Avoiding Monochromatic Rectangles Using Shift Patterns

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Introduction

Ramsey Theory (Graham and Rothschild 1990) deals with patterns that cannot be avoided indefinitely. In this paper we focus on a pattern of coloring a n by m grid with k colors: Consider all possible rectangles within the grid whose length and width are at least 2. Try to color the grid using k colors so that no such rectangle has the same color for its four corners. When this is possible, we say that the n by m grid is k-colorable while avoiding monochromatic rectangles.

Many results regarding this problem have been derived by pure combinatorial approach: for example, a generalization of Van der Waerden's Theorem can give an upper bound; it was shown (Fenner et al. 2010) that for each prime power k, a $k^2 + k$ by k^2 grid is k-colorable but adding a row makes it not k-colorable. However, these results are unable to decide many grid sizes: whether an 18 by 18 grid is 4-colorable is an example. This grid had been the last missing piece of the question of 4-colorability, and a challenge prize was raised to close the gap (Hayes 2009). Three years later, a valid 4-coloring of that grid was found by encoding the problem into propositional logic and applying SAT-solving techniques (Steinbach and Posthoff 2012). That solution has highly symmetric color assignments by construction: assignments of red are obtained by rotating the assignments of white around the center by 90 degrees, blue by 180 degrees, and so on. By now, the k-colorability has been decided for $k \in \{2, 3, 4\}$ for all grids.

Therefore, it is natural to ask, what about 5 colors? Applying the aforementioned theorem (Fenner et al. 2010), the 25 by 30 grid is 5-colorable, but for other grids such as 26 by 26 the problem remains open. Like many combinatorial search problems, the rectangle-free grid coloring problem is characterized by enormous search space and rich symmetries. Symmetry breaking is a common technique to trim down the search space while preserving satisfiability. While breaking symmetries between different solutions is definitely helpful, breaking the so-called "internal symmetries" that is within a specific solution has also been proved to be effective (Heule and Walsh 2010). Enforcing observed patterns is also known as "streamlining" (Gomes and Sellmann 2004) and "resolution tunnels" (Kouril and Franco 2005) and has been effective to improve lower bounds of various combinatorial problems including Van der Waerden numbers (Kouril and Franco 2005; Heule 2017), Latin squares (Gomes and Sellmann 2004), and graceful graphs (Heule and Walsh 2010).

However, the rotation internal symmetry that Steinbach and Posthoff applied cannot translate to 5 colors. In finding a 4-coloring of the 18 by 18 grid, Steinbach and Posthoff generated a "cyclic reusable assignment" for one color, and rotated the solution by 90, 180, and 270 degrees to assign to the remaining three. Rotation by 90 degrees does not apply naturally when the number of colors are not multiples of 4.

Thus, to find a 5-coloring of 26 by 26, or rather, to find a valid coloring for any number of colors k in general, an internal symmetry that is applicable to all k is very desirable. We found a novel internal symmetry that is unrestricted by the number of colors k. Further analysis on this symmetry gives further constraints on the number of occurrences of each color. Factoring in these constraints, the search time for $G_{24,24}$ and $G_{25,25}$ can be reduced to a few minutes. We also attempted to solve the 26 by 26 grid; many attempts came down to only 2 or 3 unsatisfied clauses, but none succeeded.

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Classifying colorings of smaller grid examples

We consider classifying colorings of simpler grids to be a good starting point that enables us to gain insight into how many "essentially different" solutions there are. As an example, we will analyze $G_{4,4}$ and $G_{10,10}$, the maximal 2-colorable and 3-colorable squares.

For square grids $G_{n,n}$ and number of colors k, there are 3 kinds of symmetries that transform a valid coloring to another: (i) permutation of colors (ii) permutation of rows or columns (iii) transposition, i.e. a flip along the diagonal. We define symmetries by sequences of these operations and define a natural equivalence relation between colorings.

Let $Grid(n, n, k)$ be the set of all valid k-colorings of $G_{n,n}$. We are interested in counting the equivalence classes of $Grid(10, 10, 3)$. Our approach is to convert grid colorings to graph colorings, and use `Bliss` (Junttila and Kaski 2007) graph isomorphism tool to assign to each graph a canonical labeling. In particular, we identify each member of $Grid(n, n, k)$ with a graph on n^2 vertices, where each vertex corresponds to a cell in the grid and has the same la-

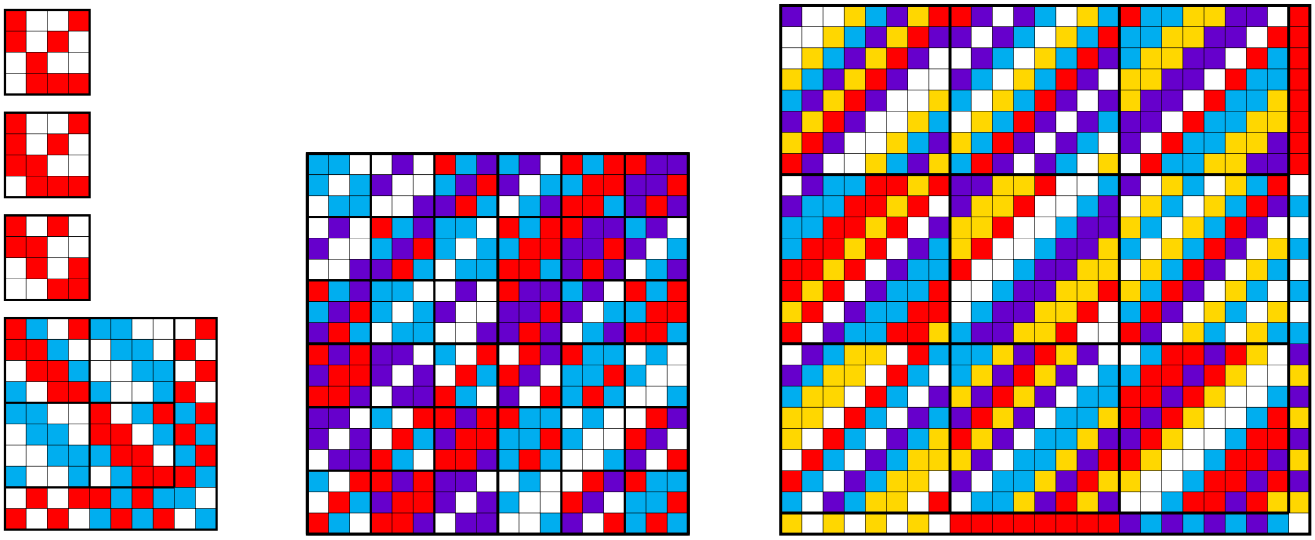

Figure 1: Rectangle-free colorings with shift patterns. Top-left: canonical 2-colorings of $G_{4,4}$; bottom-left: 3-coloring of $G_{10,10}$ with 4-subgrids; middle: 4-coloring of $G_{18,18}$ with 3-subgrids and 9-midgrids; right: 5-coloring of $G_{25,25}$ with 8-subgrids

bel and color. Two distinct vertices $(a, b), (c, d)$ are joined by an edge if and only if $a = c$ or $b = d$ (but not both).

For $Grid(4, 4, 2)$, we generated all 840 solutions and compared the canonical labelings of their graphs up to color permutations. There are three non-isomorphic solutions as shown in Figure 1 (upper-left).

For $Grid(10, 10, 3)$, the CNF formula yielded 35 solutions after adding symmetry breaking clauses by `Shatter` (Aloul, Sakallah, and Markov 2006). Comparison of graph representatives output by `Bliss` showed that all 35 solutions are isomorphic. That is, there is only one equivalence class in $Grid(10, 10, 3)$ shown in Figure 1 (lower-left).

Generalizable pattern and new results

The representative of $Grid(10, 10, 3)$ in Figure 1 (left) admits a shift pattern: within each 4 by 4 subgrid, the second row is a copy of the first row except shifted right (or left) by 1; the third row shifted by 2; the last row shifted by 3. All the shifts wrap around on the edge of the subgrid.

The shift pattern greatly reduces search space, as only the first row in each subgrid needs to be chosen. This pattern can be iterated for one more layer: by adding "midgrids" that are made up of smaller subgrids and enforcing that subgrids on subsequent rows are shifted copies of those on the first row in a similar fashion, the search space can be further reduced. This is extremely helpful to solving larger grids, and significantly reduces the running time. Figure 1 shows a 4-coloring of $G_{18,18}$ with 3-subgrids and 9-midgrids. `CaDiCaL` found this solution in under 1 second; previously (Steinbach and Posthoff 2012), it took the SAT solver `clasp` roughly 7 hours to find a cyclic-reusable assignment of $G_{18,18}$. For a fair comparison, we ran the same formula on `CadiCaL` and it terminated in 4 hours and 40 minutes.

In addition, the 4-coloring of $G_{18,18}$ in Figure 1 is a "new" solution, in the sense that it is not isomorphic to the previous solution (Steinbach and Posthoff 2012).

Infeasible case: Shift pattern on 26 by 26 If there are no remaining columns and rows, the only choices for subgrid size are 2, 13 and 26. The subgrid size cannot be 26 by the Pigeonhole Principle. If a 5-coloring of subgrid size 2 or 13 exists, then it must satisfy certain necessary conditions. For the case of size 2 and 13, integer constraints can be placed on number of each color in each subgrid. `Z3` Theorem Prover (De Moura and Bjørner 2008) reported unsatisfiable for both cases. Therefore, no 5-colorings of such shift pattern exist.

Thus, we turn our attention to finding solutions that have shift pattern for the upper-left 25 by 25 or 24 by 24 part, and constrain the remaining column(s) and row(s) appropriately.

Shift pattern on 25 by 25 In the case of 25 by 25, the possible subgrid sizes are 5 and 25. For subgrid size 5, an obvious color distribution is 5 cells of each color in each subgrid. We attempted to find a coloring of $G_{26,26}$ which has this pattern on the 25 by 25 part and left the last row and column without additional constraints. `PalSAT` was unable to find a satisfying assignment in 24 hours, with only one unsatisfied clause. This instance is unlikely to be satisfiable because of the cell at the bottom-right corner.

Shift pattern on 24 by 24, under further constraints This brings us to 24 by 24, which has more possible subgrid sizes. Directly solving $G_{24,24}$ for subgrid sizes ranging from 3 to 10 shows that subgrid sizes $\{3, 4, 5, 6, 8, 10\}$ are satisfiable. However, `PalSAT` was unable to solve $G_{26,26}$ with the similar shift pattern and diagonal patterns in 24 hours.

Therefore, we sought to constrain them with a "partial shift pattern": e.g., take the upper-left 26 by 26 part of a 32 by 32 grid, for subgrid size 8. Now, the 25th row under each subgrid is a shifted copy of the 24th row, and the columns are also constrained to be alternating colors, as observed in the solution of $G_{25,25}$ in Figure 1 (right). However, this was not enough: `PalSAT` could not get under 2 unsatisfied clauses.

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