Lacon- and Shrub-Decompositions: A New Characterization of First-Order Transductions of Bounded Expansion Classes

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***Abstract*—The concept of bounded expansion provides a robust way to capture sparse graph classes with interesting algorithmic properties. Most notably, every problem definable in first-order logic can be solved in linear time on bounded expansion graph classes. First-order interpretations and transductions of sparse graph classes lead to more general, dense graph classes that seem to inherit many of the nice algorithmic properties of their sparse counterparts.**

In this work we introduce lacon- and shrub-decompositions and use them to characterize transductions of bounded expansion graph classes and other graph classes. If one can efficiently compute sparse shrub- or lacon-decompositions of transductions of bounded expansion classes then one can solve every problem definable in first-order logic in linear time on these classes.

I. Introduction

Many hard graph problems become easier if we assume the input to be *well-structured*. Usually, this is enforced by assuming that the input belongs to a certain kind of (infinite) graph class. This can for example be the class of all planar graphs or the class of all graphs with maximal degree at most d for some number d. Many of the established tractable graph classes are *sparse* in the sense that their graphs have relatively few edges compared to their number of possible edges.

Algorithmic meta-theorems state that whole families of problems become more tractable if we restrict the input to certain graph classes. The best known example is Courcelle's theorem [1], which states that every problem definable in monadic second-order logic can be solved in linear time on bounded treewidth classes. It has been shown in a series of papers that all problems definable in first-order logic can be solved in linear time on bounded degree [2], excluded minor [3], locally bounded treewidth [4] and more general sparse graph classes [5], [6]. These results are obtained by solving the *first-order model-checking problem* on those graph classes. The input to this problem is a first-order sentence φ and a graph G and the task is to decide whether φ is satisfied on G. This problem can trivially be solved in time $O(|G|^{|\varphi|})$, which is optimal under complexity theoretic assumptions [7]. A first-order model-checking algorithm is considered *efficient* on some graph class if it solves the problem in time $f(|\varphi|)|G|^{O(1)}$ for some function f, i.e., in *fpt time*.

Nešetřil and Ossona de Mendez introduce *bounded expansion* and *nowhere dense* graph classes. These are very robust notions that generalize the previously mentioned graph classes (see left side of Figure 1) and have interesting algorithmic properties (e.g., [8], [9], [10], [11], [6], [5]). Most notably, Dvořák, Král and Thomas [5] solve the first-order model-checking problem in linear fpt time on bounded expansion classes, and Grohe, Kreutzer and Siebertz [6] solve this problem in almost linear fpt time on nowhere dense graph classes. For sparse graphs, this is in a sense the best possible result of this type: If a graph class is monotone (i.e., closed under taking subgraphs) and *somewhere dense* (i.e., not nowhere dense) then the first-order model-checking problem is as hard as on all graphs [6].

We therefore have reached a natural barrier in the study of meta-theorems for sparse graphs. But there are other well-structured graph classes that do not fit into the framework of sparsity. One of the current *main goals* in this area is to push the theory to account for such *dense* (or non-monotone) classes as well. In particular, we want to find dense, but *structurally simple* graph classes on which one can solve the first-order model-checking problem in fpt time.

An excellent tool to capture such graph classes are *first-order interpretations and transductions*. For a given first-order formula $\varphi(x, y)$, the corresponding *first-order interpretation* is a function that maps an input a graph G to the graph $I_\varphi(G)$ with vertex set $V(G)$ and edge set $\{uv \mid u, v \in V(G), u \neq v, G \models \varphi(u, v) \lor \varphi(v, u)\}$. For example with $\varphi(x, y) = \neg \mathit{edge}(x, y)$, the interpretation complements the input graph. With $\varphi(x, y) = \exists z\, \mathit{edge}(x, z) \land \mathit{edge}(z, y)$ it computes the square of a graph. For a given formula $\varphi(x, y)$, the *first-order interpretation of a graph class* $\mathcal{G}$ is the graph class $\{I_\varphi(G) \mid G \in \mathcal{G}\}$. *First-order transductions* are slightly more powerful than interpretations, since they also have the ability to copy, delete and nondeterministically color vertices. As the precise definition of transductions is not important for now, we delay it until Section II. We say a graph class $\mathcal{G}$ has *structurally property* X if it is the transduction of a graph class with property X (see right side of Figure 1 for examples). For the properties considered in this paper, this is equivalent to saying that $\mathcal{G}$ consists of induced subgraphs of an interpretation of a class with property X. This definition has nice closure properties in the sense that applying a transduction to a structurally property X class yields again a structurally property X class.

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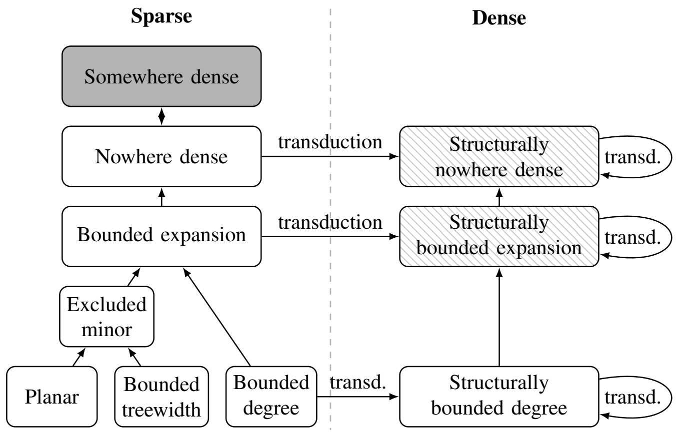

Fig. 1. Hierarchy of selected sparse graph classes and transductions thereof. The complexity of the first-order model-checking problem encoded using colors. White means that there is an fpt algorithm. Gray means that there is no fpt algorithm for monotone graph classes unless FPT = AW[$*$]. Striped means that it is unknown whether there is an fpt algorithm.

There have been recent efforts to lift the meta-theorems for sparse graph classes to transductions of these classes. Gajarský et al. [12] solve the first-order model-checking problem for structurally bounded degree graphs using a two-step approach:

1) First they provide a new structural characterization of these graphs. They define *near-uniform graph classes* and show that they are equivalent to structurally bounded degree graph classes.
2) Then they use this new characterization to compute for a graph G from a structurally bounded degree class a bounded degree graph G' that transduces to G via some fixed transduction.

It is then trivial to reduce the problem of evaluating a first-order formula φ on G to evaluating a modified formula φ' on the bounded degree graph G'. If the transduction from step 2) is based on the formula $\psi(x, y)$ then φ' is obtained from φ by replacing every edge relation *edge*(x, y) with $\psi(x, y)$. Now the existing meta-theorems for sparse graphs immediately give us an fpt model-checking algorithm for structurally bounded degree graphs. It is widely believed that the best approach to obtain meta-theorems for transductions of more general sparse graph classes is a similar two step approach: First find new structural characterizations of the dense graph class and then use them as a tool to compute a sparse preimage, i.e., a graph from a sparse class that transduces to a given dense graph. Most notably, Oum used this approach to solve the MSO model-checking problem on bounded cliquewidth [13].

Given the existing result for structurally bounded degree graph classes [12], and an extension thereof [14], a *significant milestone* would be an fpt model-checking algorithm for *structurally bounded expansion*. Bounded expansion can be characterized in many ways, one of which is via *low treedepth covers* [15]. Analogously, Gajarský et al. show that structurally bounded expansion can be characterized using so-called *low shrubdepth covers* and that one can solve the first-order model-checking problem if one can compute a low shrubdepth 2-cover [16]. Shrubdepth [17], [18] can be understood as a dense equivalent to treedepth. We have a large and flexible toolset to build algorithms on bounded expansion (grad, low treedepth covers, transitive-fraternal-augmentations, generalized coloring numbers, ...) [19], but low shrubdepth covers are at this point the only such tool for structurally bounded expansion. So far, they have not yet led to an fpt model-checking algorithm for these graph classes. In this paper, we add two new tools to our toolbox for structurally bounded expansion: *lacon-decompositions* and *shrub-decompositions*. We show that the first-order model-checking problem on structurally bounded expansion can be reduced to computing sparse lacon- or shrub-decompositions.

We should further mention another interesting approach to obtain dense, structurally simple graph classes. Bonnet, Kim, Thomassé and Watrigant [20] introduce a graph parameter called *twinwidth* that captures many interesting non-monotone graph classes. The authors provide an algorithm that — given an appropriate twinwidth contraction sequence — solves the first-order model-checking problem in linear fpt time [20]. The notion of bounded twinwidth is orthogonal to structurally bounded expansion or structurally nowhere denseness.

A. Introducing Lacon- and Shrub-Decompositions

A lacon- or shrub-decomposition of a graph G is a labeled graph G' that encodes G in a possibly much sparser way. These decompositions are not tied to any fixed sparsity measure, instead one can characterize different dense graph classes by applying different sparsity requirements to G'. In this paper, we use them to capture structurally bounded expansion, structurally bounded treewidth and structurally bounded treedepth, i.e., shrubdepth (Theorem 1, 2, 3). Due to their flexibility, it seems quite likely that they can be used to capture transductions of other sparse classes as well. Generally speaking, we show that a graph class has structurally property X if and only if each graph in the graph class has a lacon- or shrub-decomposition with property X.

A decomposition of a graph G consists of the so-called *target vertices* $V(G)$ and additional *hidden vertices*. The key difference between both types of decompositions is that in a lacon-decomposition the hidden vertices are colored (using two labels "0" and "1"), while in a shrub-decomposition the target vertices are colored (using a constant number of colors). "Lacon" is an acronym for **LA**rgest **CO**mmon **N**eighbor, since there is an order on the hidden vertices and the label of the largest common neighbor of two vertices (either "0" or "1") determines whether they are adjacent or not. Formally, this is defined as follows. A linear order π on the vertices of a graph G is represented by an injective function $\pi\colon V(G) \to \mathbf{N}$.

Definition 1 (Lacon-decomposition). *A* lacon-decomposition *is a tuple* (L, π) *satisfying the following properties.*

1) *L is a bipartite graph with sides T, H. We say T are the* target *vertices and H are the* hidden *vertices of the decomposition.*
2) *Every hidden vertex is labeled with either "0" or "1".*

3) *π is a linear order on the vertices of L with $\pi(h) < \pi(t)$ for all $t \in T$, $h \in H$.*
4) *For all $t, t' \in T$ holds $N(t) \cap N(t') \neq \emptyset$. The largest vertex with respect to π in $N(t) \cap N(t')$ is called the* dominant vertex *between t and t'.*

We say (L, π) is the lacon-decomposition of a graph *G if*

5) *$V(G) = T$,*
6) *for all $t \neq t' \in T$ there is an edge between t and t' in G if and only if the dominant vertex of t and t' is labeled with "1".*

This definition leads to a nice illustrative process that reconstructs a graph G from its lacon-decomposition (L, π): In the beginning, all edges in G are unspecified. Then we reveal the hidden vertices of L one by one in ascending order by π. If we encounter a hidden vertex h with label "1" we fully connect its neighborhood $N(h)$ in G. On the other hand, if h has label "0" we remove all edges between vertices from $N(h)$ in G. The edges inserted or removed by h "overwrite" the edges that were inserted or removed by a previous hidden vertex. After all hidden vertices have been revealed, the resulting graph is exactly G. See the left side of Figure 2 for an example of a lacon-decomposition.

Before we use lacon-decompositions to characterize transductions of different sparse graph classes, let us also introduce shrub-decompositions. Our definition closely follows the wording used by Ganian et al. [18], [17] to define shrubdepth.

Definition 2 (Shrub-decomposition)**.** *Let m and d be nonnegative integers. A* shrub-decomposition of m colors and diameter *d of a graph G is a pair (F, S) consisting of an undirected graph F and a set $S \subseteq \{1, 2, \ldots, m\}^2 \times \{1, 2, \ldots, d\}$ (called* signature*) such that*

1) *the distance in F between any two vertices is at most d,*
2) *the set of pendant vertices (i.e., degree-one vertices) of F is exactly the set $V(G)$ of vertices of G,*
3) *each pendant vertex of F is assigned one of the colors $\{1, 2 \ldots, m\}$, and*
4) *for any i, j, l it holds $(i, j, l) \in S$ iff $(j, i, l) \in S$ (symmetry in the colors), and for any two vertices $u, v \in V(G)$ such that u is colored with i and v is colored with j and the distance between u, v in F is l, the edge uv exists in G if and only if $(i, j, l) \in S$.*

Item 4) says that the existence of a G-edge between $u, v \in V(G)$ depends only on the colors of u, v and their distance in the decomposition. See the right side of Figure 2 for an example. A shrub-decomposition is a generalization of a *tree-model* used in the definition of shrubdepth. In fact, we can define shrubdepth (i.e., structurally bounded treedepth) using shrub-decompositions.

Definition 3 (Shrubdepth [17], [18])**.** *We say a* tree-model of *m* colors and depth *d is a shrub-decomposition (F, S) of m colors where F is a rooted tree and the length of every root-to-leaf path in F is exactly d. A graph class $\mathcal{G}$ is said to have* bounded shrubdepth *if there exist numbers m and d such that every graph in $\mathcal{G}$ has a tree-model of m colors and depth d.*

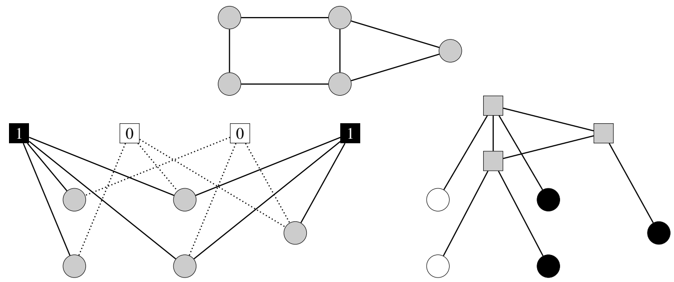

Fig. 2. Top: A graph. Left: A lacon-decomposition of the top graph. The hidden vertices are aligned in ascending order from left to right. Right: A shrub-decomposition of the top graph with two colors and diameter three. Vertices from G are adjacent if they have distance two or distance three and the same color.

Instead of requiring F to be a tree, we can require F to come from a bounded expansion class to obtain a characterization of structurally bounded expansion, as we will see soon.

B. Generalized Coloring Numbers and Bounded Expansion

Before we present our results, we introduce the so-called *generalized coloring numbers*, as we use them to pose a sparsity requirement on lacon-decompositions. These numbers can be used to characterize bounded expansion, treewidth and treedepth and have numerous algorithmic applications (e.g., [21], [22], [23], [24]). Let G be an undirected graph and π be an ordering on the vertices of G. We say a vertex u is *r-reachable* from v with respect to π if $\pi(u) \leq \pi(v)$ and there is a path of length at most r from v to u and for all vertices w on the path either $w = u$ or $\pi(w) \geq \pi(v)$. A vertex u is *weakly r-reachable* from v with respect to π if there is a path of length at most r from v to u and $\pi(u) \leq \pi(w)$ for all vertices w on that path. If G is a directed graph, we say u is (weakly) r-reachable from v if and only if it is (weakly) r-reachable in the underlying undirected graph.

Let $\mathrm{Reach}_r(G, \pi, v)$ be the set of vertices that are r-reachable from v with respect to π. We similarly define $\mathrm{WReach}_r(G, \pi, v)$ be the set of weakly r-reachable vertices. We set

$$\mathrm{col}_r(G, \pi) = \max_{v \in V(G)} |\mathrm{Reach}_r(G, \pi, v)|,$$

$$\mathrm{wcol}_r(G, \pi) = \max_{v \in V(G)} |\mathrm{WReach}_r(G, \pi, v)|.$$

We define $\Pi(G)$ to be the set of all linear orders on G. Finally, the *r-coloring number* and *weak r-coloring number* of a graph G is defined as

$$\mathrm{col}_r(G) = \min_{\pi \in \Pi(G)} \mathrm{col}_r(G, \pi),$$

$$\mathrm{wcol}_r(G) = \min_{\pi \in \Pi(G)} \mathrm{wcol}_r(G, \pi).$$

For small r, the two flavors of generalized coloring numbers are strongly related. It holds that $\mathrm{col}_r(G) \leq \mathrm{wcol}_r(G) \leq \mathrm{col}_r(G)^r$ [25]. For large r, the generalized coloring numbers converge to treewidth ($\mathrm{tw}(G)$) and treedepth ($\mathrm{td}(G)$), respectively [19], [26]

$$\mathrm{col}_1(G) \leq \cdots \leq \mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1,$$

$$\mathrm{wcol}_1(G) \leq \cdots \leq \mathrm{wcol}_\infty(G) = \mathrm{td}(G).$$

A graph H is an *r-shallow minor* of a graph G if it is the result of a contraction of mutually disjoint connected subgraphs with radius at most r in G. A graph class $\mathcal{G}$ has *bounded expansion* if there exists a function $f(r)$ such that $\frac{|E(H)|}{|V(H)|} \leq f(r)$ for all $G \in \mathcal{G}$ and r-shallow minors H of G [19]. Zhu first observed that generalized coloring numbers can also be used to characterize bounded expansion [27]. In this paper, we rely heavily on the following characterization by van den Heuvel and Kierstead [28], which is slightly stronger than Zhu's [27] original characterization.

Definition 4 (Bounded Expansion [28])**.** *A graph class $\mathcal{G}$ has* bounded expansion *if there exists a function $f(r)$ such that every graph $G \in \mathcal{G}$ has an ordering π of its vertices with $\mathrm{col}_r(G,\pi) \leq f(r)$ for all r.*

As illustrated in Figure 1, examples of bounded expansion graph classes include planar graphs or any graph class of bounded degree.

C. Main Result

This paper presents a new characterization of structurally bounded expansion based on lacon- and shrub-decompositions. As a side result, we also obtain characterizations of structurally bounded treewidth and treedepth. For the definition of *first-order transductions* and *structurally bounded expansion* see Section II.

The following theorem contains the main contribution of this paper. To the best of the author's knowledge, it lists all currently known characterizations of structurally bounded expansion. The equivalence of 1), 2), 3) is proven in this paper. For the sake of completeness, we also list 4) due to Gajarský et al. [16] and 5) due to Nešetřil et al. [29], [30].

Theorem 1. *Let $\mathcal{G}$ be a graph class. The following statements are equivalent.*

1) *$\mathcal{G}$ has structurally bounded expansion, i.e., there exists a graph class $\mathcal{G}'$ with bounded expansion and a first-order transduction τ such that $\mathcal{G} \subseteq \tau(\mathcal{G}')$.*
2) *There exists a function $f(r)$ such that every graph in $\mathcal{G}$ has a lacon-decomposition (L,π) with $\mathrm{col}_r(L,\pi) \leq f(r)$ for all r (this implies that L comes from a bounded expansion class).*
3) *There exist a signature S, numbers m, d and a graph class $\mathcal{G}'$ with bounded expansion such that every graph in $\mathcal{G}$ has a shrub-decomposition (F,S) with m colors, diameter d and $F \in \mathcal{G}'$.*
4) *$\mathcal{G}$ has low shrubdepth covers [16].*
5) *$\mathcal{G}$ has low (linear) cliquewidth covers and a stable edge relation [29], [30].*

Note that if we are given a graph G from a structurally bounded expansion class $\mathcal{G}$ and compute a lacon- or shrub-decomposition satisfying the conditions listed in 2) or 3) then we effectively compute a bounded expansion graph that transduces to G (i.e., a preimage), thereby solving the first-order model-checking problem for structurally bounded expansion graph classes. Thus, lacon- and shrub-decompositions may be useful for obtaining meta-theorems for structurally sparse graph classes.

Theorem 1 can further be understood as a limit on the expressive power of transductions on bounded expansion classes. For example, it implies that transductions based on boolean combinations of local, purely existential formulas have the same expressive power as general first-order transductions: Assume a graph class $\mathcal{G}$ was obtained from a graph class $\mathcal{G}'$ with bounded expansion via a (possibly very complicated) transduction. We replace $\mathcal{G}'$ with the class of shrub-decompositions of $\mathcal{G}$, as described in 3). Now $\mathcal{G}$ can be expressed as a very simple transduction of $\mathcal{G}'$. We merely have to check the color of two vertices and their (bounded) distance. This can be done using a boolean combination of local, existential formulas. Other bounds on the expressive power of transductions on sparse graphs (orthogonal to ours) have been obtained in the literature [16], [31]. As side results, we obtain the following characterizations of structurally bounded treedepth and treewidth. Item 4) of both Theorem 2 and 3 is not proven in this paper, but is listed for the sake of completeness [18], [29], [30].

Theorem 2. *Let $\mathcal{G}$ be a graph class. The following statements are equivalent.*

1) *$\mathcal{G}$ has structurally bounded treedepth, i.e., there exists a graph class $\mathcal{G}'$ with bounded treedepth and a first-order transduction τ such that $\mathcal{G} \subseteq \tau(\mathcal{G}')$.*
2) *There exists a number d such that every graph in $\mathcal{G}$ has a lacon-decomposition (L,π) with $\mathrm{wcol}_\infty(L,\pi) \leq d$ (this implies L has treedepth at most d).*
3) *There exist a number d and a signature S such that every graph in $\mathcal{G}$ has a shrub-decomposition (F,S) with d colors, diameter d and treedepth at most d.*
4) *$\mathcal{G}$ has bounded shrubdepth [18].*

Theorem 3. *Let $\mathcal{G}$ be a graph class. The following statements are equivalent.*

1) *$\mathcal{G}$ has structurally bounded treewidth, i.e., there exists a graph class $\mathcal{G}'$ with bounded treewidth and a first-order transduction τ such that $\mathcal{G} \subseteq \tau(\mathcal{G}')$.*
2) *There exists a number t such that every graph in $\mathcal{G}$ has a lacon-decomposition (L,π) with $\mathrm{col}_\infty(L,\pi) \leq t-1$ (this implies L has treewidth at most t).*
3) *There exist a number t and a signature S such that every graph in $\mathcal{G}$ has a shrub-decomposition (F,S) with t colors, diameter t and treewidth at most t.*
4) *$\mathcal{G}$ has bounded cliquewidth and a stable edge relation [29], [30].*

The results of Theorem 1, 2, and 3 are consequences of a more general statement, Theorem 5, which shows that for every transduction of a graph G, we can find an equivalent lacon-decomposition whose generalized coloring numbers are not too far off from the numbers of G.

D. Localized Feferman–Vaught Composition Theorem

The Feferman–Vaught theorem [32] states that the validity of first-order formulas on the disjoint union or Cartesian product of two graphs is uniquely determined by the value of first-order formulas on the individual graphs. Makowsky adjusted the theorem for algorithmic use in the context of MSO model-checking [33]. It has numerous applications and in the area of meta-theorems it leads for example to an especially concise proof of Courcelle's theorem [34].

A highly useful property of first-order logic is *locality*, for example in the form of Hanf's [35] or Gaifman's [36] theorem. Intuitively, these theorems state that first-order logic can only express local properties. Locality is a key ingredient in many first-order model-checking algorithms [37], [6], [38], [2].

A central building block in our proofs is a localized variant of the Feferman–Vaught theorem (also proven in [21, Lemma 15]). The essence of this result is the following: Assume we have a graph G, a first-order formula $\varphi(x, y)$ and two vertices v, w and want to know whether $G \models \varphi(v, w)$. Further assume that we have some kind of "local separator" between v, w, i.e., a tuple of vertices $\bar{u}$ such that all short paths between v and w pass through $\bar{u}$. We show that knowing whether $G \models \psi(\bar{u}, v)$, $G \models \psi(\bar{u}, w)$ for certain formulas ψ this gives us enough information to compute whether $G \models \varphi(v, w)$. The original Feferman–Vaught theorem claims this only if $\bar{u}$ is an actual separator between v and w, i.e., *all* paths between v and w pass through $\bar{u}$. To formalize our result, we need to define so-called *q-types*.

Definition 5 (q-type [34])**.** *Let G be a labeled graph and $\bar{v} = (v_1, \dots, v_k) \in V(G)^k$. The q-type of $\bar{v}$ in G is the set $\mathrm{tp}_q(G, \bar{v})$ of all first-order formulas $\psi(x_1 \dots x_k)$ of rank at most q such that $G \models \psi(v_1 \dots v_k)$.*

We syntactically normalize formulas so that there are only finitely many formulas of fixed quantifier rank and with a fixed set of free variables. Therefore q-types are finite sets. For a tuple $\bar{u} = (u_1, \dots, u_k)$, we write $\{\bar{u}\}$ for the set $\{u_1, \dots, u_k\}$. We follow Grohe's presentation of the Feferman–Vaught theorem [34].

Proposition 1 ([34, Lemma 2.3])**.** *Let G, H be labeled graphs and $\bar{u} \in V(G)^k$, $\bar{v} \in V(G)^l$, $\bar{w} \in V(H)^m$, such that $V(G) \cap V(H) = \{\bar{u}\}$. Then for all $q \in \mathbf{N}$, $\mathrm{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ is determined by $\mathrm{tp}_q(G, \bar{u}\bar{v})$ and $\mathrm{tp}_q(H, \bar{u}\bar{w})$.*

In the previous statement, $\bar{u}$ separates $\bar{v}$ and $\bar{w}$ by splitting $G \cup H$ into the subgraphs G and H. We extend this result using a more general notion of separation that goes as follows. The *length* of a path equals its number of edges.

Definition 6. *Let G be a graph, $r \in \mathbf{N}$, and $\bar{u}$, $\bar{v}_1, \dots, \bar{v}_k$ be tuples of vertices from G. We say $\bar{u}$ r-separates $\bar{v}_i$ and $\bar{v}_j$ if every path of length at most r between a vertex from $\bar{v}_i$ and a vertex from $\bar{v}_j$ contains at least one vertex from $\bar{u}$. We say $\bar{u}$ r-separates $\bar{v}_1, \dots, \bar{v}_k$ if it r-separates $\bar{v}_i$ and $\bar{v}_j$ for all $i \neq j$.*

Based on this notion of separation, we use the following Feferman–Vaught-inspired result.

Theorem 4. *There exists a function $f(q, l)$ such that for all labeled graphs G, every $q, l \in \mathbf{N}$, and all tuples $\bar{u}$, $\bar{v}_1, \dots, \bar{v}_k$ of vertices from G such that $\bar{u}$ 4^q-separates $\bar{v}_1, \dots, \bar{v}_k$ and $|\bar{u}| + |\bar{v}_1| + \dots + |\bar{v}_k| \le l$, the type $\mathrm{tp}_q(G, \bar{u}\bar{v}_1 \dots \bar{v}_k)$ depends only on the types $\mathrm{tp}_{f(q,l)}(G, \bar{u}\bar{v}_1), \dots, \mathrm{tp}_{f(q,l)}(G, \bar{u}\bar{v}_k)$. Furthermore, $\mathrm{tp}_q(G, \bar{u}\bar{v}_1 \dots \bar{v}_k)$ can be computed from $\mathrm{tp}_{f(q,l)}(G, \bar{u}\bar{v}_1), \dots, \mathrm{tp}_{f(q,l)}(G, \bar{u}\bar{v}_k)$.*

E. Techniques and Outline

The two cornerstones of our proofs are a localized Feferman–Vaught composition theorem [21] and generalized coloring numbers. We do not use low shrubdepth covers or any of the techniques used in their proof of existence [16]. It should be possible to derive the results of this paper also directly via low shrubdepth covers, but we refrain from doing so since we believe our direct approach has a better chance of being liftable to structurally nowhere dense classes. We show the following circular sequence of implications.

- (i) We start with a graph G that is a transduction of a graph G', where G' has bounded generalized coloring numbers.
- (i) $\implies$ (ii) Then we construct a so-called *directed lacon-decomposition* (L, π) of G whose coloring numbers are bounded as well. This is a generalization of a lacon-decomposition where we allow L to be a directed graph. This step forms the central part of the paper and here we use the localized Feferman–Vaught theorem extensively.
- (ii) $\implies$ (iii) Next, we convert (L, π) into a normal lacon-decomposition without increasing the generalized coloring numbers too much.
- (iii) $\implies$ (iv) We transform the lacon-decomposition into an equivalent shrub-decomposition (F, S), also with bounded coloring numbers.
- (iv) $\implies$ (i) This implies that G is a transduction of F, and since the generalized coloring numbers of F are bounded, this brings us back to the start.

Afterwards, we use the fact that generalized coloring numbers can describe bounded expansion, bounded treewidth and bounded treedepth. Therefore, Theorem 1, 2 and 3 follow by posing different bounds on the coloring numbers.

The proofs of (i) $\implies$ (ii) and (ii) $\implies$ (iii) are the most involved part in the above process and form the core of this paper.

We sketch some of the ideas we use for (i) $\implies$ (ii). We have a graph G' with an ordering π of its vertices and want to obtain a lacon-decomposition of an interpretation $I_\varphi(G')$ of G'. Consider vertices $v, w \in V(G')$ and a tuple $\bar{u}$ consisting of the vertices in $\mathrm{WReach}_r(G', \pi, v) \cap \mathrm{WReach}_r(G', \pi, w)$. It is a basic property of generalized coloring numbers that every path of length at most r between v and w passes through $\bar{u}$, i.e., $\bar{u}$ *r-separates* v and w. The localized decomposition theorem states that $\varphi(v,w)$ depends only on q-types of $\bar{u}v$ and $\bar{u}w$. For every possible tuple $\bar{u}$ and every combination of q-types, we introduce a hidden vertex that we label with "1" if and only if its two q-types together imply φ to be true. We then connect the hidden vertices with the vertices of G' if their q-types match.

The implication (i) $\Longrightarrow$ (ii) is proven in Section III and (ii) $\Longrightarrow$ (iii) is proven in Section IV. This is then combined in Section V to show (i) $\Longrightarrow$ (iii). In Section VI we prove (iii) $\Longrightarrow$ (iv). In Section VII we combine all these implications to prove our main results Theorem 1, 2 and 3. The localized Feferman–Vaught theorem has been proven before [21, Lemma 15]. We nevertheless finish the paper in Section VIII with a self sustained proof of the theorem.

II. Preliminaries

We use standard graph notation and first-order logic over labeled and unlabeled graphs. An unlabeled graph is a relational structure with a binary edge relation. A labeled graph may additionally have vertex labels (or colors), represented by unary relations. Let $\varphi(x,y)$ be a first-order formula. For a given graph G, the *first-order interpretation* of G under φ, denoted by $I_\varphi(G)$, is the undirected unlabeled graph with vertex set $V(G)$ and edge set $\{uv \mid u,v \in V(G), u \neq v, G \models \varphi(u,v) \vee \varphi(v,u)\}$. *First-order transductions* extend interpretations with the ability to delete vertices, as well as copy and color the input graph. We use the same notation as [12]. The most interesting building block for us are *basic transductions.* A basic transduction is a triple $\tau_0 = (\chi, \nu, \varphi)$ of first-order formulas of arity zero, one and two. If $G \not\models \chi$, then $\tau_0(G)$ is undefined. Otherwise $\tau_0(G)$ is the graph with vertex set $\{v \mid v \in V(G), G \models \nu(v)\}$ and edge set $\{uv \mid u,v \in V(G), G \models \varphi(u,v) \vee \varphi(v,u)\}$. There are two more building blocks, which are less important for this work. A *p-parameter expansion* is an operation that maps each graph G to the set of all graphs that can be created by adding p unary predicates (i.e., colors or labels) to G. An *m-copy operation* maps a graph G to a graph G^m with $V(G^m) = V(G) \times \{1,\dots,m\}$ and $E(G^m) = \{(v,i)(w,i) \mid vw \in E(G), 1 \le i \le m\}$. Thus G^m consists of m disjoint copies of G. Furthermore, G^m has a binary relation $\sim$ with $(u,i) \sim (v,j)$ iff $u = v$ and unary relations $Q_1,\dots,Q_m$ with $Q_i = \{(v,i) \mid v \in V(G)\}$. Finally, a *transduction* τ is an operation of the form $\tau = \tau_0 \circ \gamma \circ \varepsilon$ where ε is a p-parameter expansion, γ is a m-copy operation and τ_0 is a basic transduction. Notice that the output $\tau(G)$ is a *set of graphs* because of the parameter expansion. For a class $\mathcal{G}$, we define $\tau(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} \tau(G)$ to be the *transduction of $\mathcal{G}$*. This now leads us the definition of structurally bounded expansion and related graph classes.

Definition 7 (Structurally bounded expansion [16])**.** *A graph class $\mathcal{G}$ has* structurally bounded expansion *if there exists a graph class $\mathcal{G}'$ with bounded expansion and a transduction τ such that $\mathcal{G} \subseteq \tau(\mathcal{G}')$. Generally speaking, $\mathcal{G}$ has* structurally property *X if $\mathcal{G}'$ has property X.*

III. Interpretations of Bounded Expansion Have Directed Lacon-Decompositions

This section contains the central idea of this paper. We consider a generalization of lacon-decompositions called *directed lacon-decompositions* and construct such a decomposition with bounded generalized coloring numbers. In the following definition we denote the in- and out-neighborhoods of a vertex h by $N^-(h)$ and $N^+(h)$, respectively.

Definition 8 (Directed Lacon-decomposition)**.** *A* directed lacon-decomposition *is a tuple (L,π) satisfying the following properties.*

1) *L is a directed bipartite graph with sides T, H. We say T are the* target *vertices and H are the* hidden *vertices of the decomposition.*
2) *Every hidden vertex is labeled with either "0" or "1".*
3) *π is a linear order on the vertices of L with $\pi(h) < \pi(t)$ for all $t \in T$, $h \in H$.*
4) *For all $t,t' \in T$ holds $\big(N^-(t) \cap N^+(t')\big) \cup \big(N^+(t) \cap N^-(t')\big) \neq \emptyset$. The largest vertex with respect to π in $\big(N^-(t) \cap N^+(t')\big) \cup \big(N^+(t) \cap N^-(t')\big)$ is called the* dominant vertex *between t and t'.*

We say (L,π) is the directed lacon-decomposition *of an* undirected graph G *if*

5) *$V(G) = T$,*
6) *for all $t \neq t' \in T$ there is an edge between t and t' in G if and only if the dominant vertex of t, t' is labeled "1".*

Lemma 1. *Let $\varphi(x,y)$ be a first-order formula. There exists a function g such that for every labeled graph G and ordering σ on the vertices of G there exists a directed lacon-decomposition (L,π) of $I_\varphi(G)$ with*

- *$\mathrm{col}_r(L,\pi) \le g\big(|\varphi| + \mathrm{col}_{2\cdot 4^{|\varphi|}}(G,\sigma)\big) \cdot \mathrm{col}_{4^{|\varphi|}r}(G,\sigma)$ for all r,*
- *$\mathrm{wcol}_r(L,\pi) \le g\big(|\varphi| + \mathrm{col}_{2\cdot 4^{|\varphi|}}(G,\sigma)\big) \cdot \mathrm{wcol}_{4^{|\varphi|}r}(G,\sigma)$ for all r.*

In our upcoming proof of Lemma 1 we need the following facts about generalized coloring numbers. As they can be easily derived from their basic definition, we omit a proof.

Proposition 2. *Let G be a graph with ordering σ and $r \in \mathbf{N}$.*

1) *For $v \in V(G)$ and $u,w \in \mathrm{WReach}_r(G,\sigma,v)$ with $\sigma(w) < \sigma(u)$ holds $w \in \mathrm{WReach}_{2r}(G,\sigma,u)$.*
2) *Let $v,w \in V(G)$ and $S = \mathrm{WReach}_r(G,\sigma,v) \cap \mathrm{WReach}_r(G,\sigma,w)$. Then every path of length at most r between v and w contains a vertex from S.*
3) *Let $v_1,\dots,v_k$ be vertices with $\sigma(v_1) \le \sigma(v_k)$ and $\min_{i=2}^k \sigma(v_i) = \sigma(v_k)$ such that for all $1 \le i < k$ either $v_i \in \mathrm{WReach}_r(G,\sigma,v_{i+1})$ or $v_{i+1} \in \mathrm{WReach}_r(G,\sigma,v_i)$. Then there exists $w \in \mathrm{Reach}_{rk}(G,\sigma,v_k)$ with $v_1 \in \mathrm{WReach}_r(G,\sigma,w)$.*
4) *Let $v_1,\dots,v_k$ be vertices with $\sigma(v_1) = \min_{i=1}^k \sigma(v_i)$ such that for all $1 \le i < k$ either $v_i \in \mathrm{WReach}_r(G,\sigma,v_{i+1})$ or $v_{i+1} \in \mathrm{WReach}_r(G,\sigma,v_i)$. Then $v_1 \in \mathrm{WReach}_{rk}(G,\sigma,v_k)$.*
5) *$\mathrm{wcol}_r(G,\sigma) \le \mathrm{col}_r(G,\sigma)^r$.*

Proof of Lemma 1: For technical reasons, it is easier to prove the result of this lemma under the additional assumption that the input graph G has an apex vertex. We do so first, and at the end of the proof we generalize the result also to graphs without an apex vertex. Our main proof is outlined as follows: We first construct a directed lacon-decomposition, then prove that the construction is correct and at last bound the coloring numbers.

Constructing a Lacon-Decomposition. Let us fix a graph G (with an apex vertex) and an ordering σ. Let $q = |\varphi|$, $l = \mathrm{wcol}_{2\cdot 4^q}(G,\sigma) + 2$ and let $f(q,l)$ be the function from Theorem 4. We define the hidden vertices H of L to be all tuples $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ such that

- $u \in V(G)$,
- $\bar{u}$ is the tuple of all vertices in $\mathrm{WReach}_{2\cdot 4^q}(G,\sigma,u)$, ordered in ascending order by σ,
- type_1, type_2 are $f(q,l)$-types containing formulas with $|\bar{u}|+1$ free variables.

Let us fix one such hidden vertex $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$. Assume vertices $v_1, v_2 \in V(G)$ that are 4^q-separated by $\bar{u}$ and have types $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_1) = \mathit{type}_1$ and $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_2) = \mathit{type}_2$. The fact whether $G \models \varphi(v_1, v_2) \vee \varphi(v_2, v_1)$ is determined by the type $\mathrm{tp}_q(G, \bar{u}v_1v_2)$. And according to the localized Feferman–Vaught variant in Theorem 4, $\mathrm{tp}_q(G, \bar{u}v_1v_2)$ is in turn determined by type_1 and type_2. With this in mind, we iterate over all hidden vertices $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ and distinguish two cases:

- *Case 1:* For all $v_1, v_2 \in V(G)$ that are 4^q-separated by $\bar{u}$ with $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_1) = \mathit{type}_1$ and $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_2) = \mathit{type}_2$ holds $G \models \varphi(v_1, v_2) \vee \varphi(v_2, v_1)$. In this case we give $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ the label "1".
- *Case 2:* For all $v_1, v_2 \in V(G)$ that are 4^q-separated by $\bar{u}$ with $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_1) = \mathit{type}_1$ and $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_2) = \mathit{type}_2$ holds $G \not\models \varphi(v_1, v_2) \vee \varphi(v_2, v_1)$. In this case we give $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ the label "0".

Now, every hidden vertex is labeled with either "1" or "0". We define the arc set of our directed lacon-decomposition as follows: Let $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ be a hidden vertex. For every v_1 with $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_1) = \mathit{type}_1$ such that $u \in \mathrm{WReach}_{4^q}(G,\sigma,v_1)$ we add an arc from v_1 to $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$. Similarly, for every v_2 with $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_2) = \mathit{type}_2$ such that $u \in \mathrm{WReach}_{4^q}(G,\sigma,v_2)$ we add an arc from $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ to v_2.

As the last step of our construction, we fix an ordering π on the vertices of L such that for all hidden vertices $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2), (u', \bar{u}', \mathit{type}'_1, \mathit{type}'_2) \in H$ with $\sigma(u) < \sigma(u')$ holds $\pi((u, \bar{u}, \mathit{type}_1, \mathit{type}_2)) < \pi((u', \bar{u}', \mathit{type}'_1, \mathit{type}'_2))$. We further require for every $h \in H$ and every $t \in V(G)$ that $\pi(h) < \pi(t)$. Such an ordering trivially exists.

Correctness of Construction. At first, we need to show that (L, π) is a directed lacon-decomposition as defined in Definition 8. One can easily verify that L is in fact a directed bipartite graph, every hidden vertex is either labeled with "1" or "0" and that π is an ordering on the vertices of L with $\pi(h) < \pi(t)$ for all $h \in H$, $t \in T = V(G)$. What is left to do, is fix some vertices $v_1, v_2 \in T$ and show that the set $\big(N^-(v_1) \cap N^+(v_2)\big) \cup \big(N^+(v_1) \cap N^-(v_2)\big)$ is non-empty. We define $R_1 = \mathrm{WReach}_{4^q}(G,\sigma,v_1)$ and $R_2 = \mathrm{WReach}_{4^q}(G,\sigma,v_2)$. Since G has an apex vertex, there is a path of length at most two between v_1 and v_2. We can also generally assume that $4^q \geq 2$ and therefore, by 2) of Proposition 2, $R_1 \cap R_2 \neq \emptyset$. We choose some vertex $u' \in R_1 \cap R_2$ and consider the hidden vertex $h' = (u', \bar{u}', \mathrm{tp}_{f(q,l)}(G, \bar{u}'v_1), \mathrm{tp}_{f(q,l)}(G, \bar{u}'v_2))$ where $\bar{u}'$ is the tuple of all vertices in $\mathrm{WReach}_{2\cdot 4^q}(G,\sigma,u')$, ordered in ascending order by σ. We constructed L such that there is an arc from v_1 to h' and an arc from h' to v_2. We conclude that $\big(N^-(v_1) \cap N^+(v_2)\big) \cup \big(N^+(v_1) \cap N^-(v_2)\big)$ is non-empty and therefore that (L, π) is a directed lacon-decomposition.

Next, we show that (L, π) also is a directed lacon-decomposition of $I_\varphi(G)$. L was constructed such that its target vertices are $T = V(G) = V(I_\varphi(G))$. It remains to show that the dominant vertex of two arbitrary vertices $v_1, v_2 \in T$ is labeled with "1" if and only if $v_1v_2 \in E(I_\varphi(G))$. Let h be the dominant vertex of v_1 and v_2. By construction, h is either of the form $h = (u, \bar{u}, \mathrm{tp}_{f(q,l)}(G, \bar{u}v_1), \mathrm{tp}_{f(q,l)}(G, \bar{u}v_2))$ or $h = (u, \bar{u}, \mathrm{tp}_{f(q,l)}(G, \bar{u}v_2), \mathrm{tp}_{f(q,l)}(G, \bar{u}v_1))$. W.l.o.g. we assume it is the former form.

We show that $R_1 \cap R_2 \subseteq \{\bar{u}\}$. To this end, remember our previously defined vertex $u' \in R_1 \cap R_2$ and corresponding hidden vertex h'. If $\sigma(u') > \sigma(u)$ then this would mean that $\pi(h') > \pi(h)$, a contradiction to the fact that we chose h to be the largest vertex in $\big(N^-(v_1) \cap N^+(v_2)\big) \cup \big(N^+(v_1) \cap N^-(v_2)\big)$. Therefore $\sigma(u') < \sigma(u)$. We constructed L such that $u \in \mathrm{WReach}_{4^q}(G,\sigma,v_1)$ and we chose $u' \in \mathrm{WReach}_{4^q}(G,\sigma,v_1)$. Thus by 1) of Proposition 2, $u' \in \mathrm{WReach}_{2\cdot 4^q}(G,\sigma,u) = \{\bar{u}\}$. This implies that $R_1 \cap R_2 \subseteq \{\bar{u}\}$.

Thus, 2) of Proposition 2 states that $\bar{u}$ 4^q-separates v_1 and v_2. Hence, as discussed earlier, the fact whether $G \models \varphi(v_1, v_2) \vee \varphi(v_2, v_1)$ only depends on $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_1)$ and $\mathrm{tp}_{f(q,l)}(G, \bar{u}v_2)$. We constructed L such that $h = (u, \bar{u}, \mathrm{tp}_{f(q,l)}(G, \bar{u}v_1), \mathrm{tp}_{f(q,l)}(G, \bar{u}v_2))$ is labeled with "1" if and only if $G \models \varphi(v_1, v_2) \vee \varphi(v_2, v_1)$, i.e., $v_1v_2 \in E(I_\varphi(G))$. This implies that (L, π) is in fact a directed lacon-decomposition of $I_\varphi(G)$.

Bounding the Coloring Numbers. For a hidden vertex $(u, \bar{u}, \mathit{type}_1, \mathit{type}_2)$ in L, we say u is its *corresponding vertex in* G. If t is a target vertex, we say the *corresponding vertex in* G is t itself. The corresponding vertex of a vertex $x \in V(L)$ is denoted by $u(x)$.

We start by bounding the number of hidden vertices that have the same corresponding vertex in G. To do so, we need to determine the number of possible types $\mathit{type}_1, \mathit{type}_2$ a hidden vertex can have. The size of $\bar{u}$ is bounded by $\mathrm{wcol}_{2\cdot 4^q}(G,\sigma)$. Thus, the number of $f(q,l)$-types containing formulas with at most $|\bar{u}|+1$ free variables is bounded by some function of q and l. This means, there exists a function g' (independent of G and σ) such that

$$\text{for all } v \in V(G) \text{ holds } |\{x \in V(L) \mid u(x) = v\}| \leq g'(q,l). \tag{1}$$

Let h be a hidden vertex and $h' \in \mathrm{Reach}_r(L, \pi, h)$ for some r. Since all target vertices are larger than h with respect to π, we know that h' is a hidden vertex. Remember that L is bipartite. By definition of the generalized coloring numbers, there is a path $h_0 t_0 h_1 t_1 \dots t_{k-1} h_k$ such that $h_0 = h'$, $h_k = h$, $k \le r/2$, $\pi(h_0) < \pi(h_k)$, and $\pi(h_k) = \min_{i=2}^k \pi(h_i)$. Here, the h_i are hidden vertices and the t_i are target vertices. The order π was constructed such that $\sigma(u(h_i)) > \sigma(u(h_j))$ implies $\pi(h_i) > \pi(h_j)$ for all hidden vertices h_i, h_j. Therefore, $\sigma(u(h_0)) \le \sigma(u(h_k))$, and $\sigma(u(h_k)) = \min_{i=2}^k \sigma(u(h_i))$. We constructed L such that $u(h_i), u(h_{i+1}) \in \mathrm{WReach}_{4^q}(G, \sigma, t_i)$. Thus by 1) of Proposition 2, either $u(h_i) \in \mathrm{WReach}_{2 \cdot 4^q}(G, \sigma, u(h_{i+1}))$ or $u(h_{i+1}) \in \mathrm{WReach}_{2 \cdot 4^q}(G, \sigma, u(h_i))$. By 3) of Proposition 2, there exists $w \in \mathrm{Reach}_{4^q r}(G, \sigma, u(h))$ with $u(h') \in \mathrm{WReach}_{2 \cdot 4^q}(G, \sigma, w)$. In other words, $|\mathrm{Reach}_r(L, \pi, h)|$ is bounded from above by the number of tuples (w, v, h') with $w \in \mathrm{Reach}_{4^q r}(G, \sigma, u(h))$, $v \in \mathrm{WReach}_{2 \cdot 4^q}(G, \sigma, w)$ and $u(h') = v$. Using (1), we bound the number of such tuples by

$$|\mathrm{Reach}_r(L, \pi, h)| \le \mathrm{col}_{4^q r}(G, \sigma) \cdot l \cdot g'(q, l). \tag{2}$$

Let us now consider a target vertex t. Observe that

$$|\mathrm{Reach}_r(L, \pi, t)| \le 1 + \sum_{h \in N(t) \setminus \{t\}} |\mathrm{Reach}_r(L, \pi, h)|. \tag{3}$$

Since for all $h \in N(t)$ holds $u(h) \in \mathrm{WReach}_{4^q}(G, \sigma, t)$, we can again use (1) to bound

$$|N(t)| \le g'(q, l) \cdot l. \tag{4}$$

By 5) of Proposition 2, $l \le \mathrm{col}_{2 \cdot 4^q}(G, \sigma)^{2 \cdot 4^q} + 2$. Combining equations (2), (3) and (4) yields a function g with

$$\mathrm{col}_r(L, \pi) \le g(q + \mathrm{col}_{2 \cdot 4^q}(G, \sigma)) \cdot \mathrm{col}_{4^q r}(G, \sigma) \text{ for all } r.$$

The statement of this lemma also requires a bound on the weak coloring numbers $\mathrm{wcol}_r(L, \pi)$. This bound can be proven in almost the same way as for $\mathrm{col}_r(L, \pi)$ and therefore we only describe the main difference. Let h be a hidden vertex and $h' \in \mathrm{WReach}_r(L, \pi, h)$. By definition, there is a path $h_0 t_0 h_1 t_1 \dots t_{k-1} h_k$ such that $h_0 = h'$, $h_k = h$, $k \le r/2$, $\pi(h_0) = \min_{i=1}^k \pi(h_i)$. Then by Proposition 2, Item 4), $u(h') \in \mathrm{WReach}_{4^q r}(G, \sigma, u(h))$, i.e., $|\mathrm{WReach}_r(L, \pi, h)|$ is bounded by the number of tuples (v, h') with $v \in \mathrm{WReach}_{4^q r}(G, \sigma, u(h))$ and $u(h') = v$. The rest proceeds as for the strong coloring numbers.

No Apex Vertex. It remains to prove this result for the case that the input graph has no apex vertex. We reduce this case to the previously covered case with an apex. Let G be a graph without an apex and σ be an ordering of $V(G)$. We construct a graph G' from G by adding an additional apex vertex and let σ' be the ordering of $V(G')$ that preserves the order of σ but whose minimal element is the new apex vertex. Then $\mathrm{col}_r(G', \sigma') \le \mathrm{col}_r(G, \sigma) + 1$ and $\mathrm{wcol}_r(G', \sigma') \le \mathrm{wcol}_r(G, \sigma) + 1$. Assume we have a directed lacon-decomposition (L', π') of $I_\varphi(G')$. We obtain a directed lacon-decomposition (L, π) of $I_\varphi(G)$ by removing the apex vertex from (L', π'). Then $\mathrm{col}_r(L, \pi) \le \mathrm{col}_r(L', \pi')$ and $\mathrm{wcol}_r(L, \pi) \le \mathrm{wcol}_r(L', \pi')$. These observations reduce the no-apex case to the apex case. ■

IV. Directed and Undirected Lacon-Decompositions Have Same Expressive Power

In this section we show that directed and normal lacon-decompositions are equally powerful.

Lemma 2. *Assume a graph G has a directed lacon-decomposition (L, π). Then it also has an (undirected) lacon-decomposition (L', π') with*

- $\mathrm{col}_r(L', \pi') \le 4^{\mathrm{col}_2(L, \pi)} \cdot \mathrm{col}_r(L, \pi)$ *for all r,*
- $\mathrm{wcol}_r(L', \pi') \le 4^{\mathrm{col}_2(L, \pi)} \cdot \mathrm{wcol}_r(L, \pi)$ *for all r.*

Proof: Assume we have a directed lacon-decomposition (L, π) of a graph G with target vertices T and hidden vertices H. We construct an undirected lacon-decomposition (L', π') of G with target and hidden vertices T, H' as follows: We start with $H' = \emptyset$. We process the vertices in H one by one in ascending order by π and add in each processing step various new vertices to H'. The ordering π' is thereby defined implicitly as follows (in ascending order): First come the vertices of H' in the order of insertion and then come the vertices of T in the same order as in π.

We will make sure that after a vertex $h \in H$ has been processed the following invariant holds for all pairs of vertices $v_1 \neq v_2 \in T$. If v_1, v_2 have a dominant vertex d in L such that $\pi(d) \le \pi(h)$ then we guarantee that v_1, v_2 have a dominant vertex d' in L' and that d' is labeled with "1" if and only if d is. After every vertex in H has been processed, this guarantees that (L', π') is an undirected lacon-decomposition of G.

While processing a vertex $h \in H$, certain new vertices are inserted into H'. The newly inserted vertices are in a sense either a copy of h or copies of vertices that have already been inserted into H'. These copies may later again be copied, and so forth. This leads to exponential growth. To keep track of these copies and to show that while being exponential, this grow is nevertheless bounded, every newly inserted vertex h' will have a *memory*, denoted by $m(h')$. This memory consists of a sequence of vertices from H. If h' is derived from h, then we set $m(h') = h$. However, if h' is derived both from h and another vertex h'' that has previously been inserted into H', then we set $m(h') = m(h'') + h$, i.e., $m(h')$ is obtained by appending h to the memory of h''. Let us now get into the details of the construction and prove our invariant.

Construction Of Lacon-Decomposition. We describe the steps that we do in order to process a vertex $h \in H$. If $N^-(h) \cup N^+(h) = \emptyset$ we do nothing. Otherwise, we add a vertex h' to H' that has the same label ("0" or "1") as h. We further add edges to L' such that $N(h') = N^-(h) \cup N^+(h)$ (here $N(h')$ refers to the neighborhood in L', while $N^-(h)$, $N^+(h)$ refer to the in- and out-neighborhoods in L). Since h' is derived only from h, we set $m(h') = h$.

We observe a problem. Let $v_1 \neq v_2 \in N^-(h) \setminus N^+(h)$. Currently, h' is the dominant vertex for v_1 and v_2 in L', even though h is not dominant for these vertices in L. Thus h'

connects too many target vertices and we have to "undo" the effect of h' on the edges within the set $N^-(h) \setminus N^+(h)$. The same holds for the set $N^+(h) \setminus N^-(h)$. We do so as follows. We iterate over all vertices $l \in \mathrm{Reach}_2(L', \pi', h')$ in order of insertion into H'. If $N(l) \cap N^+(h) \setminus N^-(h) \neq \emptyset$ we add a vertex l' with $N(l') = N(l) \cap N^+(h) \setminus N^-(h)$. Similarly, if $N(l) \cap N^-(h) \setminus N^+(h) \neq \emptyset$ we add a vertex l'' with $N(l'') = N(l) \cap N^-(h) \setminus N^+(h)$. These two vertices get the same label as l and since they were derived from both l and h, we set $m(l') = m(l'') = m(l) + h$. These new vertices l' and l'' undo the undesired effects of inserting h', as we will prove soon. This completes the processing round of h.

By induction, we know that the invariant was satisfied in the last round for all v_1, v_2. We show that it also holds after h is processed. Let $v_1 \neq v_2$ be two vertices with dominant vertex d in L such that $\pi(d) \leq \pi(h)$. We distinguish five cases.

- $v_1 \notin N^-(h) \cup N^+(h)$ or $v_2 \notin N^-(h) \cup N^+(h)$. In this case, neither h in (L, π) nor any of the newly inserted vertices in (L', π') is dominant for v_1, v_2. Since the invariant for v_1, v_2 held in the previous round, it also holds in this round.
- $v_1 \in N^-(h)$ and $v_2 \in N^+(h)$. In this case, h and h' are dominant for v_1, v_2 in (L, π) and (L', π'), respectively. The vertex h' is labeled with "1" if and only if h is. Thus, the invariant is fulfilled.
- $v_1 \in N^+(h)$ and $v_2 \in N^-(h)$. As above.
- $v_1, v_2 \in N^+(h) \setminus N^-(h)$. Then $\pi(d) < \pi(h)$. Thus, in the previous round, there was a vertex that was dominant for v_1, v_2 in (L', π') that has the same label as d. Since $v_1, v_2 \in N(h')$, this dominant vertex is contained in $\mathrm{Reach}_2(L', \pi', h')$. In the construction, we iterate over all vertices $l \in \mathrm{Reach}_2(L', \pi', h')$ in order of insertion into H' (i.e., ascending order by π') and insert vertices l' that satisfy $v_1, v_2 \in N(l')$ iff $v_1, v_2 \in N(l)$. The last vertex l with $v_1, v_2 \in N(l)$ that we encounter during this procedure was the dominant vertex of the previous round. The corresponding vertex l' has the same label as l and is the dominant vertex now.
- $v_1, v_2 \in N^-(h) \setminus N^+(h)$. As above with l'' instead of l'.

Bounding the Coloring Numbers. For a vertex $h' \in H'$ with $m(h') = h_1 \dots h_k$, we say that h_k is *the corresponding vertex of h' in L*. If t is a target vertex, we say the *corresponding vertex in L* is t itself. The corresponding vertex of a vertex $x \in V(L)$ is denoted by $u(x)$.

We fix a vertex $h \in H$ and ask: How many vertices $h' \in H'$ can there be with $u(h') = h$? We pick a vertex $h' \in H'$ with $u(h') = h$. Then it has a memory $m(h') = h_1 \dots h_k$ with $h_k = h$. The following observations follow from the construction of (L', π').

- $N(h') \neq \emptyset$
- $N(h') \subseteq N^-(h_i) \cup N^+(h_i)$ for all $1 \leq i \leq k$
- $\pi(h_1) < \dots < \pi(h_k)$

These three observations together imply that $h_1, \dots, h_k \in \mathrm{Reach}_2(L, \pi, h)$. Thus $m(h')$ consists of a subset of $\mathrm{Reach}_2(L, \pi, h)$, written in ascending order by π. We can therefore bound the number of possible memories $m(h')$ of h' by $2^{\mathrm{col}_2(L,\pi)}$.

Furthermore, for each vertex $l \in H'$ with memory $l_1 \dots l_{k-1}$ and every vertex $l_k \in H$ there are at most two vertices (denoted by l' and l'' in the above construction) with the memory $l_1 \dots l_k$. This means for a fixed memory of length k, there are at most 2^k vertices that have this particular memory. Combining the previous observations, we can conclude that for a fixed vertex $h \in H$ there are at most $2^{\mathrm{col}_2(L,\pi)} \cdot 2^{\mathrm{col}_2(L,\pi)} = 4^{\mathrm{col}_2(L,\pi)}$ vertices $h' \in H'$ with $u(h') = h$.

The fact that $N(h') \subseteq N^-\big(u(h')\big) \cup N^+\big(u(h')\big)$for all $h' \in H'$ (see second item in enumeration above) implies the following for all $x, y \in V(L')$: If x, y are adjacent in L' then $u(x), u(y)$ are adjacent in L. Also, the order π' was constructed such that $\pi(u(x)) > \pi(u(y))$ implies $\pi'(x) > \pi'(y)$ for all $x, y \in V(L')$. The last two points together mean

- if $y \in \mathrm{Reach}_r(L', \pi', x)$ then $u(y) \in \mathrm{Reach}_r(L, \pi, u(x))$ for all r,
- if $y \in \mathrm{WReach}_r(L', \pi', x)$ then $u(y) \in \mathrm{WReach}_r(L, \pi, u(x))$ for all r.

Combining this with the observation that for a fixed $u \in V(L)$ there are at most $4^{\mathrm{col}_2(L,\pi)}$ vertices $y \in V(L')$ with $u(y) = u$ gives us the bounds on $\mathrm{col}_r(L', \pi')$ and $\mathrm{wcol}_r(L', \pi')$ required by this lemma. ∎

V. Transductions of Bounded Expansion Have Lacon-Decompositions

Now we combine the previous results into a more concise statement. We show that for every transduction of a graph G, we can find a lacon-decomposition whose coloring numbers are not too far off from the coloring numbers of G.

Theorem 5. *Let τ be a transduction. There exist a constant c and a function f such that for every graph G, ordering σ on the vertices of G, and $D \in \tau(G)$ there exists a lacon-decomposition (L, π) of D with*

- $\mathrm{col}_r(L, \pi) \leq f(\mathrm{col}_c(G, \sigma)) \cdot \mathrm{col}_{cr}(G, \sigma)$ *for all r,*
- $\mathrm{wcol}_r(L, \pi) \leq f(\mathrm{col}_c(G, \sigma)) \cdot \mathrm{wcol}_{cr}(G, \sigma)$ *for all r.*

All the key ideas needed to prove this result have already been established in Lemma 1 and Lemma 2. In this section, we merely combine them. First, we need to following technical but unexciting lemma.

Lemma 3. *Let τ be a transduction. There exists a constant c and a basic transduction τ_0 such that for every graph G, ordering π on the vertices of G, and $D \in \tau(G)$ there exists a labeled graph G' with $\tau_0(G') = D$ and ordering π' on the vertices of G' such that*

- $\mathrm{col}_r(G', \pi') \leq c \cdot \mathrm{col}_r(G, \pi)$ *for all r,*
- $\mathrm{wcol}_r(G', \pi') \leq c \cdot \mathrm{wcol}_r(G, \pi)$ *for all r.*

Proof: We break τ down into $\tau = \tau_0 \circ \gamma \circ \varepsilon$. Let $G' \in (\gamma \circ \varepsilon)(G)$ such that $\tau_0(G') = D$. Note that G' is the result of applying the copy operation ε and a coloring from γ to

G. We extend π into an ordering π' on G' by placing the copied vertices next to the corresponding original vertices. If ε copies the graph c times then $\mathrm{col}_r(G', \pi') \leq c \cdot \mathrm{col}_r(G, \pi)$ and $\mathrm{wcol}_r(G', \pi') \leq c \cdot \mathrm{wcol}_r(G, \pi)$. ■

Now we prove Theorem 5 by chaining Lemma 3, 1 and 2 in this order.

Proof of Theorem 5: Assume we have a graph G, an ordering σ on the vertices of G, and $D \in \tau(G)$. Lemma 3 gives us a basic transduction τ_0, a graph G' with $\tau_0(G') = D$ and an ordering σ' such that

- $\mathrm{col}_r(G', \sigma') \leq c' \cdot \mathrm{col}_r(G, \sigma)$ for all r,
- $\mathrm{wcol}_r(G', \sigma') \leq c' \cdot \mathrm{wcol}_r(G, \sigma)$ for all r.

for some constant c' depending only on τ. Assume τ_0 to be of the form (χ, ν, φ). By Lemma 1, there exists a function g and a directed lacon-decomposition (L', π') of $I_\varphi(G')$ with

- $\mathrm{col}_r(L', \pi') \leq$ $g(|\varphi| + \mathrm{col}_{2\cdot 4^{|\varphi|}}(G', \sigma')) \cdot \mathrm{col}_{4^{|\varphi|}r}(G', \sigma')$ for all r,
- $\mathrm{wcol}_r(L', \pi') \leq$ $g(|\varphi| + \mathrm{col}_{2\cdot 4^{|\varphi|}}(G', \sigma')) \cdot \mathrm{wcol}_{4^{|\varphi|}r}(G', \sigma')$ for all r.

We know $G' \models \chi$, since otherwise $\tau_0(G')$ would be undefined. Since ν describes the vertex set of the transduction, we update L' by removing all target vertices t with $L' \not\models \nu(t)$. (L', π') is now a directed lacon-decomposition of $\tau_0(G') = D$. Since we only deleted vertices, the coloring numbers of (L', π') did not increase, thus the above two bounds remain true. At last, we use Lemma 2 to construct an undirected lacon-decomposition (L, π) of D with

- $\mathrm{col}_r(L, \pi) \leq 4^{\mathrm{col}_2(L', \pi')} \cdot \mathrm{col}_r(L', \pi')$ for all r,
- $\mathrm{wcol}_r(L, \pi) \leq 4^{\mathrm{col}_2(L', \pi')} \cdot \mathrm{wcol}_r(L', \pi')$ for all r.

Combining the previous 6 bounds then proves the result. ■

VI. Converting Lacon- to Shrub-Decompositions

We now convert a lacon-decomposition into an equivalent shrub-decomposition while maintaining a bound on the generalized coloring numbers.

Lemma 4. *Let G be a graph with a lacon-decomposition (L, π) such that $\mathrm{col}_2(L, \pi) \leq \alpha$ for some $\alpha \in \mathbf{N}$. Then there exists a shrub-decomposition (F, S) of G whose signature S, number of colors m, and diameter d depend only on α and*

- $\mathrm{col}_r(F) \leq d + \mathrm{col}_r(L, \pi)$ *for all r,*
- $\mathrm{wcol}_r(F) \leq d + \mathrm{wcol}_r(L, \pi)$ *for all r.*

Proof: Let (L, π) be a lacon-decomposition of G. We color the hidden vertices such that for every target vertex t there are no two hidden vertices with the same color in the neighborhood $N(t)$. Note that these colors are *additional* to the labels "0" or "1" that the hidden vertices already have. This can be done only using $C \leq \mathrm{col}_2(L, \pi)$ colors by the following standard greedy approach: Iterate over the hidden vertices in ascending order by π and give a vertex h a color that does not yet occur in $\mathrm{Reach}_2(L, \pi, h)$. We assume the colors are integers from 1 to C and we indicate the color of a vertex h by $c(h)$.

For every target vertex t and $1 \leq i < |N(t)|$ we define $\nu_i(t)$ to be the ith hidden neighbor in L, counted in descending order by π. We construct a graph F from L by performing the following two operations for every vertex $t \in V(G)$ (see Figure 3).

- Add a vertex t' and make t' the only neighbor of t.
- For each $1 \leq i < |N(t)|$ add a path of length $3Ci + c(\nu_i(t)) - 1$ from t' to $\nu_i(t)$.

We iteratively remove pendant vertices in $V(F) \setminus V(G)$ from F until the pendant vertices are exactly $V(G)$. This last operation does not change the distance between any two vertices from $V(G)$ in F, since no path can pass through a pendant vertex. Let us fix some $t_1 \neq t_2 \in V(G)$ and let $N = N^L(t_1) \cap N^L(t_2)$. Each path between t_1 and t_2 in F is uniquely determined by the vertex from N that it passes through. Also, if we iterate over N in ascending order by π, the corresponding paths decrease in length. The central property of this construction is that for any two target vertices $t_1 \neq t_2$ the following two statements are equivalent.

- The dominant vertex of t_1 and t_2 in L has color c.
- The distance between t_1 and t_2 in F modulo $3C$ is $2c$.

Note that knowing the color of the dominant vertex is as good as knowing the dominant vertex itself, since within a neighborhood, the color uniquely identifies the vertex. For all $t \in V(G)$ and $1 \leq c \leq C$ we define $s_c(t) \in \{0, 1\}$ such that $s_c(t) = 1$ iff t is adjacent in L to a (unique) hidden vertex with color c and label "1". Now, there is an edge t_1t_2 in G if and only if the distance between t_1 and t_2 in F modulo $3C$ is $2c$ and $s_c(t_1) = 1$ for some number c.

We use this observation to construct the coloring and signature of F. We iterate over all vertices $t \in V(G)$ and assign them the color $(s_1(t), \ldots, s_C(t))$. We define the signature S to be all tuples $\big((s_1, \ldots, s_C), (s'_1, \ldots, s'_C), 3Ci + 2c\big)$ such that $(s_1, \ldots, s_C), (s'_1, \ldots, s'_C) \in \{0, 1\}^C$, $2 \leq i \leq 2\mathrm{col}_1(L, \pi)$, $1 \leq c \leq C$ and $s_c = s'_c = 1$. Now (F, S) connects two vertices t_1, t_2 iff their distance modulo $3C$ is $2c$ and $s_c(t_1) = s_c(t_2) = 1$. Thus, (F, S) is a shrub-decomposition of G. The diameter of F is at most $6C\mathrm{col}_1(L, \pi) + 2C$.

The bound on the coloring numbers of F can be proven as follows. Let H be the hidden vertices of L. Every connected component of $F[V(F) \setminus H]$ consists of a vertex $t \in V(G)$, a vertex t' and a bounded number of short paths attached to t'. Each component is bounded in size by a function of $\mathrm{col}_2(L, \pi)$. We extend the ordering π of L into an ordering π' of F by making the vertices in $V(F) \setminus V(L)$ larger than all vertices in $V(L)$. Now for every $h \in H$ holds

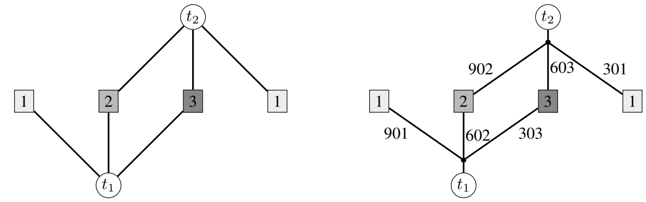

Fig. 3. Left: Part of a lacon-decomposition with colored hidden vertices. Right: Corresponding shrub-decomposition, including distances between round and square vertices (we assume $C = 100$).

$\mathrm{Reach}_r(F, \pi', h) \subseteq \mathrm{Reach}_r(L, \pi, h)$. For all other vertices $v \in V(F) \setminus H$ holds $\mathrm{Reach}_r(F, \pi', v) \subseteq \mathrm{Reach}_r(L, \pi, t) \cup X$, where X is the connected component of v in $F[V(F) \setminus H]$ and $t \in V(G)$ is the unique target vertex contained in X. An equivalent bound can be obtained for the weak reachability. ∎

VII. Proof of Theorem 1, 2 and 3

We finally prove the main result of this paper by combining all results of the previous sections.

Proof of Theorem 1: The equivalence of 1), 4), 5) is due to Gajarský et al. [16] and Nešetřil et al. [29], [30]. We prove the equivalence of the remaining statements by showing 1) $\implies$ 2) $\implies$ 3) $\implies$ 1).

- 1) $\implies$ 2). Pick any graph $D \in \mathcal{G}$. By 1) there exists a graph $G \in \mathcal{G}'$ with $D \in \tau(G)$. Since $\mathcal{G}'$ has bounded expansion, Definition 4 states that there exists an ordering σ with $\mathrm{col}_r(G, \sigma) \leq g(r)$ for all r (where the function $g(r)$ depends only on $\mathcal{G}'$). Now, 2) directly follows from Theorem 5.
- 2) $\implies$ 3). This follows from Lemma 4 and Definition 4.
- 3) $\implies$ 1). In first-order logic we can compute the (bounded) distance between two vertices, check their color and implement the lookup table S. Thus, we can easily construct a transduction τ with $\tau(\mathcal{G}') \subseteq \mathcal{G}$.

∎

We proceed in a similar manner to prove our characterizations of structurally bounded treedepth and treewidth.

Proof of Theorem 2: The equivalence of 1) and 4) is proven by Gajarský et al. [18]. We prove the remaining statements again in the order 1) $\implies$ 2) $\implies$ 3) $\implies$ 1).

- 1) $\implies$ 2). As mentioned in the introduction, the weak coloring numbers converge to treedepth, i.e., $\mathrm{wcol}_1(G) \leq \cdots \leq \mathrm{wcol}_\infty(G) = \mathrm{td}(G)$ [26]. Assume every graph in $\mathcal{G}'$ has treedepth at most d'. Thus, for a graph $D \in \mathcal{G}$ there exists a graph $G \in \mathcal{G}'$ with $D \in \tau(G)$ and $\mathrm{wcol}_\infty(G) \leq d'$. Theorem 5 then gives us a lacon-decomposition (L, π) of D with $\mathrm{wcol}_\infty(L, \pi) \leq f(\mathrm{col}_c(G)) \cdot \mathrm{wcol}_\infty(G) \leq f(d') \cdot d'$.
- 2) $\implies$ 3). This follows from Lemma 4, again using $\mathrm{wcol}_1(G) \leq \cdots \leq \mathrm{wcol}_\infty(G) = \mathrm{td}(G)$.
- 3) $\implies$ 1). As discussed in the proof of Theorem 1, we can easily construct a transduction τ with $\tau(\mathcal{G}') \subseteq \mathcal{G}$.

∎

Proof of Theorem 3: Item 4) is due to Nešetřil et al. [29], [30]. The implications 1) $\implies$ 2) $\implies$ 3) $\implies$ 1) can be proven in exactly the same way as in Theorem 2, except this time using $\mathrm{col}_1(G) \leq \cdots \leq \mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1$ [26]. ∎

VIII. Proof of Localized Feferman–Vaught Composition Theorem

We give a self sustained proof of Theorem 4, using central ideas from the proof of Gaifman's theorem [34]. An alternative proof exists in [21, Lemma 15].

Proof of Theorem 4: Our result follows from proving the following claim via structural induction. Let $\varphi(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k)$ be a first-order formula with quantifier rank q. Then one can compute a boolean combination $\Phi(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k)$ of first-order formulas of the form $\xi(\bar{x}, \bar{y}_i)$ such that for all graphs G and all tuples $\bar{v}_1, \ldots, \bar{v}_k$ that are 4^q-separated by a tuple $\bar{u}$ in G holds

$$G \models \varphi(\bar{u}, \bar{v}_1, \ldots, \bar{v}_k) \iff G \models \Phi(\bar{u}, \bar{v}_1, \ldots, \bar{v}_k).$$

We call such a boolean combination Φ a *separated expression*. The claim holds for atomic formulas because $\bar{u}$ 4^0-separates $\bar{v}_1, \ldots, \bar{v}_k$, i.e., there are no edges between $\{\bar{v}_i\} \setminus \{\bar{u}\}$ and $\{\bar{v}_j\} \setminus \{\bar{u}\}$ in G for $i \neq j$. It also holds for boolean combinations and negations. It remains to prove our claim for the case that $\varphi(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) = \exists x \psi$ for some formula ψ.

For $t \in \mathbf{N}$ and $v \in V(G)$, we define $N_t^{\bar{u}}(v)$ to be the set of all vertices in G that are reachable from v via a path of length at most t that contains no vertex from $\bar{u}$. For a tuple $\bar{v}$, we define $N_t^{\bar{u}}(\bar{v}) = \bigcup_{v \in \{\bar{v}\}} N_t^{\bar{u}}(v)$. With slight abuse of notation, we write $z \in N_t^{\bar{x}}(\bar{y})$ as a short-hand for a first-order formula $\nu_t(\bar{x}, \bar{y}, z)$ such that for all graphs G, vertex-tuples $\bar{u}$, $\bar{v}$ and vertices w holds $G \models \nu_t(\bar{u}, \bar{v}, w)$ iff $w \in N_t^{\bar{u}}(\bar{v})$. The formula $\nu_t(\bar{x}, \bar{y}, z)$ can be constructed using $t-1$ existential quantifiers.

Let $r = 4^{q-1}$. We rewrite φ by distinguishing two cases: Either the existentially quantified variable x is contained in the neighborhood $N_r^{\bar{x}}(\bar{y}_i)$ for some i or for none. This gives us

$$\varphi(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) \equiv \varphi^*(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) \vee \bigvee_{i=1}^{k} \varphi_i(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) \tag{5}$$

with

$$\varphi_i(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) = \exists x\ x \in N_r^{\bar{x}}(\bar{y}_i) \wedge \psi(\bar{x}, x, \bar{y}_1, \ldots, \bar{y}_k),$$

$$\varphi^*(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) = \exists x\ x \notin \bigcup_{i=1}^{k} N_r^{\bar{x}}(\bar{y}_i) \wedge \psi(\bar{x}, x, \bar{y}_1, \ldots, \bar{y}_k).$$

We will proceed by finding separated expressions for φ^* and for each φ_i. Since $4^q = 4r$, we consider a graph G and tuples $\bar{v}_1, \ldots, \bar{v}_k$ that are $4r$-separated by a tuple $\bar{u}$ in G. We start with φ_i. This formula asks whether there exists an $x \in N_r^{\bar{x}}(\bar{y}_i)$ satisfying ψ. For every $u \in N_r^{\bar{u}}(\bar{v}_i)$ holds that $\bar{u}$ r-separates the tuples $\bar{v}_i u$ and $\bar{v}_j$ for $i \neq j$, since otherwise there would be a short path from $\bar{v}_i$ via u to $\bar{v}_j$ that contains no vertex from $\bar{u}$. By the induction hypothesis, there exists a separated expression $\Psi_i(\bar{x}, \bar{y}_1, \ldots, \bar{y}_i x, \ldots \bar{y}_k)$ such that for all $u \in N_r^{\bar{u}}(\bar{v}_i)$, $G \models \psi(\bar{u}, u, \bar{v}_1, \ldots, \bar{v}_k) \iff G \models \Psi_i(\bar{u}, \bar{v}_1, \ldots, \bar{v}_i u, \ldots \bar{v}_k)$. Thus $G \models \varphi_i(\bar{u}, \bar{v}_1, \ldots, \bar{v}_k) \iff G \models \exists x\ x \in N_r^{\bar{u}}(\bar{v}_i) \wedge \Psi_i(\bar{u}, \bar{v}_1, \ldots, \bar{v}_i x, \ldots \bar{v}_k)$. If we assume Ψ_i to be of the form

$$\Psi_i = \bigvee_{l=1}^{m} \Big(\xi_{li}(\bar{x}, \bar{y}_i x) \wedge \bigwedge_{\substack{j=1 \\ j \neq i}}^{k} \xi_{lj}(\bar{x}, \bar{y}_j) \Big)$$

then $G \models \varphi_i(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \iff G \models \Phi_i(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$ with the separated expression

$$\Phi_i(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) = \bigvee_{l=1}^{m} \Big(\exists x\ x \in N_r^{\bar{x}}(\bar{y}_i) \wedge \xi_{li}(\bar{x}, \bar{y}_i x) \wedge \bigwedge_{\substack{j=1 \\ j \neq i}}^{k} \xi_{lj}(\bar{x}, \bar{y}_j) \Big).$$

Next, we want to obtain a separated expression for φ^*. Let therefore $u \in V(G)$ with $u \notin N_r^{\bar{u}}(\bar{v}_i)$ for all i. Then the tuples $u, \bar{v}_1, \dots, \bar{v}_k$ are r-separated by $\bar{u}$. By the induction hypothesis, there exists a separated expression $\Psi^*(\bar{x}, x, \bar{y}_1, \dots, \bar{y}_k)$ such that $G \models \psi(\bar{u}, u, \bar{v}_1, \dots, \bar{v}_k) \iff G \models \Psi^*(\bar{u}, u, \bar{v}_1, \dots, \bar{v}_k)$. We can assume Ψ^* to be of the form

$$\Psi^* = \bigvee_{l=1}^{m} \Big(\xi_l(\bar{x}, x) \wedge \bigwedge_{j=1}^{k} \xi_{lj}(\bar{x}, \bar{y}_j) \Big).$$

Then $G \models \varphi^*(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \iff G \models \Phi^*(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$ using the separated expression

$$\Phi^*(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) = \bigvee_{l=1}^{m} \Big(\exists x\ x \notin \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i) \wedge \xi_l(\bar{x}, x) \bigwedge_{j=1}^{k} \xi_{lj}(\bar{x}, \bar{y}_j) \Big).$$

Note that Φ^* is not yet a separated expression. Nevertheless, substituting Φ_i and Φ^* into equation (5) yields

$$G \models \varphi(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \iff G \models \bigvee_{i=1}^{k} \Phi_i(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$$
$$\vee \bigvee_{l=1}^{m} \Big(\exists x\ x \notin \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i) \wedge \xi_l(\bar{u}, x) \bigwedge_{j=1}^{k} \xi_{lj}(\bar{u}, \bar{v}_j) \Big).$$

The remaining problematic subformulas are those of the form

$$\gamma(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) = \exists x\ x \notin \bigcup_{i=1}^{k} N_r^{\bar{x}}(\bar{y}_i) \wedge \xi(\bar{x}, x).$$

To complete the proof, it is sufficient to find a separated expression equivalent to γ. We therefore define the separated expression

$$\Gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) = \bigvee_{i=1}^{k} \exists x\ x \in N_{3r}^{\bar{u}}(\bar{v}_i) \wedge x \notin N_r^{\bar{u}}(\bar{v}_i) \wedge \xi(\bar{u}, x)$$

and make a case distinction based on it.

- *Case 1:* $G \models \Gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$. Since $\bar{v}_1, \dots, \bar{v}_k$ are $4r$-separated by $\bar{u}$, we have $N_{3r}^{\bar{u}}(\bar{v}_i) \cap N_r^{\bar{u}}(\bar{v}_j) = \emptyset$ for $i \neq j$. This means $G \models \gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$.
- *Case 2:* $G \not\models \Gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$. We define the $\bar{u}$*-distance* between two vertices to be the length of the shortest path between them that contains no vertex from $\bar{u}$. For $V' \subseteq V(G)$, we define an $(2r, \bar{u})$*-scattered subset* of V' to be a set $S \subseteq V'$ such that all vertices in S pairwise have $\bar{u}$-distance greater than $2r$ in G. The size of the largest $(2r, \bar{u})$-scattered subset of V' in G is denoted by $s(V')$.

Let further $R(\bar{u}) = \{u \mid u \in V(G), G \models \xi(\bar{u}, u)\}$. Using this new notation, we observe that $G \models \gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$ if and only if $R(\bar{u}) \setminus \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i) \neq \emptyset$. Since we assume $G \not\models \Gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$, this means that the sets $R(\bar{u}) \setminus \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i)$, $R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_1), \dots, R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_k)$ are $2r$-separated by $\bar{u}$. Therefore

$$s\big(R(\bar{u})\big) = s\big(R(\bar{u}) \setminus \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i)\big) + \sum_{i=1}^{k} s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big).$$

Thus, $G \models \gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k)$ iff $R(\bar{u}) \setminus \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i) \neq \emptyset$ iff $s\big(R(\bar{u}) \setminus \bigcup_{i=1}^{k} N_r^{\bar{u}}(\bar{v}_i)\big) \neq 0$ iff $\sum_{i=1}^{k} s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big) < s\big(R(\bar{u})\big)$. The previous case distinction implies that

$$G \models \gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \iff$$
$$G \models \Gamma(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \text{ or } \sum_{i=1}^{k} s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big) < s\big(R(\bar{u})\big).$$

It now remains to find a separated expression $\Delta(\bar{x}, \bar{y}_1, \dots, \bar{y}_k)$ such that $G \models \Delta(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \iff \sum_{i=1}^{k} s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big) < s\big(R(\bar{u})\big)$. For arbitrary vertices a, b, c such that $a, b \in N_r^{\bar{u}}(c)$, we see that a, b have $\bar{u}$-distance at most $2r$. Since $s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big)$ is the size of a subset of $N_r^{\bar{u}}(\bar{v}_i)$ where all vertices pairwise have $\bar{u}$-distance greater than $2r$, we have $s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big) \leq |\bar{v}_i|$.

For $1 \leq i \leq k$ and $0 \leq h \leq |\bar{v}_i|$ we define a first-order formula $\delta_{ih}(\bar{x}, \bar{y}_i)$ which is true iff $s\big(R(\bar{x}) \cap N_r^{\bar{x}}(\bar{y}_i)\big) \leq h$:

$$\delta_{ih}(\bar{x}, \bar{y}_i) = \neg \exists s_1 \dots \exists s_{h+1} \bigwedge_{j=1}^{h+1} \Big(\xi(\bar{x}, s_j) \wedge s_j \in N_r^{\bar{x}}(\bar{y}_i) \wedge \bigwedge_{l=j+1}^{h+1} s_j \notin N_{2r}^{\bar{x}}(s_l) \Big).$$

We further define for $0 \leq h \leq \sum_{i=1}^{k} |\bar{v}_i|$ a first-order formula $\delta_h(\bar{x})$ which is true iff $s\big(R(\bar{x})\big) > h$. This formula can be constructed in a similar way as the formulas $\delta_{ih}(\bar{x}, \bar{y}_i)$ above. We use these formulas to construct the separated expression

$$\Delta(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) = \bigvee_{h_1=0}^{|v_1|} \cdots \bigvee_{h_k=0}^{|v_k|} \delta_{1h_1}(\bar{x}, \bar{y}_1) \wedge \cdots \wedge \delta_{kh_k}(\bar{x}, \bar{y}_k) \wedge \delta_{\sum_{i=1}^{k} h_i}(\bar{x})$$

with $G \models \Delta(\bar{u}, \bar{v}_1, \dots, \bar{v}_k) \iff \sum_{i=1}^{k} s\big(R(\bar{u}) \cap N_r^{\bar{u}}(\bar{v}_i)\big) < s\big(R(\bar{u})\big)$. This completes our decomposition of φ into separated expressions. ∎

IX. Conclusion

In this work we present lacon- and shrub-decompositions and use them to characterize graph classes with structurally bounded expansion, treewidth and treedepth.

This paper yields a new approach to solving the first-order model-checking problem on graph classes with structurally bounded expansion by reducing it to the problem of finding sparse lacon- or shrub-decompositions. The central question that arises in this work is how such sparse decompositions can be efficiently computed.

Furthermore, first-order transductions define a complex but interesting hierarchy of dense graph classes [31] and the decompositions presented in this paper have the potential to improve our understanding of this hierarchy beyond the three classes discussed in this work. In particular, it would be interesting to see whether lacon- or shrub-decompositions can also characterize structurally nowhere dense graph classes. In this setting, the coloring numbers may grow sub-polynomially with the size of the graph and therefore, we would need a stronger version of Theorem 4 where the function $f(q, l)$ grows only polynomially in l. Such a result is proven in [21] (see also [39, Lemma 10]).

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