PERFECT SUBTREE PROPERTY FOR WEAKLY COMPACT CARDINALS

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Abstract. In this paper we investigate the consistency strength of the statement: κ is weakly compact and there is no tree on κ with exactly κ^+ many branches. We show that this property fails strongly (there is a *sealed tree*) if there is no inner model with a Woodin cardinal. On the other hand, we show that this property as well as the related Perfect Subtree Property for κ, implies the consistency of $\mathrm{AD}_{\mathbb{R}}$.

1. Introduction

Trees and their collections of branches play an important role in topology and infinitary combinatorics. Indeed, closed subsets of the Cantor Set 2^ω are exactly the collections of branches of subtrees of $2^{<\omega}$. By the Cantor-Bendixson theorem, subtrees of the binary tree on ω satisfy a dichotomy - either the tree has countably many branches or there is a perfect subtree (and in particular, the number of branches of the tree is going to be the continuum, regardless of the size of the continuum). Equivalently, if a tree $T \subseteq 2^{<\omega}$ has uncountably many branches and $\mathbb{P}$ is a forcing notion that adds a new real then $\mathbb{P}$ adds a new branch to T.

Definition 1. Let κ be a regular cardinal. The *Branch Spectrum* of κ is the set

$$\mathfrak{S}_\kappa = \{|[T]| \mid T \text{ is a normal } \kappa\text{-tree}\}.$$

For the definition of normal trees see the beginning of Section 2. A first example is $\mathfrak{S}_\omega = \{2^{\aleph_0}\}$. The spectrum $\mathfrak{S}_{\omega_1}$ was first studied by Silver [Si71], showing the independence of the Kurepa Hypothesis. Later, questions about the possible values of this spectrum were addressed by Shelah and Jin in [ShJi92] and more recently an almost complete characterization was given by [Po]. By [SiSo], the branch spectrum is related to the model theoretical spectrum of maximal models of $\mathcal{L}_{\omega_1,\omega}$-sentences.

For uncountable cardinals κ, the assertion $\max(\mathfrak{S}_\kappa) = \kappa$ (i.e., there are no κ-Kurepa trees) has the consistency strength of an inaccessible cardinal. The assertion $\min(\mathfrak{S}_\kappa) = \kappa$ (i.e., the tree property at κ for κ with uncountable cofinality) has the consistency strength of a weakly compact cardinal. We are interested in the consistency strength of the combination of these two properties. Since we are interested in inaccessible cardinals, we replace the first assertion by the weaker assertion $\kappa^+ \notin \mathfrak{S}_\kappa$. As usual in these types of properties, where individually each one of them has a mild consistency strength, their combination is very strong. In Section 6 we will show that the statement $\kappa^+ \notin \mathfrak{S}_\kappa$ for a weakly compact cardinal κ implies the consistency of $\mathrm{AD}_{\mathbb{R}}$.

Trees with (somewhat) absolute sets of branches play a role in the derivation of consistency strength from unnatural cases of the branch spectrum in the context of some covering lemma. Indeed, if κ is a regular cardinal and L computes κ^+ correctly then there is a tree $T \subseteq 2^{<\kappa}$ with exactly κ^+ many branches, and moreover any model in which κ has uncountable cofinality does not have any non-constructible

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cofinal branch through T. This result, which is a variant of a construction due to Solovay, generalizes to K, replacing absoluteness by set forcing absoluteness.

Definition 2. Let κ be a regular cardinal. A normal tree T of height κ is *strongly sealed* if the set of branches of T cannot be modified by set forcing that forces $\mathrm{cf}(\kappa) > \omega$.

Strongly sealed trees with κ many branches exist in ZFC: take $T \subseteq 2^{<\kappa}$ to be the tree of all x such that $\{\alpha \in \mathrm{dom}(x) \mid x(\alpha) = 1\}$ is finite. Thus, our main interest is in strongly sealed κ-trees with at least κ^+ many branches. Our constructions are very inner model theoretical, and thus can only produce κ-trees with κ^+ many branches, where κ^+ is computed correctly by some inner model.

Question 1. Is it consistent that there is a strongly sealed κ-tree with κ^{++} many branches?

Note that strongly sealed κ-trees cannot exist in the context of a Woodin cardinal $\delta > \kappa$, since Woodin's stationary tower forcing with critical point κ^+ will introduce new branches through any κ-tree T, while preserving the regularity of κ, as well as many large cardinal properties of κ. Thus, in order to get a meaningful notion of sealed trees in the presence of large cardinals we will use a weaker notion. The weakest notion of sealed tree is arguably having no perfect subtree (a perfect subtree is a continuous copy of $2^{<\kappa}$).

Lemma 3 (Folklore)**.** *Let κ be a cardinal. The following are equivalent for a tree T of height κ:*

(1) *T has a perfect subtree.*
(2) *Every set forcing that adds a fresh subset to κ also adds a branch through T.*
(3) *There is a κ-closed forcing that adds a branch through T.*

See Lemma 10 for the argument for (3) $\implies$ (1).

Definition 4. Let κ be an uncountable cardinal. The *Perfect Subtree Property* (PSP) for κ is the statement that every κ-tree with at least κ^+ many branches has a perfect subtree.

The Perfect Subtree Property can easily be violated by small forcing by Hamkins' Gap Forcing argument, [Ha01]. Nevertheless, as we will see in Section 6, in the presence of some covering lemma there is a natural counterexample to the PSP in an inner model, providing a lower bound for the consistency of this statement as well as a natural intermediate version for the sealing property which is compatible with the existence of Woodin cardinals:

Definition 5. Let κ be a regular cardinal. A normal tree T of height κ is *sealed* if the set of branches of T cannot be modified by set forcing $\mathbb{P}$ satisfying the following properties:

(1) $\mathbb{P} \times \mathbb{P}$ does not collapse κ,
(2) $\mathbb{P} \times \mathbb{P}$ preserves $\mathrm{cf}(\kappa) > \omega$, and
(3) $\mathbb{P}$ does not add any new sets of reals.

Note that Woodin's stationary tower forcing with arbitrary critical point does not satisfy these properties. The class of forcings $\mathbb{P}$ is designed to preserve inner model theoretical properties such as iterability of mice and some form of condensation. This class of forcings contains κ-closed forcing such as $\mathrm{Add}(\kappa, 1)$ and $\mathrm{Col}(\kappa, \lambda)$ and in particular, if a tree T is sealed then it has no perfect subtree.

The structure of the paper is as follows. In Section 2 we will give a few forcing constructions providing an upper bound for the consistency of PSP at a weakly compact cardinal as well as $\kappa^+ \notin \mathfrak{S}_\kappa$. These forcing constructions are mostly folklore. In Section 3 we review some inner model theoretic notions and show a technical lemma which we will use for our main results. In Section 4 we will revisit Solovay's theorem on the existence of a Kurepa tree in L from [Je71, Section 4], which is the main ingredient in his proof for the consistency strength of the Kurepa Hypothesis. A variant of Solovay's construction provides a strongly sealed tree. We conclude that if $0^\#$ does not exist then every weakly compact cardinal carries a strongly sealed tree with κ^+ many branches. The argument extends to K below a Woodin cardinal:

Theorem 6. *Assume that there is no inner model with a Woodin cardinal. Then for every regular cardinal κ, there is a strongly sealed tree of height κ with exactly $(\kappa^+)^K$ many branches. In particular, if κ is weakly compact, then there is a strongly sealed κ-tree with κ^+ many branches.*

Using the construction of this theorem we conclude that in K every κ-tree for inaccessible κ is a projection of a strongly sealed κ-tree. This is done in Section 5. Finally, in Section 6 we prove:

Theorem 7. *Let κ be a weakly compact cardinal and assume that there is no non-domestic premouse*[1]*. Then there is a sealed κ-tree with exactly κ^+ many branches.*

By Lemma 3 this yields the following corollary.

Corollary 8. *Let κ be a weakly compact cardinal and suppose the Perfect Subtree Property holds at κ. Then there is a non-domestic premouse. In particular, there is an inner model of* $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$.

Our definitions are mostly standard. For facts about forcing and trees we refer the reader to [Je03, Chapters 9, 14]. For basic facts, definitions and notions related to inner model theory, we refer the reader to [St10] with the exception that we are using *Jensen indexing* as in [Je].

2. Trees and upper bounds

Definition 9. Let κ be a cardinal and let T be a tree of height κ. We will denote the set of all cofinal branches through T by $[T]$. T is called *κ-Kurepa* if it is a κ-tree and $|[T]| \geq \kappa^+$.

We say that a tree of height κ is *binary* if it is a subtree of $2^{<\kappa}$, the *full binary tree.* We say that a tree is *normal* if every node splits, every node has an arbitrarily high node above it and at every limit level, the nodes are uniquely determined from the branch below them. When κ is strongly inaccessible, the full binary tree is a normal Kurepa tree. Thus, in those cases the notion of *slim Kurepa trees* is more suitable, but we will not pursue this direction here. We remark that if T is a κ-tree then there is a normal tree $S \subseteq T$ (the pruned subtree of T) such that $|[S]| + \kappa \geq |[T]|$. In particular, focusing on trees with at most κ many branches, S and T have the same number of branches.

In [Si71], Silver showed that if $\mu < \kappa$ are cardinals where μ is a successor cardinal and κ is inaccessible, then there are no μ-Kurepa trees in the generic extension by the Levy collapse $\mathrm{Col}(\mu, {<}\kappa)$. We review the argument here for the reader's convenience. Let $\dot{T}$ be a name for a μ-Kurepa tree in the generic extension. Then, using the chain condition of the Levy collapse, $\dot{T}$ can be represented as a name with respect to some initial segment of the collapse, $\mathrm{Col}(\mu, {<}\bar{\kappa})$, for $\bar{\kappa} < \kappa$. In the intermediate model, $V^{\mathrm{Col}(\mu, <\bar{\kappa})}$, $2^\mu = \mu^+ \leq \bar{\kappa}^+$, and thus $\dot{T}$ has at most $\bar{\kappa}^+$ many

[1]See Definition 23 below.

branches. Since this cardinal is collapsed in the full generic extension, in order for $\dot{T}$ to have more than μ many branches, the quotient forcing $\mathrm{Col}(\mu, [\bar{\kappa}, \kappa))$ has to introduce them. The following lemma is the crucial step in the proof:

Lemma 10. *Let T be a tree of height μ and let $\mathbb{P}$ be a μ-closed forcing. If $\mathbb{P}$ introduces a branch through T then there is an order preserving injection from the full binary tree $2^{<\mu}$ to T.*

Proof. Let $\dot{b}$ be a name for the new branch. Let us construct by induction for every $\eta \in 2^{<\mu}$ a pair (p_η, t_η) where $p_\eta \in \mathbb{P}$ is a condition, $t_\eta \in T$ and $p_\eta \Vdash t_\eta \in \dot{b}$. We will assume, by induction, that for every $\eta \trianglelefteq \eta'$, p'_η is stronger than p_η, $t_{\eta'}$ is above t_η in the tree and $t_{\eta\frown\langle 0\rangle}, t_{\eta\frown\langle 1\rangle}$ are on the same level and incompatible.

At limit steps, we use the closure of the forcing in order to define p_η as a condition stronger than $p_{\eta\restriction\alpha}$, for all $\alpha < \mathrm{lh}(\eta)$. Since p_η is a condition in $\mathbb{P}$ and it forces $t_{\eta\restriction\alpha} \in \dot{b}$ for all $\alpha < \mathrm{lh}(\eta)$, there is an element of T above all $t_{\eta\restriction\alpha}$ and p_η forces the smallest such element of T to be in $\dot{b}$.

At successor steps, we use the assumption that $\dot{b}$ is new in order to find two different extensions of p_η that force different values for the element of the branch at some level above $\mathrm{lh}(t_\eta)$. □

Now, we can finish the proof by noticing that for a successor cardinal μ, there is no such embedding of the full binary tree into a μ-tree. Indeed, let ρ be the least cardinal such that $2^\rho \geq \mu$ (then $\rho < \mu$ since μ is a successor cardinal). Let us look at the ρ-th level of the binary tree. By our assumption, it consists of $2^\rho \geq \mu$ many different elements. But since $2^{<\rho} < \mu$, the levels of the images of the elements of the binary tree below ρ are bounded and by continuity so are their limits, which is a contradiction.

The proof shows that in this model, even without assuming that μ is a successor cardinal, every tree of height μ and width $\leq\mu$ which has μ^+ many branches contains a perfect subtree. By essentially the same argument, one can show that if μ is strongly inaccessible then after forcing with $\mathrm{Col}(\mu, {<}\kappa) \times \mathrm{Add}(\mu, \kappa^+)$, every tree of width μ either has $\leq \mu$ many branches or a perfect subtree. In particular, in this model $\mathfrak{S}_\mu \subseteq \mu \cup \{\mu, 2^\mu\}$ and $2^\mu > \mu^+$. If we further assume that μ is still weakly compact in this extension, then $\mathfrak{S}_\mu = \{\mu, 2^\mu\}$.

The only known way to preserve weak compactness after the Levy collapse is to start with a supercompact cardinal and force with some preparation forcing. There is some evidence that this is necessary, see [NeSt16], [ApHa01]. Altogether, we showed the following:

Proposition 11. *Let μ be ${<}\kappa$-supercompact, where κ is strongly inaccessible. Then there is a forcing extension in which μ is weakly compact, $\mathfrak{S}_\mu = \{\mu, \mu^{++}\}$ and the Perfect Subtree Property holds at μ.*

Now we aim to combine this behaviour of the branch spectrum with a failure of the Perfect Subtree Property. We say that a cardinal μ is *σ-closed* if $\rho^\omega < \mu$ for every $\rho < \mu$.

Proposition 12. *Let μ be a σ-closed regular cardinal. Adding a single Cohen real forces that the Perfect Subtree Property fails at μ. If we further assume that $\mathfrak{S}_\mu$ consists only of σ-closed cardinals then the forcing does not change it.*

This immediately gives the following corollary.

Corollary 13. *Let $\mu < \kappa$ be cardinals such that κ is inaccessible and μ is ${<}\kappa$-supercompact. Then there is a generic extension in which $\mathfrak{S}_\mu = \{\mu, \mu^{++}\}$ but the Perfect Subtree Property fails at μ.*

Proof of Proposition 12. The first assertion follows from [Ha01]. For the preservation of the spectrum, let us show that there is a way to transfer a name for a new μ-tree to a tree in the ground model. We remark that the ground model tree might have more branches than the generic one. Moreover, since $\mu \in \mathfrak{S}_\mu$, we only deal with trees with at least μ^+ many branches.

First, note that a Cohen real cannot add a new branch through an old tree of height $\geq \omega_1$. Thus, $\mathfrak{S}_\mu^{V[G]} \supseteq \mathfrak{S}_\mu^V$. We need to show that the other inclusion holds as well.

Let $\dot{T}$ be a name for a μ-tree with λ branches, where $\lambda > \mu$. For simplicity, let us assume that $\dot{T}$ is a binary tree. Let $\tilde{T}$ be the collection of all names $\dot{\eta}$ for elements of $\dot{T}$. We identify two names if they are forced to be equal and we order them by $\dot{\eta} \leq \dot{\eta}'$ if and only if the weakest condition forces $\dot{\eta} \trianglelefteq \dot{\eta}'$.

We may assume that every $\dot{\eta} \in \tilde{T}$ is a function from some ordinal $\alpha < \mu$ to a full name for $\dot{\epsilon}$, where each element of $\dot{\epsilon}$ is forced to be 0 or 1. This way $\dot{\eta}$ is below $\dot{\eta}'$ in $\tilde{T}$ iff $\dot{\eta} = \dot{\eta}' \restriction \mathrm{lh}\, \dot{\eta}$.

Let us assume that $\dot{T}$ is forced to have λ many branches where $\lambda \geq \mu^+$. Let $\{\dot{b}_\alpha \mid \alpha < \lambda\}$ be an enumeration of these branches. By the definition of $\tilde{T}$, for each $\alpha < \lambda$ and $\zeta < \mu$, $\dot{b}_\alpha \restriction \zeta$ is a member of $\tilde{T}$. By the definition of the tree order, $\langle \dot{b}_\alpha \restriction \zeta \mid \zeta < \mu \rangle$ is a cofinal branch.

We need to show that $\tilde{T}$ is a μ-tree and that it does not have more than λ many different branches. Both proofs are similar:

Let us assume that there are μ many different elements $\dot{\eta}_i$ in the ζ-th level of the tree $\tilde{T}$. Since $\dot{T}$ is forced to be a μ-tree, we can enumerate the elements in the ζ-th level of $\dot{T}$ by $\langle \dot{\rho}_i \mid i < i_* \rangle$, $i_* < \mu$. By the chain condition of Cohen forcing, for every ordinal $i < \mu$ there is a countable set $B_i \subseteq i_*$ such that it is forced by the trivial condition that $\dot{\eta}_i = \dot{\rho}_j$ for some $j \in B_i$. Since $i_*^\omega < \mu$, there is a set $A \subseteq \mu$, of size μ such that for all $i \in A$, $B_i = B_*$ (where B_* is some fixed value). For every $i, i' \in A$ there is a condition p that forces $\dot{\eta}_i$ and $\dot{\eta_{i'}}$ to be different. In particular, there is a condition that forces $\dot{\eta}_i = \dot{\rho}_\zeta$, $\dot{\eta_{i'}} = \dot{\rho}_{\zeta'}$ where $\zeta \neq \zeta'$ in B_*.

Let us apply the Erdős-Rado Theorem and obtain a set $H \subseteq A$ of size $\aleph_1$ and a fixed condition p such that for every $i, i' \in H$, p forces $\dot{\eta}_i \neq \dot{\eta_{i'}}$ and moreover decides their index in B_*. But B_* is countable, so there must be $i \neq i'$ in H which are assigned by p to the same index - a contradiction.

Let us now show that $\tilde{T}$ does not have more than λ many branches. Let $\tilde{\lambda}$ be the number of branches of $\tilde{T}$ and let us assume that $\tilde{\lambda} > \lambda$. By our assumption, $\tilde{\lambda}$ is a σ-closed cardinal. Note that any branch in $\tilde{T}$ is indeed a name for a branch in $\dot{T}$. Thus, we can apply the same argument, compare the branches through $\tilde{T}$ to the enumeration of the branches through $\dot{T}$, use the chain condition and the assumption that $\lambda^\omega < \tilde{\lambda}$ in order to get at least $\left(2^{\aleph_0}\right)^+$ many different branches through $\tilde{T}$ which are forced to belong to the same countable set of branches in the generic extension. Applying Erdős-Rado, we obtain the same contradiction. □

The proof shows that if the tree property holds at μ then it holds after adding a single Cohen real, assuming that μ is σ-closed. We remark that the same statement for the non-σ-closed case (e.g. $\mu = \aleph_2 = 2^{\aleph_0}$ or $\mu = \aleph_{\omega+1}$) is open.

3. Iteration trees and mice

In the main theorems of the paper, we will derive consistency strength from the non-existence of trees with somewhat absolute sets of branches. The construction of these trees will follow a fixed template which is described in Section 4. In order for the tree to have the desired properties, we will use a mixture of comparison and maximality of the inner model in which we construct the tree.

Recall that a premouse M is called *countably iterable* if all countable elementary substructures N of M are (ω_1, ω_1+1)-iterable. For a premouse $M = (J^{\vec{E}}_\alpha, \in, \vec{E}, E_\alpha)$ and an ordinal $\gamma \leq \alpha$ we let $M|\gamma = (J^{\vec{E}}_\gamma, \in, \vec{E} \restriction \gamma, E_\gamma)$ and we write $M||\gamma$ for the initial segment of M of height γ without the top extender, i.e., $M||\gamma = (J^{\vec{E}}_\gamma, \in, \vec{E} \restriction \gamma, \emptyset)$. The following lemma encapsulates the properties of the iteration trees which we are going to use.

Lemma 14. *Let M be a countably iterable premouse and let $\mathcal{T}$ be an iteration tree on $M = M_0$. Let us assume that there is an ordinal $\alpha \in M$ such that whenever F is an extender which is applied in $\mathcal{T}$ and* $\operatorname{crit}(F) < \alpha$ *then* $j_F(\operatorname{crit}(F)) \geq \alpha$. *Then for every $\eta \in \mathcal{T}$,*

$$M_\eta \parallel \left(\alpha^+\right)^{M_\eta} = M_0 \parallel \left(\alpha^+\right)^{M_\eta}.$$

Proof. We proceed by induction on the tree. Let $M_{\zeta+1}$ be a successor step in the tree. So $M_{\zeta+1}$ is obtained by $\mathrm{Ult}_n(M^*_{\zeta+1}, F)$ where $F \in M_\zeta$, $M^*_{\zeta+1}$ might be a proper initial segment of $M_{\eta'}$ for $\eta' = \mathrm{T}-\mathrm{pred}(\zeta+1)$ (if there was a drop in model) and n might be finite (if there was a drop in degree).

Since F coheres with the extenders on M_ζ, in M_ζ one can compute the ultrapower of some initial segment of M_ζ by F. Let $\rho = \operatorname{crit}(F)$. By the assumption of the lemma, $j_F(\rho) \geq \alpha$ (since either $\rho < \alpha$ and then we apply the assumption or that $\rho \geq \alpha$ and then $j_F(\rho) > \rho \geq \alpha$). Let β be the index of F, which is $j_F(\rho)^+$ of the ultrapower. By coherence,

$$\mathrm{Ult}_n(M^*_{\zeta+1}, F)||\beta = \mathrm{Ult}(M_\zeta||\gamma, F) = M_\zeta||\beta,$$

for $\gamma < \beta$ such that ρ is the largest cardinal in $M_\zeta||\gamma$. Let us argue that the successor of α in $M_{\zeta+1}$ is $\leq \beta$. Since $\alpha \leq j_F(\rho)$, every subset of α in $M_{\zeta+1}$ is represented by a function from ρ to the power set of ρ (which can be coded as a subset of ρ). Now $M^*_{\zeta+1} \cap \mathcal{P}(\rho) \subseteq M_\zeta||\gamma \cap \mathcal{P}(\rho)$ as $M^*_{\zeta+1}$ is the largest initial segment of $M_{\eta'}$ such that F is a pre-extender on $M^*_{\zeta+1}$. Moreover $M_\zeta||\gamma \cap \mathcal{P}(\rho) \subseteq M^*_{\zeta+1} \cap \mathcal{P}(\rho)$ since $M_{\eta'}$ and M_ζ agree below $\mathrm{lh}(F_{\eta'}) > \lambda(F_{\eta'}) > \operatorname{crit}(F)$. Therefore, any subset of α in $M_{\zeta+1}$ already appears in the ultrapower $\mathrm{Ult}(M_\zeta||\gamma, F)$. Since β is defined to be the successor of $j_F(\rho)$ in the ultrapower and $i_F^{M_\zeta||\gamma}$ and $i_F^{M^*_{\zeta+1}}$ agree on $\mathrm{dom}(F)$, we conclude that it is also the successor of $j_F(\rho)$ in $M_{\zeta+1}$.

Let us now deal with limit steps. Let M_ζ be the direct limit of a cofinal well founded branch b through the iteration tree. Then, we claim that there are at most two steps in the branch in which the critical point of the extender is $< (\alpha^+)^{M_\zeta}$. Indeed, since the extenders on the branch do not overlap and the critical points are increasing, the worst case is that at the first step of the branch we applied some extender F with $\operatorname{crit}(F) < \alpha$. Then, by our assumption, $\mathrm{lh}(F) \geq \alpha$ and therefore in the next step, we have to apply an extender with critical point $\geq \alpha$. Since the critical points of the used extenders are inaccessible cardinals in the corresponding ultrapower, each other extender which is applied on the branch is going to have critical point high above the successor of α in the limit model. Thus, $M_\zeta|| \left(\alpha^+\right)^{M_\zeta}$ is fixed by the first two steps in the branch. □

4. Abstract Construction

The following lemma is a reformulation of Solovay's argument for the consistency strength of the Kurepa Hypothesis, [Je71, Section 4].

Lemma 15. *Let κ be a regular, uncountable cardinal and let us assume that $\left(\kappa^+\right)^L = \kappa^+$. Then there is a strongly sealed tree of height κ.*

Proof. Let T be the following tree. An element of T is a pair $\langle M, \bar{x}\rangle$ where $M \cong \mathrm{Hull}^{L_{\kappa^+}}(\rho \cup \{x\})$, $\rho < \kappa$, $x \subseteq \kappa$ in L, M is transitive and $\bar{x}$ is the image of x under the transitive collapse. We order the tree by $\langle M, \bar{x}\rangle \leq_T \langle M', \bar{x}'\rangle$ if M is the transitive collapse of $\mathrm{Hull}^{M'}(\rho \cup \{\bar{x}'\})$ for some ordinal ρ and $\bar{x}$ is the image of $\bar{x}'$ under the transitive collapse.

Let $b = \{\langle M_\alpha, x_\alpha\rangle \mid \alpha < \kappa\}$ be a branch through T in some set forcing extension of L which preserves $\mathrm{cf}(\kappa) > \omega$. Let R_b be the direct limit. Then R_b is well founded as it is a limit of uncountable cofinality of well founded models. Moreover, $R_b \models$ "$V = L_\delta$" for some δ and therefore $R_b \cong L_\delta$, for some $\delta < \kappa^+$. Let $\tilde{x}$ be the limit of the values of x_α along the branch and $\tilde{\kappa}$ be the limit of the values of κ_α along the branch, where κ_α is the largest cardinal in M_α. Note that $\tilde{\kappa} \geq \kappa$ and it is the largest cardinal in R_b. Since $\tilde{x} \in R_b$, it is constructible. In particular, the branch b is in L, since it is definable by the transitive collapses of the models in

$$\{\mathrm{Hull}^{L_\delta}(\rho \cup \{\tilde{x}\}) \mid \rho < \kappa\}.$$

□

If κ is inaccessible this tree is clearly a κ-tree.

Definition 16. Let M be a transitive model of some fragment of ZFC with a definable well order of ordinal height at least $\kappa + 1$. Let $\mathbb{T}(M)$ be the tree that consists of pairs of the form $\langle \bar{M}, \bar{x}\rangle$ where $\bar{M}$ is the transitive collapse of $\mathrm{Hull}^M(\rho \cup \{x\})$ for some $\rho < \kappa$, $x \in {}^\kappa 2 \cap M$ and $\bar{x}$ is the image of x under the transitive collapse. The order of $\mathbb{T}(M)$ is defined by $\langle M_0, x_0\rangle \leq_{\mathbb{T}(M)} \langle M_1, x_1\rangle$ if there is some ordinal ρ such that M_0 is the transitive collapse of $\mathrm{Hull}^{M_1}(\rho \cup \{x_1\})$ and x_0 is the image of x_1 under the transitive collapse.

Lemma 17. *Let M be a model of* ZFC $-$ (*Power set*). *Then $\mathbb{T}(M)$ is a tree of height κ with at least $(2^\kappa)^M$ many branches.*

Proof. First, let us verify that $\mathbb{T}(M)$ is a tree. Indeed, if

$$\langle M_0, x_0\rangle \leq_{\mathbb{T}(M)} \langle M_1, x_1\rangle \leq_{\mathbb{T}(M)} \langle M_2, x_2\rangle,$$

as witnessed by the ordinals ρ_0, ρ_1 respectively, then we claim that the transitive collapse of $\mathrm{Hull}^{M_2}(\rho_0 \cup \{x_2\})$ is M_0 and x_0 is the image of x_2 under this collapse map. This is true since the Skolem hull of a Skolem hull is a hull.

Before proceeding, it is convenient to attach to each node in the tree an ordinal (that behaves like the projectum), $\rho(\bar{M}, \bar{x})$, which is the least ordinal ρ such that $\bar{M} = \mathrm{Hull}^{\bar{M}}(\rho \cup \{\bar{x}\})$. We remark that in general this is not the fine structural projectum, since we restrict the parameter. It is possible to modify the definition of the tree to include the standard parameter as well, assuming that M is a pre-mouse. In this case, $\rho(\bar{M}, \bar{x})$ will be the final projectum of $\bar{M}$, independently of $\bar{x}$.

Claim 18. *Let $\langle M_0, x_0\rangle \leq_{\mathbb{T}(M)} \langle M_1, x_1\rangle$. Then M_0 is the transitive collapse of $\mathrm{Hull}^{M_1}(\rho(M_0, x_0) \cup \{x_1\})$ and x_1 is sent to x_0 by the transitive collapse.*

Proof. By definition, there is some ρ such that

$$M_0 \cong \mathrm{Hull}^{M_1}(\rho \cup \{x_1\}),$$

and x_1 is sent to x_0 by the collapse. Clearly, $M_0 = \mathrm{Hull}^{M_0}(\rho \cup \{x_0\})$, so $\rho \geq \rho(M_0, x_0)$. On the other hand, if we let $M_0' \cong \mathrm{Hull}^{M_1}(\rho(M_0, x_0) \cup \{x_1\})$, then we get that there is an elementary embedding $\iota\colon M_0' \to M_0$ (by first applying the uncollapse map to M_0' and then applying the restriction of the collapse map witnessing $M_0 \cong \mathrm{Hull}^{M_1}(\rho \cup \{x_1\})$), with critical point $\geq \rho(M_0, x_0)$ sending x_0' to

x_0. But every element in M_0 is definable from x_0 and a sequence of ordinals below $\rho(M_0, x_0)$, so $\iota = id$. □

Let us assume now that $\langle M_0, x_0\rangle$ and $\langle M_1, x_1\rangle$ are both below $\langle N, y\rangle$ in the tree. We want to claim that they are compatible. We have, $M_0 \cong \mathrm{Hull}^N(\rho(M_0, x_0) \cup \{y\})$ and $M_1 \cong \mathrm{Hull}^N(\rho(M_1, x_1) \cup \{y\})$. Assume without loss of generality that $\rho(M_0, x_0) \le \rho(M_1, x_1)$. Then $M_0 \cong \mathrm{Hull}^{M_1}(\rho(M_0, x_0) \cup \{x_1\})$.

By the same argument, we can verify that below every element in the tree, the branch is well ordered and of length $< \kappa$. The tree $\mathbb{T}(M)$ in general might not be splitting. Indeed, if one can define x from an ordinal below κ in M then for every high enough pair $\langle \bar{M}, \bar{x}\rangle$ such that $\bar{x}$ is the collapse of x, there is only one possible extension. Nevertheless, this is the only restriction.

Claim 19. *Let $\langle \bar{M}, \bar{x}\rangle$ be in $\mathbb{T}(M)$ and let $\iota\colon \bar{M} \to M$ be the inverse of the transitive collapse. Let p be the following type:*

$$p(z) = \{\varphi(z, \iota(r)) \mid r \in \bar{M}, \bar{M} \models \varphi(\bar{x}, r)\}.$$

If there is some $y \in M$ with $y \neq x$ such that $p(y)$ holds then there are two incompatible nodes in the tree above $\langle \bar{M}, \bar{x}\rangle$.

Proof. Let $\delta < \kappa$ be sufficiently large so that $x(\delta) \neq y(\delta)$. Let us claim that $\langle N_0, \bar{x}'\rangle, \langle N_1, \bar{y}\rangle$ which are the transitive collapses of $\mathrm{Hull}^M((\delta{+}1) \cup \{x\}), \mathrm{Hull}^M((\delta + 1) \cup \{y\})$ respectively are both above $\langle \bar{M}, \bar{x}\rangle$ and incompatible.

First, $\bar{M} \cong \mathrm{Hull}^M(\rho(\bar{M}, \bar{x}) \cup \{y\})$, by the definition of y as every set in $\bar{M}$ which is definable from x is definable from y by the same formula. Moreover, the image of y under the transitive collapse is $\bar{x}$. Finally, $\rho(\bar{M}, \bar{x}) \le \delta$, as otherwise, $x(\delta)$ would be decided by the type p.

Now we argue that $\langle N_0, \bar{x}'\rangle$ and $\langle N_1, \bar{y}\rangle$ are incompatible. Otherwise, without loss of generality, $\langle N_0, \bar{x}'\rangle \le_{\mathbb{T}(M)} \langle N_1, \bar{y}\rangle$. But then, there is an elementary embedding $\iota\colon N_0 \to N_1$ sending $\bar{x}'$ to $\bar{y}$. But the critical point of this embedding is $> \delta$, and $\bar{x}'(\delta) = x(\delta) \neq y(\delta) = \bar{y}(\delta)$. □

Finally, note that for every $x \in {}^{\kappa}2$, there is a branch b_x defined by the transitive collapses of $\mathrm{Hull}^M(\rho \cup \{x\})$ for $\rho < \kappa$. It is clear that there are κ many different elements in this chain and that one can reconstruct x from the branch (as the limit of the $\bar{x}$'s). □

During the proof of Theorems 6 and 7, we will construct trees of the form $\mathbb{T}(M)$ and we will claim that they are sealed. In order to do this we will use iterability and maximality of M. The next lemma will help us to exploit these properties.

Lemma 20. *Let M be a countably iterable premouse. Let b be a branch in the tree $\mathbb{T}(M)$ and let R_b be the direct limit of the models on the branch. Then R_b is a countably iterable premouse. Moreover, $\rho_\omega(R_b) = \kappa$.*

Proof. R_b is well founded as a direct limit of uncountable cofinality of well founded models. Since each of the models is a premouse, so is R_b. Before be show iterability, let us observe that the projectum of R_b is κ.

First, note that $\rho_\omega(R_b) \ge \kappa$ as for any $\rho < \kappa$ there is some $\langle M, x\rangle \in b$ such that $\rho \subseteq M$. Now, for $\langle M, x\rangle \in b$, consider $\rho(M, x)$ as in the proof of Lemma 17, i.e. $\rho(M, x)$ is the least ordinal ρ such that $M = \mathrm{Hull}^M(\rho \cup \{x\})$.

For $\alpha < \kappa$, write $\langle M_\alpha, x_\alpha\rangle$ for the element of b on level α of the tree. First, we show that $\rho(M_\alpha, x_\alpha) \le \alpha$ for all $\alpha < \kappa$. Let $\alpha < \kappa$ and $z \in M_\alpha$ be arbitrary and suppose inductively that $\rho(M_\beta, x_\beta) \le \beta$ for all $\beta < \alpha$. We need to argue that $z \in \mathrm{Hull}^{M_\alpha}(\alpha \cup \{x_\alpha\})$. Since in the successor case it is easy to see that $\rho(M_{\alpha+1}, x_{\alpha+1}) \le \alpha + 1$, we focus on the case that α is a limit ordinal, i.e. M_α is the direct limit of M_β, $\beta < \alpha$, together with the natural hull embeddings along the branch. Write $\pi_{\beta,\alpha}\colon M_\beta \to M_\alpha$ for the direct limit embedding. In particular, there is some $\beta < \alpha$ and $\bar{z} \in M_\beta$ such that $\pi_{\beta,\alpha}(\bar{z}) = z$. Write $X = \mathrm{Hull}^{M_\alpha}(\rho(M_\beta, x_\beta) \cup \{x_\alpha\})$. Then by Claim 18, $\mathrm{Hull}^{M_\beta}(\rho(M_\beta, x_\beta) \cup \{x_\beta\}) = M_\beta = \mathrm{trcl}(X)$. So $\bar{z}$ is definable in M_β from ordinals below $\rho(M_\beta, x_\beta)$ and x_β. By elementarity of the inverse of the collapse embedding $\pi\colon M_\beta \to X \prec M_\alpha$ and since $\pi(\bar{z}) = z$, z is definable in M_α from ordinals below $\rho(M_\beta, x_\beta) \leq \alpha$ and x_α, as desired.

By pressing down, there is no stationary set of ordinals $\alpha < \kappa$ with the property that $\rho(M_\alpha, x_\alpha) < \alpha$ (consider the function $f\colon \kappa \to \kappa$ given by $f(\alpha) = \rho(M_\alpha, x_\alpha)$, if there would be such a stationary set, then this would be regressive on a stationary set and therefore constant on a stationary set, which contradicts the fact that for all ordinals $\gamma < \kappa$ there is a tail of ordinals $\alpha < \kappa$ such that $\rho(M_\alpha, x_\alpha) > \gamma$). So there is a club of $\alpha < \kappa$ such that $\rho(M_\alpha, x_\alpha) = \alpha$. Therefore the direct limit along the branch satisfies $\rho(R_b, x_b) \leq \kappa$. Altogether we have $\rho_\omega(R_b) \leq \rho(R_b, x_b) \leq \kappa$ and hence $\rho_\omega(R_b) = \kappa$.

Finally, we argue that R_b is countably iterable. Let θ be a sufficiently large regular cardinal and let $N \prec H(\theta)$ be a countable elementary substructure that contains all relevant objects. Then $\delta = \sup(N \cap \kappa) \in \kappa$ (since $\mathrm{cf}(\kappa) > \omega$). We would like to construct an iteration strategy for the model $\tilde{N} = R_b \cap N$. This would be enough, by the definition of being countably iterable.

For $\langle M_\alpha, x_\alpha\rangle$ and $\langle M_\beta, x_\beta\rangle$ on the branch b with $\langle M_\alpha, x_\alpha\rangle <_{\mathbb{T}(M)} \langle M_\beta, x_\beta\rangle$ let $j_{\alpha,\beta}\colon M_\alpha \to M_\beta$ be the canonical embedding sending x_α to x_β. Moreover, let $j_{\alpha,\infty}\colon M_\alpha \to R_b$ be the direct limit embedding. Now let us consider the δ-th element of the branch $\tilde{M} = \mathrm{trcl}(\mathrm{Hull}^M(\delta \cup \{x_\delta\}))$. $\tilde{M}$ is (ω_1, ω_1+1)-iterable as an elementary substructure of the countably iterable premouse M. Let $\pi\colon R_b \cap N \to \tilde{M}$ be the following embedding. Let $a \in R_b \cap N$. Then, by the definition of the direct limit, there is some model $X = b(\alpha)$ for $\alpha \in N$ and some $\bar{a} \in X$ such that $a = j_{\alpha,\infty}(\bar{a})$. Let $\pi(a) = j_{\alpha,\delta}(\bar{a})$.

Let us verify that π is well defined and fully elementary. So, let $a \in R_b \cap N$ and let $\bar{a}, \bar{a}'$ be two elements such that $j_{\alpha,\infty}(\bar{a}) = j_{\beta,\infty}(\bar{a}') = a$ for some ordinals α, β. Then, $\alpha, \beta < \sup(N \cap \kappa) = \delta$ and in particular $j_{\alpha,\delta}(\bar{a}) = j_{\beta,\delta}(\bar{a}')$. Thus, π is well defined. Let us show that π is elementary. Indeed, let $a \in R_b \cap N$ and let us assume that $\varphi(a)$ holds in $\tilde{N} = R_b \cap N$. Then $\varphi(a)$ holds in R_b (by the Tarski-Vaught criterion, $\tilde{N}$ is an elementary substructure of R_b). Therefore $\varphi(\bar{a})$ holds in $b(\alpha)$, using the elementarity of $j_{\alpha,\infty}$. Finally, $\varphi(\pi(a)) = \varphi(j_{\alpha,\delta}(\bar{a}))$ holds in $\tilde{M}$.

We conclude that there is an elementary embedding from $\tilde{N}$ to $\tilde{M}$ and therefore $\tilde{N}$ is $(\omega_1, \omega_1 + 1)$-iterable by the pullback strategy. □

The above proof might suggest that when R_b is really an initial segment of M then x_b should be a member of ${}^{\kappa}2$ of M, thus enabling us to get a full characterization of the branches of $\mathbb{T}(M)$. Unfortunately, this is not the case in general. For example, if κ is measurable in M and U is a normal filter on κ, we can consider $\mathrm{Ult}(M, U)$ and the sequence $x \in \mathrm{Ult}(M, U)$ such that $x(\kappa) = 1$ and $x(\alpha) = 0$ for all $\alpha \neq \kappa$. Then we can obtain a branch by taking hulls, but it is clearly not generated by any $x \in {}^{\kappa}2$.

5. Trees in K

Let us prove Theorem 6.

Proof. Using the anti-large cardinal hypothesis that there is no inner model with a Woodin cardinal, we can construct the core model K as in [JS13] (building on [St96]). Let κ be an uncountable cardinal and let us consider the tree $\mathbb{T}(K|\kappa^+)$.

Here we are referring to κ^+ as computed in K. Let b be a branch in the tree in some generic extension $V[G]$. The direct limit along the branch, R_b, is a countably iterable premouse. By the forcing absoluteness of K, $K^{V[G]} = K^V$ and in particular, K^V is still universal, as in [St96]. Thus, when we compare it with the premouse R_b, the comparison finishes successfully after set many steps and the R_b-side looses. We now argue that R_b does not move in this comparison. As by Lemma 20 $\rho_\omega(R_b) = \kappa$, the R_b-side can only use extenders with critical point $<\kappa$. Since $R_b|\kappa = K|\kappa$ this means that if the iteration on the R_b-side moves, it consists of applying one extender F overlapping κ. The argument for iterability of R_b in Lemma 20 shows that R_b can be elementarily embedded into a hull of $K|\kappa^+$. Hence (K, F) is weakly countably certified in the sense of [SchSt99, Definition 2.2] and [SchSt99, Theorem 2.3] implies that F is on the K-sequence. Therefore, there was no need to use F in the comparison and R_b does not move.

Let $\mathcal{T}$ be the comparison tree based on K. We denote the α-th model in the tree $\mathcal{T}$ by $\mathcal{M}^{\mathcal{T}}_\alpha$ and the direct limit model by $\mathcal{M}^{\mathcal{T}}_\infty$. Let us analyse the possible extenders E which are used during the iteration. Since $R_b|\kappa = K|\kappa$, if we apply an extender E with critical point below κ, then $\mathrm{lh}(E) \geq \kappa$. As κ is an uncountable cardinal, in fact $\mathrm{lh}(E) > \kappa$ and $j_E(\mathrm{crit}(E)) \geq \kappa$. So, we can apply Lemma 14 and conclude that $\mathcal{M}^{\mathcal{T}}_\infty || (\kappa^+)^{\mathcal{M}^{\mathcal{T}}_\infty}$ is an initial segment of K. Since R_b is an initial segment of $\mathcal{M}^{\mathcal{T}}_\infty$, the branch b itself is going to be constructed before step $(\kappa^+)^{R_b} \leq (\kappa^+)^{\mathcal{M}^{\mathcal{T}}_\infty}$. Therefore the tree $\mathbb{T}(K|\kappa^+)$ has exactly $(2^\kappa)^K = (\kappa^+)^K$ many branches in $V[G]$. If κ is weakly compact, the covering lemma [SchSt99, Theorem 3.1] implies that $(\kappa^+)^K = (\kappa^+)^V$. □

Let κ be an inaccessible cardinal. By applying the Löwenheim-Skolem Theorem, we can see that below an element $(\bar{M}, \bar{x}) \in \mathbb{T}(M)$, with $\rho = \rho(\bar{M}, \bar{x})$ uncountable, there are at least $|\rho|$ many predecessors. Thus, if $(\bar{M}, \bar{x})$ is an element of level α in the tree then it has size at most $|\alpha| + \aleph_0$ and in particular, there are at most $2^{|\alpha|+\aleph_0} < \kappa$ such elements.

On the other hand, when κ is a successor cardinal, the tree $\mathbb{T}(K|\kappa^+)$ is not a κ-tree since it contains κ many elements at bounded levels. Clearly, one can obtain a sealed Kurepa tree on ω_1 by starting with an inaccessible cardinal κ in K and collapsing it to be $\aleph_1$, but it is still unclear how to obtain the same object in K.

Question 2. Is there a strongly sealed Kurepa tree on ω_1 in K?

By slightly modifying the construction of $\mathbb{T}(M)$, we obtain the following strong sealing property for κ-trees in K, where κ is inaccessible.

Definition 21. Let T, S be trees and let $h\colon T \to S$ be an order preserving function. We say that h is *full* if for every branch $b \in [S]$ there is a branch $\tilde{b} \in [T]$ such that $h``\,\tilde{b}$ generates b.

Let M be a model of ZFC − (Power set). Let $T \in M$ be a κ-tree (in our case, $M = K|\kappa^+$). We define $\mathbb{T}(M, T)$ to be the collection of transitive collapses of models of the form $\mathrm{Hull}^M(\rho \cup \{x, T\})$ together with the images of x and T under the transitive collapse, where $\rho < \kappa$ and x is a cofinal branch of T in M. The tree order is defined accordingly.

Theorem 22. *Let $T \in K$ be a subtree of $2^{<\kappa}$. Then there is a full homomorphism h from $\mathbb{T}(K|\kappa^+, T)$ to T and the tree $\mathbb{T}(K|\kappa^+, T)$ is strongly sealed.*

Proof. Let us define h to send an element of the form $(M, \bar{x}, \bar{T})$ to $\bar{x} \restriction \rho(M, \bar{x}, \bar{T})$, where $\rho(M, \bar{x}, \bar{T})$ is defined analogously to $\rho(M, \bar{x})$ in the proof of Lemma 17. $\bar{x} \restriction \rho(M, \bar{x}, \bar{T}) \in T$ since the critical point of the inverse of the collapse from M to

$K|\kappa^+$ is at least $\rho(M, \bar{x}, \bar{T})$. Indeed, M is the transitive collapse of a model of the form $\mathrm{Hull}(\rho \cup \{x, T\})$ for some branch x of T. By the definition $\rho(M, \bar{x}, \bar{T}) \leq \rho$ and the critical point of the inverse of the transitive collapse is $\geq \rho$. We conclude that $\bar{x}$ is the same as x up to ρ and in particular, belongs to T.

A similar argument shows that h respects the structure of the tree. Every branch of T is represented by a branch of $\mathbb{T}(K|\kappa^+, T)$ by the definition.

Let us show that the tree is strongly sealed. Up to a standard coding, $\mathbb{T}(K|\kappa^+, T)$ is a subtree of $\mathbb{T}(K|\kappa^+)$ and any unbounded branch through it corresponds to an unbounded branch through $\mathbb{T}(K|\kappa^+)$. Thus, if b is a new branch of $\mathbb{T}(K|\kappa^+, T)$ added by the forcing then we already have $b \in K$, which is absurd as being a cofinal branch is absolute. □

Let us remark that in a model of PFA, every ω_1-tree has a most $\aleph_1$ many branches and it is (strongly) sealed, in the sense that it is specialized. In this model there are obviously no Kurepa trees.

Question 3. Is it possible to obtain a model with a strongly sealed κ-Kurepa tree using forcing?

6. Stacking mice

In this section we prove Theorem 7. We first recall the definition of domestic premouse from [ANS01].

Definition 23. A premouse M is called *domestic* if there is no initial segment $N \trianglelefteq M$ with non-empty top extender F^N such that $\mathrm{crit}(F^N)$ is a limit of Woodin cardinals in N and $\mathrm{crit}(F^N)$ is a limit of strong cardinals in $N||\,\mathrm{crit}(F^N)$.

Moreover, we will use Jensen's notion of a stack of mice from [JSSS09].

Definition 24. Let $\mathcal{N}$ be a premouse such that $\mathcal{N} \cap \mathrm{Ord}$ is an uncountable regular cardinal. Then, if it exists, $\mathcal{S}(\mathcal{N})$ denotes the unique premouse $\mathcal{S}$ such that $\mathcal{M} \trianglelefteq \mathcal{S}$ iff there is a sound countably iterable premouse $\mathcal{M}^* \trianglerighteq \mathcal{N}$ with $\rho_\omega(\mathcal{M}^*) = \mathcal{N} \cap \mathrm{Ord}$ such that $\mathcal{M} \trianglelefteq \mathcal{M}^*$.

In the context that K^c as in [ANS01] exists, there is no premouse with a superstrong extender, and κ is a regular uncountable cardinal $\mathcal{S}(K^c||\kappa)$ as defined above exists by [JSSS09, Lemma 3.1].

Proof. Assume that there is no non-domestic premouse. Then K^c as in [ANS01] exists and there is no premouse with a superstrong extender. So we can consider the tree $\mathbb{T}(\mathcal{S})$ for $\mathcal{S} = \mathcal{S}(K^c||\kappa)$ the stack of mice on $K^c||\kappa$. We claim that $\mathbb{T}(\mathcal{S})$ is a sealed tree with exactly κ^+ many branches.

Claim 25. *$\mathbb{T}(\mathcal{S})$ has exactly $(\kappa^+)^V$ many branches.*

Proof. By Lemma 17, $\mathbb{T}(\mathcal{S})$ has at least $(\kappa^+)^{\mathcal{S}} := \mathcal{S} \cap \mathrm{Ord}$ many branches. We argue that every branch b through $\mathbb{T}(\mathcal{S})$ in V is already in $\mathcal{S}$. By our anti-large cardinal hypothesis, covering holds for $\mathcal{S}$ in the sense of [JSSS09, Lemma 5.1], i.e., $\mathcal{S} \cap \mathrm{Ord} = (\kappa^+)^V$. Therefore it follows that $\mathbb{T}(\mathcal{S})$ has $(\kappa^+)^V$ many branches, as desired.

Let b be an arbitrary branch through $\mathbb{T}(\mathcal{S})$ and consider the direct limit $\langle R_b, x_b \rangle$ along the branch b given by the natural hull embeddings $\pi_{\bar{M},M} \colon \bar{M} \to M$ witnessing $\langle \bar{M}, \bar{x} \rangle \leq_{\mathbb{T}(\mathcal{S})} \langle M, x \rangle$ for $\langle \bar{M}, \bar{x} \rangle, \langle M, x \rangle \in b$. We will identify R_b with its transitive collapse and argue that $R_b \trianglelefteq \mathcal{S}$. This suffices since we can recover b from R_b as in the proof of Lemma 15. $\mathcal{S}$ is countably iterable by [JSSS09, Corollary 2.11]. So Lemma 20 yields that R_b is a countably iterable premouse with $\rho_\omega(R_b) = \kappa$. Therefore $R_b \trianglelefteq \mathcal{S}$ by definition of the stack $\mathcal{S}$. □

Claim 26. $\mathbb{T}(\mathcal{S})$ *is sealed.*

Proof. Let $\mathbb{P}$ be a partial order satisfying the conditions in Definition 5 and let G be $\mathbb{P}$-generic over V. By [NeSt16, Corollary 3.4], building on Jensen's results in [JSSS09], we have that $\mathcal{S} = \mathcal{S}^{V[G]}((K^c||\kappa)^V)$, the stack of mice as constructed in $V[G]$ on top of $(K^c||\kappa)^V$. Now let b be an arbitrary branch through $\mathbb{T}(\mathcal{S})$ in $V[G]$ and consider the direct limit $\langle R_b, x_b\rangle$. Then R_b is a premouse and $K^c||\kappa \trianglelefteq R_b$. Note that $\mathcal{S}^{V[G]}((K^c||\kappa)^V)$ is countably iterable in $V[G]$ by construction. Therefore R_b is countably iterable in $V[G]$ and $\rho_\omega(R_b) = \kappa$ by Lemma 20. So by definition of the stack, $R_b \trianglelefteq \mathcal{S}^{V[G]}((K^c||\kappa)^V) = \mathcal{S}$ and hence $b \in V$, as desired. $\square$

Finally, let us consider the following lemma which is a strengthening of the corresponding lemma in [JSSS09].

Lemma 27. *Let κ be a weakly compact and assume that the K^c construction, in the sense of* [JSSS09], *works up to κ and $\mathcal{S}(K^c||\kappa) \cap \mathrm{Ord} < \kappa^+$. Then κ is $(\kappa^+)^{\mathcal{K}}$-Π^1_1-subcompact in $\mathcal{K} = L[\mathcal{S}(K^c||\kappa)]$.*

Proof. Let $\mathcal{S} = \mathcal{S}(K^c||\kappa)$ be such that $\eta = \mathcal{S} \cap \mathrm{Ord} < \kappa^+$ and let $\mathcal{K} = L[\mathcal{S}]$. We claim that for every predicate $R^* \subseteq H(\eta)^{\mathcal{K}}$ and Π^1_1-statement Ψ there is some $\bar\kappa$, $\bar{R}^*$ and an elementary embedding $j\colon H(\bar\kappa^+)^{\mathcal{K}} \to H(\eta)^{\mathcal{K}}$, sending $\bar{R}^*$ to R^*, such that the smaller model satisfies Ψ as well. Let us focus on the case that Ψ is trivial and that R^* is empty. The general case is proved similarly, only with additional notational complexity.

Working in V, let $R \subseteq \kappa$ code $\mathcal{S}$, η and their bijections with κ, as well as V_κ and let Φ be the Π^1_1-statement over the model $\langle V_\kappa, \in, R\rangle$, saying that $\mathcal{S}$ is the stack of mice over $K^c||\kappa$. Strictly speaking, this formula is Π^1_1 over the logic $\mathcal{L}_{\omega_1,\omega_1}$ (as it refers also to well foundedness and iterability) but one can easily verify that weakly compact cardinals reflect such statements as well. Informally, the statement is: For every mouse $\mathcal{M}$ that projects to κ, $\mathcal{M} \trianglelefteq \mathcal{S}$. Formally, let Φ be as follows: For every $X \subseteq V_\kappa$ such that X codes a well founded premouse $\mathcal{M}$ with projectum κ, either there is a countable non-iterable premouse $\mathcal{N}$ which embeds into $\mathcal{M}$ or $\mathcal{M} \trianglelefteq \mathcal{S}$.

Let us analyse the complexity of this statement. The statement that X is well founded is expressible in $\mathcal{L}_{\omega_1,\omega_1}$. The statement that it is a premouse with projectum κ is first order and finally the iterability is again expressible in $\mathcal{L}_{\omega_1,\omega_1}$ by including in the parameter R the collection of all iterable countable mice. For simplicity, we will include in Φ the information that κ is inaccessible and add a parameter coding the elementary diagram of $\mathcal{S}$.

Let $\bar\kappa < \kappa$ be reflecting Φ. So $\bar\kappa$ is an inaccessible cardinal and there is some ordinal $\bar\eta$ (reflecting η), which is the height of $\bar{\mathcal{S}}$. Moreover, the restriction defines an elementary embedding j from $\bar{\mathcal{S}}$ to $\mathcal{S}$, sending $\bar\kappa$ to κ.

Let us claim that $\bar{\mathcal{S}}$ is exactly $K^c||\bar\eta$ and $\bar\eta$ is the successor of $\bar\kappa$ in K^c. Indeed, over $V_{\bar\kappa}$, Φ still says that $\bar{\mathcal{S}}$ is a maximal iterable premouse over $K^c||\bar\kappa$. In particular, no new subsets of $\bar\kappa$ are introduced to $\mathcal{S}$ above $\bar\eta$.

Let us now derive an extender F from the embedding j, with generators $a \in \kappa^{<\omega}$. As in [JSSS09], we show that any initial segment of it is in $\mathcal{S}$, and that F is certified by a collapse and thus conclude that it appears in the sequence of $\mathcal{S}$ (one can verify that using F one can compute j completely). Let us consider the following diagram.

$$\begin{tikzcd} K^c||\bar\eta = \mathcal{S}||\bar\eta \arrow[r, "j"] \arrow[d, "\iota_{F\restriction\alpha}"'] & \mathcal{S} \\ \tilde{\mathcal{K}} \arrow[ur, "k_\alpha"] & \end{tikzcd}$$

Here $\iota_{F\restriction\alpha}$ is the ultrapower using the extender $F \restriction \alpha$ and k_α is the quotient map. Consider successor ordinals α. The critical point of k_α is at least $i_\alpha = (\alpha^+)^{\tilde{K}}$. But this is exactly the supposed index of $F \restriction (\alpha + 1)$ in the sequence of extenders, thus $\tilde{\mathcal{K}}||i_\alpha = \mathcal{K}||i_\alpha$, showing that the extender coheres. F (and its initial segments) is clearly certified by a collapse. Thus, for cofinally many α, $F \restriction \alpha$ is in $\mathcal{K}$. □

□

While Lemma 27 is an overkill, as the assumption of the proof currently does not permit even the existence of mice with superstrong cardinals it is of some interest. Indeed, if there is a construction of K^c with extremely large cardinals on the level of supercompactness, and similar indexing scheme, then the arguments of the lemma will work with no significant modification. Thus, the following conjecture seems reasonable.

Conjecture. Let κ be a weakly compact cardinal with the Perfect Subtree Property. Then there is an inner model with a pair of cardinals $\lambda < \mu$ such that λ is $<\mu$-supercompact and μ is inaccessible.

In order to justify the above conjecture, let us use the notations and definitions from [NeSt16]. In this paper, Neeman and Steel construct, using a different certification assumption (allowing long extenders) and the iterability hypothesis SBH_δ, a fine structural mouse $\mathcal{W}$ which behaves much like $K^c||\delta$ in terms of its maximality. They show that if δ is a Woodin cardinal, then the corresponding stack $\mathcal{S}(\mathcal{W})$ enjoys a very weak form of covering, namely, its height has cofinality $\geq \delta$ in V. For the full details about the definition of certification as well as SBH_δ, we refer to [NeSt16]. Following Neeman and Steel, we use the following hypothesis which is tailored in order to enable us to certify (in the strong sense) elementary embeddings which are obtained from weakly compact embeddings.

Definition 28. A cardinal κ is Π^1_1-Woodin, if for every $A \subseteq V_\kappa$, and for every Π^1_1-statement Φ for which $\langle V_\kappa, \in, A\rangle \models \Phi$, there is a $<\kappa$-A-strong cardinal μ such that $\langle V_\mu, \in, A \cap V_\mu\rangle \models \Phi$.

As discussed in [NeSt16], the least Π^1_1-Woodin cardinal is smaller than the least Shelah cardinal and is a limit of measurable Woodin cardinals.

Theorem 29. *Let κ be Π^1_1-Woodin and assume SBH_κ. Then either κ is Π^1_1-κ^+-subcompact in an inner model or there is a sealed κ-tree on κ with κ^+ many branches.*

Proof. Let $\mathcal{W}$ be the mouse obtained from [NeSt16, Lemma 2.4] and let $\mathcal{M} = \mathcal{S}(\mathcal{W})$. Let $\gamma = \mathcal{S}(\mathcal{W}) \cap \mathrm{Ord}$. By the covering lemma, [NeSt16, Claim 4.1], $\mathrm{cf}(\gamma) \geq \kappa$ and $\mathcal{M}$ is an iterable mouse. Moreover, in $L[\mathcal{M}]$, $\gamma = \kappa^+$.

Let us split into two cases. Either $\gamma = (\kappa^+)^V$ and in this case the tree $\mathbb{T}(\mathcal{M})$ has κ^+ many branches. Then by the argument of Claim 26, this tree is sealed.

Now let us assume that $\gamma < (\kappa^+)^V$. We want to show that in $L[\mathcal{M}]$, κ is γ-Π^1_1-subcompact. We follow the same arguments as in Lemma 27, but we need to verify a different certification assumption about our extender: in this construction, a certification using a V-extender is used, instead of an extender on a weaker model as in [JSSS09].

At this point we use the stronger assumption on κ:

Claim 30. *Let κ be a weakly compact Woodin cardinal. Then for every $A \subseteq V_\kappa$ there is a cardinal μ which is κ-A-strong.*

Proof. Since κ is Woodin, there is some μ which is $<\kappa$-A-strong. Let S be the tree of all extenders with critical point μ which are α-A-strong and of length α for some inaccessible $\alpha < \kappa$, ordered by end extension. Even though not all nodes in the tree can be extended, the tree can be pruned to a normal κ-tree and by weak compactness it has a branch. It is clear that a cofinal branch corresponds to an extender of length κ which is κ-A-strong. $\square$

Claim 31. *Let κ be Π^1_1-Woodin, let $\mathcal{M}$ be as above, let $A \subseteq (\kappa^+)^{\mathcal{M}}$ be in $L[\mathcal{M}]$, and fix a Π^1_1-statement Φ such that $\langle \mathcal{M}, \in, A\rangle \models \Phi$. Then there is an elementary embedding $j\colon \mathcal{M}||\,(\bar\kappa^+)^{\mathcal{M}} \to \mathcal{M}$, certified by a short V-extender F^*, such that $\langle \mathcal{M}||\,(\bar\kappa^+)^{\mathcal{M}}, \in, A \cap (\bar\kappa^+)^{\mathcal{M}}\rangle \models \Phi$.*

Proof. In order to obtain the elementary embedding j we follow the same construction as in Lemma 27. The coherence of the extender derived from j follows from the same arguments. The only difference is that we have to make sure that j is certified by a V-extender. For this, we follow the arguments from [NeSt16].

Working in V, let $\langle H_\alpha \mid \alpha < \kappa\rangle$ be a continuous chain of elementary submodels of $\langle \mathcal{M}, \in, A\rangle$, $|H_\alpha| < \kappa$, $H_\alpha \cap \kappa \in \kappa$ and at least α. Let $\bar H_\alpha$ be the transitive collapse of H_α and let $\sigma_{\alpha,\kappa}$ be the anti-collapse elementary embedding. Let $\sigma_{\alpha,\beta} = \sigma^{-1}_{\beta,\kappa} \circ \sigma_{\alpha,\kappa}$ be the corresponding elementary embedding from $\bar H_\alpha$ to $\bar H_\beta$. Let $\tilde A$ be a predicate on κ coding all relevant information, including the sequence $\langle \bar H_\alpha, \sigma_{\alpha,\beta} \mid \alpha < \beta < \kappa\rangle$, A, and any additional information which is required for the definition of $\mathcal{M}$ and $\mathcal{W}$. Let $\tilde\Phi$ be a combination of the assertion that the Π^1_1 statement Φ holds in $L_{\kappa^+}[\mathcal{M}]$ and the Π^1_1-statement stating that $\mathcal{M}$ is the stack above $\mathcal{W}$.

At this point, we use the Π^1_1-Woodinness of κ and assume that the cardinal $\bar\kappa$ which reflects the corresponding $\tilde\Phi$-statement is also κ-$\tilde A$-strong. By the choice of $\tilde A$, $\bar H_{\bar\kappa}$ is the stack above $\mathcal{W}||\bar\kappa$. By condensation, $\bar H_{\bar\kappa}$ is $\mathcal{W}||\,(\bar\kappa^+)^{\mathcal{W}}$. Write $j = \sigma_{\bar\kappa,\kappa}$.

Let F^* be an extender on $\bar\kappa$ as above, and let $X \in \mathcal{P}(\bar\kappa)^{\mathcal{M}}$. We need to verify that $i_{F^*}(X) \cap \kappa = j(X)$. Indeed, $i_{F^*}(\langle \sigma_{\alpha,\beta} \mid \alpha < \beta < \kappa\rangle) \restriction \kappa = \langle \sigma_{\alpha,\beta} \mid \alpha < \beta < \kappa\rangle$ and $i_{F^*}(\bar H)_{\bar\kappa} = \bar H_{\bar\kappa}$. Since $\mathrm{crit}(i_{F^*}) = \bar\kappa$, for every $Y \in \bar H_\alpha$, $\alpha < \bar\kappa$, $i_{F^*}(Y) = Y$.

Let $\lambda = i_{F^*}(\bar\kappa) > \kappa$. Let us denote $i_{F^*}(\bar H_{\bar\kappa}) = \bar H^*_\lambda$. By the elementarity of i_{F^*}, there is an elementary embedding $\sigma^*_{\kappa,\lambda}\colon \bar H^*_\kappa \to \bar H^*_\lambda$. Since both $\bar H^*_\kappa$ and $\mathcal{M}$ are transitive direct limits of the system $\langle \bar H_\alpha, \sigma_{\alpha,\beta} \mid \alpha < \beta < \kappa\rangle$, necessarily $\bar H^*_\kappa = \mathcal{M}$. Let $X \in \bar H_{\bar\kappa}$. We claim that $i_{F^*}(X) = \sigma^*_{\kappa,\lambda} \circ \sigma_{\bar\kappa,\kappa}(X)$ (and since the critical point of $\sigma^*_{\kappa,\lambda}$ is $\geq \kappa$, this is enough to certify j). Indeed, since $\bar\kappa$ is a limit ordinal, any $X \in \bar H_{\bar\kappa}$ is of the form $\sigma_{\alpha,\bar\kappa}(\bar X)$ for some $\bar X \in \bar H_\alpha$. Thus,

$$\begin{aligned} i_{F^*}(X) &= i_{F^*}(\sigma_{\alpha,\bar\kappa}(\bar X)) \\ &= \sigma^*_{\alpha,\lambda}(i_{F^*}(\bar X)) \\ &= \sigma^*_{\kappa,\lambda} \circ \sigma_{\bar\kappa,\kappa} \circ \sigma_{\alpha,\bar\kappa}(\bar X) \\ &= \sigma^*_{\kappa,\lambda} \circ \sigma_{\bar\kappa,\kappa}(X). \end{aligned}$$

$\square$

This finishes the proof of Theorem 29. $\square$

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