omega}$ have to be treated separately. This is not addressed in [4] where the proof relies on $\gamma_1 < \gamma_0$, and is carried out by repeating adaptations of the technical proofs in the setting of [5], see also [12, Sec. 4.5]. In the approach presented here, it is enough to interchange the roles of a_1 and a_2 to get to the topological derivative for the other scenario.

Finally, we will also show how to treat objective functionals of the form

$$\int_D |u - u_d|^2 \, \mathrm{d}x,$$

in Section 5, which is not covered by the analysis shown in [4] and [5].

The rest of this paper is organized as follows: In Section 2 we state the main assumptions and the main result. The remaining sections are devoted to the proof of this result. In Section 3, we recall and extend results from an abstract Lagrangian framework that will be used to derive the topological derivative. In Section 4 we show that the hypotheses of the abstract theorem are satisfied and obtain the final formula. In Section 5 we compare the Lagrangian framework of Section 3 with the averaged framework.

2 Assumptions and main results

2.1 Preliminaries: notation and definitions

Function spaces Standard L^p spaces and Sobolev spaces on an open set $\mathsf{D} \subset \mathbf{R}^d$ are denoted $L_p(\mathsf{D})$ and $W_p^k(\mathsf{D})$, respectively, where $p \geq 1$ and $k \geq 1$. In case $p = 2$ and $k \geq 1$ we set as usual $H^k(\mathsf{D}) := W_2^k(\mathsf{D})$. Vector valued spaces are denoted $L_p(\mathsf{D})^d := L_p(\mathsf{D},\mathbf{R}^d)$ and $W_p^k(\mathsf{D})^d := W_p^k(\mathsf{D},\mathbf{R}^d)$. We denote by $H_0^1(\mathsf{D})$ the subspace of functions in $H^1(\mathsf{D})$ with vanishing trace on $\partial\mathsf{D}$. Given a normed vector space V we denote by $\mathscr{L}(V,\mathbf{R})$ the space of linear and continuous functions on V. We denote by $B_\delta(x)$ the ball centred at x with radius $\delta > 0$ and set $\bar{B}_\delta(x) := \overline{B_\delta(x)}$. For the ball centered at $x = 0$ we write $B_\delta := B_\delta(0)$.

For $d \geq 1$ and $1 \leq p < \infty$, we set $BL_p(\mathbf{R}^d) := \{u \in W^1_{p,\mathrm{loc}}(\mathbf{R}^d) : \nabla u \in L_p(\mathbf{R}^d)^d\}$ and define the *Beppo-Levi space* as the quotient space $\dot{BL}_p(\mathbf{R}^d) := BL_p(\mathbf{R}^d)/\mathbf{R}$, where $/\mathbf{R}$ means that we quotient out the constant functions. We denote by $[u]$ the equivalence classes of $\dot{BL}(\mathbf{R}^d)$. Equipped with the norm

$$\|[u]\|_{\dot{BL}_p(\mathbf{R}^d)} := \|\nabla u\|_{L_p(\mathbf{R}^d)^d}, \quad u \in [u], \tag{2.1}$$

the Beppo-Levi space is a Banach space (see [9, 19]) and $C_c^\infty(\mathbf{R}^d)/\mathbf{R}$ is dense in $\dot{BL}_p(\mathbf{R}^d)$. In case $p = 2$ the space $\dot{BL}(\mathbf{R}^d)$ becomes a Hilbert space and we abbreviate this space simply with $\dot{BL}(\mathbf{R}^d)$.

Moreover, we write $⨍_A f\, dx := \frac{1}{|A|}\int_A f\, dx$ to indicate the average of f over a measurable set A with measure $|A| < \infty$. We equip $\mathbf{R}^d$ with the Euclidean norm $|\cdot|$ and use the same notation for the corresponding matrix (operator) norm.

Definition of topological derivative Before we state our main result we recall the definition of the topological derivative. We restrict ourselves to the special case as it was introduced in [20] and refer the reader to [17, pp. 4] for the more general definition.

Definition 2.1 (Topological derivative)**.** Let $\mathsf{D} \subset \mathbf{R}^3$ be an open set and $\Omega \subset \mathsf{D}$ an open subset. Let $\omega \subset \mathbf{R}^3$ be open with $0 \in \omega$. Define for $z \in \mathbf{R}^3$, $\omega_\varepsilon(z) := z + \varepsilon\omega$. Then the topological derivative of J at Ω at the point $z \in \mathsf{D} \setminus \partial\Omega$ is defined by

$$dJ(\Omega)(z) = \begin{cases} \lim_{\varepsilon\searrow 0} \frac{J(\Omega\setminus\omega_\varepsilon(z))-J(\Omega)}{|\omega_\varepsilon(z)|} & \text{if } z \in \Omega, \\ \lim_{\varepsilon\searrow 0} \frac{J(\Omega\cup\omega_\varepsilon(z))-J(\Omega)}{|\omega_\varepsilon(z)|} & \text{if } z \in \mathsf{D}\setminus\overline{\Omega}. \end{cases} \tag{2.2}$$

Without loss of generality, we will restrict ourselves to the second case and will always assume $z \in D \setminus \overline{\Omega}$. The derivation for the case $z \in \Omega$ is analogous, cf. Remark 2.3.

2.2 Main results

We need the following assumptions:

Assumption A. There are constants c_1, c_2, c_3 such that the functions $a_i : \mathbf{R}^d \to \mathbf{R}^d$, $i = 1, 2$ are differentiable and satisfy:

(i) $(a_i(x) - a_i(y)) \cdot (x - y) \geq c_1|x - y|^2 \quad$ for all $x, y \in \mathbf{R}^d$.

(ii) $|a_i(x) - a_i(y)| \leq c_2|x - y| \quad$ for all $x, y \in \mathbf{R}^d$.

(iii) $|\partial a_i(x) - \partial a_i(y)| \leq c_3|x - y| \quad$ for all $x, y \in \mathbf{R}^d$.

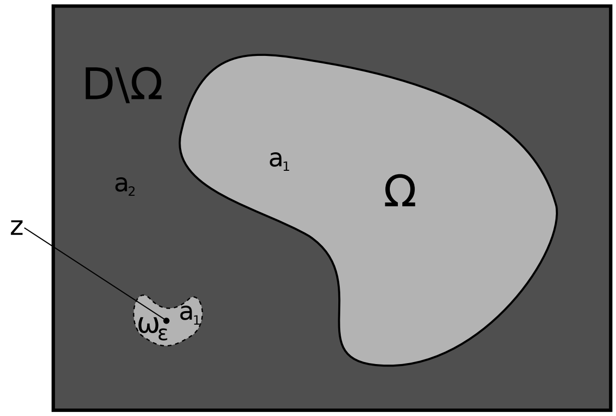

Figure 1: Setting for topological derivative: Inclusion ω_ε of radius $\varepsilon > 0$ containing material a_1 around point $z \in \mathsf{D} \setminus \overline{\Omega}$ (where material a_2 is present).

Remark 2.2. By using the inverse triangle inequality and choosing $y = 0$, we get from Assumption A(ii) and (iii) that

$$|a_i(x)| \le |a_i(0)| + c_2|x|, \tag{2.3}$$

$$|\partial a_i(x)| \le |\partial a_i(0)| + c_3|x|, \tag{2.4}$$

for $i = 1, 2$ and for all $x \in \mathbf{R}^d$. Notice also that using (ii), we get

$$|\partial a_i(x)v| = \lim_{t \searrow 0} |a_i(x + tv) - a_i(x)|/t \le c_2|v|, \tag{2.5}$$

for $i = 1, 2$ and all $x, v \in \mathbf{R}^d$.

Properties (i) and (ii) of Assumption A imply that the operator $A_\Omega : H^1_0(\mathsf{D}) \to (H^1_0(\mathsf{D}))^*$ defined by $\langle A_\Omega \varphi, \psi \rangle := \int_{\mathsf{D}} \mathscr{A}_\Omega(x, \nabla\varphi) \cdot \nabla\psi \, dx$ is Lipschitz continuous and strongly monotone for all measurable $\Omega \subset \mathsf{D}$. Hence the state equation (1.2) admits a unique solution by the theorem of Zarantonello; see [22, p.504, Thm. 25.B].

We restrict ourselves to Dirichlet boundary conditions but also other boundary conditions, e.g., Neumann boundary conditions, could be considered. In what follows at many places we extend functions $u \in H^1_0(\Omega)$ to function $\tilde{u} \in H^1(\mathbf{R}^d)$ by setting u to zero outside of D. When dealing with other boundary conditions, we can replace this extension by the standard Sobolev extension operator $E : H^1(\mathsf{D}) \to H^1(\mathbf{R}^d)$.

Before we state our main result we introduce the adjoint $p \in H^1_0(\mathsf{D})$ as the solution to

$$\int_{\mathsf{D}} \partial_u \mathscr{A}_\Omega(x, \nabla u)(\nabla\varphi) \cdot \nabla p \, dx = -\int_{\mathsf{D}} 2\nabla(u - u_d) \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in H^1_0(\mathsf{D}). \tag{2.6}$$

In view of the monotonicity of $\mathscr{A}_\Omega$ the previous equation has according to Lax-Milgram a unique solution in $H^1_0(\mathsf{D})$.

We fix the following setting for the topological perturbation (cf. Figure 1):

- an open and bounded set $\omega \subset \mathbf{R}^d$ with $0 \in \omega$,

- an open set $\Omega \Subset \mathsf{D}$ and the inclusion point $z := 0 \in \mathsf{D} \setminus \overline{\Omega}$,
- the perturbation $\omega_\varepsilon(z) := \varepsilon\omega$ and $\varepsilon \in [0,\tau]$, where $\tau > 0$ is such that $\omega_\varepsilon(z) \Subset \mathsf{D} \setminus \overline{\Omega}$ for all $\varepsilon \in [0,\tau]$.
- the perturbed shape $\Omega_\varepsilon(z) := \Omega \cup \omega_\varepsilon(z)$
- $T_\varepsilon(x) := \varepsilon x$, $x \in \mathbf{R}^d$, $\varepsilon \geq 0$

To simplify notation we will often write ω_ε instead of $\omega_\varepsilon(z)$, Ω_ε instead of $\Omega_\varepsilon(z)$ and x_ε instead of $T_\varepsilon(x)$. For $\varepsilon > 0$ we introduce the notation $\varepsilon^{-1}\mathsf{D} := T_\varepsilon^{-1}(\mathsf{D})$.

Let $\ell(\varepsilon) := |\omega_\varepsilon|$, and introduce the Lagrangian $G : [0,\tau] \times H_0^1(\mathsf{D}) \times H_0^1(\mathsf{D}) \to \mathbf{R}$ associated with the perturbation ω_ε by

$$G(\varepsilon,u,p) := \int_{\mathsf{D}} |\nabla(u-u_d)|^2\,dx + \int_{\mathsf{D}} \mathscr{A}_{\Omega_\varepsilon}(x,\nabla u)\cdot\nabla p\,dx - \int_{\mathsf{D}} f p\,dx. \tag{2.7}$$

Here, the operator $\mathscr{A}_{\Omega_\varepsilon}$ is defined according to (1.3) with $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon$.

Now we can state our main result of this paper:

Main Theorem. Let Assumption A be satisfied. Let $\Omega \subset D$ open and u_0 the solution to (1.2) and p_0 the solution to (2.6). Let $z \in \mathsf{D} \setminus \overline{\Omega}$ and assume that $u_0 \in C^{1,\alpha}(\overline{B_\delta(z)})$ and $p_0 \in C^1(\overline{B_\delta(z)})$ for some $\delta > 0$ and $0 < \alpha < 1$. Assume further that $\nabla p_0 \in L_\infty(\mathsf{D})^d$.

(a) Then the assumptions of Theorem 3.4 are satisfied for the Lagrangian G given by (2.7) and hence the topological derivative at $z \in \mathsf{D} \setminus \overline{\Omega}$ is given by

$$dJ(\Omega)(z) = \partial_\ell G(0,u_0,p_0) + R_1(u_0,p_0) + R_2(u_0,p_0) \tag{2.8}$$

(b) We have

$$\partial_\ell G(0,u_0,p_0) = ((a_1(U_0) - a_2(U_0))\cdot P_0 \tag{2.9}$$

and

$$R_1(u_0,p_0) = \frac{1}{|\omega|}\bigg(\int_{\mathbf{R}^d} \big[\mathscr{A}_\omega(x,\nabla K + U_0) - \mathscr{A}_\omega(x,U_0) - \partial_u\mathscr{A}_\omega(x,U_0)(\nabla K)\big]\cdot P_0\,dx + \int_{\mathbf{R}^d} |\nabla K|^2\,dx\bigg) \tag{2.10}$$

and

$$R_2(u_0,p_0) = \frac{1}{|\omega|}\int_\omega [\partial_u a_1(U_0) - \partial_u a_2(U_0)](\nabla K)\cdot P_0\,dx \tag{2.11}$$

where $U_0 := \nabla u_0(z)$, $P_0 := \nabla p_0(z)$ and $\mathscr{A}_\omega(x,y) := a_1(y)\chi_\omega(x) + a_2(y)\chi_{\mathbf{R}^d\setminus\omega}(x)$. Here $K \in \dot{B}L(\mathbf{R}^d)$ is the unique solution to

$$\begin{aligned}\int_{\mathbf{R}^d} &(\mathscr{A}_\omega(x,\nabla K + U_0) - \mathscr{A}_\omega(x,U_0))\cdot\nabla\varphi\,dx \\ &= -\int_\omega (a_1(U_0) - a_2(U_0))\cdot\nabla\varphi\,dx \quad \text{for all } \varphi \in BL(\mathbf{R}^d).\end{aligned} \tag{2.12}$$

Remark 2.3. We restrict ourselves to the case where $z \in D \setminus \overline{\Omega}$ without loss of generality. However, the exact same proof can be conducted in the case where $z \in \Omega$. In that case, the formula for the topological derivative is obtained by just switching the roles of a_1 and a_2 in the theorem above (in particular also in the definition of $\mathscr{A}_\omega$).

The assumption $z = 0$ is without loss of generality, too. In the general case, this situation can be obtained by a simple change of the coordinate system.

Remark 2.4. Although we assume $f \in L_2(\mathsf{D})$, also more general right hand sides, such as $f_\Omega := \chi_\Omega f_1 + \chi_{\mathsf{D}\setminus\Omega} f_2$ with $f_1, f_2 \in L_2(\mathsf{D})$ could be considered with minor changes.

Remark 2.5. We note that in [4] the topological derivative for a quasi-linear problem in L_p spaces is considered. We believe that our analysis can also be transferred to this setting.

Remark 2.6. Although we did not treat the limiting case where a_1 or a_2 is zero, this can be done in a similar fashion. We refer to Section 5 in [21]. Dirichlet conditions on the inclusion using our approach have to be studied in different manner and deserve further research.

3 Lagrangian framework

In this section we recall results on a Lagrangian framework, which is a suitable refinement of [7]. These abstract results will be used to derive the topological derivative for our quasi-linear model problem. We begin with the definition of a Lagrangian function; see also [8].

Definition 3.1 (parametrised Lagrangian)**.** Let X and Y be vector spaces and $\tau > 0$. A parametrised Lagrangian (or short Lagrangian) is a function

$$(\varepsilon, u, p) \mapsto G(\varepsilon, u, p) : [0, \tau] \times X \times Y \to \mathbf{R},$$

satisfying,

$$p \mapsto G(\varepsilon, u, p) \quad \text{is affine on } Y. \tag{3.1}$$

Definition 3.2 (state and adjoint state)**.** Let $\varepsilon \in [0, \tau]$. We define the state equation by: find $u_\varepsilon \in X$, such that

$$\partial_p G(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in Y. \tag{3.2}$$

The set of states is denoted $E(\varepsilon)$. We define the adjoint state by: find $p_\varepsilon \in Y$, such that

$$\partial_u G(\varepsilon, u_\varepsilon, p_\varepsilon)(\varphi) = 0 \quad \text{for all } \varphi \in X. \tag{3.3}$$

The set of adjoint states associated with $(\varepsilon, u_\varepsilon)$ is denoted $Y(\varepsilon, u_\varepsilon)$.

Definition 3.3 (ℓ-differentiable Lagrangian)**.** Let X and Y be vector spaces and $\tau > 0$. Let $\ell : [0, \tau] \to \mathbf{R}$ be a given function satisfying $\ell(0) = 0$ and $\ell(\varepsilon) > 0$ for $\varepsilon \in (0, \tau]$. An ℓ-differentiable parametrised Lagrangian is a parametrised Lagrangian $G : [0, \tau] \times X \times Y \to \mathbf{R}$, satisfying,

(a) for all $v, w \in X$ and $p \in Y$,

$$s \mapsto G(\varepsilon, v + sw, p) \text{ is continuously differentiable on } [0, 1]. \tag{3.4}$$

(b) for all $u_0 \in E(0)$ and $p_0 \in Y(0,u_0)$ the limit

$$\partial_\ell G(0,u_0,p_0) := \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon,u_0,p_0) - G(0,u_0,p_0)}{\ell(\varepsilon)} \quad \text{exists.} \tag{3.5}$$

Assumption (H0). (i) We assume that for all $\varepsilon \in [0,\tau]$, the set $E(\varepsilon) = \{u_\varepsilon\}$ is a singleton.

(ii) We assume that the adjoint equation for $\varepsilon = 0$, $\partial_u G(0,u_0,p_0)(\varphi) = 0$ for all $\varphi \in E$, admits a unique solution.

We now give sufficient conditions when the function

$$\begin{aligned} [0,\tau] &\to \mathbf{R} \\ \varepsilon &\mapsto g(\varepsilon) := G(\varepsilon,u_\varepsilon,0), \end{aligned} \tag{3.6}$$

is one sided ℓ-differentiable, that means, when the limit

$$d_\ell g(0) := \lim_{\varepsilon \searrow 0} \frac{g(\varepsilon) - g(0)}{\ell(\varepsilon)} \tag{3.7}$$

exists, where $\ell : [0,\tau] \to \mathbf{R}$ is a given function satisfying $\ell(0) = 0$ and $\ell(\varepsilon) > 0$ for $\varepsilon \in (0,\tau]$.

The following theorem is a refinement of [7, Thm. 3.3]. Instead of having one R-term we obtain two terms, which simplifies the later analysis.

Theorem 3.4. Let $G : [0,\tau] \times X \times Y \to \mathbf{R}$ be an ℓ-differentiable parametrised Lagrangian satisfying Hypothesis (H0). Define for $\varepsilon > 0$,

$$R_1^\varepsilon(u_0,p_0) := \frac{1}{\ell(\varepsilon)} \int_0^1 (\partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, p_0) - \partial_u G(\varepsilon,u_0,p_0))(u_\varepsilon - u_0)\, ds \tag{3.8}$$

and

$$R_2^\varepsilon(u_0,p_0) := \frac{1}{\ell(\varepsilon)} (\partial_u G(\varepsilon,u_0,p_0) - \partial_u G(0,u_0,p_0))(u_\varepsilon - u_0). \tag{3.9}$$

If $R_1(u_0,p_0) := \lim_{\varepsilon\searrow 0} R_1^\varepsilon(u_0,p_0)$ and $R_2(u_0,p_0) := \lim_{\varepsilon\searrow 0} R_2^\varepsilon(u_0,p_0)$ exist, then

$$d_\ell g(0) = \partial_\ell G(0,u_0,p_0) + R_1(u_0,p_0) + R_2(u_0,p_0).$$

Proof. Using $\partial_u G(0,u_0,p_0)(\varphi) = 0$ for all $\varphi \in E$ and the fundamental theorem of calculus, we obtain

$$\begin{aligned} g(\varepsilon) - g(0) &= G(\varepsilon,u_\varepsilon,p_0) - G(0,u_0,p_0) = G(\varepsilon,u_\varepsilon,p_0) - G(\varepsilon,u_0,p_0) + G(\varepsilon,u_0,p_0) - G(0,u_0,p_0) \\ &= \int_0^1 \partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, p_0)(u_\varepsilon - u_0)\, ds + G(\varepsilon,u_0,p_0) - G(0,u_0,p_0) \\ &= \int_0^1 (\partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, p_0) - \partial_u G(\varepsilon,u_0,p_0))(u_\varepsilon - u_0)\, ds \\ &\quad + (\partial_u G(\varepsilon,u_0,p_0) - \partial_u G(0,u_0,p_0))(u_\varepsilon - u_0) \\ &\quad + G(\varepsilon,u_0,p_0) - G(0,u_0,p_0). \end{aligned}$$

Notice that the fundamental theorem of calculus is applicable in view of assumption (3.4). Now dividing by $\ell(\varepsilon)$, using Hypothesis (H0) and that $R_1(u_0,p_0)$ and $R_2(u_0,p_0)$ exist, we can pass to the limit $\varepsilon \searrow 0$. This finishes the proof. □

Remark 3.5. In the next section, we will apply the abstract result of Theorem 3.4 to the Lagrangian introduced in (2.7). There, it holds that $g(\varepsilon) = J(\Omega_\varepsilon)$ and, when using $\ell(\varepsilon) = |\omega_\varepsilon|$, the derivative (3.7) corresponds to the topological derivative defined in (2.2).

4 The topological derivative

Let $X = Y = H^1_0(\mathsf{D})$ and let the Lagrangian G be defined as in (2.7). We are now going to verify that the hypotheses of Theorem 3.4 are satisfied for this G with $\ell(\varepsilon) = |\omega_\varepsilon|$.

4.1 Analysis of the perturbed state equation

We introduce the abbreviation $\mathscr{A}_\varepsilon(x,y) := \mathscr{A}_{\Omega_\varepsilon}(x,y)$ for $x,y \in \mathbf{R}^d$. The perturbed state equation reads: find $u_\varepsilon \in H^1_0(\mathsf{D})$ such that

$$\partial_p G(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in H^1_0(\mathsf{D}), \tag{4.1}$$

or equivalently $u_\varepsilon \in H^1_0(\mathsf{D})$ satisfies

$$\int_{\mathsf{D}} \mathscr{A}_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla\varphi \, dx = \int_{\mathsf{D}} f\varphi \, dx \quad \text{for all } \varphi \in H^1_0(\mathsf{D}). \tag{4.2}$$

Since (4.2) admits a unique solution we have that $E(\varepsilon) = \{u_\varepsilon\}$ is a singleton. Together with the previous observation that (2.6) admits a unique solution, we have that Hypothesis (H0) is satisfied.

Lemma 4.1. Let Assumption A(i),(ii) be satisfied. There is a constant $C > 0$, such that for all small $\varepsilon > 0$,

$$\|u_\varepsilon - u_0\|_{H^1(\mathsf{D})} \le C\varepsilon^{d/2}. \tag{4.3}$$

Proof. Subtracting (4.2) for $\varepsilon > 0$ and $\varepsilon = 0$ yields

$$\begin{aligned}\int_{\mathsf{D}} (\mathscr{A}_\varepsilon(x,\nabla u_\varepsilon) - \mathscr{A}_\varepsilon(x, \nabla u_0)) \cdot \nabla\varphi \, dx \\ = -\int_{\omega_\varepsilon} (a_1(\nabla u_0) - a_2(\nabla u_0)) \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in H^1_0(\mathsf{D}).\end{aligned} \tag{4.4}$$

Therefore testing (4.4) with $\varphi := u_\varepsilon - u_0$, then applying Hölder's inequality and using the monotonicity of $\mathscr{A}_\varepsilon$ leads to

$$\|\nabla(u_\varepsilon - u_0)\|^2_{L_2(\mathsf{D})^d} \le C\sqrt{|\omega_\varepsilon|}(\|\nabla u_0\|_{C(\bar{B}_\delta(z))^d} + 1)\|\nabla(u_\varepsilon - u_0)\|_{L_2(\mathsf{D})^d}, \tag{4.5}$$

where $0 < \varepsilon < \delta$ and C is a generic constant. Here, we also used (2.3). Now the result follows from $|\omega_\varepsilon| = |\omega|\varepsilon^d$ and the Poincaré inequality. □

Definition 4.2. We define the variation of the state by

$$K_\varepsilon := \frac{(u_\varepsilon - u_0)\circ T_\varepsilon}{\varepsilon} \in H^1_0(\varepsilon^{-1}\mathsf{D}), \quad \varepsilon > 0. \tag{4.6}$$

By extending u_ε and u_0 by zero outside of D, we can view K_ε as an element of $BL(\mathbf{R}^d)$ (and its equivalence class $[K_\varepsilon]$ as element of $\dot{BL}(\mathbf{R}^d)$).

Our main result of this section is the following theorem:

Theorem 4.3. Let Assumption A(i),(ii) be satisfied.

(i) There exists a unique solution $K \in \dot{BL}(\mathbf{R}^d)$ to

$$\begin{aligned}\int_{\mathbf{R}^d} (\mathscr{A}_\omega(x, \nabla K + U_0) - \mathscr{A}_\omega(x, U_0)) \cdot \nabla \varphi \, dx \\ = -\int_\omega (a_1(U_0) - a_2(U_0)) \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in BL(\mathbf{R}^d),\end{aligned} \tag{4.7}$$

where $U_0 := \nabla u_0(z)$ and $\mathscr{A}_\omega(x, y) := a_1(y)\chi_\omega(x) + a_2(y)\chi_{\mathbf{R}^d \setminus \omega}(x)$.

(ii) We have $\nabla K_\varepsilon \to \nabla K$ strongly in $L_2(\mathbf{R}^d)^d$ as $\varepsilon \searrow 0$.

Proof of (i): Thanks to Assumption A the operator $B_\omega : \dot{BL}(\mathbf{R}^d) \to \dot{BL}(\mathbf{R}^d)^*$ defined by $\langle B_\omega \varphi, \psi \rangle := \int_{\mathbf{R}^d} (\mathscr{A}_\omega(x, \nabla \varphi + U_0) - \mathscr{A}_\omega(x, U_0)) \cdot \nabla \psi \, dx$ is a strongly monotone and Lipschitz continuous and hence the unique solvability follows by the theorem of Zarantonello; see [22, p.504, Thm. 25.B].

Proof of (ii): We split the proof into two lemmas. The idea is as follows:

(a) introduce the intermediate quantity H_ε and split $K - K_\varepsilon = K - H_\varepsilon + H_\varepsilon - K_\varepsilon$,

(b) show $K - H_\varepsilon \to 0$,

(c) show $H_\varepsilon - K_\varepsilon \to 0$.

This splitting is not necessary, but simplifies the presentation. Note that changing variables in (4.3) gives

$$\|\nabla K_\varepsilon\|_{L_2(\mathbf{R}^d)} \le C \quad \text{for all } \varepsilon > 0. \tag{4.8}$$

We start by changing variables in (4.4) to obtain an equation for K_ε:

$$\begin{aligned}\int_{\mathbf{R}^d} (\mathscr{A}_\omega(x, \nabla K_\varepsilon + \nabla u_0(x_\varepsilon)) - \mathscr{A}_\omega(x, \nabla u_0(x_\varepsilon))) \cdot \nabla \varphi \, dx \\ = -\int_\omega (a_1(\nabla u_0(x_\varepsilon)) - a_2(\nabla u_0(x_\varepsilon))) \cdot \nabla \varphi \, dx\end{aligned} \tag{4.9}$$

for all $\varphi \in H^1_0(\varepsilon^{-1}\mathsf{D})$ where we recall the notation $x_\varepsilon = T_\varepsilon(x) = \varepsilon x$. Similarly as in [4, 5] we approximate K_ε by $H_\varepsilon \in H^1_0(\varepsilon^{-1}\mathsf{D})$ solution to

$$\begin{aligned}\int_{\mathbf{R}^d} (\mathscr{A}_\omega(x, \nabla H_\varepsilon + U_0) - \mathscr{A}_\omega(x, U_0)) \cdot \nabla \varphi \, dx \\ = -\int_\omega (a_1(U_0) - a_2(U_0)) \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in H^1_0(\varepsilon^{-1}\mathsf{D}).\end{aligned} \tag{4.10}$$

This equation is simply (4.9) with $\nabla u(x_\varepsilon)$ replaced by U_0. We now introduce the projection of K into the space $H^1_0(\varepsilon^{-1}\mathsf{D})$: For this we consider the more general situation of $\dot{BL}_p(\mathbf{R}^d)$.

Definition 4.4. Let $\varepsilon > 0$ and $1 < p < \infty$. For every $K \in \dot{BL}_p(\mathbf{R}^d)$ we define its projection $P_\varepsilon(K) \in W^1_{p,0}(\mathbf{R}^d)$ as the minimiser of

$$\min_{\varphi \in W^1_{p,0}(\varepsilon^{-1}\mathsf{D})} \|\nabla(\varphi - K)\|_{L_p(\varepsilon^{-1}\mathsf{D})^d}. \tag{4.11}$$

So $P_\varepsilon : \dot{BL}_p(\mathbf{R}^d) \to W^1_{p,0}(\varepsilon^{-1}\mathsf{D}) \subset W^1_{p,0}(\mathbf{R}^d)$ is a nonlinear operator.

The next lemma shows that the operator P_ε is continuous with respect to ε.

Lemma 4.5. For every $K \in \dot{BL}_p(\mathbf{R}^d)$ it holds that

$$\nabla(P_\varepsilon(K)) \to \nabla K \quad \text{strongly in } L_p(\mathbf{R}^d)^d \text{ as } \varepsilon \searrow 0. \tag{4.12}$$

Proof. Since $\varphi \mapsto \|\nabla(\varphi - K)\|^p_{L_p(\varepsilon^{-1}\mathsf{D})^d}$ is strictly convex on $W^1_{p,0}(\varepsilon^{-1}\mathsf{D})$ it follows that (4.11) admits a unique solution which is denoted by $P_\varepsilon(K)$. We have by definition

$$\|\nabla(P_\varepsilon(K)-K)\|_{L_p(\varepsilon^{-1}\mathsf{D})^d} \le \|\nabla(\varphi - K)\|_{L_p(\varepsilon^{-1}\mathsf{D})^d} \quad \text{for all } \varphi \in W^1_{p,0}(\varepsilon^{-1}\mathsf{D}). \tag{4.13}$$

Choosing a function $\varphi \in W^1_{p,0}(\varepsilon^{-1}\mathsf{D})$ with fixed support in some compact set $K \subset \mathsf{D}$, we see that we find $C > 0$, such that $\|\nabla(P_\varepsilon(K))\|_{L_p(\mathbf{R}^d)^d} \le C$ for all $\varepsilon \in (0,1)$. Now let $\tilde{\varepsilon} > 0$ be arbitrary. Let (ε_n) be a null-sequence, such that $\varepsilon_n < \tilde{\varepsilon}$ for all $n \ge 1$. We obtain from (4.13) that for all $n \ge 1$:

$$\|\nabla(P_{\varepsilon_n}(K)-K)\|_{L_p(\varepsilon_n^{-1}\mathsf{D})^d} \le \|\nabla(\varphi - K)\|_{L_p(\varepsilon_n^{-1}\mathsf{D})^d} \quad \text{for all } \varphi \in W^1_{p,0}(\tilde{\varepsilon}^{-1}\mathsf{D}). \tag{4.14}$$

Here we extended φ to $\varepsilon_n^{-1}\mathsf{D}$ by zero. Since $P_{\varepsilon_n}(K)$ is bounded in $\dot{BL}_p(\mathbf{R}^d)$ we find a weakly converging subsequence (denoted the same) and an element $\hat{K} \in \dot{BL}_p(\mathbf{R}^d)$, such that $\liminf_{n\to\infty} \|\nabla(P_{\varepsilon_n}(K))\|_{L_p(\mathbf{R}^d)^d} \ge \|\nabla\hat{K}\|_{L_p(\mathbf{R}^d)^d}$. Hence it follows from (4.14) that

$$\|\nabla(\hat{K}-K)\|_{L_p(\mathbf{R}^d)^d} \le \|\nabla(\varphi - K)\|_{L_p(\mathbf{R}^d)^d} \quad \text{for all } \varphi \in W^1_{p,0}(\tilde{\varepsilon}^{-1}\mathsf{D}). \tag{4.15}$$

But since $\tilde{\varepsilon} > 0$ was arbitrary and since $C_c^\infty(\mathbf{R}^d)/\mathbf{R}$ is dense in $\dot{BL}_p(\mathbf{R}^d)$ it follows

$$\|\nabla(\hat{K}-K)\|_{L_p(\mathbf{R}^d)^d} \le \liminf_{n\to\infty} \|\nabla(P_{\varepsilon_n}(K)-K)\|_{L_p(\varepsilon_n^{-1}\mathsf{D})^d} \le \|\nabla(\varphi - K)\|_{L_p(\mathbf{R}^d)^d} \tag{4.16}$$

for all $\varphi \in \dot{BL}_p(\mathbf{R}^d)$. However, by choosing $\varphi = K$, this implies $\hat{K} = K$. It follows in particular that $P_\varepsilon(K) \rightharpoonup K$ weakly in $L_p(\mathbf{R}^d)$ as $\varepsilon \searrow 0$. In addition it follows from (4.16) the norm convergence of $\nabla(P_\varepsilon(K))$ in $L_p(\mathbf{R}^d)^d$. Hence by the theorem of Radon-Riesz (see [10, p.264, Thm.5.10]) we have $\|\nabla(P_\varepsilon(K)-K)\|_{L_p(\mathbf{R}^d)^d} \to 0$ as $\varepsilon \searrow 0$. □

We now let $\hat{K}_\varepsilon := P_\varepsilon(K) \in H^1_0(\varepsilon^{-1}\mathsf{D})$ be the solution to (4.11) with $p = 2$.

As for K_ε, we can also view H_ε and $\hat{K}_\varepsilon$ as elements of $BL(\mathbf{R}^d)$ by extending them by 0 outside $\varepsilon^{-1}\mathsf{D}$.

Remark 4.6. In [4, 5] the proof of $\nabla K_\varepsilon \to \nabla K$ strongly in $L_2(\mathbf{R}^d)^d$ as $\varepsilon \searrow 0$ was given using a cut-off argument of K. The reason is that one cannot directly work with K since $K \notin H^1_0(\varepsilon^{-1}\mathsf{D})$ for every $\varepsilon > 0$. This cut-off technique lead to technical arguments which required additional smoothness of the operators, some restrictions on the non-linearity and also to restrict to $\omega = B_1(0)$. As we will see by introducing the projection $\hat{K}_\varepsilon$ this step is simplified substantially.

Lemma 4.7. We have

$$\nabla H_\varepsilon \to \nabla K \quad \text{strongly in } L_2(\mathbf{R}^d)^d \text{ as } \varepsilon \searrow 0. \tag{4.17}$$

Proof. Subtracting (4.10) from (4.7) yields after rearranging:

$$\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla\hat{K}_\varepsilon+U_0)-\mathscr{A}_\omega(x,\nabla H_\varepsilon+U_0))\cdot\nabla\varphi\,dx=\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla\hat{K}_\varepsilon+U_0)-\mathscr{A}_\omega(x,\nabla K+U_0))\cdot\nabla\varphi\,dx \tag{4.18}$$

for all $\varphi\in H^1_0(\varepsilon^{-1}\mathrm{D})$. Now we test this equation with $\varphi=\hat{K}_\varepsilon-H_\varepsilon\in H^1_0(\varepsilon^{-1}\mathrm{D})$, use the monotonicity of $\mathscr{A}_\omega$ and Hölder's inequality:

$$\begin{aligned} C\|\nabla(\hat{K}_\varepsilon-H_\varepsilon)\|^2_{L_2(\mathbf{R}^d)^d} &\le \int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla\hat{K}_\varepsilon+U_0)-\mathscr{A}_\omega(x,\nabla H_\varepsilon+U_0))\cdot\nabla(\hat{K}_\varepsilon-H_\varepsilon)\,dx\\ &\overset{(4.18)}{=}\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla\hat{K}_\varepsilon+U_0)-\mathscr{A}_\omega(x,\nabla K+U_0))\cdot\nabla(\hat{K}_\varepsilon-H_\varepsilon)\,dx\\ &\le\int_{\mathbf{R}^d}|\nabla(\hat{K}_\varepsilon-K)||\nabla(\hat{K}_\varepsilon-H_\varepsilon)|\,dx\\ &\le\|\nabla(\hat{K}_\varepsilon-K)\|_{L_2(\mathbf{R}^d)^d}\|\nabla(\hat{K}_\varepsilon-H_\varepsilon)\|_{L_2(\mathbf{R}^d)^d}. \end{aligned} \tag{4.19}$$

Since in view of Lemma 4.5, we have $\nabla\hat{K}_\varepsilon\to\nabla K$ strongly in $L_2(\mathbf{R}^d)^d$ it follows from (4.19) that $\nabla(\hat{K}_\varepsilon-H_\varepsilon)\to 0$ strongly in $L_2(\mathbf{R}^d)^d$ and therefore also $\|\nabla(H_\varepsilon-K)\|_{L_2(\mathbf{R}^d)^d}\le\|\nabla(H_\varepsilon-\hat{K}_\varepsilon)\|_{L_2(\mathbf{R}^d)^d}+\|\nabla(\hat{K}_\varepsilon-K)\|_{L_2(\mathbf{R}^d)^d}\to 0$ as $\varepsilon\searrow 0$. □

We now prove that $\nabla(H_\varepsilon-K_\varepsilon)\to 0$ strongly in $L_2(\mathbf{R}^d)^d$.

Lemma 4.8. We have

$$\nabla(H_\varepsilon-K_\varepsilon)\to 0 \quad \text{strongly in } L_2(\mathbf{R}^d)^d \text{ as } \varepsilon\searrow 0. \tag{4.20}$$

Proof. Subtracting (4.9) and (4.10) we obtain

$$\begin{aligned} &\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla K_\varepsilon+\nabla u_0(x_\varepsilon))-\mathscr{A}_\omega(x,\nabla H_\varepsilon+U_0))\cdot\nabla\varphi\,dx\\ &\quad+\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,U_0)-\mathscr{A}_\omega(x,\nabla u_0(x_\varepsilon)))\cdot\nabla\varphi\,dx\\ &\quad=-\int_\omega(a_1(\nabla u_0(x_\varepsilon))-a_2(\nabla u_0(x_\varepsilon)))\cdot\nabla\varphi+(a_1(U_0)-a_2(U_0))\cdot\nabla\varphi\,dx \end{aligned} \tag{4.21}$$

for all $\varphi\in H^1_0(\varepsilon^{-1}\mathrm{D})$. In order to be able to use the monotonicity of $\mathscr{A}_\omega$ we rewrite this as follows

$$\begin{aligned} &\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla K_\varepsilon+\nabla u_0(x_\varepsilon))-\mathscr{A}_\omega(x,\nabla H_\varepsilon+\nabla u_0(x_\varepsilon))))\cdot\nabla\varphi\,dx\\ &=\underbrace{-\int_{\mathbf{R}^d}((\mathscr{A}_\omega(x,\nabla H_\varepsilon+\nabla u_0(x_\varepsilon))-(\mathscr{A}_\omega(x,\nabla H_\varepsilon+U_0)\cdot\nabla\varphi\,dx}_{=:I_1(\varepsilon,\varphi)}\\ &\quad\underbrace{-\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,U_0)-\mathscr{A}_\omega(x,\nabla u_0(x_\varepsilon)))\cdot\nabla\varphi\,dx}_{=:I_2(\varepsilon,\varphi)}\\ &\quad\underbrace{-\int_\omega(a_1(\nabla u_0(x_\varepsilon))-a_2(\nabla u_0(x_\varepsilon)))\cdot\nabla\varphi\,dx+(a_1(U_0)-a_2(U_0))\cdot\nabla\varphi\,dx}_{=:I_3(\varepsilon,\varphi)}. \end{aligned} \tag{4.22}$$

Since a_i are Lipschitz continuous and $u_0 \in C^{1,\alpha}(\overline{B_\delta(z)})$ with $\alpha, \delta > 0$, we immediately obtain that $|I_3(\varepsilon,\varphi)| \le C\varepsilon^\alpha \|\nabla\varphi\|_{L_2(\mathbf{R}^d)^d}$ for a suitable constant $C > 0$. We now show that also $|I_1(\varepsilon,\varphi)+I_2(\varepsilon,\varphi)| \le C(\varepsilon)\|\nabla\varphi\|_{L_2(\mathbf{R}^d)^d}$ and $C(\varepsilon) \to 0$ as $\varepsilon \searrow 0$. We write for arbitrary $r \in (0,1)$,

$$
\begin{aligned}
I_1(\varepsilon,\varphi)+I_2(\varepsilon,\varphi) = &-\int_{B_{\varepsilon^{-r}}} ((\mathscr{A}_\omega(x,\nabla H_\varepsilon + \nabla u_0(x_\varepsilon)) - (\mathscr{A}_\omega(x,\nabla H_\varepsilon + U_0)\cdot\nabla\varphi\, dx \\
&-\int_{B_{\varepsilon^{-r}}} (\mathscr{A}_\omega(x,U_0) - \mathscr{A}_\omega(x,\nabla u_0(x_\varepsilon)))\cdot\nabla\varphi\, dx \\
&-\int_{\mathbf{R}^d\setminus B_{\varepsilon^{-r}}} ((\mathscr{A}_\omega(x,\nabla H_\varepsilon + \nabla u_0(x_\varepsilon)) - (\mathscr{A}_\omega(x,\nabla u_0(x_\varepsilon))\cdot\nabla\varphi\, dx \\
&+\int_{\mathbf{R}^d\setminus B_{\varepsilon^{-r}}} ((\mathscr{A}_\omega(x,\nabla H_\varepsilon + U_0) - \mathscr{A}_\omega(x,U_0))\cdot\nabla\varphi\, dx.
\end{aligned}
\tag{4.23}
$$

As in [4, Prop. 6.7] the idea of choosing a power ε^{-r} is to let the ball $B_{\varepsilon^{-r}}(0)$ expand slower than $B_{\varepsilon^{-1}}(0)$ by choosing $r \in (0,1)$ appropriately. Now we can estimate the right hand side of (4.23) using the Lipschitz continuity of a_i (see Assumption A(ii)) as follows

$$
\begin{aligned}
|I_1(\varepsilon,\varphi)+I_2(\varepsilon,\varphi)| &\le 2C\int_{B_{\varepsilon^{-r}}} |U_0 - \nabla u_0(x_\varepsilon)||\nabla\varphi|\, dx + 2C\int_{\mathbf{R}^d\setminus B_{\varepsilon^{-r}}} |\nabla H_\varepsilon||\nabla\varphi|\, dx \\
&\le C\int_{B_{\varepsilon^{-r}}} |x_\varepsilon|^\alpha|\nabla\varphi| \mathrm{d}x + 2C\int_{\mathbf{R}^d\setminus B_{\varepsilon^{-r}}} |\nabla H_\varepsilon||\nabla\varphi|\, dx \\
&\le \varepsilon^{-r\alpha}\varepsilon^\alpha\varepsilon^{-rd/2} C\|\nabla\varphi\|_{L_2(\mathbf{R}^d)^d} + 2C\|\nabla H_\varepsilon\|_{L_2(\mathbf{R}^d\setminus B_{\varepsilon^{-r}})^d}\|\nabla\varphi\|_{L_2(\mathbf{R}^d\setminus B_{\varepsilon^{r}})^d}
\end{aligned}
\tag{4.24}
$$

For r sufficiently close to 0, we have $\varepsilon^{-r\alpha}\varepsilon^\alpha\varepsilon^{-rd/2} = \varepsilon^{\alpha - r(\frac{d}{2}+\alpha)} \to 0$. Moreover, by the triangle inequality we have

$$
\|\nabla H_\varepsilon\|_{L_2(\mathbf{R}^d\setminus B_{\varepsilon^{-r}})} \le \|\nabla(H_\varepsilon - K)\|_{L_2(\mathbf{R}^d\setminus B_{\varepsilon^{-r}})} + \|\nabla K\|_{L_2(\mathbf{R}^d\setminus B_{\varepsilon^{-r}})}.
\tag{4.25}
$$

The first term on the right hand side goes to zero in view of Lemma 4.7. The second term goes to zero since $\nabla K \in L_2(\mathbf{R}^d)^d$ thus $\|\nabla K\|_{L_2(\mathbf{R}^d\setminus B_{\varepsilon^{-r}})^d} \to 0$ as $\varepsilon \searrow 0$. Using $K_\varepsilon - H_\varepsilon$ as test function in (4.22), using the monotonicity of $\mathscr{A}$ and employing $|I_1(\varepsilon,\varphi)+I_2(\varepsilon,\varphi)+I_3(\varepsilon,\varphi)| \le C(\varepsilon)\|\nabla\varphi\|_{L_2(\mathbf{R}^d)^d}$ with $C(\varepsilon) \to 0$ as $\varepsilon \searrow 0$, shows the result. □

Combining Lemma 4.7 and Lemma 4.8 proves Theorem 4.3(ii). *square*
We get the following properties of the sequence $(\varepsilon K_\varepsilon)$:

Corollary 4.9. We have

$$
\varepsilon K_\varepsilon \to 0 \quad \begin{cases} \text{strongly in } L_p(\mathbf{R}^d) & \text{for } d = 2, \quad p \in (2,4], \\ \text{strongly in } L_p(\mathbf{R}^d) & \text{for } d \ge 3, \quad p \in (2,2^*], \\ \text{weakly in } L_2(\mathbf{R}^d) & \text{for } d \ge 2, \end{cases}
\tag{4.26}
$$

where $2^* := 2d/(d-2)$ denotes the Sobolev exponent of 2 for $d \ge 3$.

Proof. Let $d = 2$. From the Ladyzhenskaya inequality (see [15]) we obtain the estimate $\|\varepsilon K_\varepsilon\|_{L_4(\mathbf{R}^2)} \le C\varepsilon^{1/2}\|\varepsilon K_\varepsilon\|^{1/2}_{L_2(\mathbf{R}^2)}\|\nabla K_\varepsilon\|^{1/2}_{L_2(\mathbf{R}^2)^2}$. Hence for $d = 2$, we conclude $\varepsilon K_\varepsilon \to 0$ in $L_4(\mathbf{R}^2)$ as $\varepsilon \searrow 0$. Let now $p \in (2,4)$. Then in view of the interpolation inequality $\|\varepsilon K_\varepsilon\|_{L_p(\mathbf{R}^2)} \le \|\varepsilon K_\varepsilon\|^{\theta}_{L_2(\mathbf{R}^2)} \|\varepsilon K_\varepsilon\|^{1-\theta}_{L_4(\mathbf{R}^2)}$ for all $\theta \in (0,1)$ and $\frac{1}{p} = \frac{\theta}{2} + \frac{(1-\theta)}{4}$ it follows $\varepsilon K_\varepsilon \to 0$ in $L_p(\mathbf{R}^2)$ as $\varepsilon \searrow 0$.

Let now $d \ge 3$. By the Gagliardo-Nirenberg inequality (see [16]) we obtain $\|\varepsilon K_\varepsilon\|_{L_{2_*}(\mathbf{R}^d)} \le C\varepsilon \|\nabla K_\varepsilon\|_{L_2(\mathbf{R}^d)^d}$ and hence $\varepsilon K_\varepsilon \to 0$ strongly in $L_{2^*}(\mathbf{R}^d)$ as $\varepsilon \searrow 0$. The convergence $\varepsilon K_\varepsilon \to 0$ strongly in $L_p(\mathbf{R}^d)$ as $\varepsilon \searrow 0$ for all $p \in (2, 2^*)$ follows by the interpolation argument as in the previous step.

The convergence $\varepsilon K_\varepsilon \rightharpoonup 0$ weakly in $L_2(\mathbf{R}^d)$ as $\varepsilon \searrow 0$ can be proved using the same arguments as in [21, Thm. 4.14].

□

4.2 Computation of $R_1(u_0,p_0)$ and $R_2(u_0,p_0)$

It is easily seen from the continuity of a_1, a_2, ∇u_0 and ∇p_0 that G is ℓ-differentiable with

$$\partial_\ell G(0,u_0,p_0) = \frac{1}{|\omega|}(a_1(U_0) - a_2(U_0)) \cdot \int_\omega P_0 \, dx. \tag{4.27}$$

It remains to check that the limits of $R_1(u_0,p_0)$ and $R_2(u_0,p_0)$ exist. For this we use Assumption A(i)-(iii). Using the change of variables T_ε, we have

$$\begin{aligned} R_1^\varepsilon(u_0,p_0) &= \frac{1}{\ell(\varepsilon)} \int_0^1 \int_{\mathrm{D}} (\partial_u \mathscr{A}_\varepsilon(x, \nabla(s u_\varepsilon + (1-s)u_0)) - \partial_u \mathscr{A}_\varepsilon(x, \nabla u_0))(\nabla(u_\varepsilon - u_0)) \cdot \nabla p_0 \, dx\, ds \\ &\quad + \frac{1}{\ell(\varepsilon)} \int_{\mathrm{D}} |\nabla(u_\varepsilon - u_0)|^2 \, dx \\ &= \frac{1}{|\omega|} \int_0^1 \int_{\mathbf{R}^d} (\partial_u \mathscr{A}_\omega(x, s\nabla K_\varepsilon + \nabla u_0(x_\varepsilon)) - \partial_u \mathscr{A}_\omega(x, \nabla u_0(x_\varepsilon)))(\nabla K_\varepsilon) \cdot \nabla p_0(x_\varepsilon) \, dx\, ds \\ &\quad + \frac{1}{|\omega|} \int_{\mathbf{R}^d} |\nabla K_\varepsilon|^2 \, dx \\ &\to \frac{1}{|\omega|} \int_0^1 \int_{\mathbf{R}^d} (\partial_u \mathscr{A}_\omega(x, s\nabla K + U_0) - \partial_u \mathscr{A}_\omega(x, U_0))(\nabla K) \cdot P_0 \, dx\, ds + \frac{1}{|\omega|} \int_{\mathbf{R}^d} |\nabla K|^2 \, dx. \end{aligned} \tag{4.28}$$

Here, we used that $\nabla K_\varepsilon \to \nabla K$ strongly in $L_2(\mathbf{R}^d)^d$ as $\varepsilon \searrow 0$ for the limit of the second term. To see the convergence of the first term, we may write

$$\begin{aligned} &\int_0^1 \int_{\mathbf{R}^d} (\partial_u \mathscr{A}_\omega(x, s\nabla K_\varepsilon + \nabla u_0(x_\varepsilon)) - \partial_u \mathscr{A}_\omega(x, \nabla u_0(x_\varepsilon)))(\nabla K_\varepsilon) \cdot \nabla p_0(x_\varepsilon) \, dxds = \\ &\qquad + \int_0^1 \int_{\mathbf{R}^d} (\partial_u \mathscr{A}_\omega(x, s\nabla K_\varepsilon + \nabla u_0(x_\varepsilon)) - \partial_u \mathscr{A}_\omega(x, s\nabla K + \nabla u_0(x_\varepsilon)))(\nabla K_\varepsilon) \cdot \nabla p_0(x_\varepsilon) \, dxds \\ &\qquad + \int_0^1 \int_{\mathbf{R}^d} (\partial_u \mathscr{A}_\omega(x, s\nabla K + \nabla u_0(x_\varepsilon)) - \partial_u \mathscr{A}_\omega(x, \nabla u_0(x_\varepsilon)))(\nabla(K_\varepsilon - K)) \cdot \nabla p_0(x_\varepsilon) \, dxds \\ &\qquad + \int_0^1 \int_{\mathbf{R}^d} (\partial_u \mathscr{A}_\omega(x, s\nabla K + \nabla u_0(x_\varepsilon)) - \partial_u \mathscr{A}_\omega(x, \nabla u_0(x_\varepsilon)))(\nabla K) \cdot \nabla p_0(x_\varepsilon) \, dxds. \end{aligned}$$

Using Assumption A(iii) and $\nabla p_0 \in L^\infty(\mathrm{D})^d$, we see that the absolute value of the first and second term on the right hand side can be bounded by $C\|\nabla(K_\varepsilon - K)\|_{L_2(\mathbf{R}^d)^d} \|\nabla K\|_{L_2(\mathbf{R}^d)^d}$ and

$C\|\nabla(K_\varepsilon - K)\|_{L_2(\mathbf{R}^d)^d}\|\nabla K_\varepsilon\|_{L_2(\mathbf{R}^d)^d}$, respectively, and hence using $\nabla K_\varepsilon \to \nabla K$ in $L_2(\mathbf{R}^d)^d$ as $\varepsilon \searrow 0$ they disappear in the limit. The last term converges to the desired limit by using Lebesgue's dominated convergence theorem. Using the fundamental theorem, we obtain the expression in (2.10). Similarly, using (2.4), the continuity of ∇u_0 and ∇p_0 at z, the continuity of $\partial_u a_1, \partial_u a_2$, and again $\nabla K_\varepsilon \to \nabla K$ strongly in $L_2(\mathbf{R}^d)^d$, we obtain by Lebesgue's dominated convergence theorem

$$\begin{aligned} R_2^\varepsilon(u,p) &= \frac{1}{\ell(\varepsilon)}\int_{\omega_\varepsilon}(\partial_u a_1(\nabla u_0) - \partial_u a_2(\nabla u_0))(\nabla(u_\varepsilon - u_0))\cdot\nabla p_0\,dx \\ &= \frac{1}{|\omega|}\int_\omega(\partial_u a_1(\nabla u_0(x_\varepsilon)) - \partial_u a_2(\nabla u_0(x_\varepsilon)))(\nabla K_\varepsilon)\cdot\nabla p_0(x_\varepsilon)\,dx \\ &\to \frac{1}{|\omega|}\int_\omega(\partial_u a_1(U_0) - \partial_u a_2(U_0))(\nabla K)\cdot P_0\,dx. \end{aligned} \tag{4.29}$$

This finishes the proof of the Main Theorem.

Remark 4.10. We remark that, while the problem considered in this paper is in an L_2 setting, the projection trick of Definition 4.4 is also possible in W_p^1 spaces. Thus, an extension of our results for the topological derivative to PDE constraints posed in an L_p setting as considered in [4] with $1 < p < \infty$, $p \neq 2$ seems possible.

Remark 4.11. The obtained formula for the topological derivative coincides with the formulas obtained in [4, Thm. 4.4] and [5, Thm. 2 and Thm. 3] for the respective special cases, which can be seen as follows: Introducing the problem defining the variation of the adjoint state $\tilde{Q} \in \dot{B}L(\mathbf{R}^d)$,

$$\int_{\mathbf{R}^d}\partial_u\mathscr{A}_\omega(x,U_0)(\nabla\varphi)\cdot\nabla\tilde{Q}\,dx = -\int_\omega(\partial_u a_1(U_0) - \partial_u a_2(U_0))(\nabla\varphi)\cdot P_0\,dx \tag{4.30}$$

for all $\varphi \in BL(\mathbf{R}^d)$, and adding the left and right hand side of (4.7) tested with the solution $\tilde{Q}$ of (4.30), the term $R_2(u_0,p_0)$ can be rewritten as

$$\begin{aligned} R_2(u_0,p_0) =& -\frac{1}{|\omega|}\int_{\mathbf{R}^d}\partial_u\mathscr{A}_\omega(x,U_0)(\nabla K)\cdot\nabla\tilde{Q}\,dx \\ =& \frac{1}{|\omega|}\int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla K + U_0) - \mathscr{A}_\omega(x,U_0) - \partial_u\mathscr{A}_\omega(x,U_0)(\nabla K))\cdot\nabla\tilde{Q}\,dx \\ &+ \frac{1}{|\omega|}\int_\omega(a_1(U_0) - a_2(U_0))\cdot\nabla\tilde{Q}\,dx. \end{aligned} \tag{4.31}$$

Together with the terms $\partial_\ell G(0,u_0,p_0)$ and $R_1(u_0,p_0)$, the topological derivative reads

$$\begin{aligned} dJ(\Omega)(z) = \frac{1}{|\omega|}\Big[&(a_1(U_0) - a_2(U_0))\cdot\int_\omega P_0 + \nabla\tilde{Q}\,dx \\ &+ \int_{\mathbf{R}^d}(\mathscr{A}_\omega(x,\nabla K + U_0) - \mathscr{A}_\omega(x,U_0) - \partial_u\mathscr{A}_\omega(x,U_0)(\nabla K))\cdot(P_0 + \nabla\tilde{Q})\,dx \\ &+ \int_{\mathbf{R}^d}|\nabla K|^2\,dx\Big] \end{aligned} \tag{4.32}$$

which is, up to a scaling by $1/|\omega|$ the same formula as obtained in [4] and [5]. The different scaling is due to a different definition of the topological derivative in these publications.

Remark 4.12. It can be seen from (4.30) that $\nabla\tilde{Q}$ depends linearly on P_0. Thus, it can be shown that there exists a matrix $\mathcal{M} = \mathcal{M}(\omega, \partial_u a_1(U_0), \partial_u a_2(U_0))$, which is related to the concept of polarization matrices [1], such that $\int_\omega \nabla\tilde{Q}\, dx = \mathcal{M} P_0$, see also [5, Sec. 6] for the special setting of two-dimensional magnetostatics.

For a discussion on the efficient numerical evaluation of the second integral in (4.32) involving K, see [5, Sec. 7].

5 Comparison with the averaged adjoint approach and more general cost functions

In this section we compare the Lagrangian approach of the previous section with the averaged adjoint approach; see [21]. We demonstrate that the averaged adjoint approach has some advantages at the price of being more technically involved. In fact, with the averaged adjoint approach we are able to treat a cost function of the type:

$$J(\Omega) := a \int_{\mathsf{D}} (u - u_d)^2 \, dx + b \int_{\mathsf{D}} |\nabla(u - u_d)|^2 \, dx \tag{5.1}$$

with $a, b \geq 0$ and u the solution to (4.2) with $\varepsilon = 0$ for $\Omega \subset \mathsf{D}$. It can be checked that the first term cannot be directly be handled with the Lagrangian technique of Section 3. In fact in order to pass to the limit $\varepsilon \searrow 0$ in (4.28), we would need $\varepsilon K_\varepsilon \to 0$ strongly in $L_2(\mathbf{R}^d)^d$, which does not directly follow (see Corollary 4.9). Note that this term is not covered by the analysis in [4].

5.1 Averaged adjoint

We use the same setting as in Section 3 and let G be a Lagrangian (see Definition 3.3) defined on $[0, \tau] \times X \times Y$ with X, Y being vector spaces.

The key ingredient of the averaged adjoint approach is the averaged adjoint equation. In addition to G as in Definition 3.3 we assume that G satisfies:

Assumption (H1)**.** For all $t \in [0, \tau]$, $\varphi, \tilde{\varphi}, p \in X$ and $\psi \in Y$ the derivative $[0,1] \to \mathbf{R} : s \mapsto \partial_\varphi G(t, \varphi + s\tilde{\varphi}, \psi)(p)$ is well-defined and integrable on $[0,1]$.

For a Lagrangian satisfying the previous assumption we can introduce the averaged adjoint equation.

Definition 5.1. Given $\varepsilon \in [0, \tau]$ and $(u_0, u_\varepsilon) \in E(0) \times E(\varepsilon)$, the *averaged adjoint state equation* is defined as follows: find $p_\varepsilon \in X$, such that

$$\int_0^1 \partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, p_\varepsilon)(\varphi)\, ds = 0 \quad \text{for all } \varphi \in X. \tag{5.2}$$

For every triplet $(\varepsilon, u_0, u_\varepsilon)$ the set of solutions of (5.2) is denoted by $Y(\varepsilon, u_0, u_\varepsilon)$ and its elements are referred to as *adjoint states* for $\varepsilon = 0$ and *averaged adjoint states* for $\varepsilon > 0$.

By construction of the averaged adjoint equation we have for $\varepsilon \in [0,\tau]$,

$$G(\varepsilon, u_\varepsilon, p_\varepsilon) = G(\varepsilon, u_0, p_\varepsilon). \tag{5.3}$$

The following is an alternative to Theorem 3.4 (see [21]).

Theorem 5.2. Let G be an ℓ-differentiable Lagrangian function satisfying Assumption (H1). Assume further that for all $\varepsilon \in [0,\tau]$

(i) the set $E(\varepsilon) = \{u_\varepsilon\}$ is a singleton,

(ii) for $u_0 \in E(0)$, $u_\varepsilon \in E(\varepsilon)$ the set of averaged adjoint states $Y(\varepsilon, u_0, u_\varepsilon) = \{p_\varepsilon\}$ is a singleton,

(iii) the limit

$$R(u_0, p_0) := \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u_0, p_\varepsilon) - G(\varepsilon, u_0, p_0)}{\ell(\varepsilon)} \quad \text{exists.} \tag{5.4}$$

Then we have

$$d_\ell g(0) = \partial_\ell G(0, u_0, p_0) + R(u_0, p_0). \tag{5.5}$$

5.2 Analysis of the averaged adjoint equation

We now apply Theorem 5.2 to $X = Y = H_0^1(\mathrm{D})$ with Lagrangian G given by

$$G(\varepsilon, u, p) := a \int_{\mathrm{D}} (u - u_d)^2 \, dx + b \int_{\mathrm{D}} |\nabla(u - u_d)|^2 \, dx + \int_{\mathrm{D}} \mathscr{A}_{\Omega_\varepsilon}(x, \nabla u) \cdot \nabla p \, dx - \int_{\mathrm{D}} f p \, dx. \tag{5.6}$$

We use the same setting as in Subsection 2.2. The averaged adjoint $p_\varepsilon \in H_0^1(\mathrm{D})$ is defined by

$$\int_0^1 \partial_u G(\varepsilon, s u_\varepsilon + (1-s) u_0, p_\varepsilon)(\varphi) \, ds = 0 \quad \text{for all } \varphi \in H_0^1(\mathrm{D}). \tag{5.7}$$

This is equivalent to

$$\begin{aligned}\int_0^1 \int_{\mathrm{D}} \partial_u \mathscr{A}_\varepsilon(x, \nabla(s u_\varepsilon + (1-s) u_0))(\nabla\varphi) \cdot \nabla p_\varepsilon \, dx \, ds \\ = -\int_{\mathrm{D}} (u_\varepsilon + u_0 - 2u_d)\varphi \, dx - \int_{\mathrm{D}} \nabla(u_\varepsilon + u_0 - 2u_d) \cdot \nabla\varphi \, dx\end{aligned} \tag{5.8}$$

for all $\varphi \in H_0^1(\mathrm{D})$. As noted earlier, Problem (4.2) admits a unique solution under Assumption A. Moreover, problem (5.8) has a unique solution due to Assumption A and Lax-Milgram. Therefore, the assumptions (i) and (ii) of Theorem 5.2 are satisfied. The ℓ-differentiability of G follows again as in the previous section. It remains to show the existence of the limit term $R(u_0, p_0)$.

The following analysis is similar to the study of the perturbation of the state equation. Since the model problem is quasi-linear it is crucial that we have the strong convergence of $\nabla K_\varepsilon \to \nabla K$.

Lemma 5.3. There is a constant $C > 0$, such that

$$\|p_\varepsilon - p_0\|_{H^1(\mathrm{D})} \le C(\varepsilon^{d/2} + \|u_\varepsilon - u_0\|_{H^1(\mathrm{D})}) \quad \text{for all } \varepsilon > 0. \tag{5.9}$$

Proof. Using (5.8) for $\varepsilon > 0$ and $\varepsilon = 0$ we obtain

$$\begin{aligned} &\int_0^1 \int_{\mathsf{D}} \partial_u \mathscr{A}_\varepsilon(x, \nabla(su_\varepsilon + (1-s)u_0))(\nabla\varphi)\cdot\nabla(p_\varepsilon - p_0)\,dx\,ds \\ &+ \int_0^1 \int_{\mathsf{D}} \Big(\partial_u \mathscr{A}_\varepsilon(x, \nabla(su_\varepsilon + (1-s)u_0)) - \partial_u \mathscr{A}_\varepsilon(x, \nabla u_0))(\nabla\varphi)\Big)\cdot \nabla p_0\,dx\,ds \\ &+ \int_{\mathsf{D}} \Big(\partial_u \mathscr{A}_\varepsilon(x, \nabla u_0) - \partial_u \mathscr{A}_0(x, \nabla u_0)(\nabla\varphi)\Big)\cdot\nabla p_0\,dx \\ &\qquad + a\int_{\mathsf{D}} (u_\varepsilon - u_0)\varphi\,dx + b\int_{\mathsf{D}} \nabla(u_\varepsilon - u_0)\cdot\nabla\varphi\,dx = 0 \end{aligned} \tag{5.10}$$

for all $\varphi \in H_0^1(\mathsf{D})$. Testing with $\varphi = p_\varepsilon - p_0$, using the boundedness of ∇p_0, Hölder's inequality from Assumption A gives the result. □

Definition 5.4. We consider again the variation of the adjoint state

$$Q_\varepsilon := \frac{(p_\varepsilon - p_0)\circ T_\varepsilon}{\varepsilon} \in H_0^1(\varepsilon^{-1}\mathsf{D}),\ \varepsilon > 0. \tag{5.11}$$

Note that Lemma 5.3 together with Lemma 4.1 implies that

$$\int_{\mathbf{R}^d} (\varepsilon Q_\varepsilon)^2 + |\nabla Q_\varepsilon|^2\,dx \le C \quad \text{for all } \varepsilon > 0. \tag{5.12}$$

This means that (Q_ε) is bounded in the Beppo-Levi space $BL(\mathbf{R}^d)$. We now show the weak convergence $Q_\varepsilon \rightharpoonup Q$ in $BL(\mathbf{R}^d)$ to some $Q \in BL(\mathbf{R}^d)$. It can also be shown that $\varepsilon Q_\varepsilon \to 0$ in $L_2(\mathbf{R}^d)$.

Theorem 5.5. We have

$$\nabla Q_\varepsilon \rightharpoonup \nabla Q \quad \text{weakly in } L_2(\mathbf{R}^d)^d \text{ as } \varepsilon \searrow 0, \tag{5.13}$$

where $Q \in BL(\mathbf{R}^d)$ is the unique solution to

$$\begin{aligned} &\int_{\mathbf{R}^d} \int_0^1 \partial_u \mathscr{A}_\omega(x, s\nabla K + U_0)(\nabla\psi)\cdot\nabla Q\,ds\,dx = \\ &\quad -\int_{\mathbf{R}^d}\int_0^1 \big(\partial_u \mathscr{A}_\omega(x, s\nabla K + U_0)(\nabla\psi) - \partial_u \mathscr{A}_\omega(x, U_0)(\nabla\psi)\big)\cdot P_0\,dx \\ &\quad -\int_\omega (\partial_u a_1(U_0) - \partial_u a_2(U_0))(\nabla\psi)\cdot P_0\,dx - b\int_{\mathbf{R}^d} \nabla K\cdot\nabla\psi\,dx \quad \text{for all } \psi \in BL(\mathbf{R}^d), \end{aligned} \tag{5.14}$$

with $P_0 := \nabla p_0(z)$ and K defined in (4.7).

Proof. Changing variables in (5.10) and rearranging yields

$$\begin{aligned} &\int_0^1 \int_{\mathbf{R}^d} \partial_u \mathscr{A}_\omega(x, s\nabla K_\varepsilon + \nabla u_0(x_\varepsilon))(\nabla\psi)\cdot\nabla Q_\varepsilon\,dx\,ds = \\ &-\int_0^1\int_{\mathbf{R}^d} \Big(\partial_u \mathscr{A}_\omega(x, s\nabla K_\varepsilon + \nabla u_0(x_\varepsilon))) - \partial_u \mathscr{A}_\omega(x, \nabla u_0(x_\varepsilon)))(\nabla\psi)\Big)\cdot\nabla p_0(x_\varepsilon)\,dx\,ds \\ &\quad -\int_\omega \Big(\partial_u a_1(\nabla u_0(x_\varepsilon)) - \partial_u a_2(\nabla u_0(x_\varepsilon))(\nabla\psi)\Big)\cdot\nabla p_0\,dx\,ds \\ &\qquad\qquad - a\int_{\mathbf{R}^d} \varepsilon K_\varepsilon \psi\,dx - b\int_{\mathbf{R}^d} \nabla K_\varepsilon\cdot\nabla\psi\,dx = 0 \end{aligned} \tag{5.15}$$

for all $\psi \in H_0^1(\varepsilon^{-1}\mathsf{D})$. Using $\nabla K_\varepsilon \to \nabla K$ strongly in $L_2(\mathbf{R}^d)^d$ and $\varepsilon K_\varepsilon \rightharpoonup 0$ weakly in $L_2(\mathbf{R}^d)$, we can use the Lebesgue dominated convergence theorem pass to the limit in (5.15) (for a subsequence) and obtain that the weak limit of the subsequence of (Q_ε) satisfies (5.14). Since the solution to (5.14) is unique we conclude that $Q_\varepsilon \rightharpoonup Q$ weakly in $BL(\mathbf{R}^d)$. □

5.3 Computation of $R(u_0, p_0)$

Lemma 5.6. We have

$$R(u_0,p_0) = (a_1(U_0) - a_2(U_0)) \cdot \fint_\omega \nabla Q \, dx, \tag{5.16}$$

where Q is the solution to (5.14).

Proof. Testing (4.2) for $\varepsilon = 0$ with $\varphi := p_\varepsilon - p_0$ yields

$$\int_{\mathsf{D}} \mathscr{A}_0(x,\nabla u_0)\cdot\nabla(p_\varepsilon - p_0)\, dx = \int_{\mathsf{D}} f(p_\varepsilon - p_0)\, dx. \tag{5.17}$$

Therefore

$$\begin{aligned} G(\varepsilon,u_0,p_\varepsilon) - G(\varepsilon,u_0,p_0) &= \int_{\mathsf{D}} \mathscr{A}_\varepsilon(x,\nabla u_0)\cdot\nabla(p_\varepsilon - p_0)\, dx - \int_{\mathsf{D}} f(p_\varepsilon - p_0)\, dx \\ &\overset{(5.17)}{=} \int_{\mathsf{D}} (\mathscr{A}_\varepsilon(x,\nabla u_0) - \mathscr{A}_0(x,\nabla u_0))\cdot\nabla(p_\varepsilon - p_0)\, dx \\ &= \int_{\omega_\varepsilon} (a_1(\nabla u_0) - a_2(\nabla u_0))\cdot\nabla(p_\varepsilon - p_0)\, dx. \end{aligned} \tag{5.18}$$

Therefore invoking the change of variables T_ε in (5.18) leads to

$$\frac{G(\varepsilon,u,p_\varepsilon) - G(\varepsilon,u,p)}{|\omega_\varepsilon|} = \frac{1}{|\omega|}\int_\omega (a_1(\nabla u_0(x_\varepsilon)) - a_2(\nabla u_0(x_\varepsilon)))\cdot\nabla Q_\varepsilon \, dx. \tag{5.19}$$

In view of the continuity of $a_1, a_2, \nabla u$ and the weak convergence $\nabla Q_\varepsilon \rightharpoonup \nabla Q$ in $L_2(\mathbf{R}^d)^d$, we see that the right hand side converges to the expression (5.16). □

5.4 The final expression of the topological expansion

So we see that all conditions of Theorem 5.2 are satisfied and we have

$$dJ(\Omega)(z) = \partial_\ell G(0,u_0,p_0) + R(u_0,p_0), \tag{5.20}$$

with $R(u_0,p_0)$ given by (5.16). We see that the second term on the right hand side still depends on Q, which we can express through u_0 and p_0 as follows. First we test (5.14) with $\psi := K$ and use the fundamental theorem to obtain

$$\begin{aligned} &\int_{\mathbf{R}^d} (\mathscr{A}_\omega(x,\nabla K + U_0) - \mathscr{A}_\omega(x,U_0)))\cdot\nabla Q\, dx = \\ &-\int_{\mathbf{R}^d}\int_0^1 \big(\partial_u\mathscr{A}_\omega(x,s\nabla K + U_0)(\nabla K) - \partial_u\mathscr{A}_\omega(x,U_0)(\nabla K)\big)\cdot P_0\, ds\, dx \\ &-\int_\omega (\partial_u a_1(U_0) - \partial_u a_2(U_0))(\nabla K)\cdot P_0\, dx - b\int_{\mathbf{R}^d} |\nabla K|^2\, dx \end{aligned} \tag{5.21}$$

and testing (4.7) with $\varphi = Q$ yields

$$\int_{\mathbf{R}^d} (\mathscr{A}_\omega(x, \nabla K + U_0) - \mathscr{A}_\omega(x, U_0)) \cdot \nabla Q \, dx = -\int_\omega (a_1(U_0) - a_2(U_0)) \cdot \nabla Q \, dx. \tag{5.22}$$

Combining these two equations we obtain

$$\begin{aligned} R(u,p) &= (a_1(U_0) - a_2(U_0)) \cdot \fint_\omega \nabla Q \, dx \\ &= \frac{1}{|\omega|} \int_{\mathbf{R}^d} \int_0^1 \left(\partial_u \mathscr{A}_\omega(x, s\nabla K + U_0)(\nabla K) - \partial_u \mathscr{A}_\omega(x, U_0)(\nabla K) \right) \cdot P_0 \, ds \, dx \\ &+ \frac{1}{|\omega|} \int_\omega (\partial_u a_1(U_0) - \partial_u a_2(U_0))(\nabla K) \cdot P_0 \, dx + \frac{1}{|\omega|} b \int_{\mathbf{R}^d} |\nabla K|^2 \, dx. \end{aligned} \tag{5.23}$$

In particular we see that for $a = 1$ and $b = 0$ we retrieve the formula (2.8), that is, $R_1(u_0, p_0) + R_2(u_0, p_0) = R(u_0, p_0)$.

Conclusion

In this paper we derived topological sensitivities for a class of quasi-linear problems under more general assumptions than previous results. Moreover, we simplified many of the previous calculations, which can be helpful when dealing with other types of nonlinear problems. In fact our analysis of $K_\varepsilon \to K$ is not restricted to elliptic problems and is extendable to other types of equations, such as Maxwell's equation, see [13].

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