Geometric Dominating Sets - A Minimum Version of the No-Three-In-Line Problem

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Abstract

We consider a minimizing variant of the well-known *No-Three-In-Line Problem*, the *Geometric Dominating Set Problem*: What is the smallest number of points in an $n \times n$ grid such that every grid point lies on a common line with two of the points in the set? We show a lower bound of $\Omega(n^{2/3})$ points and provide a constructive upper bound of size $2\lceil n/2 \rceil$. If the points of the dominating sets are required to be in general position we provide optimal solutions for grids of size up to 12×12. For arbitrary n the currently best upper bound remains the obvious $2n$. Finally, we discuss some further variations of the problem.

1 Introduction

The well-known *No-Three-In-Line Problem* asks for the largest point set in an $n \times n$ grid without three points in a line. This problem has intrigued many mathematicians including e.g. Paul Erdős for roughly 100 years now. Few results are known and explicit solutions obtaining the trivial upper bound of $2n$ only exist for n up to 46 and $n = 48, 50, 52$ (See e.g. [3]). Providing general bounds seems to be notoriously hard to solve; see [4, 6] for some history of this problem.

In this note we concentrate on an interesting minimizing variant of the No-Three-In-Line problem, which we call the *Geometric Dominating Set Problem*: What is the smallest number of points (or points in general position) in an $n \times n$ grid such that every grid point lies on a common line with two of the points in the set? This problem arose during the 2018 Bellairs Winter Workshop on Computational Geometry. Later we found out that already in 1976 in Martin Gardner's Mathematical Games column [4] the minimization version has been mentioned. Gardner wrote: "*Instead of asking for the maximum number of counters that can be put on an order-n board, no three in line, let us ask for the minimum that can be placed such that adding one more counter on any vacant cell will produce three in line.*" According to Gardner, the problem had already been mentioned briefly in a paper by Adena, Holton and Kelly [1]. He mentioned their best results which they obtained by hand for $3 \leq n \leq 10$. These are $4, 4, 6, 6, 8, 8, 12, 12$. Surprisingly, up to $n = 8$, their solutions are indeed optimal solutions as we will see in Section 3. However, it seems that no progress has been made since then, except for the special case where lines are restricted to vertical, horizontal and 45° diagonal lines [2].

This minimum version might remind one less of the No-Three-In-Line Problem, which itself is based on a mathematical chess puzzle, and more of the *Queens Domination Problem* that asks for a placement of five queens on a chessboard such that every square of the board is attacked by a queen. In a more general setting this problem asks for the domination

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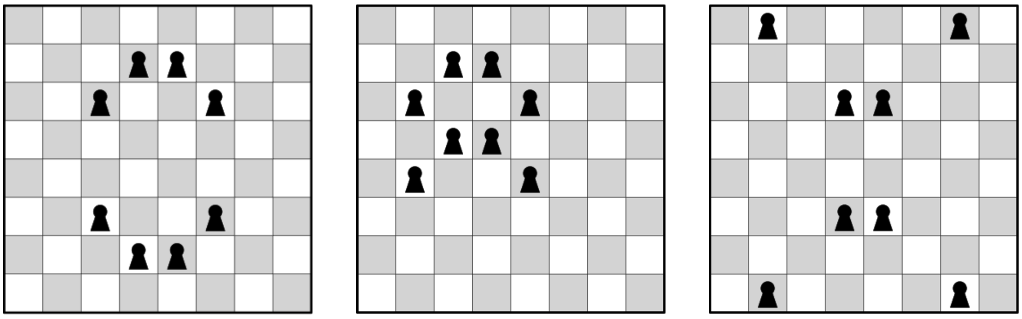

Figure 1 Three out of 228 solutions: Every square lies on a line defined by two pawns where no three pawns are allowed to lie on a common line.

number of the $n \times n$ queen graph. Inspired by that, we call the smallest size of a solution for the Geometric Dominating Set Problem the *geometric domination number* $\mathcal{D}_n$.

After introducing the Geometric Dominating Set Problem formally, we will prove non-trivial asymptotic upper and lower bounds and provide further computational results.

1.1 Dominating Sets

In the spirit of mathematical chess puzzles, the Geometric Dominating Set Problem can be formulated in two variants as

How many pawns do we have to place on a chessboard such that every square lies on a straight line defined by two pawns? How many pawns do we need if no three pawns are allowed to lie on a common line?

We will see in Section 3, the answer for a chessboard is eight and some solutions are the placements shown in Figure 1. In fact, there are 228 possibilities to do so, and 44 if we cancel out rotation and reflection symmetries.

▶ **Definition 1.1.** Three points are called *collinear*, if they lie on a common line. Conversely, a set S is called *in general position* if no three points in S are collinear.

We call a point p in the $n \times n$ grid *dominated (by a set S)*, if $\mathsf{p} \in S$ or there exist $\mathsf{x}, \mathsf{y} \in S$ such that $\{\mathsf{x}, \mathsf{y}, \mathsf{p}\}$ are collinear. Similarly, we say p is dominated by a line L if p lies on L.

A subset S of the $n \times n$ grid is called a *(geometric) dominating set* or simply *dominating* if every point in the grid is dominated by S.

We call the smallest size of a dominating set of the $n \times n$ grid the *(geometric) domination number* and denote it by $\mathcal{D}_n$. The smallest size of a dominating set in general position (an *independent dominating set*) is called the *independent (geometric) domination number* and denoted by $\mathcal{I}_n$. Note that every point in an independent dominating set is only dominated by pairs that include the point itself.

1.2 Summary Of Results

We will show that

- $\mathcal{I}_n \geq \mathcal{D}_n = \Omega(n^{2/3})$ (Subsection 2, Theorem 2.3)
- $\mathcal{D}_n \leq 2\lceil n/2 \rceil$ (Subsection 3, Theorem 3.1)

and present several computational results.

2 Lower Bounds

For a lower bound on $\mathcal{D}_n$, let us consider a set S of s points in the $n \times n$ grid. Any pair of points in S can dominate at most n points, so it has to hold that $\binom{s}{2}n \geq n^2$ which is the case if $s \geq \frac{1}{2} + \sqrt{\frac{1}{4} + 2n} \geq \sqrt{2n}$. Therefore, $\mathcal{D}_n = \Omega(n^{1/2})$.

However, hardly any lines in the $n \times n$ grid dominate n points. In fact, we can prove a significantly better bound by using the following theorem, where φ denotes the Euler-Phi function.

▶ **Theorem 2.1.** *Let $n = 2k + 1$ and S be a subset of the $n \times n$ grid with $|S| \leq 4 \sum_{i=1}^{m} \varphi(i)$, where $1 \leq m \leq k$. Then the number of points dominated by lines incident to a fixed point $\mathsf{x} \in S$ and the other points in S is bounded by*

$$1 + 8 \sum_{i=1}^{m} \left\lfloor \frac{n}{i} \right\rfloor \varphi(i) \ \leq \ \frac{48}{\pi^2} nm + O\left(n \log m\right).$$

The proof of this theorem can be found in [6] and uses the following well known number theoretic result.

▶ **Theorem 2.2** (Arnold Walfisz [7])**.**

$$\sum_{i=1}^{k} \varphi(i) = \frac{3}{\pi^2} k^2 + O\left(k (\log k)^{\frac{2}{3}} (\log \log k)^{\frac{4}{3}}\right)$$

$$\sum_{i=1}^{m} \frac{\varphi(i)}{i} = \frac{6}{\pi^2} m + O\left((\log m)^{\frac{2}{3}} (\log \log m)^{\frac{4}{3}}\right)$$

▶ **Theorem 2.3** (A lower bound on $\mathcal{D}_n$)**.** *For $n \in \mathbb{N}$, it holds that $\mathcal{D}_n = \Omega(n^{2/3})$.*

Proof. First, let $n = 2k + 1$, $k \in \mathbb{N}$ and let S be a set of s points in the grid, where $\sqrt{2n} \leq s \leq 2n$. (Recall that $2n$ is a trivial upper bound on $\mathcal{D}_n$ and $\sqrt{2n}$ a lower bound.) Let m be the smallest positive integer such that $s \leq 4 \cdot \sum_{i=1}^{m} \varphi(i)$. Then $s \sim \frac{12}{\pi^2} m^2$ by Theorem 2.2.

By Theorem 2.1, the number of points dominated by lines incident to a fixed point p and one of $s - 1$ additional points is bounded by $\frac{48}{\pi^2} nm + O\left(n \log m\right)$. To dominate all points in the grid, we thus need

$$n^2 \leq s \left(\frac{48}{\pi^2} nm + O\left(n \log m\right)\right).$$

Next, we plug in the asymptotic expression for s, such that the inequality simplifies to

$$n^2 \leq \left(\frac{12}{\pi^2} m^2 + O\left(m \log m\right)\right)\left(\frac{48}{\pi^2} nm + O\left(n \log m\right)\right) = \frac{576}{\pi^4} nm^3 + O\left(nm^2 \log m\right)$$

If we divide by n, we can see that $m = \Omega(n^{1/3})$ and consequently $s = \Omega(n^{2/3})$ which proves the claim for n odd.

For n even we embed the $n \times n$ grid into the $(n + 1) \times (n + 1)$ grid and obtain the same asymptotic results. ◀

▶ **Corollary 2.4.** $\mathcal{I}_n = \Omega(n^{2/3})$.

Proof. Since any independent dominating set is a dominating set, $\mathcal{I}_n \geq \mathcal{D}_n$. ◀

3 Upper Bounds

▶ **Theorem 3.1** (An upper bound on $\mathcal{D}_n$). *For $n \in \mathbb{N}$, it holds that $\mathcal{D}_n \leq 2\left\lceil \frac{n}{2} \right\rceil$.*

The proof is based on the construction in Figure 2 and can be found in [6]. If $n = k^2$ is an odd square the result can be slightly improved to $n - 1$ by a construction similar to the one depicted for $k = 3$ and $n = 9$ in the leftmost drawing of Figure 6.

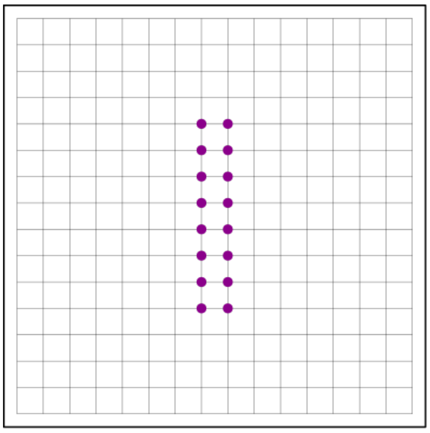

Figure 2 Dominating set construction for $n = 16$.

So far, for $\mathcal{I}_n$ there is no better upper bound known than the obvious $2n$.

4 Small Cardinalities and Examples

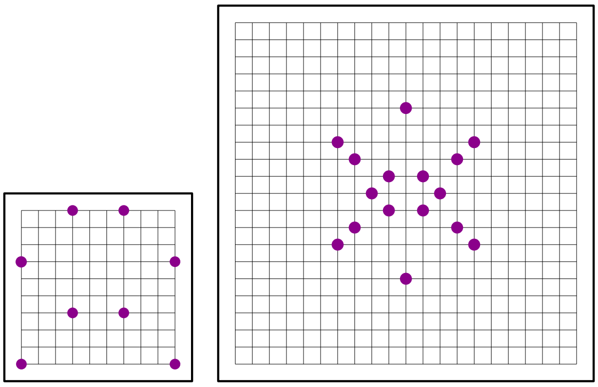

Figure 3 The unique (up to symmetry) minimal independent dominating set of size 8 for the 10×10 board and a small independent dominating set of size 16 for the 21×21 board. The latter gives the currently best known ratio (number of points / grid size) of $16/21 < 0{,}762$.

To obtain results for small grids we developed a search algorithm based on the classic backtracking approach. To speed up the computation, both symmetries – rotation and reflection – were taken into account. For $n = 2, \ldots, 12$, we made an exhaustive enumeration of all independent dominating sets, and the obtained results are summarized in Table 1. For larger sets upper bounds on $\mathcal{I}_n$ are given in Table 2. Figure 3 gives two examples of small independent dominating sets.

n	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{I}_n$	4	4	4	6	6	8	8	8	8	10	10
non sym. sets	1	2	2	26	2	573	44	3	1	19	2
all sets	1	5	2	152	8	4136	228	11	4	108	12

Table 1 Size of smallest independent dominating sets for $n = 2, \ldots, 12$ and number of different sets, considering symmetry (rotation and reflection), and all sets.

n	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\mathcal{I}_n \leq$	12	12	14	14	15	16	16	16	16	18	20	20	22	24	24	24	24	25

Table 2 Currently best upper bounds for smallest independent dominating sets for $n > 12$.

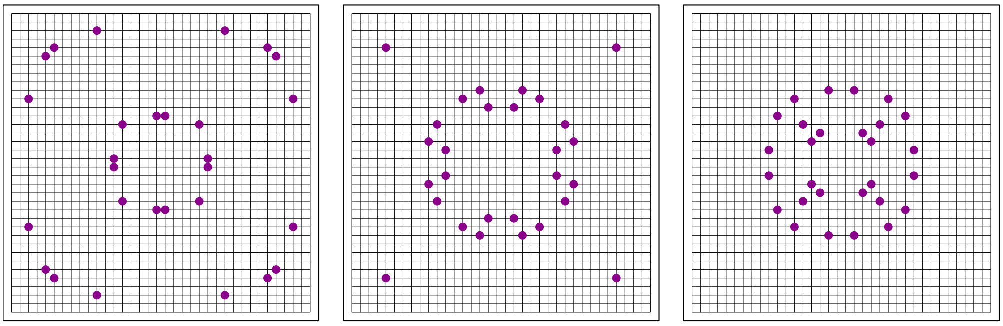

Figure 4 Three independent dominating sets of cardinality 28 for a 36×36 board.

We also obtained results for larger sets, but there is no evidence that our sets are (near the) optimal solutions. Most of these examples are rather symmetric, but that might be biased due to the approach we used to generate larger sets from smaller sets by adding symmetric groups of points. Figure 4 shows three drawings for $n = 36$ with independent dominating sets of size 28.

Figure 5 shows different dominating sets for $n = 7$. The best dominating sets that contain collinear points are smaller than the best solutions in general position. For $n \leq 12$ this is the only board size where allowing collinear points leads to smaller dominating sets. Figure 6(left) shows some nicely symmetric dominating sets with collinear points.

5 Variations of the Problem and Conclusion

We have already seen in the previous section that minimal examples can get smaller if we allow dominating sets to contain collinearities, cf. Figures 5 and 6. We can also release the restriction that the points of the dominating set have to lie within the grid, that is, the points can have coordinates smaller than one, or larger than n. In Figure 6(right) we depict two examples where the shown dominating sets are smaller than the best bounded solutions in general position. So far we have not been able to find any examples where this idea combined with collinear points in the dominating set provided even better solutions.

Another interesting variant is a game version: Two players alternatingly place a point on the $n \times n$-grid such that no three points are collinear. The last player who can place a valid point wins the game (called normal play in game theory). It is not hard to see that for

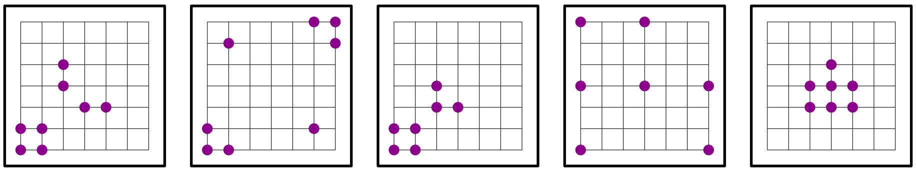

Figure 5 Five different dominating sets for a 7×7-board. The first two sets are in general position and have size 8, while the remaining three sets have size 7 but contain collinear points.

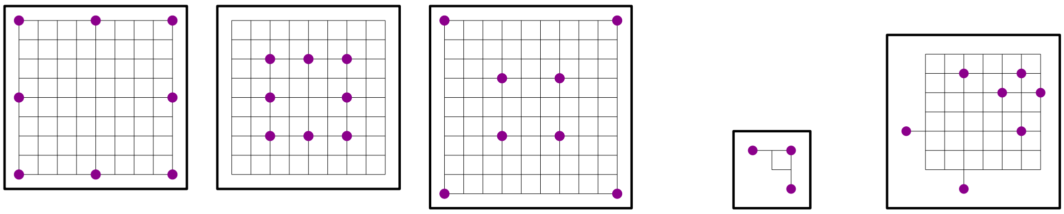

Figure 6 Left: Symmetric dominating sets with collinear points for $n = 9$ and $n = 10$. Right: Smaller dominating sets for a 2×2-board and a 7×7-board if points are allowed to be outside of the board. These solutions are unique up to symmetry.

any even n the second player has a winning strategy. She just always sets the point which is center mirrored to the previous move of the first player. By symmetry arguments this move is always valid, as long as the first player made a valid move. For n odd the situation is more involved. If the first player does not start by placing the central point in her first move, then we have again a winning strategy for the second player by the same reasoning (note that the central place can not be used after the first two points have been placed, as this would cause collinearity). So if the first player starts by placing the central point it can be shown that for $n = 3$ she can also win the game. But for $n = 5, 7, 9$ still the second player has a winning strategy. For odd n we currently do not know the outcome for games on grids of size $n \geq 11$.

Several open problems arise from our considerations:

- Is there a constant $c > 0$ such that $\mathcal{D}_n \leq \mathcal{I}_n \leq (2 - c)n$ holds for large enough n?
- Do $\mathcal{I}_n$ and $\mathcal{D}_n$ grow in a monotone way, that is, is $\mathcal{I}_{n+1} \geq \mathcal{I}_n$ and $\mathcal{D}_{n+1} \geq \mathcal{D}_n$?
- Is there some n_0 such that $\mathcal{I}_n > \mathcal{D}_n$ for all $n \geq n_0$?
- Do minimal dominating sets in general position always have even cardinality? For $n \leq 12$ this is the case, but the currently best example for $n = 17$ might be a counterexample.
- How much can the size of dominating sets (with or without collinear points) be improved if the points are allowed to lie outside the grid?
- Which player has a winning strategy in the game version for boards of size $n \geq 11$, n odd.

In [6], the problem was also considered on the discrete $n \times n$ torus. By extending the probabilistic approach of Guy and Kelly to the No-3-In-Line problem [5] an upper bound of $O(\sqrt{n \log n})$ holds, which remarkably is below the lower bound of the regular grid. We can show a lower bound of $\Omega(\sqrt{n})$ for the torus if n is prime, but if n is a power of 2, then actually 4 points are sufficient. We will provide detailed results in the full version.

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