# Free Overfall Flow


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**Abstract**

Many works have considered two-dimensional free-surface flow over the edge of a plate, forming a waterfall, and with uniform horizontal flow far upstream. The flow is assumed to be steady and irrotational, whilst the fluid is assumed to be inviscid and incompressible, and gravity is taken into account. In particular, amongst these works, numerical solutions for both supercritical and subcritical flows are computed by Dias and Tuck (1991), utilising conformal mappings as well as a series truncation and collocation method. We present an extension to this work where a more appropriate expression is taken for the assumed form of the complex velocity. The justification of this lies in the behaviour of the waterfall flow far downstream and how the parabolic nature of such a free-falling jet can be better encapsulated. New numerical results will be presented, demonstrating the difference in the shape of the new free surface profiles. Comparisons with the asymptotic solutions found by Clarke (1965) will also be made to validate these numerical solutions.


## Introduction

Numerical solutions for two-dimensional free-surface flows, where two free surfaces form a jet far downstream, were computed by Dias and Tuck (1991) . However, the parabolic nature of such a free falling jet can be better encapsulated. Many works have considered such flows over the edge of a plate, forming a waterfall, and with uniform horizontal flow far upstream. The flow is assumed to be steady and irrotational, whilst the fluid is assumed to be inviscid and incompressible, and gravity is taken into account. Numerical solutions are also obtained by Chow and Han (1979), Smith and Abd-el-Malek (1983), and Goh and Tuck (1985) using a finite difference method, integral equations, and integral equations for a waterfall from a channel with an upper wall, respectively. Dias and Tuck (1991) utilised conformal mappings as well as a series truncation and collocation method. An extension to that work will be presented here, where a more appropriate expression is taken for the assumed form of the complex velocity. The justification of this lies in the waterfall flow far downstream, where we should look to include more terms for the behaviour of the jet to avoid having spillway-like flow. New numerical results will be presented, demonstrating the improvement in the shape of the new free surface profiles. Comparisons with the asymptotic solutions found by Clarke (1965) will also be made, validating these numerical solutions. Enhanced decay of the coefficients that are obtained through the series truncation and collocation method is also achieved.

## Formulation

We define the Froude number, $F$, by $F = U/(gH)^{1/2}$, where $U$ is the far upstream velocity, $g$ is the acceleration due to gravity and $H$ is the far upstream depth of the flow. In the calculations here, we are concerned with $F > 1$, i.e. supercritical flow. We work in non-dimensional variables and so far upstream we have unit depth and velocity. Figure 1 shows the $z$-plane of the waterfall problem, where $z$ is the complex variable defined by $z = x + \mathrm{i}y$. Note that the origin is at the corner $C$.

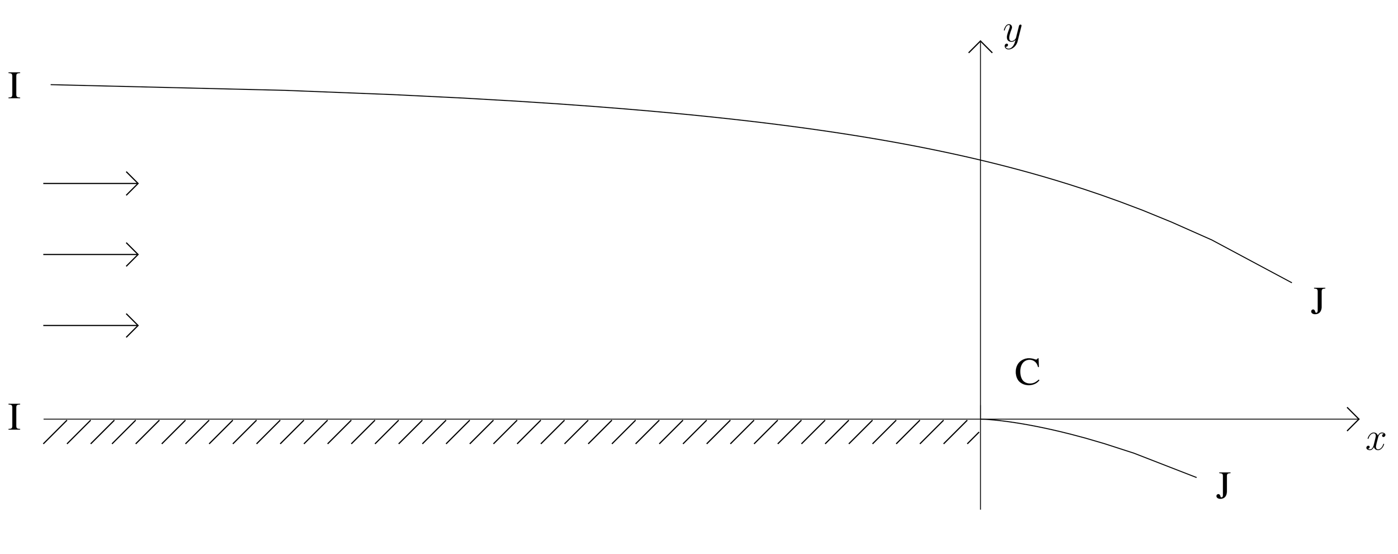


**Figure 1:** $z$-plane

Assuming equal pressure on both the upper ($IJ$) and lower ($CJ$) free surfaces, using the Bernoulli equation we arrive at

$$\frac{1}{2}q^2 + \frac{y}{F^2} = \frac{1}{2} + \frac{1}{F^2}, \tag{1}$$

along both free surfaces, where $q$ is the magnitude of the velocity. The complex potential is defined $f = \phi + \mathrm{i}\psi$, and we set $\phi = 0$ at $C$, and $\psi = 0$ and $\psi = 1$ along the lower and upper free surfaces, respectively. Then, the $f$-plane is as shown in Figure 2, the semi-infinite, horizontal strip of width 1.

We now introduce the $t$-plane which is defined by

$$f = \frac{1}{\pi}\log\frac{(t+1)^2}{2(t^2+1)}. \tag{2}$$

This maps the $f$-plane to the upper half of a unit semi-circle centred at the origin of the $t$-plane. The interior of the semi-infinite strip maps into the interior of the semi-circle, whilst the upper free surface $IJ$ maps to the left-hand arc of the semi-circle and the lower free surface $CJ$ maps to the right-hand arc of the semi-circle (c.f. Figure 3).

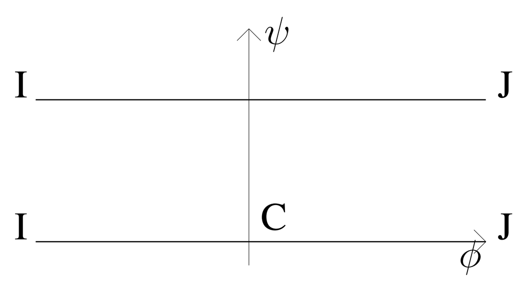


**Figure 2:** $f$-plane

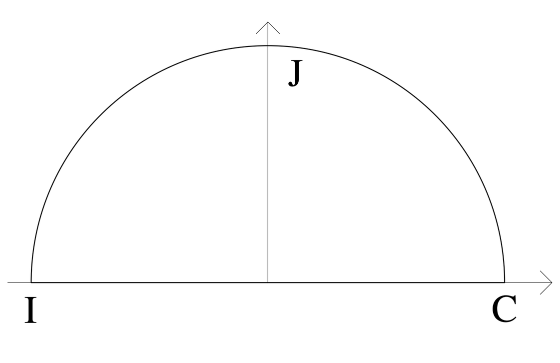


**Figure 3:** $t$-plane

The complex velocity is defined by $\zeta = u - \mathrm{i}v$, where $u$ and $v$ are the horizontal and vertical components of velocity, respectively. The aim now is to find $\zeta$ as an analytic function of the complex potential, $f$. The complex velocity must satisfy the following:

- $\zeta \sim (1 + ae^{\lambda f})$ as $\phi \to -\infty$, where $a$ is an unknown constant and $\lambda$ is the smallest positive root of $\lambda - F^{-2}\tan\lambda = 0$,
- $v = 0$ on $\psi = 0$, $\phi < 0$,
- $\zeta \sim \mathrm{i}(3G)^{1/3}f^{1/3} + 1 + \frac{G}{2} + Cf^{-1/3}$ as $\phi \to +\infty$, where $G = F^{-2}$ and $C$ is an unknown constant.

The first and second conditions listed above are as utilised by Dias and Tuck (1991). The first describes the upstream condition that as $\phi \to -\infty$, the flow approaches a uniform horizontal stream of constant unit velocity. The second condition simply ensures no through-flow along the horizontal wall. The third condition listed above is to capture the behaviour of the jet. This three-term expansion differs to the condition on the far downstream flow employed by Dias and Tuck (1991) - i.e. $\zeta \sim f^{1/3}$.

## Method

The following form for $\zeta$ satisfies the aforementioned conditions:

$$\zeta(t) = A + (1+t)^{2\lambda/\pi}B(t), \tag{3}$$

where

$$B(t) = \left(\frac{3G}{\pi}\right)^{\frac{1}{3}}\left(-\log\left(c(1+t^2)\right)\right)^{\frac{1}{3}} l_1(t) + \frac{G}{2}l_2(t) + \sum_{n=0}^{\infty} a_n t^n \left(-\log\left(c(1+t^2)\right)\right)^{-\frac{1}{3}}, \tag{4}$$

and $l_1$ and $l_2$ are the following linear functions:

$$l_1(t) = 2^{-\lambda/\pi}[\sin(\lambda/2) + t\cos(\lambda/2)],\; l_2(t) = 2^{-\lambda/\pi}[\cos(\lambda/2) - t\sin(\lambda/2)]. \tag{5}$$

The constants $A$ and $a_n$, $n = 0, 1, 2, ...$ are to be found; and $c$ is a real constant such that $0 < c < \frac{1}{2}$.

It remains to satisfy the Bernoulli condition on both free surfaces which will, for a given Froude number, enable us to find the unknowns. We truncate the infinite series in Equation 3 after $N-1$ terms. For the image of the free surfaces in the $t$-plane, we can use $t = e^{i\sigma}$, for $0 < \sigma < \pi/2$. We introduce $N$ mesh points $\sigma_I = \pi/2N + (\pi/N)(I-1)$, for $I = 1, ..., N$ for the collocation method. For the $N$ mesh points, we obtain $N$ equations in $N$ unknowns ($A$ and the $N-1$ unknown coefficients), which can then be solved numerically by iteration.

## Results

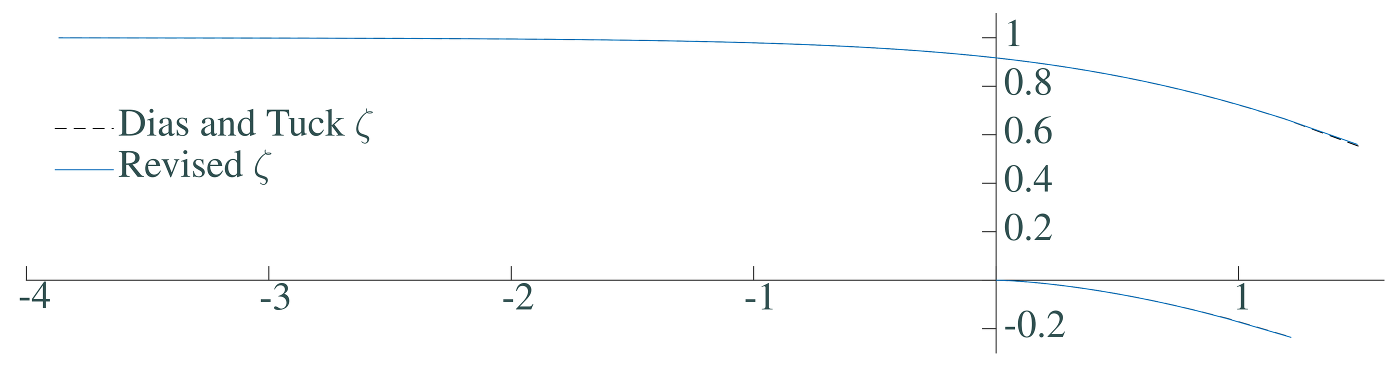


**Figure 4:** Free surface profile: $F = 2$, $N = 400$ and $c = 0.2$

Figure 4 shows a comparison of the free surface profiles obtained using the Dias and Tuck complex velocity and the revised form. The profiles are the same to order $10^{-3}$. The effect of the altered form for the complex velocity can better be seen in Figure 5. Here, the system has been solved with 400 equations in 400 unknowns, as before, but 40000 mesh points have been used to plot the free surface profiles. In the work of Dias and Tuck, the assumed form for the complex velocity far downstream (i.e. $\zeta \sim f^{1/3}$) means that the flow will approach a jet of constant slope. The new waterfall appears to approach a more parabolic shape, as hoped for.

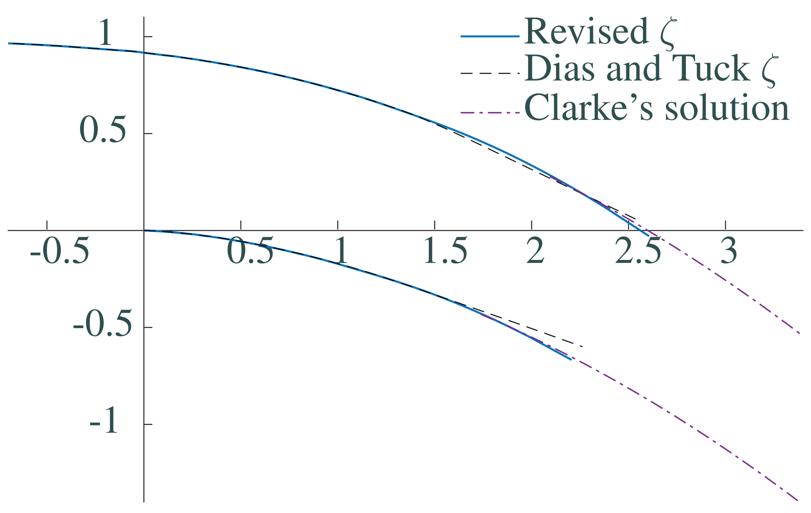


**Figure 5:** Waterfall free surface profiles: $F = 2$, $c = 0.2$, $N = 400$, extra plotting points and Clarke's asymptotic solution.

Figure 5 also includes Clarke's asymptotic outer solution which agrees well with the free surface profile obtained via the revised complex velocity form. Further comparisons with Clarke's outer solution have been obtained for different Froude numbers. The agreement improves as the Froude number increases. It is more appropriate to utilise the numerical method described here with the revised complex velocity (than Clarke's asymptotic solution) for smaller Froude numbers.

The final point to note is that the last few coefficients of the truncated series are very small. In particular, in the case discussed above in Figure 4, the last few coefficients are of order $10^{-7}$. Leaving the constant $A$ unknown in Equation 2 (as opposed to setting $A = 1$) enables there to be such significant decay of the coefficients; and the resulting value for $A$ is 1, to order $10^{-6}$.

## Conclusion

It has been demonstrated that the revised form for the complex velocity better encapsulates the behaviour of the free-falling jet. Visually, this is evident in Figure 5 where the new free surface profiles are compared with that of Dias and Tuck and the asymptotic solution of Clarke. The effectiveness of the overall numerical method is greatly improved by the addition of the unknown constant $A$ in the complex velocity, c.f. Equation 2), improving the decay of coefficients of the truncated series.

## Forthcoming Research

The improved three-term expansion for the behaviour of $\zeta$ for the jet can be applied to other flows, e.g. weir flows or a waterfall impeded by an obstacle upstream. Furthermore, the use of the unknown constant $A$ can be applied to these revised complex velocity expressions to improve coefficient decay.

## References


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