

# Topological sensitivities via a Lagrangian approach for semilinear problems

Kevin Sturm \*

## Abstract

In this paper we present a methodology that allows the efficient computation of the topological derivative for semilinear elliptic problems within the averaged adjoint Lagrangian framework. The generality of our approach should also allow the extension to evolutionary and other nonlinear problems. Our strategy relies on a rescaled differential quotient of the averaged adjoint state variable which we show converges weakly to a function satisfying an equation defined in the whole space. A unique feature and advantage of this framework is that we only need to work with weakly converging subsequences of the differential quotient. This allows the computation of the topological sensitivity within a simple functional analytic framework under mild assumptions.

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## 1 Introduction

*Shape functions* (also called shape functionals) are real valued functions defined on sets of subsets of the Euclidean space  $\mathbf{R}^d$ . The field of mathematics dealing with the minimisation of shape functions that are constrained by a partial differential equation is called *PDE constrained shape optimisation*. Numerous applications in the engineering and life sciences, such as the aircraft and car design or electrical impedance/magnetic induction tomography, underline its importance; [24, 25]. Among other approaches [9, 12, 16, 31, 35] the topological derivative approach [10, 19, 34] constitutes an important tool to solve shape optimisation problems for which the final topology of the shape is unknown. We refer to the monograph [31] and references therein for applications of this approach.

The idea of the *topological derivative* is to study the local behaviour of a shape function  $J$  with respect to a family of singular perturbations  $(\Omega_\varepsilon)$ . Two important singular perturbations are obtained by translating and scaling of an inclusion  $\omega$  which contains the origin by  $\omega_\varepsilon(z) := z + \varepsilon\omega$ ; then the singular perturbations are given by  $\Omega_\varepsilon := \Omega \cup \omega_\varepsilon(z)$  for  $z \in \Omega^c$  and  $\Omega_\varepsilon := \Omega \setminus \overline{\omega_\varepsilon(z)}$  for  $z \in \Omega$ . Both singular perturbations are examples of the class of perturbations

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\*Technische Universität Wien, Wiedner Hauptstr. 8-10, 1040 Vienna, Austria, E-Mail: kevin.sturm@tuwien.ac.at

called dilatations that are considered in [13]. The topological derivative of a shape function  $J$  with respect to perturbations  $(\Omega_\varepsilon)$  is defined by

$$\partial J(\Omega) := \lim_{\varepsilon \searrow 0} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\ell(\varepsilon)}, \quad (1.1)$$

where  $\ell : [0, \tau] \rightarrow \mathbf{R}$ ,  $\tau > 0$ , is an appropriate function depending on the perturbation chosen. If  $\Omega$  is perturbed by a family of transformations  $\Phi_\varepsilon := \text{Id} + \varepsilon \mathbf{V} : \mathbf{R}^d \rightarrow \mathbf{R}^d$  generated by a Lipschitz vector field  $\mathbf{V} : \mathbf{R}^d \rightarrow \mathbf{R}^d$ , that is,  $\Omega_\varepsilon := \Phi_\varepsilon(\Omega)$ , then we can choose  $\ell(\varepsilon) = \varepsilon$  and (1.1) reduces to the definition of the shape derivative [35]. So the topological derivative can be seen as an generalisation of the shape derivative. In some cases, notably when shape functions are constrained by elliptic partial differential equations, the topological derivative can be obtained as the singular limit of the shape derivative as presented in the monograph [31, pp. 12]. While the shape derivative can be interpreted as the Lie derivative on a manifold, the topological derivative is merely a semi-differential defined on a cone, which makes its computation a challenging topic; see [13].

The goal of this paper is to give a coincide way to compute topological sensitivities for the following class of semilinear problems. Given a bounded domain  $D \subset \mathbf{R}^d$ ,  $d \in \{2, 3\}$ , with Lipschitz boundary  $\partial D$  we want to find the topological derivative of the objective function

$$J(\Omega) := \int_D j(x, u(x)) dx \quad \} \quad (C)$$

in an open set  $\Omega \subset D$  subject to  $u = u_\Omega$  solves the semilinear transmission problem

$$\left. \begin{aligned} -\operatorname{div}(\beta_1 \nabla u^+) + \varrho_1(u^+) &= f_1 && \text{in } \Omega \\ -\operatorname{div}(\beta_2 \nabla u^-) + \varrho_2(u^-) &= f_2 && \text{in } D \setminus \bar{\Omega} \\ u^- &= 0 && \text{on } \partial D \\ (\beta_1 \nabla u^+) \nu &= (\beta_2 \nabla u^-) \nu && \text{on } \partial \Omega \\ u^+ &= u^- && \text{on } \partial \Omega \end{aligned} \right\} \quad (S)$$

where  $u^+, u^-$  denote the restriction of  $u$  to  $\Omega$  and  $D \setminus \bar{\Omega}$ , respectively. The function  $\nu$  denotes the outward pointing unit normal field along  $\partial \Omega$ . The technical assumptions for the matrix valued functions  $\beta_1, \beta_2$  and the scalar functions  $j, \varrho_1, \varrho_2, f_1, f_2$  will be introduced in Section 4. A related work is [31, Ch. 10, pp. 277], which is based on the research article [26], where a semilinear problem without transmission conditions in a Hölder space setting is studied.

There are two main approaches to compute topological derivatives for PDE constrained shape functions. A typical and general strategy to obtain the topological sensitivity is to derive the asymptotic expansion of the partial differential equation with respect to the singular perturbation of the shape [29, 30]. For our problem above this would amount to prove that an expansion of the form (see [31, p. 280])

$$u_\varepsilon(x) = u(x) + \varepsilon K_1(\varepsilon^{-1}x) + \varepsilon^2(K_2(\varepsilon^{-1}x) + u'(x)) + \mathcal{R}_\varepsilon(x) \quad (1.2)$$

exists. Here  $K_1, K_2$  are so-called called boundary layer correctors, which solve certain exterior boundary value problems and  $u'$  is called regular corrector and solves a linearised system. The

function  $u_\varepsilon$  denotes the solution to (S) for the singular perturbed domain  $\Omega_\varepsilon$  and  $\mathcal{R}_\varepsilon(x)$  is an appropriate remainder. However, the proof of an expansion like (1.2) can technically involved and depends very much on the problem; [26].

A second approach to compute the topological derivative is presented in [5] and based on a perturbed adjoint equation, see also [5, 6, 11, 22, 23] and [28]. A key of this method is to prove

$$\begin{aligned} u_\varepsilon(x) &= u(x) + \varepsilon K_1(\varepsilon^{-1}x) + \mathcal{R}_\varepsilon^1(x), \\ p_\varepsilon(x) &= p(x) + \varepsilon Q(\varepsilon^{-1}x) + \mathcal{R}_\varepsilon^2(x), \end{aligned} \quad (1.3)$$

where  $K_1$  is the same as in (1.2),  $Q$  is the solution to an exterior problem, and  $\mathcal{R}_\varepsilon^1, \mathcal{R}_\varepsilon^2$  are appropriate remainder that have to go to zero in some function space. Here  $p_\varepsilon$  is the solution to a certain perturbed adjoint equation depending on the derivative of  $J$ ; see [5]. As a by-product of this approach one obtains the topological sensitivity for non-transmission type problems where Neumann boundary conditions on the inclusion are imposed. However, the proof of the expansions (1.3), particularly for nonlinear problems, can be technically involved and necessitate knowledge of the asymptotic behaviour of  $Q$  and  $K_1$  at infinity.

In this paper we will show that neither the expansion (1.2) nor (1.3) are necessary to obtain the topological sensitivity for (S). For this purpose, we use a Lagrangian approach which uses the averaged adjoint variable  $q_\varepsilon$  [15, 36, 37]. The key ingredient, which leads to the existence of the topological derivative of (C), is the convergence property

$$\nabla \left( \frac{q_\varepsilon(z + \varepsilon x) - q(z + \varepsilon x)}{\varepsilon} \right) \rightharpoonup \nabla Q \quad \text{weakly in } L_2(\mathbf{R}^d)^d, \quad (1.4)$$

where  $Q$  is the same function as in (1.3). The averaged adjoint variable reduces to the usual adjoint in the unperturbed situation, that is,  $q_0 = q = p = p_0$ . We emphasise that the weak convergence property (1.4) is a relaxation of (1.2) and (1.3), since no remainder estimates are necessary. In addition no further knowledge about the asymptotic behaviour of  $Q$  at infinity is needed. We will demonstrate that the proof of (1.4) is constructive in that it reveals the equation  $Q$  must satisfy. This is particularly important when dealing with other more complicated nonlinear equations, e.g., quasilinear equations. We will show that our strategy also allows, with minor changes, to treat the extremal case where  $\beta_1, \varrho_1, f_1 = 0$ , i.e., the transmission problem (S) reduces to a semilinear equation with homogeneous Neumann boundary conditions on  $\partial\Omega$ . Compared to previous works we can prove the existence of the topological derivative under milder assumptions on the regularity of the inclusion.

## Notation and definitions

**Notation for derivatives** Let  $(\varepsilon, u, q) \mapsto G(\varepsilon, u, q) : [0, \tau] \times X \times Y \rightarrow \mathbf{R}$  be a function defined on real normed vector spaces  $X, Y$ , and  $\tau > 0$ . When the limits exist we use the following notation:

$$v \in X, \quad \partial_u G(\varepsilon, u, q)(v) := \lim_{t \searrow 0} \frac{G(\varepsilon, u + tv, q) - G(\varepsilon, u, q)}{t} \quad (1.5)$$

$$w \in Y, \quad \partial_q G(\varepsilon, u, q)(w) := \lim_{t \rightarrow 0} \frac{G(\varepsilon, u, q + tw) - G(\varepsilon, u, q)}{t}. \quad (1.6)$$

The notation  $t \searrow 0$  means that  $t$  goes to 0 by strictly positive values.

**Miscellaneous notation** Standard  $L^p$  spaces and Sobolev spaces on an open set  $\Omega \subset \mathbf{R}^d$  are denoted  $L_p(\Omega)$  and  $W_p^k(\Omega)$ , respectively, where  $p \geq 1$  and  $k \geq 1$ . In case  $p = 2$  and  $k \geq 1$  we set as usual  $H^k(\Omega) := W_2^k(\Omega)$ . Vector valued spaces are denoted  $L_p(\Omega)^d := L_p(\Omega, \mathbf{R}^d)$  and  $W_p^k(\Omega)^d := W_p^k(\Omega, \mathbf{R}^d)$ . We write  $\int_A f dx := \frac{1}{|A|} \int_A f dx$  to indicate the average of  $f$  over a measurable set  $A$  with measure  $|A| < \infty$ . The Hölder conjugate of  $p \in [1, \infty)$  is defined by  $p' := p/(p-1)$ . For  $1 \leq p < d$  we denote by  $p^* := dp/(d-p)$  the Sobolev conjugate of  $p$ . Given a normed vector space  $V$  we denote by  $\mathcal{L}(V, \mathbf{R})$  the space of linear and continuous functions on  $V$ . We denote by  $B_\delta(x)$  the ball centred at  $x$  with radius  $\delta > 0$  and set  $\bar{B}_\delta(x) := \overline{B_\delta(x)}$ .

## 2 Abstract averaged adjoint framework

### 2.1 Lagrangians and infimum

The following material can be found in [15]. We begin with the definition of a Lagrangian function.

**Definition 2.1.** Let  $X$  and  $Y$  be vector spaces and  $\tau > 0$ . A *parametrised Lagrangian* (or short Lagrangian) is a function

$$(\varepsilon, u, q) \mapsto G(\varepsilon, u, q) : [0, \tau] \times X \times Y \rightarrow \mathbf{R},$$

satisfying for all  $(\varepsilon, u) \in [0, \tau] \times X$ ,

$$q \mapsto G(\varepsilon, u, q) \quad \text{is affine on } Y. \quad (2.1)$$

The next definition formalises the notion of state and perturbed state variable associated with  $G$ .

**Definition 2.2.** For  $\varepsilon \in [0, \tau]$  we define the *state equation* by: find  $u_\varepsilon \in X$ , such that

$$\text{find } u_\varepsilon \in X \text{ such that } \partial_q G(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in X. \quad (2.2)$$

The set of solution of (2.2) (for  $\varepsilon$  fixed) is denoted by  $E(\varepsilon)$ . For  $\varepsilon = 0$ , the elements of  $E(\varepsilon)$  are called *unperturbed states* (or short states) and for  $\varepsilon > 0$  they are referred to as *perturbed states*.

**Definition 2.3.** We introduce for  $\varepsilon \in [0, \varepsilon]$  the set of minimisers

$$X(\varepsilon) = \{u_\varepsilon \in E(\varepsilon) : \inf_{u \in E(\varepsilon)} G(\varepsilon, u, 0) = G(\varepsilon, u_\varepsilon, 0)\}. \quad (2.3)$$

Notice that  $X(\varepsilon) \subset E(\varepsilon)$  and that  $X(\varepsilon) = E(\varepsilon)$  whenever  $E(\varepsilon)$  is a singleton. We associate with the *parameter*  $\varepsilon$  the *parametrised infimum*

$$\varepsilon \mapsto g(\varepsilon) := \inf_{u \in E(\varepsilon)} G(\varepsilon, u, 0) : [0, \tau] \rightarrow \mathbf{R}. \quad (2.4)$$

We now recall sufficient conditions introduced in [15] under which the limit

$$d_\ell g(0) := \lim_{\varepsilon \searrow 0} \frac{g(\varepsilon) - g(0)}{\ell(\varepsilon)} \quad (2.5)$$

exists, where  $\ell : [0, \tau] \rightarrow \mathbf{R}$  is a given function satisfying  $\ell(0) = 0$  and  $\ell(\varepsilon) > 0$  for  $\varepsilon \in (0, \tau]$ . The key ingredient is the so-called *averaged adjoint equation*. The definition of the averaged adjoint equation requires that the set of states is nonempty:

**Assumption (H0).** For all  $\varepsilon \in [0, \tau]$  the set  $X(\varepsilon)$  is nonempty.

Before we can introduce the averaged adjoint equation we need the following hypothesis.

**Assumption (H1).** For all  $\varepsilon \in [0, \tau]$  and  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$  we assume:

- (i) For all  $q \in Y$ , the mapping  $s \mapsto G(\varepsilon, su_\varepsilon + (1-s)u_0, q) : [0, 1] \rightarrow \mathbf{R}$  is absolutely continuous.
- (ii) For all  $(\varphi, q) \in X \times Y$  and almost all  $s \in (0, 1)$  the function

$$s \mapsto \partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, q)(\varphi) : [0, 1] \rightarrow \mathbf{R} \quad (2.6)$$

is well-defined and belongs to  $L_1(0, 1)$ .

**Remark 2.4.** Notice that item (i) implies that for all  $\varepsilon \in [0, \tau]$ ,  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$  and  $q \in Y$ ,

$$G(\varepsilon, u_\varepsilon, q) = G(\varepsilon, u_0, q) + \int_0^1 \partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, q)(u_\varepsilon - u_0) ds. \quad (2.7)$$

This follows at once by applying the fundamental theorem of calculus to  $s \mapsto G(\varepsilon, su_\varepsilon + (1-s)u_0, q)$  on  $[0, 1]$ .

The following gives the definition of the averaged adjoint equation; see [38].

**Definition 2.5.** Given  $\varepsilon \in [0, \tau]$  and  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$ , the *averaged adjoint state equation* is defined as follows: find  $q_\varepsilon \in X$ , such that

$$\int_0^1 \partial_u G(\varepsilon, su_\varepsilon + (1-s)u_0, q_\varepsilon)(\varphi) ds = 0 \quad \text{for all } \varphi \in X. \quad (2.8)$$

For every triplet  $(\varepsilon, u_0, u_\varepsilon)$  the set of solutions of (2.8) is denoted by  $Y(\varepsilon, u_0, u_\varepsilon)$  and its elements are referred to as *adjoint states* for  $\varepsilon = 0$  and *averaged adjoint states* for  $\varepsilon > 0$ .

Notice that  $Y(0, u_0) := Y(0, u_0, u_0)$  is the usual set of *adjoint states* associated with  $u_0$ ,

$$Y(0, u_0) = \{q \in Y : \forall \varphi \in X, \partial_u G(0, u_0, q)(\varphi) = 0\}. \quad (2.9)$$

An important consequence of the introduction of the averaged adjoint state is the following identity: for all  $\varepsilon \in [0, \tau]$ ,  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$  and  $q_\varepsilon \in Y(\varepsilon, u_0, u_\varepsilon)$ ,

$$g(\varepsilon) = G(\varepsilon, u_\varepsilon, q_\varepsilon) = G(\varepsilon, u_0, q_\varepsilon). \quad (2.10)$$

This is readily seen by substituting  $q_\varepsilon$  into equation (2.7). The following result is an extension of [15, Thm. 3.1]. We refer the reader to [14, 38] for further results on the averaged adjoint approach and [36] for more examples involving the shape derivative.

**Theorem 2.6** ([15]). Let Hypotheses (H0) and (H1) and the following conditions be satisfied.

(H2) For all  $\varepsilon \in [0, \tau]$  and  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$  the set  $Y(\varepsilon, u_0, u_\varepsilon)$  is nonempty.

(H3) For all  $u_0 \in X(0)$  and  $q_0 \in Y(0, u_0)$  the limit

$$\partial_\ell G(0, u_0, q_0) := \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u_0, q_0) - G(0, u_0, q_0)}{\ell(\varepsilon)} \quad \text{exists.} \quad (2.11)$$

(H4) There exist sequences  $(u_\varepsilon)$  and  $(q_\varepsilon)$ , where  $u_\varepsilon \in X(\varepsilon)$  and  $q_\varepsilon \in Y(\varepsilon, u_0, u_\varepsilon)$ , such that the limit

$$R := \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u_0, q_\varepsilon) - G(\varepsilon, u_0, q_0)}{\ell(\varepsilon)} \quad \text{exists.} \quad (2.12)$$

Then we have

$$d_\ell g(0) = \partial_\ell G(0, u_0, q_0) + R. \quad (2.13)$$

Moreover,  $R = R(u_0, q_0)$  does not depend on the choice of the sequences  $(u_\varepsilon)$  and  $(q_\varepsilon)$ , but only on  $u_0$  and  $q_0$ .

*Proof.* Thanks to Hypotheses (H0)-(H2) the sets  $X(\varepsilon)$  and  $Y(\varepsilon, u_0, u_\varepsilon)$  are nonempty for all  $\varepsilon$ . Therefore in view of (2.10) we have for all  $\varepsilon \in [0, \tau]$ ,  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$  and  $q_\varepsilon \in Y(\varepsilon, u_\varepsilon, u_0)$ ,

$$\begin{aligned} g(\varepsilon) - g(0) &= G(\varepsilon, u_0, q_\varepsilon) - G(0, u_0, q_0) \\ &= G(\varepsilon, u_0, q_\varepsilon) - G(\varepsilon, u_0, q_0) + G(\varepsilon, u_0, q_0) - G(0, u_0, q_0). \end{aligned} \quad (2.14)$$

Thus selecting  $(u_\varepsilon)$  and  $(q_\varepsilon)$  from Hypothesis (H4) and using Hypothesis (H3) we obtain

$$\begin{aligned} d_\ell g(0) &= \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u_0, q_0) - G(0, u_0, q_0)}{\ell(\varepsilon)} + \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u_0, q_\varepsilon) - G(\varepsilon, u_0, q_0)}{\ell(\varepsilon)} \\ &= \partial_\ell G(0, u_0, q_0) + R. \end{aligned} \quad (2.15)$$

It follows from (2.15) that  $R$  only depends on  $u_0$  and  $q_0$ . □

**Remark 2.7.** An important application of Theorem 2.6 is the computation of shape derivatives for which one chooses  $\ell(\varepsilon) = \varepsilon$ , see e.g., [36, 38]. In this case one typically has  $R(u_0, q_0) = 0$ , which means

$$d_\varepsilon g(0) = \partial_\varepsilon G(0, u_0, q_0). \quad (2.16)$$

However for the topological derivative, in which case  $\ell(\varepsilon) \neq \varepsilon$ , the term  $R(u_0, q_0)$  is typically not equal to zero as shown by the one dimensional example of [14].

### 3 Linear elliptic equations in $\mathbf{R}^d$

In preparation for the study of the semilinear problem (S), we first recall existence and uniqueness results for the following exterior problem. Let  $\omega \subset \mathbf{R}^d$  be an open and bounded set, and let  $\zeta \in \mathbf{R}^d$  be a vector. Given a suitable vector space  $V$  of functions  $\mathbf{R}^d \rightarrow \mathbf{R}$  we consider: find  $Q_\zeta \in V$  such that

$$\int_{\mathbf{R}^d} A \nabla \psi \cdot \nabla Q_\zeta \, dx = \int_{\omega} \zeta \cdot \nabla \psi \, dx \quad \text{for all } \psi \in V. \quad (3.1)$$

Here  $A : \mathbf{R}^d \rightarrow \mathbf{R}^{d \times d}$  is a measurable, uniformly coercive (not necessarily symmetric) matrix-valued functions, that is, there are constants  $M_1, M_2 > 0$ , such that

$$M_1 |v|^2 \leq A(x)v \cdot v \leq M_2 |v|^2 \quad \text{for a.e } x \in \mathbf{R}^d \text{ and all } v \in \mathbf{R}^d. \quad (3.2)$$

The well-posedness of (3.1) can be achieved by several choices of  $V$ . The most popular ones are weighted Sobolev spaces; see [17]. In the next section we discuss a more straight forward choice for  $V$ .

### 3.1 Solution in the Beppo-Levi space

**Definition 3.1.** For  $d \geq 1$  define

$$BL(\mathbf{R}^d) := \{u \in H_{loc}^1(\mathbf{R}^d) : \nabla u \in L_2(\mathbf{R}^d)^d\}. \quad (3.3)$$

Then the *Beppo-Levi space* is defined by

$$\dot{B}L(\mathbf{R}^d) := BL(\mathbf{R}^d)/\mathbf{R}, \quad (3.4)$$

where  $/\mathbf{R}$  means that we quotient out the constant functions. We denote by  $[u]$  the equivalence classes of  $\dot{B}L(\mathbf{R}^d)$ . The Beppo-Levi space is equipped with the norm

$$\|[u]\|_{\dot{H}^1(\mathbf{R}^d)} := \|\nabla u\|_{L_2(\mathbf{R}^d)^d}, \quad u \in [u]. \quad (3.5)$$

The Beppo-Levi space is a Hilbert space (see [17, 32] and also [8]) and  $C_c^\infty(\mathbf{R}^d)/\mathbf{R}$  is dense in  $\dot{B}L(\mathbf{R}^d)$ .

**Lemma 3.2.** Let  $d \geq 1$  and suppose that  $A$  satisfies (3.1). Then there exists a unique equivalence class  $[Q] \in \dot{B}L(\mathbf{R}^d)$  solving

$$\int_{\mathbf{R}^d} A \nabla \psi \cdot \nabla Q \, dx = \int_{\omega} \zeta \cdot \nabla \psi \, dx \quad \text{for all } \psi \in BL(\mathbf{R}^d). \quad (3.6)$$

*Proof.* This is a direct consequence of the theorem of Lax-Milgram.  $\square$

As shown in [32], in dimension  $d \geq 3$  every equivalence class  $[u]$  of  $\dot{B}L(\mathbf{R}^d)$  contains an element  $u_0 \in [u]$  that is in turn contained in the Banach space

$$\mathbf{E}_2(\mathbf{R}^d) := \{u \in L_{2^*}(\mathbf{R}^d) : \nabla u \in L_2(\mathbf{R}^d)^d\} \quad (3.7)$$

equipped with the norm

$$\|u\|_{\mathbf{E}_2} := \|u\|_{L_{2^*}} + \|\nabla u\|_{(L_2)^d}. \quad (3.8)$$

This follows at once since  $C_c^\infty(\mathbf{R}^d)/\mathbf{R}$  is dense in  $\dot{B}L(\mathbf{R}^d)$  and from the Gagliardo-Nirenberg-Sobolev inequality; see [32]. As a result for  $d \geq 3$  we can replace the Beppo-Levi space by  $\mathbf{E}_2(\mathbf{R}^d)$  and can even consider a more general problem.

**Lemma 3.3.** Let  $d \geq 3$ . Suppose that  $A$  satisfies (3.1) and  $A = A^\top$  a.e. on  $\mathbf{R}^d$ . Then for every  $F \in \mathcal{L}(\mathbf{E}_2(\mathbf{R}^d), \mathbf{R})$  there exists a unique solution  $Q \in \mathbf{E}_2(\mathbf{R}^d)$  to

$$\int_{\mathbf{R}^d} A \nabla \psi \cdot \nabla Q \, dx = F(\psi) \quad \text{for all } \psi \in \mathbf{E}_2(\mathbf{R}^d). \quad (3.9)$$

*Proof.* A proof can be found in the appendix.  $\square$

So for  $d \geq 3$  equation (3.1) admits a unique solution in  $V := \mathbf{E}_2(\mathbf{R}^d)$  since obviously  $F(\varphi) := \int_{\omega} \zeta \cdot \nabla \varphi \, dx \in \mathcal{L}(\mathbf{E}_2(\mathbf{R}^d), \mathbf{R})$ .

### 3.2 Relation to weighted Sobolev spaces

Since the exterior equation (3.1) is, as we will see later, of paramount importance for the first topological derivative we review here an alternative choice for the space  $V$ , namely, a weighted Sobolev Hilbert space. We follow the presentation of [7], where a more general situation than the following is considered.

For this purpose we introduce the weight function  $w \in L_1(\mathbf{R}^d)$  defined by

$$w(x) := (1 + |x|^2)^{-\gamma_d} \quad (3.10)$$

where  $\gamma_d := \frac{d}{2} + \delta$  and  $\delta \in (0, 1/2)$  is arbitrary, but fixed. Since the weight satisfies  $|w|^p \leq |w|$  on  $\mathbf{R}^d$  for  $p \in [1, \infty)$  it also follows that  $w \in L_p(\mathbf{R}^d)$  for all  $p \in [1, \infty)$ .

**Definition 3.4.** The weighted Hilbert Sobolev space  $H_w^1(\mathbf{R}^d)$  is defined by

$$H_w^1(\mathbf{R}^d) := \{u : \mathbf{R}^d \rightarrow \mathbf{R} \text{ measurable} : \sqrt{w}u \in L_2(\mathbf{R}^d), \quad \nabla u \in L_2(\mathbf{R}^d)^d\}. \quad (3.11)$$

The norm on  $H_w^1(\mathbf{R}^d)$  is given by  $\|u\|_{H_w^1} := \|\sqrt{w}u\|_{L_2} + \|\nabla u\|_{(L_2)^d}$ .

The weight  $w$  is chosen in such a way that the set of constant functions on  $\mathbf{R}^d$  are contained in  $H_w^1(\mathbf{R}^d)$ . Therefore it is clear that (3.1) can only be uniquely solvable in  $H_w^1(\mathbf{R}^d)$  up to a constant. A remedy is to consider the quotient space

$$\dot{H}_w^1(\mathbf{R}^d) := H_w^1(\mathbf{R}^d)/\mathbf{R} \quad (3.12)$$

and equip this space, as in [7], with the quotient norm

$$\|[u]\|_{\dot{H}_w^1} := \inf_{c \in \mathbf{R}} \|u + c\|_{H_w^1(\mathbf{R}^d)}, \quad (3.13)$$

where  $[u]$  denote the equivalence classes of  $\dot{H}_w^1(\mathbf{R}^d)$ . In [7, Cor. C.5, p. 23] it is shown that there is a constant  $c > 0$ , such that,  $\|[u]\|_{\dot{H}_w^1} \leq c \|\nabla u\|_{(L_2)^d}$  for all  $u \in H_w^1(\mathbf{R}^d)$ . Therefore existence of a solution to (3.1) follows directly from the theorem of Lax-Milgram.

In the following lemma let us agree that the Sobolev conjugate of 2 in dimension two is given by  $\infty$ , i.e.  $2^* := \infty$  if  $d = 2$ .

**Lemma 3.5.** We have  $\mathbf{E}_2(\mathbf{R}^d) \hookrightarrow H_w^1(\mathbf{R}^d)$  for all  $d \geq 2$ , i.e., there is a constant  $C > 0$ , such that

$$\|u\|_{H_w^1} \leq C \|u\|_{\mathbf{E}_2} \quad \text{for all } u \in \mathbf{E}_2(\mathbf{R}^d). \quad (3.14)$$

*Proof.* Let  $u \in \mathbf{E}_2(\mathbf{R}^d)$  be given so that  $u \in L_{2^*}(\mathbf{R}^d)$ . In case  $d \neq 2$ , we have  $2^* = \frac{2d}{d-2}$ . Therefore the Hölder conjugate of  $2^*/2$  is given by  $\frac{2^*}{2^*-2} = d/2$  and Hölder's inequality yields

$$\int_{\mathbf{R}^d} wu^2 dx \leq \|w\|_{L_{d/2}(\mathbf{R}^d)} \|u\|_{L_{2^*}(\mathbf{R}^d)}^2. \quad (3.15)$$

Since  $d \geq 2$  we deduce  $\sqrt{w}u \in L_2(\mathbf{R}^d)$  and since by definition also  $\nabla u \in L_2(\mathbf{R}^d)^d$  we deduce  $\mathbf{E}_2(\mathbf{R}^d) \subset H_w^1(\mathbf{R}^d)$  and the continuity of the embedding follows from (3.15). In case  $d = 2$  we have  $2^* = \infty$  and thus Hölder's inequality directly gives (3.15) and thus the continuous embedding.  $\square$



## 4 The topological derivative via Lagrangian

In this section we show how Theorem 2.6 of Section 2 can be used to compute the topological derivative for a semilinear transmission problem. Our approach is related to the one of [5] (see also [4]), where also a perturbed adjoint equation is used, too. However the main difference here is that we only need to work with weakly converging subsequences and do not need to know any asymptotic behaviour of the limiting function.

### 4.1 Weak formulation and a priori estimates

In the following exposition we restrict ourselves to the shape function

$$J(\Omega) = \int_{\mathbf{D}} u^2 dx, \quad (4.1)$$

where  $u = u_{\Omega} \in H_0^1(\mathbf{D}) \cap L_{\infty}(\mathbf{D})$  is the weak solution of (S):

$$\int_{\mathbf{D}} \beta_{\Omega} \nabla u \cdot \nabla \varphi + \varrho_{\Omega}(u) \varphi dx = \int_{\mathbf{D}} f_{\Omega} \varphi dx \quad \text{for all } \varphi \in H_0^1(\mathbf{D}), \quad (4.2)$$

where  $\beta_{\Omega} : \mathbf{R}^d \rightarrow \mathbf{R}^{d \times d}$  and  $f_{\Omega} : \mathbf{R}^d \rightarrow \mathbf{R}$  are defined by

$$\beta_{\Omega}(x) := \begin{cases} \beta_1(x) & \text{for } x \in \Omega \\ \beta_2(x) & \text{for } x \in \mathbf{R}^d \setminus \bar{\Omega} \end{cases}, \quad f_{\Omega}(x) := \begin{cases} f_1(x) & \text{for } x \in \Omega \\ f_2(x) & \text{for } x \in \mathbf{R}^d \setminus \bar{\Omega} \end{cases}, \quad (4.3)$$

and similarly  $\varrho_{\Omega}$  is defined by

$$\varrho_{\Omega}(u) := \begin{cases} \varrho_1(u) & \text{for } x \in \Omega \\ \varrho_2(u) & \text{for } x \in \mathbf{R}^d \setminus \bar{\Omega}. \end{cases} \quad (4.4)$$

Notice that  $\beta_{\Omega} = \beta_1 \chi_{\Omega} + \beta_2 \chi_{\mathbf{R}^d \setminus \Omega}$ ,  $f_{\Omega} = f_1 \chi_{\Omega} + f_2 \chi_{\mathbf{R}^d \setminus \Omega}$  and  $\varrho_{\Omega}(u) = \varrho_1(u) \chi_{\Omega} + \varrho_2(u) \chi_{\mathbf{R}^d \setminus \Omega}$ .

It can be checked that the following proofs remain true when the shape function (4.1) is replaced by (C) from the introduction under the assumption that  $j$  is sufficiently smooth. However, in favour of a clearer presentation we use the simplified cost function (4.1). The functions  $\beta_i$ ,  $\varrho_i$  and  $f_i$  are specified in the following assumption. The extremal case where  $\beta_1$ ,  $\varrho_1$ ,  $f_1$  are zero will be discussed in the last section.

**Assumption 1.** (a) For  $i = 1, 2$ , we assume that  $\beta_i \in C^1(\mathbf{R}^d)^{d \times d}$  and that there are constants  $\beta_m, \beta_M > 0$ , such that

$$\beta_m |v|^2 \leq \beta_i(x) v \cdot v \leq \beta_M |v|^2 \quad \text{for all } x \in \mathbf{R}^d, v \in \mathbf{R}^d. \quad (4.5)$$

(b) For  $i = 1, 2$ , we assume  $\varrho_i \in C^1(\mathbf{R})$ ,  $\varrho_i(0) = 0$  and the monotonicity condition

$$(\varrho_i(x) - \varrho_i(y))(x - y) \geq 0 \quad \text{for all } x, y \in \mathbf{R}. \quad (4.6)$$

(c) For  $i = 1, 2$ , we assume  $f_i \in H^1(\mathbf{D})$  if  $f_1 = f_2$  and  $f_i \in H^1(\mathbf{D}) \cap C(\mathbf{D})$  if  $f_1 \neq f_2$ .

Notice that since for  $x \in D$  the matrix  $\beta_\Omega(x)$  is either equal to  $\beta_1(x)$  or  $\beta_2(x)$  and in view of the bound (4.5), we have

$$\beta_m |v|^2 \leq \beta_\Omega(x) v \cdot v \quad \text{for all } x \in \mathbf{R}^d, v \in \mathbf{R}^d. \quad (4.7)$$

Similarly, in view of the monotonicity property (4.6) and  $\varrho_i(0) = 0$ , we get

$$0 \leq \varrho_\Omega(x)x \quad \text{for all } x \in \mathbf{R}^d. \quad (4.8)$$

**Lemma 4.1.** (i) Let  $f \in L_r(D)$ ,  $r > d/2$ . Then for every measurable set  $\Omega \subset D$  there is a unique solution  $u_\Omega$  of (4.2). Moreover, there is a constant  $C$  independent of  $\Omega$ , such that

$$\|u_\Omega\|_{L^\infty(D)} + \|u_\Omega\|_{H_0^1(D)} \leq C \|f\|_{L_r(D)}. \quad (4.9)$$

(ii) For every  $z \in D \setminus \bar{\Omega}$ , we find  $\delta > 0$ , such that  $u_\Omega \in H^3(B_\delta(z))$ .

*Proof.* (i) Our assumptions imply that we can apply [39, Theorem 4.5] which gives the existence of a solution to (4.2) and also the a priori bound (4.9). As pointed out in this reference the constant  $C$  is independent of the nonlinearity  $\varrho_\Omega$ .

(ii) Let  $U := D \setminus \bar{\Omega}$  and  $z \in U$ . The restriction of  $u$  to  $U$  solves

$$\int_U \beta_2 \nabla u \cdot \nabla \varphi \, dx = \int_U \tilde{f} \varphi \, dx \quad \text{for all } \varphi \in H_0^1(U), \quad (4.10)$$

with right-hand side  $\tilde{f}(x) := f_2(x) - \varrho_2(u(x))$ . Since  $\nabla \tilde{f} = \nabla f_2 - \varrho_2'(u) \nabla u \in L_2(U)^d$  we have  $\tilde{f} \in H^1(U)$ . Hence  $u \in H_{\text{loc}}^3(U)$  by standard regularity theory for elliptic PDEs; see, e.g., [20, Thm. 2, p. 314]. Since  $U$  is open we can choose  $\delta > 0$  such that  $B_\delta(z) \Subset U$ . This finishes the proof.  $\square$

**Remark 4.2.** Although we restrict ourselves to Dirichlet boundary conditions in (S) other boundary conditions, e.g., Neumann boundary conditions, can be considered as well. This only requires minimal changes in the following analysis and we will make remarks at the relevant places.

## 4.2 The parametrised Lagrangian

From now on we fix:

- an open and bounded set  $\omega \subset \mathbf{R}^d$  with  $0 \in \omega$ ,
- an open set  $\Omega \Subset D$  and a point  $z \in D \setminus \bar{\Omega}$ ,
- the perturbation  $\Omega_\varepsilon := \Omega \cup \omega_\varepsilon(z)$ , where  $\omega_\varepsilon(z) := z + \varepsilon \omega$  and  $\varepsilon \in [0, \tau]$ ,  $\tau > 0$ .

To simplify notation we will often write  $\omega_\varepsilon$  instead of  $\omega_\varepsilon(z)$ . Let  $X = Y = H_0^1(D)$  and introduce the Lagrangian  $G : [0, \tau] \times X \times Y \rightarrow \mathbf{R}$  associated with the perturbation  $\Omega_\varepsilon$  by

$$G(\varepsilon, u, q) := \int_D u^2 \, dx + \int_D \beta_\varepsilon \nabla u \cdot \nabla q + \varrho_\varepsilon(u)q \, dx - \int_D f_\varepsilon q \, dx, \quad (4.11)$$

where we use the abbreviations

$$\beta_\varepsilon := \beta_1 \chi_{\Omega_\varepsilon} + \beta_2 \chi_{\mathbf{R}^d \setminus \Omega_\varepsilon}, \quad f_\varepsilon := f_1 \chi_{\Omega_\varepsilon} + f_2 \chi_{\mathbf{R}^d \setminus \Omega_\varepsilon}, \quad \varrho_\varepsilon(u) := \varrho_1(u) \chi_{\Omega_\varepsilon} + \varrho_2(u) \chi_{\mathbf{R}^d \setminus \Omega_\varepsilon}. \quad (4.12)$$

We are now going to verify that Hypotheses (H0)-(H4) are satisfied with  $\ell(\varepsilon) = |\omega_\varepsilon|$ . Moreover, we will determine the explicit form of  $R(u, p)$ .

**Remark 4.3** (Removing an inclusion). We only treat the case of "adding" a hole here, i.e.,  $\Omega_\varepsilon := \Omega \cup \omega_\varepsilon(z)$  for  $z \in D \setminus \bar{\Omega}$ . The second case of "removing" a hole, i.e.,  $\Omega_\varepsilon := \Omega \setminus \bar{\omega}_\varepsilon(z)$  for  $z \in \Omega$  can be dealt with in the same way.

### 4.3 Analysis of the perturbed state equation

The *perturbed state equation* reads: find  $u_\varepsilon \in H_0^1(D)$  such that

$$\partial_q G(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in H_0^1(D), \quad (4.13)$$

or equivalently  $u_\varepsilon \in H_0^1(D)$  satisfies,

$$\int_D \beta_\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi + \varrho_\varepsilon(u_\varepsilon) \varphi \, dx = \int_D f_\varepsilon \varphi \, dx \quad \text{for all } \varphi \in H_0^1(D). \quad (4.14)$$

Henceforth we write  $u := u_0$  to simplify notation. Since (4.14) is precisely (4.2) with  $\Omega = \Omega_\varepsilon$ , we infer from Lemma 4.1 that (4.14) admits a unique solution. This means that  $E(\varepsilon) = \{u_\varepsilon\}$  is a singleton and thus  $E(\varepsilon) = X(\varepsilon)$  and Hypothesis (H0) is satisfied. From this and Assumption 1 we also infer that Hypothesis (H1) is satisfied. We proceed by showing a Hölder-type estimate for  $(u_\varepsilon)$ .

**Lemma 4.4.** There is a constant  $C > 0$ , such that for all small  $\varepsilon > 0$ ,

$$\|u_\varepsilon - u\|_{H^1(D)} \leq C \varepsilon^{d/2}. \quad (4.15)$$

*Proof.* We obtain from (4.14)

$$\begin{aligned} \int_D \beta_\varepsilon \nabla(u_\varepsilon - u) \cdot \nabla \varphi \, dx + \int_D (\varrho_\varepsilon(u_\varepsilon) - \varrho_\varepsilon(u)) \varphi \, dx &= - \underbrace{\int_{\omega_\varepsilon} (\beta_1 - \beta_2) \nabla u \cdot \nabla \varphi \, dx}_{=: I(\varepsilon, \varphi)} \\ &\quad - \underbrace{\int_{\omega_\varepsilon} (\varrho_1(u) - \varrho_2(u)) \varphi \, dx}_{=: II(\varepsilon, \varphi)} \\ &\quad + \underbrace{\int_{\omega_\varepsilon} (f_1 - f_2) \varphi \, dx}_{=: III(\varepsilon, \varphi)} \end{aligned} \quad (4.16)$$

for all  $\varphi \in H_0^1(D)$ . Hence, since  $u \in C^1(\bar{B}_\delta(z))$  for  $\delta > 0$  sufficiently small, we can apply Hölder's inequality to obtain

$$\begin{aligned} |I(\varepsilon, \varphi)| &\leq \|\beta_1 - \beta_2\|_{C(\bar{B}_\delta(z))^{d \times d}} \|\nabla u\|_{C(\bar{B}_\delta(z))^d} \sqrt{|\omega_\varepsilon|} \|\nabla \varphi\|_{L_2(D)^d} \\ |II(\varepsilon, \varphi)| &\leq \|\varrho_1(u) - \varrho_2(u)\|_{C(\bar{B}_\delta(z))} \sqrt{|\omega_\varepsilon|} \|\varphi\|_{L_2(D)} \end{aligned} \quad (4.17)$$

and

$$|III(\varepsilon, \varphi)| \leq \|f_1 - f_2\|_{L_\infty(B_\delta(z))} \sqrt{|\omega_\varepsilon|} \|\varphi\|_{L_2(D)}. \quad (4.18)$$

Now testing (4.16) with  $\varphi = u_\varepsilon - u$  and using (4.17) together with Assumption 1, (a)-(b) lead to the desired estimate.  $\square$

#### 4.4 Analysis of the averaged adjoint equation

We introduce for  $\varepsilon \in [0, \tau]$  the (not necessarily symmetric) bilinear form  $b_\varepsilon : H_0^1(D) \times H_0^1(D) \rightarrow \mathbf{R}$  by

$$b_\varepsilon(\psi, \varphi) := \int_D \beta_\varepsilon \nabla \psi \cdot \nabla \varphi + \left( \int_0^1 \varrho'_\varepsilon(su_\varepsilon + (1-s)u) ds \right) \varphi \psi dx, \quad (4.19)$$

where  $\varrho'_\varepsilon(u) := \varrho'_1(u)\chi_{\Omega_\varepsilon} + \varrho'_2(u)\chi_{\mathbf{R}^d \setminus \Omega_\varepsilon}$ . Then the averaged adjoint equation (2.8) for the Lagrangian  $G$  given by (4.11) reads: find  $q_\varepsilon \in H_0^1(D)$  such that

$$b_\varepsilon(\psi, q_\varepsilon) = - \int_D (u + u_\varepsilon) \psi dx \quad (4.20)$$

for all  $\psi \in H_0^1(D)$ . In view of Assumption 1 we have  $\varrho'_\varepsilon \geq 0$  and  $\beta_\varepsilon \geq \beta_m I$  and thus  $b_\varepsilon$  is coercive,

$$b_\varepsilon(\psi, \psi) \geq \beta_m \|\nabla \psi\|_{L_2(D)^d}^2 \quad \text{for all } \psi \in H_0^1(D), \varepsilon \in [0, \tau]. \quad (4.21)$$

As for the state equation, we use the notation  $q := q^0$ .

**Lemma 4.5.** (i) For each  $\varepsilon \in [0, \tau]$  equation (4.20) admits a unique solution.

(ii) We find for every  $z \in D \setminus \bar{\Omega}$  a number  $\delta > 0$ , such that  $q \in H^3(B_\delta(z)) \subset C^1(\bar{B}_\delta(z))$  for  $d \in \{2, 3\}$ .

*Proof.* (i) Since  $b_\varepsilon$  is coercive and continuous on  $H_0^1(D)$ , the theorem of Lax-Milgram shows that (4.20) admits a unique solution.

(ii) The proof is the same as the one for item (ii) of Lemma 4.1 and therefore omitted.  $\square$

The previous lemma shows that  $Y(\varepsilon, u, u_\varepsilon) = \{q_\varepsilon\}$  is a singleton and therefore Hypothesis (H2) is satisfied. We proceed with a Hölder-type estimate for  $\varepsilon \mapsto q_\varepsilon$ .

**Lemma 4.6.** There is a constant  $C > 0$ , such that for all small  $\varepsilon > 0$ ,

$$\|q_\varepsilon - q\|_{H^1(D)} \leq C(\|u_\varepsilon - u\|_{L_2(D)} + \varepsilon^{d/2}). \quad (4.22)$$

*Proof.* Using (4.20) we obtain

$$\begin{aligned} b_\varepsilon(\psi, q_\varepsilon - q) &= b_\varepsilon(\psi, q_\varepsilon) - b_\varepsilon(\psi, q) \\ &\stackrel{(4.20)}{=} - \int_D (u_\varepsilon - u) \psi dx - (b_\varepsilon - b_0)(\psi, q) \end{aligned} \quad (4.23)$$

for all  $\psi \in H_0^1(D)$ . Since furthermore

$$(b_\varepsilon - b_0)(\psi, q) = - \int_{\omega_\varepsilon} (\beta_1 - \beta_2) \nabla q \cdot \nabla \psi dx - \int_{\omega_\varepsilon} \left( \int_0^1 (\varrho'_1 - \varrho'_2)(su_\varepsilon + (1-s)u) ds \right) q \psi dx, \quad (4.24)$$

we obtain using Hölder's inequality and  $q \in C^1(\bar{B}_\delta(z))$ ,

$$\begin{aligned} |(b_\varepsilon - b_0)(\psi, q)| &\leq \|\beta_1 - \beta_2\|_{C(\bar{B}_\delta(z))^{d \times d}} \|\nabla q\|_{C(\bar{B}_\delta(z))^d} \sqrt{|\omega_\varepsilon|} \|\nabla \psi\|_{L_2(D)^d} \\ &\quad + \|\varrho'_1 - \varrho'_2\|_{L_\infty(B_C(0))} \|q\|_{L_2(D)} \|\psi\|_{L_2(D)}. \end{aligned} \quad (4.25)$$

where  $C > 0$  is a constant, such that  $\|u_\varepsilon\|_{L^\infty(D)} \leq C$  for all  $\varepsilon \in [0, \tau]$ . So inserting  $\psi = q_\varepsilon - q$  as test function in (4.23) and using (4.25) yields

$$\beta_m \|\nabla(q_\varepsilon - q)\|_{L_2(D)^d} \leq b_\varepsilon(q_\varepsilon - q, q_\varepsilon - q) \leq C(\sqrt{\omega_\varepsilon} + \|u_\varepsilon - u\|_{L_2(D)}) \|q_\varepsilon - q\|_{H^1(D)}. \quad (4.26)$$

Now the result follows from the Poincaré inequality and  $|\omega_\varepsilon| = \varepsilon^d |\omega|$ .  $\square$

**Remark 4.7.** The proof of estimate (5.22) requires  $q \in C^1(\bar{B}_\delta(z))$ , but not  $q_\varepsilon \in C^1(\bar{B}_\delta(z))$ , which is false in general, since  $\nabla q_\varepsilon$  has a jump across  $\partial \omega_\varepsilon$ .

Let us finish this section with the verification of Hypothesis (H3).

**Lemma 4.8.** We have

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u, q) - G(0, u, q)}{|\omega_\varepsilon|} &= (\beta_1 - \beta_2)(z) \nabla u(z) \cdot \nabla q(z) \\ &\quad + (\varrho_1(u(z)) - \varrho_2(u(z)))q(z) \\ &\quad - ((f_1 - f_2)q)(z). \end{aligned} \quad (4.27)$$

*Proof.* The change of variables  $T_\varepsilon$  shows that for  $\varepsilon > 0$ ,

$$\begin{aligned} \frac{G(\varepsilon, u, q) - G(0, u, q)}{|\omega_\varepsilon|} &= \frac{1}{|\omega|} \int_\omega ((\beta_1 - \beta_2) \nabla u \cdot \nabla q)(T_\varepsilon(x)) dx \\ &\quad + \frac{1}{|\omega|} \int_\omega ((\varrho_1(u) - \varrho_2(u))uq)(T_\varepsilon(x)) dx \\ &\quad - \frac{1}{|\omega|} \int_\omega ((f_1 - f_2)q)(T_\varepsilon(x)) dx. \end{aligned} \quad (4.28)$$

Recalling that  $f_1, f_2 \in C(\bar{B}_\delta(z))$  and  $u, q \in C^1(\bar{B}_\delta(z))$  for a small  $\delta > 0$  and since  $T_\varepsilon(\omega) \subset \bar{B}_\delta(z)$  for all small  $\varepsilon > 0$ , we can pass to the limit in (4.28) to obtain (4.27).  $\square$

## 4.5 Variation of the averaged adjoint equation and its weak limit

The goal of this section is to verify Hypothesis (H4), that is, to show that

$$R(u, q) := \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u, q_\varepsilon) - G(\varepsilon, u, q)}{|\omega_\varepsilon|} \quad (4.29)$$

exists and, if possible, to determine its explicit form. In contrast to previous works we consider the variation of the averaged adjoint state variable which we will show converges weakly to a function  $Q$  defined on the whole space  $\mathbf{R}^d$ . For this purpose we need the following definition.

**Definition 4.9.** The *inflation* of  $D \setminus \bar{\Omega}$  around  $z \in D \setminus \bar{\Omega}$  is defined by  $D_\varepsilon := T_\varepsilon^{-1}(D \setminus \bar{\Omega})$ , where the transformation  $T_\varepsilon$  is defined by  $T_\varepsilon(x) := \varepsilon x + z$ .

Notice that  $\cup_{\varepsilon > 0} D_\varepsilon = \mathbf{R}^d$  and that  $\varepsilon \mapsto D_\varepsilon$  is monotonically decreasing, that is,  $\varepsilon_1 < \varepsilon_2 \Rightarrow D_{\varepsilon_2} \subset D_{\varepsilon_1}$ .

**Lemma 4.10.** For  $\varepsilon > 0$  we have  $\varphi \in H^1(D \setminus \bar{\Omega})$  if and only if  $\varphi \circ T_\varepsilon \in H^1(D_\varepsilon)$ .

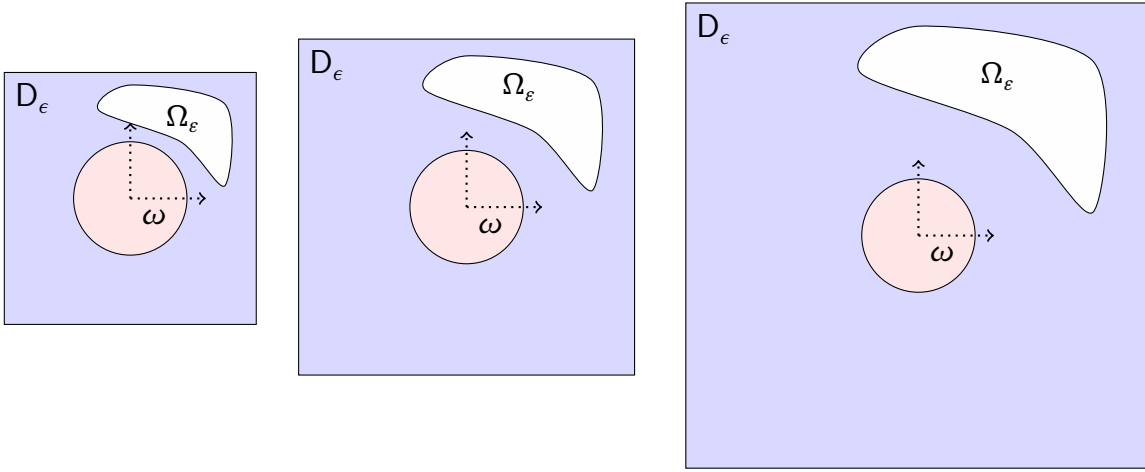


Figure 1: Depicted are several inflated domains  $D_\varepsilon = T_\varepsilon^{-1}(D \setminus \overline{\Omega})$  and  $\Omega_\varepsilon := T_\varepsilon^{-1}(\Omega)$  with  $\varepsilon$  decreasing from left to right. The original inclusion  $\omega_\varepsilon$  appears as the fixed inclusion  $\omega$  centered at the origin in the inflated domain. It can be seen that the domain  $\Omega_\varepsilon$  is gradually pushed to infinity the smaller  $\varepsilon$  gets.

*Proof.* Since  $T_\varepsilon$  is bi-Lipschitz continuous for  $\varepsilon > 0$ , this follows from [40, Thm. 2.2.2, p.52].  $\square$

The next step is to consider the variation of the averaged adjoint state. For this purpose let us extend  $q_\varepsilon$  to zero outside of  $D$ , that is,

$$\tilde{q}_\varepsilon(x) := \begin{cases} q_\varepsilon(x) & \text{for a.e. } x \in D, \\ 0 & \text{for a.e. } x \in \mathbf{R}^d \setminus D. \end{cases} \quad (4.30)$$

In the same way we extend  $u_\varepsilon$  to a function  $\tilde{u}_\varepsilon : \mathbf{R}^d \rightarrow \mathbf{R}$ . Notice that  $\tilde{u}_\varepsilon, \tilde{q}_\varepsilon \in H^1(\mathbf{R}^d)$  for all  $\varepsilon > 0$ . We will use the notation  $q^\varepsilon := \tilde{q}_\varepsilon \circ T_\varepsilon$ .

**Remark 4.11** (Neumann boundary conditions). If we had imposed Neumann conditions in (S), then it would be sufficient to replace (4.30) by  $\tilde{q}_\varepsilon := E q_\varepsilon$ , where  $E : H^1(D) \rightarrow H^1(\mathbf{R}^d)$  is a continuous extension operator; see [20, Thm. 1, pp. 254]. The subsequent analysis were still the same.

**Definition 4.12.** The variation of the averaged adjoint state  $q_\varepsilon$  is defined pointwise a.e. in  $\mathbf{R}^d$  by

$$Q^\varepsilon(x) := \frac{\tilde{q}_\varepsilon(T_\varepsilon(x)) - \tilde{q}(T_\varepsilon(x))}{\varepsilon}. \quad (4.31)$$

Notice that for every  $\varepsilon > 0$  we have  $Q^\varepsilon \in H^1(\mathbf{R}^d)$ .

Our next task is to show that  $(Q^\varepsilon)$  converges in  $\dot{B}L(\mathbf{R}^d)$  to a equivalence class of functions  $[Q]$  and determine an equation for it. The first step is to prove the following apriori estimates.

**Lemma 4.13.** There is a constant  $C > 0$ , such that for all small  $\varepsilon > 0$ ,

$$\int_{\mathbf{R}^d} (\varepsilon Q^\varepsilon)^2 + |\nabla Q^\varepsilon|^2 dx \leq C. \quad (4.32)$$

*Proof.* Obviously, the Lemmas 4.6/4.4 imply that there is a constant  $C > 0$  such that  $\|q_\varepsilon - q\|_{H^1(D)} \leq C\varepsilon^{d/2}$  for all small  $\varepsilon > 0$ . This and definition (4.30) imply

$$\int_{\mathbf{R}^d} (\tilde{q}_\varepsilon - \tilde{q})^2 + |\nabla(\tilde{q}_\varepsilon - \tilde{q})|^2 dx \leq C\varepsilon^d. \quad (4.33)$$

Hence invoking the change of variables  $T_\varepsilon$  in (4.33) yields the bound (4.32).  $\square$

Notice that for  $\varepsilon > 0$  the function  $Q^\varepsilon$  belongs to  $H^1(\mathbf{R}^d)$ , but it is not bounded with respect to  $\varepsilon$ . However, the bound (4.32) is sufficient to show the following key theorem.

**Theorem 4.14.** For  $d \in \{2, 3\}$ , we have

$$\begin{aligned} \nabla Q^\varepsilon &\rightharpoonup \nabla Q && \text{in } L_2(\mathbf{R}^d)^d, \\ \varepsilon Q^\varepsilon &\rightharpoonup 0 && \text{in } H^1(\mathbf{R}^d), \end{aligned} \quad (4.34)$$

where  $[Q] \in \dot{B}L(\mathbf{R}^d)$  is the unique solution to

$$\int_{\mathbf{R}^d} A \nabla \psi \cdot \nabla Q dx = \int_{\omega} \zeta \cdot \nabla \psi dx \quad \text{for all } \psi \in BL(\mathbf{R}^d), \quad (4.35)$$

where  $A := \beta_1(z)\chi_\omega + \beta_2(z)\chi_{\mathbf{R}^d \setminus \omega}$  and  $\zeta := -(\beta_1(z) - \beta_2(z))\nabla q(z)$ ; see (3.1).

*Proof.* Fix  $\bar{\varepsilon} > 0$  and let  $0 < \varepsilon < \bar{\varepsilon}$ . We first notice that using (4.20) we have

$$b_\varepsilon(\psi, q_\varepsilon - q) = - \int_D (u_\varepsilon - u)\psi dx - (b_\varepsilon - b_0)(\psi, q) \quad (4.36)$$

for all  $\psi \in H_0^1(D)$ . The idea is now to choose appropriate test functions in (4.36) and then pass to the limit. For this purpose let  $\bar{\psi} \in H_0^1(D_{\bar{\varepsilon}})$  be arbitrary and define  $\psi := \varepsilon \bar{\psi} \circ T_\varepsilon^{-1}$ . Thanks to Lemma 4.10 we have  $\psi \in H_0^1(D \setminus \bar{\Omega})$  and the latter space embeds via (4.30) into  $H_0^1(D)$ . Hence we readily check that for such a test function, using a change of variables, we have

$$b_\varepsilon(\psi, q_\varepsilon - q) = \varepsilon^d \int_{D_{\bar{\varepsilon}}} A_\varepsilon \nabla \bar{\psi} \cdot \nabla Q^\varepsilon dx + \underbrace{\varepsilon^{d+1} \int_{D_{\bar{\varepsilon}}} (\varrho'_\varepsilon(u) \circ T_\varepsilon) \varepsilon Q^\varepsilon \bar{\psi} dx}_{=: I(\varepsilon, \bar{\psi})} \quad (4.37a)$$

$$(b_\varepsilon - b_0)(\psi, q) = \underbrace{\varepsilon^d \int_{\omega} (\beta_1 - \beta_2)(T_\varepsilon(x)) \nabla \bar{\psi} \cdot \nabla q(T_\varepsilon(x)) dx}_{=: II(\varepsilon, \bar{\psi})} \quad (4.37b)$$

$$+ \underbrace{\varepsilon^{d+1} \int_{D_{\bar{\varepsilon}}} \left( \int_0^1 (\varrho'_\varepsilon(su_\varepsilon + (1-s)u) \circ T_\varepsilon - \varrho'_0(u) \circ T_\varepsilon ds \right) q(T_\varepsilon(x)) \bar{\psi} dx}_{=: III(\varepsilon, \bar{\psi})}$$

$$\int_D (u_\varepsilon - u)\psi dx = \underbrace{\varepsilon^{d+1} \int_{D_{\bar{\varepsilon}}} (u_\varepsilon \circ T_\varepsilon - u \circ T_\varepsilon) \bar{\psi} dx}_{=: IV(\varepsilon, \bar{\psi})}, \quad (4.37c)$$

where  $A_\varepsilon(x) := \beta_1(T_\varepsilon(x))\chi_\omega(x) + \beta_2(T_\varepsilon(x))\chi_{\mathbf{R}^d \setminus \omega}$ . Therefore inserting (4.37a)-(4.37c) into (4.36) we obtain

$$\int_{D_\varepsilon} A_\varepsilon \nabla \bar{\psi} \cdot \nabla Q^\varepsilon dx + \int_\omega (\beta_1 - \beta_2)(T_\varepsilon(x)) \nabla \bar{\psi} \cdot \nabla q(T_\varepsilon(x)) dx = -\varepsilon(I - II - III + IV)(\varepsilon, \bar{\psi}) \quad (4.38)$$

for all  $\varepsilon < \bar{\varepsilon}$  and all  $\bar{\psi} \in H_0^1(D_{\bar{\varepsilon}})$ . The next step is to show that I-IV are bounded. Using the boundedness of  $u_\varepsilon$  on  $D$  we see that  $\varrho'_\varepsilon(s\tilde{u}_\varepsilon + (1-s)\tilde{u}) \circ T_\varepsilon$  and  $\varrho'_0(\tilde{u}) \circ T_\varepsilon$  are bounded (independently of  $\varepsilon$ ) on  $\mathbf{R}^d$ , too. Therefore Hölder's inequality yields

$$|I(\varepsilon, \bar{\psi})| \leq c \|\varepsilon Q^\varepsilon\|_{L_2(\mathbf{R}^d)} \|\bar{\psi}\|_{L_2(\mathbf{R}^d)} \stackrel{(4.32)}{\leq} C \|\bar{\psi}\|_{L_2(\mathbf{R}^d)}, \quad (4.39)$$

$$|II(\varepsilon, \bar{\psi})| \leq c \|\nabla q\|_{C(\bar{B}_{\bar{\varepsilon}}(z))} \|\nabla \bar{\psi}\|_{L_2(\mathbf{R}^d)^d} \stackrel{(4.32)}{\leq} C \|\nabla \bar{\psi}\|_{L_2(\mathbf{R}^d)^d}, \quad (4.40)$$

$$|III(\varepsilon, \bar{\psi})| \leq c \|q\|_{C(\bar{B}_{\bar{\varepsilon}}(z))} \|\bar{\psi}\|_{L_2(\mathbf{R}^d)} \quad (4.41)$$

$$|IV(\varepsilon, \bar{\psi})| \leq c \|\tilde{u}_\varepsilon \circ T_\varepsilon - \tilde{u} \circ T_\varepsilon\|_{L_2(\mathbf{R}^d)} \|\bar{\psi}\|_{L_2(\mathbf{R}^d)} \stackrel{(4.15)}{\leq} C \|\bar{\psi}\|_{L_2(\mathbf{R}^d)} \quad (4.42)$$

for all  $\bar{\psi} \in H_0^1(D_{\bar{\varepsilon}})$  and  $\varepsilon \in [0, \bar{\varepsilon}]$ . Thanks to Lemma 4.13 the family  $(Q^\varepsilon)$  is bounded in  $\dot{B}L(\mathbf{R}^d)$ . The latter space is a Hilbert space and therefore for every null-sequence  $(\varepsilon_n)$  we find a subsequence  $(\varepsilon_{n_k})$  and  $[Q] \in \dot{B}L(\mathbf{R}^d)$ , such that  $\nabla Q^{\varepsilon_{n_k}} \rightharpoonup \nabla Q$  in  $L_2(\mathbf{R}^d)^d$ , where  $Q \in [Q]$ . Hence selecting  $\varepsilon = \varepsilon_{n_k}$  in (4.38) and taking into account (4.39)-(4.41) we can pass to the limit  $k \rightarrow \infty$  and obtain

$$\int_{D_{\bar{\varepsilon}}} A \nabla \bar{\psi} \cdot \nabla Q dx = -(\beta_1(z) - \beta_2(z)) \nabla q(z) \cdot \int_\omega \nabla \bar{\psi} dx \quad \text{for all } \bar{\psi} \in H_0^1(D_{\bar{\varepsilon}}). \quad (4.43)$$

The mapping  $\bar{\varepsilon} \mapsto D_{\bar{\varepsilon}}$  is monotonically decreasing and we have  $H_0^1(D_{\bar{\varepsilon}}) \subset H_0^1(\mathbf{R}^d)$ . This shows, recalling that  $\bar{\varepsilon} > 0$  is arbitrary, that  $D_{\bar{\varepsilon}}$  appearing in (4.43) may be replaced by  $\mathbf{R}^d$ . But this means that  $Q$  is the unique solution of (4.35).

Let us now show that  $\varepsilon Q^\varepsilon \rightharpoonup 0$  in  $H^1(\mathbf{R}^d)$  as  $\varepsilon \searrow 0$ . From the first part of the proof it is clear that  $\nabla(\varepsilon Q^\varepsilon) \rightharpoonup 0$  in  $L_2(\mathbf{R}^d)^d$ . To see the weak convergence of  $(\varepsilon Q^\varepsilon)$  in  $L_2(\mathbf{R}^d)$  we fix  $r > 0$ . Then Poincaré's inequality for a ball yields

$$\|(\varepsilon Q^\varepsilon)_r - \varepsilon Q^\varepsilon\|_{L_2(B_r(0))} \leq \varepsilon C(r) \|\nabla Q^\varepsilon\|_{L_2(B_r(0))^d}, \quad (4.44)$$

where  $(\varepsilon Q^\varepsilon)_r := \int_{B_r(0)} \varepsilon Q^\varepsilon dx$  denotes the average over the ball  $B_r(0)$ . Since the gradient  $\|\nabla Q^\varepsilon\|_{L_2(\mathbf{R}^d)^d}$  is uniformly bounded (see Lemma 4.13), the right hand side of (4.44) goes to zero as  $\varepsilon \searrow 0$ . But also  $\varepsilon Q^\varepsilon$  is bounded in  $L_2(\mathbf{R}^d)$  and therefore we find for any null-sequence  $(\varepsilon_n)$  a subsequence  $(\varepsilon_{n_k})$  and  $\hat{Q} \in L_2(\mathbf{R}^d)$ , such that  $\varepsilon_{n_k} Q^{\varepsilon_{n_k}} \rightharpoonup \hat{Q}$  in  $L_2(\mathbf{R}^d)$ . It is clear that  $(\varepsilon_{n_k} Q^{\varepsilon_{n_k}})_{B_r(0)} \rightarrow (\hat{Q})_{B_r(0)}$  in  $\mathbf{R}$ . Therefore we obtain from (4.44) together with the weak lower semi-continuity of the  $L_2$ -norm

$$\|(\hat{Q})_r - \hat{Q}\|_{L_2(B_r(0))} \leq \liminf_{k \rightarrow \infty} \|(Q^{\varepsilon_{n_k}})_{B_r(0)} - Q^{\varepsilon_{n_k}}\|_{L_2(B_r(0))} \leq 0. \quad (4.45)$$

This shows that  $\hat{Q} = (\hat{Q})_r$  a.e. on  $B_r(0)$  and thus  $\hat{Q}$  is constant on  $B_r(0)$ . Since  $r > 0$  was arbitrary,  $\hat{Q}$  must be constant on  $\mathbf{R}^d$ . Further  $\hat{Q} \in L_2(\mathbf{R}^d)$  implies  $\hat{Q} = 0$  and this finishes the proof.  $\square$



We are now ready to compute  $R(u, q)$  and thereby verify the second part of Hypothesis (H2).

**Lemma 4.15.** We have

$$R(u, q) = (\beta_1(z) - \beta_2(z)) \nabla u(z) \cdot \int_{\omega} \nabla Q \, dx, \quad (4.46)$$

where  $[Q]$  is the solution to (4.35).

*Proof.* Testing the state equation (4.14) (for  $\varepsilon = 0$ ) with  $\varphi = q_\varepsilon - q$  gives

$$\int_{\text{D}} \beta_0 \nabla u \cdot \nabla (q_\varepsilon - q) + \varrho_0(u)(q_\varepsilon - q) \, dx = \int_{\text{D}} f_0(q_\varepsilon - q) \, dx. \quad (4.47)$$

Therefore we can write for  $\varepsilon > 0$ ,

$$\begin{aligned} G(\varepsilon, u, q_\varepsilon) - G(\varepsilon, u, q) &= \int_{\text{D}} \beta_\varepsilon \nabla u \cdot \nabla (q_\varepsilon - q) + \varrho_\varepsilon(u)(q_\varepsilon - q) \, dx - \int_{\text{D}} f_\varepsilon(q_\varepsilon - q) \, dx \\ &\stackrel{(4.47)}{=} \int_{\omega_\varepsilon} (\beta_1 - \beta_2) \nabla u \cdot \nabla (q_\varepsilon - q) + [(\varrho_1 - \varrho_2)(u) - (f_1 - f_2)](q_\varepsilon - q) \, dx. \end{aligned} \quad (4.48)$$

Invoking the change of variables  $T_\varepsilon$  in (4.48) we obtain for  $\varepsilon > 0$

$$\begin{aligned} \frac{G(\varepsilon, u, q_\varepsilon) - G(\varepsilon, u, q)}{|\omega_\varepsilon|} &= \frac{1}{|\omega|} \int_{\omega} [(\varrho_1 - \varrho_2)(u(T_\varepsilon(x))) - (f_1 - f_2)(T_\varepsilon(x))] \varepsilon Q^\varepsilon \, dx \\ &\quad + \frac{1}{|\omega|} \int_{\omega} ((\beta_1 - \beta_2) \nabla u)(T_\varepsilon(x)) \cdot \nabla Q^\varepsilon \, dx \\ &\rightarrow (\beta_1(z) - \beta_2(z)) \nabla u(z) \cdot \int_{\omega} \nabla Q \, dx, \end{aligned} \quad (4.49)$$

where in the last step we used Theorem 4.14,  $f_1, f_2 \in C(\overline{B_\delta}(z))$ , and  $u \in C^1(\overline{B_\delta}(z))$  for  $\delta > 0$  small.  $\square$

## 4.6 Topological derivative and polarisation matrix

**Topological derivative** Now we are in a position to formulate our main result. In the previous sections we have checked that Hypotheses (H0)-(H4) of Theorem 2.6 are satisfied for the Lagrangian  $G$  given by (4.11). Therefore Theorem 4.16 can be applied and we obtain the following result.

**Theorem 4.16.** The topological derivative of  $J$  at  $\Omega$  in  $z \in \text{D} \setminus \overline{\Omega}$  is given by

$$\lim_{\varepsilon \searrow 0} \frac{J(\Omega \cup \omega_\varepsilon(z)) - J(\Omega)}{|\omega_\varepsilon(z)|} = \partial_\ell G(0, u, q) + R(u, q), \quad (4.50)$$

where

$$\partial_\ell G(0, u, q) = ((\beta_1 - \beta_2) \nabla u \cdot \nabla q + (\varrho_1(u) - \varrho_2(u))q - (f_1 - f_2)q)(z) \quad (4.51)$$

and

$$R(u, q) = (\beta_1(z) - \beta_2(z)) \nabla u(z) \cdot \int_{\omega} \nabla Q \, dx, \quad (4.52)$$

where  $Q$  depends on  $z$  and is the solution to (4.35).

Next we rewrite the term  $R(u, q)$  with the help of the so-called polarisation matrix. For this purpose we fix  $z \in D \setminus \bar{\Omega}$  in the following and denote by  $[Q_\zeta]$ ,  $\zeta \in \mathbf{R}^d$ , the solution to (3.6) with  $A := A_\omega := \beta_1(z)\chi_\omega + \beta_2(z)\chi_{\mathbf{R}^d \setminus \omega}$ . Also we denote by  $Q_\zeta$  an arbitrary representative of  $[Q_\zeta]$ .

**Definition 4.17.** The matrix representing the linear *averaging operator*

$$\zeta \mapsto \int_\omega \nabla Q_\zeta \, dx, \quad \mathbf{R}^d \mapsto \mathbf{R}^d \quad (4.53)$$

is called *weak polarisation matrix* and will be denoted  $\mathcal{P}_z \in \mathbf{R}^{d \times d}$ . Notice that this matrix depends on  $\beta_1(z)$  and  $\beta_2(z)$ .

We use the term weak polarisation matrix here, because it is defined via the weak formulation (3.6) and therefore does not require any regularity assumptions on  $\partial\omega$  or  $\Omega$ . We give another definition of a polarisation matrix later and relate it to the weak polarisation matrix. We also refer to [33] and the monograph [31, Sec. 9.4.4, pp. 273].

**Corollary 4.18.** We have

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{J(\Omega \cup \omega_\varepsilon(z)) - J(\Omega)}{|\omega_\varepsilon|} &= ((\beta_1 - \beta_2)\nabla u) \cdot (I - \mathcal{P}_z(\beta_1 - \beta_2))\nabla q(z) \\ &+ ((\varrho_1(u) - \varrho_2(u))q - (f_1 - f_2)q)(z). \end{aligned} \quad (4.54)$$

*Proof.* This follows at once from (4.50) noting that  $\mathcal{P}_z \zeta = \int_\omega \nabla Q_\zeta \, dx$ , where  $\zeta := -(\beta_1(z) - \beta_2(z))\nabla q(z)$ .  $\square$

**Further properties of the polarisation matrix** Next we derive further properties of the polarisation matrix. Furthermore we relate our polarisation matrix to previous definitions. We refer the reader to [2] for further information on polarisation matrices.

**Lemma 4.19.** If  $\beta_2(z) = \beta_2^\top(z)$  and  $\beta_1(z) = \beta_1^\top(z)$ , then the polarisation matrix is symmetric, that is,  $\mathcal{P}_z = \mathcal{P}_z^\top$ .

*Proof.* We compute for the  $(i, j)$ -entry of the polarisation matrix:

$$\begin{aligned} e_i \cdot \mathcal{P}_z e_j &= e_i \cdot \int_\omega \nabla Q_{e_j} \, dx \stackrel{(4.35)}{=} \int_{\mathbf{R}^d} A_\omega \nabla Q_{e_j} \cdot \nabla Q_{e_i} \, dx \\ &\stackrel{\text{sym. of } A_\omega}{=} \int_{\mathbf{R}^d} \nabla Q_{e_j} \cdot A_\omega \nabla Q_{e_i} \, dx \\ &\stackrel{(4.35)}{=} e_j \cdot \int_\omega \nabla Q_{e_i} \, dx = e_j \cdot \mathcal{P}_z e_i. \end{aligned} \quad (4.55)$$

This shows the symmetry.  $\square$

The polarisation matrix is also positive definite (even in the nonsymmetric case).

**Lemma 4.20.** The matrix  $\mathcal{P}_z$  is positive definite.

*Proof.* Let  $w = (w_1, \dots, w_d) \in \mathbf{R}^d$  be an arbitrary vector. Set  $W := \sum_{i=1}^d w_i Q_{e_i}$ . Then we readily check using (4.55),

$$w \cdot \mathcal{P}_z w = \int_{\mathbf{R}^d} A_\omega \nabla W \cdot \nabla W \, dx \geq \beta_m \int_{\mathbf{R}^d} |\nabla W|^2 \, dx. \quad (4.56)$$

This shows that  $\mathcal{P}_z$  is positive semidefinite. Suppose now  $w$  is such that  $w \cdot \mathcal{P}_z w = 0$ . Then, in view of (4.56), we must have  $[W] = [0]$ . Hence (4.35) gives

$$w \cdot \int_{\omega} \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in \text{BL}(\mathbf{R}^d). \quad (4.57)$$

Let  $V \subset \mathbf{R}^d$  be a bounded and open set, such that  $\omega \Subset V$ . Choose a smooth function  $\rho$ , such that  $\rho = 1$  on  $\omega$ ,  $0 \leq \rho \leq 1$  on  $V \setminus \omega$  and  $\rho = 0$  outside of  $V$ . Then we define  $\varphi(x) := e_i \cdot x \rho(x)$  for  $i \in \{1, \dots, d\}$ , which belongs to  $\text{BL}(\mathbf{R}^d)$ . Hence we may test (4.57) with this function and conclude  $w_i = 0$ . This shows  $w = 0$  and finishes the proof.  $\square$

Suppose from now on  $\beta_1 = \gamma_1 I$  and  $\beta_2 = \gamma_2 I$  for  $\gamma_1, \gamma_2 > 0$ . We select  $Q_\zeta \in [Q_\zeta]$  and suppose that it can be represented by a single layer potential: there is a function  $h_\zeta \in C(\partial\omega)$ , such that

$$Q_\zeta(x) = \int_{\partial\omega} h_\zeta(y) E(x-y) \, ds(y), \quad \int_{\partial\omega} h_\zeta \, ds = 0, \quad (4.58)$$

where  $E$  denotes the fundamental solution of  $u \mapsto -\Delta u$ ; [21, Chap. 3]. It is readily checked using (4.58) that  $|Q_\zeta(x)| = O(|x|^{1-d})$ .

**Definition 4.21.** The *strong polarisation matrix* is the matrix  $\tilde{\mathcal{P}}_z = (\tilde{\mathcal{P}}_z)_{ij} \in \mathbf{R}^{d \times d}$  with entries

$$(\tilde{\mathcal{P}}_z)_{ij} = \int_{\partial\omega} x_j h_{e_i} \, ds. \quad (4.59)$$

The strong and weak polarisation matrices are related as shown in the following lemma.

**Lemma 4.22.** Assume that  $\partial\omega$  is  $C^2$ . Then we have

$$\mathcal{P}_z = -\frac{1}{|\omega|} \frac{\beta_2}{\beta_1 - \beta_2} \tilde{\mathcal{P}}_z + \frac{1}{\beta_1 - \beta_2} I. \quad (4.60)$$

*Proof.* At first we obtain by partial integration, noting that  $e_i = \nabla x_i$ ,

$$e_i \cdot \mathcal{P}_z e_j = \int_{\omega} \nabla x_i \cdot \nabla Q_{e_j} \, ds = \frac{1}{|\omega|} \int_{\partial\omega} x_i \partial_\nu Q_{e_j} \, ds - \underbrace{\int_{\omega} \Delta Q_{e_j} \, dx}_{=0, \text{ in view of (4.35)}}. \quad (4.61)$$

Next we express  $\partial_\nu Q_{e_j}$  in terms of  $h_{e_j}$ . For this recall (see, e.g., [21]) that the jump condition

$$\partial_\nu Q_{e_i}^+ - \partial_\nu Q_{e_i}^- = h_{e_i} \quad \text{on } \partial\omega \quad (4.62)$$

is satisfied. In addition we get from (4.35),

$$\beta_1 \partial_\nu Q_{e_i}^+ - \beta_2 \partial_\nu Q_{e_i}^- = e_i \cdot \nu \quad \text{on } \partial\omega. \quad (4.63)$$

Combining (4.62) and (4.63) we obtain

$$\partial_\nu Q_{e_i}^+ = -\frac{\beta_2}{\beta_1 - \beta_2} h_{e_i} + \frac{1}{\beta_1 - \beta_2} e_i \cdot \nu. \quad (4.64)$$

Inserting this expression into (4.61) yields

$$e_i \cdot \mathcal{P}_z e_j = -\frac{\beta_2}{\beta_1 - \beta_2} \frac{1}{|\omega|} \int_{\partial\omega} x_i h_{e_j} ds + \frac{1}{\beta_1 - \beta_2} \frac{1}{|\omega|} \int_{\partial\omega} (e_i \cdot \nu) x_j ds. \quad (4.65)$$

This is equivalent to formula (4.60), since by Gauss's divergence theorem

$$\frac{1}{|\omega|} \int_{\partial\omega} (e_i \cdot \nu) x_j ds = \underbrace{\int_{\omega} \operatorname{div}(e_i x_j) dx}_{=\delta_{ij}} = \delta_{ij}. \quad (4.66)$$

□

**Remark 4.23.** In some cases, see, e.g., [3, 5, 27], the polarisation matrix can be computed explicitly: for instance when  $\beta_1 = \gamma_1 I$ ,  $\beta_2 = \gamma_2 I$ ,  $\beta_1, \beta_2 > 0$ , and  $\omega$  is a circle or more generally an ellipse. However for general inclusions  $\omega$  the exterior equation (4.35) has to be solved numerically in order to evaluate formula (4.50).

## 5 The extremal case of void material

In this last section we discuss the extremal situation in which  $\beta_1 = 0$ ,  $\varrho_1 = 0$  and  $f_1 = 0$  in (4.2). This case corresponds to the insertion of a hole with Neumann boundary conditions imposed on the inclusion; see [26]. Since the extremal case is similar to the considerations from the previous section, we will only work out the main differences in detail.

### 5.1 Problems setting

We suppose as before that  $D \subset \mathbf{R}^d$  is a bounded Lipschitz domain. For a simply connected domain  $\Omega \Subset D$  with Lipschitz boundary  $\partial\Omega$ , we consider the shape function

$$J(\Omega) = \int_{D \setminus \bar{\Omega}} u^2 dx \quad (5.1)$$

subject to  $u = u_\Omega \in H_{\partial D}^1(D \setminus \bar{\Omega})$  solves

$$\int_{D \setminus \bar{\Omega}} \beta_2 \nabla u \cdot \nabla \varphi + \varrho_2(u) \varphi dx = \int_{D \setminus \bar{\Omega}} f_2 \varphi dx \quad \text{for all } \varphi \in H_{\partial D}^1(D \setminus \bar{\Omega}), \quad (5.2)$$

where  $H_{\partial D}^1(D \setminus \bar{\Omega}) := \{v \in H^1(D \setminus \bar{\Omega}) : v = 0 \text{ on } \partial D\}$ . This setting corresponds to the limiting case of (4.2) in which  $\beta_1 = 0$ ,  $\varrho_1 = 0$  and  $f_1 = 0$ .

The rest of this section is dedicated to the computation of the topological sensitivity of  $J$  at  $\Omega = \emptyset$  with respect to the inclusion  $\omega$  (which will be specified below), i.e.,

$$\lim_{\varepsilon \searrow 0} \frac{J(\omega_\varepsilon) - J(\emptyset)}{|\omega_\varepsilon|}. \quad (5.3)$$

We will see that almost all steps are the same as in the last section with two main differences. The first main difference being that  $X(\varepsilon)$  is not a singleton and that we have to introduce a new equation on the inclusion, which requires a more detailed explanation and a thorough analysis. The second difference concerns the required assumptions on the regularity of the inclusion  $\omega$ . While in the previous section it was sufficient to assume that  $\omega$  is merely an open set, here we strengthen the assumption and assume that  $\omega$  is a simply connected Lipschitz domain.

**Assumption 2.** We assume that either

- (a)  $\beta_2 \in \mathbf{R}^{d \times d}$  is symmetric, positive definite and  $\varrho_2$  satisfies Assumption 1, (b) and it is bounded, or
- (b)  $\beta_2$  satisfies Assumption (1), (a) and  $\varrho_2$  satisfies Assumption 1, (b) and additionally  $\varrho_2' > \lambda$  for some  $\lambda > 0$ ,

is satisfied. In both cases we assume  $f_2 \in H^1(D) \cap C(D)$ .

Under these assumptions we can prove, using similar arguments as in Lemma 4.1, that (5.2) admits a unique solution and that there is a constant  $C > 0$  (depending on  $\Omega$ ), such that

$$\|u_\Omega\|_{L^\infty(D \setminus \bar{\Omega})} + \|u_\Omega\|_{H_0^1(D \setminus \bar{\Omega})} \leq C \|f_2\|_{L^r(D \setminus \bar{\Omega})} \quad (5.4)$$

for  $r > d/2$  close enough to  $d/2$ . Moreover, for every  $z \in D \setminus \bar{\Omega}$ , we find  $\delta > 0$ , such that  $u_\Omega \in H^3(B_\delta(z))$ .

## 5.2 The parametrised Lagrangian

From now on we fix:

- a simply connected Lipschitz domain  $\omega \subset \mathbf{R}^d$  with  $0 \in \omega$ ,
- a point  $z \in D$ ,
- the perturbation  $\Omega_\varepsilon := \omega_\varepsilon := \omega_\varepsilon(z)$ , where  $\omega_\varepsilon(z) := z + \varepsilon \omega$  and  $\varepsilon \in [0, \tau]$ ,  $\tau > 0$ .

Let  $X = Y = H_0^1(D)$  and introduce the Lagrangian  $G : [0, \tau] \times X \times Y \rightarrow \mathbf{R}$  associated with the perturbation  $\Omega_\varepsilon$  by

$$G(\varepsilon, u, q) := \int_{D \setminus \bar{\omega}_\varepsilon} u^2 dx + \int_{D \setminus \bar{\omega}_\varepsilon} \beta_2 \nabla u \cdot \nabla q + \varrho_2(u) q dx - \int_{D \setminus \bar{\omega}_\varepsilon} f_2 q dx. \quad (5.5)$$

We will verify that Hypotheses (H0)-(H4) are satisfied with  $\ell(\varepsilon) = |\omega_\varepsilon|$ .

### 5.3 Analysis of the perturbed state equation

The *perturbed state equation* reads: find  $u_\varepsilon \in H_0^1(D)$  such that  $\partial_p G(\varepsilon, u_\varepsilon, 0)(\varphi) = 0$  for all  $\varphi \in H_0^1(D)$ , or equivalently  $u_\varepsilon \in H_0^1(D)$  satisfies,

$$\int_{D \setminus \bar{\omega}_\varepsilon} \beta_2 \nabla u_\varepsilon \cdot \nabla \varphi + \varrho_2(u_\varepsilon) \varphi \, dx = \int_{D \setminus \bar{\omega}_\varepsilon} f_2 \varphi \, dx \quad \text{for all } \varphi \in H_0^1(D). \quad (5.6)$$

Henceforth we write  $u := u_0$  to simplify notation. Since (5.2) admits a unique solution  $\bar{u}_\varepsilon$  for  $\Omega = \omega_\varepsilon$ , which can be extended to  $H_0^1(D)$ , (5.6) admits a solution, too, whose restriction to  $D \setminus \bar{\Omega}$  is unique. This means that

$$E(\varepsilon) = \{u \in H_0^1(D) : u = \bar{u}_\varepsilon \text{ a.e. on } D \setminus \bar{\omega}_\varepsilon\}, \quad (5.7)$$

where  $\bar{u}_\varepsilon$  is the unique solution to (5.2). It also follows that  $X(\varepsilon) = E(\varepsilon)$  since the Lagrangian only depends on the restriction of functions to  $D \setminus \bar{\omega}_\varepsilon$ . Note that the set  $X(0)$  is a singleton. Moreover for all  $\varepsilon \in [0, \tau]$ ,

$$g(\varepsilon) = \inf_{u \in E(\varepsilon)} G(\varepsilon, u, 0) = \int_{D \setminus \bar{\omega}_\varepsilon} \bar{u}_\varepsilon^2 \, dx. \quad (5.8)$$

This shows that Hypothesis (H0) and, in view of Assumption 2, also Hypothesis (H1) is satisfied.

The next step deviates from the transmission problem case (of Section 4). We construct functions  $u_\varepsilon \in X(\varepsilon)$  and  $q_\varepsilon \in Y(\varepsilon, u_0, u_\varepsilon)$  that satisfy Hypothesis (H4). For this purpose we associate with  $u_\varepsilon \in H_{\partial D}^1(D \setminus \bar{\omega}_\varepsilon)$  a function  $u_\varepsilon^+ \in H^1(\omega_\varepsilon)$  defined as the unique weak solution to the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(\beta_2 \nabla u_\varepsilon^+) + \varrho_2(u) &= f_2 && \text{in } \omega_\varepsilon \\ u_\varepsilon^+ &= u_\varepsilon && \text{on } \partial \omega_\varepsilon. \end{aligned} \quad (5.9)$$

With this function we can extend  $u_\varepsilon$  to a function  $u_\varepsilon \in H_0^1(D)$  by setting

$$u_\varepsilon := \begin{cases} u_\varepsilon^+ & \text{in } \omega_\varepsilon \\ u_\varepsilon & \text{in } D \setminus \bar{\omega}_\varepsilon \end{cases}. \quad (5.10)$$

Now we prove the following analogue of Lemma 4.4.

**Lemma 5.1.** There is a constant  $C > 0$ , such that for all small  $\varepsilon > 0$ ,

$$\|u_\varepsilon - u\|_{H^1(D)} \leq C \varepsilon^{d/2}. \quad (5.11)$$

*Proof.* We first establish an estimate for  $u_\varepsilon - u$  on  $\omega_\varepsilon$ . For this purpose we fix a bounded domain  $S \subset D$  containing  $\omega$ . We note that the difference  $e_\varepsilon(x) := u_\varepsilon(T_\varepsilon(x)) - u(T_\varepsilon(x))$  satisfies  $-\operatorname{div}(\beta_2 \circ T_\varepsilon \nabla e_\varepsilon) = 0$  on  $\omega$  and  $e_\varepsilon = u_\varepsilon(T_\varepsilon(x)) - u(T_\varepsilon(x))$  on  $\partial \omega$ . Hence by standard elliptic regularity and the trace theorem we find

$$\|e_\varepsilon + \lambda\|_{H^1(\omega)} \leq c \|e_\varepsilon + \lambda\|_{H^{1/2}(\partial \omega)} \leq c \|e_\varepsilon + \lambda\|_{H^1(S \setminus \bar{\omega})} \quad (5.12)$$

for all  $\lambda \in \mathbf{R}$ . Since the quotient norms on the spaces  $H^1(\omega)/\mathbf{R}$  and  $H^1(S \setminus \bar{\omega})$  are equivalent to the seminorms  $|v|_{H^1(\omega)} := \|\nabla v\|_{L_2(\omega)^d}$  and  $|v|_{H^1(S \setminus \bar{\omega})} := \|\nabla v\|_{L_2(S \setminus \bar{\omega})^d}$ , respectively, we conclude

$\|\nabla e_\varepsilon\|_{L_2(\omega)^d} \leq c\|\nabla e_\varepsilon\|_{L_2(S \setminus \omega)^d}$ . Therefore estimating the right hand side and changing variables shows

$$\|\nabla(u_\varepsilon - u)\|_{L_2(\omega_\varepsilon)^d} \leq c\|\nabla(u_\varepsilon - u)\|_{L_2(D \setminus \bar{\omega}_\varepsilon)^d}. \quad (5.13)$$

A fortiori using (5.13) and a similar argument shows that (5.12) implies

$$\|u_\varepsilon - u\|_{L_2(\omega_\varepsilon)} \leq c(\varepsilon\|\nabla(u_\varepsilon - u)\|_{L_2(D \setminus \bar{\omega}_\varepsilon)^d} + \|u_\varepsilon - u\|_{L_2(D \setminus \bar{\omega}_\varepsilon)}). \quad (5.14)$$

This finishes the first step of the proof. We now establish an estimate for the right hand side of (5.13). Following the steps of Lemma 4.4 we find

$$\begin{aligned} & \int_{D \setminus \bar{\omega}_\varepsilon} \beta_2 \nabla(u_\varepsilon - u) \cdot \nabla \varphi \, dx + \int_{D \setminus \bar{\omega}_\varepsilon} (\varrho_2(u_\varepsilon) - \varrho_2(u)) \varphi \, dx \\ &= \int_{\omega_\varepsilon} \beta_2 \nabla u \cdot \nabla \varphi \, dx + \int_{\omega_\varepsilon} \varrho_2(u) \varphi - f_2 \varphi \, dx \end{aligned} \quad (5.15)$$

for all  $\varphi \in H_0^1(D)$ . Let us first assume that Assumption 2, (a) holds. Fix  $\bar{\varepsilon} > 0$  and let  $0 < \varepsilon < \bar{\varepsilon}$ . Changing variables in (5.15) yields (recalling that we denote by  $\tilde{u}_\varepsilon$  the extension of  $u_\varepsilon$  to  $\mathbf{R}^d$ )

$$\begin{aligned} & \int_{\mathbf{R}^d \setminus \bar{\omega}} \beta_2 \nabla K_\varepsilon \cdot \nabla \varphi \, dx = -\varepsilon^2 \underbrace{\int_{\mathbf{R}^d \setminus \bar{\omega}} (\varrho_2(\tilde{u}_\varepsilon(T_\varepsilon)) - \varrho_2(\tilde{u}(T_\varepsilon))) \varphi \, dx}_{\rightarrow 0, \text{ since } \varrho_2 \text{ is bounded}} \\ & + \varepsilon \underbrace{\int_{\omega} \beta_2 \nabla u(T_\varepsilon) \cdot \nabla \varphi \, dx + \varepsilon^2 \int_{\omega} \varrho_2(u(T_\varepsilon)) \varphi - f_2(T_\varepsilon) \varphi \, dx}_{\rightarrow 0, \text{ since } u \in C^1(\bar{B}_\delta(z)), f_2 \in C(D) \text{ and } \varrho_2 \in C(\mathbf{R})}, \end{aligned} \quad (5.16)$$

for all  $\varphi \in H_{\partial D}^1(D_{\bar{\varepsilon}} \setminus \bar{\omega})$ , where  $K_\varepsilon := (u_\varepsilon - u) \circ T_\varepsilon$ . Since  $\bar{\varepsilon} > 0$  is arbitrary, this shows that  $K_\varepsilon \rightharpoonup 0$  weakly in  $\text{BL}(\mathbf{R}^d \setminus \bar{\omega})$ . But this means that  $K_\varepsilon$  must be bounded in  $\text{BL}(\mathbf{R}^d \setminus \bar{\omega})$  and hence we find  $C > 0$ , such that  $\|\nabla K_\varepsilon\|_{L_2(\mathbf{R}^d \setminus \bar{\omega})^d} \leq C$  or equivalently after changing variables  $\|\nabla(u_\varepsilon - u)\|_{L_2(D \setminus \bar{\omega}_\varepsilon)} \leq C\varepsilon^{d/2}$ . Combining this estimate with (5.13) and using Poincaré's inequality gives (5.11).

Let us now assume that Assumption 2, (b) is satisfied. Testing (5.15) with  $\varphi = u_\varepsilon - u$ , using  $\varrho_2' > \lambda$  and applying Hölder's inequality yield

$$C\|u_\varepsilon - u\|_{H^1(D \setminus \bar{\omega}_\varepsilon)}^2 \leq \sqrt{|\omega_\varepsilon|} (\|\nabla u\|_{C(B_\delta(z))^d} \|\nabla(u_\varepsilon - u)\|_{L_2(\omega_\varepsilon)^d} + \|\varrho(u) - f_2\|_{C(\bar{B}_\delta(z))} \|u_\varepsilon - u\|_{L_2(\omega_\varepsilon)}).$$

Using (5.13) and (5.14) to estimate the right hand side and noting  $|\omega_\varepsilon| = |\omega|\varepsilon^d$ , we infer  $\|u_\varepsilon - u\|_{H^1(D \setminus \bar{\omega}_\varepsilon)} \leq C\varepsilon^{d/2}$ . Again combining this estimate with (5.13) yields (5.11).  $\square$

## 5.4 Analysis of the averaged adjoint equation

We introduce for  $\varepsilon \in [0, \tau]$  the (not necessarily symmetric) bilinear form  $b_\varepsilon : H_0^1(D) \times H_0^1(D) \rightarrow \mathbf{R}$  by

$$b_\varepsilon(\psi, \varphi) := \int_{D \setminus \bar{\omega}_\varepsilon} \beta_2 \nabla \psi \cdot \nabla \varphi + \left( \int_0^1 \varrho_2'(s u_\varepsilon + (1-s)u) \, ds \right) \varphi \psi \, dx. \quad (5.17)$$

Then the averaged adjoint equation (2.8) for the Lagrangian  $G$  given by (5.5) reads: for  $(u_0, u_\varepsilon) \in X(0) \times X(\varepsilon)$  find  $q_\varepsilon \in H_0^1(D)$ , such that

$$b_\varepsilon(\psi, q^\varepsilon) = - \int_{D \setminus \bar{\omega}_\varepsilon} (u + u_\varepsilon) \psi \, dx \quad (5.18)$$

for all  $\psi \in H_0^1(D)$ . In view of Assumption 1 we have  $\varrho'_2 \geq 0$  and  $\beta_2 \geq \beta_m I$  and thus  $b_\varepsilon$  satisfies,

$$b_\varepsilon(\psi, \psi) \geq \beta_m \|\nabla \psi\|_{L_2(D \setminus \bar{\omega}_\varepsilon)}^2 \quad (5.19)$$

for all  $\psi \in H_0^1(D)$  and  $\varepsilon \in [0, \tau]$ . As for the state equation, we use the notation  $q := q^0$ .

**Lemma 5.2.** (i) For each  $\varepsilon \in [0, \tau]$  equation (5.18) admits a solution whose restriction to  $D \setminus \bar{\omega}_\varepsilon$  is unique.

(ii) For every  $z \in D \setminus \bar{\Omega}$  we find a number  $\delta > 0$ , such that  $q \in H^3(B_\delta(z)) \subset C^1(\bar{B}_\delta(z))$  for  $d \in \{2, 3\}$ .

The previous lemma shows that  $Y(\varepsilon, u, u_\varepsilon) = \{q \in H_0^1(D) : q = q_\varepsilon \text{ a.e. on } D \setminus \bar{\omega}_\varepsilon\}$  and thus Hypothesis (H2) is satisfied. In the same way as done in (5.9) we extend the restriction  $q_\varepsilon|_{D \setminus \bar{\omega}_\varepsilon}$  (which is unique) to a function  $q_\varepsilon \in H_0^1(D)$  by solving the following Dirichlet problem: find  $q_\varepsilon^+ \in H^1(\omega_\varepsilon)$ , such that

$$\begin{aligned} -\operatorname{div}(\beta_2^\top \nabla q_\varepsilon^+) + \varrho'_2(u)q &= -2u \quad \text{in } \omega_\varepsilon \\ q_\varepsilon^+ &= q_\varepsilon \quad \text{on } \partial \omega_\varepsilon. \end{aligned} \quad (5.20)$$

With this function we define again

$$q_\varepsilon := \begin{cases} q_\varepsilon^+ & \text{in } \omega_\varepsilon \\ q_\varepsilon & \text{in } D \setminus \bar{\omega}_\varepsilon \end{cases}. \quad (5.21)$$

It is clear that  $q_\varepsilon \in Y(\varepsilon, u, u_\varepsilon)$ . We proceed with a Hölder-type estimate for the extension  $\varepsilon \mapsto q_\varepsilon$ .

**Lemma 5.3.** There is a constant  $C > 0$ , such that for all small  $\varepsilon > 0$ ,

$$\|q_\varepsilon - q\|_{H^1(D)} \leq C(\|u_\varepsilon - u\|_{L_2(D)} + \varepsilon^{d/2}). \quad (5.22)$$

*Proof.* The proof is the same as the one of Lemma 4.6 and therefore omitted.  $\square$

It is readily checked that Hypothesis (H3) is satisfied, too.

**Lemma 5.4.** We have

$$\lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u, q) - G(0, u, q)}{\ell(\varepsilon)} = (-\beta_2 \nabla u \cdot \nabla q - \varrho_2(u)q + f_2 q)(z). \quad (5.23)$$

*Proof.* Since  $f_2 \in C(\bar{B}_\delta(z))$  and  $u, q \in C^1(\bar{B}_\delta(z))$  for a small  $\delta > 0$ , we can repeat the steps of the proof of Lemma 4.8.  $\square$



## 5.5 Variation of the averaged adjoint equation and its weak limit

The next step is to consider the variation of the averaged adjoint state. The variation of the averaged adjoint variable, denoted  $Q^\varepsilon$ , is defined as in Definition 4.12.

The following is the analogue of Lemma 4.13 with the main difference that we have an additional equation which gives information of  $Q$  inside the inclusion  $\omega$ .

**Lemma 5.5.** There is a constant  $C > 0$ , such that for all small  $\varepsilon > 0$ ,

$$\int_{\mathbf{R}^d} (\varepsilon Q^\varepsilon)^2 + |\nabla Q^\varepsilon|^2 dx \leq C. \quad (5.24)$$

*Proof.* We follow the steps of Lemma 4.13, but use Lemmas 5.3,5.1 instead Lemmas 4.6,4.4.  $\square$

**Theorem 5.6.** We have

$$\nabla Q^\varepsilon \rightharpoonup \nabla Q \quad \text{weakly in } L_2(\mathbf{R}^d)^d, \quad (5.25a)$$

$$\varepsilon Q^\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\mathbf{R}^d), \quad (5.25b)$$

where  $[Q] \in \dot{B}L(\mathbf{R}^d)$  is the unique solution to

$$\int_{\mathbf{R}^d \setminus \overline{\omega}} \beta_2(z) \nabla \psi \cdot \nabla Q dx = \int_{\omega} \zeta \cdot \nabla \psi dx \quad \text{for all } \psi \in BL(\mathbf{R}^d), \quad (5.26a)$$

$$\int_{\omega} \beta_2(z) \nabla \psi \cdot \nabla Q dx = 0 \quad \text{for all } \psi \in H_0^1(\omega), \quad (5.26b)$$

where  $\zeta := -(\beta_1(z) - \beta_2(z)) \nabla q(z)$ .

*Proof.* It follows from Lemma 5.5 that for every null-sequence  $(\varepsilon_n)$  there is a subsequence (indexed the same) and  $Q \in BL(\mathbf{R}^d)$  such that (5.25a) and (5.25b) holds for this subsequence. Now using the same arguments as in the proof of Theorem 4.14 shows that  $Q$  satisfies (5.26a). The uniqueness of  $Q|_{\mathbf{R}^d \setminus \overline{\omega}}$  follows directly from (5.26a). To prove (5.26b) note that  $Q^{\varepsilon_n}$  satisfies

$$\int_{\omega} \beta(T_{\varepsilon_n}(x)) \nabla \psi \cdot \nabla Q^{\varepsilon_n} dx = 0 \quad \text{for all } \psi \in H_0^1(\omega). \quad (5.27)$$

Using (5.25a) and (5.25b) we may pass to the limit  $n \rightarrow \infty$  which shows that  $Q$  satisfies (5.26b). Since  $Q|_{\partial\omega}$  is uniquely determined from (5.26a) also (5.26b) admits a unique solution, because it is a Dirichlet problem with boundary values  $Q|_{\partial\omega}$ .  $\square$

We are now ready to compute  $R(u, q)$ .

**Lemma 5.7.** We have

$$R(u, q) = -\beta_2(z) \nabla u(z) \cdot \int_{\omega} \nabla Q dx, \quad (5.28)$$

where  $[Q]$  is the solution to (5.26b).

*Proof.* The proof follows the lines of Lemma 4.15 and Lemma 5.1.  $\square$

Collecting all previous results we see that Theorem 2.6 can be applied and we obtain the following result.

**Theorem 5.8.** The topological derivative of  $J$  given by (5.1) in  $z \in D$  is given by

$$\lim_{\varepsilon \searrow 0} \frac{J(\omega_\varepsilon(z)) - J(\Omega)}{|\omega_\varepsilon|} = \partial_\ell G(0, u, q) + R(u, q), \quad (5.29)$$

where

$$\partial_\ell G(0, u, q) = (-\beta_2 \nabla u \cdot \nabla q - \varrho_2(u)q + f_2 q)(z) \quad (5.30)$$

and

$$R(u, q) = -\beta_2(z) \nabla u(z) \cdot \int_\omega \nabla Q \, dx \quad (5.31)$$

and  $Q$  depends on  $z$  and is the unique solution to (5.26a).

Let  $Q_\zeta$  denote the solution to (5.26a)-(5.26b) for fixed  $z \in D$  and for  $\zeta \in \mathbf{R}^d$ . Since  $Q_\zeta$  depends linearly on  $\zeta$  we can proceed as in Subsection 4.6 and introduce a polarisation matrix  $\mathcal{P} \in \mathbf{R}^{d \times d}$  (depending on  $\beta_2(z)$ ) such that  $\mathcal{P}\zeta = \int_\omega \nabla Q_\zeta \, dx$  to simplify (5.29). Finally in the same way done as in Lemmas 4.19, 4.20 we can show that  $\mathcal{P}$  is symmetric if  $\beta_2$  is symmetric and that it is always positive definite. Since the considerations are almost identical with the ones of Subsection 4.6 the details are left to the reader.

## Concluding remarks

In this paper we showed that the Lagrangian averaged adjoint framework of [15] provides an efficient and fairly simple tool to compute topological derivatives for semilinear problems. We illustrated that using standard apriori estimates for the perturbed states and averaged adjoint variables are sufficient to obtain the topological sensitivity under comparatively mild assumptions on the inclusion. Our work also provides a second examples (the first was given by [14]) for which the  $R$  term in [15, Thm. 3.1] is not equal to zero and thus underlines the flexibility of this theorem.

There are several problems that remain open for further research. Firstly, it would be interesting to consider quasilinear equations, but also other types of equations, such as Maxwell's equation. Secondly, an important point we have not addressed here is the topological derivative when Dirichlet boundary conditions are imposed on the inclusion. This case is known to be much different from the Neumann case and needs further investigations.

## 6 Appendix

### 6.1 The space $E_p(\mathbf{R}^d)$

Define for  $1 < p < d$  the space

$$E_p(\mathbf{R}^d) := \{u \in L_{p^*}(\mathbf{R}^d) : \nabla u \in L_2(\mathbf{R}^d)^d\} \quad (6.1)$$

with the norm

$$\|u\|_{E_p} := \|u\|_{L_{p^*}(\mathbf{R}^d)} + \|\nabla u\|_{L_2(\mathbf{R}^d)^d}. \quad (6.2)$$

**Lemma 6.1.** Let  $d \geq 3$ . Let  $A$  satisfy (3.1) and  $A = A^\top$  a.e. on  $\mathbf{R}^d$ . Then for every  $F \in \mathcal{L}(\mathbf{E}_2(\mathbf{R}^d), \mathbf{R})$ , there is a unique solution  $Q \in \mathbf{E}_2(\mathbf{R}^d)$  to

$$\int_{\mathbf{R}^d} A \nabla \varphi \cdot \nabla Q \, dx = F(\varphi) \quad \text{for all } \varphi \in \mathbf{E}_2(\mathbf{R}^d). \quad (6.3)$$

*Proof.* Let us introduce the energy  $\mathcal{E} : \mathbf{E}_2(\mathbf{R}^d) \rightarrow \mathbf{R}$  by

$$\mathcal{E}(\varphi) := \frac{1}{2} \int_{\mathbf{R}^d} A \nabla \varphi \cdot \nabla \varphi \, dx - F(\varphi). \quad (6.4)$$

We are now going to prove that the minimisation problem

$$\inf_{\varphi \in \mathbf{E}_2(\mathbf{R}^d)} \mathcal{E}(\varphi), \quad (6.5)$$

admits a unique solution. We have to show that  $\mathcal{E}$  is coercive on  $\mathbf{E}_2(\mathbf{R}^d)$ , that is,

$$\lim_{\|\varphi\|_{\mathbf{E}_2} \rightarrow \infty} \mathcal{E}(\varphi) = +\infty \quad \text{for } \varphi \in \mathbf{E}_2(\mathbf{R}^d), \quad (6.6)$$

and that the energy is lower semi-continuous; see [18, Prop. 1.2, p.35]. For the coercivity it is sufficient to show that there are constants  $C_1, C_2 > 0$  such that

$$\mathcal{E}(\varphi) \geq C_1 \|\varphi\|_{\mathbf{E}_2(\mathbf{R}^d)}^2 - C_2 \|\varphi\|_{\mathbf{E}_2(\mathbf{R}^d)} \quad \text{for all } \varphi \in \mathbf{E}_2(\mathbf{R}^d). \quad (6.7)$$

Using the NSG inequality we can estimate as follows

$$\begin{aligned} \frac{1}{2} \int_{\mathbf{R}^d} A \nabla \varphi \cdot \nabla \varphi \, dx &\geq \frac{1}{2} M_1 \|\nabla \varphi\|_{(L_2)^d}^2 \\ &\geq \frac{1}{4} C_N^2 M_1 \|\varphi\|_{L_2^*}^2 + \frac{1}{4} M_1 \|\nabla \varphi\|_{(L_2)^d}^2 \\ &\geq C (\|\varphi\|_{L_2^*}^2 + \|\nabla \varphi\|_{(L_2)^d}^2), \end{aligned} \quad (6.8)$$

where  $C := \min\{\frac{1}{4} C_N^2 M_1, \frac{1}{4} M_1\}$ . On the other hand using again the NSG inequality yields

$$\begin{aligned} (\|\varphi\|_{L_2^*} + \|\nabla \varphi\|_{(L_2)^d})^2 &= \|\varphi\|_{L_2^*}^2 + \|\nabla \varphi\|_{(L_2)^d}^2 + 2\|\varphi\|_{L_2^*} \|\nabla \varphi\|_{(L_2)^d} \\ &\leq \|\varphi\|_{L_2^*}^2 + \|\nabla \varphi\|_{(L_2)^d}^2 + 2\frac{1}{C_N} \|\varphi\|_{L_2^*}^2 \\ &\leq \tilde{C} (\|\varphi\|_{L_2^*}^2 + \|\nabla \varphi\|_{(L_2)^d}^2) \end{aligned} \quad (6.9)$$

where  $\tilde{C} := \min\{1 + \frac{2}{C_N}, 1\}$ . Combining (6.8) and (6.9) yields

$$\frac{1}{2} \int_{\mathbf{R}^d} A \nabla \varphi \cdot \nabla \varphi \, dx \geq \frac{C}{\tilde{C}} \|\varphi\|_{\mathbf{E}_2(\mathbf{R}^d)}^2. \quad (6.10)$$

Finally the continuity of  $F$  gives

$$F(\varphi) \geq -\|F\|_{\mathcal{L}(\mathbf{E}_2, \mathbf{R})} \|\varphi\|_{\mathbf{E}_2(\mathbf{R}^d)}. \quad (6.11)$$

Combining (6.10) and (6.11) yields (6.7) with  $C_1 = C/\tilde{C}$  and  $C_2 = \|F\|_{\mathcal{L}(\mathbf{E}_2, \mathbf{R})}$ .  $\square$

Recall the Gagliardo-Nirenberg-Sobolev inequality (short NSG inequality)

$$\|u\|_{L_{p^*}(\mathbf{R}^d)} \leq C_N \|\nabla u\|_{L_p(\mathbf{R}^d)} \quad (6.12)$$

valid for all  $u \in C_c^\infty(\mathbf{R}^d)$ . The constant  $C_N$  does not depend on the support of the function  $u$ . Notice also that for  $p = d$  the inequality fails. Thanks to Lemma 6.2 we know that  $C_c^\infty(\mathbf{R}^d)$  is dense in  $\mathbf{E}_p(\mathbf{R}^d)$ . Hence it follows that (6.12) holds for all test functions  $u \in \mathbf{E}_p(\mathbf{R}^d)$ . For instance for  $d = 3$  and  $E_2(\mathbf{R}^3)$  we have

$$\|u\|_{L_6(\mathbf{R}^3)} \leq C \|\nabla u\|_{L_2(\mathbf{R}^3)}. \quad (6.13)$$

**Lemma 6.2.** For all  $1 < p < d$  the space  $(\mathbf{E}_p(\mathbf{R}^d), \|\cdot\|_{\mathbf{E}_p})$  is a Banach space. For every sequence  $(u_n)$  in  $\mathbf{E}_p(\mathbf{R}^d)$  we find a subsequence  $(u_{n_k})$  and an element  $u \in \mathbf{E}_p(\mathbf{R}^d)$ , such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u \quad \text{weakly in } L_{p^*}(\mathbf{R}^d) \quad \text{as } n \rightarrow \infty, \\ \nabla u_{n_k} &\rightharpoonup \nabla u \quad \text{weakly in } L_p(\mathbf{R}^d)^d \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (6.14)$$

Moreover,  $C_c^\infty(\mathbf{R}^d)$  is dense in  $\mathbf{E}_p(\mathbf{R}^d)$ .

*Proof.* Let  $(u_n)$  be a bounded sequence in  $\mathbf{E}_p(\mathbf{R}^d)$ . Since the  $L_p(\mathbf{R}^d)$ -spaces are reflexive for all  $p \in (1, \infty)$ , we find elements  $\eta \in L_{p^*}(\mathbf{R}^d)$  and  $\zeta \in L_p(\mathbf{R}^d)^d$  and a subsequence  $(u_{n_k})$ , such that

$$\begin{aligned} u_{n_k} &\rightharpoonup \eta \quad \text{weakly in } L_{p^*}(\mathbf{R}^d) \quad \text{as } n \rightarrow \infty, \\ \nabla u_{n_k} &\rightharpoonup \zeta \quad \text{weakly in } L_p(\mathbf{R}^d)^d \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (6.15)$$

Now we claim that  $\zeta = \nabla \eta$ , which then implies  $\eta \in \mathbf{E}_p(\mathbf{R}^d)$ . To see this notice that by definition of the weak derivative

$$\int_{\mathbf{R}^d} \partial_{x_i} \varphi u_{n_k} dx = - \int_{\mathbf{R}^d} \varphi \partial_{x_i} u_{n_k} dx \quad (6.16)$$

for all  $\varphi \in C_c^\infty(\mathbf{R}^d)$ . Now we pass to the limit in (6.16) and obtain

$$\int_{\mathbf{R}^d} \partial_{x_i} \varphi \eta dx = - \int_{\mathbf{R}^d} \varphi \zeta dx \quad (6.17)$$

for all  $\varphi \in C_c^\infty(\mathbf{R}^d)$ , which proves the claim. Since a linear and continuous functional on a Banach space is continuous if and only if it is weakly continuous the claim follows.

To prove the completeness of  $\mathbf{E}_p(\mathbf{R}^d)$  let  $(u_n)$  be a Cauchy sequence in  $\mathbf{E}_p(\mathbf{R}^d)$ . Then  $(u_n)$  is a Cauchy sequence in  $L_{p^*}(\mathbf{R}^d)$  and  $(\nabla u_n)$  is a Cauchy sequence in  $L_p(\mathbf{R}^d)^d$ . Since  $(u_n)$  is a Cauchy sequence in  $L_{p^*}(\mathbf{R}^d)$  and  $(\nabla u_n)$  is a Cauchy sequence in  $L_p(\mathbf{R}^d)^d$  we find elements  $\eta \in L_{p^*}(\mathbf{R}^d)$  and  $\zeta \in L_p(\mathbf{R}^d)^d$  so that  $u_n \rightarrow \eta$  strongly in  $L_{2^*}(\mathbf{R}^d)$  and  $\nabla u_n \rightarrow \zeta$  strongly in  $L_p(\mathbf{R}^d)^d$ . Now we can follow the previous argumentation to show that  $\nabla \eta = \zeta$  which shows that  $\eta \in \mathbf{E}_p(\mathbf{R}^d)$  and thus shows that  $\mathbf{E}_p(\mathbf{R}^d)$  is complete.

Let us now show the density of  $C_c^\infty(\mathbf{R}^d)$  in  $\mathbf{E}_p(\mathbf{R}^d)$ . As shown in [1, Thm. 3.22, p. 68] it suffices to show every  $u \in \mathbf{E}_p(\mathbf{R}^d)$  with bounded support can be approximated by function in  $C_c^\infty(\mathbf{R}^d)$ . Suppose that the support of  $u$  is compact. Denote by  $u_\varepsilon := \varrho_\varepsilon * u$  the standard mollification of  $u$  with a mollifier  $\varrho_\varepsilon \in C^\infty(\mathbf{R}^d)$ ,  $\varepsilon > 0$ ; see [1, pp. 36]. Then  $u_\varepsilon \in L_{p^*}(\mathbf{R}^d)$  and  $\partial_i u_\varepsilon = \varrho_\varepsilon * \partial_i u \in L_2(\mathbf{R}^d)$ . Moreover according to [1, Thm. 2.29, p. 36] we have  $\lim_{\varepsilon \searrow 0} \|u_\varepsilon - u\|_{L_{2^*}(\mathbf{R}^d)} = 0$  and  $\lim_{\varepsilon \searrow 0} \|\partial_{x_i}(u_\varepsilon - u)\|_{L_2(\mathbf{R}^d)} = 0$ . This finishes the proof.  $\square$

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