

## LIMIT BEHAVIOR OF WEYL COEFFICIENTS

© R. PRUCKNER, H. WORACEK

The sets of radial or nontangential limit points towards  $i\infty$  of a Nevanlinna function  $q$  are studied. Given a nonempty, closed, and connected subset  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$ , a Hamiltonian  $H$  is constructed explicitly such that the radial and outer angular cluster sets towards  $i\infty$  of the Weyl coefficient  $q_H$  are both equal to  $\mathcal{L}$ . The method is based on a study of the continuous group action of rescaling operators on the set of all Hamiltonians.

### §1. Introduction

A Nevanlinna function is an analytic function in the open upper half-plane  $\mathbb{C}_+$  whose values lie in  $\mathbb{C}_+ \cup \mathbb{R}$ . Such functions are intensively studied for various reasons; we mention two of them.

▷ In complex analysis they occur as regularised Cauchy transforms of positive Poisson integrable measures, see e.g., [Lev80, KK68, GT00]. Namely, a function  $q$  is a Nevanlinna function if and only if it is of the form

$$q(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x), \quad z \in \mathbb{C}_+, \quad (1.1)$$

where  $a \in \mathbb{R}$ ,  $b \geq 0$ , and  $\mu$  is a positive Borel measure on the real line with

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty.$$

▷ In the spectral theory of differential operators they occur as Weyl coefficients whenever H. Weyl's nested disks method is applicable, see e.g., [Wey10, Tit46, Atk64, BHS20].

The relationship between these two instances is that (for simplicity we suppress some technical issues and exceptional cases) the measure  $\mu$  in the integral

---

*Ключевые слова:* Weyl coefficient, canonical system, cluster set, Nevanlinna function.

This work was supported by the project P 30715-N35 of the Austrian Science Fund. The second author was supported by the joint project I 4600 of the Austrian Science Fund (FWF) and the Russian Foundation of Basic Research (RFBR).

representation (1.1) of the Weyl coefficient of an equation is a spectral measure for the corresponding selfadjoint model operator.

The natural context of Weyl's method is the framework of two-dimensional canonical systems

$$y'(t) = zJH(t)y(t), \quad t \in (0, \infty), \quad (1.2)$$

where  $z \in \mathbb{C}$  is the eigenvalue parameter,  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and the Hamiltonian  $H$  of the system is assumed to satisfy  $H(t) \geq 0$  and  $\operatorname{tr} H(t) = 1$  a.e., see, e.g. [Bra68, HSW00, Rom14, Rem18]. It is a deep theorem due to L. de Branges that the map assigning to each Hamiltonian  $H$  the Weyl coefficient  $q_H$  of the equation (1.2) is a bijection between the set of all Hamiltonians

$$\mathbb{H} := \{H: (0, \infty) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ measurable, } H(t) \geq 0, \operatorname{tr} H(t) = 1 \text{ a.e.}\} \quad (1.3)$$

up to equality a.e., and the set of all Nevanlinna functions including the function identically equal to  $\infty$

$$\mathcal{N} := \{q: \mathbb{C}_+ \rightarrow \overline{\mathbb{C}} \mid q \text{ analytic, } q(\mathbb{C}_+) \subseteq \overline{\mathbb{C}_+}\}.$$

Here  $\overline{\mathbb{C}}$  denotes the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  regarded as a Riemann surface in the usual way, and  $\overline{\mathbb{C}_+}$  denotes the closure of  $\mathbb{C}_+$  in the sphere, explicitly,  $\overline{\mathbb{C}_+} = \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$ . The assignment  $H \mapsto q_H$  is also called the de Branges correspondence.

Having available this bijection, it is a natural task to relate properties of  $H$  to properties of  $q_H$ . For many properties of Hamiltonians or Nevanlinna functions it turns out to be quite involved (or even quite impossible) to find their counterpart on the other side of de Branges' correspondence. One type of properties where some explicit relations are known is the high-energy behavior of  $q_H$ , i.e., its behavior towards  $i\infty$ . It is a frequently instantiated intuition, going back at least to B.M. Levitan [Lev52], that the high-energy behavior of  $q_H$  corresponds to the local behavior of  $H$  at 0. For example it was shown in [EKT18] that the nontangential limit<sup>1</sup>  $\lim_{z \xrightarrow{*} i\infty} q_H(z)$  exists in  $\overline{\mathbb{C}}$  if and only if the limit  $\lim_{t \searrow 0} \frac{1}{t} \int_0^t H(s) ds$  exists in  $\mathbb{R}^{2 \times 2}$ . Moreover, if these limits exist, they are related by simple formulas.

In this paper we investigate the situation when the Weyl coefficient does not necessarily have a limit. Natural substitutes for a limit value are cluster sets. We consider two variants, which are fitted to the nontangential approach. For

---

<sup>1</sup>We write  $z_n \xrightarrow{*} i\infty$  for:  $|z_n| \rightarrow \infty$  while  $\arg z_n \in [\alpha, \pi - \alpha]$  for some  $\alpha \in (0, \frac{\pi}{2}]$ . And we write  $\lim_{z \xrightarrow{*} i\infty} q(z) = \zeta$  if  $\lim_{n \rightarrow \infty} q(z_n) = \zeta$  for every sequence  $z_n \xrightarrow{*} i\infty$ . Convergence on the Riemann sphere is understood with respect to the chordal metric.

$\alpha \in (0, \frac{\pi}{2}]$  denote by  $\Gamma_\alpha$  the Stolz angle

$$\Gamma_\alpha := \{z \in \mathbb{C}_+ \mid \arg z \in [\alpha, \pi - \alpha]\}.$$

(i) Let  $M \subseteq \mathbb{C}_+$  be such that

$$\begin{aligned} M \text{ is unbounded, there is } \alpha \in (0, \frac{\pi}{2}] \text{ with } M \subseteq \Gamma_\alpha, \\ \{z \in M \mid |z| \geq r\} \text{ is connected for all sufficiently large } r. \end{aligned} \tag{1.4}$$

For a Nevanlinna function  $q$  we consider the cluster set

$$\mathcal{C}(q, M) := \{\zeta \in \overline{\mathbb{C}} \mid \exists z_n \in M: |z_n| \rightarrow \infty \wedge q(z_n) \rightarrow \zeta\}.$$

(ii) The outer angular cluster set of a Nevanlinna function  $q$  is

$$\mathcal{C}_\times(q) := \bigcup_{\alpha \in (0, \frac{\pi}{2}]} \mathcal{C}(q, \Gamma_\alpha) = \{\zeta \in \overline{\mathbb{C}} \mid \exists z_n \in \mathbb{C}_+: z_n \xrightarrow{\times} \infty \wedge q(z_n) \rightarrow \zeta\}.$$

We do not consider an arbitrary — possibly tangential — approach to infinity.

The cluster sets  $\mathcal{C}(q, M)$  and  $\mathcal{C}_\times(q)$  are both nonempty and connected. They show different behavior in the sense that  $\mathcal{C}(q, M)$  is always closed, while  $\mathcal{C}_\times(q)$  need not have this property, cf. [CL66, Nos60] (see also Remark 3.7 below).

It is known from [BCP85] (by using a fractional linear transformation to pass from the half-plane to the unit disk) that for every nonempty, closed, and connected subset  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$  there exists a Nevanlinna function  $q$  such that the radial cluster set  $\mathcal{C}(q, i[1, \infty))$  equals  $\mathcal{L}$ . In fact, in [BCP85, Theorem] the radial boundary interpolation problem was solved for countably many interpolation nodes, and the given solution is a Blaschke product. Variants of this result for singular inner functions can be found in [Dec94, Theorem 9], or [Don01, Theorem 3]. For smaller classes of functions, e.g., interpolating or thin Blaschke products, the radial boundary interpolation problem is in general not anymore solvable, cf. [GM05, GS11]. The outer angular cluster set is a countable increasing union of nonempty, closed, and connected subsets, and P. M. Gauthier conjectured in [Gau21] (personal communication) that every set of this form can be realised as an outer angular cluster set; a proof has not yet been given.

Our main result in the present paper is Theorem 4.1, where we give an explicit solution to the following inverse spectral problem.

*Given a nonempty, closed, and connected subset  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$ , find a Hamiltonian  $H$  such that  $\mathcal{C}(q_H, M) = \mathcal{C}_\times(q_H) = \mathcal{L}$  (for arbitrary  $M$  as in (1.4))*

The Hamiltonian  $H$  constructed in the proof of Theorem 4.1 has the property that  $q_H$  (transferred to the unit disk) is a Blaschke product.

Our method of proof is based on a rescaling trick, which goes back at least to Y. Kasahara [Kas75], who applied it on the level of Krein strings, and which was exploited further in [KW10], and in [EKT18] and its forthcoming extension [LPW]. Namely, given a Hamiltonian  $H \in \mathbb{H}$ , one considers rescaled Hamiltonians

$$(\mathcal{A}_r H)(t) := H\left(\frac{t}{r}\right), \quad t \in (0, \infty), \quad r > 0. \quad (1.5)$$

The operators  $\mathcal{A}_r$  blow up the scale and thereby zoom into the vicinity of 0. We will see that cluster sets of  $q_H$  are related to cluster sets of the family  $(\mathcal{A}_r H)_{r \geq 1}$  where the set  $\mathbb{H}$  is appropriately topologized, cf. Propositions 3.5 and 3.6. In fact, one may say that the continuous group action of rescaling operators on  $\mathbb{H}$  is responsible for the mentioned intuition that the high-energy behavior of  $q_H$  relates to the local behavior of  $H$  at 0.

In [EKT18, Kas75, KW10] a simple continuity property of de Branges' correspondence was sufficient to obtain the desired conclusions. This property goes back at least to [Bra61], where it formed a step in the existence proof of an inverse spectral theorem. Despite being used in the literature ever since, an explicit presentation was given only recently in [Rem18]. In the presently considered general situation, when limits do not necessarily exist, finer arguments and a thorough understanding of the topology on  $\mathbb{H}$  are necessary.

After this introduction, the article is structured in three more sections. In §2 we study the appropriate topology on  $\mathbb{H}$ ; this section is to a certain extent of expository nature. Contrasting the presentation in [Rem18], we introduce the topology from a higher level viewpoint, namely, as an inverse limit of weak topologies on sets of Hamiltonians defined on finite intervals ( $T \in (0, \infty)$ ),

$$\mathbb{H}_T := \{H: (0, T) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ is measurable, } H(t) \geq 0, \operatorname{tr} H(t) = 1 \text{ a.e.}\}. \quad (1.6)$$

By this approach the most important features, namely compactness and metrizability, are readily built into the construction. Besides offering structural clarity, it also simplifies matters by avoiding the unnecessary passage from  $L^1$  to the space of complex Borel measures done in [Bra61, Rem18]. For the convenience of the nonspecialist reader, we include a complete and concise derivation of the required continuity of de Branges' correspondence  $H \leftrightarrow q_H$ .

In §3 we study the group action of rescaling operators  $\{\mathcal{A}_r \mid r > 0\}$  on  $\mathbb{H}$ , and relate limit points of  $q_H$  with limit points of  $(\mathcal{A}_r H)_{r \geq 1}$ . The case when limits exist, which has been studied in [EKT18], was revisited in the extended preprint version of this article, cf. [PW19].

§4 is devoted to the proof of the main result of the paper. In Theorem 4.1 we give the aforementioned explicit construction of Hamiltonians whose Weyl coefficient has prescribed cluster set. We close the paper with stating some open problems related to Theorem 4.1.

§2. Topologizing the set of Hamiltonians

Thoroughly understanding the convergence of Hamiltonians is crucial for our present investigation. We shall first consider Hamiltonians defined on a finite interval and then pass to Hamiltonians on the half-line by a limiting process.

**2.1. Hamiltonians on a finite interval.** Recall the notation (1.6):

**Definition 2.1.** For  $T > 0$  we denote the set of all Hamiltonians on the interval  $(0, T)$  by  $\mathbb{H}_T$ , i.e.,

$$\mathbb{H}_T := \{H: (0, T) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ is measurable, } H(t) \geq 0, \text{ tr } H(t) = 1 \text{ a.e.}\}.$$

We shall always tacitly identify two Hamiltonians that coincide almost everywhere.

Let  $\|\cdot\|$  denote the  $\ell^1$ -norm on  $\mathbb{C}^{2 \times 2}$ . For every positive semidefinite matrix  $A = (a_{ij})_{i,j=1}^2$ , we have  $|a_{ij}| \leq \|A\| \leq 2 \text{tr } A$ . This shows that all  $H \in \mathbb{H}_T$  are entrywise (equivalently, with respect to  $\|\cdot\|$ ) essentially bounded by 2. In particular, we have

$$\mathbb{H}_T \subseteq L^1((0, T), \mathbb{C}^{2 \times 2}).$$

The space  $L^1((0, T), \mathbb{C}^{2 \times 2})$ , and with it its subset  $\mathbb{H}_T$ , carries several natural topologies. We will work with its norm and weak topology,  $\mathcal{T}_{\|\cdot\|}$  and  $\mathcal{T}_w$ .

**Remark 2.2.** In order to work with the weak topology, we recall the following representation of continuous functionals. We have (linearly and homeomorphically)

$$\begin{aligned} L^1((0, T), \mathbb{C}^{2 \times 2})' &\cong [L^1(0, T)^4]' \\ &\cong [L^1(0, T)']^4 \cong [L^\infty(0, T)]^4 \cong L^\infty((0, T), \mathbb{C}^{2 \times 2}). \end{aligned}$$

A linear homeomorphism is given by the assignment

$$\left\{ \begin{array}{l} L^\infty((0, T), \mathbb{C}^{2 \times 2}) \rightarrow L^1((0, T), \mathbb{C}^{2 \times 2})' \\ (f_{ij})_{i,j=1}^2 \mapsto \left[ (h_{ij})_{i,j=1}^2 \mapsto \sum_{i,j=1}^2 \int_0^T h_{ij}(t) f_{ij}(t) dt \right] \end{array} \right.$$

Sometimes it is practical to note that  $L^1((0, T), \mathbb{C}^{2 \times 2})'$  is spanned by the set of functionals

$$\left\{ H \mapsto \int_0^T e_1^* H(t) e_2 \cdot f(t) dt \mid e_1, e_2 \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, f \in L^\infty(0, T) \right\}.$$

The weak topology on  $\mathbb{H}_T$  has striking properties.

**Lemma 2.3.** *Let  $T > 0$ . The weak topology  $\mathcal{T}_w|_{\mathbb{H}_T}$  is compact and metrizable.*

**Proof.** Since  $\mathbb{H}_T$  is uniformly bounded, it is also uniformly integrable. The Dunford–Pettis theorem (see, e.g., [Bog07, Theorem 4.7.18]) implies that  $\mathbb{H}_T$  is relatively compact in the weak topology of  $L^1((0, T), \mathbb{C}^{2 \times 2})$ . Since every  $\|\cdot\|_1$ -convergent sequence has a subsequence that converges pointwise a.e., the set  $\mathbb{H}_T$  is  $\|\cdot\|_1$ -closed. Since it is convex, it follows that it is weakly closed. Hence  $\mathbb{H}_T$  is indeed weakly compact.

Since  $L^1((0, T), \mathbb{C}^{2 \times 2})$  is  $\|\cdot\|_1$ -separable, the weak topology on a weakly compact subset is metrizable (see, e.g., [Fab+01, Proposition 3.2.9]).  $\square$

We come to a variant of continuity in de Branges’ correspondence for Hamiltonians on finite intervals. To this end, we need some notation. First, denote by  $\mathcal{E}$  the set of all entire  $(2 \times 2)$ -matrix functions endowed with the topology  $\mathcal{T}_{\text{lu}}$  of locally uniform convergence. Second, we introduce a notation for the (transpose of the) fundamental solution of a canonical system.

**Definition 2.4.** Let  $T > 0$ . For  $H \in \mathbb{H}_T$  we denote by  $W(H; t, z)$  a unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(H; t, z) J = z W(H; t, z) H(t), & t \in [0, T], \\ W(H; 0, z) = I, \end{cases} \quad (2.1)$$

where  $I$  is the  $(2 \times 2)$ -identity matrix.

For every fixed  $t \in [0, T]$ , the matrix  $W(H; t, \cdot)$  is an entire function, i.e.,  $W(H; t, \cdot)$  belongs to  $\mathcal{E}$ .

**Definition 2.5.** Let  $T > 0$ . We denote by  $\Psi_T$  the map

$$\Psi_T: \begin{cases} \mathbb{H}_T & \rightarrow \mathcal{E}, \\ H & \mapsto W(H; T, \cdot). \end{cases}$$

The continuity result announced above now reads as follows.

**Theorem 2.6 (Continuity; fundamental solution).**

*Let  $T > 0$ . Then  $\Psi_T$  is  $\mathcal{T}_w$ -to- $\mathcal{T}_{\text{lu}}$ -continuous.*

This theorem is implicit in [Bra61], and, up to identification of topologies, explicit in [Rem18, Theorem 5.7]. For convenience of the reader we give a complete proof.

**Proof of Theorem 2.6.** Let

$$W(H; t, z) = \sum_{l=0}^{\infty} W_l(H; t) z^l \quad (2.2)$$

be the power series expansion of  $W(H; t, \cdot)$ . Plugging this in equation (2.1), we deduce that the coefficients  $W_l(H; t)$  satisfy the recurrence

$$W_0(H, t) = I, \quad W_{l+1}(H; t) = - \int_0^t W_l(H; s) H(s) J \, ds, \quad l \in \mathbb{N}.$$

From this one inductively obtains

$$\|W_l(H; t)\| \leq \frac{(2t)^l}{l!}, \quad H \in \mathbb{H}_T, \quad t \in [0, T], \quad l \in \mathbb{N}. \tag{2.3}$$

Therefore, for each compact set  $K \subseteq \mathbb{C}$ , the series (2.2) converges uniformly on  $\mathbb{H}_T \times [0, T] \times K$ , and we have the global growth estimate

$$\|W(H; t, z)\| \leq e^{2t|z|}, \quad (H, t, z) \in \mathbb{H}_T \times [0, T] \times K.$$

Now let  $(H_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{H}_T$  that converges weakly to some  $H \in \mathbb{H}_T$ . Since the series (2.2) converges uniformly in  $H$ , it suffices to show that

$$\forall l \in \mathbb{N}: \lim_{n \rightarrow \infty} W_l(H_n; T) = W_l(H; T)$$

in order to conclude that  $\lim_{n \rightarrow \infty} W(H_n; T, \cdot) = W(H; T, \cdot)$  locally uniformly on  $\mathbb{C}$ . We use induction to show the stronger statement

$$\forall l \in \mathbb{N}: \lim_{n \rightarrow \infty} W_l(H_n; t) = W_l(H; t) \text{ uniformly for } t \in [0, T].$$

For  $l = 0$  this is trivial. Assume that it has already been established for some  $l \in \mathbb{N}$ . Using the recurrence gives

$$\begin{aligned} & \|W_{l+1}(H_n; t) - W_{l+1}(H; t)\|_\infty \\ &= \left\| \int_0^t W_l(H_n; s) H_n(s) J \, ds - \int_0^t W_l(H; s) H(s) J \, ds \right\|_\infty \\ &\leq \left\| \int_0^t (W_l(H_n; s) - W_l(H; s)) H_n(s) J \, ds \right\|_\infty \\ &\quad + \underbrace{\left\| \int_0^t W_l(H; s) (H_n(s) - H(s)) J \, ds \right\|_\infty}_{=: g_n(t)}. \end{aligned} \tag{2.4}$$

The first summand is estimated as

$$\left\| \int_0^t (W_l(H_n; s) - W_l(H; s)) H_n(s) J \, ds \right\|_\infty \leq T \cdot \|W_l(H_n; s) - W_l(H; s)\|_\infty \cdot 2,$$

and tends to 0 by the inductive hypothesis. The functions  $g_n$  tend to 0 pointwise on  $[0, T]$  because

$$\|g_n(t)\| = \left\| \int_0^T \mathbf{1}_{(0,t)}(s) W_l(H; s) \cdot (H_n(s) - H(s)) \cdot J \, ds \right\|$$

and  $\lim_{n \rightarrow \infty}^w H_n = H$ . We see that

$$g_n(0) = 0, \quad \|g_n(t) - g_n(t')\| \leq |t - t'| \cdot \frac{(2T)^l}{l!} \cdot 4,$$

and by the Arzela–Ascoli theorem the family  $\{g_n \mid n \in \mathbb{N}\}$  is relatively compact in  $C([0, T], \mathbb{C}^{2 \times 2})$ . Thus pointwise convergence upgrades to uniform convergence, and we deduce that also the second summand in (2.4) tends to 0.  $\square$

**2.2. Hamiltonians on the half-line.** We turn to Hamiltonians defined on the whole half-line. Recall the notation (1.3).

**Definition 2.7.** We denote the set of all Hamiltonians on the half-line  $(0, \infty)$  by  $\mathbb{H}$ , i.e.,

$$\mathbb{H} := \{H : (0, \infty) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ is measurable, } H(t) \geq 0, \operatorname{tr} H(t) = 1 \text{ a.e.}\}.$$

Again we tacitly identify two Hamiltonians that coincide almost everywhere.

We consider the set of functions on the half-line as the inverse limit of the sets of functions on finite intervals in the usual way. For  $T > 0$  let  $\rho_T$  be the restriction map

$$\rho_T : \begin{cases} \mathbb{H} & \rightarrow \mathbb{H}_T, \\ H & \mapsto H|_{(0,T)} \end{cases}$$

and let  $\iota$  be the map

$$\iota : \begin{cases} \mathbb{H} & \rightarrow \prod_{T>0} \mathbb{H}_T, \\ H & \mapsto (\rho_T H)_{T>0}. \end{cases}$$

Then  $\iota$  is injective and

$$\iota(\mathbb{H}) = \left\{ (H_T)_{T>0} \in \prod_{T>0} \mathbb{H}_T \mid \forall 0 < T < T' : H_{T'}|_{(0,T)} = H_T \right\}.$$

We use  $\iota$  to pull back the topology of the product. That is, we define a topology on  $\mathbb{H}$  by the demand that  $\iota$  become a homeomorphism of  $\mathbb{H}$  onto  $\iota(\mathbb{H})$ , where the codomain is topologized in the canonical way.

**Definition 2.8.** Let  $\mathcal{T}$  be the initial topology on  $\mathbb{H}$  with respect to the one-element family  $\{\iota\}$  from the product topology of the weak topologies on  $\mathbb{H}_T$ .

This construction automatically implies the following crucial properties.



**Lemma 2.9.** *The topology  $\mathcal{T}$  is compact and metrizable.*

**Proof.** By Tychonoff’s theorem and Lemma 2.3 the product topology of the weak topologies on  $\mathbb{H}_T$  is compact. Each restriction map

$$\rho_T^{T'} : \begin{cases} L^1((0, T'), \mathbb{C}^{2 \times 2}) & \rightarrow L^1((0, T), \mathbb{C}^{2 \times 2}), \\ F & \mapsto F|_{(0, T)} \end{cases}$$

is  $\|\cdot\|_1$ -to- $\|\cdot\|_1$ -continuous, and hence also  $w$ -to- $w$ -continuous. Thus  $\iota(\mathbb{H})$  is a closed subset of the product, and hence also compact.

Consider the map

$$\kappa : \begin{cases} \prod_{T>0} \mathbb{H}_T & \rightarrow \prod_{n \in \mathbb{N}} \mathbb{H}_n, \\ (H_T)_{T>0} & \mapsto (H_n)_{n \in \mathbb{N}}. \end{cases}$$

Then  $\kappa$  is clearly continuous when both products are endowed with the product topology of the weak topologies. Moreover,  $\kappa|_{\iota(\mathbb{H})}$  is injective. Since  $\iota(\mathbb{H})$  is compact, it is therefore a homeomorphism of  $\iota(\mathbb{H})$  onto  $(\kappa \circ \iota)(\mathbb{H})$ . Lemma 2.3 implies that the countable product  $\prod_{n \in \mathbb{N}} \mathbb{H}_n$  is metrizable. It follows that  $\mathbb{H}$ , being homeomorphic to a subspace of this product, is metrizable.  $\square$

**Remark 2.10.** The topology  $\mathcal{T}$  constructed above coincides with the topology defined in [Rem18, Chapter 5.2]. This follows by writing out our definition and applying the argument that gave the metrizability of  $\mathcal{T}$ , and by remembering Remark 2.2.

In [EKT18, Proposition 2.3] convergence of Hamiltonians was introduced in yet another form. To see that this form coincides with convergence with respect to  $\mathcal{T}$ , one has to note that step functions are dense in  $L^1$ .

We turn to the continuity of de Branges’ correspondence. Recall that  $\mathcal{N}$ , as a subset of the space of all analytic functions of  $\mathbb{C}_+$  into the Riemann sphere, naturally carries the topology  $\mathcal{T}_{\text{lu}}$  of locally uniform convergence.

**Definition 2.11.** We denote by  $\Psi$  the map

$$\Psi : \begin{cases} \mathbb{H} & \rightarrow \mathcal{N}, \\ H & \mapsto q_H. \end{cases}$$

**Theorem 2.12 (Continuity; Weyl coefficients).**

*The map  $\Psi$  is  $\mathcal{T}$ -to- $\mathcal{T}_{\text{lu}}$ -homeomorphic.*

Also this theorem is implicit in [Bra61] and explicit in [Rem18, Theorem 5.7], and we provide a complete derivation for convenience of the reader.

The proof of the “finite interval variant” of Theorem 2.6 relied on the uniform estimate (2.3) of power series coefficients. The proof of the present “half-line variant” will follow from a uniform estimate of the size of Weyl disks.

Recall that for  $H \in \mathbb{H}$  and  $T > 0$  the Weyl disk  $\Omega_{T,z}(H)$  at  $z \in \mathbb{C}_+$  is the image of  $\overline{\mathbb{C}_+}$  under the fractional linear transformation with coefficient matrix  $W(H; T, z)$ . Moreover, recall that the inverse stereographic projection is Lipschitz continuous. In fact, viewing the Riemann sphere as the unit sphere whose south pole lies at the origin of the complex plane, the chordal distance  $\chi$  of two points  $\zeta, \xi \in \mathbb{C}$  (we suppress the explicit notation of the stereographical projection) is

$$\chi(\zeta, \xi) = \frac{2|\zeta - \xi|}{\sqrt{1 + |\zeta|^2}\sqrt{1 + |\xi|^2}},$$

and hence  $\chi(\zeta, \xi) \leq 2|\zeta - \xi|$ ,  $\zeta, \xi \in \mathbb{C} \subseteq \overline{\mathbb{C}}$ .

**Lemma 2.13.** *Suppose that  $H \in \mathbb{H}$ ,  $T > 0$ , and  $z \in \mathbb{C}_+$ . The diameter of the Weyl disk  $\Omega_{T,z}$  with respect to the chordal metric can be estimated as*

$$\text{diam}_\chi \Omega_{T,z}(H) \leq \frac{8}{T \cdot \text{Im } z}.$$

**Proof.** Write  $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ , and assume first that  $\int_0^T h_2(s) \, ds \geq \frac{T}{2}$ . Then  $\infty \notin \Omega_{T,z}(H)$ . By the usual formula for the the Euclidean radius of  $\Omega_{T,z}(H)$ , see, e.g., [Rem18, Lemma 3.11], the monotonicity result [Bra61, Lemma 4], and the differential equation (2.1), we find

$$\text{diam}_\chi \Omega_{T,z}(H) \leq 2 \text{diam}_{|\cdot|} \Omega_{T,z}(H) \leq 2 \cdot \frac{2}{\text{Im } z \cdot \int_0^T h_2(t) \, dt} \leq \frac{8}{T \cdot \text{Im } z}.$$

Now consider the case where

$$\int_0^T h_2(s) \, ds < \frac{T}{2}.$$

Then we must have

$$\int_0^T h_1(s) \, ds \geq \frac{T}{2},$$

and the already established estimate applies to  $\tilde{H} := -JHJ$ . A computation shows that  $W(\tilde{H}; T, z) = -JW(H; T, z)J$ , and hence the Weyl disk  $\Omega_{T,z}(\tilde{H})$  is the image of  $\Omega_{T,z}(H)$  under the fractional linear transformation with coefficient

matrix  $J$ . Since  $J$  is unitary, this is a rotation of the sphere, and hence is isometric with respect to the chordal metric. We obtain

$$\text{diam}_\chi \Omega_{T,z}(H) = \text{diam}_\chi \Omega_{T,z}(\tilde{H}) \leq \frac{8}{T \cdot \text{Im } z}. \quad \square$$

**Proof of Theorem 2.12.** Let  $(H_n)_{n \in \mathbb{H}}$  be a sequence in  $\mathbb{H}$  that converges to some  $H \in \mathbb{H}$ . By the definition of the topology of  $\mathbb{H}$ , this means that  $\lim_{n \rightarrow \infty}^w \rho_T(H_n) = \rho_T(H)$  for every  $T > 0$ .

Write  $W(H; T, z) = (w_{ij}(H; t, z))_{i,j=1}^2$ , and denote

$$Q_{n,T}(z) := \frac{w_{12}(H_n; T, z)}{w_{22}(H_n; T, z)}, \quad Q_T(z) := \frac{w_{12}(H; T, z)}{w_{22}(H; T, z)}, \quad z \in \mathbb{C}_+.$$

Throughout the following, all limits of complex numbers are understood with respect to the chordal metric  $\chi$ .

Let  $K \subseteq \mathbb{C}_+$  satisfy  $\inf_{z \in K} \text{Im } z > 0$ . Lemma 2.13 shows that the limit

$$q_{\tilde{H}}(z) = \lim_{T \rightarrow \infty} \frac{w_{12}(\tilde{H}; T, z)}{w_{22}(\tilde{H}; T, z)}$$

defining the Weyl coefficient of a Hamiltonian  $\tilde{H}$  is attained uniformly for  $(\tilde{H}, z) \in \mathbb{H} \times K$ . This implies

- ▷  $\lim_{T \rightarrow \infty} Q_{n,T}(z) = q_{H_n}(z)$  uniformly for  $(n, z) \in \mathbb{N} \times K$ ;
- ▷  $\lim_{T \rightarrow \infty} Q_T(z) = q_H(z)$  uniformly for  $z \in K$ .

Theorem 2.6 says that

- ▷ for each  $T > 0$  we have  $\lim_{n \rightarrow \infty} Q_{n,T}(z) = Q_T(z)$  locally uniformly for  $z \in \mathbb{C}_+$ .

Together, we obtain

$$q_H(z) = \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} Q_{n,T}(z) = \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} Q_{n,T}(z) = \lim_{n \rightarrow \infty} q_{H_n}(z)$$

locally uniformly for  $z \in \mathbb{C}_+$ .

Being a continuous bijection of a compact space onto a Hausdorff space,  $\Psi$  is a homeomorphism. □

We often use the continuity of  $\Psi$  in another form.

**Definition 2.14.** We denote by  $\Phi$  the map

$$\Phi: \begin{cases} \mathbb{H} \times \mathbb{C}_+ & \rightarrow \overline{\mathbb{C}_+}, \\ (H, w) & \mapsto q_H(w). \end{cases}$$

The following reformulations of the continuity of  $\Psi$  are obtained by elementary arguments; an explicit proof is deferred to the preprint version [PW19] of this article.

**Corollary 2.15 (Continuity; Weyl coefficients — a variant).** *Each of the properties (i) and (ii) below is equivalent to the  $\mathcal{T}$ -to- $\mathcal{T}_{\text{lin}}$ -continuity of  $\Psi$ , and hence holds true.*

- (i) *The map  $\Phi$  is continuous when  $\mathbb{H} \times \mathbb{C}_+$  is endowed with the product topology of  $\mathcal{T}$  and the Euclidean topology.*
- (ii) *For every compact set  $K \subseteq \mathbb{C}_+$  the family  $\{\Phi(\cdot, w) \mid w \in K\}$  is equicontinuous.*

**2.3. Constant Hamiltonians.** A particular role is played by Hamiltonians  $H \in \mathbb{H}$  that are constant a.e. on  $(0, \infty)$ . We denote the set of all them by  $\mathbb{CH}$ .

Constant Hamiltonians can be identified with the points of  $\overline{\mathbb{C}_+}$ .

**Definition 2.16.** Let  $\Theta: \overline{\mathbb{C}_+} \rightarrow \mathbb{CH}$  be the map acting as

$$\Theta(\zeta) := \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix},$$

where

$$h_1 := \frac{|\zeta|^2}{|\zeta|^2 + 1}, \quad h_2 := \frac{1}{|\zeta|^2 + 1}, \quad h_3 := \frac{\operatorname{Re} \zeta}{|\zeta|^2 + 1},$$

if  $\zeta \neq \infty$ , and

$$\Theta(\infty) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The map  $\Theta$  is bijective. Its inverse  $\Theta^{-1}: \mathbb{CH} \rightarrow \overline{\mathbb{C}_+}$  is given by

$$\Theta^{-1} \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix} = \frac{h_3 + i\sqrt{h_1 h_2 - h_3^2}}{h_2}$$

if  $h_2 \neq 0$ , and

$$\Theta^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \infty.$$

Note that  $\det \Theta(\zeta) = 0$  if and only if  $\zeta \in \overline{\mathbb{R}}$ , and that  $\Theta(\zeta)$  is diagonal if and only if  $\zeta \in i\overline{\mathbb{R}_+}$ .

From the defining formulas it is obvious that for each  $T > 0$  the map  $\rho_T \circ \Theta: \overline{\mathbb{C}_+} \rightarrow \langle \mathbb{H}_T, \mathcal{T}_{\|\cdot\|_1} \rangle$  is continuous. Thus  $\rho_T \circ \Theta$  is also continuous into  $\mathcal{T}_w$ , and hence  $\Theta$  is continuous into  $\langle \mathbb{H}, \mathcal{T} \rangle$ . Since  $\overline{\mathbb{C}_+}$  is compact, each of

$$\langle \rho_T(\mathbb{CH}), \mathcal{T}_{\|\cdot\|_1} \rangle, \quad \langle \rho_T(\mathbb{CH}), \mathcal{T}_w \rangle, \quad \langle \mathbb{CH}, \mathcal{T} \rangle$$

is homeomorphic to  $\overline{\mathbb{C}_+}$ . In particular, these spaces are all compact.

**Remark 2.17.** The definition of  $\Theta$  is formulated in such a way that

$$q_{\Theta(\zeta)}(z) = \zeta, \quad z \in \overline{\mathbb{C}_+},$$

in other words,  $\Phi(\Theta(\zeta), w) = \zeta, w \in \mathbb{C}_+$ . This is shown by a simple calculation, see e.g., [EKT18, §2.2, Example 1]<sup>2</sup>.

For later use we introduce a separate notation for constant Hamiltonians corresponding to boundary points of  $\overline{\mathbb{C}_+}$ , namely,

$$\mathbb{C}\mathbb{H}_0 := \Theta(\overline{\mathbb{R}}) = \{H \in \mathbb{C}\mathbb{H} \mid \det H = 0\}.$$

### §3. The rescaling method

We have already mentioned the rescaling operation  $\mathcal{A}_r: H(\cdot) \mapsto H(\frac{1}{r} \cdot)$  on Hamiltonians in (1.5). In this section we put this in an appropriate framework and establish the relationship between cluster sets of  $\mathcal{A}_r H$  for  $r \rightarrow \infty$  and  $q_H(z)$  for  $z \rightarrow i\infty$ .

Clearly,  $\mathcal{A}_r$  maps  $\mathbb{H}$  into itself and satisfies the computation rules

$$\mathcal{A}_1 = \text{id}, \quad \forall r, s > 0: \mathcal{A}_r \circ \mathcal{A}_s = \mathcal{A}_s \circ \mathcal{A}_r = \mathcal{A}_{rs}. \tag{3.1}$$

This means precisely that the map

$$\begin{cases} \mathbb{R}_+ \times \mathbb{H} & \rightarrow \mathbb{H}, \\ (r, H) & \mapsto \mathcal{A}_r H. \end{cases} \tag{3.2}$$

is a group action of  $\mathbb{R}_+$  on  $\mathbb{H}$ .

**Lemma 3.1.** *The map (3.2) is continuous.*

**Proof.** Assume we are given  $H_n, H \in \mathbb{H}$  with  $H_n \rightarrow H$  and  $r_n, r \in \mathbb{R}_+$  with  $r_n \rightarrow r$ , and assume without loss of generality that  $\frac{r}{2} \leq r_n \leq 2r$  for all  $n$ . We have to show that

$$\forall T > 0: \rho_T \mathcal{A}_{r_n} H_n \xrightarrow{w} \rho_T \mathcal{A}_r H.$$

Recall Remark 2.2 and let  $e_1, e_2 \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$  and  $f \in L^\infty(0, T)$  be given. Denote by  $\tilde{f}$  the extension of  $f$  to the element of  $L^\infty(0, \infty)$  with  $\tilde{f}(t) = 0, t \geq T$ . Then we have

$$\begin{aligned} & \int_0^T e_1^* \left( (\rho_T \mathcal{A}_{r_n} H_n)(t) - (\rho_T \mathcal{A}_r H)(t) \right) e_2 \cdot f(t) dt \\ &= \int_0^T e_1^* \left( H_n\left(\frac{t}{r_n}\right) - H\left(\frac{t}{r_n}\right) \right) e_2 \cdot f(t) dt + \int_0^T e_1^* \left( H\left(\frac{t}{r_n}\right) - H\left(\frac{t}{r}\right) \right) e_2 \cdot f(t) dt \end{aligned}$$

---

<sup>2</sup>Caution: notation in [EKT18] is different.

$$\begin{aligned}
&= r_n \int_0^{\frac{2T}{r}} e_1^*(H_n(s) - H(s))e_2 \cdot \tilde{f}(r_n s) \, ds + \int_0^T e_1^*(H(\frac{t}{r_n}) - H(\frac{t}{r}))e_2 \cdot f(t) \, dt \\
&= \int_0^{\frac{2T}{r}} e_1^*(H_n(s) - H(s))e_2 \cdot (\tilde{f}(r_n s) - \tilde{f}(rs)) \, ds \\
&\quad + \int_0^{\frac{2T}{r}} e_1^*(H_n(s) - H(s))e_2 \cdot \tilde{f}(rs) \, ds + \int_0^T e_1^*(H(\frac{t}{r_n}) - H(\frac{t}{r}))e_2 \cdot f(t) \, dt.
\end{aligned}$$

The first summand tends to 0 because

$$\|e_1^*(H_n(s) - H(s))e_2\|_\infty \leq 2$$

and

$$\|\tilde{f}(r_n s) - \tilde{f}(rs)\|_1 \rightarrow 0,$$

the second summand also tends to zero because

$$H_n \rightarrow H \text{ in } \mathbb{H},$$

and the third because

$$\|H(\frac{t}{r_n}) - H(\frac{t}{r})\|_1 \rightarrow 0. \quad \square$$

The fact that (3.2) is a continuous group action has some immediate consequences. In our context, the following two are of interest.

**Remark 3.2.**

- (i) For every  $H \in \mathbb{H}$  and  $s > 0$  the map  $\mathcal{A}_s$  leaves  $\mathcal{C}l[\mathcal{A}_r H]$  invariant. Hence, (3.2) induces a continuous group action on the cluster set  $\mathcal{C}l[\mathcal{A}_r H]$ .
- (ii) For every  $H \in \mathbb{H}$  the stabiliser

$$(\mathbb{R}_+)_H := \{r \in \mathbb{R}_+ \mid \mathcal{A}_r H = H\}$$

is a closed subgroup of  $\mathbb{R}_+$ .

Item (ii) of the above remark shows that  $(\mathbb{R}_+)_H$  is either equal to  $\{1\}$  or  $\mathbb{R}_+$ , or is of the form  $\{p^n \mid n \in \mathbb{Z}\}$  for some  $p > 1$ . We have  $(\mathbb{R}_+)_H = \mathbb{R}_+$  if and only if  $H \in \mathbb{C}\mathbb{H}$ , and  $(\mathbb{R}_+)_H$  is a nontrivial subgroup if and only if  $H$  is nonconstant and multiplicatively periodic.

**Remark 3.3.** The case of a nontrivial stabiliser is particularly simple: if  $H$  is multiplicatively periodic with primitive period  $p > 1$ , then

$$\mathcal{C}l[\mathcal{A}_r H] = \{\mathcal{A}_r H \mid 1 \leq r \leq p\}. \quad (3.3)$$

For the inclusion “ $\subseteq$ ” note that

$$\{\mathcal{A}_r H \mid r > 0\} = \{\mathcal{A}_r H \mid 1 \leq r \leq p\},$$

and hence the orbit of  $H$  is compact. The reverse inclusion holds since  $\mathcal{A}_{sp^n} H = \mathcal{A}_s H$  for all  $n \in \mathbb{N}$  and  $s > 0$ , and hence

$$\mathcal{A}_s H = \lim_{n \rightarrow \infty} \mathcal{A}_{sp^n} H \in \mathcal{C}l[\mathcal{A}_r H].$$

Rescaling operators have a rescaling effect on fundamental solutions. This is a particular case of [EKT18, Lemma 2.7]. For convenience of the reader, we recall the argument.

**Lemma 3.4.** *Let  $H \in \mathbb{H}$  and let  $r > 0$ . Then the fundamental solutions, Weyl disks, and Weyl coefficients of  $H$  and  $\mathcal{A}_r H$  are related as follows ( $t \geq 0$ ,  $z \in \mathbb{C}_+$ ):*

$$W(\mathcal{A}_r H; t, z) = W(H; \frac{t}{r}, rz), \quad \Omega_{t,z}(\mathcal{A}_r H) = \Omega_{\frac{t}{r}, rz}(H), \quad q_{\mathcal{A}_r H}(z) = q_H(rz).$$

If we use the notation  $\Phi$  from Definition 2.14, the relation between Weyl coefficients writes as

$$\forall H \in \mathbb{H}, r > 0, z \in \mathbb{C}_+ : \Phi(\mathcal{A}_r H, z) = \Phi(H, rz). \tag{3.4}$$

**Proof.** Set  $\widetilde{W}(t, z) := W(H; \frac{t}{r}, rz)$ . Then

$$\begin{aligned} \frac{\partial}{\partial t} \widetilde{W}(t, z) J &= \frac{1}{r} \frac{\partial}{\partial t} W(H; \frac{t}{r}, rz) \\ &= \frac{1}{r} \cdot rz \cdot W(H; \frac{t}{r}, rz) H\left(\frac{t}{r}\right) = z \widetilde{W}(t, z) (\mathcal{A}_r H)(t). \end{aligned}$$

Thus  $\widetilde{W}(t, z)$  is the fundamental solution of  $\mathcal{A}_r H$ .

The relation between Weyl disks follows immediately, and the relation between Weyl coefficients follows by letting  $t \rightarrow \infty$ . □

The next proposition is the basis for translating cluster sets of  $\mathcal{A}_r H$  to those of  $q_H$ .

Given a subset  $M \subseteq \mathbb{C}_+$  as in (1.4), we denote the limiting directions of  $M$  by

$$D(M) := \{\theta \in [0, \pi] \mid \exists z_n \in M : |z_n| \rightarrow \infty \wedge \arg z_n \rightarrow \theta\}.$$

Note that  $D(M)$  is closed and contained in  $(0, \pi)$ .

**Proposition 3.5.** *Let  $M \subseteq \mathbb{C}_+$  be as in (1.4) and let  $H \in \mathbb{H}$ . Then*

$$\mathcal{C}(q_H, M) \subseteq \Phi(\mathcal{C}l[\mathcal{A}_r H] \times e^{iD(M)}) = \mathcal{C}(q_H, e^{iD(M)}[1, \infty)). \tag{3.5}$$

**Proof.** To show the inclusion on the left of (3.5), let  $w \in \mathcal{C}(q_H, M)$ . Choose  $z_n \in M$  with  $|z_n| \rightarrow \infty$  and  $q_H(z_n) \rightarrow w$ . By the compactness of  $\mathbb{H}$  and  $[0, \pi]$ , we can choose a subsequence such that both limits

$$\tilde{H} := \lim_{k \rightarrow \infty} \mathcal{A}_{|z_{n_k}|} H, \quad \theta := \lim_{k \rightarrow \infty} \arg z_{n_k},$$

exist. Then  $\theta \in D(M)$ , and the continuity of  $\Phi$  implies that

$$w = \lim_{k \rightarrow \infty} q_H(z_{n_k}) = \lim_{k \rightarrow \infty} \Phi(\mathcal{A}_{|z_{n_k}|} H, e^{i \arg z_{n_k}}) = \Phi(\tilde{H}, e^{i\theta}).$$

The inclusion “ $\supseteq$ ” of the claimed identity on the right of (3.5) readily follows because  $D(M)$  is closed and hence

$$D(e^{iD(M)}[1, \infty)) = D(M).$$

To prove the reverse inclusion, let  $w \in \Phi(\mathcal{C}l[\mathcal{A}_r H] \times e^{iD(M)})$  be given. Write  $w = \Phi(\tilde{H}, e^{i\theta})$  with some  $\tilde{H} \in \mathcal{C}l[\mathcal{A}_r H]$  and  $\theta \in D(M)$ , and choose  $r_n \rightarrow \infty$  with  $\tilde{H} = \lim_{n \rightarrow \infty} \mathcal{A}_{r_n} H$ . Then

$$w = \Phi(\tilde{H}, e^{i\theta}) = \lim_{n \rightarrow \infty} \Phi(\mathcal{A}_{r_n} H, e^{i\theta}) = \lim_{n \rightarrow \infty} q_H(r_n e^{i\theta}) \in \mathcal{C}(q_H, e^{i\theta[1, \infty)}). \quad \square$$

We also obtain some knowledge about outer angular cluster sets.

**Proposition 3.6.** *Let  $H \in \mathbb{H}$ . Then*

- (i)  $\mathcal{C}_\times(q_H) = \Phi(\mathcal{C}l[\mathcal{A}_r H] \times \mathbb{C}_+)$ ;
- (ii)  $\mathcal{C}l[\mathcal{A}_r H] \subseteq \mathbb{C}\mathbb{H} \Rightarrow$  for all  $M$  as in (1.4) we have

$$\mathcal{C}(q_H, M) = \mathcal{C}_\times(q_H) = \Theta^{-1}(\mathcal{C}l[\mathcal{A}_r H]);$$

- (iii)  $\mathcal{C}l[\mathcal{A}_r H] \cap \mathbb{C}\mathbb{H} = \emptyset \Rightarrow \mathcal{C}_\times(q_H)$  is open.

**Proof.** Using Remark 3.2 (i) and (3.5) we find

$$\begin{aligned} \Phi(\mathcal{C}l[\mathcal{A}_r H] \times \mathbb{C}^+) &= \Phi(\mathcal{C}l[\mathcal{A}_r H] \times e^{i(0, \pi)}) \\ &= \bigcup_{\alpha \in (0, \frac{\pi}{2}]} \Phi(\mathcal{C}l[\mathcal{A}_r H] \times e^{i[\alpha, \pi - \alpha]}) = \bigcup_{\alpha \in (0, \frac{\pi}{2}]} \mathcal{C}(q_H, \Gamma_\alpha) = \mathcal{C}_\times(q_H). \end{aligned}$$

Assume now that  $\mathcal{C}l[\mathcal{A}_r H] \subseteq \mathbb{C}\mathbb{H}$ , and set  $K := \Theta^{-1}(\mathcal{C}l[\mathcal{A}_r H])$ . Then

$$\mathcal{C}_\times(q_H) = \Phi(\mathcal{C}l[\mathcal{A}_r H] \times \mathbb{C}_+) = \bigcup_{\tilde{H} \in \mathcal{C}l[\mathcal{A}_r H]} q_{\tilde{H}}(\mathbb{C}_+) = \bigcup_{\tilde{H} \in \mathcal{C}l[\mathcal{A}_r H]} \{\Theta^{-1}(\tilde{H})\} = K.$$

The inclusion  $\mathcal{C}(q_H, M) \subseteq \mathcal{C}_\times(q_H)$  is trivially true. Let  $\xi \in K$ , and choose  $r_n \rightarrow \infty$  with  $\mathcal{A}_{r_n} H \rightarrow \Theta(\xi)$ . Since  $\{z \in M \mid |z| > r\}$  is connected for all sufficiently large  $r$ , for all sufficiently large  $n$  we can choose points  $z_n \in M$  with



$|z_n| = r_n$ . Choose a subsequence such that  $\arg z_{n_k} \rightarrow \theta$  for some  $\theta \in (0, \pi)$ . Then

$$q_H(z_n) = \Phi(\mathcal{A}_{|z_{n_k}|} H, e^{i \arg z_{n_k}}) \rightarrow \Phi(\Theta(\xi), e^{i\theta}) = \xi,$$

and therefore  $\xi \in \mathcal{C}(q_H, M)$ .

Finally, assume that  $\mathcal{C}l[\mathcal{A}_r H] \cap \mathbb{C}\mathbb{H} = \emptyset$ . Then

$$\mathcal{C}_\star(q_H) = \Phi(\mathcal{C}l[\mathcal{A}_r H] \times \mathbb{C}_+) = \bigcup_{\tilde{H} \in \mathcal{C}l[\mathcal{A}_r H]} q_{\tilde{H}}(\mathbb{C}_+),$$

and each set in the union on the right is open. □

Let us revisit the multiplicatively periodic situation.

**Remark 3.7.** Let  $H \in \mathbb{H}$  be nonconstant and multiplicatively periodic. Then (3.3) and Proposition 3.6 (iii) imply that  $\mathcal{C}_\star(q_H)$  is open. In particular, the outer angular cluster set is not equal to any of the cluster sets  $\mathcal{C}(q_H, M)$ .

#### §4. Weyl coefficients with prescribed cluster set

In the theorem below we give an explicit construction of Hamiltonians  $H$  for which the cluster set of  $q_H$  can be computed. These Hamiltonians are piecewise constant on quickly shrinking intervals that accumulate only at the initial point.

In the formulation of the theorem we denote the cluster set of a sequence  $(\zeta_n)_{n \in \mathbb{N}}$  in  $\overline{\mathbb{C}_+}$  by

$$\mathcal{C}l[\zeta_n] := \{ \zeta \in \overline{\mathbb{C}_+} \mid \exists n_k \in \mathbb{N}: n_k \rightarrow \infty \wedge \lim_{k \rightarrow \infty} \zeta_{n_k} = \zeta \}.$$

Moreover, recall that  $\chi$  denotes the chordal metric on  $\overline{\mathbb{C}_+}$ .

**Theorem 4.1.** *Let  $(t_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers with*

$$1 = t_1 > t_2 > t_3 > \dots, \quad \lim_{n \rightarrow \infty} t_n = 0, \quad \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 0,$$

*and let  $(\zeta_n)_{n \in \mathbb{N}}$  be a sequence of points on  $\overline{\mathbb{C}_+}$  with*

$$\lim_{n \rightarrow \infty} \chi(\zeta_{n+1}, \zeta_n) = 0.$$

*Define  $H$  to be the piecewise constant Hamiltonian*

$$H(t) := \begin{cases} \Theta(\zeta_n), & t \in (t_{n+1}, t_n], n \in \mathbb{N}, \\ \Theta(0), & t \in (1, \infty). \end{cases} \tag{4.1}$$

*Then, for every  $M$  as in (1.4),*

$$\mathcal{C}_\star(q_H) = \mathcal{C}(q_H, M) = \mathcal{C}l[\zeta_n].$$

An elementary argument shows that for every nonempty, closed, and connected subset  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$  there exists a sequence  $(\zeta_n)_{n \in \mathbb{N}}$  with  $\mathcal{E}l[\zeta_n] = \mathcal{L}$  (an explicit proof can be found in [PW19]). Thus we obtain an explicit solution of an inverse problem dealing with boundary interpolation.

**Corollary 4.2.** *Let  $\mathcal{L} \subseteq \overline{\mathbb{C}_+}$  be nonempty, closed, and connected. Then we can construct a Hamiltonian  $H$  for whose Weyl coefficient  $q_H$  the outer angular and radial cluster sets at  $i\infty$  are both equal to  $\mathcal{L}$ .*

We turn to the proof of Theorem 4.1. The crucial step is presented in the next lemma. Here we denote by  $d_{\|\cdot\|}$  the metric induced by the  $L^1$ -norm.

**Lemma 4.3.** *Let  $H \in \mathbb{H}$  and assume that*

$$\lim_{r \rightarrow \infty} d_{\|\cdot\|}(\rho_1 \mathcal{A}_r H, \rho_1 \Theta(\mathbb{C}\mathbb{H})) = 0. \quad (4.2)$$

Moreover, denote

$$K := \{\xi \in \overline{\mathbb{C}_+} \mid \rho_1 \Theta(\xi) \in \mathcal{E}l_{\|\cdot\|}[\rho_1 \mathcal{A}_r H]\}. \quad (4.3)$$

Then

- (i)  $\mathcal{E}l_{\|\cdot\|}[\rho_T \mathcal{A}_r H] = \rho_T \Theta(K)$ , for all  $T > 0$ ,
- (ii)  $\mathcal{E}l[\mathcal{A}_r H] = \Theta(K)$ .

**Proof.** Let  $T > 0$  and set

$$K_T := \{\xi \in \overline{\mathbb{C}_+} \mid \rho_T \Theta(\xi) \in \mathcal{E}l_{\|\cdot\|}[\rho_T \mathcal{A}_r H]\}.$$

The relation

$$\begin{aligned} \|\rho_T \mathcal{A}_r H - \rho_T \Theta(\xi)\|_1 &= \int_0^T \|H(\frac{t}{r}) - \Theta(\xi)\| dt \\ &= T \int_0^1 \|H(\frac{T}{r} \cdot t) - \Theta(\xi)\| dt = T \|\rho_1 \mathcal{A}_{\frac{r}{T}} H - \rho_1 \Theta(\xi)\|_1, \end{aligned}$$

which is true for all  $H \in \mathbb{H}$  and  $\xi \in \overline{\mathbb{C}_+}$ , shows that

$$\liminf_{r \rightarrow \infty} \|\rho_T \mathcal{A}_r H - \rho_T \Theta(\xi)\|_1 = T \cdot \liminf_{r \rightarrow \infty} \|\rho_1 \mathcal{A}_{\frac{r}{T}} H - \rho_1 \Theta(\xi)\|_1, \quad (4.4)$$

$$d_{\|\cdot\|}(\rho_T \mathcal{A}_r H, \rho_T \Theta(\mathbb{C}\mathbb{H})) = T \cdot d_{\|\cdot\|}(\rho_1 \mathcal{A}_{\frac{r}{T}} H, \rho_1 \Theta(\mathbb{C}\mathbb{H})). \quad (4.5)$$

Relation (4.4) implies that  $K_T = K$  for all  $T > 0$ , and (4.5) that

$$\forall T > 0: \lim_{r \rightarrow \infty} d_{\|\cdot\|}(\rho_T \mathcal{A}_r H, \rho_T \Theta(\mathbb{C}\mathbb{H})) = 0. \quad (4.6)$$

Since  $\rho_T\Theta(\mathbb{C}\mathbb{H})$  is compact with respect to  $\|\cdot\|_1$ , and hence closed, (4.6) in turn implies that

$$\mathcal{E}l_{\|\cdot\|_1}[\rho_T\mathcal{A}_rH] \subseteq \rho_T\Theta(\mathbb{C}\mathbb{H}).$$

Item (i) of the present assertion follows.

The inclusion “ $\supseteq$ ” in item (ii) holds true because of a general argument. Namely, it is fulfilled for every Hamiltonian  $H \in \mathbb{H}$  that

$$\{\tilde{H} \in \mathbb{H} \mid \forall T > 0: \rho_T\tilde{H} \in \mathcal{E}l_{\|\cdot\|_1}[\rho_T\mathcal{A}_rH]\} \subseteq \mathcal{E}l[\mathcal{A}_rH].$$

To show this, assume that  $\tilde{H}$  belongs to the set on the left. We choose inductively numbers  $r_n > 0$ , such that

$$\forall n \in \mathbb{N}: r_{n+1} \geq r_n + 1 \wedge \|\rho_n\mathcal{A}_{r_n}H - \rho_n\tilde{H}\|_1 \leq \frac{1}{n}.$$

Given  $T > 0$ , for all  $n \geq T$  we have

$$\|\rho_T\mathcal{A}_{r_n}H - \rho_T\tilde{H}\|_1 \leq \|\rho_n\mathcal{A}_{r_n}H - \rho_n\tilde{H}\|_1 \leq \frac{1}{n},$$

and hence  $\rho_T\mathcal{A}_{r_n}H \xrightarrow{\|\cdot\|_1} \tilde{H}$ . This clearly implies that  $\mathcal{A}_{r_n}H \rightarrow \tilde{H}$ .

The reverse inclusion “ $\subseteq$ ” in item (ii) relies on the assumption (4.2). Assume that  $\tilde{H} \in \mathcal{E}l[\mathcal{A}_rH]$  and choose a sequence  $r_n \rightarrow \infty$  such that  $\mathcal{A}_{r_n}H \rightarrow \tilde{H}$ . Then

$$\rho_T\mathcal{A}_{r_n}H \xrightarrow{w} \tilde{H} \text{ for all } T > 0. \tag{4.7}$$

Let  $T > 0$ . Since (4.6) holds and  $\rho_T\Theta(\mathbb{C}\mathbb{H})$  is compact with respect to  $\|\cdot\|_1$ , we find a point  $\xi \in \overline{\mathbb{C}_+}$  and a subsequence  $(r_{n_k})_{k \in \mathbb{N}}$  (both depending on  $T$ ) such that

$$\rho_T\mathcal{A}_{r_{n_k}}H \xrightarrow{\|\cdot\|_1} \Theta(\xi). \tag{4.8}$$

Combining this with (4.7), we see that  $\rho_T\tilde{H} = \rho_T\Theta(\xi)$ . It follows that  $\xi$  is independent of  $T$  and that  $\tilde{H} = \Theta(\xi)$ . By (4.8), used for  $T = 1$ , we have  $\xi \in K$ . □

**Proof of Theorem 4.1.** The function

$$\rho_1 \circ \Theta: \overline{\mathbb{C}_+} \rightarrow L^1((0, 1), \mathbb{C}^{2 \times 2})$$

is  $\chi$ -to- $\|\cdot\|_1$ -continuous and injective. Since  $\overline{\mathbb{C}_+}$  is compact, it is therefore uniformly continuous and a homeomorphism onto its image. Let  $\omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the modulus of continuity of  $\rho_1 \circ \Theta$ , so that

$$\lim_{\delta \rightarrow 0} \omega(\delta) = 0 \wedge \forall \zeta, \xi \in \overline{\mathbb{C}_+}: \|\rho_1\Theta(\zeta) - \rho_1\Theta(\xi)\|_1 \leq \omega(\chi(\zeta, \xi)).$$

We show that

$$\forall n \in \mathbb{N}, r \in \left[\frac{1}{t_n}, \frac{1}{t_{n+1}}\right]: \|\rho_1\mathcal{A}_rH - \rho_1\Theta(\zeta_n)\|_1 \leq 4\frac{t_{n+2}}{t_{n+1}} + \omega(\chi(\zeta_{n+1}, \zeta_n)). \tag{4.9}$$

To see this, estimate

$$\begin{aligned} \int_0^{rt_{n+2}} \|H(\frac{t}{r}) - \Theta(\zeta_n)\| dt &\leq 4rt_{n+2} \leq 4\frac{rt_{n+2}}{rt_{n+1}}, \\ \int_{rt_{n+2}}^{rt_{n+1}} \|H(\frac{t}{r}) - \Theta(\zeta_n)\| dt &= \int_{rt_{n+2}}^{rt_{n+1}} \|\Theta(\zeta_{n+1}) - \Theta(\zeta_n)\| dt \\ &\leq \|\rho_1\Theta(\zeta_{n+1}) - \rho_1\Theta(\zeta_n)\|_1 \leq \omega(\chi(\zeta_{n+1}, \zeta_n)), \\ \int_{rt_{n+1}}^1 \|H(\frac{t}{r}) - \Theta(\zeta_n)\| dt &= 0. \end{aligned}$$

For  $r \geq 1$  let  $n(r) \in \mathbb{N}$  be a unique number with  $[\frac{1}{t_n}, \frac{1}{t_{n+1}})$ . Then  $\lim_{r \rightarrow \infty} n(r) = \infty$ . The right-hand side of (4.9) tends to 0 as  $n$  tends to  $\infty$ , and hence for every sequence  $r_k \rightarrow \infty$  we have

$$\lim_{k \rightarrow \infty} \|\rho_1 \mathcal{A}_{r_k} H - \rho_1 \Theta(\zeta_{n(r_k)})\|_1 = 0.$$

This shows that (4.2) holds true and that

$$\mathcal{E}l_{\|\cdot\|}[\rho_1 \mathcal{A}_r H] \subseteq \mathcal{E}l_{\|\cdot\|}[\rho_1 \Theta(\zeta_n)].$$

If  $n_k \rightarrow \infty$ , then (4.9) shows that

$$\lim_{k \rightarrow \infty} \|\rho_1 \mathcal{A}_{\frac{1}{t_{n_k}}} H - \rho_1 \Theta(\zeta_{n_k})\|_1 = 0,$$

and it follows that

$$\rho_1 \Theta(\mathcal{E}l[\zeta_n]) \subseteq \mathcal{E}l_{\|\cdot\|}[\rho_1 \mathcal{A}_r H].$$

Since  $\rho_1 \circ \Theta$  is a homeomorphism between compact sets,

$$\rho_1 \Theta(\mathcal{E}l[\zeta_n]) = \mathcal{E}l_{\|\cdot\|}[\rho_1 \Theta(\zeta_n)],$$

and we see that the set  $K$  from (4.3) is equal to  $\mathcal{E}l[\zeta_n]$ .

The asserted properties of  $q_H$  now follow from Lemma 4.3 and Proposition 3.6 (ii). □

Let us pass from the half-plane to the unit disk with the fractional linear transformation  $\beta(z) := \frac{z-i}{z+i}$ , which maps  $\overline{\mathbb{C}^+}$  onto the closed unit disk  $\overline{\mathbb{D}}$  with  $\beta(\infty) = 1$ .

**Remark 4.4.** Consider a Hamiltonian of the form (4.1). Since  $H$  is constant equal to  $\Theta(0)$  on the interval  $(1, \infty)$ , the Weyl coefficient  $q_H$  is given by  $q_H = \frac{w_{12}}{w_{22}}$ , where

$$w_{12}(z) := (1, 0)W(H; 1, z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad w_{22}(z) := (0, 1)W(H; 1, z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since  $\det H$  is constant equal to 0, the entire function  $W(H; 1, z)$  is of zero exponential type. The function

$$B(z) := \beta \circ q_H \circ \beta^{-1} = \frac{w_{12} - iw_{22}}{w_{12} + iw_{22}} \circ \beta^{-1}$$

is thus a Blaschke product whose zeroes have no finite accumulation point.

Cluster sets of  $q_H$  towards  $i\infty$  clearly correspond to cluster sets of  $B$  towards 1. Thus we reobtain the fact that for every nonempty, closed, and connected subset  $\mathcal{L}$  of  $\overline{\mathbb{D}}$ , there exists a Blaschke product whose outer angular and radial cluster sets at 1 are equal to  $\mathcal{L}$ .

In the context of the present construction and its consequences for functions on the disk some open questions occur.

- (i) We do not know if the function constructed in the above way has also cluster set  $\mathcal{L}$  when  $z$  is allowed to approach 1 in an unrestricted, possibly tangential, way.
- (ii) We do not know if our construction method can be modified so as to yield results about simultaneous boundary interpolation at more than one point (as done for radial cluster sets in [BCP85, Dec94, Don01]).
- (iii) We do not know if our construction method can be modified to produce an approach to cluster values along a prescribed curve when  $z$  approaches the point 1 radially (as in [Don01, Theorem 1]).
- (iv) We do not know an analog of Theorem 4.1 for outer angular cluster sets that realises any countable increasing union of nonempty closed connected sets (and by this proves Gauthiers conjecture).

Concerning the third question we have some preliminary results indicating that the answer is affirmative.

## References

- [Atk64] Atkinson F. V., *Discrete and continuous boundary problems*, Math. Sci. Eng., vol. 8, Acad. Press, New York, 1964.
- [BHS20] Behrndt J., Hassi S., de Snoo H., *Boundary value problems, Weyl functions, and differential operators*, Monogr. Math., vol. 108, Birkhäuser/Springer, Cham, 2020.
- [BCP85] Belna C. L., Colwell P., Piranian G., *The radial behavior of Blaschke products*, Proc. Amer. Math. Soc. **93** (1985), no. 2, 267–271.

- [Bog07] Богачев В. И., Основы теории меры. Т.1, 2-е изд., НИЦ Регулярная и хаотическая динамика, Москва-Ижевск, 2006.
- [Bra61] de Branges L., *Some Hilbert spaces of entire functions*. II, Trans. Amer. Math. Soc. **99** (1961), 118–152.
- [Bra68] de Branges L., *Hilbert spaces of entire functions*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1968.
- [CL66] Collingwood E. F., Lohwater A. J., *The theory of cluster sets*, Cambridge Tracts Math. Math. Phys., vol. 56, Cambridge Univ. Press, Cambridge, 1966.
- [Dec94] Decker E., *On the boundary behavior of singular inner functions*, Michigan Math. J. **41** (1994), no. 3, 547–562.
- [Don01] Donaire J. J., *Radial behaviour of inner functions in  $B_0$* , J. London Math. Soc. (2) **63** (2001), no. 1, 141–158.
- [EKT18] Eckhardt J., Kostenko A., Teschl G., *Spectral asymptotics for canonical systems*, J. Reine Angew. Math. **736** (2018), 285–315.
- [Fab+01] Fabian M. et al., *Functional analysis and infinite-dimensional geometry*, CMS Books Math., vol. 8, Springer-Verlag, New York, 2001.
- [Gau21] Gauthier P. M., *A characterization of non-tangential cluster sets for holomorphic functions  $f: \mathbb{D} \rightarrow \mathbb{D}^n$* , lecture given at *Recent Advances in Complex and Harmonic Analysis*, Réunion dété SMC, 2021.
- [GT00] Gesztesy F., Tsekanovskii E., *On matrix-valued Herglotz functions*, Math. Nachr. **218** (2000), 61–138.
- [GS11] Girela D., Suárez D., *On Blaschke products, Bloch functions and normal functions*, Rev. Mat. Complut. **24** (2011), no. 1, 49–57.
- [GM05] Gorkin P., Mortini R., *Cluster sets of interpolating Blaschke products*, J. Anal. Math. **96** (2005), 369–395.
- [HSW00] Hassi S., de Snoo H., Winkler H., *Boundary-value problems for two-dimensional canonical systems*, Integral Equations Operator Theory **36** (2000), no. 4, 445–479.
- [KK68] Кац И. С., Крейн М. Г., *R-функции — аналитические функции, отображающие верхнюю полуплоскость в себя*, В кн.: Аткинсон Ф., Дискретные и непрерывные граничные задачи, Мир, М., 1968, с. 629–647.
- [Kas75] Kasahara Y., *Spectral theory of generalized second order differential operators and its applications to Markov processes*, Japan. J. Math. (N.S.) **1** (1975), no. 1, 67–84.
- [KW10] Kasahara Y., Watanabe S., *Asymptotic behavior of spectral measures of Krein's and Kotani's strings*, Kyoto J. Math. **50** (2010), no. 3, 623–644.
- [LPW] Langer M., Pruckner R., Woracek H., *Canonical systems whose Weyl coefficients have regularly varying asymptotics*, manuscript in preparation.
- [Lev80] Левин Б. Я., *Распределение корней целых функций*, Гостехиздат, М., 1956.
- [Lev52] Левитан Б. М., *Об асимптотическом поведении спектральной функции самосопряженного дифференциального уравнения второго порядка*, Изв. АН СССР. Сер. мат. **16** (1952), №4, 325–352.
- [Nos60] Noshiro K., *Cluster sets*, Ergeb. Math. Grenzgeb., vol. 28, Springer-Verlag, Berlin, 1960.
- [PW19] Pruckner R., Woracek H., *Limit behaviour of Nevanlinna functions*, ASC Report 16, 29 pp., <http://www.asc.tuwien.ac.at/preprint/2019/asc16x2019.pdf>, Vienna Univ. Technology, 2019.
- [Rem18] Remling C., *Spectral theory of canonical systems*, De Gruyter Stud. Math., vol 70, De Gruyter, Berlin, 2018.

- 
- [Rom14] Romanov R., *Canonical systems and de Branges spaces*, version 1, Aug. 26, 2014, [arXiv:1408.6022v1](https://arxiv.org/abs/1408.6022v1) [math.SP].
- [Tit46] Titchmarsh E. C., *Eigenfunction expansions associated with second-order differential equations*, Clarendon Press, Oxford, 1946.
- [Wey10] Weyl H., *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. **68** (1910), no. 2, 220–269.

Institute for Analysis  
and Scientific Computing  
Vienna University of Technology  
Wiedner Hauptstrasse  
8–10/101, 1040 Wien, Austria  
*E-mail:* [raphael.pruckner@tuwien.ac.at](mailto:raphael.pruckner@tuwien.ac.at)  
*E-mail:* [harald.woracek@tuwien.ac.at](mailto:harald.woracek@tuwien.ac.at)

Поступило 11 июня 2019 г.