THE UNDULAR HYDRAULIC JUMP IN INVISCID AXISYMMETRIC FLOW

Dominik Murschenhofer*[1] and Wilhelm Schneider[1]
[1]Institute of Fluid Mechanics and Heat Transfer, TU Wien, Vienna, Austria

Summary Steady inviscid free-surface flow over a horizontal bottom is considered. The results of an asymptotic analysis for slightly supercritical upstream Froude numbers show that undular hydraulic jumps can exist if the flow is axisymmetric. Provided the origin of the jump is located at a very large radius, the free surface is described by the steady-state version of an extended Korteweg–de Vries (KdV) equation. A numerical solution of the extended KdV equation is shown in comparison with the corresponding one-dimensional flow approximation, indicating that the supercritical branch of the latter is prone to develop into an undular hydraulic jump.

INTRODUCTION AND PROBLEM DESCRIPTION

In hydraulics the transition from slightly supercritical to subcritical free-surface flow is characterized by a wavy surface known as the undular hydraulic jump. In *plane* flow, e.g. in open channels, the undular hydraulic jump is inherently associated with dissipation and cannot exist in inviscid flow [1]. In the present paper it is shown that in the case of *axisymmetric* flow the situation changes, and the governing equations of inviscid flow permit an undular hydraulic jump.

Steady inviscid axisymmetric free-surface flow over a horizontal bottom is considered. The basic equations are the continuity equation for incompressible flow and the Euler equations. Cylindrical coordinates are introduced, with r and z in radial and vertical direction, respectively. The free surface is defined by a streamline, and the ambient pressure is set to zero. The bottom is assumed to be impermeable for the liquid.

ASYMPTOTIC ANALYSIS AND HYDRAULIC APPROXIMATIONS

An asymptotic analysis of the governing equations for slightly supercritical upstream Froude numbers is performed, i.e. $(2/3)(\mathrm{Fr_r}-1)=\varepsilon\ll 1$, with the reference Froude number $\mathrm{Fr_r}:=Q/(r_\mathrm{r}\sqrt{gh_\mathrm{r}^3})$. Here Q, g and h denote, in this order, the constant discharge per unit azimuth angle, the acceleration due to gravity and the height of the free surface above the bottom. The subscript r refers to a reference state. A non-dimensional radius $R=\delta r/h_\mathrm{r}$ is introduced, with $\delta=3\varepsilon^{1/2}$ serving as a contraction parameter. The non-dimensional reference radius R_r is assumed to be very large such that R can be decomposed in $R=R_\mathrm{r}+\eta$ with $\mathrm{d}R=\mathrm{d}\eta$. In the present analysis R_r is chosen as $R_\mathrm{r}=\tilde{R}\varepsilon^{-5/2}$ with $\tilde{R}=\mathrm{const}=\mathrm{O}(1)$. The analysis for reference radii of other orders of magnitude, e.g. $\mathrm{O}(\varepsilon^{-2})$, is beyond the scope of the present paper and will be discussed elsewhere. Expanding all dependent variables in terms of powers of ε, e.g. $H(\eta)=h/h_\mathrm{r}=1+\varepsilon H_1(\eta)+\ldots$ for the non-dimensional height of the free surface, leads to a set of first-order equations that allow to express all unknowns in terms of the free-surface elevation H_1. The latter remains undetermined in the framework of the first-order equations. This requires an investigation of the second-order equations, leading to the solvability condition

$$H_1'''+H_1'(H_1-1)=\varepsilon^{1/2}/3\tilde{R}\,,\tag{1}$$

i.e. an ODE for $H_1(\eta)$. Equation (1) is a steady-state version of an extended Korteweg–de Vries (KdV) equation, with the constant term of order $\varepsilon^{1/2}$ on the right-hand side representing the extension due to axisymmetric flow. According to the present analysis, (1) can be expected to be uniformly valid.

The asymptotic analysis leads to the interesting result $U(\eta,Z)=u/u_\mathrm{r}=1+\varepsilon[c_1(Z)-H_1(\eta)]+\ldots$ for the non-dimensional radial velocity. The function of integration $c_1(Z)$ defines the velocity profile and can be chosen freely, as it does not affect the final result for H_1. If the reference velocity u_r is defined as the volumetric mean velocity, the integral of $c_1(Z)$ over the non-dimensional vertical coordinate $Z=z/h_\mathrm{r}$ has to vanish, i.e. $\int_0^1 c_1(Z)\mathrm{d}Z=0$.

For the purpose of comparison it is of interest to consider a one-dimensional flow approximation, known as the "hydraulic approximation". It is based on the equation of motion in radial direction with hydrostatic pressure distribution. Continuity is expressed as $urh=Q=\mathrm{const}$. Combining the continuity equation and the equation of motion gives

$$\frac{r}{r^*}=\frac{R}{R^*}=\frac{(2+\mathrm{Fr}^2)^{3/2}}{3\sqrt{3}\mathrm{Fr}}\,,\tag{2}$$

i.e. an implicit equation for the local Froude number, which is defined as $\mathrm{Fr}(r):=Q/(r\sqrt{gh^3})$. The asterisk refers to the critical state, where $\mathrm{Fr}=1$. Note that the relation (2) is "universal", as it is free of parameters describing the upstream state.

A near-critical version of the hydraulic approximation is obtained by expanding (2) for $|\mathrm{Fr}-1|\ll 1$. The result is

$$\mathrm{Fr}=1\pm\sqrt{3(R/R^*-1)/2}\,.\tag{3}$$

*Corresponding author. E-mail: dominik.murschenhofer@tuwien.ac.at.

Introducing the reference radius R_r instead of R^* in (3), using the relation $\mathrm{Fr}(\eta) = 1 + 3\varepsilon(1 - H_1)/2$ and expanding for $\varepsilon \ll 1$ gives

$$H_1 = 1 \mp \sqrt{1 + 2\varepsilon^{1/2}\eta/3\tilde{R}}\,. \tag{4}$$

The upper and lower signs in (3) and (4) correspond to the supercritical and subcritical branch, respectively. Both (3) and (4) are near-critical hydraulic approximations. They differ from each other due to the different reference states, i.e. $\mathrm{Fr} = 1$ at $R = R^*$ for (3) and $H_1 = 0$ at $\eta = 0$ for (4). It follows from (4) that the condition $H_1 = \mathrm{O}(1)$ for the validity of the asymptotic expansion is only satisfied if $\varepsilon^{1/2}\eta = \mathrm{O}(1)$. It may be of interest to observe that differentiating (4) with respect to η leads to an equation that is equal to (1) without the term H_1''', i.e. the extended KdV equation without the third-order term is equivalent to the near-critical hydraulic approximation.

RESULTS

In Fig. 1 a solution of the extended KdV equation (1) is shown in black. The red curve represents the solution of the hydraulic approximation, (2). In Fig. 1a) the sub- and supercritical branches of the near-critical hydraulic approximation according to (4) are shown as blue dotted and dashed lines, respectively. The initial conditions for the black curve are chosen such that initial value, initial slope and initial curvature are in accord with the *super*critical branch of (4) in the reference state, i.e. $\eta = 0$. Within a short distance from the reference radius the solution of (1) develops into a wavy solution oscillating around the *sub*critical branch of (4). Such a transition is a characteristic property of undular hydraulic jumps. In Fig. 1b) the radial coordinate is referred to the critical radius. The solution of the near-critical hydraulic approximation according to (3) is shown in green. In order to plot the solution of (1), the critical radius is determined from (2) using the parameter values of the reference state $\mathrm{Fr_r} = 1.15$ and $R_\mathrm{r} = 2214$, corresponding to $\varepsilon = 0.1$ and $\tilde{R} = 0.7$. If, for instance, the discharge is chosen to be $Q = 10$ l/s, these parameters give the reference values $h_\mathrm{r} = 10.72$ mm, $r_\mathrm{r} = 2.50$ m, $u_\mathrm{r} = 0.37$ m/s, which appear reasonable for applications.

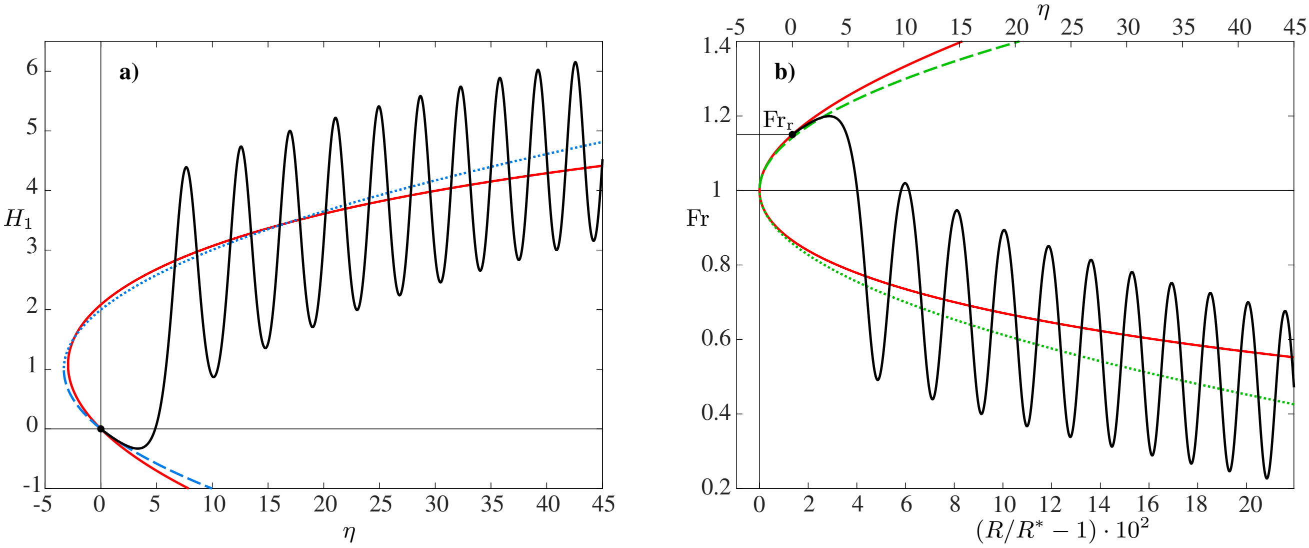

Figure 1: Results for an undular hydraulic jump in inviscid axisymmetric flow over a horizontal bottom; $\varepsilon = 0.1$, $\tilde{R} = 0.7$. a) Non-dimensional surface elevation, H_1. b) Local Froude number, Fr. Black: Numerical solution of (1) for $H_1(0) = 0$, $H_1'(0) = -0.15$, $H_1''(0) = 2.27 \cdot 10^{-2}$. Red: Full hydraulic approximation according to (2). Dotted and dashed: Sub- and supercritical branches, respectively, of the near-critical hydraulic approximation according to (3) in green, according to (4) in blue.

CONCLUSIONS

The present analysis demonstrates that – in contrast to inviscid plane flow – undular hydraulic jumps are possible in inviscid axisymmetric flow. The free-surface elevation is described by an extended steady-state version of the KdV equation. It turns out that the velocity profile can be chosen freely via a function of integration $c_1(Z)$, provided $c_1(Z) = \mathrm{O}(1)$. The velocity profile does not affect the final result for the surface elevation. The numerical solution of (1) shows that the supercritical branch of the hydraulic approximation is prone to develop into an undular hydraulic jump. The validity of the results is limited to non-dimensional distances from the reference radius up to $\eta = \mathrm{O}(\varepsilon^{-1/2})$.

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References

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