

## Surface tension and energy conservation in a moving fluid

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The transport of energy in a moving fluid with a simply connected free surface is analyzed, taking into account the contribution of surface tension. This is done by following a “control volume” with arbitrary, specified velocity, independent of the flow velocity, and determining the rates of energy passing through the boundaries, as well as the energy dissipation in the bulk. In particular, a simple conservation equation for the surface area is given, which clearly shows the contribution of the Laplace pressure at the free surface and the tangential surface tension forces at its boundary. It emerges as the mechanical conservation law for the surface energy in its general form. For a static control volume, all contributions from surface tension disappear, except that the pressure has to be modified by the Laplace contribution.

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Capillary effects have been included in flows with free surfaces or interfaces for around 150 years. Some famous pioneers are Rayleigh [1,2], Landau and Levich [3], Taylor [4], Culick [5], and Bretherton [6], and it is still an active field of research with outstanding survey papers [7,8] and the textbook [9]. In contrast to flows having free surfaces with objects protruding, where surface tension gives rise to tangential forces at the boundaries, or Marangoni flows, where gradients of the surface tension contribute to the surface-stress balance, free-surface flows with uniform surface tension experience only a discontinuity of the normal component of the fluid stresses on the interface, commonly referred to as Laplace pressure and described, e.g., carefully by Batchelor [10] (Sec. 1.9).

Although this is merely a consequence of the governing equations of motion and conventional kinematic and dynamic boundary conditions, this procedure has been challenged recently in *Journal of Fluid Mechanics* by Bhagat *et al.* [11] and Bhagat and Linden [12]. These authors claim that the tangential surface tension contributes to a power term in the integral energy budget and should, therefore, be taken into account explicitly. For a *static control volume* in a steady (stationary) flow, this power is claimed to be

$$\oint_{C_s} \mathbf{f}_\gamma \cdot \mathbf{u} dC_s. \quad (1)$$

Here the closed curve  $C_s$  has the infinitesimal arc length  $dC_s$  and encloses a portion of the free surface that forms the control area and confines the control volume,  $\mathbf{u}$  is the flow velocity on  $C_s$ , and  $\mathbf{f}_\gamma$  is the external surface tension force (per length). Indeed, inclusion of such a power term,

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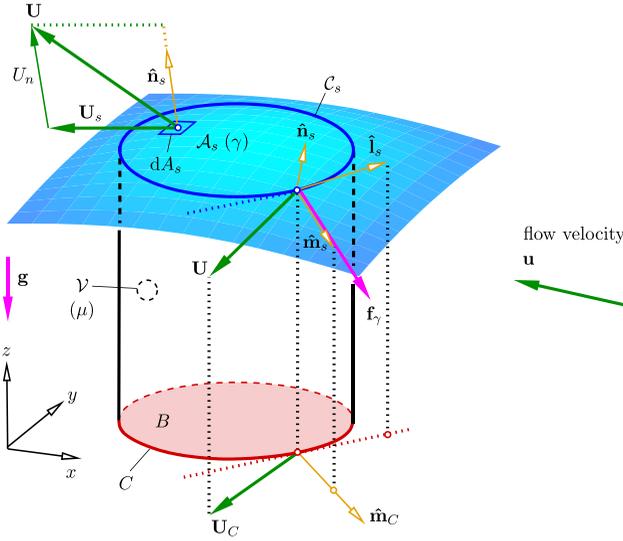


FIG. 1. Control volume  $\mathcal{V}$  of bulk flow, moving with speed  $U$  in a flow with velocity  $\mathbf{u}$ , bounded by control area  $A_s$  of free surface (top face, blue) enclosed by  $C_s$ , projections  $B$ ,  $C$  of  $A_s$ ,  $C_s$  onto  $(x, y)$  plane (bottom, red), and projection lines (bold black); essential bulk and surface properties ( $\mu$ ,  $\gamma$ ), body and surface force densities (magenta), velocities (green), unit vectors (empty arrows, orange); dotted: tangents and projection lines of vectors.

37 referring to the external work of surface tension, seems intuitively correct and its omission therefore  
 38 a serious error in all the preceding works. However, in this paper we demonstrate that within the  
 39 continuum hypothesis, where the interface is taken as infinitely thin and mass- and inertialess, this  
 40 is not the case. Such a power contribution is present only when the control surface is moving, and  
 41 then the velocity  $\mathbf{u}$  in (1) should be replaced by the velocity of the control surface,  $\mathbf{U}$ , transporting  
 42  $C_s$ . Even in the general situation of an unsteady flow, the appearance of this term is inconsistent  
 43 with the advection of surface energy combined with the standard work-energy balance for the bulk  
 44 flow, derived from the momentum (Navier-Stokes) equations. This was already noticed by Duchesne  
 45 *et al.* [13] and Scheichl [14], but in the present paper we show how this can be seen directly from  
 46 the conservation law for a moving surface area that we derive below.

47 We consider a free-surface flow of an incompressible fluid (liquid) of uniform density  $\rho$ , as  
 48 sketched in Fig. 1. The free surface forming the interface with the (gaseous) environment is  
 49 characterized by a uniform surface tension  $\gamma$ . The environment is typically taken as nominally  
 50 at rest and under a constant ambient pressure  $p_0$ . We assume that the dynamic viscosity  $\mu$  for the  
 51 gas is much smaller than that of the liquid, so the deviatoric stresses from the external gas flow  
 52 can be neglected (for simplicity). Let  $t$  denote the time and  $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = (x, y, z)$  the space  
 53 vector expressed in static Cartesian laboratory coordinates  $x, y, z$  and the associated unit vectors  
 54 (indicated with hats). For partial derivatives we use  $\nabla = (\partial_x, \partial_y, \partial_z)$  and  $\partial_t$  as well as subscripts  
 55  $t, x, y, z$  of scalars that cannot be confused with indices. We assume gravity to be the only body  
 56 force at play, with the constant gravitational acceleration  $\mathbf{g} = (0, 0, -g)$ . In the bulk of the flow,  
 57 the Eulerian “material” velocity field  $\mathbf{u}(\mathbf{x}, t) = (u, v, w)$ , the pressure  $p(\mathbf{x}, t)$ , and the (symmetric)  
 58 tensor  $\mathbf{T}$  of the deviatoric (viscous) Cauchy stresses, subject to a constitutive relationship, then  
 59 satisfy the continuity and momentum equations

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p - \rho g \hat{\mathbf{z}} + \nabla \cdot \mathbf{T}. \quad (3)$$

For a Newtonian liquid with the dynamic viscosity  $\mu$ ,  $\mathbf{T} = \mu \mathbf{S}$  where  $\mathbf{S} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$  is twice the strain-rate tensor.

We look at the simply connected control area  $\mathcal{A}_s$  of the moving fluid surface with the boundary  $\mathcal{C}_s = \partial \mathcal{A}_s$  and consider also their projections  $B$  and  $C = \partial B$  on to the  $(x, y)$  plane (the “bottom”). The free surface is described simplistically as a function  $z = h(x, y, t)$ . The value of  $\mathbf{u}$  on the surface can be decomposed into a component *in* the surface (i.e., in the tangent plane of the instantaneous surface)  $\mathbf{u}_s(x, y, t)$  and a normal component  $u_n(x, y, t)$  along the outward unit normal vector

$$\hat{\mathbf{n}}_s(x, y, t) = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + |\nabla h|^2}} = \frac{\hat{\mathbf{z}} - \nabla h}{\sqrt{1 + |\nabla h|^2}}. \quad (4)$$

We furthermore introduce the vectors

$$\hat{\mathbf{l}}_s = d\mathbf{x}/d\mathcal{C}_s, \quad \hat{\mathbf{m}}_s = \hat{\mathbf{l}}_s \times \hat{\mathbf{n}}_s, \quad \mathbf{f}_\gamma = \gamma \hat{\mathbf{m}}_s, \quad (5)$$

where  $d\mathcal{C}_s$  is the infinitesimal arc length on  $\mathcal{C}_s$ ,  $\hat{\mathbf{l}}_s$  a unit tangent vector on  $\mathcal{C}_s$ ,  $\hat{\mathbf{m}}_s$  is a unit vector perpendicular to  $\mathcal{C}_s$ , and  $\mathbf{f}_\gamma$  is the tangential capillary force per length from the surface region outside  $\mathcal{C}_s$ . Similarly, in the  $(x, y)$  plane, with infinitesimal arc length  $dC$  of  $C$ , its outer normal  $\hat{\mathbf{m}}_C$ , and the projection of  $\hat{\mathbf{n}}_s$  are

$$\hat{\mathbf{m}}_C = \frac{(dy, -dx, 0)}{dC}, \quad \mathbf{n}_b = \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}}. \quad (6)$$

Further  $\mathbf{u}$  satisfies the kinematic free-surface boundary condition

$$h_t = w - uh_x - vh_y = u_n \sqrt{1 + |\nabla h|^2} \quad (7)$$

on  $\mathcal{A}_s$ , so  $u_n$  is only nonzero when  $h$  is explicitly time-dependent. The dynamic boundary condition expresses continuity of the total stress across  $\mathcal{A}_s$  in the form (see, e.g., [15], § 61)

$$(p - p_0) \hat{\mathbf{n}}_s - \hat{\mathbf{n}}_s \cdot \mathbf{T} - \Delta p_L \hat{\mathbf{n}}_s, \quad (8)$$

where  $\Delta p_L$  is the Laplace pressure involving (twice) the local mean curvature  $\kappa$  of  $\mathcal{A}_s$  as

$$\Delta p_L = \gamma \kappa = \gamma \nabla \cdot \hat{\mathbf{n}}_s = \gamma \nabla \cdot \mathbf{n}_b \quad (9)$$

(where the sign of  $\kappa$  is chosen such that  $\kappa > 0$  and  $\nabla^2 h < 0$  for a convex, droplet-shaped surface). The equivalence of the local stress equilibrium (8) on  $\mathcal{A}_s$  with its global counterpart

$$\oint_{\mathcal{A}_s} [(p - p_0) \hat{\mathbf{n}}_s - \hat{\mathbf{n}}_s \cdot \mathbf{T}] dA_s = - \oint_{\mathcal{C}_s} \mathbf{f}_\gamma d\mathcal{C}_s \quad (10)$$

is obtained using Stokes’ theorem in the variant

$$\oint_{\mathcal{C}_s} \hat{\mathbf{m}}_s d\mathcal{C}_s = \oint_{\mathcal{C}_s} \hat{\mathbf{l}}_s \times \hat{\mathbf{n}}_s d\mathcal{C}_s = - \int_{\mathcal{A}_s} (\nabla \cdot \hat{\mathbf{n}}_s) \hat{\mathbf{n}}_s dA_s \quad (11)$$

as shown, e.g., in [16] (p. 235), with the counterintuitive result that an integral over tangential vectors points in the direction of the normals.

It is our goal to combine the (hitherto unappreciated) conservation law for the surface energy contained in  $\mathcal{A}_s$  with that for the mechanical energy of the bulk flow in an arbitrarily moved, simply connected control volume  $\mathcal{V}$  bounded by  $\mathcal{A}_s$  at the free surface. We denote the velocity field moving the closed boundary  $\partial \mathcal{V}$  as  $\mathbf{U}(\mathbf{x}, t) = (U, V, W)$  and denote its component normal to  $\mathcal{A}_s$  as  $U_n$ . Analogously,  $C$  moves with  $\mathbf{U}_C$ , the projection of  $\mathbf{U}$  evaluated on  $\mathcal{C}_s$  on the  $(x, y)$  plane. Most importantly,  $\mathbf{U}$  must be kinematically compatible to the material flow field  $\mathbf{u}$  in the sense that any point on the free surface considered to move with  $\mathbf{U}$  remains in the surface for all times. That is,  $\mathbf{U}$  satisfies

$$h_t = W - Uh_x - Vh_y = U_n \sqrt{1 + |\nabla h|^2} \quad (12)$$

89 on  $\mathcal{A}_s$ , in analogy to (7). Thus a completely static control volume  $\mathcal{V}$  is of course possible only for a  
 90 steady flow.

91 The surface energy stored in  $\mathcal{A}_s$  is simply  $E_s = \gamma A_s$  where

$$A_s = \int_{\mathcal{A}_s} dA_s = \int_B \sqrt{1 + |\nabla h|^2} dx dy \quad (13)$$

92 is the time-dependent area of  $\mathcal{A}_s$ .

93 The integral conservation law for the surface energy is found by analyzing  $\gamma dA_s/dt$  in terms of  
 94  $\mathbf{U}$ , which transports  $\mathcal{V}$  and thus  $\mathcal{A}_s$ . Using Leibnitz's integral rule, the time derivative of (13) is

$$\frac{dA_s}{dt} = \int_B \partial_t \sqrt{1 + |\nabla h|^2} dx dy + \oint_C \sqrt{1 + |\nabla h|^2} \mathbf{U}_C \cdot \hat{\mathbf{m}}_C dC. \quad (14)$$

95 The first term to the right can be rewritten as

$$\begin{aligned} \int_B \frac{\nabla h_t \cdot \nabla h}{\sqrt{1 + |\nabla h|^2}} dx dy &= \int_B \left[ -h_t \nabla \cdot \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) + \nabla \cdot \left( \frac{h_t \nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \right] dx dy \\ &= \int_B h_t \kappa dx dy + \oint_C \frac{h_t}{\sqrt{1 + |\nabla h|^2}} \nabla h \cdot \hat{\mathbf{m}}_C dC. \end{aligned} \quad (15)$$

96 The last equality follows from (9) and the divergence theorem (or, more precisely, Green's theorem,  
 97 since the integral of the divergence is over a surface). All in all, we have

$$\frac{dA_s}{dt} = \int_B h_t \kappa dx dy + \oint_C \left( \sqrt{1 + |\nabla h|^2} \mathbf{U}_C + \frac{h_t}{\sqrt{1 + |\nabla h|^2}} \nabla h \right) \cdot \hat{\mathbf{m}}_C dC. \quad (16)$$

98 The expressions (5) combined with (4) yield the explicit representation

$$\hat{\mathbf{m}}_s = \frac{[(1 + h_y^2) dy, -(1 + h_x^2) dx, h_x dy - h_y dx]}{\sqrt{1 + |\nabla h|^2} dC_s}, \quad (17)$$

99 and together with the kinematic boundary condition (12) this leads to the simple result

$$\frac{dA_s}{dt} = \int_{\mathcal{A}_s} U_n \kappa dA_s + \oint_{\mathcal{C}_s} \mathbf{U} \cdot \hat{\mathbf{m}}_s dC_s. \quad (18)$$

100 Actually, this result could have been obtained more directly and intuitively by considering  
 101 independently the three infinitesimally small temporal (linear) variations of the shape of the material  
 102 surface element  $dA_s$ : (1) its dilatation normal to  $\mathcal{A}_s$  (inflation), (2) its distortion in its tangential  
 103 directions (stretching and shearing), and (3) its tangential rigid-body movement. Since the last  
 104 has no effect, this leaves us with the two contributions on the right side of (18), where the first  
 105 term originates in the relative differential variation of  $dA_s$  in the normal direction,  $(\delta_n dA_s)/dA_s$   
 106 (see, e.g., [17] and [15], § 61), and the last term describes the tangential net change of  $A_s$  as the  
 107 annihilation/generation of area adjacent to  $\mathcal{C}_s$ . In fact, these considerations immediately confirm the  
 108 extension of the well-known Leibnitz-Reynolds transport theorem to the general transport of some  
 109 scalar  $f(\mathbf{x}, t)$  on a moving surface  $\mathcal{A}_s$ . Following, e.g., [18] [Eq. (3.14)] or [19] (Theorem 4.2), this  
 110 can be written in the form

$$\frac{d}{dt} \int_{\mathcal{A}_s} f(\mathbf{x}, t) dA_s = \int_{\mathcal{A}_s} (\partial_t + U_n \partial_n + U_n \kappa) f dA_s + \oint_{\mathcal{C}_s} f \mathbf{U} \cdot \hat{\mathbf{m}}_s dC_s, \quad (19)$$

111 where  $\partial_n = \hat{\mathbf{n}}_s \cdot \nabla$  is the normal component of  $\nabla$  on  $\mathcal{A}_s$ . The first and the second integral on the  
 112 right side of (19) account for, respectively, the surface-normal and tangential variations of both  $A_s$   
 113 and  $f$ . Therefore, the first includes the area-fixed time derivative normal to the surface  $(\partial_t + U_n \partial_n)f$ .  
 114 The succinct derivation of (18) follows from taking  $f = \text{const}$ .



133 results help us to rewrite (18) in this case as

$$\begin{aligned}
 \frac{dL_s}{dt} &= \int_{x_1}^{x_2} h_t \kappa_1 dx + \left[ \sqrt{1 + h_x^2} \dot{x} + \frac{h_t h_x}{\sqrt{1 + h_x^2}} \right]_{x=x_1}^{x=x_2} \\
 &= \int_{\mathcal{L}_s} U_n \kappa_1 dL_s + [\mathbf{U} \cdot \hat{\mathbf{m}}_s]_{x=x_1}^{x=x_2}
 \end{aligned} \tag{25}$$

134 in direct analogy with (16) by noticing the projection  $\mathbf{U}_C = \dot{x}_{1,2} \hat{\mathbf{x}}$  of  $\mathbf{U}$  onto the endpoints. Alternatively, one obtains (25) from (18) in straightforward manner under the assumptions of a planar flow and  $\mathcal{C}$  chosen as a rectangle with a side of length  $x_2 - x_1$  parallel to the  $x$  axis.

137 We next remark some important consequences of (18). First, we derive the differential version, using (22) for  $f$  as a constant, and expanding  $\nabla_s \cdot \mathbf{U} = \nabla_s \cdot \mathbf{U}_s + U_n \nabla_s \cdot \hat{\mathbf{n}}_s = \nabla \cdot \mathbf{U} - \hat{\mathbf{n}}_s \cdot (\hat{\mathbf{n}}_s \cdot \nabla \mathbf{U})$ . Then (18) takes on the form

$$\frac{dA_s}{dt} = \int_{\mathcal{A}_s} [U_n \nabla \cdot \hat{\mathbf{n}}_s + \nabla_s \cdot \mathbf{U}_s] dA_s = \int_{\mathcal{A}_s} [\nabla \cdot \mathbf{U} - \hat{\mathbf{n}}_s \cdot (\nabla \mathbf{U}) \cdot \hat{\mathbf{n}}_s] dA_s. \tag{26}$$

140 As  $\mathcal{A}_s$  is chosen arbitrarily, the relative temporal rate of change of an infinitesimal area element must be

$$(\delta dA_s / \delta t) / dA_s = U_n \nabla \cdot \hat{\mathbf{n}}_s + \nabla_s \cdot \mathbf{U}_s = \nabla \cdot \mathbf{U} - \hat{\mathbf{n}}_s \cdot (\nabla \mathbf{U}) \cdot \hat{\mathbf{n}}_s. \tag{27}$$

142 For  $\mathbf{U}$  identical to the material flow field  $\mathbf{u}$ , this local form of (26) was already derived by Batchelor [10], Eqs. (3.1.5)–(3.1.8), without taking notice of its integral formulation. For an incompressible material flow field, it takes the form  $(\delta dA_s / \delta t) / dA_s = -\hat{\mathbf{n}}_s \cdot (\nabla \mathbf{u}) \cdot \hat{\mathbf{n}}_s$  and these local forms makes explicit the required invariance under the Galilean boost, i.e., for (27):  $\mathbf{U} \mapsto \mathbf{U} + \mathbf{C}$ ,  $\mathbf{x} \mapsto \mathbf{x} + \mathbf{C}t$ , which is evident neither from (18) nor from (16).

147 Finally, for a stationary control volume velocity  $\mathbf{U}$ , the curvature term in (18) is zero. But interestingly and counterintuitively,  $dA_s/dt$  does not necessarily vanish for a control volume following the flow (i.e.,  $\mathbf{U} = \mathbf{u}$ ) even for a stationary incompressible flow. Incompressibility suppresses the dilatation of the fluid particles but not their stretching and shearing *in* the surface governed by the surface divergence in (20). Therefore, the streamlines lying in the surface can diverge/converge, which explains the in general nonvanishing  $dA_s/dt$ .

153 In the remainder of this letter, we discuss the dynamic implications of the kinematic relationship (18). Multiplying (18) with  $\gamma$  and using the expressions for the Laplace pressure in (8) and the vectorial surface tension,  $\mathbf{f}_\gamma = \gamma \hat{\mathbf{m}}_C$ , results in the integral transport equation for the surface energy convected by  $\mathbf{U}$ :

$$\frac{dE_s}{dt} = \int_{\mathcal{A}_s} \Delta p_L U_n dA_s + \oint_{\mathcal{C}_s} \mathbf{f}_\gamma \cdot \mathbf{U} dC_s. \tag{28}$$

157 This has an appealing and convincing physical interpretation as a law of mechanical energy conversion for the piece  $\mathcal{A}_s$  of the inertialess free fluid surface: the work done by the Laplace pressure on  $\mathcal{A}_s$  and by the tangential surface tension forces at its boundary  $\mathcal{C}_s$ , on the surface moving with the imposed velocity field  $\mathbf{U}$ , changes the potential internal energy, i.e., the surface energy  $E_s$ , contained in  $\mathcal{A}_s$ . Therefore, (28) represents the precise justification for the Laplace pressure and the existence of the tangential surface tension forces. Note that the inclusion of the Marangoni effect (inhomogeneous  $\gamma$ ) would lead to an extension of (28) incorporating the transport of  $\gamma$  as a nonuniform passive scalar: see (19), (21), and (22) with  $f = \gamma(\mathbf{x}, t)$ . Differential versions are given in Eqs. (5) and (6) of [21] describing the transport of a surfactant on a deforming fluid surface: a nonconstant surface tension then is a field quantity defined only on the free surface, so  $\partial_n \gamma$  must be taken as zero.

168 The conservation law (28) is the basis for the subsequent careful re-examination of the work-energy balance for the entire volume  $\mathcal{V}$  including its top face  $\mathcal{A}_s$  once  $\mathbf{U}$  is specified appropriately. To do this, we first briefly recall the derivation of the classical work-energy theorem as stated, e.g.,

by Eq. (16.2) in [15]. More precisely, it denotes the transport of the overall mechanical energy of the bulk flow,

$$E_m = \int_{\mathcal{V}} e_m dV, \quad e_m = \frac{\rho|\mathbf{u}|^2}{2} + \rho g z, \quad (29)$$

where the energy density  $e_m$  consists of the kinetic and the potential energy densities. Taking the inner product of  $\mathbf{u}$  and the momentum equation (3) together with the incompressibility condition (2) and the identity  $w = \nabla \cdot (z\mathbf{u})$  leads to the differential form of the energy equation:

$$\partial_t e_m = \nabla \cdot [\mathbf{T} \cdot \mathbf{u} - (p + e_m)\mathbf{u}] - \mathbf{T} : \nabla \mathbf{u}. \quad (30)$$

Integration of (30) over  $\mathcal{V}$  and applying the divergence theorem gives

$$\int_{\mathcal{V}} \partial_t e_m dV = \oint_{\partial\mathcal{V}} [\mathbf{T} \cdot \mathbf{u} - (p + e_m)\mathbf{u}] \cdot \hat{\mathbf{n}} dA - \int_{\mathcal{V}} \mathbf{T} : \nabla \mathbf{u} dV, \quad (31)$$

where the first term represents the advected energy, and the last term represents the (non-negative) integral dissipation per time unit due to the deviatoric stresses, commonly denoted with  $\dot{\Phi}$ . We have also introduced the infinitesimal area element  $dA$  of  $\partial\mathcal{V}$  and its outward unit normal  $\hat{\mathbf{n}}$ . We note that, according to the derivation of (31), the stresses acting on  $\partial\mathcal{V}$  are taken as their limiting values approached from inside  $\mathcal{V}$ . For the bulk flow this makes no difference due to continuity, but when we include the free surface it becomes important. The transport equation then follows from the classical Leibnitz-Reynolds transport theorem [cf. [22] and Eqs. (3.1.3)–(3.1.5) in [10]]:

$$\begin{aligned} \frac{dE_m}{dt} &= \int_{\mathcal{V}} \partial_t e_m dV + \oint_{\partial\mathcal{V}} e_m \mathbf{U} \cdot \hat{\mathbf{n}} dA \\ &= \oint_{\partial\mathcal{V}} [e_m(\mathbf{U} - \mathbf{u}) + \mathbf{T} \cdot \mathbf{u} - p\mathbf{u}] \cdot \hat{\mathbf{n}} dA - \dot{\Phi}. \end{aligned} \quad (32)$$

This work-energy principle states that the temporal change of  $E_m$  is compensated by the flux of mechanical energy into  $\mathcal{V}$  and the external work done by the total stresses acting on  $\partial\mathcal{V}$ , due to the material velocity  $\mathbf{u}$ , and the associated internal work of the deviatoric (viscous) stresses, equal to the negative mechanical dissipation. The left side of (32) equals zero for a stationary flow.

For many applications, especially for stationary flows, it is valuable to consider a cylindrical control volume  $\mathcal{V}$  aligned with  $z$  direction and having the patch  $\mathcal{A}_s$  of the free surface as its top boundary, as indicated in Fig. 1. In the general, nonstationary flow situation its lateral sides facing the bulk of the flow and  $\mathcal{A}_s$  are both moving. In the important case of a static control volume, however, its projection  $B$  and thus  $C$  stay stationary, while its top face  $\mathcal{A}_s$  is moving with the normal component  $u_n$  of the material flow (when the flow is not stationary), following from comparison of (12) with (7). Thus,  $\mathbf{U} = \mathbf{0}$  on the lateral sides of  $\mathcal{V}$ ,  $\mathbf{U}_C = (U, V, 0) \equiv \mathbf{0}$ , but  $\mathcal{C}_s$  moves vertically with  $\mathbf{U} = h_t \hat{\mathbf{z}}$ , see (12), and  $\hat{\mathbf{n}} = \hat{\mathbf{n}}_s$ ,  $dA = dA_s$ ,

$$U_n = u_n, \quad (33)$$

on the interior of  $\mathcal{A}_s$ . In this case, (32) is written as

$$\begin{aligned} \frac{dE_m}{dt} &= \frac{d}{dt} \int_0^{h(x,y,t)} dz \int_B e_m dx dy \\ &= - \int_{\mathcal{A}_s} \Delta p_L u_n dA_s + \int_{\partial\mathcal{V}_r} [\mathbf{T} \cdot \mathbf{u} - (e_m + p - p_0)\mathbf{u}] \cdot \hat{\mathbf{n}} dA - \dot{\Phi}, \end{aligned} \quad (34)$$

where  $\partial\mathcal{V}_r$  denotes the remaining boundary, i.e., the boundary excluding the free surface on which we have used the boundary condition (8), and where we have subtracted the constant  $p_0$  from the pressure everywhere since  $\oint_{\partial\mathcal{V}} \mathbf{u} \cdot \hat{\mathbf{n}} dA = 0$  by continuity and incompressibility. To determine the total energy budget, we note that the Laplace pressure acts on both sides of the interface separating

201  $\mathcal{A}_s$  and  $\mathcal{V}$ . But the powers it exerts on  $\mathcal{A}_s$  [see (28)] and on  $\mathcal{V}$  [see (34)] differ in their signs and  
 202 therefore cancel. Since the purely vertical movement of  $\mathcal{C}_s$  yields

$$\oint_{\mathcal{C}_s} \mathbf{f}_\gamma \cdot \mathbf{U} d\mathcal{C}_s = \gamma \oint_{\mathcal{C}_s} h_t \hat{\mathbf{m}}_s \cdot \hat{\mathbf{z}} d\mathcal{C}_s \quad (35)$$

203 [which can also be found from comparing (18) with (16) using (17) for  $\mathbf{U}_C \equiv \mathbf{0}$ ], we obtain the total  
 204 energy budget in the form

$$\frac{d(E_m + E_s)}{dt} = \int_{\partial\mathcal{V}_r} (\mathbf{T} \cdot \mathbf{u} - (e_m + p - p_0)\mathbf{u}) \cdot \hat{\mathbf{n}} dA - \dot{\Phi} + \gamma \oint_{\mathcal{C}_s} h_t \hat{\mathbf{m}}_s \cdot \hat{\mathbf{z}} d\mathcal{C}_s. \quad (36)$$

205 For stationary flow, the time derivatives vanish in (36) as does  $u_n$ ; see (7). In that case, the total  
 206 energy budget (36), combining that of the bulk and the surface flow, becomes indistinguishable from  
 207 that just governing the bulk flow (34). Correspondingly, the only trace of capillarity then comes from  
 208 the pressure  $p$ , which must include the Laplace pressure to respect the boundary condition formed  
 209 by (8) and (9). This is the standard approach, and its validity was recently emphasized in [13]  
 210 and [14]. If this seems strange, one should keep in mind that surface area is not advected like mass  
 211 or kinetic energy. In the bulk of the flow, each fluid particle carries a particular mass and kinetic  
 212 energy with it, but it does *not* carry a particular surface area with it.

213 The additional energy term (1) for a static control volume in a steady flow was originally  
 214 postulated by [11] without explanation. In their follow-up paper [12] they argue that it actually  
 215 stems from the viscous advection terms in (32). Writing the global stress balance (10) in the form  
 216 [their Eq. (3.4) in our notation]

$$\int_{\mathcal{A}_s} (p - p_0) \hat{\mathbf{n}}_s dA_s - \mu \int_{\mathcal{A}_s} \hat{\mathbf{n}}_s \cdot \mathbf{S} dA_s + \gamma \oint_{\mathcal{C}_s} d\mathbf{l} \times \hat{\mathbf{n}}_s = 0, \quad (37)$$

217 they then take the inner product with  $\mathbf{u}$  *inside* both integrals to get

$$\int_{\mathcal{A}_s} (p - p_0) \mathbf{u} \cdot \hat{\mathbf{n}}_s dA_s - \mu \int_{\mathcal{A}_s} \mathbf{u} \cdot \mathbf{S} \cdot \hat{\mathbf{n}}_s dA_s + \gamma \oint_{\mathcal{C}_s} \mathbf{u} \cdot (d\mathbf{l} \times \hat{\mathbf{n}}_s) = 0. \quad (38)$$

218 Since  $\mathbf{u} \cdot \hat{\mathbf{n}}_s = 0$  (for a stationary flow), they furthermore conclude that

$$\mu \int_{\mathcal{A}_s} \mathbf{u} \cdot \mathbf{S} \cdot \hat{\mathbf{n}}_s dA_s = \gamma \oint_{\mathcal{C}_s} \mathbf{u} \cdot (d\mathbf{l} \times \hat{\mathbf{n}}_s), \quad (39)$$

219 relating viscous and capillary effects and reinstating the viscous term that was dropped from the  
 220 free surface part of the energy equation (36). However, multiplying by  $\mathbf{u}$  inside the integral, which  
 221 might seem a legal operation since the area of integration is arbitrary, is actually *illegal* since one  
 222 of the integrals is a loop integral. Rewriting the loop integral in (37) as a surface integral, using the  
 223 identity (11), we get [as in (10) with (11)]

$$\int_{\mathcal{A}_s} [(p - p_0) \hat{\mathbf{n}}_s - \mu \mathbf{S} \cdot \hat{\mathbf{n}}_s - \gamma (\nabla \cdot \hat{\mathbf{n}}_s) \hat{\mathbf{n}}_s] dA_s = 0, \quad (40)$$

224 and we can now take  $\mathcal{A}_s$  as any shape we want and conclude—since the first and last terms are  
 225 explicitly orthogonal to  $\mathbf{u}$ —that

$$\int_{\mathcal{A}_s} \mathbf{u} \cdot \mathbf{S} \cdot \hat{\mathbf{n}}_s dA_s = 0. \quad (41)$$

226 Indeed, the integrand  $\mathbf{u}_s \cdot \mathbf{S} \cdot \hat{\mathbf{n}}_s$  [which we removed from the free surface to get (36)] must be  
 227 zero everywhere on the interface, which is simply the shear part of the usual dynamic boundary  
 228 condition (8) on a free surface, and does *not* involve the surface tension.

229 The fact that creation of surface energy through the breakup of intermolecular bonds is a  
 230 *reversible*, nondissipative process, was emphasized by Batchelor in 1970, who warned that “changes

in the surface energy should not be counted as ‘dissipation’” ([23], p. 570). From a historical perspective, it is interesting that the error of attributing power to the surface tension forces acting on a static control volume was made already by Hagen in the context of Savart sheets, and referenced faithfully by Bouasse in his textbook *Jets, Tubes and Canals* from 1923 ([24], Sec. 107; see also [25], p. 23). Bouasse noted—perhaps with some scepticism (?)—that the theory predicted a minimum for the thickness profile of the sheet not near the edge, but halfway between the center and the edge.

As we mentioned above, it is perhaps strange that the surface tension forces  $\mathbf{f}_\gamma$  in (28) do no work when the control volume does not move, since these forces are still acting on a moving fluid. The reason for this is that the surface energy, being proportional to the surface area (13), does *not* depend on the fluid velocity  $\mathbf{u}$ . In other words, the surface energy has no kinetic energy term, which is due to the fact that the surface, in our idealized continuum approach, is infinitely thin and thus has no mass. But how can it then have an energy? The reason for this is found in the fact that the surface energy, coming from the breaking of bonds between the liquid molecules, is at least of the order of  $kT$  per particle or  $RT$  per mol, i.e., around 2 kJ/mol ( $k$ ,  $R$ , and  $T$  denote the Boltzmann, the universal gas constant, and the absolute temperature, respectively), whereas the kinetic energy of matter, say, water, moving at 1 m/s is about 0.009 J/mol. Hence, the ratio of these two energies of around  $2 \times 10^5$  allows us to neglect the kinetic energy of the organized motion represented by the velocity field  $\mathbf{u}$  compared to the surface energy for the same amount of matter—as opposed to the kinetic energy of the random thermal motion, which is also of the order of  $kT$  per particle.

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