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Numerical treatment of the spike formation stage in marginally separated flows

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The method of matched asymptotic expansions is used to describe the so-called 'by-pass' transition in marginally separated boundary layer flows. Such flows may typically be observed e.g. on the suction side of a slender airfoil if the angle of attack is raised above a critical value. As a consequence, transition to turbulent flow is triggered by the repeated bursting of a laminar separation bubble. For asymptotically large Reynolds numbers the bursting of a laminar separation bubble can be described by various consecutive stages. The present work addresses the numerical solution of the triple deck stage succeeding a finite-time blow-up of the marginal separation stage. Although the corresponding evolution equations of the triple deck stage have been studied extensively by Elliott and Smith, a reliable numerical solution has not yet been presented. Special emphasis is placed on the formulation of the matching condition to the terminal structure of the preceding marginal separation stage.

1 Introduction

Among many possible routes from laminar to turbulent flow we consider the so-called 'by-pass' transition which is initiated by the repeated bursting of a laminar separation bubble (LSB). The asymptotic theory of marginal separation and its extensions describe the bursting process by various successive stages, Fig. 1.

The initial phase (i) is represented by a laminar wall-bounded shear layer on the verge of separation for which classical boundary layer theory ceases to provide an uniformly valid description. In the subsequent stage of marginal separation (ii) the formation of a LSB is described by an interactive triple deck formulation. The emergence of finite-time singularities in the solution of the corresponding model equations lead to a breakdown of the marginal separation stage and initiate the spike formation stage (iii). The present work addresses a numerical solution of the Prandtl-type evolution equation governing the spike formation stage with the focus on the connection to the preceding marginal separation stage.

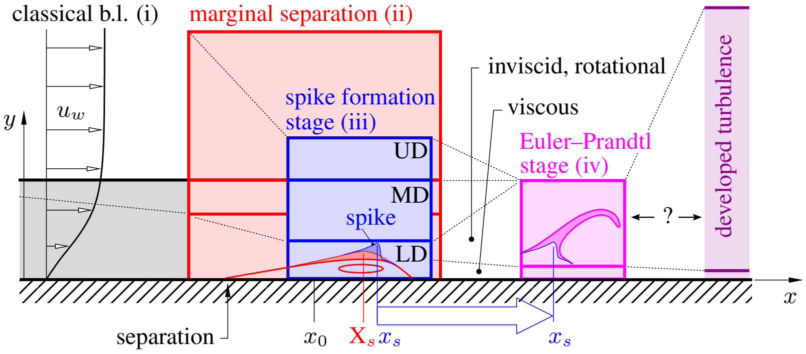

Fig. 1 Schematic view of the asymptotic layer structure of the early stages of the laminar-turbulent transition process in laminar separation bubbles. The classical boundary layer structure (i) is composed of an inviscid outer flow (transparent) and a viscous region (grey). The consecutive interactive stages of marginal separation (ii) and spike formation (iii) are followed by the Euler–Prandtl stage (iv) (shifted to the right for illustration). Eventually a stage of developed turbulence may be reached.

2 Spike formation stage

For two-dimensional incompressible flows the fundamental problem of the triple deck stage is governed by a fully nonlinear, unsteady viscous-inviscid interaction, [1], [2]. In terms of the stream function $\psi(x, y, t)$ one obtains the modified Prandtl boundary layer equation supplemented with the interaction law between the induced pressure $\mathcal{P}(x, t)$ and the displacement function $\mathcal{A}(x, t)$,

$$\frac{\partial^2\psi}{\partial y\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -\left(1 + \frac{\partial\mathcal{P}}{\partial x}\right) + \frac{\partial^3\psi}{\partial y^3}, \qquad \mathcal{P} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial\mathcal{A}/\partial\xi}{x-\xi}\mathrm{d}\xi, \tag{1}$$

subject to the no-slip conditions $\psi = \partial\psi/\partial y = 0$ at the solid wall $y = 0$ and the far-field conditions $\psi \sim (y + A)^3/6 + \ldots$ as $y \to \infty$ as well as $\psi \to y^3/6, (A, P) \to 0$, as $|x| \to \infty$. Here all variables are non-dimensionalized, suitably scaled and x, y and t denote the stream-wise, wall-normal coordinates and the time, respectively. The connection to the self-similar blow-up

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structure of the preceding marginal separation stage is ensured by specifying the matching or equivalently initial condition as $t \to -\infty$ with the similarity scalings $x = |t|^{4/9}\hat{X}$ and $y = |t|^{1/9}\hat{Y}$:

$$\begin{aligned}
&\psi \sim |t|^{\frac{1}{3}} \left(\frac{\hat{Y}^3}{6} + |t|^{-\frac{7}{9}} \hat{A} \frac{\hat{Y}^2}{2} + |t|^{-\frac{14}{9}} \hat{\psi} \right), \\
&\hat{\psi} \sim \left(\hat{\psi}_2(\hat{X}, \hat{Y}) + \hat{A}_1^2 \frac{\hat{Y}}{2} \right) + |t|^{-\frac{4}{9}} \left(\hat{\psi}_{e1}(\hat{X}, \hat{Y}) + \hat{A}_1 \hat{e}_1 \hat{Y} \right) + |t|^{-\frac{7}{9}} \left(\hat{\psi}_3(\hat{X}, \hat{Y}) + \hat{A}_1 \hat{A}_2 \hat{Y} + \hat{A}_1 \frac{\partial \hat{\psi}_2}{\partial \hat{Y}} + \frac{\hat{A}_1^3}{6} \right), \\
&[\hat{A}, \hat{P}] \sim [\hat{A}_1, \hat{P}_1](\hat{X}) + |t|^{-\frac{4}{9}} [\hat{e}_1, \hat{p}_{e_1}](\hat{X}) + |t|^{-\frac{7}{9}} [\hat{A}_2, \hat{p}_2](\hat{X}) + |t|^{-1} [\hat{e}_2, \hat{p}_{e_2}] + \cdots .
\end{aligned} \tag{2}$$

Here the leading order term $\hat{Y}^3/6$ represents the leading order term of the separation profile of the classical boundary layer and the (partly unique) blow-up profiles $[\hat{A}_1, \hat{P}_1]$ as well as $[\hat{e}_1, \hat{p}_{e_1}]$, etc. form Hilbert pairs according to the second equation (1). Furthermore, the eigenfunctions $\hat{e}_1$, $\hat{e}_2$ with arbitrary amplitudes carry the history of the flow and enable the embedding of the local into a global solution. The unique stream functions $\hat{\psi}_2$, $\hat{\psi}_3$,... etc. are determined accordingly, [3].

To solve the initial value problem (1) and (2) for the intrinsic unknowns ψ and $\mathcal{A}$ numerically, the infinite (half) space domain of the asymptotic formulation is mapped onto a bounded domain and discretized with Gauß-Lobatto points. A spectral collocation method based on Chebyshev polynomials of the first kind is used for interpolation of the unknown field variables, [4]. Singular behaviour of ψ as $y \to \infty$ and its similarity properties as $x \to |\infty|$ are taken into account by the ansatz $\psi(x, y, t) = (bd)^3\{[\eta + \mathcal{A}/(b^7 d)]^3/6 + g\}$, $\mathcal{A} = b^{-6}A$, $\mathcal{P} = b^{-10}P$ and the scalings $x = b^4\xi$, $y = bd\eta$. Here, $g(\xi, \eta, t)$ denotes the modified stream function and the auxiliary functions $b(t)$, $d(\xi)$ accomplish the properties $b \sim |t|^{1/9}$ as $t \to -\infty$, $b \sim O(1)$ if $t \sim O(1)$ and $d \sim |\xi|^{1/4}$ as $\xi \to \pm\infty$, $d \sim O(1)$ if $\xi \sim O(1)$. Time derivatives are formulated analytically based on the matching/initial condition (2) in the asymptotic regime $-t \gg 1$ and otherwise approximated by a third order accurate backward finite differencing scheme capable of adaptive time stepping on a mapped domain. The resulting system of nonlinear equations is solved in a time-marching manner by means of a modified Powell hybrid method for each instant of time. The results of the numerical investigation are presented in terms of the boundary layer characteristics, both the displacement thickness $\delta^* \sim \delta_0^*(x_0) + Re^{-1/14} c\,\mathcal{A}$ and the wall shear stress $\tau_w \sim Re^{-1/14}\partial^2\psi/\partial y^2|_{y=0}$, with the local displacement thickness of the classical boundary layer theory denoted as $\delta_0^*(x_0)$ and an arbitrary constant $c < 0$, Fig. 2. The

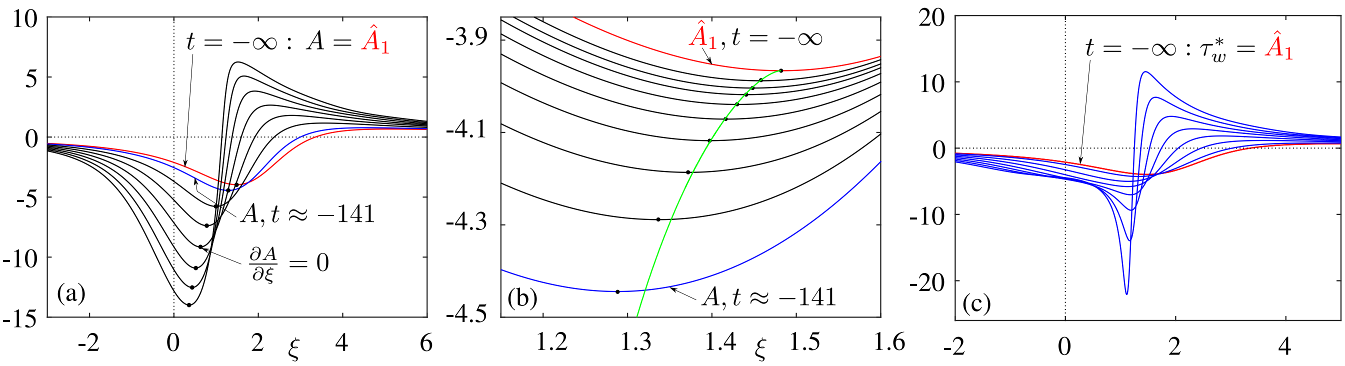

Fig. 2: Evolution of the displacement function $A(\xi, t)$ (a) and the wall shear stress $\tau_w^*(\xi, t) = b^6\partial^2\psi/\partial y^2|_{y=0}$ (c) for $t = -(\infty, 141.2, 34.3, 19.7, 14.3, 11.6, 10.0, 8.98)$. (b) depicts $\hat{A}_1$ (red), the minima of $\hat{A}$ (green) in the limit as $t \to -\infty$ and the minimum of $A(\xi, t)$ (black dots) at additional instants of time, $t = -(\infty, 6366.2, 3183.1, 2122.1, 1384.0, 889.1, 565.6, 357.5, 225.0, 141.2)$.

temporal evolution of the minimum of A for $-t \gg 1$, (black dots) in Fig. 2 (b), compared with the minimum of $\hat{A}$ of the asymptotic description, (2), as $t \to -\infty$ (green line) clearly shows a smooth connection to the preceding marginal separation stage. The growing amplitudes of A and τ_w^* indicate the emergence of yet another finite-time blow-up. However, the currently manageable maximal degrees of the underlying polynomials ($n_\xi \times n_\eta \leq 250 \times 125$) fail to accurately approximate the sought functions as the cut-off Chebyshev polynomials of degree $k > n_\xi, n_\eta$ become relevant. Further investigations will focus on the terminal (blow-up) structure of (1) which represents the central building block of the initial or equivalently matching condition for the subsequent Euler–Prandtl stage.

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