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Why (some) abnormal problems are "normal"

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ABSTRACT

In abnormal optimal control problems it is necessary to basically ignore the objective for certain state values in order to be able to determine the optimal control. In the past, abnormal problems were considered to be degenerated problems that did not fit to any real application. In the present paper we discuss reasons for the occurrence of abnormality. We show that abnormality can be an integral part of a meaningful problem rather than to be a sign for degeneracy.

1. Introduction

In his often-quoted paper Halkin [1] presents a relatively simple infinite time horizon problem with free end-state that is abnormal. In particular, this means that the Lagrange multiplier corresponding to the objective function in the Hamiltonian, which is often denoted by λ_0, is equal to zero. In finite dimensional optimization it was John [2] who formulated Lagrange's rule in case that no conditions on the constraints are specified. Since then much effort has been undertaken to find so called constraint qualifications, cf. [3–7], that guarantee the existence of regular Lagrange-multipliers, i.e. $\lambda_0 = 1$. Results on this topic can also be found for the infinite dimensional optimization problems and abstract constraints, see e.g. [8–10]. But in its generality these conditions are hard to verify.

In optimal control theory normality can be guaranteed for finite time and free end-state problems. However, normality does not need to hold for infinite time horizon problems with free end-state. Halkin [1] shows that even relatively simple problems can be abnormal, if the time horizon is increased to infinity.

Nonetheless, abnormality in real applications seems to occur only in degenerate cases and it was thought that it relates to an ill posed model. The fact that in the abnormal case the objective function does not play any role in the optimization process seems convincing for the latter viewpoint. In this note we design and analyze an optimal control model about the optimal accumulation of reputation. We show that abnormality is an integral part and not a sign for degeneracy.

The paper is structured as follows. Section 2 discusses the occurrence of abnormality in Halkin's example. In Section 3 we consider a model dealing with the accumulation of reputation. Section 4 concludes.

2. Halkin's example

We start with Halkin's example and carry out the details leading to abnormality. We also shortly discuss a possible modification of this example, which reveals more clearly the cause for abnormality.

In Halkin [1] the following stylized model

$$\max_{u(\cdot)} \left\{ \int_0^\infty (u(t) - x(t))\mathrm{d}t \right\} \tag{1a}$$

$$\text{s.t.} \quad \dot{x}(t) = u(t)^2 + x(t) \tag{1b}$$

$$-1 \le u(t) \le 1, \quad \text{for all} \quad t \tag{1c}$$

$$x(0) = x_0, \tag{1d}$$

is used to show that optimal control problems over an infinite time horizon with free end state are not necessarily normal contrary to finite time horizon problems. Note that in Halkin [1] the problem is only considered for $x_0 = 0$.

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To derive the necessary optimality conditions we consider the Hamiltonian function

$$\mathcal{H}(x,u,\lambda,\lambda_0) := \lambda_0(u-x)+\lambda(u^2+x), \tag{2a}$$

and the Lagrangian

$$\mathcal{L}(x,u,\lambda,\nu_1,\nu_2,\lambda_0) := \mathcal{H}(x,u,\lambda,\lambda_0)+\nu_1(u+1)+\nu_2(1-u), \tag{2b}$$

together with the derivatives

$$\mathcal{H}_u(x,u,\lambda,\lambda_0) = \lambda_0 + 2\lambda u, \tag{2c}$$

$$\mathcal{H}_x(x,u,\lambda,\lambda_0) = -\lambda_0 + \lambda. \tag{2d}$$

For an optimal solution $(x^*(\cdot),u^*(\cdot))$ the maximizing condition

$$u^*(t) = \underset{-1\le u\le 1}{\operatorname{argmax}}\, \mathcal{H}(x^*(t),u,\lambda(t),\lambda_0) \tag{2e}$$

yields

$$u^*(t) = \begin{cases} -1 & \text{for } \lambda_0 - 2\lambda(t) \le 0 \\ -\dfrac{\lambda_0}{2\lambda(t)} & \text{for } -1 \le \dfrac{\lambda_0}{2\lambda(t)} \le 1 \\ 1 & \text{for } \lambda_0 + 2\lambda(t) \ge 0. \end{cases} \tag{2f}$$

For $x_0 = 0$ it is immediately clear that the optimal solution is $(x^*(\cdot),u^*(\cdot)) = (0,0)$. The reason is that for this solution the objective value is equal to zero, whereas for every other choice of the control u the objective value is strictly negative.

Now it is important to realize that, when taking Eq. (2f) into account, we find that $u=0$ can only be achieved for $\lambda_0 = 0$ or $\lambda(\cdot) = -\infty$. The latter choice is not an absolutely continuous function, as is required by the necessary optimality conditions. Then we are left with $\lambda_0 = 0$, implying that the problem is abnormal.

For $x_0 > 0$ the optimal solution is $(x^*(\cdot),u^*(\cdot)) = (e^t x_0, 0)$.[1] The same argument as for $x_0 = 0$ yields that the problem is abnormal.

It follows that for problem (1) the objective value depends on x_0, where it is discontinuous at $x_0 = 0$:

$$V^*(x_0) = \begin{cases} 0 & x_0 = 0 \\ -\infty & x_0 > 0. \end{cases} \tag{3}$$

This model is degenerate in the sense that the state diverges and the objective value immediately jumps from zero to minus infinity.

3. A model with self-enforcing reputation

Let $x(t)$ be the reputation of the decision maker at time t. The decision maker wants his or her reputation to be high and therefore the objective is to maximize the discounted stream of reputation values over time:

$$\max_{u(\cdot)} \left\{ \int_0^\infty e^{-rt} x(t)dt \right\}, \tag{4a}$$

in which r is the discount rate and the control variable $u(t)$ stands for networking efforts by which the decision maker can improve reputation over time.

The development of reputation over time is influenced by three effects. First, reputation is positively influenced by the networking efforts with the amount of $u(t)x(t)$, which are thus assumed to be more effective when reputation is already large. Second, there is a depreciation or forgetfulness effect due to which reputation decreases by the amount $ax(t)$, in which a is the constant depreciation rate or rate of forgetfulness. Third, reputation has a self-enforcing effect, $x(t)^2$ see, e.g. [11], due to which a sufficiently large reputation grows without any efforts by the decision maker (e.g. due to word-of-mouth propagation or the so-called Matthew effect), namely when x exceeds the depreciation rate a. Adding up the three effects results in the following state equation[2]:

$$\dot{x}(t) = x(t)(x(t) - a + u(t)). \tag{4b}$$

The decision maker's capacity related to networking efforts is bounded, for instance because there are only a restricted number of hours per day that can be spend on networking. In order to limit the number of possible scenarios we assume that networking capacity falls below the rate of depreciation, implying that for the control variable $u(t)$ it holds that

$$0 \le u(t) \le u_{\max} < a, \quad \text{for all} \quad t \tag{4c}$$

If reputation gets to very high levels, at some point a situation arises where reputation cannot increase further. Then the decision maker is known to "everybody being relevant". To account for this in the model we introduce a fixed upper bound A so that

$$0 \le x(t) \le A, \quad \text{for all} \quad t \tag{4d}$$

The optimal control model consists of the expressions (4a)-(4d). To guarantee that the maximum reputation level A can be reached, we introduce the additional assumption

$$0 \le a - A \le u_{\max}. \tag{4e}$$

Noting that the usage of control is costless[3] and that it is advantageous to stay at the highest possible state value, an optimal control is

$$u(t) = \begin{cases} u_{\max} & 0 \le x(t) < A \\ a - A & x(t) = A. \end{cases} \tag{5}$$

From this result the following proposition follows directly.

Proposition 1. *Problem (4) can be reformulated as a free end-time problem with the objective function*

$$V(u(\cdot),x_0) := \int_0^T e^{-rt} x(t)dt + \frac{e^{-rT}}{r} x(T)$$

$$V^*(x_0) := \max_{u(\cdot)} V(u(\cdot),x_0) \tag{6a}$$

satisfying the state dynamics (4b), control constraint (4c) and the end constraint

$$x(T) \le A. \tag{6b}$$

For initial values satisfying $a - u_{\max} < x_0 \le A$ the optimal solution $(x^*(\cdot),u^*(\cdot))$ is

$$x^*(t) = \frac{x_0(a-u_{\max})}{x_0 + e^{(a-u_{\max})t}((a-u_{\max})-x_0)}, \quad 0 \le t \le T \tag{7}$$

and

$$u^*(t) = u_{\max}, \quad 0 \le t \le T$$

with

$$T = \frac{1}{(a-u_{\max})} \ln\left(\frac{x_0(u_{\max}-a+A)}{A(u_{\max}-a+x_0)}\right).$$

[1] The technical details can be requested from the authors.

[2] In case the decision maker is a scientist, the state variable x represents the goodwill that a scientist receives within his or her peer community. If we link our model to a marketing application, we can argue that Eq. (4b) extends the state equation of the classic Nerlove–Arrow model for goodwill accumulation, see [12], by a term accounting for the self-enforcing effect.

[3] On the one hand networking, for instance by visiting a conference, is costly, but on the other hand it is also rewarding meeting old friends and so on. So implicitly we assume that costs and rewards cancel out in our model.

For initial values satisfying $0 \le x_0 \le a - u_{\max}$ *the optimal solution is*

$$x^*(t) = \frac{x_0(a-u_{\max})}{x_0 + \mathrm{e}^{(a-u_{\max})t}((a-u_{\max}) - x_0)}, \quad 0 \le t < \infty$$

and

$$u^*(t) = u_{\max}, \quad 0 \le t < \infty.$$

Note that for the free end-time problem (6) two cases have to be distinguished, namely $T < \infty$ and $T = \infty$, yielding different transversality conditions, see also [13] and [14].

Proof. We already argued that the optimal control is $u_{\max}$ as long as $x(t) < A$ is satisfied. Solving state equation (4b) with this control value we find

$$x(t) = \frac{x_0(a-u_{\max})}{x_0 + \mathrm{e}^{(a-u_{\max})t}((a-u_{\max}) - x_0)}, \quad t \ge 0. \tag{8}$$

For $0 \le x_0 < a - u_{\max}$ the dynamics (4b) is strictly negative. Thus, $x(t)$ in Eq. (8) converges to zero and exists for the every $t \ge 0$. For $x_0 = a - u_{\max}$ Eq. (8) shows that the solution stays put at $x(t) = a - u_{\max}$.

For $a - u_{\max} < x(0) \le A$, there exists a time $T \ge 0$ with

$$T = \frac{1}{(a-u_{\max})} \ln\left(\frac{x_0(u_{\max} - a + A)}{A(u_{\max} - a + A)}\right)$$

and $x(T) = A$.[4] Plugging this solution into the objective function (4a) yields Eq. (6a) under the condition (6b). □

3.1. Necessary optimality conditions

We start out presenting the current value Hamiltonian

$$\mathcal{H}(x, u, \lambda, \lambda_0) := \lambda_0 x + \lambda x(x - a + u),$$

and the Lagrangian

$$\mathcal{L}(x, u, \lambda, \nu_1, \nu_2, \lambda_0) := \mathcal{H}(x, u, \lambda, \lambda_0) + \nu_1 u + \nu_2(u_{\max} - u) \tag{9a}$$

together with the derivatives

$$\mathcal{H}_u(x, u, \lambda, \lambda_0) = x\lambda,$$
$$\mathcal{H}_x(x, u, \lambda, \lambda_0) = \lambda_0 + \lambda(2x - a + u).$$

Then the Maximum Principle, see e.g. Seierstad and Sydsæter [15], yields

$$u^*(t) = \begin{cases} 0 & \text{for } \nu_1 = -\mathcal{H}_u > 0 \\ [0, u_{\max}] & \text{for } \mathcal{H}_u = 0 \\ u_{\max} & \text{for } \nu_2 = \mathcal{H}_u > 0 \end{cases} \tag{9b}$$

$$\dot\lambda(t) = \lambda(t)(r - 2x(t) - u(t) + a) - \lambda_0. \tag{9c}$$

Using Proposition 1 the necessary optimality conditions for $A \ge x(0) > \tilde{x} := a - u_{\max}$ are those for a finite time horizon problem with fixed end state $x(T) = A$. Therefore the costate at time T is free. Since the optimal control is $u_{\max}$ the costate at T has to satisfy condition (9b) yielding

$$\lambda(T) \ge 0. \tag{9d}$$

Specifically we can chose

$$\lambda(T) = 0. \tag{9e}$$

For the optimal solution over an infinite time horizon $T = \infty$ the following limiting transversality condition has to hold

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\lambda(t) = 0. \tag{9f}$$

A more general formulation of the limiting transversality condition can be found in Aseev and Veliov [16].

Next we analyze the geometric properties of the *Stalling Equilibrium* $\tilde{x} > 0$, which is a steady state at which effort is at its maximum (see Feichtinger et al. [11]).

Due to the assumption $u_{\max} < a$, it follows from equation (4b) that the Stalling Equilibrium with $\tilde{x} = a - u_{\max} > 0$ always exists.

3.2. Stalling equilibrium

In this section we consider problem (4) for $x(0) = \tilde{x}$ and the according equilibrium solution. We see that the adjoint equation (9c) exhibits an equilibrium $\tilde\lambda$ at $\tilde{x}$. The properties of the equilibrium $(\tilde{x}, \tilde\lambda)$ in the state–costate space are:

$$\tilde{x} := a - u_{\max} \tag{10a}$$

$$\tilde\lambda := \frac{\lambda_0}{r - \tilde{x}}. \tag{10b}$$

At the Stalling Equilibrium the maximizing condition (9b) yields for $\lambda_0 > 0$:

$$\mathcal{H}_u(\tilde{x}, u_{\max}, \tilde\lambda) = \frac{\lambda_0 \tilde{x}}{r - \tilde{x}} \begin{cases} < 0 & \text{for } r < \tilde{x} \\ \text{undefined} & \text{for } r = \tilde{x} \\ > 0 & \text{for } r > \tilde{x}. \end{cases} \tag{10c}$$

Thus, for $r \ne \tilde{x}$ the equilibrium $(\tilde{x}, \tilde\lambda)$ exists and the according Jacobian $\tilde{J}$ is given as

$$\tilde{J} = \begin{pmatrix} \tilde{x} & 0 \\ -2\tilde\lambda & r - \tilde{x} \end{pmatrix}. \tag{10d}$$

This matrix exhibits the eigenvalues

$$\xi_1 = \tilde{x} > 0, \quad \xi_2 = r - \tilde{x} \lesseqqgtr 0 \tag{10e}$$

and eigenvectors

$$\nu_1 = \begin{pmatrix} (r - \tilde{x})(r - 2\tilde{x}) \\ 2 \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10f}$$

The eigenvalue $\xi_1 > 0$ is always strictly positive and hence corresponds to an unstable direction. The sign of the eigenvalue ξ_2 depends on the relation between the discount rate r and the size of the Stalling Equilibrium $\tilde{x}$.

3.3. Solution structure

From (10c) we obtain that the relationship between the Stalling Equilibrium $\tilde{x}$ and the discount rate r is crucial. Therefore, we have to distinguish between the cases where r is larger or smaller than $\tilde{x}$. The following proposition states necessary and sufficient conditions for abnormality of problem (6).

Proposition 2. *Problem (6) is abnormal iff* $x(0) = \tilde{x}$ *and* $r - \tilde{x} \le 0$.

Proof. We already carried out, why for every $0 \le x(0) \le A$ an optimal control is $u^*(\cdot) \equiv u_{\max}$. Therefore $x(\cdot)$ satisfies a logistic equation, converging to zero for $x(0) < \tilde{x}$, or (for infinite T) diverging to infinity for $x(0) > \tilde{x}$ and staying at $\tilde{x}$ for $x(0) = \tilde{x}$. The costate path corresponding to $x(\cdot)$ is given as

$$\lambda(t) = \mathrm{e}^{\int_0^t (r+\tilde{x}-2x(s))\mathrm{d}s}\left(\lambda(0) - \lambda_0 \int_0^t \mathrm{e}^{\int_0^s (r+\tilde{x}-2x(z))\mathrm{d}z}\,\mathrm{d}s\right). \tag{11}$$

Moreover we note that an admissible equilibrium $\left(0, \frac{\lambda_0}{r+\tilde{x}}\right)$ exists, with the Jacobian

$$\hat{J} = \begin{pmatrix} -\tilde{x} & 0 \\ -2\frac{\lambda_0}{r+\tilde{x}} & r + \tilde{x} \end{pmatrix}.$$

[4] For the numerical example we set $a = A = 2$.

$\hat{J}$ has a negative $\xi_1 = -\tilde{x}$ and positive $\xi_2 = r + \tilde{x}$ eigenvalue and hence exhibits a stable saddle path $(x_s(\cdot), \lambda_s(\cdot))$, which exists on the interval $[0, \tilde{x})$. The stable path is the only solution that satisfies the limiting transversality condition (9f). Due to the representation of the costate path (11) $(x_s(\cdot), \lambda_s(\cdot))$ with $x_s(0) < \tilde{x}$ satisfies

$$\lim_{t\to\infty} \lambda_s(t) = \frac{\lambda_0}{r+\tilde{x}} > 0$$

and

$$\lambda_s(0) = \lim_{t\to\infty} \lambda_0 \int_0^t e^{\int_0^s (r+\tilde{x}-2x_s(z))\mathrm{d}z}\, \mathrm{d}s,$$

where the term on the RHS is strictly increasing, implying

$$\lambda_s(0) > \lambda_0 \int_0^t e^{\int_0^s (r+\tilde{x}-2x_s(z))\mathrm{d}z}\, \mathrm{d}s, \quad \text{for all} \quad t \geq 0.$$

Therefore the optimality conditions

$$\nu_2(t) = H_u(x_s(t), u_{\max}, \lambda_s(t), \lambda_0) = x_s(t)\lambda_s(t) > 0$$

are satisfied for all $t \geq 0$, yielding that for every $0 \leq x(0) < \tilde{x}$ the optimal solution is normal.

For $\tilde{x} < x(0) \leq A$ the constraint value A is reached in finite time. Again using the representation (11) we can choose $\lambda(0)$ such that the adjoint equation (9c) together with the transversality condition (9e) for $\lambda_0 = 1$ is satisfied and hence the problem is normal.

Finally we have to analyze the case $x(0) = \tilde{x}$ and therefore we consider the cases $r - \tilde{x} \lessgtr 0$ and $r - \tilde{x} = 0$.

Case $r - \tilde{x} < 0$. The equilibrium $(\tilde{x}, \tilde{\lambda})$ is not admissible for $\lambda_0 > 0$. This is because $\tilde{\lambda} < 0$ and hence the maximizing condition (9b) for $u_{\max}$ is violated since Eq. (10c) yields for $\lambda_0 = 1$

$$\nu_2 = H_u(\tilde{x}, u_{\max}, \tilde{\lambda}, \lambda_0) < 0. \tag{12}$$

In Section 3.2 we showed that for $r - \tilde{x} < 0$ the equilibrium is a saddle point (see Eq. (10e)) and the vertical line is the stable manifold (see Eq. (10f)). This situation is depicted in Fig. 1(a). The maximized objective value for different initial state values is shown in Fig. 2(a).

To show that the problem is abnormal we have to prove that no costate path $\lambda(\cdot)$ exists that satisfies the adjoint equation (9c) and the maximizing condition (9b) for $u = u_{\max}$. To the contrary let us assume that $\lambda(\cdot)$ exists such that

$$\nu_2(t) = H_u(\tilde{x}, u_{\max}, \lambda(t), \lambda_0) = \lambda(t)\tilde{x} \geq 0, \quad \text{for all} \quad t \geq 0. \tag{13a}$$

Representation (11) reduces to

$$\lambda(t) = e^{(r-\tilde{x})t}\left(\lambda(0) + \frac{\lambda_0}{r-\tilde{x}}\left(e^{(\tilde{x}-r)t} - 1\right)\right)$$

thus, time t, such that $\lambda(t) = 0$ for some $\lambda(0) > 0$ yields

$$t = \frac{1}{\tilde{x}-r}\ln\left(1 + \frac{\lambda(0)(\tilde{x}-r)}{\lambda_0}\right). \tag{13b}$$

To satisfy the non-negativity condition (13a) expression (13b) has to be infinite. Due to $r - \tilde{x} < 0$ this can only be satisfied if $\lambda_0 = 0$ or $\lambda(0) = \infty$. Analogous to Halkin's example this yields that the problem is abnormal.

Case $r - \tilde{x} = 0$. In that case the costate dynamics (9c) reduces to

$$\dot{\lambda}(t) = -\lambda_0 \quad \text{yielding the solution} \quad \lambda(t) = \lambda(0) - \lambda_0 t$$

$$\text{and therefore} \quad t_s = \frac{\lambda(0)}{\lambda_0},$$

where t_s is the time, when $\lambda(t_s)$ becomes zero and hence is not admissible for $t > t_s$ in the sense that inequality (13a) is violated. Thus for any $\lambda(0) > 0$ and $\lambda_0 = 1$ the solution is not admissible for a large enough t. Therefore the necessary optimality conditions are only satisfied if $\lambda_0 = 0$ or $\lambda(0) = \infty$. This implies that the problem is abnormal.

Case $r - \tilde{x} > 0$. In that case the equilibrium $(\tilde{x}, \tilde{\lambda})$ is admissible and is an unstable node, see Fig. 1(b). Thus, it is a threshold point separating the solutions converging to the origin or moving to A and staying there and the necessary optimality conditions hold for $\lambda_0 = 1$, yielding the normal case, which finishes the proof. □

Note that results in Basco et al. [17] would suggest that the necessary conditions are normal for each $x(0)$, thus also when $x(0) = \tilde{x}$. However, the analysis of Basco et al. [17] is based on Lipschitz continuity.

In the next proposition we show that for sufficiently small values of r and for $x_0 = \tilde{x}$ the value function $V^*(x_0)$ of problem (6) is not Lipschitz continuous at $\tilde{x}$. This result is a consequence of the abnormality at $x_0 = \tilde{x}$ and is illustrated in Fig. 2.

Proposition 3. *For $0 \leq r \leq \tilde{x}$ the value function $V^*(x_0)$ is not Lipschitz continuous at $x_0 = \tilde{x}$.*

Proof. To show that the value function $V^*(x_0)$ is not Lipschitz continuous at $\tilde{x}$ we calculate the left side derivative of Eq. (7). This yields for $0 \leq x_0 \leq \tilde{x}$

$$x(t) = \frac{x_0\tilde{x}}{x_0 + e^{\tilde{x}t}(\tilde{x} - x_0)}$$

and hence

$$V^*(x_0) = \int_0^\infty \frac{e^{-rt}\, x_0\tilde{x}}{x_0 + e^{\tilde{x}t}(\tilde{x} - x_0)}\mathrm{d}t.$$

This value is finite for every (admissible) x_0 satisfying

$$\lim_{x_0\to\tilde{x}} V^*(x_0) = V^*(\tilde{x}) = \frac{1}{r}\tilde{x}.$$

For $x_0 < \tilde{x}$ the derivative of $V^*(x_0)$ with respect to x_0 exists and is given by

$$\frac{\partial}{\partial x_0}V^*(x_0) = \int_0^\infty \frac{e^{(\tilde{x}-r)t}\, \tilde{x}^2}{x_0 + e^{\tilde{x}t}(\tilde{x} - x_0)^2}\mathrm{d}t.$$

For $x_0 = \tilde{x}$ and $r \leq \tilde{x}$ we find

$$\lim_{x_0\to\tilde{x}} \frac{\partial}{\partial x_0}V^*(x_0) = \int_0^\infty e^{(\tilde{x}-r)t}\, \tilde{x}\mathrm{d}t = \infty. \quad \square$$

The next proposition shows that for $r > \tilde{x}$ the first variation of the objective function $V(\tilde{x}, u(\cdot))$ at the Stalling Equilibrium $\tilde{x}$ exists, and the first order necessary optimality conditions can be applied. For $r \leq \tilde{x}$ the first variation does not exist and hence the first order necessary optimality conditions cannot be applied in its normal form.

Proposition 4. *The first variation $\delta V(\tilde{x}, \delta u(\cdot))$ with*

$$\delta V(\tilde{x}, \delta u(\cdot)) := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}V(\tilde{x}, u_{\max} + \varepsilon\delta u(\cdot))\right|_{0^+}, \quad \delta u(\cdot) \in \mathcal{U}$$

with

$$\mathcal{U} := \{v(\cdot)\colon [0,\infty) \to \mathbb{R}_0^-,\ \textit{measurable and}\ \operatorname{ess\,inf} v(\cdot) > -\infty\}, \tag{14}$$

satisfies for $\delta u(\cdot)$ being essentially different from zero

$$\delta V(\tilde{x}, \delta u(\cdot)) \in \begin{cases} \mathbb{R}^- & r > \tilde{x} \\ \{-\infty\} & r \leq \tilde{x}. \end{cases} \tag{15}$$

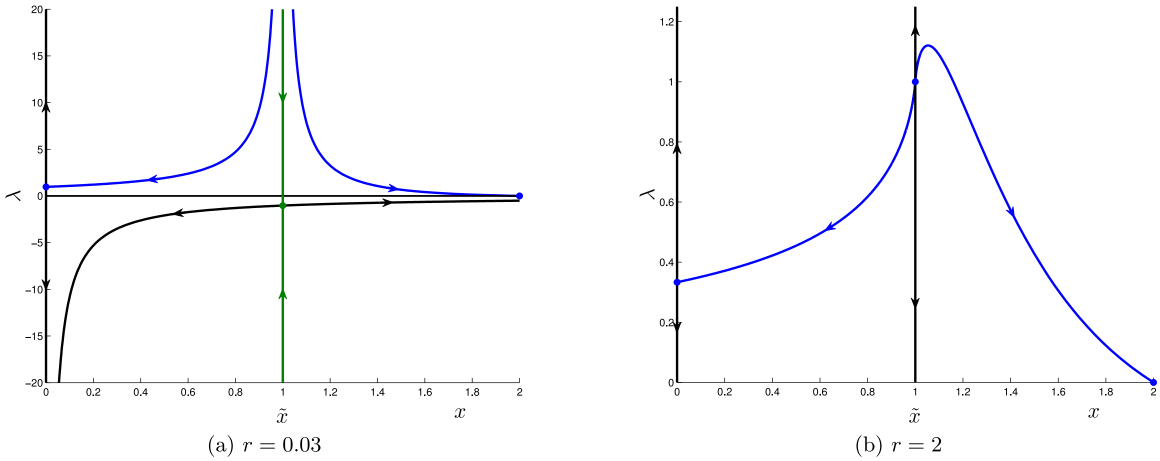


Fig. 1. Phase portrait of the solution paths (blue) in the state–costate space for $r = 0.03$ in panel 1(a) and $r = 2$ in panel 1(b) with $A = a = 2$. The black curves denote the unstable paths of the equilibria and green shows the stable path, that is non-admissible for $\lambda_0 > 0$, the normal case. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. Let $x(\cdot, \tilde{x}, \varepsilon)$ be the solution of the state equation (4b) with $u_{\max} + \varepsilon\delta u(\cdot)$, starting at $\tilde{x}$, i.e.

$$\dot{x}(t, \tilde{x}, \varepsilon) = x(t, \tilde{x}, \varepsilon)(x(t, \tilde{x}, \varepsilon) - \tilde{x} + \varepsilon\delta u), \quad x(0, \tilde{x}, \varepsilon) = \tilde{x}. \tag{16}$$

Due to the continuous differentiability of the dynamics (16), $x(\cdot, \tilde{x}, \varepsilon)$ is differentiable with respect to ε and satisfies the ODE

$$\dot{x}_\varepsilon(t, \tilde{x}, \varepsilon) = (2x(t, \tilde{x}, \varepsilon) - \tilde{x} + \varepsilon\delta u)x_\varepsilon(t, \tilde{x}, \varepsilon) + \delta u.$$

Evaluated at $\varepsilon = 0$ this yields the ODE

$$\dot{x}_\varepsilon(t, \tilde{x}, 0) = \tilde{x}x_\varepsilon(t, \tilde{x}, 0) + \delta u. \tag{17}$$

Solving ODE (17) we find for $x_\varepsilon(0, \tilde{x}, 0) = 0$

$$x_\varepsilon(t, \tilde{x}, 0) = \mathrm{e}^{\tilde{x}t} \int_0^t \mathrm{e}^{-\tilde{x}s}\, \delta u(s)\mathrm{d}s.$$

Since the integral (4a) is finite for every T and the integrand is continuously differentiable, integration and differentiation can be interchanged yielding

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^T \mathrm{e}^{-rt}\, x(t, \tilde{x}, \varepsilon)\mathrm{d}t = \int_0^T \mathrm{e}^{-rt}\, x_\varepsilon(t, \tilde{x}, \varepsilon)\mathrm{d}t.$$

Evaluated at $\varepsilon = 0$ and for $\tilde{x} - r > 0$ this yields

$$\int_0^T \mathrm{e}^{-rt}\, x_\varepsilon(t, \tilde{x}, 0)\mathrm{d}t = \int_0^T \mathrm{e}^{(\tilde{x}-r)t} \int_0^t \mathrm{e}^{-\tilde{x}s}\, \delta u(s)\mathrm{d}s.$$

Partially integrating the last integral we get

$$\int_0^T \mathrm{e}^{(\tilde{x}-r)t} \int_0^t \mathrm{e}^{-\tilde{x}s}\, \delta u(s)\mathrm{d}s = \frac{\mathrm{e}^{(\tilde{x}-r)T}}{\tilde{x}-r} \int_0^T \mathrm{e}^{-\tilde{x}t}\, \delta u(t)\mathrm{d}t - \frac{1}{\tilde{x}-r} \int_0^T \mathrm{e}^{-rt}\, \delta u(t)\mathrm{d}t.$$

Since according to Eq. (14) $\delta u(\cdot)$ is essentially bounded, the integrals converge and the limiting behavior depends on the term in front of the first integral, i.e.

$$\lim_{T\to\infty} \frac{\mathrm{e}^{(\tilde{x}-r)T}}{\tilde{x}-r} = \begin{cases} \infty & \tilde{x} - r > 0 \\ 0 & \tilde{x} - r < 0. \end{cases} \tag{18}$$

For $\tilde{x} - r = 0$ we argue analogously

$$\int_0^T \mathrm{e}^{-rt}\, x_\varepsilon(t, \tilde{x}, 0)\mathrm{d}t = \int_0^T \int_0^t \mathrm{e}^{-rs}\, \delta u(s)\mathrm{d}s = T\int_0^T \mathrm{e}^{-rt}\, \delta u(t)\mathrm{d}t - \int_0^T t\, \mathrm{e}^{-rt}\, \delta u(t)\mathrm{d}t. \tag{19}$$

The integrals converge since $\delta u(\cdot)$ is essentially bounded.

Summing up we find for $\delta u(\cdot)$ being essentially different from zero

$$\lim_{T\to\infty} \int_0^T \mathrm{e}^{-rt}\, x_\varepsilon(t, \tilde{x}, 0)\mathrm{d}t \in \begin{cases} \mathbb{R}^- & r > \tilde{x} \\ \{-\infty\} & r \le \tilde{x} \end{cases}$$

which completes the proof. □

The first remark stresses the importance of the infinite time horizon for the appearance of an abnormal solution.

Remark 3.1. The infinite time horizon is crucial. Otherwise, for some fixed finite time T, we could chose $\lambda(0)$ large enough, such that $\lambda(T) \ge 0$ satisfies the transversality condition and hence yields $H_u(t) \ge 0$ for $t \in [0, T]$.

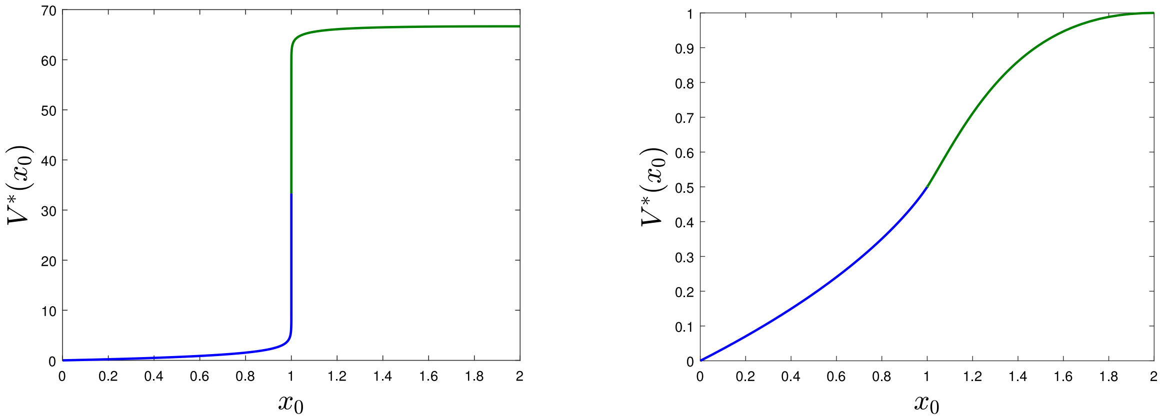
In the second remark we give a more intuitive explanation for abnormality in problem (4) and its difference to the normal case.

Remark 3.2. Fig. 2(a) shows that a solution ending at A gives a significantly higher value than a solution ending up at zero. Due to the control constraint $u \le u_{\max}$, however, reaching A is only possible for $x(0) > \tilde{x}$. Exactly at $\tilde{x}$, setting $u \le u_{\max}$ is just sufficient to keep x equal to $\tilde{x}$ forever. This implies that an infinitesimal increase of x at $\tilde{x}$ would make a solution of reaching A possible, which would result in a significant value increase. This value increase translates into an infinite value of the costate, as is confirmed by the blue trajectories in Fig. 1(a). The maximum principle does not allow infinite costate values, which is the reason that the abnormal problem applies for $x(0) = \tilde{x}$.

An important difference with Fig. 2(a) is that in Fig. 2(b) the value function is smooth, especially also at the Stalling Equilibrium $\tilde{x}$. Still it is the case that solutions ending at A have a higher value but differences with the alternative solutions, like staying at $\tilde{x}$ or converging to zero are not that large. The reason is that future proceeds are to a large extent discounted away. Note that what distinguishes the scenario of Fig. 2(b) from Fig. 2(a) is the large discount rate. The smoothness of the value function implies that the costate value is finite at $\tilde{x}$, so that considering the abnormal solution is not needed here.

4. Conclusion

In the present paper we saw that the presence of constraints can lead to the occurrence of abnormal behavior. The essential feature of these problems is that at a certain point in the state



(a) $r = 0.03$ (b) $r = 2$

Fig. 2. The graphs of this figure show the maximized objective value for $r = 0.03$ and $r = 2$ in dependence of x_0 with $A = a = 2$. The solutions of the different cases, ending at zero and ending at a are represented by the different colors blue and green. This graphs are connected by the solution staying at $\tilde{x} = 1$, the value of the Stalling Equilibrium. The corresponding derivatives of the objective function with respect to the state yielding the shadow price denoted by the costate λ are depicted in Fig. 1. For $r = 0.03$ the optimal objective value is not differentiable in the Stalling Equilibrium $\tilde{x}$, and since the derivative diverges, it is not Lipschitz continuous in this point (see Proposition 3). For $r = 2$ the optimal objective value is continuously differentiable for every initial x_0 in the state space $[0, A]$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

space the decision maker would like to steer the system into a favorable direction, but is not able to as the control does not have the desired impact on the state dynamics. This property is something that can occur in economically meaningful problems e.g. in environmental or health economics, marketing, capital accumulation etc. Therefore, the possibility of abnormality must not be neglected in models exhibiting this feature as it is a central part of the underlying problem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] H. Halkin, Necessary conditions for optimal control problems with infinite horizons, Econometrica 42 (2) (1974) 267–272, http://dx.doi.org/10.2307/1911976.

[2] F. John, Extremum problems with inequalities as subsidiary conditions, in: Studies and Essays: Presented To R. Courant on His 60th Birthday, Interscience Publishers, 1948, pp. 187–204.

[3] H.W. Kuhn, A.W. Tucker, Nonlinear programming, in: J. Neyman (Ed.), Proceedings of the Second Berkeley Symposium, 1950, pp. 481–492.

[4] O.L. Mangasarian, S. Fromovitz, The fritz john necessary conditions in the presence of equality and inequality constraints, J. Math. Anal. Appl. 17 (1967) 37–47.

[5] J.P. Evans, On constraint qualifications in nonlinear programming, Nav. Res. Logist. Q. 17 (3) (1970) 281–286, http://dx.doi.org/10.1002/nav.3800170304.

[6] M.S. Bazaraa, J.J. Goode, C.M. Shetty, Constraint qualifications revisited, Manage. Sci. 18 (9) (1972) 567–573, http://dx.doi.org/10.1287/mnsc.18.9.567.

[7] D.W. Peterson, A review of constraint qualifications in finite-dimensional spaces, SIAM Rev. 15 (3) (1973) 639–654.

[8] F.J. Gould, J.W. Tolle, Optimality conditions and constraint qualifications in Banach space, J. Optim. Theory Appl. 15 (6) (1975) 667–684, http://dx.doi.org/10.1007/BF00935506.

[9] A.V. Arutyunov, Optimality Conditions: Abnormal and Degenerate Problems, Kluwer Academic Publishers, Dordrecht, 2000, http://dx.doi.org/10.1007/978-94-015-9438-7.

[10] A.V. Arutyunov, A.F. Izmailov, Abnormal equality-constrained optimization problems: sensitivity theory, Math. Program. 100 (3) (2004) 485–515, http://dx.doi.org/10.1007/s10107-003-0494-3.

[11] G. Feichtinger, D. Grass, P.M. Kort, A. Seidl, On the Matthew effect in research careers, J. Econom. Dynam. Control 123 (2021) 104058, http://dx.doi.org/10.1016/j.jedc.2020.104058.

[12] M. Nerlove, K.J. Arrow, Optimal advertising policy under dynamic conditions, Economica 29 (114) (1962) 129–142, http://dx.doi.org/10.2307/2551549.

[13] J.P. Caulkins, G. Feichtinger, D. Grass, R.F. Hartl, P.M. Kort, A. Seidl, Skiba points in free end-time problems, J. Econom. Dynam. Control 51 (2015) 404–419, http://dx.doi.org/10.1016/j.jedc.2014.11.003.

[14] D. Grass, J.P. Caulkins, G. Feichtinger, G. Tragler, D.A. Behrens, Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror, Springer, Berlin, 2008.

[15] A. Seierstad, K. Sydsæter, Optimal Control Theory with Economic Applications, North-Holland, Amsterdam, 1987.

[16] S.M. Aseev, V.M. Veliov, Maximum principle for infinite-horizon optimal control problems under weak regularity assumptions, Proc. Steklov Inst. Math. 291 (1) (2015) 22–39, http://dx.doi.org/10.1134/S0081543815090023.

[17] V. Basco, P. Cannarsa, H. Frankowska, Necessary conditions for infinite horizon optimal control problems with state constraints, Math. Control Relat. Fields 8 (3) (2018) 535–555, http://dx.doi.org/10.3934/mcrf.2018022.