The One-More Discrete Logarithm Assumption in the Generic Group Model

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Abstract. The one more-discrete logarithm assumption (OMDL) underlies the security analysis of identification protocols, blind signature and multi-signature schemes, such as blind Schnorr signatures and the recent MuSig2 multi-signatures. As these schemes produce standard Schnorr signatures, they are compatible with existing systems, e.g. in the context of blockchains. OMDL is moreover assumed for many results on the impossibility of certain security reductions.

Despite its wide use, surprisingly, OMDL is lacking any rigorous analysis; there is not even a proof that it holds in the generic group model (GGM). (We show that a claimed proof is flawed.) In this work we give a formal proof of OMDL in the GGM. We also prove a related assumption, the one-more computational Diffie-Hellman assumption, in the GGM. Our proofs deviate from prior proofs in the GGM and replace the use of the Schwartz-Zippel Lemma by a new argument.

Keywords: One-more discrete logarithm, generic group model, blind signatures, multi-signatures

1 Introduction

Provable security is the prevailing paradigm in present-day cryptography. To analyze the security of a cryptographic scheme, one first formally defines what it means to break it and then gives a rigorous proof that this is infeasible assuming that certain computational problems are hard.

Classical hardness assumption like RSA and the discrete logarithm assumption in various groups have received much scrutiny over the years, but there are now myriads of less studied assumptions. This has attracted criticism [KM07, KM10], as the value of a security proof is unclear when it is by reduction from an (often newly introduced) assumption that is not well understood. A sanity check that is considered a minimum requirement for assumptions in cyclic groups is a proof in the generic group model (GGM), which guarantees that there are no efficient solvers that work for any group.

In this work we give the first proof that the *one-more discrete logarithm assumption*, a widely used hardness assumption, holds in the GGM. While prior proofs in the GGM have followed a common blueprint, the nature of OMDL differs from that of other assumptions and its proof requires a different approach, which we propose in this paper. We then extend our proof so that it also covers the *one-more Diffie-Hellman assumption*.

GGM. The *generic group model* [Nec94, Sho97] is an idealized model for the security analysis of hardness assumptions (as well as cryptographic schemes themselves) that are defined over cyclic groups. It models a "generic group" by not giving the adversary any group elements, but instead abstract "handles" (or *encodings*) for them. To compute the group operation, the adversary has access to an oracle which given handles for group elements X and Y returns the handle of the group element $X + Y$ (we denote groups additively).

OMDL. The one-more discrete logarithm problem, introduced by Bellare et al. [BNPS03], is an extension of the discrete logarithm (DL) problem. Instead of being given one group element X of which the adversary must compute the discrete logarithm w.r.t. some basis G, for OMDL the adversary can ask for as many challenges X_i as it likes. Moreover, it has access to an oracle that returns the discrete logarithm of any group element submitted by the adversary. The adversary's goal is to compute the DL of all challenges X_i, of which there must be (at least) *one more* than the number of calls made to the DL oracle.

Applications of OMDL

Blind signatures. Blind signature schemes [Cha82] let a user obtain a signature from a signer without the latter learning the message it signed. Their security is formalized by *one-more unforgeability*, which requires that after q signing interactions with the signer, the user should not be able to compute signatures on more than q messages.

The signatures in the *blind Schnorr signature scheme* [CP93] are standard Schnorr signatures [Sch91], which, in the form of EdDSA [BDL+12] are increasingly used in practice and considered for standardization by NIST [NIS19]. They are now used in OpenSSL, OpenSSH, GnuPG and considered to be supported by Bitcoin [WNR20], which will enable drastic scalability improvements due to signature aggregation [BDN18, MPSW19] (see below). Blind Schnorr signatures will moreover enable new privacy-preserving applications such as *blind coin swaps* and *trustless tumbler services* [Nic19].

One-more unforgeability of blind Schnorr signatures was proven by Schnorr and Jakobsson [SJ99, Sch01] directly in the GGM, also assuming the random-oracle model (ROM) and that the so-called *ROS problem* is hard. While unforgeability of blind Schnorr signatures cannot, even in the ROM, be proved from standard assumptions [FS10, Pas11, BL13], Fuchsbauer et al. [FPS20] give a proof in the *algebraic group model* (AGM) [FKL18], a model between the standard model and the GGM.

In the AGM, adversaries are assumed to be *algebraic*, meaning that for every group element Z they output, they must know a "representation" $\vec{z} = (z_1, \ldots, z_n)$ such that $Z = \sum_{i=1}^{n} z_i X_i$, where $X_1, \ldots, X_n$ are the group elements received so far. The authors prove unforgeability of blind Schnorr in the AGM+ROM assuming ROS and OMDL [FPS20].

While there has been evidence [Wag02] that the ROS problem was easier than initially assumed, Benhamouda et al. [BLOR20] recently presented a polynomial-time solver for ROS. This leads to forgeries of blind Schnorr signatures when the attacker is allowed to run *concurrent* executions of the signing protocol. To overcome these issues, Fuchsbauer et al. [FPS20] define a new signing protocol and introduce a modified ROS assumption, against which there are no known attacks. Their *Clause blind Schnorr* signature scheme is proven unforgeable in the AGM+ROM assuming hardness of their modified ROS problem and OMDL.

Multi-signatures. *Multi-signature schemes* [IN83] allow a group of signers, each having individual verification and signing keys, to sign a message on behalf of all of them via a single signature. In recent work, Nick et al. [NRS20] present a (concurrently secure) two-round multi-signature scheme called *MuSig2* (a variant of the *MuSig* scheme [MPSW19]), which they prove secure under the OMDL assumption. The resulting signatures are ordinary Schnorr signatures (under an *aggregated* verification key, which is of the same form as a key for Schnorr); they are thus fully compatible with blockchain systems already using Schnorr and will help ease scalability issues, as a single aggregate signature can replace a set of individual signatures to be stored on the blockchain.

Earlier, Bellare and Neven [BN06] instantiated another signature primitive called *transitive signatures* [MR02] assuming OMDL.

Identification schemes. Bellare and Palacio [BP02] assume OMDL to prove that the Schnorr identification protocol is secure against active and concurrent attacks, and Gennaro et al. [GLSY04] use it for a batched version of the scheme. Bellare and Shoup [BS07] prove that the Schnorr identification scheme verifies *special soundness* under concurrent attack from OMDL. Bellare et al. [BNN04] assume OMDL to prove their ID-based identification protocol secure against impersonation under concurrent attacks.

Negative results. OMDL has also been assumed in numerous proofs of impossibility results. Paillier and Vergnaud [PV05] prove that unforgeability of Schnorr signatures cannot be proven under the discrete logarithm assumption. Specifically, they show that there is no *algebraic* reduction to DL in the standard model if OMDL holds. Seurin [Seu12] shows that, assuming OMDL, the security bound for Schnorr signatures by Pointcheval and Stern [PS96] using the *forking lemma* is optimal in the ROM under the DL assumption. More precisely, the paper shows that if the OMDL assumption holds, then any algebraic reduction of Schnorr signatures must lose the same factor as a proof via the forking lemma. Fischlin and Fleischhacker [FF13] generalize this impossibility result to a large class of reductions they call *single-instance reductions*, again assuming OMDL. There are further negative results on the security of Schnorr signatures that assume OMDL [GBL08, FJS14, FH21].

Finally, Drijvers et al. [DEF+19] show under the OMDL assumption that many multi-signature schemes, such as *CoSi* [STV+16], *MuSig* [MPSW19], *BCJ* [BCJ08] and *MWLD* [MWLD10], cannot be proven secure from DL or OMDL.

The Generic Security of OMDL

Despite its wide use, surprisingly, OMDL is lacking any rigorous analysis, apart from a comparison to DL in certain groups: while the OMDL problem is obviously easer than DL, using the index calculus algorithm, OMDL appears to be strictly easier than DL in Jacobian groups of genus 3 or more [KM08]. This result does thus not apply to elliptic-curve groups, which would typically underlie instantiations of schemes based on OMDL.

The only analysis of OMDL in the GGM is a relatively recent proof sketch by Coretti, Dodis, and Guo [CDG18, eprint version], which we show is flawed.[4] (The authors confirmed this.)

Their analysis follows the blueprint of earlier GGM proofs, which goes back to Shoup's [Sho97] proof of the hardness of DL in the GGM. However, as we explain below, the adversary can make their simulation of the GGM OMDL game fail with overwhelming probability. The particularity of OMDL compared to other assumptions, which lend themselves more easily to a GGM proof, is that via its DL oracle, the adversary can obtain information about the secret values chosen by the experiment.

Bauer et al. [BFL20] have recently given further evidence that the analysis of the generic security of OMDL must differ from that of other assumptions. They show that in the algebraic group model, a large class of assumptions, captured by an extension of the *uber assumption* framework [BBG05, Boy08], is implied by the hardness of q-DLog, a parametrized DL problem where the adversary is given $(xG, x^2G, \ldots, x^qG)$ and must find x. While in the AGM q-DLog implies assumptions as diverse as the strong Diffie-Hellman [BB08], the Gap Diffie-Hellman [OP01], and the LRSW assumption [LRSW99], this is not the case for OMDL. Using the meta-reduction technique, Bauer et al. [BFL20] show that it is impossible to prove OMDL from q-DLog, for any q, in the AGM.

[4] The authors study assumptions (including OMDL) and schemes in an extension of the GGM that models preprocessing attacks. They give a proof sketch for the security of OMDL with preprocessing. While we show that their sketch is flawed (see p. 4), their preprocessing techniques can be adapted to our proof. Their result for OMDL in the preprocessing GGM thus still holds, except for a change of bounds.

PROOFS IN THE GGM. To explain the challenges in proving OMDL in the GGM, we start by recalling how proofs in the GGM typically proceed. In the GGM the adversary does not see actual group elements of the form xG, with $x \in \mathbb{Z}_p$ and G a fixed generator; instead it gets encodings $\Xi(x)$ of them, where Ξ is a random injective function. As the adversary cannot compute the encoding of $(x+y)G$ from encodings of xG and yG, it is provided with an oracle that on input $(\Xi(x), \Xi(y))$ returns $\Xi(x+y)$.

In proofs in the GGM, instead of choosing secret values in the security game, the challenger represents them by indeterminates. For concreteness, consider the GGM game for the DL assumption: the adversary is given the challenge $\Xi(x)$ and must compute the discrete logarithm $x \in \mathbb{Z}_p$. In the proof, the challenger *simulates* this game by using the variable X instead of x and encodes the *polynomial* X instead of x. That is, the challenger gives $\Xi(\mathsf{X})$ to the adversary, who is oblivious to this change. If then the adversary asks, for example, for the addition of $\Xi(1)$ and $\Xi(\mathsf{X})$, the challenger replies $\Xi(\mathsf{X}+1)$, that is, the encoding of a polynomial of degree 1.

This allows the challenger to simulate the game without actually defining a challenge. In the proof, *after* the adversary output its answer, the challenger picks a value x uniformly at random, which the adversary can only guess with negligible probability.

There is however a caveat: $\Xi(\mathsf{X})$ represents $\Xi(x)$, and, more generally, for any polynomial P that the adversary constructed via its queries, $\Xi(P)$ represents $\Xi(P(x))$. So the simulation would be inconsistent if for some polynomials $P \neq Q$ computed by the adversary we had $P(x) = Q(x)$. Indeed, if such a *collision* occurs, then the simulated game would give the adversary $\Xi(P) \neq \Xi(Q)$ instead of $\Xi(P(x)) = \Xi(Q(x))$.

In order to bound the probability that the simulation fails due to such collisions, the standard technique is to apply the Schwartz-Zippel Lemma, which states that for a non-zero degree-d (multivariate) polynomial $P \in \mathbb{Z}_p[\mathsf{X}_1, \ldots, \mathsf{X}_n]$ the probability that $P(x_1, \ldots, x_n) = 0$ for a uniformly chosen $\vec{x} \xleftarrow{\$} \mathbb{Z}_p^n$ is $\frac{d}{p}$.

Since x is picked uniformly *after* the adversary has defined the polynomials P and Q, the probability that $P(x) - Q(x) = 0$ is bounded by $\frac{1}{p}$. Applying this to all pairs of polynomials generated by the adversary via its group-operation oracle during the game then yields the final bound. This was precisely how Shoup [Sho97] proved the security of DL in the GGM and it was followed by many subsequent GGM proofs. The technique easily extends to games where there are several secrets $x_1, \ldots, x_n$.

CHALLENGES IN THE GGM PROOF OF OMDL. We will follow Shoup [Sho97] in that we replace all challenges x_i in the OMDL game by corresponding polynomials $\mathsf{X}_i \in \mathbb{Z}_p[\mathsf{X}_1, \ldots, \mathsf{X}_n]$. It seems tempting to then deduce, like for DL, that the probability that $P(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n)$ for any $P \neq Q$ generated during the game is at most $\frac{1}{p}$ by Schwartz-Zippel. (This is what Coretti et al. [CDG18] do in their proof sketch.) This argument however ignores the fact that, via the discrete logarithm oracle $\textsc{Dlog}(\cdot)$, the adversary can obtain (a lot of) information on the challenges x_i and can thereby easily cause collisions. In more detail, such a straightforward proof has the following issues:

First, in the game simulated via polynomials, the adversary's oracle $\textsc{Dlog}(\cdot)$ must be simulated carefully. For example, suppose the adversary asks for the discrete logarithm of the first challenge by querying $\textsc{Dlog}(\Xi(\mathsf{X}_1))$. Since x_1 is not defined yet, the challenger samples it uniformly and gives it to the adversary. Now if the adversary later asks for $\Xi(\mathsf{X}_1 + 1)$ (via its group-operation oracle) and queries $\textsc{Dlog}$ on it, it expects the answer $x_1 + 1$, and not a random value. (In [CDG18], the $\textsc{Dlog}$ oracle always returns random values; the adversary can thus easily detect that it is not playing the OMDL game in the GGM.)

Second, there is a more subtle issue. Again suppose that the adversary queried $\textsc{Dlog}(\Xi(\mathsf{X}_1))$ and was given x_1. Let $P := \mathsf{X}_1$. Using the group-operation oracle, the adversary can compute (an encoding of) the constant polynomial $Q := x_1$, that is, it can obtain $\Xi(Q)$. Since $P(x_1) = Q(x_1) = x_1$, this means that the adversary can in fact construct polynomials P and Q such that

$P(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n)$ and $P \neq Q$. Note that this situation cannot occur in prior GGM proofs for other assumptions, because as long as there is no simulation failure, the adversary's polynomials are *independent* of $\vec{x}$, which is a prerequisite for applying Schwartz-Zippel (SZ) in the end. This standard use of SZ (followed by [CDG18]) is thus not possible for OMDL, as the adversary can obtain information on the challenge $(x_1, \ldots, x_n)$ even when there is no simulation failure, from its DLog oracle.

Instead of Shoup's GGM model [Sho97], one could also use Maurer's model [Mau05], which is an abstraction of the former, but all the issues persist. In his model, all (logarithms of) group elements remain in a "black box", and the adversary can ask for creating new entries in the box that are either the sum of existing entries or for values of its choice. For OMDL one would have to extend the model and allow the adversary to ask for values from the box to capture the DLog oracle. In proofs in Maurer's model the adversary wins if it creates a collision between values in the black box [Mau05] (which is what lets Maurer assume *non-adaptive* adversaries). However, an OMDL adversary *is* adaptive and can easily create collisions (e.g., get x_1 from the DLog oracle, then insert the constant x_1 into the black box).

Our GGM proof of OMDL. In our proof we simulate the OMDL game in the GGM using polynomials, but we take into account all the issues just described. That is, the challenger monitors what the adversary has learned about the challenge and defines the simulation considering this knowledge, thus preventing the adversary to trivially distinguish the real game from the simulation.

Still, there might be simulation failures due to "bad luck", which corresponds precisely to the event whose probability previous proofs bound via Schwartz-Zippel. As OMDL requires a different approach, we propose a new lemma that bounds the probability that our simulation of the OMDL game fails. After modifying the game by aborting in case of a simulation failure, we give a formally defined sequence of game hops showing that the game is equivalent to a game that the adversary cannot win. Given the pitfalls in previous approaches and the intricacies outlined so far (and the importance of OMDL), we believe that such a rigorous approach is justified for OMDL.

Our first step is comparable to how Yun [Yun15] analyzed the generic security of the *multiple discrete logarithm* assumption, where the adversary must solve multiple DL challenges (but is not given a DLog oracle, which is what makes OMDL so different from other assumptions). Like Yun, we formalize the knowledge about the challenge that the adversary accumulates by affine hyperplanes in $\mathbb{Z}_p^n$.

Possible alternative approaches. One might wonder if it was possible to nonetheless rely on the Schwartz-Zippel lemma (SZ) for proving OMDL. We have already argued that applying it once and at the end of the game, as in previous proofs, is not possible. But can SZ be used earlier in the game?

A first idea could be to apply SZ at each DLog call. Consider a call $\mathrm{DLog}(\Xi(\mathsf{X}_1 + \mathsf{X}_2))$, answered with a uniform $v \leftarrow \mathbb{Z}_p$. One could now formally replace the indeterminate X_1 by the expression $\mathsf{X}_2 - v$ in all polynomials P generated so far and use SZ to bound the probability that this creates a collision. A first issue is that since P is a multivariate polynomial, SZ does not directly imply a bound on $\Pr[P(\mathsf{X}_2 - v, \mathsf{X}_2, \ldots, \mathsf{X}_n) = 0]$. Indeed, $P(\mathsf{X}_2 - v, \mathsf{X}_2, \ldots, \mathsf{X}_n)$ is the evaluation of the polynomial $\hat{P}(\mathsf{X}_1) := P(\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_n)$ for $\mathsf{X}_1 = \mathsf{X}_2 - v$, so we need to bound $\Pr[\hat{P}(\mathsf{X}_2 - v) = 0]$ for a polynomial $\hat{P}$ with coefficients in the ring $\mathbb{Z}_p[\mathsf{X}_2, \ldots, \mathsf{X}_n]$, whereas SZ is defined for polynomials over fields.

Moreover, when the query $\mathrm{DLog}(\Xi(P(\mathsf{X}_1, \ldots, \mathsf{X}_n))$ involves a more complex polynomial than $P = \mathsf{X}_1 + X_2$ then the substitution of one variable by a linear expression of the others is even cumbersome to describe notationally. We avoid these problems in our proof by using our lemma instead of (a variant of) SZ, which also lets us keep notation simple.

Another idea would be to apply SZ each time a new encoding is computed. Indeed, assuming no collisions have occurred so far, one could use SZ to bound the probability that the new encoding introduces a collision and then proceed by induction. But the resulting proof would require one game hop for every newly computed encoding: In the j-th hybrid of this game the first j encodings are chosen all different independently of the real value of the challenge; the challenge $\vec{x}$ is picked by the game just before the $(j+1)$-th encoding, when P_{j+1} is defined. Using SZ, we can show that the probability that $P_{j+1}(\vec{x}) = P_i(\vec{x})$ for all $i \leq j$ is negligible.

However, we need to be more cautious. To prevent the attack in which the adversary queries $\text{DLOG}(\Xi(\mathsf{X}_1))$, obtains x_1 and then generates the constant polynomial $P_{j+1} = x_1$, we need to adapt all polynomials defined so far to reflect the information revealed by $\text{DLOG}(\cdot)$. In this example, this is easy to formalize: update every polynomial by evaluating X_1 on x_1 and replace $P_k(x_1, \mathsf{X}_2, \ldots, \mathsf{X}_n)$ by some $P'_k(\mathsf{X}_2, \ldots, \mathsf{X}_n)$; the updated challenge $\vec{x}$ would be of size $n-1$. To generalize this, we would have to apply an affine transformation to all variables of the polynomials at each call to $\text{DLOG}(\cdot)$. After as many game hops as there are queries by the adversary, we would arrive at a game in which all encodings are random and the challenge is defined after the adversary output its solution.

We believe that both approaches just sketched lead to more complicated (and error-prone) proofs than the one we propose. In our proof, in the first game hop we abort if our simulation fails and we bound this probability by our new lemma. The remaining 3 game hops are purely syntactical and do not change the adversary's winning probability.

One-More CDH

Another "one-more" assumption is the *one-more computational Diffie-Hellman* assumption [BNN04], also known as 1-*MDHP* [KM08, KM10], which is similar to the *chosen-target CDH* assumption [Bol03]. Here, the adversary receives q pairs of group elements (X, Y_i), all with the same first component $X = xG$, and its task is to compute xY_i for all i. It is given an oracle $\text{CDH}_1(\cdot)$, which on input Y returns xY, and which it can query fewer than q times.

It turns out that this assumption can be proved to hold in the generic group model using standard techniques. Following the original GGM proof of DL [Sho97], we modify the simulation for the adversary from encoding logarithms to encoding polynomials in $\mathbb{Z}_p[\mathsf{X}, \mathsf{Y}_1, \ldots, \mathsf{Y}_n]$. The challenges that the adversary receives are the monomials $\mathsf{X}, \mathsf{Y}_1, \ldots, \mathsf{Y}_n$, and when the adversary queries its oracle $\text{CDH}_1(\cdot)$ on an encoding of a polynomial P, it receives an encoding of $\mathsf{X}P$, i.e., its polynomial multiplied by the indeterminate X. To win this "ideal" game, the adversary must construct encodings of $(\mathsf{XY}_1, \ldots, \mathsf{XY}_n)$. Making q calls to its $\text{CDH}_1(\cdot)$ oracle and using its group-operation oracle, it can only construct (encodings of) polynomials from $\mathsf{Span}(1, \mathsf{X}, \mathsf{Y}_1, \ldots, \mathsf{Y}_n, \mathsf{X}P_1, \ldots, \mathsf{X}P_q)$.

Ignoring polynomials of degree less than 2, the adversary wins if $\mathsf{Span}(\mathsf{XY}_1, \ldots, \mathsf{XY}_n) \subseteq \mathsf{Span}(\mathsf{X}P_1, \ldots, \mathsf{X}P_q)$. But it must also solve more challenges than it makes $\text{CDH}_1(\cdot)$ oracle queries; that is $q < n$. Using a dimension argument, we deduce that the above condition cannot be satisfied, and thus the adversary cannot win this game.

This "ideal" game is indistinguishable from the real game if the adversary does not create two distinct polynomials that agree on $x, y_1, \ldots, y_n$, the secret values of the real game. Because the degree of all polynomials is upper-bounded by $q+1$, we can use the Schwartz-Zippel Lemma (as, e.g., done in [Boy08]) to upper-bound the statistical distance between the two games by $\mathcal{O}\left(\frac{(q+1)(m+q)^2}{p}\right)$, where m is the number of group operations made by the adversary. This establishes the generic security of this assumption. (An alternative is to cast the assumption as an uber-assumption in the algebraic group model and apply [BFL20, Theorem 4.1].)

The situation is quite different for a variant of the above problem, in which the first component of the challenge pairs is not fixed. That is, the adversary can request challenges, which are random

pairs (X_i, Y_i) and is provided with an oracle $\mathrm{CDH}(\cdot)$, which on input any pair $(X = xG, Y)$ returns the CDH solution of X and Y, that is xY. The adversary must compute the CDH solutions of the challenge pairs while making fewer queries to $\mathrm{CDH}(\cdot)$. In this paper we will refer to this assumption as OMCDH.

For this problem the standard proof methodology in the GGM fails for the following reason. Providing the adversary with an oracle $\mathrm{CDH}_1(\cdot)$, as in the one-more Diffie-Hellman assumption with one component fixed (or a DLog oracle in OMDL) lets the adversary only construct polynomials of degree at most $q + 1$. In contrast, the $\mathrm{CDH}(\cdot)$ oracle in OMCDH leads to a multiplication of the degrees, which enables the adversary to "explode" the degrees and makes arguments à la Schwartz-Zippel impossible, since they rely on low-degree polynomials.

To get around this problem, we prove the following, *stronger* assumption: as in OMCDH, the adversary still has to compute CDH solutions, but now it is given a discrete-logarithm oracle. This hybrid assumption implies both OMDL (for which the goal is harder) and OMCDH (in which the oracle is less powerful) and we prove it in the GGM by extending our proof of OMDL.

2 Preliminaries

General Notation. We denote the (closed) integer interval from a to b by $[a, b]$. A function $\mu\colon \mathbb{N} \to [0, 1]$ is *negligible* (denoted $\mu = \mathsf{negl}$) if for all $c \in \mathbb{N}$ there exists $\lambda_c \in \mathbb{N}$ such that $\mu(\lambda) \leq \lambda^{-c}$ for all $\lambda \geq \lambda_c$. A function ν is *overwhelming* if $1 - \nu = \mathsf{negl}$. Given a non-empty finite set S, we let $x \xleftarrow{\$} S$ denote sampling an element x from S uniformly at random. A list $\vec{z} = (z_1, \ldots, z_n)$, also denoted $(z_i)_{i\in[n]}$, is a finite sequence. The length of a list $\vec{z}$ is denoted $|\vec{z}|$. The empty list is denoted $(\,)$.

All algorithms are probabilistic unless stated otherwise. By $y \leftarrow \mathcal{A}(x_1, \ldots, x_n)$ we denote running algorithm $\mathcal{A}$ on inputs $(x_1, \ldots, x_n)$ and uniformly random coins and letting y denote the output. If $\mathcal{A}$ has oracle access to some algorithm Oracle, we write $y \leftarrow \mathcal{A}^{\textsc{Oracle}}(x_1, \ldots, x_n)$. A *security game* $\mathrm{GAME}_{par}(\lambda)$ indexed by a set of parameters *par* consists of a main procedure and a collection of oracle procedures. The main procedure, on input the security parameter λ, generates input on which an adversary $\mathcal{A}$ is run. The adversary interacts with the game by calling oracles provided by the game and returns some output, based on which the game computes its own output bit b, which we write $b \leftarrow \mathrm{GAME}^{\mathcal{A}}_{par}(\lambda)$. We identify **false** with 0 and **true** with 1. As all games in this paper are computational, we define the *advantage* of $\mathcal{A}$ in $\mathrm{GAME}_{par}(\lambda)$ as $\mathbf{Adv}^{\mathrm{GAME}}_{par,\mathcal{A}} := \Pr[1 \leftarrow \mathrm{GAME}^{\mathcal{A}}_{par}(\lambda)]$. We say that GAME_{par} is *hard* if $\mathbf{Adv}^{\mathrm{GAME}}_{par,\mathcal{A}} = \mathsf{negl}$ for any probabilistic polynomial-time (p.p.t.) adversary $\mathcal{A}$.

Algebraic Notation. A *group description* is a tuple $\Gamma = (p, \mathbb{G}, G)$ where p is an odd prime, $\mathbb{G}$ is an abelian group of order p, and G is a generator of $\mathbb{G}$. We use additive notation for the group law and denote group elements with uppercase letters. We assume the existence of a p.p.t. algorithm GrGen which, on input the security parameter 1^λ in unary, outputs a group description $\Gamma = (p, \mathbb{G}, G)$ where p is of bit-length λ. For $X \in \mathbb{G}$, we let $\log_G(X)$ denote the discrete logarithm of X with respect to the generator G, i.e., the unique $x \in \mathbb{Z}_p$ such that $X = xG$.

For multivariate polynomials $P \in \mathbb{Z}_p[\mathsf{X}_1, \ldots, \mathsf{X}_n]$ we write $\vec{\mathsf{X}} := (\mathsf{X}_1, \ldots, \mathsf{X}_n)$ and $P(\vec{x}) := P(x_1, \ldots, x_n)$ for $\vec{x} \in \mathbb{Z}_p^n$. We consider subspaces of $\mathbb{Z}_p[\mathsf{X}_1, \ldots, \mathsf{X}_n]$: for a set $L = \{P_1, \ldots, P_q\}$ of polynomials, $\mathsf{Span}(L) := \{ \sum_{i\in[1,q]} \alpha_i P_i \mid \vec{\alpha} \in \mathbb{Z}_p^q \}$ is the smallest vector space containing the elements of L. If $L = \emptyset$ then $\mathsf{Span}(L) = \{0\}$. By $\mathsf{dim}(A)$ we denote the dimension of vector spaces or affine spaces.

By $\langle \vec{x}, \vec{y} \rangle = \sum_{i\in[1,n]} x_i y_i$ we denote the scalar product of vectors $\vec{x}$ and $\vec{y}$ of length n. In this work, polynomials are typically of degree 1, so we can write $P = \rho_0 + \sum_{i=1}^n \rho_i \mathsf{X}_i$ as a scalar product: $P(\vec{\mathsf{X}}) = \rho_0 + \langle \vec{P}, \vec{\mathsf{X}} \rangle$, where we define $\vec{P} := (\rho_i)_{i\in[1,n]}$, that is the vector of *non-constant* coefficients of P.

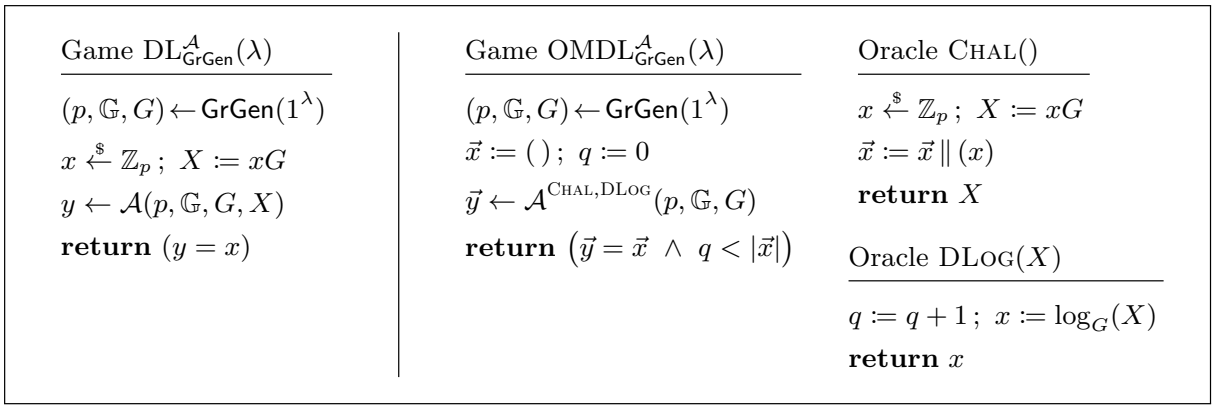

Fig. 1: The DL and the OMDL problem

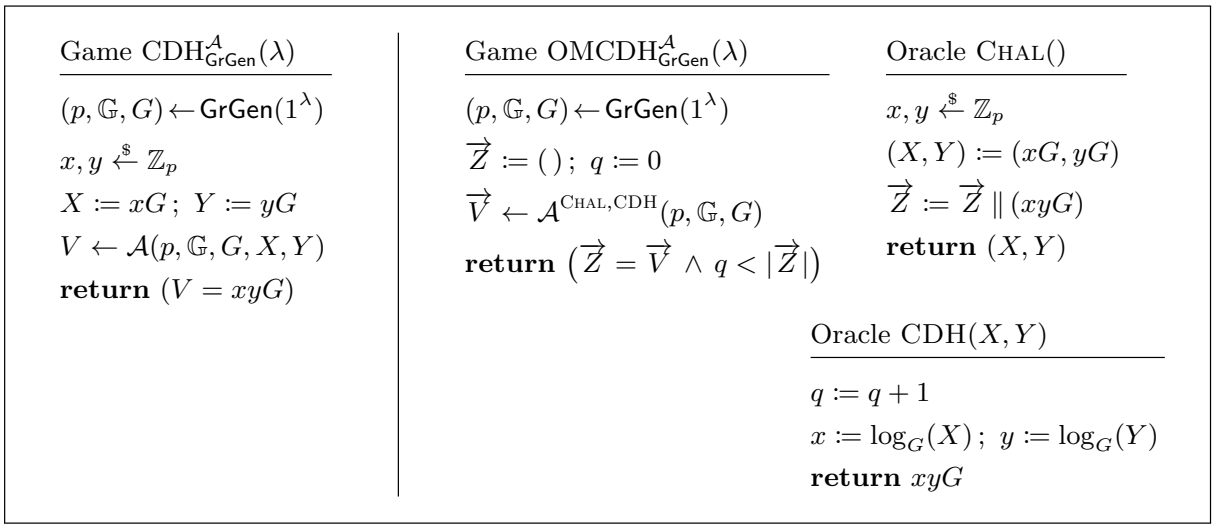

Fig. 2: The CDH and the OMCDH problem

Discrete Logarithm and Diffie-Hellman problems. In in Figures 1 and 2 we recall the discrete logarithm (DL) problem and the computational Diffie-Hellman (CDH) problem and define the one-more discrete logarithm (OMDL) problem and the one-more computational Diffie-Hellman (OMCDH) problem.

3 OMDL in the GGM

3.1 A Technical Lemma

While a standard argument in GGM proofs uses the Schwartz-Zippel lemma, it does not work for OMDL, where the adversary obtains information on the challenge $\vec{x}$ *not* only when the simulation fails. So we cannot argue that $\vec{x}$ looks uniformly random to the adversary, which is a precondition for applying Schwartz-Zippel. We therefore use a different lemma, which bounds the probability that for a given polynomial P, we have $P(\vec{x}) = 0$ when $\vec{x}$ is chosen uniformly from a set $\mathcal{C}$. This set $\mathcal{C} \subseteq \mathbb{Z}_p^n$ represents the knowledge the adversary has gained on the challenge $\vec{x}$ during the OMDL game.

The Schwartz-Zippel lemma applies when $\mathcal{C} = S^n$ with S a subset of $\mathbb{Z}_p$, whereas our lemma is for the case that P has degree 1 and $\mathcal{C}$ is defined by an intersection of affine hyperplanes $\mathcal{Q}_j$ from which we remove other affine hyperplanes $\mathcal{D}_i$, that is $\mathcal{C} := \left(\bigcap_{j\in[1,q]} \mathcal{Q}_j\right) \setminus \left(\bigcup_{i\in[1,m]} \mathcal{D}_i\right)$.

We start with some notations. Consider m polynomials $D_i \in \mathbb{Z}_p[\mathsf{X}_1, \ldots, \mathsf{X}_n]$ of degree 1, and $q+1$ polynomials $Q_j \in \mathbb{Z}_p[X_1, \ldots, \mathsf{X}_n]$ also of degree 1. We can write them as

$$D_i(\vec{\mathsf{X}}) = D_{i,0} + \sum_{k=1}^{n} D_{i,k}\mathsf{X}_k = D_{i,0} + \langle \vec{D_i}, \vec{\mathsf{X}} \rangle \tag{1}$$

with $\vec{D_i} \coloneqq (D_{i,k})_{1 \le k \le n}$, and similarly for Q_j. We define the sets of roots of these polynomials, which are hyperplanes of $\mathbb{Z}_p^n$:

$$\begin{aligned} \forall i \in [1,m] :\ \mathcal{D}_i &\coloneqq \{\vec{x} \in \mathbb{Z}_p^n \mid D_i(\vec{x}) = 0\} \\ \forall j \in [1,q+1] :\ \mathcal{Q}_j &\coloneqq \{\vec{x} \in \mathbb{Z}_p^n \mid Q_j(\vec{x}) = 0\} \ . \end{aligned} \tag{2}$$

From (1), we see that the vector $\vec{D_i}$ of non-constant coefficients defines the direction of the hyperplane $\mathcal{D}_i$. It contains the coefficients of the polynomial $D_i - D_i(0) = \sum_{k=1}^{n} D_{i,k}\mathsf{X}_k$.

We define the set

$$\mathcal{C} \coloneqq \Big(\bigcap_{j \in [1,q]} \mathcal{Q}_j \Big) \setminus \Big(\bigcup_{i \in [1,m]} \mathcal{D}_i \Big) \ , \tag{3}$$

that is, the set of points at which all Q_i's vanish but none of the D_i's do. The following lemma will be the heart of our proofs of one-more assumptions in the GGM.

Lemma 1. *Let $D_1, \ldots, D_m, Q_1, \ldots, Q_{q+1} \in \mathbb{Z}_p[\mathsf{X}_1, \ldots, \mathsf{X}_n]$ be of degree 1; let $\mathcal{C}$ be as defined in (2) and (3). Assume $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$ and $\vec{Q}_{q+1}$ is linearly independent of $(\vec{Q}_j)_{j \in [1,q]}$. If $\vec{x}$ is picked uniformly at random from $\mathcal{C}$ then*

$$\frac{p-m}{p^2} \le \ \Pr\left[Q_{q+1}(\vec{x}) = 0\right] \ \le \frac{1}{p-m} \ .$$

Proof. Since $\vec{x}$ is picked uniformly in $\mathcal{C}$, we have $\Pr[\,\vec{x} \in \mathcal{Q}_{q+1}] = |\mathcal{Q}_{q+1} \cap \mathcal{C}|/|\mathcal{C}|$.

We first bound $|\mathcal{C}|$. We define the affine space $\mathcal{Q} \coloneqq \bigcap_{j \in [1,q]} \mathcal{Q}_j$ and let $d \coloneqq \mathsf{dim}(\mathcal{Q})$ denote its dimension. Thus, $\mathcal{Q}$ contains p^d elements. We rewrite $\mathcal{C}$:

$$\mathcal{C} = \mathcal{Q} \setminus \Big(\bigcup_{i \in [1,m]} (\mathcal{D}_i \cap \mathcal{Q}) \Big) \ .$$

Now for a fixed $i \in [1,m]$ we bound the size of $\mathcal{D}_i \cap \mathcal{Q}$. Since the polynomial D_i has degree 1 by definition, $\mathcal{D}_i$ is a hyperplane. There are three cases: either $\mathcal{Q} \subseteq \mathcal{D}_i$, which means $\mathcal{C} = \emptyset$. This contradicts the premise of the lemma, namely $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$. Since $\mathcal{D}_i$ is an hyperplane, the remaining cases are $\mathcal{Q} \cap \mathcal{D}_i = \emptyset$ and $\mathcal{Q} \cap \mathcal{D}_i$ has dimension $\mathsf{dim}(\mathcal{Q}) - 1 = d - 1$. In both cases $\mathcal{D}_i \cap \mathcal{Q}$ contains at most p^{d-1} elements.

When we remove the sets $(\mathcal{D}_i)_{i \in [1,m]}$ from $\mathcal{Q}$, we remove at most mp^{d-1} elements, which yields

$$p^d - mp^{d-1} \le |\mathcal{C}| \le p^d \ . \tag{4}$$

We now use the same method to bound $|\mathcal{C} \cap \mathcal{Q}_{q+1}|$. We define $\mathcal{Q}' = \mathcal{Q}_{q+1} \cap \mathcal{Q}$. Since $\vec{Q}_{q+1}$ is linearly independent of $(\vec{Q}_j)_{j \in [1,m]}$, we get $\mathsf{dim}(\mathcal{Q}') = d - 1$.

For a fixed $i \in [1,m]$, since by assumption $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$, we can proceed as with $\mathcal{Q}$ above: either $\mathcal{Q}' \cap \mathcal{D}_i = \emptyset$ or $\mathcal{Q}' \cap \mathcal{D}_i$ has dimension $d-2$, which yields

$$p^{d-1} - mp^{d-2} \le |\mathcal{Q}_{q+1} \cap \mathcal{C}| \le p^{d-1} \ . \tag{5}$$

Combining equations (4) and (5) we obtain the following, which concludes the proof:

$$\frac{p^{d-1} - mp^{d-2}}{p^d} \le \frac{|\mathcal{Q}_{q+1} \cap \mathcal{C}|}{|\mathcal{C}|} \le \frac{p^{d-1}}{p^d - mp^{d-1}} \ . \qquad \square$$

3.2 Proof Overview

THE GENERIC GAME. We prove a lower bound on the computational complexity of the OMDL game in generic groups in the sense of Shoup [Sho97]. We follow the notation developed by Boneh and Boyen [BB08] for this proof.

In the generic group model, elements of $\mathbb{G}$ are encoded as arbitrary unique strings, so that no property other than equality can be directly tested by the adversary. The adversary performs operations on group elements by interacting with an oracle called GCMP.

To represent and simulate the working of the oracles, we model the opaque encoding of the elements of $\mathbb{G}$ using an injective function $\Xi\colon \mathbb{Z}_p \to \{0,1\}^{\lceil \log_2(p) \rceil}$ where p is the group order. Internally, the simulator represents the elements of $\mathbb{G}$ by their discrete logarithms relative to a fixed generator G. This is captured by Ξ, which maps any integer a to the string $\xi := \Xi(a)$ representing $a \cdot G$. In the game we will use an encoding procedure ENC to implement Ξ.

We specify the game OMDL in the GGM in Figure 3. In contrast to Figure 1 there are no more group elements. The game instead maintains discrete logarithms $a \in \mathbb{Z}_p$ and gives the adversary their encodings $\Xi(a)$, which are computed by the procedure ENC. The challenger uses the variable j to represent the number of created group elements, which is incremented before each call to ENC. The procedure ENC then encodes the latest scalar a_j. If a_j has already been assigned a string ξ, then ENC() outputs ξ, else it outputs a random string different from all previous ones. For this, the game maintains a list $(a_i, \xi_i)_{0 \le i \le j}$ of logarithms and their corresponding encodings.

OMDLGGM initializes $j = 0$ and $a_0 = 1$, and runs the adversary on input $\xi_0 \leftarrow$ ENC() (ξ_0 is thus the encoding of the group generator). The oracle CHAL increments a counter of challenges n, samples a new value x_n and returns its encoding by calling ENC(). Since it creates a new element, it first increments j and defines the $a_j := x_n$. The oracle DLOG is called with a string ξ and returns $\perp$ if the string is not in the set of assigned strings $\{\xi_i\}_{i \in [0,j]}$. Else, it picks an index i (concretely: the smallest such index) such that $\xi_i = \xi$ and returns a_i, which is the Ξ-preimage of ξ (and thus the logarithm of the group element encoded by ξ).

The adversary also has access to the oracle GCMP for group operations, which takes as input two strings ξ and ξ' and a bit b, which indicates whether to compute the addition or the subtraction of the group elements. The oracle gets the (smallest) indexes i and i' such that $\xi = \xi_i$ and $\xi' = \xi_{i'}$, it increments j, sets $a_j := a_i + (-1)^b a_{i'}$ and returns ENC(), which computes the encoding of a_j.

PROOF OVERVIEW. The goal of our proof is to simulate the game without ever computing scalars a_i by replacing them with polynomials P_i and show that with overwhelming probability this does not affect the game. Game_0 (defined by ignoring all the boxes, except the dashed ones, in Figure 4) is the same game as OMDLGGM, except for two syntactical changes, which will be useful in the proof. The main modification is that we now make n calls to the oracle DLOG after $\mathcal{A}$ outputs its answer $\vec{y}$: for $i \in [1,n]$ we set $x_i := \text{DLOG}(\xi_{j_i})$, where indices j_i are defined in the oracle CHAL so that $a_{j_i} = x_i$; thus $\text{DLOG}(\xi_{j_i})$ always outputs $a_{j_i} = x_i$, meaning this does not affect the game. Second, as calls to DLOG increase q, we put the condition "**if** $q < n$ **then return** 0" before those calls.

INTRODUCING POLYNOMIALS. Game_1, defined in Figure 4 by only ignoring the gray boxes, introduces the polynomials P_i, where $P_0 = 1$ represents $a_0 = 1$. In the n-th call to CHAL, the game defines a new polynomial $P_j = \mathsf{X}_n$, which represents the value x_n. We thus have

$$P_i(\vec{x}) = a_i \ , \tag{6}$$

and in this sense the polynomial P_i represents the scalar a_i (and thus implicitly the group element $a_i G$). The group-operation oracle maintains this invariant; when computing $a_j := a_i + (-1)^b a_{i'}$, it also sets $P_j := P_i + (-1)^b P_{i'}$.

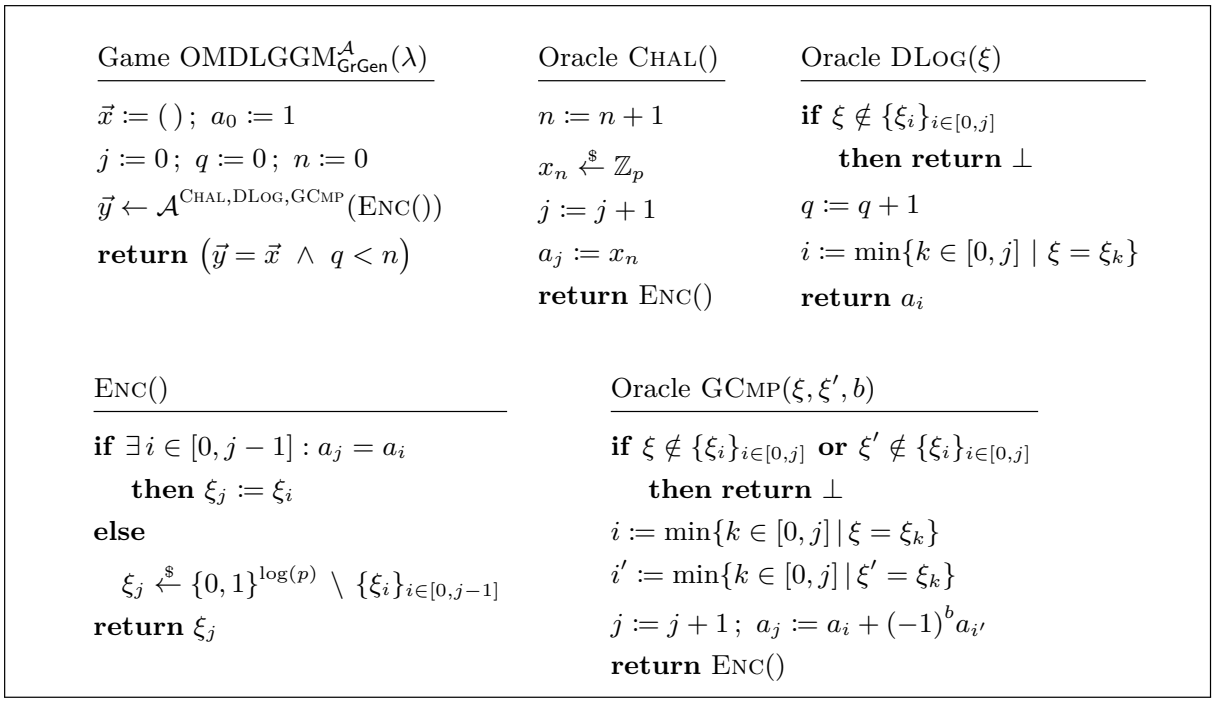

Fig. 3: The OMDL game in the GGM

Note that there are many ways to represent a group element aG by a polynomial. E.g., the first challenge $x_1 G$ is represented by both the polynomial X_1 and the constant polynomial x_1. Intuitively, since x_1 is a challenge, it is unknown to $\mathcal{A}$, and as long as $\mathcal{A}$ does not query $\textsc{DLog}(\xi)$, with $\xi \coloneqq \Xi(x_1)$, it does not know that the polynomials X_1 and x_1 represent the same group element. Game_1 introduces a list L that represents this knowledge of $\mathcal{A}$. E.g., when $\mathcal{A}$ calls $\textsc{DLog}(\Xi(x_1))$, the game will append the polynomial $\mathsf{X}_1 - x_1$ to the list L. More generally, on call $\textsc{DLog}(\xi_i)$ it appends $P_i - P_i(\vec{x})$ to L, which represents the fact that $\mathcal{A}$ knows that the polynomial $P_i - P_i(\vec{x})$ represents the scalar 0 and the group element $0_{\mathbb{G}}$. The list L will be used to ensure consistency when we replace scalars by polynomials in the game.

Recall that our goal is to have the challenger only deal with polynomials when simulating the game for $\mathcal{A}$. As this can be done without actually defining the challenge $\vec{x}$, the challenger could then select $\vec{x}$ *after* $\mathcal{A}$ gave its output, making it impossible for $\mathcal{A}$ to predict the right answer.

This is done in the final game Game_4, defined in Figure 6, where the challenger is in the same position as $\mathcal{A}$: it does not know that x_1 is the answer to the challenge represented by the polynomial X_1 until $\textsc{DLog}(\xi)$ is called with $\xi \coloneqq \Xi(x_1)$. In fact, x_1 is not even defined before this call, and, more generally, $\vec{x}$ does not exist until the proper DLog queries are made.

To get to Game_4, we define two intermediate games. We will modify procedure Enc so that it later deals with polynomials only (instead of their evaluations, as $\vec{x}$ will not exist). Because of this, it will be unknown whether $P_j(\vec{x}) = P_i(\vec{x})$ for some $i \in [0, j-1]$, *unless* $P_j - P_i \in \mathsf{Span}(L)$, since both the challenger and the adversary are aware that all polynomials in L evaluate to 0 at $\vec{x}$.

However, it can happen that, when $\vec{x}$ is defined later, $P_j(\vec{x}) = P_i(\vec{x})$. That is, in the original game, we would have had $a_j = a_i$, but in the final game, Enc is not aware of this. This is precisely when the simulation fails, and we abort the game. We will then bound the probability of this event, using Lemma 1.

In "typical" GGM proofs an abort happens when $P_j(\vec{x}) = P_i(\vec{x})$ and $P_j \neq P_i$. For OMDL, because the adversary might have information on the $\vec{x}$ (and the challenger is aware of this), we allow that there are $P_j \neq P_i$ for which the current knowledge on $\vec{x}$ lets us deduce $P_j(\vec{x}) = P_i(\vec{x})$.

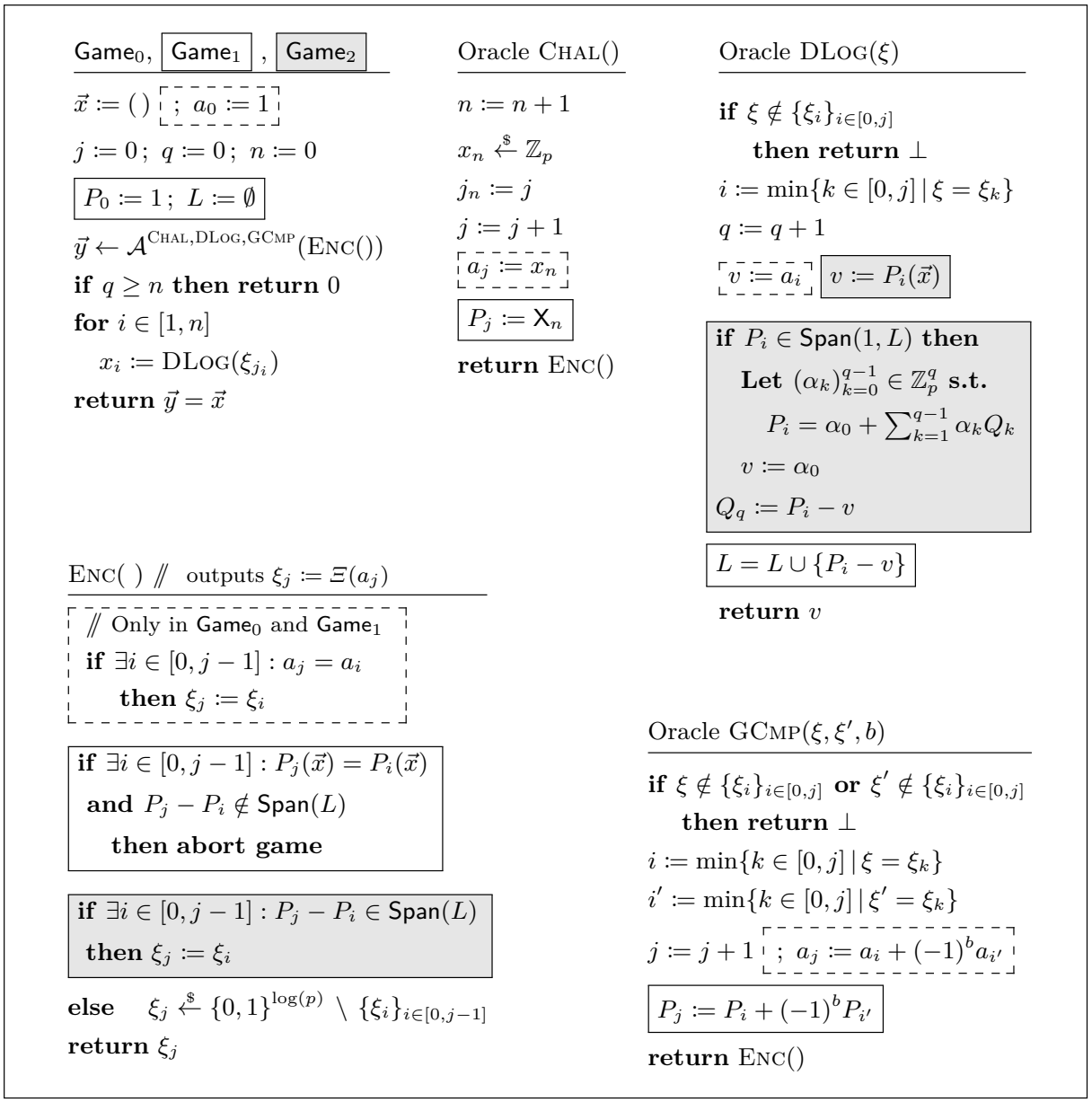

Fig. 4: Game_0 (which only includes the dashed boxes) is the GGM version of OMDL. Game_1 (including all but the gray boxes) introduces the polynomials that represent the information that $\mathcal{A}$ obtains, and aborts when Game_0 cannot be simulated with polynomials. In Game_2 (including all but the dashed boxes) we eliminate the use of scalars (except for the abort condition) in oracles ENC and DLOG.

With the formalism introduced above this corresponds exactly to the situation that $P_i - P_j \in \mathsf{Span}(L)$. We introduce this abort condition in the procedure ENC in Game_1 (Figure 4). Because in the "ideal" game Game_4 (Figure 6), there are no more values a_i, we will express the abort condition differently (namely in oracle DLOG) and argue that the two conditions are equivalent.

ELIMINATING USES OF SCALARS. Using the abort condition in Game_1, we can replace some uses of the scalars a_i by their representations as polynomials P_i. This is what we do in Game_2, (Figure 4, including all boxes *except* the dashed box), which eliminates all occurrences of a_i's. In ENC, since the game aborts when $P_j(\vec{x}) = P_i(\vec{x})$ and $P_j - P_i \notin \mathsf{Span}(L)$, and because when $P_j - P_i \in \mathsf{Span}(L)$

then $P_j(\vec{x}) = P_i(\vec{x})$, we can replace the event $P_j(\vec{x}) = P_i(\vec{x})$ by $P_j - P_i \in \mathsf{Span}(L)$. Intuitively, we can now think of ENC() as encoding the polynomial P_j instead of the scalar a_j.

We next modify the oracle DLOG. The first change is that instead of returning a_i the oracle uses $P_i(\vec{x})$, which is equivalent by (6). The second change is that on input ξ, oracle DLOG checks if $\mathcal{A}$ already knows the answer to its query, in which case it computes the answer without using $\vec{x}$. E.g., assume $\mathcal{A}$ has only made one query CHAL(), and thus $q = 0$ and $L = \emptyset$: if $\mathcal{A}$ now queries DLOG(ξ) with $\xi \coloneqq \Xi(x_1)$, the oracle first checks if $P_i = \mathsf{X}_1 \in \mathsf{Span}(1, L)$, (where i is the current number of group elements seen by the adversary), which is not the case, and so it computes $v \coloneqq P_i(\vec{x}) = x_1$. It then adds the polynomial $Q_1 \coloneqq \mathsf{X}_1 - x_1$ to L and returns x_1. If for example $\mathcal{A}$ makes another call DLOG(ξ') with $\xi' \coloneqq \Xi(2x_1 + 2)$, then it knows that the answer should be $2x_1 + 2$. And indeed, the oracle DLOG checks if $2\mathsf{X}_1 + 2 \in \mathsf{Span}(1, L)$, and since this is the case, it gets the decomposition

$$2\mathsf{X}_1 + 2 = (2x_1 + 2) + 2Q_1 = \alpha_0 + \alpha_1 Q_1$$

with $\alpha_0 = 2x_1 + 2$ and $\alpha_1 = 2$. The oracle uses this decomposition to compute its answer $v \coloneqq \alpha_0 = 2x_1 + 2$.

More generally, on input ξ_i, the oracle DLOG checks if $P_i \in \mathsf{Span}(1, L)$. If so, it computes the answer using the decomposition of P_i in $\mathsf{Span}(1, L)$; else it uses $\vec{x}$ and outputs $a_i = P_i(\vec{x})$.

We have now arrived at a situation close to the "ideal" game, where the challenger only uses polynomials. The only uses of scalars are the abort condition in ENC (since it compares $P_j(\vec{x})$ and $P_i(\vec{x})$) and in DLOG, when computing the logarithm of an element that is not already known to $\mathcal{A}$. Towards our goal of simulating the game without defining $\vec{x}$, we modify those two parts next.

CHANGING THE ABORT CONDITION. The aim of Game_3 is precisely to modify the abort condition so that it does not use $\vec{x}$ anymore. Figure 5 recalls Game_2 and defines Game_3 by not including the dashed and the gray box. In Game_3 the challenger does not abort in the procedure ENC. This means that if $P_j - P_i \notin \mathsf{Span}(L)$ for some i, the challenger creates a string $\xi_j \neq \xi_i$ even when $P_j(\vec{x}) = P_i(\vec{x})$. This means that the simulation of the game is not correct anymore; but we will catch these inconsistencies and abort in the oracle DLOG.

For concreteness consider the following example: let $\vec{x} = (x_1)$ and suppose $\mathcal{A}$ built the polynomials $P_{i_1} = x_1$ using the oracle GCMP and $P_{i_2} = \mathsf{X}_1$ using the oracle CHAL; suppose also that $\mathcal{A}$ has not queried DLOG yet, thus $L = \emptyset$. If $i_1 < i_2$ then Game_2 aborts on the call ENC() which encodes P_{i_2}, since $P_{i_1}(\vec{x}) = P_{i_2}(\vec{x})$ and $P_{i_2} - P_{i_1} \notin \mathsf{Span}(L)$. In contrast, in Game_3 the challenger defines $\xi_{i_1} \neq \xi_{i_2}$, which is inconsistent. But the abort will now happen during a call to DLOG.

Suppose $\mathcal{A}$ queries DLOG(ξ_{i_3}), with $\xi_{i_3} = \Xi(2\mathsf{X}_1 + 2)$. Game_3 now adds the polynomial $Q_1 = 2\mathsf{X}_1 + 2 - (2x_1 + 2) = 2(\mathsf{X}_1 - x_1)$ to L and checks for an inconsistency of this answer with all the polynomials that $\mathcal{A}$ computed. Since it finds that $P_{i_1} - P_{i_2} = x_1 - \mathsf{X}_1 \in \mathsf{Span}(L)$ but $\xi_{i_1} \neq \xi_{i_2}$, the game aborts. But Game_3 should also abort even if $\mathcal{A}$ does not query the oracle DLOG. This was precisely the reason for adding the final calls of the game to the oracle DLOG in Game_0. Since $P_{j_i} = \mathsf{X}_i$ and the challenger calls $x_i \leftarrow$ DLOG(ξ_{j_i}) for $i \in [1, n]$ at the end, the challenger makes the query DLOG(ξ_{j_1}), which adds $\mathsf{X}_1 - x_1$ to L, after which we have $P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$ and therefore an abort.

More generally, in Game_3 the oracle DLOG aborts if there exists $(i_1, i_2) \in [0, j]^2$ such that $P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$ and $\xi_{i_1} \neq \xi_{i_2}$. In the proof of Theorem 1 we show that this abort condition is equivalent to the abort condition in Game_2.

ELIMINATING ALL USES OF $\vec{x}$. In Game_3 the only remaining part that uses $\vec{x}$ is the operation $v \coloneqq P_i(\vec{x})$ in oracle DLOG. Our final game hop will replace this by an equivalent operation.

In Game_4, also presented in Figure 5, the challenger samples v uniformly from $\mathbb{Z}_p$ instead of evaluating P_i on the challenge. In the proof of Theorem 1, we will show that since the distribution of $P_i(\vec{x})$ is uniform for a fixed P_i, this change does not affect the game.

This is the only difference between Game_4 and Game_3, but since this modification removes all uses of $\vec{x}$ for the challenger, we rewrite Game_4 explicitly in Figure 6, where we define $\vec{x}$ only after $\mathcal{A}$ outputs $\vec{y}$. Game_4 is thus easily seen to be impossible (except with negligible probability) to win for $\mathcal{A}$. The reason is that $\mathcal{A}$ cannot make enough queries to DLog to constrain the construction of $\vec{x}$ at the end of the game and therefore cannot predict the challenge $\vec{x}$. We now make the intuition given above formal in the following theorem.

3.3 Formal Proof

Theorem 1. *Let $\mathcal{A}$ be an adversary that solves OMDL in a generic group of prime order p, making at most m oracle queries. Then*

$$\mathbf{Adv}^{\mathrm{OMDLGGM}}_{\mathcal{A}} \leq \frac{m^2}{p-m^2} + \frac{1}{p} \ .$$

Proof of Theorem 1. The proof will proceed as follows: we first compute the statistical distance between Game_0, which is OMDLGGM, and Game_1 (Figure 4); we then show that Game_1, Game_2, Game_3 and Game_4 (Figures 4 and 5) are equivalently distributed; and finally we upper-bound the probability of winning Game_4 (Figure 6).

Preliminary results. We start with proving three useful invariants of the polynomials P_i and the set L which are introduced in Game_1. The first one is:

$$\forall\, i \in [0,j] : \ P_i(\vec{x}) = a_i \ . \tag{7}$$

This holds in Game_1 and justifies replacing all occurrences of a_i by $P_i(\vec{x})$ in Game_2 in Figure 5. To prove this, we show that each time the games introduce a new polynomial P_j, we have $P_j(\vec{x}) = a_j$.

We prove this by induction. Initially, $P_0 = 1$ and $a_0 = 1$ so the statement holds for $j = 0$. Now suppose it is true for all $i \in [0, j-1]$. We show it is true for j. Polynomial P_j can be built either by oracle Chal or by oracle GCmp:

- In oracle Chal, $P_j := \mathsf{X}_n$ and $a_j := x_n$ so we have $P_j(\vec{x}) = x_n = a_j$.
- In oracle GCmp, $P_j := P_i + (-1)^b P_{i'}$ and $a_j := a_i + (-1)^b a_{i'}$ so we have $P_j(\vec{x}) := P_i(\vec{x}) + (-1)^b P_{i'}(\vec{x}) = a_i + (-1)^b a_{i'} = a_j$.

This proves (7).

We next show that the following holds in Game_1, Game_2 and Game_3:

$$\forall\, Q \in \mathsf{Span}(L),\ Q(\vec{x}) = 0 \tag{8}$$

(in the other games either L or $\vec{x}$ are not defined). For $L = \{Q_1, \ldots, Q_q\}$ if $Q \in \mathsf{Span}(L)$ then $Q = \sum_{k=1}^{q} \alpha_k Q_k$. To show (8), it suffices to show that for all $k \in [1,q]$ we have $Q_k(\vec{x}) = 0$.

For $k \in [0,q]$, Q_k is defined during the k-th call to DLog on some input ξ. In Game_1, the oracle finds i such that $\xi_i = \xi$ and sets $v := a_i$ and $Q_k := P_i - v$, so we get $Q_k(\vec{x}) = P_i(\vec{x}) - a_i$. Using the first result (7), we get that (8) holds. In Game_2 and Game_3 the oracle sets $v := P_i(\vec{x})$ so we directly get $Q_k(\vec{x}) = P_i(\vec{x}) - P_i(\vec{x}) = 0$

The third result we will use holds (assuming the game did not abort) in Game_1, Game_2, Game_3 and Game_4:

$$\forall j \geq 1\ \forall i \in [0, j-1] : \xi_j = \xi_i \Leftrightarrow P_j - P_i \in \mathsf{Span}(L) \ . \tag{9}$$

Game_2, $\boxed{\mathsf{Game}_3}$, $\boxed{\mathsf{Game}_4}$

$\vec{x} \coloneqq ()$
$j \coloneqq 0\,;\ q \coloneqq 0\,;\ n \coloneqq 0$
$P_0 \coloneqq 1\,;\ L \coloneqq \emptyset$
$\vec{y} \leftarrow \mathcal{A}^{\textsc{Chal},\textsc{DLog},\textsc{GCmp}}(\textsc{Enc}())$
if $q \geq n$ **then return** 0
for $i \in [1, n]$
  $x_i \coloneqq \textsc{DLog}(\xi_{j_i})$
return $\vec{y} = \vec{x}$

Oracle $\textsc{Chal}()$

$j \coloneqq j + 1\,;\ n \coloneqq n + 1$
$x_n \xleftarrow{\$} \mathbb{Z}_p\,;\ j_n \coloneqq j$
$P_j \coloneqq \mathsf{X}_n$
return $\textsc{Enc}()$

Oracle $\textsc{DLog}(\xi)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$ **then return** $\perp$
$i \coloneqq \min\{k \in [0, j] \mid \xi = \xi_k\}$
$q \coloneqq q + 1$
$v \coloneqq P_i(\vec{x})$ $\boxed{\ ;\ v \xleftarrow{\$} \mathbb{Z}_p}$
if $P_i \in \mathsf{Span}(1, L)$ **then**
  let $(\alpha_k)_{k=0}^{q-1} \in \mathbb{Z}_p^q$ **s.t.** $P_i = \alpha_0 + \sum_{k=1}^{q-1} \alpha_k Q_k$
  $v \coloneqq \alpha_0$
$Q_q \coloneqq P_i - v\,;\ L = L \cup \{P_i - v\}$

// Abort condition in Game_3 and Game_4 only
if $\exists (i_1, i_2) \in [0, j]^2$ **:** $P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$
  and $\xi_{i_1} \neq \xi_{i_2}$
  then abort game

return v

$\textsc{Enc}()$ // outputs ξ_j which encodes P_j

// Abort condition in Game_2 only
if $\exists i \in [0, j-1]$ **:** $P_j(\vec{x}) = P_i(\vec{x})$
  and $P_j - P_i \notin \mathsf{Span}(L)$
  then abort game

if $\exists i \in [0, j-1] : P_j - P_i \in \mathsf{Span}(L)$
  then $\xi_j \coloneqq \xi_i$
else
  $\xi_j \xleftarrow{\$} \{0,1\}^{\log(p)} \setminus \{\xi_i\}_{i\in[0,j-1]}$
return ξ_j

Oracle $\textsc{GCmp}(\xi, \xi', b)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$ **or** $\xi' \notin \{\xi_i\}_{i\in[0,j]}$
  then return $\perp$
$i \coloneqq \min\{k \in [0, j] \mid \xi = \xi_k\}$
$i' \coloneqq \min\{k \in [0, j] \mid \xi' = \xi_k\}$
$j \coloneqq j + 1$
$P_j \coloneqq P_i + (-1)^b P_{i'}$
return $\textsc{Enc}()$

Fig. 5: In Game_3 we move the abort condition from Enc to the oracle DLog, so it can be checked without using scalars. The only remaining use is then "$v \coloneqq P_i(\vec{x})$" in oracle DLog. Game_4 instead pick the output x uniformly at random.

We first prove

$$\forall j \geq 1\, \forall i \in [0, j-1] :\ \xi_j = \xi_i \Rightarrow P_j - P_i \in \mathsf{Span}(L)$$

by induction. We show that this holds for $j = 1$ and all other $j > 0$ and suppose that for some $i^* \in [0, j-1]$, $\xi_j = \xi_{i^*}$. We show that $P_j - P_{i^*} \in \mathsf{Span}(L)$.

– In Game_2, Game_3 and Game_4, since ξ_j is not a new random string when it is defined, thus for some $i_1 \in [0, j-1]$ we had $P_j - P_{i_1} \in \mathsf{Span}(L)$ and so the game defined $\xi_j \coloneqq \xi_{i_1}$. This implies that $\xi_{i_1} = \xi_{i^*}$, and since $i_1 < j$, using the induction hypothesis, we get that $P_{i_1} - P_{i^*} \in \mathsf{Span}(L)$ and furthermore

$$P_j - P_{i^*} = (P_j - P_{i_1}) - (P_{i_1} - P_{i^*}) \in \mathsf{Span}(L)\ .$$

Game_4

$j \coloneqq 0\,;\ q \coloneqq 0\,;\ n \coloneqq 0$
$P_0 \coloneqq 1\,;\ L \coloneqq \emptyset$
$\vec{y} \leftarrow \mathcal{A}^{\textsc{Chal},\textsc{DLog},\textsc{GCmp}}(\textsc{Enc}())$
if $q \geq n$ **then return** 0
for $i \in [1,n]$
  $x_i \coloneqq \textsc{DLog}(\xi_{j_i})$
return $\vec{y} = \vec{x}$

Oracle $\textsc{Chal}()$

$j \coloneqq j+1\,;\ n \coloneqq n+1$
$P_j \coloneqq \mathsf{X}_n\,;\ j_n \coloneqq j$
return $\textsc{Enc}()$

Oracle $\textsc{DLog}(\xi)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$ **then return** $\perp$
$i \coloneqq \min\{k \in [0,j] \mid \xi = \xi_k\}$
$q \coloneqq q+1\,;\ v \xleftarrow{\$} \mathbb{Z}_p$
if $P_i \in \mathsf{Span}(1, L)$ **then**
  let $(\alpha_k)_{k=0}^{q-1} \in \mathbb{Z}_p^q$ **s.t.** $P_i = \alpha_0 + \sum_{k=1}^{q-1} \alpha_k Q_k$
  $v \coloneqq \alpha_0$
$Q_q \coloneqq P_i - v\,;\ L = L \cup \{P_i - v\}$
if $\exists (i_1, i_2) \in [0,j]^2 : P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$
  and $\xi_{i_1} \neq \xi_{i_2}$
  then abort game
return v

$\textsc{Enc}()$ // outputs ξ_j which encodes P_j

if $\exists i \in [0, j-1] : P_j - P_i \in \mathsf{Span}(L)$
 then $\xi_j \coloneqq \xi_i$
else
  $\xi_j \xleftarrow{\$} \{0,1\}^{\log(p)} \setminus \{\xi_i\}_{i\in[0,j-1]}$
return ξ_j

Oracle $\textsc{GCmp}(\xi, \xi', b)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$ **or** $\xi' \notin \{\xi_i\}_{i\in[0,j]}$
  then return $\perp$
$i \coloneqq \min\{k \in [0,j] \mid \xi = \xi_k\}$
$i' \coloneqq \min\{k \in [0,j] \mid \xi' = \xi_k\}$
$j \coloneqq j+1$
$P_j \coloneqq P_i + (-1)^b P_{i'}$
return $\textsc{Enc}()$

Fig. 6: Final game Game_4 does not use $\vec{x}$ in the oracles anymore. It defines the challenge $\vec{x}$ after $\mathcal{A}$ gave its output and this is what makes it simple for us to prove it is hard to win for $\mathcal{A}$.

Now the situation is simpler when $j = 1$: we must have $i_1 = i^* = 0$ so

$$P_j - P_{i_1} = P_j - P_{i^*} = P_1 - P_0 \in \mathsf{Span}(L)\ .$$

– In Game_1 the proof is almost the same: since ξ_j is not a new random string, thus for some $i_1 \in [0, j-1]$ we had $P_j(\vec{x}) = P_{i_1}(\vec{x})$, so the game defined $\xi_j \coloneqq \xi_{i_1}$. Since the game did not abort, "$P_j(\vec{x}) = P_{i_1}(\vec{x})$ **and** $P_j - P_{i_1} \notin \mathsf{Span}(L)$" does not hold, and thus $P_j - P_{i_1} \in \mathsf{Span}(L)$. From here the proof proceeds as for the other games above, and thus $P_j - P_{i^*} \in \mathsf{Span}(L)$. When $j = 1$, we have $i^* = 0$ and $P_1 - P_0 \in \mathsf{Span}(L)$, as otherwise the game aborts.

We now prove the other implication:

$$\forall j \geq 1\, \forall i \in [0, j-1] : P_j - P_i \in \mathsf{Span}(L) \Rightarrow \xi_j = \xi_i\ ,$$

again by induction. Using the same method as before we can argue that this is true for $j = 1$. For $j > 1$, when $\textsc{Enc}()$ defines ξ_j, if for some $i^* \in [0, j-1]$ we have $P_j - P_{i^*} \in \mathsf{Span}(L)$ then we show that ξ_j is assigned $\xi_j = \xi_{i^*}$.

– In Game_2, Game_3 and Game_4, since for some $i_1 \in [0, j-1] : P_j - P_{i_1} \in \mathsf{Span}(L)$, the game defines $\xi_j \coloneqq \xi_{i_1}$. And since

$$P_{i^*} - P_{i_1} = (P_{i^*} - P_j) + (P_j - P_{i_1}) \in \mathsf{Span}(L)\ ,$$

by induction we get $\xi_{i_1} = \xi_{i^*}$ which yields $\xi_j = \xi_{i^*}$.

– In Game_1, since we know that $(P_j - P_{i^*})(\vec{x}) = 0$ from the (8), we get that for some $i_1 \in [0, j-1] : P_j(\vec{x}) = P_{i_1}(\vec{x})$. Since the game did not abort, we know that $P_j - P_{i_1} \in \mathsf{Span}(L)$, so by the same argument as before, we get $\xi_j = \xi_{i^*}$.

Game_0 TO Game_1. We now compare Game_0 to Game_1. The only difference between the two is when Game_1 aborts in the procedure ENC() on event

$$\exists i \in [0, j-1] \textbf{ such that } P_j(\vec{x}) = P_i(\vec{x}) \textbf{ and } P_j - P_i \notin \mathsf{Span}(L) \ . \tag{10}$$

We call this event F. Since ENC is called at most m times, we get:

$$\mathbf{Adv}^{\mathsf{Game}_0}_{\mathcal{A}} \leq \mathbf{Adv}^{\mathsf{Game}_1}_{\mathcal{A}} + m \cdot \Pr[F] \ . \tag{11}$$

We now upper-bound $\Pr[F]$. Before a call to ENC, the oracle defines P_j. Consider a fixed $i \in [0, j-1]$ and define $P \coloneqq P_j - P_i$. We will upper-bound the probability that

$$P_j(\vec{x}) - P_i(\vec{x}) = P(\vec{x}) = 0$$

with $P \coloneqq P_j - P_i \notin \mathsf{Span}(L)$.

Since $\mathcal{A}$ does not know $\vec{x}$ one might consider applying the Schwartz-Zippel lemma. But we cannot, since $\mathcal{A}$ knows information on $\vec{x}$. From $\mathcal{A}$'s point of view, $\vec{x}$ is not uniformly chosen from $\mathbb{Z}_p^n$, since it satisfies $Q(\vec{x}) = 0$ for all $Q \in L$ (using (8)). We write $L = \{Q_1, \ldots, Q_q\}$, and using the notation from Lemma 1 $Q_{q+1} \coloneqq P$.

$\mathcal{A}$ also knows that if for some indexes i_1, i_2 it was given $\xi_{i_1} \neq \xi_{i_2}$ then $P_{i_1}(\vec{x}) \neq P_{i_2}(\vec{x})$. We can reformulate this by writing $D_{\vec{\imath}} = P_{i_1} - P_{i_2}$ for $\vec{\imath} \in I \coloneqq \{(i_1, i_2) \in [0, j-1]^2 \,|\, \xi_{i_1} \neq \xi_{i_2}\}$. $\mathcal{A}$ knows that $D_{\vec{\imath}}(\vec{x}) \neq 0$. Using the notation of Lemma 1 we get that

$$\vec{x} \in \mathcal{C} \coloneqq \Big(\bigcap_{j \in [1,q]} \mathcal{Q}_j \Big) \setminus \Big(\bigcup_{i \in I} \mathcal{D}_i \Big) \ .$$

Our goal is to apply Lemma 1 to upper-bound $\Pr_{\vec{x} \leftarrow \mathcal{C}}[P(\vec{x}) = 0]$. We need to verify that the three premises of the lemma are satisfied, which are: from $\mathcal{A}$'s point of view, $\vec{x} \in \mathcal{C}$ is picked uniformly at random, $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$ and $\vec{Q}_{q+1}$ is independent of $(\vec{Q}_i)_{i \in [1,q]}$.

$\vec{x}$ IS CHOSEN UNIFORMLY IN $\mathcal{C}$. To show this, we fix the randomness (of the challenger and the adversary) of the game (which means the order in which the ξ_i are picked is deterministic) and we consider the transcript $\pi(\vec{x})$ of what $\mathcal{A}$ sees during the game when the secret is chosen as $\vec{x}$: $\pi(\vec{x}) = (\xi_0, \ldots, \xi_{j-1}, v_1, \ldots, v_q)$ (In this transcript, the strings ξ_i are ordered and so are the v_i, but we implicitly suppose that before the query v_k there was a query v_{k-1} or ξ_{i_k} and after the query v_k there was either a query v_{k+1} or $\xi_{i'_k}$. We do not formalize this.)

The transcript π corresponds to all the output of the oracles that were given to $\mathcal{A}$: The ξ_i are the outputs of GCMP and CHAL, and the v_i are the outputs of DLOG. The transcript $\pi(\vec{x})$ only depends on the challenge $\vec{x}$. What is important to notice is that for all $\vec{y} \in \mathcal{C}$: $\pi(\vec{y}) = \pi(\vec{x})$. Indeed, if we call $\pi(\vec{y}) = (\xi'_0, \ldots, \xi'_{j-1}, v'_1, \ldots, v'_q)$ we can show by induction that $\xi'_i = \xi_i$ and $v'_k = v_k$ for all $i \in [1, j-1]$ and $k \in [1, q]$.

– Let $k \in [1, q]$; we show that $v_k = v'_k$: in both challenges $\vec{x}$ and $\vec{y}$, since the transcript $\mathcal{A}$ received is the same by the induction hypothesis, it behaves the same way and calls DLOG on input ξ. The oracle DLOG then picks $i = \min\{j \,|\, \xi_j = \xi\}$ which is the same in both cases by the induction hypothesis. DLOG computes $v_k = P_i(\vec{x})$ and defines $Q_i \coloneqq P_i - v_k$ for the challenge $\vec{x}$ while it computes $v'_k = P_i(\vec{y})$ and $Q'_i \coloneqq P_i - v'_k$ for the challenge $\vec{y}$. Now Since $\vec{y} \in \mathcal{C}$, we have in particular $\vec{y} \in \mathcal{Q}_i$, so we know that $Q_k(\vec{y}) = P_i(\vec{y}) - v_k = 0$. This gives $P_i(\vec{y}) = v'_k = v_k$ and $Q'_k = Q_k$.

– Let $k \in [1, j-1]$; we show that $\xi_k = \xi'_k$: for both challenges $\vec{x}$ and $\vec{y}$, since the transcript $\mathcal{A}$ received is the same by induction hypothesis, it behaves the same way and calls either CHAL or GCMP. In both cases the the game creates a polynomial P_k and calls the procedure ENC(), for which there are two cases:

  1: $\forall\, i \in [0, k-1] : P_k(\vec{x}) \neq P_i(\vec{x})$. The game with challenge $\vec{x}$ outputs a new random ξ_k, which means $\xi_k \neq \xi_i$ for $i \in [1, k-1]$. Since $\vec{y} \in \mathcal{C}$, we know that for all $i \in [0, k-1]$, $\vec{y} \notin \mathcal{D}_{i,k} = \{\vec{z} \ : \ (P_i - P_k)(\vec{z}) = 0 \textbf{ and } \xi_i \neq \xi_k\}$ This means that for all $i \in [0, k-1]$, since $\xi_i \neq \xi_k$, we have $P_i(\vec{y}) \neq P_k(\vec{y})$, so the game also chooses ξ'_k as a new random string. Since we fixed the randomness of the game, we get $\xi_k = \xi'_k$.

  2: $\exists\, i^* \in [0, k-1] : P_k(\vec{x}) = P_{i^*}(\vec{x})$. The game defines $\xi_k \coloneqq \xi_i$ for the challenge $\vec{x}$. Since the game did not abort for $k < j$, we know that $P_k - P_{i^*} \in \mathsf{Span}(L)$. Now since $L = (Q_i)_i$ and $\vec{y} \in \bigcap_{i \in [1,q]} \mathcal{Q}_i$, we also get $(P_k - P_{i^*})(\vec{y}) = 0$. So the game defines $\xi'_k \coloneqq \xi'_i = \xi_i = \xi_k$, by the induction hypothesis and the preliminary result (9).

  In both cases we get that $\xi_k = \xi'_k$.

Since the transcript that $\mathcal{A}$ sees is the same for all elements in $\mathcal{C}$, $\mathcal{A}$ can only make a uniform guess on which element of $\mathcal{C}$ is the challenge. Thus from $\mathcal{A}$'s point of view, $\vec{x}$ is chosen uniformly at random in $\mathcal{C}$.

$\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$. Since $\mathcal{Q}_{q+1} = \{\vec{x} \in \mathbb{Z}_p : P(\vec{x}) = 0\}$, if we had $\mathcal{C} \cap \mathcal{Q}_{q+1} = \emptyset$, then $P(\vec{x}) \neq 0$ for all $\vec{x} \in \mathcal{C}$, and thus $\Pr_{\vec{x} \xleftarrow{\$} \mathcal{C}}[P(\vec{x}) = 0] = 0$. In this case, there is no need to upper-bound the probability, which is why we assume that $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$.

$\vec{Q}_{q+1}$ IS INDEPENDENT OF $(\vec{Q}_i)_{i \in [1,q]}$. Recall that $\vec{P} = (p_k)_{k \in [1,n]}$ is the vector representing the polynomial $P - P(\vec{0}) = \sum_{k=1}^{n} p_k \mathsf{X}_k$. We assume that $\vec{Q}_{q+1}$ is dependent of $(\vec{Q}_i)_{i \in [1,q]}$ and then show that this contradicts the previous premise $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$. Assume thus that for some α:

$$Q_{q+1} - Q_{q+1}(\vec{0}) = \sum_{k=1}^{q} \alpha_k (Q_k - Q_k(\vec{0})) \ .$$

With $\alpha \coloneqq Q_{q+1}(\vec{0}) + \sum_{k=1}^{q} \alpha_k Q_k(\vec{0})$ and $Q \coloneqq \sum_{k=1}^{q} \alpha_k Q_k$, we can write this as $Q_{q+1} = \alpha + Q$ with $\alpha \in \mathbb{Z}_p$ and $Q \in \mathsf{Span}(L)$. Now since we are in event F, defined in (10), we have $Q_{q+1} = P \notin \mathsf{Span}(L)$, which implies $\alpha \neq 0$ (otherwise $P = Q \in \mathsf{Span}(L)$). Since $\mathcal{C} \subset \mathcal{Q}_i$ we have that for all $i \in [1, q]$ and all $\vec{x} \in \mathcal{C}$: $Q_i(\vec{x}) = 0$, and thus $Q(\vec{x}) = 0$. From this, we have $Q_{q+1}(\vec{x}) = \alpha + Q(\vec{x}) = \alpha$. Thus, $Q_{q+1}(\vec{x}) \neq 0$ for all $\vec{x} \in \mathcal{C}$, which implies $\mathcal{C} \cap \mathcal{Q}_{q+1} = \emptyset$, which contradicts the previous assumption. We thus proved that $\vec{Q}_{q+1}$ is independent of $(\vec{Q}_i)_{i \in [1,q]}$.

APPLYING LEMMA 1. Since all its premises are satisfied, we can apply Lemma 1 and obtain:

$$\Pr_{\vec{x} \leftarrow \mathcal{C}}\left[P(\vec{x}) = 0\right] = \Pr_{\vec{x} \leftarrow \mathcal{C}}\left[Q_{q+1}(\vec{x}) = 0\right] \leq \frac{1}{p - |I|} \ ,$$

with $|I| \leq j^2 \leq m^2$. Since we need to test this with $P = P_j - P_i$ for all $i \in [0, j-1]$, we get $\Pr[F] \leq \dfrac{m}{p - m^2}$ and from (11):

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_0} \leq \mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_1} + \frac{m^2}{p - m^2} \ . \tag{12}$$

Game_1 TO Game_2. There are three changes in Game_2, which we show do not affect the distributions of the game. First, we replace a_i by $P_i(\vec{x})$ in oracle DLOG, which is equivalent by (7).

Second, in ENC, we replace the condition

$$\textbf{if } \exists i \in [0, j-1] : P_j(\vec{x}) = P_i(\vec{x}) \textbf{ then } \xi_j \coloneqq \xi_i$$

by

$$\textbf{if } \exists i \in [0, j-1] : P_j - P_i \in \mathsf{Span}(L) \textbf{ then } \xi_j \coloneqq \xi_i \ .$$

We show that this new condition does not affect the output of ENC(). There are two cases for $P_j(\vec{x})$:

Case 1: $\exists i^* \in [0, j-1] : P_j(\vec{x}) = P_{i^*}(\vec{x})$. We have either
- $P_j - P_{i^*} \in \mathsf{Span}(L)$, and in this case Game_1 and Game_2 both set $\xi_j = \xi_{i^*}$ and output ξ_j using (9); or
- $P_j - P_{i^*} \notin \mathsf{Span}(L)$, meaning that both Game_1 and Game_2 abort since "$P_j - P_{i^*} \notin \mathsf{Span}(L)$ **and** $P_j(\vec{x}) = P_{i^*}(\vec{x})$" is the abort condition.

Case 2: $\forall i \in [0, j-1]\ P_j(\vec{x}) \neq P_i(\vec{x})$. Since, by 8, all polynomials in $\mathsf{Span}(L)$ vanish at $\vec{x}$, this implies $\forall i \in [0, j-1] : P_j - P_i \notin \mathsf{Span}(L)$. In this case both Game_1 and Game_2 output a random new string ξ_j.

The third change in Game_2, in the oracle DLOG, does not change the output either: in Game_1 the DLOG oracle always outputs $a_i = P_i(\vec{x})$. In Game_2, when $P_i \in \mathsf{Span}(L)$, the game uses the decomposition $P_i = \alpha_0 + \sum_{k=1}^{q-1} \alpha_k Q_k$, and since $Q_k(\vec{x}) = 0$ by (8), it outputs $P_i(\vec{x}) = \alpha_0$.

Together this yields:

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_1} = \mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_2} \ . \tag{13}$$

Game_2 TO Game_3. In this game hop we move the abort condition from the procedure ENC to the oracle DLOG. We show that the two abort conditions are equivalent, by showing the two implications of the equivalence:

IF Game_2 ABORTS THEN Game_3 ALSO ABORTS. If Game_2 aborts, it means that for a fixed index j^* the game found $i^* \in [0, j^*-1]$ such that $P_{j^*} - P_{i^*} \notin \mathsf{Span}(L)$ and $P_{j^*}(\vec{x}) = P_{i^*}(\vec{x})$. We show that Game_3 also aborts in this situation. Let $P := P_{j^*} - P_{i^*}$. At the end of Game_3 the challenger makes calls to DLOG on each challenge $P_{j_i} = \mathsf{X}_i$. This adds the corresponding polynomials $\mathsf{X}_i - x_i$ to L for all $i \in [1, n]$. With $P = P(\vec{0}) + \sum_{k=1}^{n} p_k \mathsf{X}_k$, we can write

$$P = \sum_{k=1}^{n} p_k(\mathsf{X}_k - x_k) + P(\vec{0}) + \sum_{k=1}^{n} p_k x_k \ .$$

Since $P(\vec{x}) = P_{j^*}(\vec{x}) - P_{i^*}(\vec{x})$, we have $P(\vec{x}) = 0$. On the other hand, by the equation above, we have $P(\vec{x}) = P(\vec{0}) + \sum_{k=1}^{n} p_k x_k$. Together, this yields $P = \sum_{k=1}^{n} p_k(\mathsf{X}_k - x_k)$, which means $P \in \mathsf{Span}(L)$ at the end of the game. At the time when Game_2 would have aborted, we had $P \notin \mathsf{Span}(L)$ and thus the game attributed two different strings $\xi_{i^*} \neq \xi_{j^*}$ to P_{i^*} and P_{j^*}, respectively. But at the end of Game_3, when L contains all $\mathsf{X}_i - x_i$ for $i \in [1, n]$, we have $P \in \mathsf{Span}(L)$. This means that one call to DLOG updated L so that $P \in \mathsf{Span}(L)$ and when this happened, since $\xi_{i^*} \neq \xi_{j^*}$, the abort condition in DLOG was satisfied and the game aborted

IF Game_3 ABORTS THEN Game_2 ALSO ABORTS. If Game_3 aborts, then on a call to DLOG we have $\exists (i_1, i_2) \in [0, j]^2$ such that $P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$ and $\xi_{i_1} \neq \xi_{i_2}$. From $P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$, using (8) we get $P_{i_1}(\vec{x}) = P_{i_2}(\vec{x})$. Suppose $i_1 < i_2$. The challenger in Game_2 used the procedure ENC() when the counter j was equal to i_2 to compute $\xi_{i_2} \neq \xi_{i_1}$. This means that at that moment, L contained fewer elements and we had $P_{i_2} - P_{i_1} \notin \mathsf{Span}(L)$. Since Game_2 aborts when $P_{i_1}(\vec{x}) = P_{i_2}(\vec{x})$ and $P_{i_2} - P_{i_1} \notin \mathsf{Span}(L)$, thus Game_2 aborts in this case.

Combining both implications yields

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_2} = \mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_3} \ . \tag{14}$$

Game_3 TO Game_4. The only difference between these games is in the oracle DLOG. Instead of computing $v \coloneqq P_i(\vec{x})$, Game_4 picks a random $v \xleftarrow{\$} \mathbb{Z}_p$. We prove that after this modification, the distribution of the outputs of oracle DLOG remains the same. The difference between the two games occurs only when $P_i \notin \mathsf{Span}(1, L)$. Let us bound $\Pr_{\vec{x} \leftarrow \mathcal{C}}\left[P_i(\vec{x}) = v \textbf{ in } \mathsf{Game}_3\right]$, where $\vec{x} \in \mathcal{C}$ represents the information that $\mathcal{A}$ knows about $\vec{x}$, which we previously used in the first game hop.

We apply Lemma 1 again to bound this probability. Now since the game does not abort immediately when the inconsistency $P_{i_1}(\vec{x}) = P_{i_2}(\vec{x})$ and $\xi_{i_1} \neq \xi_{i_2}$ occurs, the inequalities on the strings level do not give $\mathcal{A}$ any information on what the evaluation $P_i(\vec{x})$ cannot be. This means that $\mathcal{C}$ is simpler than in the first game hop, namely

$$\mathcal{C} = \bigcap_{i \in [1,q]} \mathcal{Q}_i \ .$$

We define $Q_{q+1} \coloneqq P_i - v$ and show that once again the three premises of Lemma 1 hold: $\vec{x} \in \mathcal{C}$ is picked uniformly at random, $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$ and $\vec{Q}_{q+1}$ is independent of $(\vec{Q}_i)_{i \in [1,q]}$.

$\vec{x}$ IS CHOSEN UNIFORMLY IN $\mathcal{C}$. To show this, we again fix the randomness of the game and consider the transcript π that $\mathcal{A}$ sees during the game if a particular $\vec{x}$ is chosen: $\pi(\vec{x}) = (\xi_0, \ldots, \xi_{j-1}, v_1, \ldots, v_q)$, which contains all oracle outputs given to $\mathcal{A}$. We show that for all $\vec{y} \in \mathcal{C} : \pi(\vec{y}) = \pi(\vec{x})$. Indeed, for $\pi(\vec{y}) =: (\xi'_0, \ldots, \xi'_{j-1}, v'_1, \ldots, v'_q)$ we show by induction that $\xi'_i = \xi_i$ and $v'_k = v_k$ for all $i \in [1, j-1]$ and $k \in [1, q]$.

- Let $k \in [1, q]$; then $v_k = v'_k$ is showed exactly as in the first game hop (on page 17).
- Let $k \in [1, j-1]$; we show that $\xi_k = \xi'_k$: for both challenges $\vec{x}$ and $\vec{y}$, since the transcript $\mathcal{A}$ received is the same by induction hypothesis, $\mathcal{A}$ behaves the same way and calls either CHAL or GCMP. In both cases the game creates a polynomial P_k and calls ENC(), for which there are two cases:
  1: $\forall\, i \in [0, k-1] : P_k - P_i \notin \mathsf{Span}(L)$. Since this condition is independent of $\vec{x}$ and $\vec{y}$, for both the game outputs a new random string ξ_k and ξ'_k. Since we fixed the randomness of the game, we get $\xi_k = \xi'_k$.
  2: $\exists i^* \in [0, k-1] : P_k - P_{i^*} \in \mathsf{Span}(L)$. In this case the game defines $\xi_k \coloneqq \xi_i$ and $\xi'_k \coloneqq \xi'_i$ for both challenge $\vec{x}$ and $\vec{y}$. We get $\xi'_k \coloneqq \xi'_i = \xi_i = \xi_k$ by the induction hypothesis and (9).
  In both cases we thus have $\xi_k = \xi'_k$.

As in first game hop, we conclude that $\mathcal{A}$ cannot distinguish between two different values $\vec{x} \in \mathcal{C}$ and so we can consider $\vec{x}$ to be chosen uniformly at random in $\mathcal{C}$.

$\vec{Q}_{q+1}$ IS LINEARLY INDEPENDENT OF $(\vec{Q}_i)_{i \in [1,q]}$. Recall that $P_i \notin \mathsf{Span}(1, L)$ and $Q_{q+1} \coloneqq P_i - v$. If $\vec{Q}_{q+1}$ were linearly dependent of $(\vec{Q}_i)_{i \in [0,j]}$, then (using the same method as in the first game hop) we would have $Q_{q+1} = P_i - v = \alpha + Q$ with $\alpha \in \mathbb{Z}_p$ and $Q \in \mathsf{Span}(L)$. As this contradicts $P_i \notin \mathsf{Span}(1, L)$, we conclude that $\vec{Q}_{q+1}$ is linearly independent of $(\vec{Q}_i)_{i \in [1,q]}$.

$\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$. $\mathcal{C} = \bigcup_{i \in [1,q]} \mathcal{Q}_i$ is an affine space and $\vec{Q}_{q+1}$ is linearly independent of $(\vec{Q}_i)_{i \in [0,j]}$. This implies that $\mathcal{Q}_{q+1} \cap \mathcal{C}$ has dimension $\dim(\mathcal{C}) - 1$ and thus $\mathcal{Q}_{q+1} \cap \mathcal{C} \neq \emptyset$.

APPLYING LEMMA 1. Since its three premises are satisfied, Lemma 1 with $M \coloneqq 0$ yields:

$$\Pr[\, Q_{q+1}(\vec{x}) = 0 \,]_{\vec{x} \leftarrow \mathcal{C}} = \Pr_{\vec{x} \leftarrow \mathcal{C}}\left[P_i(\vec{x}) = v \textbf{ in } \mathsf{Game}_3\right] = \frac{1}{p} \ .$$

This means that in Game_3 the distribution of $P_i(\vec{x})$ is uniform, so the change we make does not affect the overall distribution of the game. We thus have

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_3} = \mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_4} \ . \tag{15}$$

Analysis of Game_4. We prove that $\mathcal{A}$ wins Game_4 at most with negligible probability $\frac{1}{p}$. To do this, we prove that at least one component of the vector $\vec{x}$ is picked uniformly at random *after* $\mathcal{A}$ outputs $\vec{y}$.

When $\mathcal{A}$ outputs $\vec{y}$, L contains q elements, so $\mathsf{dim}(\mathsf{Span}(L)) \leq q$. Since $q < n$, $\mathsf{Span}(1, L)$ has dimension at most $q+1$ and therefore at most n when the adversary outputs the vector $\vec{y}$. Since the dimension of $\mathsf{Span}(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ is n and $1 \notin \mathsf{Span}(\mathsf{X}_1, \ldots, \mathsf{X}_n)$, we get that $\mathsf{Span}(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ is not contained in $\mathsf{Span}(1, L)$. This means that there will be at least one index $i \in [1, n]$ such that $\mathsf{X}_i \notin \mathsf{Span}(1, L)$. We choose the smallest index i that verifies this. Then the oracle DLog outputs a randomly sampled value x_i when called on ξ_{j_i}. This x_i is sampled randomly after the i-th coefficient of vector $\vec{y}$ output by $\mathcal{A}$ and we obtain: $\Pr[\vec{x} = \vec{y}] \leq \frac{1}{p}$. This yields:

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_4} \leq \frac{1}{p}. \tag{16}$$

The theorem now follows from Equations (12), (13), (14), (15), and (16) □

4 OMCDH in the GGM

The OMCDH assumption (defined in Figure 2), though, similar to the OMDL assumption, is slightly more complex. In OMDL the adversary has access to a DLog oracle and must solve DLog challenges; in OMCDH the adversary has access to a CDH oracle and must solve CDH challenges. In particular, the CDH oracle in OMCDH enables the adversary to construct (encodings of) group elements that correspond to high-degree polynomials since on input $(\Xi(x), \Xi(y))$, the oracle returns $\Xi(xy)$, which in the "ideal" game is encoded as the polynomial XY of degree 2 by the challenger. This makes using known proof techniques in the GGM impossible, since if $\mathcal{A}$ is not constrained in terms of the degree, it can build non-zero polynomials that evaluate to zero on the challenge with non-negligible probability. (E.g., $\mathsf{X}^p - \mathsf{X}$ evaluates to 0 everywhere in $\mathbb{Z}_p$.)

Given this situation, we can neither use the Schwartz-Zippel lemma (this lemma would yield a non-negligible bound on the adversary's advantage) nor Lemma 1 (since it only applies for polynomials of degree 1). In fact, some cryptanalysis in the literature (e.g., the attacks by Maurer and Wolf [MW96, Mau99]) use high-degree polynomials to break DL in group of order p (when $p-1$ is smooth) when given a CDH oracle.

Since the generic group model does not handle high-degree polynomials well, in order to analyze the hardness of OMCDH, we decided to consider a *stronger* assumption instead, which we call OMCDH^{DL} and define in Figure 7. This problem is analog to OMCDH, except that the CDH oracle is replaced by a DLog oracle. As the adversary has access to the same oracles as in the game OMDL, as seen in the OMDL proof, it can only build polynomials of degree at most 1. Actually, as we show in Appendix A, OMCDH^{DL} implies OMDL (Proposition 2) and we also prove that (modulo a polynomial number of group operations) OMCDH^{DL} implies OMCDH (Proposition 1).

In Appendix B we formally prove the hardness of OMCDH^{DL} in the generic group model. This is done following the same strategy as for OMDL in Theorem 1 (Section 3); the games hops are the same, only the final analysis of the last game is different, since the winning condition is different. This leads to a different winning probability at the end. This is summarized in Theorem 3 below.

Proposition 1 (OMCDH^{DL} implies OMCDH). *In a cyclic group of order p, let $\mathcal{A}$ be an adversary that solves OMCDH by using at most m group operations and q calls to* DLog. *We can build an adversary $\mathcal{B}$ that solves OMCDH^{DL} using at most $m + 2q\lceil\log(p)\rceil$ group operations.*

The proof is straightforward by answering the CDH oracle queries by making queries to the DLog oracle. It is formally given as Proposition 3 in Appendix A.

Game $\mathrm{OMCDHDL}^{\mathcal{A}}_{\mathsf{GrGen}}(\lambda)$	Oracle $\textsc{Chal}()$	Oracle $\textsc{DLog}(X)$
$(p, \mathbb{G}, G) \leftarrow \mathsf{GrGen}(1^\lambda)$	$x \xleftarrow{\$} \mathbb{Z}_p\,;\ X \coloneqq xG$	$q \coloneqq q + 1$
$\vec{Z} \coloneqq (\,)\,;\ q \coloneqq 0$	$y \xleftarrow{\$} \mathbb{Z}_p\,;\ Y \coloneqq yG$	$x \coloneqq \log_G(X)$
$\vec{Z}' \leftarrow \mathcal{A}^{\textsc{Chal},\textsc{DLog}}(p, \mathbb{G}, G)$	$\vec{Z} \coloneqq \vec{Z} \,\|\, (xyG)$	**return** x
return $\left(\vec{Z} = \vec{Z}' \ \wedge\ q < \lvert\vec{Z}\rvert\right)$	**return** (X, Y)	

Fig. 7: The $\mathrm{OMCDH}^{\mathrm{DL}}$ problem

Theorem 2. *Let $\mathcal{A}$ be an adversary that solves $OMCDH^{DL}$ in a generic group of order p, making at most m oracle queries. Then*

$$\boldsymbol{Adv}^{\mathit{OMCDH\ GGM}}_{\mathcal{A}} \leq \frac{1}{p-1} + \frac{2m}{p} + \frac{m^2}{p-m^2}\ .$$

A formal proof of the theorem can be found in Appendix B. Combining this with Proposition 1, we obtain the following corollary, which proves the security of OMCDH in the generic group model.

Corollary 1. *Let $\mathcal{A}$ be an adversary that solves $OMCDH^{DL}$ in a generic group of order p, making at most m oracle queries and q* CDH *oracle queries. Then*

$$\boldsymbol{Adv}^{\mathit{OMCDH\ GGM}}_{\mathcal{A}} \leq \frac{1}{p-1} + \frac{2(m+2q\lceil\log(p)\rceil)}{p} + \frac{(m+2q\lceil\log(p)\rceil)^2}{p-(m+2q\lceil\log(p)\rceil)^2}\ .$$

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Adversary $\mathcal{B}_1^{\mathcal{A}_1,\textsc{Chal},\textsc{DLog}}(p,\mathbb{G},G)$	Oracle $\textsc{Chal}'()$
$\vec{Y} \coloneqq () \,;\ \vec{x} \leftarrow \mathcal{A}_1^{\textsc{Chal}',\textsc{DLog}}(1^\lambda)$	$X, Y \coloneqq \textsc{Chal}()$
return $\big(\textsc{SqMul}(x_1, Y_1), \ldots, \textsc{SqMul}(x_n, Y_n)\big)$	$\vec{Y}, \coloneqq \vec{Y} \,\|\, (Y)$
	return X

Fig. 8: Adversary $\mathcal{B}_1$ against OMCDH^{DL}

A Comparison of OMCDH^{DL} to Other Assumptions

We want to prove that OMCDH^{DL} is easier than OMDL and OMCDH. To show these two properties, we will use the well-known square-and-multiply algorithm.

Lemma 2. *Let $\mathbb{G}$ be a group of order p. The square-and-multiply* SqMul *algorithm in $\mathbb{G}$ takes as input a scalar $a \in \mathbb{Z}_p$, and a group element $X \in G$, and returns aX after computing at most $2\lceil \log(p) \rceil$ group operations.*

Proposition 2 (OMCDH^{DL} is easier than OMDL). *In a cyclic group of order p, let $\mathcal{A}_1$ be an adversary that solves OMDL using at most m group operations and n challenge oracle calls. Then we can build an adversary $\mathcal{B}_1$ solving OMCDH^{DL} using at most $m + 2n\lceil \log(p) \rceil$ group operations.*

Proof. We define $\mathcal{B}_1$ in Figure 8. $\mathcal{A}_1$ plays in OMDL and $\mathcal{B}_1$ in OMCDH^{DL}. $\mathcal{B}_1$ gives $\mathcal{A}_1$ access to its own oracle DLog. Then for the oracle $\textsc{Chal}'$, $\mathcal{B}_1$ makes one query to its own oracle Chal, receiving the pair (X, Y), and outputs the challenge X to $\mathcal{A}_1$. Using the oracles DLog and $\textsc{Chal}'$, $\mathcal{A}_1$ can play in OMDL.

Let q be the number of queries made by $\mathcal{A}_1$ to the oracle DLog. Then $\mathcal{B}_1$ also uses the DLog oracle q times. When $\mathcal{A}_1$ wins this game we have $n > q$ and $\vec{x} = (x_1, \ldots, x_n)$ with $x_i = \log_G(X_i)$. Since n is the number of queries made by $\mathcal{A}_1$ to $\textsc{Chal}'$, it is also the number of queries made by $\mathcal{B}_1$ to Chal. Thus, $\mathcal{B}_1$ outputs

$$(x_1 Y_1, \ldots, x_n Y_n) = (x_1 y_1 G, \ldots, x_n y_n G) \ ,$$

which is the right answer to win OMCDH^{DL}. And since moreover $n > q$, it implies $\mathcal{B}_1$ wins the game OMCDH^{DL}. We upper-bound the number of group operations of $\mathcal{B}_1$ by noting that $\mathcal{B}_1$ uses group operations only trough $\mathcal{A}_1$ and SqMul and by combining Lemma 2 and the fact that the number of executions of SqMul is equal to n. □

Adversary $\mathcal{B}_2^{\mathcal{A}_2,\textsc{Chal},\textsc{DLog}}(p,\mathbb{G},G)$	Oracle $\mathrm{CDH}'(X,Y)$
$\vec{Z}' \leftarrow \mathcal{A}_2^{\textsc{Chal},\mathrm{CDH}'}(1^\lambda)$	$x \coloneqq \textsc{DLog}(X)$
return $(\vec{Z}')$	$Z \coloneqq \textsc{SqMul}(x,Y)$
	return Z

Fig. 9: Adversary $\mathcal{B}_2$ against $\mathrm{OMCDH}^{\mathrm{DL}}$

Proposition 3 ($\mathrm{OMCDH}^{\mathrm{DL}}$ is easier than OMCDH). *In a cyclic group of order p, let $\mathcal{A}_2$ be an adversary that solves OMCDH by using at most m group operations and q $\textsc{DLog}$ oracle calls. Then we can build an adversary $\mathcal{B}_2$ that solves OMCDH^{DL} using at most $m + 2q\lceil\log(p)\rceil$ group operations.*

Proof. We show in Figure 9 how to build $\mathcal{B}_2$ using $\mathcal{A}_2$. The adversary $\mathcal{B}_2$ plays in $\mathrm{OMCDH}^{\mathrm{DL}}$ and gives $\mathcal{A}_2$ access to the oracles CDH and $\textsc{Chal}$ in order to simulate game OMCDH to $\mathcal{A}_2$. The oracle $\textsc{Chal}$ is the same for $\mathcal{A}_2$, and the oracle CDH' is defined using the oracle $\textsc{DLog}$ that $\mathcal{B}_2$ has an access to. On input (X,Y), with $X \coloneqq xG$ and $Y \coloneqq yG$, the oracle CDH' should output $Z \coloneqq xyG$. It therefore uses the oracle $\textsc{DLog}$ to compute x and then uses $\textsc{SqMul}$ to compute $Z \coloneqq xY$.

Let n be the number of queries made by $\mathcal{A}_2$ to $\textsc{Chal}$. Then $\mathcal{B}_2$ also calls the $\textsc{Chal}$ oracle n times. When $\mathcal{A}_2$ wins this game, we have $n > q$ and $\vec{Z}' = (\mathrm{CDH}(X_1,Y_1),\ldots,\mathrm{CDH}(X_1,Y_n))$. Thus $\mathcal{B}_2$ wins the game $\mathrm{OMCDH}^{\mathrm{DL}}$.

We upper-bound the number of group operations of $\mathcal{B}_2$ by noting that $\mathcal{B}_2$ uses group operations only via $\mathcal{A}_2$ and $\textsc{SqMul}$ and by combining Lemma 2 and the fact that the number of executions of $\textsc{SqMul}$ is equal to q. □

B $\mathrm{OMCDH}^{\mathrm{DL}}$ in the Generic Group Model

In this section we show that $\mathrm{OMCDH}^{\mathrm{DL}}$ is hard in the GGM. We first argue that Game_0, defined in Figure 10, describes $\mathrm{OMCDH}^{\mathrm{DL}}$ in the GGM. As with OMDL, we show that all the modifications we made for convenience in Game_0 do not affect the result of the game and its behavior. In fact the only important modification in Game_0 for the initial $\mathrm{OMCDH}^{\mathrm{DL}}$ is at the end of the game, after the adversary outputs $\vec{Z}'$.

We see that since j_n is defined in the oracle $\textsc{Chal}$ such that $a_{j_n} = x_n$ and $a_{j_n+1} = y_n$, the final calls to $\textsc{DLog}$:

$$x_i \coloneqq \textsc{DLog}(\xi_{j_i}) \textbf{ and } y_i \coloneqq \textsc{DLog}(\xi_{j_i})$$

do not change the values of x_i and y_i for all $i \in [1,n]$.

The game also defines $z_i \coloneqq x_i y_i$ and $z_i' \coloneqq \textsc{DLog}(Z_i')$, which does not change its behavior since those are new elements.

The last change from the initial $\mathrm{OMCDH}^{\mathrm{DL}}$ in Game_0 is the output of the game: instead of outputting the winning condition $\vec{Z}' = (x_i y_i G)_{i\in[1,n]}$, the game outputs the condition $\vec{z'} = \vec{z}$. But those conditions are the same since for a fixed generator in a cyclic group of prime order p, comparing the discrete logarithm of two group elements is equivalent to comparing the group elements themselves.

Since $z_i' \coloneqq \textsc{DLog}(Z_i')$, we get $z_i' = a_k$ such that $\Xi(a_k) = Z_i'$ which means $\Xi(z_i') = Z_i'$. So when we compare $\vec{z'} = \vec{z}$, we do the same comparison as $\vec{Z}' = (x_i y_i G)_{i\in[1,n]}$ but on discrete-logarithm level. This means the answer is the same. This will help us prove the following theorem:

Game_0

$a_0 \coloneqq 1$
$j \coloneqq 0\,;\ q \coloneqq 0\,;\ n \coloneqq 0$
$\vec{Z}' \leftarrow \mathcal{A}^{\textsc{Chal},\textsc{DLog},\textsc{GCmp}}(\textsc{Enc}())$
if $q < n$ **then return** 0
for $i \in [1,n]$
  $z'_i \coloneqq \textsc{DLog}(Z'_i)$
for $i \in [1,n]$
  $x_i \coloneqq \textsc{DLog}(\xi_{j_i})$
  $y_i \coloneqq \textsc{DLog}(\xi_{j_i+1})$
  $z_i \coloneqq x_i y_i$
return $\vec{z'} = \vec{z}$

Oracle $\textsc{Chal}()$

$j \coloneqq j+1$
$n \coloneqq n+1$
$x_n \xleftarrow{\$} \mathbb{Z}_p\,;\ j_n \coloneqq j$
$a_j \coloneqq x_n$
$j \coloneqq j+1$
$y_n \xleftarrow{\$} \mathbb{Z}_p$
$a_j \coloneqq y_n$
return $\textsc{Enc}()$

Oracle $\textsc{DLog}(\xi)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$
  then return $\perp$
$i \coloneqq \mathsf{min}(k \in [0,j] \mid \xi = \xi_k)$
$q = q+1\,;\ v = a_i$
return v

$\textsc{Enc}(\ )$ // outputs $\xi_j \coloneqq \Xi(a_j)$

if $\exists i \in [0, j-1] : a_j = a_i$
  then $\xi_j \coloneqq \xi_i$
else
  $\xi_j \xleftarrow{\$} \{0,1\}^{\log(p)} \setminus \{\xi_i\}_{i\in[0,j-1]}$
return ξ_j

Oracle $\textsc{GCmp}(\xi, \xi', b)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$ **or** $\xi' \notin \{\xi_i\}_{i\in[0,j]}$
  then return $\perp$
$i \coloneqq \mathsf{min}(\ k \in [0,j] \mid \xi = \xi_k)$
$i' \coloneqq \mathsf{min}(\ k \in [0,j] \mid \xi' = \xi_k)$
$j \coloneqq j+1\,;\ a_j \coloneqq a_i + (-1)^b a_{i'}$
return $\textsc{Enc}()$

Fig. 10: Game_0 represents the GGM version of the game OMCDH^{DL}, with small modifications inspired by the proof of OMDL, which are useful in the proof of Theorem 3.

Theorem 3. *Let $\mathcal{A}$ be an adversary that solves $OMCDH^{DL}$ in a generic group of order p, making at most m oracle queries. Then*

$$\mathbf{Adv}^{\mathsf{OMCDH\ GGM}}_{\mathcal{A}} \leq \frac{1}{p-1} + \frac{2m}{p} + \frac{m^2}{p-m^2}\ .$$

Proof. The proof follows exactly the same method as the proof of OMDL in Section 3. It makes the same game hops with the same boundaries and uses the Lemma 1 the same way. So we skip all the game hops and proceed directly to the analysis of the final game, which we call Game_4 in Figure 11

The various game hops give us the following bound:

$$\mathbf{Adv}^{\mathsf{Game}_0}_{\mathcal{A}} \leq \mathbf{Adv}^{\mathsf{Game}_4}_{\mathcal{A}} + \frac{m^2}{p-m^2}\ . \tag{17}$$

Now we analyze Game_4 described in Figure 11. This game looks a lot like the final game of OMDL (Figure 6): the challenges x_i and y_i are no longer defined, until the adversary makes some calls to DLog or until the game defines them at the end. We show that the adversary cannot predict some of the challenges that are assigned randomly at the end.

In this analysis we need to suppose that all the challenges x_i, y_i will be different from 0. Note that in OMCDH^{DL} and also in OMCDH, if we have $x_i = 0$ or $y_i = 0$, it becomes easy

Game_4

$j \coloneqq 0\,;\ q \coloneqq 0\,;\ n \coloneqq 0$
$P_0 \coloneqq 1\,;\ L \coloneqq \emptyset$
$\vec{Z'} \leftarrow \mathcal{A}^{\textsc{Chal},\textsc{DLog},\textsc{GCmp}}(\textsc{Enc}())$
if $q < n$ **then return** 0
for $i \in [1,n]$
  $z'_i \coloneqq \textsc{DLog}(Z'_i)$
for $i \in [1,n]$
  $x_i \coloneqq \textsc{DLog}(\xi_{j_i})$
  $y_i \coloneqq \textsc{DLog}(\xi_{j_i})$
  $z_i \coloneqq x_i y_i\,;$
return $\vec{z'} = \vec{z}$

Oracle $\textsc{Chal}()$

$j \coloneqq j+1$
$n \coloneqq n+1$
$P_j \coloneqq \mathsf{X}_n$
$j_n \coloneqq j$
$\xi \coloneqq \textsc{Enc}()$
$j \coloneqq j+1$
$P_j \coloneqq \mathsf{Y}_n$
$\xi' \coloneqq \textsc{Enc}()$
return ξ, ξ'

Oracle $\textsc{DLog}(\xi)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$
  then return $\perp$
$i \coloneqq \mathsf{min}(k \in [0,j] \mid \xi = \xi_k)$
$q = q+1\,;\ v \xleftarrow{\$} \mathbb{Z}_p$
if $P_i \in \mathsf{Span}(1, L)$ **then**
  get $(\alpha_k)_k \in \mathbb{Z}_p^q$ **s.t.**
  $P_i = \alpha_0 + \sum_{k=1}^{q-1} \alpha_k L_k$
  $x \coloneqq \alpha_0$
$L_q \coloneqq P_i - v\,;\ L = L \cup \{P_i - v\}$
if $\exists (i_1, i_2) \in [0,j]^2$:
  $P_{i_1} - P_{i_2} \in \mathsf{Span}(L)$
  and $\xi_{i_1} \neq \xi_{i_2}$
  then abort game
return v

$\textsc{Enc}(\ )$

if $\exists i \in [0, j-1] : P_j - P_i \in \mathsf{Span}(L)$
 then $\xi_j \coloneqq \xi_i$
else
  $\xi_j \xleftarrow{\$} \{0,1\}^{\log(p)} \setminus \{\xi_i\}_{i\in[0,j-1]}$
return ξ_j

Oracle $\textsc{GCmp}(\xi, \xi', b)$

if $\xi \notin \{\xi_i\}_{i\in[0,j]}$ **or** $\xi' \notin \{\xi_i\}_{i\in[0,j]}$
  then return $\perp$
$i \coloneqq \mathsf{min}(\ k \in [0,j] \mid \xi = \xi_k)$
$i' \coloneqq \mathsf{min}(\ k \in [0,j] \mid \xi' = \xi_k)$
$j \coloneqq j+1$
$P_j \coloneqq P_i + (-1)^b P_{i'}$
return $\textsc{Enc}()$

Fig. 11: Game_4 is the final game in which the challenger simulates OMCDH^{DL} to the adversary by using only polynomials.

for the adversary to win since $x_i y_i = 0$ and the adversary can output $1_{\mathbb{G}}$. Since in Game_4 all the discrete logarithms are picked uniformly at random, we can deduce that x_i and y_i have a probability at most $\frac{1}{p}$ to be equal to zero. Another way to obtain this result would be to consider the fact that the distribution of the games are not modified and by doing the game hops from Game_1 to Game_4, as in OMDL. This way, in Game_4 the distribution of the challenge is the same as in Game_1, which is the same as Game_0 and in those games, x_i and y_i are defined uniformly at random, which means they are equal to 0 with probability $\frac{1}{p}$. We call E the event "$\exists i \in [1,n]$ **such that** $x_i = 0$ **or** $y_i = 0$".

Since there are n challenges x_i and n challenges y_i, we get:

$$\Pr[\,E\,] \leq 1 - \big(1 - \frac{1}{p}\big)^{2n} \leq 1 - \big(1 - \frac{1}{p}\big)^{2m}\ ,$$

and since $\frac{2m}{p}$ is small, using $\big(1 - \frac{1}{p}\big)^{2m} \geq 1 - \frac{2m}{p}$, we get:

$$Pr[\,E\,] \leq \frac{2m}{p}\ . \tag{18}$$

Now we suppose that we are not in event E. Since $q < n$, we know that L contains at most $n-1$ elements when $\mathcal{A}$ outputs its answers. Then the challenger adds n polynomials to L by making n queries to the oracle DLog, which are $z'_i \coloneqq \text{DLog}(Z'_i)$.

This means that just before defining the challenges x_i and y_i, L contains at most $2n-1$ elements, which yields that $\mathsf{Span}(1, L)$ has dimension at most $2n$. But since the dimension of $\mathsf{Span}(\mathsf{X}_1, \ldots, \mathsf{X}_n, \mathsf{Y}_1, \ldots, \mathsf{Y}_n)$ is exactly $2n$ and $1 \notin \mathsf{Span}(\mathsf{X}_1, \ldots, \mathsf{X}_n, \mathsf{Y}_1, \ldots, \mathsf{Y}_n)$, there must exist an index i such that $\mathsf{X}_i \notin \mathsf{Span}(1, L)$ or $\mathsf{Y}_i \notin \mathsf{Span}(1, L)$. Without loss of generality, we suppose that $\mathsf{X}_i \notin \mathsf{Span}(1, L)$.

This means that on the call $x_i \coloneqq \text{DLog}(\xi_{j_i})$, x_i is picked uniformly at random. Since z'_i is fixed before that we have either:

- y_i was fixed by the game with the adversary and we get $\Pr[x_i = \frac{z'_i}{y_i}] \leq \frac{1}{p-1}$ (it is an equality when $z'_i \neq 0$). (We know that $y_i \neq 0$ since we are not on event E and we get $p-1$ instead of p since we know that x_i can't be 0). Or:
- y_i is defined uniformly at random as x_i, which implies the product $x_i y_i$ is also picked uniformly at random over $\mathbb{Z}_p^*$ and we obtain $\Pr[\, x_i y_i = z'_i \,] \leq \frac{1}{p-1}$ as before.

In the end, we get:

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{Game}_4} \leq \Pr[\, E \,] + \frac{1}{p-1} \,. \tag{19}$$

Equations (17), (18) and (19) together prove the theorem. □