A REDUCED MODEL FOR PLATES ARISING AS LOW ENERGY Γ-LIMIT IN NONLINEAR MAGNETOELASTICITY

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Abstract. We investigate the problem of dimension reduction for plates in nonlinear magnetoelasticity. The model features a mixed Eulerian-Lagrangian formulation, as magnetizations are defined on the deformed set in the actual space. We consider low-energy configurations by rescaling the elastic energy according to the linearized von Kármán regime. First, we identify a reduced model by computing the Γ-limit of the magnetoelastic energy, as the thickness of the plate goes to zero. This extends a previous result obtained by the first author in the incompressible case to the compressible one. Then, we introduce applied loads given by mechanical forces and external magnetic fields and we prove that, under clamped boundary conditions, sequences of almost minimizers of the total energy converge to minimizers of the corresponding energy in the reduced model. Subsequently, we study quasistatic evolutions driven by time-dependent applied loads and a rate-independent dissipation. We prove that solutions of the approximate incremental minimization problem at the bulk converge to energetic solutions for the reduced model. This result provides a further justification of the latter in the spirit of the evolutionary Γ-convergence.

1. Introduction

Magnetoelasticity [10, 15, 16] concerns the interaction between magnetic fields and deformable solids. Indeed, magnetic materials can change their strain upon the application of magnetic fields. This peculiar behaviour is termed magnetostriction [10]. Conversely, mechanical loads may change the magnetic response of the specimen. These phenomena were first observed by J. Joule in the middle of the 19th century. Internally, magnetic materials have structures that are divided into domains, each of them has a uniform magnetization. If we apply a magnetic field, the boundaries between these domains shift and the domains themselves rotate; both of these effects result in a change of the specimen shape. The reason that a change in the magnetic domains of a material results in a change of the shape is a consequence of magnetocrystalline anisotropy: it requires more energy to magnetize a crystalline material in one direction than in another one [28]. The least energy is needed to magnetize the material along easy axes. These are three or four in cubic materials and one in uniaxial ones. If a magnetic field is applied to the material at a certain angle to an easy axis of magnetization, the material will tend to rearrange its structure so that an easy axis is aligned with the field to minimize the free energy of the system. As different crystal directions are associated with different lengths, this effect induces a strain and consequently a change of the shape in the magnetoelastic specimen.

According to the variational theory of Brown [10], the magnetoelastic energy is a function of deformations and magnetizations. Equilibrium states correspond to minima of the energy. The model contemplates finite strains. Therefore, while deformations are defined of the reference configuration (Lagrangian), magnetizations are defined on the deformed set in the actual space (Eulerian) [5, 9, 31]. Similar mixed Eulerian-Lagrangian formulations appear also in other contexts, such as the theory of liquid crystals [4, 26], phase transitions [24] and finite plasticity [27]. From the mathematical point of view, the analysis of such models is very challenging. Indeed, several standard techniques are no longer available in this setting, so that novel strategies are required. For these reasons, in recent years, mixed Eulerian-Lagrangian variational problems got the attention of the mathematical community.

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Rigorously derived lower-dimensional models of continuum mechanics play an important role in applications because they preserve the main features of the bulk model but they are usually simpler from the computational point of view [33, 34]. Fundamental results obtained in [20, 21] have initiated a remarkable progress in this area and have established the prominent role of Γ-convergence [7, 13] in the validation of reduced models for thin structures. For micromagnetics, important results have been achieved, among others, in [11, 22]. However, in the case of magnetoelasticity, few rigorous results are available. Two-dimensional models were first derived in [30] for Kirchhoff-Love plates starting from linearized magnetoelasticity, and then in [14] for non-simple materials in the fully nonlinear membrane regime. In the first case, rate-independent evolutions were also studied.

In this contribution, we derive a reduced model for plates in the linearized von Kármán regime starting from nonlinear magnetoelasticity. Our results develop the investigations initiated in [8] for incompressible materials to various extents. We consider compressible materials and we make more realistic assumptions on the elastic energy density, so that deformations and magnetizations are strongly coupled. Also, we address the effect of applied loads given by mechanical forces and external magnetic fields. Unlike [8], our analysis covers both the static the quasistatic setting. In the first case, we employ Γ-convergence techniques to study the asymptotic behaviour of minimizers of the magnetoelastic energy, as the thickness of the plate goes to zero. In the latter one, we investigate the dimension reduction in the framework of evolutionary Γ-convergence [38].

Let $h > 0$ represent the thickness of a thin magnetoelastic plate $\Omega_h := S \times hI \subset \mathbb{R}^3$, where $S \subset \mathbb{R}^2$ and $I := (-1/2, 1/2)$. Deformations are maps $\boldsymbol{\chi}\colon \Omega_h \to \mathbb{R}^3$ while magnetizations are given by maps $\boldsymbol{m}\colon \boldsymbol{\chi}(\Omega_h) \to \mathbb{S}^2$. Deformations are assumed to be continuous and injective in order to exclude the interpenetration of matter. The fact that magnetizations take values in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is due to the constraint of magnetic saturation [10], which is physically reasonable for sufficiently low constant temperature.

The magnetoelastic energy is given by the functional in (2.1) and consists of three terms: the elastic energy, which is rescaled according to the linearized von Kármán regime; the exchange energy, that penalizes spatial changes of magnetizations; and the magnetostatic energy, which involves the stray field $\boldsymbol{\Psi}_{\boldsymbol{m}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the solution of Maxwell equations (2.13). In particular, we specify the structure of the elastic energy density W_h in (2.1), which is assumed to take the form in (2.2). Similar expressions, with the nematic director in place of the magnetization, are widely accepted in the context of liquid crystals. See [2] for a specific example in the case of dimension reduction. Moreover, the expression in (2.2) fulfills the physical requirements of frame indifference (2.5) and magnetic parity (2.6).

Our main results are contained in Theorem 3.1 and Theorem 3.10 for the static setting, and in Theorem 4.3 for the quasistatic setting. The enunciation of these results requires the specification of the setting and the introduction of a considerable amount of notation. Therefore, we limit ourselves to briefly describe them and we postpone the precise statements to Sections 3 and 4.

In Theorem 3.1, we compute the Γ-limit of the magnetoelastic energy in (2.1), as $h \to 0^+$. This is computed with respect to the convergence of the averaged displacements (3.1)–(3.2) and the Lagrangian magnetizations (3.4). The first two quantities were introduced in [21], while the third one constitutes a reasonable way to pull back magnetizations to the reference configurations. Note that, by (2.2), the elastic energy depends on the magnetization only trough the quantity in (3.4). The limiting energy that we obtain is purely Lagrangian and is naturally given by integrals on the section S. In contrast with [8], the limiting elastic energy exhibits a strong coupling between elastic and magnetic variables. Also, we observe that the term corresponding to the magnetostatic energy in the reduced model simplifies substantially.

In Theorem 3.10, we consider applied loads given by mechanical forces and external magnetic fields, all dependent on the thickness $h > 0$ of the plate. In particular, the energy contribution determined by the external magnetic field, usually called Zeeman energy, is of Eulerian type. The total energy is given by the difference between the magnetoelastic energy and the work of applied loads. We prove that, assuming clamped boundary conditions on the deformations, sequence of almost minimizers of the total energy converge, as $h \to 0^+$, to minimizers of the corresponding energy functional in the reduced model.

Because of the rescaling of the elastic energy, the analysis in quite involved since the coercivity of the total energy functional is not immediate.

Finally, we address the quasistatic setting. We consider time-dependent applied loads and a rate-independent dissipation. The dissipation distance is given by the distance in L^1 among Lagrangian magnetizations (4.3). This notion of dissipation has the appreciable feature of being frame-indifferent, that is, rigid motions do not dissipate energy [9]. For every $h > 0$, we consider the approximate incremental minimization problem, a relaxed version of incremental minimization problem [37] that has been introduced in order to cope with the possible lack of minimizers of energy functionals. In Theorem 4.3, we show that, for a sequence of partitions of the time interval whose size vanish together with some tolerance constants, as $h \to 0^+$, the piecewise constant interpolants of the solutions of the approximate incremental minimization problems for suitably well prepared initial data converge, as $h \to 0^+$, to an energetic solution [37] of the reduced model. This result is inspired by [38]. As a byproduct, we deduce the existence of energetic solutions for the reduced model.

We emphasize that all these results are achieved without resorting on any regularization of the energy. However, our argument to prove the compactness of magnetizations works only under some restriction on the scalings. Precisely, the scaling of the elastic energy in (2.1) has to satisfy the condition $\beta > 6 \vee p$, where $p > 3$ is the integrability exponent of deformations, while the linearized von Kármán regime corresponds to $\beta > 4$. Also, the existence of energetic solutions for the bulk model is out of reach in our setting. The situation is pretty analogous to the one in [39] for the problem of linearization in finite plasticity.

The paper is structured as follows. In Section 2, we introduce the mathematical model and we list all the assumptions. In Section 3, we address the static setting: Theorem 3.1 and Theorem 3.10 are stated and proved in Subsection 3.1 and Subsection 3.2, respectively. Finally, Section 4 is devoted to the quasistatic setting with the corresponding main result, Theorem 4.3.

Notation. For scalars $a, b \in \mathbb{R}$, we use the notation $a \wedge b \coloneqq \min\{a, b\}$ and $a \vee b \coloneqq \max\{a, b\}$. Given $\boldsymbol{a} = (a^1, a^2, a^3)^\top \in \mathbb{R}^3$, we set $\boldsymbol{a}' \coloneqq (a^1, a^2)^\top \in \mathbb{R}^2$. The null vector in $\mathbb{R}^3$ is denoted by $\mathbf{0}$, so that $\mathbf{0}'$ is the null vector in $\mathbb{R}^2$. The same notation applied also to space variables and ∇' denotes the gradient with respect to the first two variables. Given $\boldsymbol{A} = (A^i_j)^{i=1,2,3}_{j=1,2,3} \in \mathbb{R}^{3\times 3}$, we set $\boldsymbol{A}'' \coloneqq (A^i_j)^{i=1,2}_{j=1,2} \in \mathbb{R}^{2\times 2}$. The null matrix and the identity matrix in $\mathbb{R}^{3\times 3}$ are denoted by $\boldsymbol{O}$ and $\boldsymbol{I}$, thus $\boldsymbol{O}''$ and $\boldsymbol{I}''$ are the corresponding matrices in $\mathbb{R}^{2\times 2}$. The tensor product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is given by $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathbb{R}^{3\times 3}$ where $(\boldsymbol{a} \otimes \boldsymbol{b})^i_j \coloneqq a^i b^j$ for every $i, j \in \{1, 2, 3\}$. We will denote general points in the physical (unscaled) space, in the reference space and in the actual space by $\boldsymbol{X}$, $\boldsymbol{x}$ and $\boldsymbol{\xi}$, respectively. Accordingly, the integration with respect to the three-dimensional Lebesgue measure will be denoted by $\mathrm{d}\boldsymbol{X}$, $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{\xi}$, respectively. The integration with respect to the one and the two-dimensional Hausdorff measure in the reference space will be denoted by $\mathrm{d}\boldsymbol{l}$ and $\mathrm{d}\boldsymbol{a}$, respectively. We denote by χ_A the characteristic function of a set $A \subset \mathbb{R}^k$, where $k \in \{1, 2, 3\}$. We will use standard notation for Lebesgue, Sobolev, Bochner, Bochner-Sobolev spaces and for spaces of functions of bounded variation. Given $D \subset \mathbb{R}^k$ open, where $k \in \{2, 3\}$, and an embedded submanifold $\mathcal{M} \subset \mathbb{R}^m$, where $m \in \mathbb{N}$, we denote by $W^{1,q}(D; \mathcal{M})$, where $1 \leq q < \infty$, the set of maps $\boldsymbol{v} \in W^{1,q}(D; \mathbb{R}^m)$ such that $\boldsymbol{v}(\boldsymbol{z}) \in \mathcal{M}$ for a.e. $\boldsymbol{z} \in D$. In the following, $\mathcal{M}$ will be either the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ or the special orthogonal group $SO(3) \subset \mathbb{R}^{3\times 3}$. Finally, the topological degree of a map $\boldsymbol{y} \in C^0(\overline{\Omega}; \mathbb{R}^3)$, where $\Omega \subset \mathbb{R}^3$ is open and bounded, on Ω at $\boldsymbol{\xi} \in \mathbb{R}^3$ will be denoted by $\deg(\boldsymbol{y}, \Omega, \boldsymbol{\xi})$.

We will make use of the Landau symbols 'o' and 'O'. When referred to vectors or matrices, these are to be understood with respect to the maximum of their components. We will adopt the common convention of denoting by $C, C_1, C_2 \ldots$ positive constants that can change from line to line. We will identify functions defined on the plane with functions defined on the three-dimensional space that are independent on the third variable. Also, we will drop some parentheses when these make the notation quite cumbersome. In general, we will think at the parameter $h > 0$ as varying along a sequence even if this is not mentioned. The particular sequence of thicknesses considered will be specified only in a few circumstances, when this is particularly important for the understanding.

2. Basic setting

In this section we describe the general setting of the paper. First, in Subsection 2.1, we introduce the mechanical model and we list all the assumptions. Then, in Subsection 2.2, we perform a standard change of variables in order to work on a fixed domain.

2.1. **The mechanical model.** Let $\Omega_h \coloneqq S \times hI$ represent a thin magnetoelastic plate in its reference configuration. The section $S \subset \mathbb{R}^2$ is a bounded connected Lipschitz domain, while the parameter $h > 0$ gives the thickness of the plate and $I \coloneqq (-1/2, 1/2)$. The plate is subjected to elastic deformations given by maps $\boldsymbol{\chi} \in W^{1,p}(\Omega_h; \mathbb{R}^3)$ for some fixed $p > 3$. By the Morrey embedding, any such map admits a continuous representative with whom it is systematically identified. Every deformation $\boldsymbol{\chi}$ is required to be *orientation-preserving*, namely to satisfy the constraint $\det \nabla \boldsymbol{\chi} > 0$ almost everywhere in Ω_h, and to be *almost everywhere injective*. This means that there exists a set $X \subset \Omega_h$ with $\mathscr{L}^3(X) = 0$ such that $\boldsymbol{\chi}|_{\Omega_h \setminus X}$ is injective. Recall that any such map $\boldsymbol{\chi}$ has both *Lusin properties* (N) and (N^{-1}), that is, $\mathscr{L}^3(\boldsymbol{\chi}(E)) = 0$ for every $E \subset \Omega_h$ with $\mathscr{L}^3(E) = 0$ and $\mathscr{L}^3(\boldsymbol{\chi}^{-1}(F)) = 0$ for every $F \subset \mathbb{R}^3$ with $\mathscr{L}^3(F) = 0$, and that the area formula and the change-of-variable formula hold for such a map [35].

Given a deformation $\boldsymbol{\chi}$, we define the corresponding deformed configuration as $\Omega_h^{\boldsymbol{\chi}} \coloneqq \boldsymbol{\chi}(\Omega_h) \setminus \boldsymbol{\chi}(\partial \Omega_h)$. This set is open [9, Lemma 2.1] and, by the Lusin property (N), there holds $\mathscr{L}^3(\boldsymbol{\chi}(\Omega_h) \setminus \Omega_h^{\boldsymbol{\chi}}) = 0$. Magnetizations are then defined as maps $\boldsymbol{m} \in W^{1,2}(\Omega_h^{\boldsymbol{\chi}}; \mathbb{S}^2)$.

The energy corresponding to a deformation $\boldsymbol{\chi} \in W^{1,p}(\Omega_h; \mathbb{R}^3)$ and a magnetization $\boldsymbol{m} \in W^{1,2}(\Omega_h^{\boldsymbol{\chi}}; \mathbb{S}^2)$, neglecting the material parameters, is given by

$$G_h(\boldsymbol{\varphi}, \boldsymbol{m}) \coloneqq \frac{1}{h^\beta} \int_{\Omega_h} W_h(\nabla \boldsymbol{\chi}, \boldsymbol{m} \circ \boldsymbol{\chi}) \, \mathrm{d}\boldsymbol{X} + \int_{\Omega_h^{\boldsymbol{\chi}}} |\nabla \boldsymbol{m}|^2 \, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \int_{\mathbb{R}^3} |\Psi_{\boldsymbol{m}}|^2 \, \mathrm{d}\boldsymbol{\xi}. \tag{2.1}$$

The first term in (2.1) represents the *elastic energy* and it is rescaled according to the linearized von Kármán regime [21]. Precisely, we assume $\beta > 6 \vee p$. Note that, by the Lusin property (N^{-1}), the composition $\boldsymbol{m} \circ \boldsymbol{\chi}$ is measurable and its equivalence class does not depend on the choice of the representative of $\boldsymbol{m}$. The elastic energy density $W_h \colon \mathbb{R}^{3\times 3} \times \mathbb{S}^2 \to [0, +\infty)$ is continuous and, for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$ and $\boldsymbol{\lambda} \in \mathbb{S}^2$, takes the form

$$W_h(\boldsymbol{F}, \boldsymbol{\lambda}) \coloneqq \Phi\left(\sqrt{\boldsymbol{F}^\top \boldsymbol{F}}\, \mathcal{K}_h(\boldsymbol{F}, \boldsymbol{\lambda})^{-1}\right), \tag{2.2}$$

for some function $\Phi \colon \mathbb{R}^{3\times 3} \to [0, +\infty)$. In (2.2), we set

$$\mathcal{K}_h(\boldsymbol{F}, \boldsymbol{\lambda}) \coloneqq \boldsymbol{I} + h^{\beta/2} \frac{(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}}{|(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}|} \otimes \frac{(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}}{|(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}|}. \tag{2.3}$$

Here, $\mathrm{adj}\boldsymbol{F}$ denotes the adjugate matrix, i.e. the transpose of the cofactor matrix of $\boldsymbol{F}$. Note that, for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$ and $\boldsymbol{\lambda} \in \mathbb{S}^2$, we have $(\mathrm{adj}\, \boldsymbol{F})\boldsymbol{\lambda} \neq \boldsymbol{0}$, so that $\mathcal{K}_h(\boldsymbol{F}, \boldsymbol{\lambda})$ is well defined. Moreover, this matrix is invertible and its inverse is given by

$$\mathcal{K}_h(\boldsymbol{F}, \boldsymbol{\lambda})^{-1} = \boldsymbol{I} - \frac{h^{\beta/2}}{1 + h^{\beta/2}} \frac{(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}}{|(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}|} \otimes \frac{(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}}{|(\mathrm{adj}\boldsymbol{F})\boldsymbol{\lambda}|}. \tag{2.4}$$

We remark that, thanks to (2.2)–(2.4), the elastic energy density W_h satisfies the physical requirements of *frame indifference* and *magnetic parity*, namely

$$\forall\, \boldsymbol{R} \in SO(3),\ \forall\, \boldsymbol{F} \in \mathbb{R}^{3\times 3}_+,\ \forall\, \boldsymbol{\lambda} \in \mathbb{S}^2, \quad W_h(\boldsymbol{R}\boldsymbol{F}, \boldsymbol{R}\boldsymbol{\lambda}) = W_h(\boldsymbol{F}, \boldsymbol{\lambda}) \tag{2.5}$$

and

$$\forall\, \boldsymbol{F} \in \mathbb{R}^{3\times 3}_+,\ \forall\, \boldsymbol{\lambda} \in \mathbb{S}^2, \quad W_h(\boldsymbol{F}, -\boldsymbol{\lambda}) = W_h(\boldsymbol{F}, \boldsymbol{\lambda}). \tag{2.6}$$

The function $\Phi \colon \mathbb{R}^{3\times 3}_+ \to [0, +\infty)$ is assumed to satisfy the following:

$$\Phi(\boldsymbol{I}) = 0 = \min \Phi, \tag{2.7}$$

$$\exists\, C > 0 :\ \forall\, \boldsymbol{Y} \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\boldsymbol{Y}) \geq C \mathrm{dist}^2(\boldsymbol{Y}; SO(3)) \vee \mathrm{dist}^p(\boldsymbol{Y}; SO(3)), \tag{2.8}$$

$$\Phi \text{ is continuous and of class } C^2 \text{ in a neighborhood of } SO(3), \tag{2.9}$$

By (2.7), the function W_h is minimized on the set

$$\left\{(\boldsymbol{RF}, \boldsymbol{R\lambda}):\ \boldsymbol{F} \in \mathbb{R}^{3\times 3}_+,\ \boldsymbol{\lambda} \in \mathbb{S}^2,\ \boldsymbol{R} \in SO(3),\ \boldsymbol{F} = \sqrt{\mathcal{K}_h(\boldsymbol{F}, \boldsymbol{\lambda})}\right\}.$$

From (2.8), we deduce that Φ has global p-growth and quadratic growth close to $SO(3)$. In particular, we have the following:

$$\exists\, C_1, C_2 > 0:\ \forall\, a \in \{2, p\},\ \forall\, \boldsymbol{Y} \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\boldsymbol{Y}) \geq C_1\, |\boldsymbol{Y}|^a - C_2. \tag{2.10}$$

Assumptions (2.7) and (2.9) justify the second-order Taylor expansion of Φ close to the identity. Precisely, we have the following:

$$\exists\, \delta_\Phi > 0:\ \forall\, \boldsymbol{Y} \in \mathbb{R}^{3\times 3}:\ |\boldsymbol{Y}| < \delta_\Phi, \quad \Phi(\boldsymbol{I} + \boldsymbol{Y}) = \frac{1}{2} Q_\Phi(\boldsymbol{Y}) + \omega_\Phi(\boldsymbol{Y}). \tag{2.11}$$

The quadratic form Q_Φ is defined by $Q_\Phi(\boldsymbol{Y}) \coloneqq D^2\Phi(\boldsymbol{I})(\boldsymbol{Y}, \boldsymbol{Y})$ while $\omega_\Phi(\boldsymbol{Y}) = o(|\boldsymbol{Y}|^2)$, as $|\boldsymbol{Y}| \to 0^+$. Note that Q_Φ is positive semidefinite and, in turn, convex by (2.7). Additionally, exploiting (2.8) and (2.11) and arguing as in [39, p. 927], we prove that Q_Φ is positive definite on symmetric matrices, that is

$$\exists\, C > 0:\ \forall\, \boldsymbol{Y} \in \mathbb{R}^{3\times 3}, \quad Q_\Phi(\boldsymbol{Y}) = Q_\Phi(\mathrm{sym}\boldsymbol{Y}) \geq C |\mathrm{sym}\, \boldsymbol{Y}|^2. \tag{2.12}$$

The last two terms in (2.1) are of Eulerian type. The second one is the *exchange energy*, while the third one is the *magnetostatic energy*. This last term involves the function $\Psi_{\boldsymbol{m}} \colon \mathbb{R}^3 \to \mathbb{R}^3$, called *stray field*, which is a weak solution of the *magnetostatic Maxwell equations*:

$$\begin{cases} \mathrm{curl}\, \Psi_{\boldsymbol{m}} = 0 \\ \mathrm{div}\, (\Psi_{\boldsymbol{m}} - \chi_{\Omega_h^{\boldsymbol{\chi}}} \boldsymbol{m}) = 0 \end{cases} \quad \text{in } \mathbb{R}^3. \tag{2.13}$$

The system (2.13) is equivalent to the existence of a *stray field potential* $\psi_{\boldsymbol{m}} \colon \mathbb{R}^3 \to \mathbb{R}$, so that $\boldsymbol{\Psi}_{\boldsymbol{m}} = \nabla \psi_{\boldsymbol{m}}$, which is a weak solution of the equation

$$\Delta \psi_{\boldsymbol{m}} = \mathrm{div}\, (\chi_{\Omega_h^{\boldsymbol{\chi}}} \boldsymbol{m}) \quad \text{in } \mathbb{R}^3. \tag{2.14}$$

It is proved that weak solutions of (2.14) exist and are unique up to additive constants [5, Proposition 8.8]. Therefore, $\boldsymbol{\Psi}_{\boldsymbol{m}}$ is uniquely defined.

We mention that the magnetostatic term usually comprises other terms such as the anisotropy energy or the Dzyaloshinskii-Moriya interaction energy [9] that here, for simplicity, we are neglecting.

2.2. **Change of variables and rescaling.** For $h > 0$, we introduce the map $\boldsymbol{\pi}_h$ defined by $\boldsymbol{\pi}_h(\boldsymbol{x}) \coloneqq ((\boldsymbol{x}')^\top, h x_3)^\top$ for every $\boldsymbol{x} \in \mathbb{R}^3$. Let $\Omega \coloneqq S \times I$. Given any deformation $\boldsymbol{\chi} \in W^{1,p}(\Omega_h; \mathbb{R}^3)$, we consider the map $\boldsymbol{y} \coloneqq \boldsymbol{\chi} \circ \boldsymbol{\pi}_h|_\Omega \in W^{1,p}(\Omega; \mathbb{R}^3)$. Then, $\Omega^{\boldsymbol{y}} \coloneqq \boldsymbol{y}(\Omega) \setminus \boldsymbol{y}(\partial\Omega) = \Omega_h^{\boldsymbol{\chi}}$ and, recalling (2.1) and applying the change-of-variable formula, we obtain

$$\frac{1}{h} G_h(\boldsymbol{\chi}, \boldsymbol{m}) = \frac{1}{h^\beta} \int_\Omega W_h(\nabla_h \boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y})\, \mathrm{d}\boldsymbol{x} + \frac{1}{h} \int_{\Omega^{\boldsymbol{y}}} |\nabla \boldsymbol{m}|^2\, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2h} \int_{\mathbb{R}^3} |\nabla \psi_{\boldsymbol{m}}|^2\, \mathrm{d}\boldsymbol{\xi}, \tag{2.15}$$

where the scaled gradient is defined as $\nabla_h \coloneqq (\nabla', h^{-1}\partial_3)$. Therefore, we define the class of admissible states as

$$\mathcal{Q} \coloneqq \left\{(\boldsymbol{y}, \boldsymbol{m}):\ \boldsymbol{y} \in \mathcal{Y},\ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}}; \mathbb{S}^2)\right\},$$

where admissible deformations belong to the set

$$\mathcal{Y} \coloneqq \left\{\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^3):\ \det \nabla \boldsymbol{y} > 0 \text{ a.e. in } \Omega,\ \boldsymbol{y} \text{ a.e. injective in } \Omega\right\}. \tag{2.16}$$

In view of (2.15), we consider the energy functional $E_h \colon \mathcal{Q} \to [0, +\infty)$ defined by

$$E_h(\boldsymbol{y}, \boldsymbol{m}) \coloneqq \frac{1}{h^\beta} \int_\Omega W_h(\nabla_h \boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y})\, \mathrm{d}\boldsymbol{x} + \frac{1}{h} \int_{\Omega^{\boldsymbol{y}}} |\nabla \boldsymbol{m}|^2\, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2h} \int_{\mathbb{R}^3} |\nabla \psi_{\boldsymbol{m}}|^2\, \mathrm{d}\boldsymbol{\xi} \tag{2.17}$$

and we denote the three terms on the right-hand side by $E_h^{\mathrm{el}}(\boldsymbol{y}, \boldsymbol{m})$, $E_h^{\mathrm{exc}}(\boldsymbol{y}, \boldsymbol{m})$ and $E_h^{\mathrm{mag}}(\boldsymbol{y}, \boldsymbol{m})$, respectively. Given (2.14), the function $\psi_{\boldsymbol{m}}$ in (2.17) is a weak solution of the equation

$$\Delta \psi_{\boldsymbol{m}} = \mathrm{div}\, (\chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \quad \text{in } \mathbb{R}^3. \tag{2.18}$$

More explicitly, this means that $\psi_{\boldsymbol{m}} \in V^{1,2}(\mathbb{R}^3)$ and the following holds:

$$\forall \varphi \in V^{1,2}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla \psi_{\boldsymbol{m}} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi}. \tag{2.19}$$

Here, we adopt the same notation in [40, Subsection 2.7.3] and we set

$$V^{1,2}(\mathbb{R}^3) \coloneqq \left\{ \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) : \, \nabla \varphi \in L^2(\mathbb{R}^3;\mathbb{R}^3) \right\}. \tag{2.20}$$

In particular, testing (2.19) with $\varphi = \psi_{\boldsymbol{m}}$ and applying the Hölder inequality, we obtain

$$||\nabla \psi_{\boldsymbol{m}}||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq ||\chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}. \tag{2.21}$$

Remark 2.1 (Invariance with respect to rigid motions). The functional E_h is invariant with respect to rigid motions. Let $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ and let $\boldsymbol{T}$ be a rigid motion of the form $\boldsymbol{T}(\boldsymbol{\xi}) \coloneqq \boldsymbol{R}\boldsymbol{\xi} + \boldsymbol{d}$ for every $\boldsymbol{\xi} \in \mathbb{R}^3$, where $\boldsymbol{R} \in SO(3)$ and $\boldsymbol{d} \in \mathbb{R}^3$. If we set $\widetilde{\boldsymbol{q}} = (\widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{m}}) \in \mathcal{Q}$ with $\widetilde{\boldsymbol{y}} \coloneqq \boldsymbol{T} \circ \boldsymbol{y}$ and $\widetilde{\boldsymbol{m}} \coloneqq \boldsymbol{R}\boldsymbol{m} \circ \boldsymbol{T}^{-1}$, then there holds $E_h(\widetilde{\boldsymbol{q}}) = E_h(\boldsymbol{q})$. Indeed, by (2.5), $E_h^{\mathrm{el}}(\widetilde{\boldsymbol{q}}) = E_h^{\mathrm{el}}(\boldsymbol{q})$ and, by the change-of-variable formula, $E_h^{\mathrm{exc}}(\widetilde{\boldsymbol{q}}) = E_h^{\mathrm{exc}}(\boldsymbol{q})$. Moreover, if ψ is a stray field potential corresponding to $\boldsymbol{q}$, then we check that $\psi \circ \boldsymbol{T}^{-1}$ is a stray field potential corresponding to $\widetilde{\boldsymbol{q}}$. Clearly, this yields $E_h^{\mathrm{mag}}(\widetilde{\boldsymbol{q}}) = E_h^{\mathrm{mag}}(\boldsymbol{q})$.

In the present work, we do not deal with the problem of the existence of minimizers and we do not even specify the topology on $\mathcal{Q}$. We just mention that, without further assumptions on the elastic energy density W_h, the functional E_h in (2.17) does not necessarily attains its infimum. However, if the elastic energy density W_h is assumed to be polyconvex in its first argument and some Dirichlet boundary conditions are imposed on the deformations, then the existence of minimizers can be proved, see [5, Theorem 8.9] and [9, Theorem 3.2].

3. The static setting

In this section we study the asymptotic behaviour of the energy E_h in (2.17), as $h \to 0^+$, in the static case. First, in Subsection 3.1, we compute the Γ-limit the sequence (E_h), thus identifying a reduced variational model for the plate. Then, in Subsection 3.2, we consider applied loads and we prove that sequences of almost minimizers of the total energy converge to minimizers of the corresponding energy in the reduced model.

3.1. $\boldsymbol{\Gamma}$-convergence.

We introduce some notation that is going to be used in the rest of the paper. For $h > 0$ and $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, we define the *(scaled) horizontal* and *vertical averaged displacements* and the *(scaled) first moment*, respectively

$$\mathcal{U}_h(\boldsymbol{q}) \colon S \to \mathbb{R}^2, \qquad \mathcal{V}_h(\boldsymbol{q}) \colon S \to \mathbb{R}, \qquad \mathcal{W}_h(\boldsymbol{q}) \colon S \to \mathbb{R}^3,$$

by setting

$$\mathcal{U}_h(\boldsymbol{q})(\boldsymbol{x}') \coloneqq \frac{1}{h^{\beta/2}} \int_I (\boldsymbol{y}'(\boldsymbol{x}', x_3) - \boldsymbol{x}') \, \mathrm{d}x_3, \tag{3.1}$$

$$\mathcal{V}_h(\boldsymbol{q})(\boldsymbol{x}') \coloneqq \frac{1}{h^{\beta/2-1}} \int_I y^3(\boldsymbol{x}', x_3) \, \mathrm{d}x_3, \tag{3.2}$$

$$\mathcal{W}_h(\boldsymbol{q})(\boldsymbol{x}') \coloneqq \frac{1}{h^{\beta/2}} \int_I x_3(\boldsymbol{y}(\boldsymbol{x}', x_3) - \boldsymbol{\pi}_h(\boldsymbol{x}', x_3)) \, \mathrm{d}x_3, \tag{3.3}$$

for every $\boldsymbol{x}' \in S$. With a slight abuse of notation, we will equivalently write $\mathcal{U}_h(\boldsymbol{q})$ or $\mathcal{U}_h(\boldsymbol{y})$, and analogously for $\mathcal{V}_h(\boldsymbol{q})$ and $\mathcal{W}_h(\boldsymbol{q})$. Furthermore, we define the *(scaled and normalized) Lagrangian magnetizations*

$$\mathcal{Z}_h(\boldsymbol{q}) \colon \Omega \to \mathbb{S}^2,$$

by setting

$$\mathcal{Z}_h(\boldsymbol{q}) \coloneqq \frac{(\mathrm{adj} \nabla_h \boldsymbol{y}) \boldsymbol{m} \circ \boldsymbol{y}}{|(\mathrm{adj} \nabla_h \boldsymbol{y}) \boldsymbol{m} \circ \boldsymbol{y}|}. \tag{3.4}$$

Note that this object is always well defined. Also, if we set $\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}$ and $\boldsymbol{\lambda} \coloneqq \boldsymbol{m} \circ \boldsymbol{y}$, then, recalling (2.3), there holds

$$\mathcal{K}_h(\boldsymbol{F}_h, \boldsymbol{\lambda}) = \boldsymbol{I} + h^{\beta/2} \mathcal{Z}_h(\boldsymbol{q}) \otimes \mathcal{Z}_h(\boldsymbol{q}).$$

Recall the quadratic form Q_Φ in (2.11). As in [21], the reduced quadratic form is defined by

$$Q_\Phi^{\mathrm{red}}(\boldsymbol{\Xi}) \coloneqq \min \left\{ Q_\Phi \left(\left(\begin{array}{c|c} \boldsymbol{\Xi} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + \boldsymbol{c} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{c}, \boldsymbol{\lambda} \right) : \ \boldsymbol{c} \in \mathbb{R}^3 \right\} \tag{3.5}$$

for every $\boldsymbol{\Xi} \in \mathbb{R}^{2\times 2}$. The positive definiteness and the convexity of Q_Φ^{red} follow from that of Q_Φ. Moreover, from (2.12), we deduce that Q_Φ^{red} is also positive definite on symmetric matrices, namely

$$\exists\, C > 0 : \ \forall \boldsymbol{\Xi} \in \mathbb{R}^{2\times 2}, \quad Q_\Phi^{\mathrm{red}}(\boldsymbol{\Xi}) = Q_\Phi(\mathrm{sym}\boldsymbol{\Xi}) \geq C |\mathrm{sym}\,\boldsymbol{\Xi}|^2. \tag{3.6}$$

The Γ-limit of the functionals (E_h) in (2.17), as $h \to 0^+$, is given by the functional

$$E_0 \colon W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2) \to [0,+\infty)$$

defined as

$$\begin{aligned} E_0(\boldsymbol{u}, v, \boldsymbol{\zeta}) \coloneqq \frac{1}{2} \int_S Q_\Phi^{\mathrm{red}}(\mathrm{sym}\nabla' \boldsymbol{u} - \boldsymbol{\zeta}' \otimes \boldsymbol{\zeta}')\,\mathrm{d}\boldsymbol{x}' + \frac{1}{24} \int_S Q_\Phi^{\mathrm{red}}((\nabla')^2 v)\,\mathrm{d}\boldsymbol{x}' \\ + \int_S |\nabla' \boldsymbol{\zeta}|^2\,\mathrm{d}\boldsymbol{x}' + \frac{1}{2} \int_S |\zeta^3|^2\,\mathrm{d}\boldsymbol{x}'. \end{aligned} \tag{3.7}$$

We denote the sum of the first two terms on the right-hand side of (3.7) by $E_0^{\mathrm{el}}(\boldsymbol{u}, v, \boldsymbol{\zeta})$ and the last two terms on the right-hand side of (3.7) by $E_0^{\mathrm{exc}}(\boldsymbol{\zeta})$ and $E_0^{\mathrm{mag}}(\boldsymbol{\zeta})$, respectively. Note that the limiting functional E_0 is purely Lagrangian and that it trivially admits minimizers.

Our first main result asserts the Γ-convergence of (E_h) to E_0, as $h \to 0^+$, and reads as follows.

Theorem 3.1 ($\boldsymbol{\Gamma}$-convergence). *Assume $p > 3$ and $\beta > 6 \vee p$. Suppose that the elastic energy density W_h has the form in (2.2), where the function Φ satisfies (2.7)–(2.9).*

*(1) **(Compactness and lower bound).** Let $(\boldsymbol{q}_h) \subset \mathcal{Q}$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h)$ be such that*

$$\sup_{h>0} E_h(\boldsymbol{q}_h) \leq C. \tag{3.8}$$

Then, there exist a sequence $(\boldsymbol{T}_h)$ of rigid motions of the form $\boldsymbol{T}_h(\boldsymbol{\xi}) \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{\xi} - \boldsymbol{c}_h$ for every $\boldsymbol{\xi} \in \mathbb{R}^3$ with $(\boldsymbol{Q}_h) \subset SO(3)$ and $(\boldsymbol{c}_h) \subset \mathbb{R}^3$ and maps $\widetilde{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$, $\widetilde{v} \in W^{2,2}(S)$ and $\widetilde{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ such that, setting $\widetilde{\boldsymbol{q}}_h \coloneqq (\widetilde{\boldsymbol{y}}_h, \widetilde{\boldsymbol{m}}_h)$ where $\widetilde{\boldsymbol{y}}_h \coloneqq \boldsymbol{T}_h \circ \boldsymbol{y}_h$ and $\widetilde{\boldsymbol{m}}_h \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{m}_h \circ \boldsymbol{T}_h^{-1}$, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\widetilde{\boldsymbol{u}}_h \coloneqq \mathcal{U}_h(\widetilde{\boldsymbol{q}}_h) \rightharpoonup \widetilde{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{3.9}$$

$$\widetilde{v}_h \coloneqq \mathcal{V}_h(\widetilde{\boldsymbol{q}}_h) \to \widetilde{v} \text{ in } W^{1,2}(S); \tag{3.10}$$

$$\widetilde{\boldsymbol{z}}_h \coloneqq \mathcal{Z}_h(\widetilde{\boldsymbol{q}}_h) \to \widetilde{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{3.11}$$

Moreover, the following inequality holds:

$$E_0(\widetilde{\boldsymbol{u}}, \widetilde{v}, \widetilde{\boldsymbol{\zeta}}) \leq \liminf_{h\to 0^+} E_h(\boldsymbol{q}_h). \tag{3.12}$$

*(2) **(Optimality of the lower bound).** For every $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$, $\widehat{v} \in W^{2,2}(S)$ and $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$, there exists $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ such that the following convergences hold, as $h \to 0^+$:*

$$\widehat{\boldsymbol{u}}_h \coloneqq \mathcal{U}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S); \tag{3.13}$$

$$\widehat{v}_h \coloneqq \mathcal{V}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{v} \text{ in } W^{1,2}(S); \tag{3.14}$$

$$\widehat{\boldsymbol{z}}_h \coloneqq \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{3.15}$$

Moreover, the following equality holds:

$$E_0(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) = \lim_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{3.16}$$

We mention that the injectivity of deformations is not essential in the proof of Theorem 3.1 and that the result still holds true if this requirement is dropped.

Note that Theorem 3.1 is not a proper Γ-convergence statement in the sense of the rigorous definition [7, 13] since the functionals E_h and E_0 are defined on different spaces. Also, in the first part of the statement, compactness is obtained up to composition with rigid motions. However, we mention that Theorem 3.1 can be reformulated as a rigorous Γ-convergence statement similarly to [8, Corollary 3.4].

The remainder of the subsection is devoted to the proof of Theorem 3.1. For future reference, we start by collecting some preliminary compactness results which we present in a more self-contained form.

The compactness of deformations is proved by adapting the techniques in [21] to our setting. A fundamental tool in these arguments is the celebrated rigidity estimate [20, Theorem 3.1]. For convenience, given $h > 0$ and $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$, we set

$$R_h(\boldsymbol{y}) \coloneqq \int_\Omega \mathrm{dist}^2(\boldsymbol{F}_h; SO(3)) \vee \mathrm{dist}^p(\boldsymbol{F}_h; SO(3))\,\mathrm{d}\boldsymbol{x}. \tag{3.17}$$

We will use the following slight modification of [21, Theorem 6] which was given in [8, Lemma 4.1].

Lemma 3.2 (Approximation by rotations). *Let $h > 0$ and $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$. Set $\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}$. Then, there exist $\boldsymbol{R}_h \in W^{1,p}(S;SO(3))$ and $\boldsymbol{Q}_h \in SO(3)$ such that, for $a \in \{2,p\}$, the following estimates hold:*

$$||\boldsymbol{F}_h - \boldsymbol{R}_h||_{L^s(\Omega;\mathbb{R}^{3\times 3})} \le C R_h(\boldsymbol{y})^{1/a}, \qquad ||\nabla' \boldsymbol{R}_h||_{L^a(S;\mathbb{R}^{3\times 3})} \le C h^{-1} R_h(\boldsymbol{y})^{1/a}, \tag{3.18}$$

$$||\boldsymbol{R}_h - \boldsymbol{Q}_h||_{L^a(S;\mathbb{R}^{3\times 3})} \le C h^{-1} R_h(\boldsymbol{y})^{1/a}, \qquad ||\boldsymbol{F}_h - \boldsymbol{Q}_h||_{L^a(\Omega;\mathbb{R}^{3\times 3})} \le C h^{-1} R_h(\boldsymbol{y})^{1/a}. \tag{3.19}$$

The next Proposition provides a simple reformulation of the compactness results in [21]. Henceforth, $\boldsymbol{\pi}_0$ denotes projection map defined by $\boldsymbol{\pi}_0(\boldsymbol{x}) \coloneqq ((\boldsymbol{x}')^\top, 0)^\top$ for every $\boldsymbol{x} \in \mathbb{R}^3$.

Proposition 3.3 (Compactness of deformations). *Let $(\widehat{\boldsymbol{y}}_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ and let $(r_h), (e_h) \subset \mathbb{R}$ with $r_h, e_h > 0$ be such that $r_h \le C e_h$ for every $h > 0$. For every $h > 0$, set $\widehat{\boldsymbol{F}}_h \coloneqq \nabla_h \widehat{\boldsymbol{y}}_h$ and suppose that there exists $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S;SO(3))$ satisfying*

$$||\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\sqrt{r_h}, \qquad ||\nabla' \widehat{\boldsymbol{R}}_h||_{L^2(S;\mathbb{R}^{3\times 3\times 3})} \le C h^{-1} \sqrt{r_h} \tag{3.20}$$

$$||\widehat{\boldsymbol{R}}_h - \boldsymbol{I}||_{L^2(S;\mathbb{R}^{3\times 3})} \le C h^{-1} \sqrt{r_h}, \qquad ||\widehat{\boldsymbol{F}}_h - \boldsymbol{I}||_{L^2(S;\mathbb{R}^{3\times 3})} \le C h^{-1} \sqrt{r_h}. \tag{3.21}$$

Also, for every $h > 0$, assume the following:

$$\text{either } \widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h \text{ has null average over } \Omega \text{ or } \widehat{\boldsymbol{y}}_h = \boldsymbol{\pi}_h \text{ on } \partial S. \tag{3.22}$$

Define $\widehat{\boldsymbol{u}}_h \colon S \to \mathbb{R}^2$, $\widehat{v}_h \colon S \to \mathbb{R}$ and $\widehat{\boldsymbol{w}}_h \colon S \to \mathbb{R}^3$ by setting

$$\widehat{\boldsymbol{u}}_h(\boldsymbol{x}') \coloneqq \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}} \int_I \left(\widehat{\boldsymbol{y}}'_h(\boldsymbol{x}', x_3) - \boldsymbol{x}' \right)\,\mathrm{d}x_3,$$
$$\widehat{v}_h(\boldsymbol{x}') \coloneqq \frac{h}{\sqrt{e_h}} \int_I \widehat{y}^3_h(\boldsymbol{x}', x_3)\,\mathrm{d}x_3,$$
$$\widehat{\boldsymbol{w}}_h(\boldsymbol{x}') \coloneqq \frac{1}{\sqrt{e_h}} \int_I x_3 \big(\widehat{\boldsymbol{y}}_h(\boldsymbol{x}', x_3) - \boldsymbol{\pi}_h(\boldsymbol{x}', x_3) \big)\,\mathrm{d}x_3,$$

for every $\boldsymbol{x}' \in S$. Then, for every $h > 0$, the following estimates hold:

$$||\widehat{\boldsymbol{u}}_h||_{W^{1,2}(S;\mathbb{R}^2)} \le C\sqrt{\frac{r_h}{e_h}}, \qquad ||\widehat{v}_h||_{W^{1,2}(S)} \le C\sqrt{\frac{r_h}{e_h}}. \tag{3.23}$$

Moreover, if $e_h/h^2 \to 0$, as $h \to 0^+$, then there exist $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{v} \in W^{2,2}(S)$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\widehat{\boldsymbol{y}}_h \to \boldsymbol{\pi}_0 \text{ in } W^{1,2}(\Omega;\mathbb{R}^3); \tag{3.24}$$

$$\widehat{\boldsymbol{u}}_h \rightharpoonup \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{3.25}$$

$$\widehat{v}_h \to \widehat{v} \text{ in } W^{1,2}(S); \tag{3.26}$$

$$\widehat{\boldsymbol{w}}_h \rightharpoonup \widehat{\boldsymbol{w}} \text{ in } W^{1,2}(S;\mathbb{R}^3), \text{ where } \widehat{\boldsymbol{w}} := -\frac{1}{12}\begin{pmatrix} \nabla'\widehat{v} \\ 0 \end{pmatrix}; \tag{3.27}$$

$$\widehat{\boldsymbol{F}}_h \to \boldsymbol{I} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}); \tag{3.28}$$

$$\widehat{\boldsymbol{A}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{R}}_h - \boldsymbol{I}) \rightharpoonup \widehat{\boldsymbol{A}} \text{ in } W^{1,2}(S;\mathbb{R}^{3\times 3}), \text{ where } \widehat{\boldsymbol{A}} = \left(\begin{array}{c|c} \boldsymbol{O}'' & -\nabla'\widehat{v} \\ \hline (\nabla'\widehat{v})^\top & 0 \end{array}\right). \tag{3.29}$$

Proof. Claim (3.28) is immediate from the second estimate in (3.21) and entails (3.24) by the Poincaré inequality. Set

$$\widehat{\boldsymbol{A}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{R}}_h - \boldsymbol{I}), \qquad \widehat{\boldsymbol{B}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{F}}_h - \boldsymbol{I}), \qquad \widehat{\boldsymbol{C}}_h := \frac{h^2}{e_h}\mathrm{sym}(\widehat{\boldsymbol{R}}_h - \boldsymbol{I}).$$

We claim the following:

$$||\widehat{\boldsymbol{A}}_h||_{W^{1,2}(S;\mathbb{R}^{3\times 3})} \le C\sqrt{\frac{r_h}{e_h}}, \qquad ||\widehat{\boldsymbol{B}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\sqrt{\frac{r_h}{e_h}}, \qquad ||\widehat{\boldsymbol{C}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\sqrt{\frac{r_h}{e_h}}. \tag{3.30}$$

The first two estimates in (3.30) follow from the ones in (3.21). Exploiting the identity $\widehat{\boldsymbol{C}}_h = -1/2\,\widehat{\boldsymbol{A}}_h^\top \widehat{\boldsymbol{A}}_h$, using the first estimate in (3.30) and applying the Sobolev embedding, we obtain

$$||\widehat{\boldsymbol{C}}_h||_{L^2(S;\mathbb{R}^{3\times 3})} \le ||\widehat{\boldsymbol{A}}_h||^2_{L^4(S;\mathbb{R}^{3\times 3})} \le ||\widehat{\boldsymbol{A}}_h||^2_{W^{1,2}(S;\mathbb{R}^{3\times 3})} \le C\frac{r_h}{e_h} \le C\sqrt{\frac{r_h}{e_h}},$$

so that the third estimate in (3.30) is checked.

We now prove (3.23). Using the Jensen inequality, we compute

$$\begin{aligned} ||\mathrm{sym}\nabla'\widehat{\boldsymbol{u}}_h||_{L^2(S;\mathbb{R}^{2\times 2})} &\le \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}||\mathrm{sym}(\widehat{\boldsymbol{F}}''_h - \boldsymbol{I}'')||_{L^2(\Omega;\mathbb{R}^{2\times 2})} \\ &\le \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}||\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \left(\frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\right)\frac{e_h}{h^2}||\widehat{\boldsymbol{C}}||_{L^2(S;\mathbb{R}^{3\times 3})} \\ &\le C\sqrt{\frac{r_h}{e_h}}, \end{aligned}$$

where in the last line, we employed the first estimate in (3.20) and the third estimate in (3.30). Thus, the first estimate in (3.23) follows from the previous computation and the Korn inequality. From the second estimate in (3.30), using Poincaré and Jensen inequalities, we obtain

$$||\widehat{v}_h||_{W^{1,2}(S)} \le C||\nabla'\widehat{v}_h||_{L^2(S;\mathbb{R}^2)} \le C\frac{h}{\sqrt{e_h}}||\widehat{\boldsymbol{F}}_h - \boldsymbol{I}||_{L^2(\Omega;\mathbb{R}^{3\times 3})} = C||\widehat{\boldsymbol{B}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\sqrt{\frac{r_h}{e_h}},$$

which is the second estimate in (3.23).

Now, by (3.23), the two sequences are bounded since $r_h/e_h \le C$. Thus, the convergences in (3.25)–(3.26) follow. Similarly, the first estimate in (3.30) entails the convergence in (3.29). The higher regularity of $\widehat{v}$ and the identification of the limit in (3.29) are proved arguing as in [21, Lemma 1]. Finally, (3.27) is shown as in [21, Corollary 1].

Note that assumption (3.22) is needed in order to apply Poincaré and Korn inequalities. Indeed, if $\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h$ has null average over Ω, then the same property holds for $\widehat{\boldsymbol{u}}_h$ and $\widehat{v}_h$, while, if $\widehat{\boldsymbol{y}}_h = \boldsymbol{\pi}_h$ on ∂S, then $\widehat{\boldsymbol{u}}_h$ and $\widehat{v}_h$ satisfy homogeneous Dirichlet boundary conditions. □

We will employ the following result given by [21, Lemma 2].

Lemma 3.4 (Identification of the limiting strain). *Let $(\widehat{\boldsymbol{y}}_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ and let $(e_h) \subset \mathbb{R}$ with $e_h > 0$ be such that $e_h/h^2 \to 0$, as $h \to 0^+$. For every $h > 0$, set $\widehat{\boldsymbol{F}}_h := \nabla_h \widehat{\boldsymbol{y}}_h$ and suppose that there exists $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S;SO(3))$ satisfying*

$$||\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\sqrt{e_h}.$$

Moreover, for every $h>0$, define $\widehat{\boldsymbol{u}}_h\colon S\to\mathbb{R}^2$ and $\widehat{v}_h\colon S\to\mathbb{R}$ by setting

$$\widehat{\boldsymbol{u}}_h(\boldsymbol{x}') \coloneqq \frac{h^2}{e_h}\wedge\frac{1}{\sqrt{e_h}}\int_I\left(\widehat{\boldsymbol{y}}'_h(\boldsymbol{x}',x_3)-\boldsymbol{x}'\right)\mathrm{d}x_3,$$
$$\widehat{v}_h(\boldsymbol{x}') \coloneqq \frac{h}{\sqrt{e_h}}\int_I\widehat{y}^3_h(\boldsymbol{x}',x_3)\,\mathrm{d}x_3,$$

for every $\boldsymbol{x}'\in S$, and assume that there exist $\widehat{\boldsymbol{u}}\in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{v}\in W^{2,2}(S)$ such that the following convergences hold

$$\widehat{\boldsymbol{u}}_h\rightharpoonup\widehat{\boldsymbol{u}}\ \text{in}\ W^{1,2}(S;\mathbb{R}^2);$$
$$\widehat{v}_h\to\widehat{v}\ \text{in}\ W^{1,2}(S).$$

Then, there exists $\widehat{\boldsymbol{G}}\in L^2(\Omega;\mathbb{R}^{3\times3})$ such that

$$\widehat{\boldsymbol{G}}_h\coloneqq\frac{1}{\sqrt{e_h}}(\widehat{\boldsymbol{R}}_h^\top\widehat{\boldsymbol{F}}_h-\boldsymbol{I})\rightharpoonup\widehat{\boldsymbol{G}}\ \text{in}\ L^2(\Omega;\mathbb{R}^{3\times3}).$$

Furthermore, there exists $\widehat{\boldsymbol{\Sigma}}\in L^2(\Omega;\mathbb{R}^{2\times2})$ such that, for almost every $\boldsymbol{x}\in\Omega$, there holds

$$\widehat{\boldsymbol{G}}''(\boldsymbol{x}',x_3)=\widehat{\boldsymbol{\Sigma}}(\boldsymbol{x}')-\nabla'\widehat{v}(\boldsymbol{x}')x_3.$$

Additionally, if $e_h/h^4\to0$, as $h\to0^+$, then $\mathrm{sym}\,\widehat{\boldsymbol{\Sigma}}=\mathrm{sym}\nabla'\widehat{\boldsymbol{u}}$.

The compactness of magnetizations is established by refining the techniques introduced in [8, Proposition 4.3]. We will use the following notation. Given $\varepsilon>0$, we set $S^\varepsilon\coloneqq\{\boldsymbol{x}'\in S:\ \mathrm{dist}(\boldsymbol{x}';\partial S)<\varepsilon\}$ and $S^{-\varepsilon}\coloneqq\{\boldsymbol{x}'\in\mathbb{R}^3:\ \mathrm{dist}(x;S)<\varepsilon\}$. Moreover, for $b>0$, we set $\Omega^\varepsilon_b\coloneqq S^\varepsilon\times bI$ and $\Omega^{-\varepsilon}_b\coloneqq S^{-\varepsilon}\times bI$.

Proposition 3.5 (Compactness of magnetizations). *Let $(\widehat{\boldsymbol{q}}_h)\subset\mathcal{Q}$ with $\widehat{\boldsymbol{q}}_h=(\widehat{\boldsymbol{y}}_h,\widehat{\boldsymbol{m}}_h)$ be such that*

$$\sup_{h>0}E^{\mathrm{exc}}_h(\widehat{\boldsymbol{q}}_h)\le C. \tag{3.31}$$

For every $h>0$, set $\widehat{\boldsymbol{F}}_h\coloneqq\nabla_h\widehat{\boldsymbol{y}}_h$ and suppose that, for $a\in\{2,p\}$, there holds

$$||\widehat{\boldsymbol{F}}_h-\boldsymbol{I}||_{L^a(\Omega;\mathbb{R}^{3\times3})}\le Ch^{\beta/a-1}. \tag{3.32}$$

Moreover, for every $h>0$, assume the following:

$$\text{either}\ \widehat{\boldsymbol{y}}_h-\boldsymbol{\pi}_h\ \text{has null average over}\ \Omega\ \text{or}\ \widehat{\boldsymbol{y}}_h=\boldsymbol{\pi}_h\ \text{on}\ \partial S. \tag{3.33}$$

Then, there exists $\widehat{\boldsymbol{\zeta}}\in W^{1,2}(S;\mathbb{S}^2)$ and $\widehat{\boldsymbol{\nu}}\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, up to subsequences, the following convergences hold, as $h\to0^+$:

$$\widehat{\boldsymbol{\eta}}_h\coloneqq(\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}}\widehat{\boldsymbol{m}}_h)\circ\boldsymbol{\pi}_h\to\widehat{\boldsymbol{\eta}}\ \text{in}\ L^q(\mathbb{R}^3;\mathbb{R}^{3\times3})\ \text{for every}\ 1\le q<+\infty,\ \text{where}\ \widehat{\boldsymbol{\eta}}\coloneqq\chi_\Omega\widehat{\boldsymbol{\zeta}}; \tag{3.34}$$

$$\widehat{\boldsymbol{H}}_h\coloneqq(\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}}\nabla\widehat{\boldsymbol{m}}_h)\circ\boldsymbol{\pi}_h\rightharpoonup\widehat{\boldsymbol{H}}\ \text{in}\ L^2(\mathbb{R}^3;\mathbb{R}^{3\times3}),\ \text{where}\ \widehat{\boldsymbol{H}}\coloneqq\chi_\Omega(\nabla'\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\nu}}); \tag{3.35}$$

$$\widehat{\boldsymbol{m}}_h\circ\widehat{\boldsymbol{y}}_h\to\widehat{\boldsymbol{\zeta}}\ \text{in}\ L^q(\Omega;\mathbb{R}^3)\ \text{for every}\ 1\le q<+\infty; \tag{3.36}$$

$$\widehat{\boldsymbol{z}}_h\coloneqq\mathcal{Z}_h(\widehat{\boldsymbol{q}}_h)\to\widehat{\boldsymbol{\zeta}}\ \text{in}\ L^q(\Omega;\mathbb{R}^{3\times3})\ \text{for every}\ 1\le q<+\infty. \tag{3.37}$$

Proof. For convenience of the reader, the proof is subdivided into three steps.

Step 1 (Approximation of the deformed configuration). First note that, by the assumption $\beta>6\vee p$, we have $\beta/2-1>0$ and $\beta/p-1>0$. Recall (3.32). Let $0<\rho<1$ and consider $2\le q_\rho\le p$ such that $1/q_\rho=\rho/2+(1-\rho)/p$. By the interpolation inequality [23, Proposition 1.1.14], there holds

$$||\widehat{\boldsymbol{F}}_h-\boldsymbol{I}||_{L^{q_\rho}(\Omega;\mathbb{R}^{3\times3})}\le||\widehat{\boldsymbol{F}}_h-\boldsymbol{I}||^\rho_{L^2(\Omega;\mathbb{R}^{3\times3})}||\widehat{\boldsymbol{F}}_h-\boldsymbol{I}||^{1-\rho}_{L^p(\Omega;\mathbb{R}^{3\times3})}\le Ch^{\delta_\rho}, \tag{3.38}$$

where we set $\delta_\rho\coloneqq\rho(\beta/2-1)+(1-\rho)(\beta/p-1)$. We choose ρ in order to have

$$q_\rho>3,\qquad\delta_\rho>1. \tag{3.39}$$

Note that these two conditions are equivalent to $\rho<\frac{2(p-3)}{3(p-2)}$ and $\rho>\frac{2(2p-\beta)}{\beta(p-2)}$, respectively. Therefore, as $\frac{2(2p-\beta)}{\beta(p-2)}<\frac{2(p-3)}{3(p-2)}$ if and only if $\beta>6$, such a value $0<\rho<1$ always exists.

Now, we argue as in [8, Proposition 4.3]. Recall (3.33). From (3.38), using the Poincaré inequality and the Morrey embedding, we obtain the estimate

$$||\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h||_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq C||\nabla\widehat{\boldsymbol{y}}_h - \nabla\boldsymbol{\pi}_h||_{L^{\rho_s}(\Omega;\mathbb{R}^{3\times 3})} \leq C||\widehat{\boldsymbol{F}}_h - \boldsymbol{I}||_{L^{q_\rho}(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{\delta_\rho}. \tag{3.40}$$

Note that here we implicitly exploited the first condition in (3.39). We claim the following:

$$\forall\,\varepsilon>0,\,\forall\,0<\vartheta<1,\;\;\exists\,\bar{h}(\varepsilon,\vartheta)>0:\;\;\forall\,0<h\leq\bar{h}(\varepsilon,\vartheta),\;\;\Omega^\varepsilon_{\vartheta h}\subset\Omega^{\widehat{\boldsymbol{y}}_h}, \tag{3.41}$$

and

$$\forall\,\varepsilon>0,\,\forall\,\gamma>1,\;\;\exists\,\underline{h}(\varepsilon,\gamma)>0:\;\;\forall\,0<h\leq\underline{h}(\varepsilon,\gamma),\;\;\Omega^{\widehat{\boldsymbol{y}}_h}\subset\Omega^{-\varepsilon}_{\gamma h}. \tag{3.42}$$

To see (3.41), fix $\varepsilon>0$ and $0<\vartheta<1$. Let $\boldsymbol{\xi}\in\Omega^\varepsilon_{\vartheta h}$. Given the second condition in (3.39), there exists $\bar{h}(\varepsilon,\vartheta)>0$ such that for every $h\leq\bar{h}(\varepsilon,\vartheta)$ there holds

$$\mathrm{dist}(\boldsymbol{\xi};\partial\Omega_h)\geq\varepsilon\wedge(1-\vartheta)h/2>Ch^{\delta_\rho}$$

so that, by (3.40), we obtain

$$||\widehat{\boldsymbol{y}}_h-\boldsymbol{\pi}_h||_{C^0(\overline{\Omega};\mathbb{R}^3)}<\mathrm{dist}(\boldsymbol{\xi};\partial\Omega_h)=\mathrm{dist}(\boldsymbol{\xi};\boldsymbol{\pi}_h(\partial\Omega)).$$

By the stability property of the topological degree [17, Theorem 2.3, Claim (1)], this entails $\boldsymbol{\xi}\notin\widehat{\boldsymbol{y}}_h(\partial\Omega)$ and $\deg(\widehat{\boldsymbol{y}}_h,\Omega,\boldsymbol{\xi})=\deg(\boldsymbol{\pi}_h,\Omega,\boldsymbol{\xi})=1$. Then, by the solvability property of the topological degree [17, Theorem 2.1], we deduce $\boldsymbol{\xi}\in\Omega^{\widehat{\boldsymbol{y}}_h}$. As $\boldsymbol{\xi}\in\Omega^\varepsilon_{\vartheta h}$ was arbitrary, this proves (3.41).

To see (3.42), fix $\varepsilon>0$ and $\gamma>1$. Again, by the second condition in (3.39), there exists $\underline{h}(\varepsilon,\gamma)>0$ such that for every $h\leq\underline{h}(\varepsilon,\gamma)$ there holds

$$\mathrm{dist}(\overline{\Omega}_h;\partial\Omega^{-\varepsilon}_{\gamma h})\geq\varepsilon\wedge(\gamma-1)h/2>Ch^{\delta_\rho}.$$

Thus, by (3.40), we have

$$\Omega^{\widehat{\boldsymbol{y}}_h}\subset\Omega_h+B(\boldsymbol{0},Ch^{\delta_\rho})\subset\Omega^{-\varepsilon}_{\gamma h}.$$

Step 2 (Compactness of magnetizations). Let $\varepsilon>0$ and $0<\vartheta<1$. By (3.41), for $h\leq\bar{h}(\varepsilon,\vartheta)$ the map $\widehat{\boldsymbol{\zeta}}_h\coloneqq\widehat{\boldsymbol{m}}_h\circ\boldsymbol{\pi}_h|_{\Omega^\varepsilon_\vartheta}$ is well defined and belongs to $W^{1,2}(\Omega^\varepsilon_\vartheta;\mathbb{S}^2)$. Using (3.31) and the change-of-variable formula, we obtain

$$C\geq E^{\mathrm{exc}}_h(\widehat{\boldsymbol{q}}_h)=\frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}}|\nabla\widehat{\boldsymbol{m}}_h|^2\,\mathrm{d}\boldsymbol{\xi}\geq\frac{1}{h}\int_{\Omega^\varepsilon_{\vartheta h}}|\nabla\widehat{\boldsymbol{m}}_h|^2\,\mathrm{d}\boldsymbol{\xi}=\int_{\Omega^\varepsilon_\vartheta}|\nabla_h\widehat{\boldsymbol{\zeta}}_h|^2\,\mathrm{d}\boldsymbol{x}. \tag{3.43}$$

From this bound, as the maps $\widehat{\boldsymbol{\zeta}}_h$ are sphere-valued, we deduce the existence of two maps $\widehat{\boldsymbol{\zeta}}\in W^{1,2}(\Omega^\varepsilon_\vartheta;\mathbb{R}^3)$ and $\widehat{\boldsymbol{\nu}}\in L^2(\Omega^\varepsilon_\vartheta;\mathbb{R}^3)$ such that, up to subsequences, $\widehat{\boldsymbol{\zeta}}_h\rightharpoonup\widehat{\boldsymbol{\zeta}}$ in $W^{1,2}(\Omega^\varepsilon_\vartheta)$ and $\partial_3\widehat{\boldsymbol{\zeta}}_h/h\rightharpoonup\widehat{\boldsymbol{\nu}}$ in $L^2(\Omega^\varepsilon_\vartheta;\mathbb{R}^3)$, as $h\to0^+$. By the Sobolev embedding, we actually have $\widehat{\boldsymbol{\zeta}}\in W^{1,2}(\Omega^\varepsilon_\vartheta;\mathbb{S}^2)$. Moreover, $\partial_3\widehat{\boldsymbol{\zeta}}=\boldsymbol{0}$ and, in turn, $\widehat{\boldsymbol{\zeta}}\in W^{1,2}(S^\varepsilon;\mathbb{S}^2)$. In principle, the subsequences and the weak limits $\widehat{\boldsymbol{\zeta}}$ and $\widehat{\boldsymbol{\nu}}$ depend on the parameters ε and ϑ. However, by means of a diagonal argument, we can assume $\widehat{\boldsymbol{\zeta}}\in W^{1,2}_{\mathrm{loc}}(S;\mathbb{S}^2)$ and $\widehat{\boldsymbol{\nu}}\in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ and we select a subsequence (h_ℓ), independent from the parameters, such that the following holds:

$$\forall\,\varepsilon>0,\,\forall\,0<\vartheta<1,\;\;\widehat{\boldsymbol{\zeta}}_{h_\ell}\rightharpoonup\widehat{\boldsymbol{\zeta}}\text{ in }W^{1,2}(\Omega^\varepsilon_\vartheta;\mathbb{R}^3)\text{ and a.e. in }\Omega^\varepsilon_\vartheta,\;\;\partial_3\widehat{\boldsymbol{\zeta}}_{h_\ell}/h_\ell\rightharpoonup\widehat{\boldsymbol{\nu}}\text{ in }L^2(\Omega^\varepsilon_\vartheta;\mathbb{R}^3). \tag{3.44}$$

Here, we exploited once more the Sobolev embedding. Note that the sequences in (3.44) are defined only for $\ell\gg1$ depending on ε and ϑ. By (3.44) and by lower semicontinuity, for every $\varepsilon>0$ and $0<\vartheta<1$ there holds

$$C\geq\liminf_{\ell\to\infty}\int_{\Omega^\varepsilon_\vartheta}|\nabla_{h_\ell}\widehat{\boldsymbol{\zeta}}_{h_\ell}|^2\,\mathrm{d}\boldsymbol{x}\geq\vartheta\int_{S^\varepsilon}|\nabla'\widehat{\boldsymbol{\zeta}}|^2\,\mathrm{d}\boldsymbol{x}+\int_{\Omega^\varepsilon_\vartheta}|\widehat{\boldsymbol{\nu}}|^2\,\mathrm{d}\boldsymbol{x},$$

so that, letting $\varepsilon\to0^+$ and $\vartheta\to1^-$, we deduce $\widehat{\boldsymbol{\zeta}}\in W^{1,2}(S;\mathbb{S}^2)$ and $\widehat{\boldsymbol{\nu}}\in L^2(\Omega;\mathbb{R}^3)$.

Set $\widehat{\boldsymbol{\eta}}_h\coloneqq(\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}}\widehat{\boldsymbol{m}}_h)\circ\boldsymbol{\pi}_h$ and $\widehat{\boldsymbol{H}}_h\coloneqq(\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}}\nabla\widehat{\boldsymbol{m}}_h)\circ\boldsymbol{\pi}_h$. Note that these two maps are defined on the whole space for every $h>0$. By (3.41) and (3.42), there holds $\chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})}\to\chi_\Omega$ almost everywhere. This, combined with (3.44), yields $\widehat{\boldsymbol{\eta}}_{h_\ell}\to\widehat{\boldsymbol{\eta}}$ almost everywhere, as $\ell\to\infty$, where $\widehat{\boldsymbol{\eta}}\coloneqq\chi_\Omega\widehat{\boldsymbol{\zeta}}$. Moreover, by

(3.42), for $\ell \gg 1$ the maps $\widehat{\boldsymbol{\eta}}_{h_\ell}$ are supported in a common compact set containing Ω so that, applying the Dominated Convergence Theorem, we obtain (3.34). To prove (3.35), we observe that

$$E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h) = \frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} |\nabla \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} |\nabla \widehat{\boldsymbol{m}}_h|^2 \circ \boldsymbol{\pi}_h \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} |\widehat{\boldsymbol{H}}_h|^2 \,\mathrm{d}\boldsymbol{x}, \tag{3.45}$$

where we used the change-of-variable formula. This, together with (3.31), gives the boundedness of $(\widetilde{\boldsymbol{H}}_h)$ in $L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$. In order to prove (3.35), let $\boldsymbol{\Phi} \in L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ and set $\widehat{\boldsymbol{H}} \coloneqq \chi_\Omega(\nabla'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\nu}})$. Fix $\varepsilon > 0$ and $0 < \vartheta < 1$. We compute

$$\int_{\mathbb{R}^3} (\widehat{\boldsymbol{H}}_{h_\ell} - \widehat{\boldsymbol{H}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} = \int_{\Omega^\varepsilon_\vartheta} (\widehat{\boldsymbol{H}}_{h_\ell} - \widehat{\boldsymbol{H}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3 \setminus \Omega^\varepsilon_\vartheta} (\widehat{\boldsymbol{H}}_{h_\ell} - \widehat{\boldsymbol{H}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x}. \tag{3.46}$$

The first integral on the right-hand side of (3.46) goes to zero, as $\ell \to \infty$. Indeed, by (3.41), for every $\ell \gg 1$ such that $h_\ell \leq \bar{h}(\varepsilon, \vartheta)$, we have

$$\int_{\Omega^\varepsilon_\vartheta} (\widehat{\boldsymbol{H}}_{h_\ell} - \widehat{\boldsymbol{H}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} = \int_{\Omega^\varepsilon_\vartheta} (\nabla \widehat{\boldsymbol{m}}_{h_\ell} \circ \boldsymbol{\pi}_{h_\ell} - (\nabla'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\chi}})) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} = \int_{\Omega^\varepsilon_\vartheta} (\nabla_{h_\ell} \widehat{\boldsymbol{\zeta}}_{h_\ell} - (\nabla'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\chi}})) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x},$$

where, by (3.44), the right-hand side goes to zero, as $\ell \to \infty$. The second integral on the right-hand side of (3.46) goes as well to zero, as $\ell \to \infty$. Indeed, by (3.42) and by the boundedness of $(\widehat{\boldsymbol{H}}_{h_\ell})$ in $L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$, for every $\gamma > 1$ and for every $\ell \gg 1$ such that $h_\ell \leq \underline{h}(\varepsilon, \gamma)$, there holds

$$\left| \int_{\mathbb{R}^3 \setminus \Omega^\varepsilon_\vartheta} (\widehat{\boldsymbol{H}}_{h_\ell} - \widehat{\boldsymbol{H}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} \right| = \left| \int_{\Omega^{-\varepsilon}_\gamma \setminus \Omega^\varepsilon_\vartheta} (\widehat{\boldsymbol{H}}_{h_\ell} - (\nabla'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\nu}})) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} \right| \leq C\, ||\boldsymbol{\Phi}||_{L^2(\Omega^{-\varepsilon}_\gamma \setminus \Omega^\varepsilon_\vartheta;\mathbb{R}^{3\times 3})}.$$

As the right-hand side can be made arbitrarily small by properly choosing ε, ϑ and γ according to $\boldsymbol{\Phi}$ only, this establishes (3.35).

Step 3 (Convergence of compositions). We claim that $\widehat{\boldsymbol{m}}_{h_\ell} \circ \widehat{\boldsymbol{y}}_{h_\ell} \to \widehat{\boldsymbol{\zeta}}$ in $L^2(\Omega^\varepsilon_\vartheta;\mathbb{R}^3)$ for every $\varepsilon > 0$ and $0 < \vartheta < 1$. Here, we refer to the subsequence (h_ℓ) in (3.44). As the sequence $(\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h)$ is uniformly bounded in $L^\infty(\Omega;\mathbb{R}^3)$, letting $\varepsilon \to 0^+$ and $\vartheta \to 1^-$, this entails (3.36). Then, since $\widehat{\boldsymbol{F}}_h \to \boldsymbol{I}$ in $L^p(\Omega;\mathbb{R}^{3\times 3})$ by (3.32) and the assumption $\beta > p$, claim (3.37) follows by the Dominated Convergence Theorem.

Fix $\varepsilon > 0$ and $0 < \vartheta < 1$. Recall (3.41) and consider $h \leq \bar{h}(\varepsilon, \vartheta)$. We compute

$$\int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x} \leq \int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{\zeta}}_h - \widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x}. \tag{3.47}$$

By (3.44) and by the Sobolev embedding, for $h = h_\ell$, the second integral on the rigth-hand side of (3.47) goes to zero, as $\ell \to \infty$. Thus, we focus on the first one and we show that it goes to zero, as $h \to 0^+$.

Consider $\widehat{\boldsymbol{\zeta}}_h \in W^{1,2}(\Omega^\varepsilon_\vartheta;\mathbb{S}^2)$. Since, at least for $\varepsilon \ll 1$ and $1 - \vartheta \ll 1$, the set $\Omega^\varepsilon_\vartheta$ is a Lipschitz domain, this map admits an extension $\widehat{\boldsymbol{Z}}_h \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)$, possibly dependent on ε and ϑ, which satisfies

$$||\widehat{\boldsymbol{Z}}_h||_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)} \leq C(\varepsilon, \vartheta)\, ||\widehat{\boldsymbol{\zeta}}_h||_{W^{1,2}(\Omega^\varepsilon_\vartheta;\mathbb{R}^3)}.$$

In particular, recalling (3.43), we have

$$||\nabla \widehat{\boldsymbol{Z}}_h||_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \leq \left(\int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\varepsilon_\vartheta} |\nabla \widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} \right) \leq C(\varepsilon, \vartheta). \tag{3.48}$$

Define $\widehat{\boldsymbol{M}}_h \coloneqq \widehat{\boldsymbol{Z}}_h \circ \boldsymbol{\pi}_h^{-1}$. By construction, $\widehat{\boldsymbol{M}}_h|_{\Omega^\varepsilon_{\vartheta h}} = \widehat{\boldsymbol{m}}_h|_{\Omega^\varepsilon_{\vartheta h}}$ and, by (3.48) and the change-of-variable formula, there holds

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \widehat{\boldsymbol{M}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} &= \int_{\mathbb{R}^3} |\nabla_h \widehat{\boldsymbol{Z}}_h \circ \boldsymbol{\pi}_h^{-1}|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla \widehat{\boldsymbol{Z}}_h \circ \boldsymbol{\pi}_h^{-1}|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{h} \int_{\mathbb{R}^3} |\nabla \widehat{\boldsymbol{Z}}_h|^2 \,\mathrm{d}\boldsymbol{x} \leq \frac{C(\varepsilon, \vartheta)}{h}. \end{aligned} \tag{3.49}$$

Let $\lambda > 0$. By the Lusin-type property of Sobolev maps [1], there exists a measurable set $F_{\lambda,h} \subset \mathbb{R}^3$ such that $\widehat{\boldsymbol{M}}_h|_{F_{\lambda,h}}$ is Lipschitz-continuous with constant $C\lambda > 0$, that is

$$\forall\, \boldsymbol{\xi}, \widehat{\boldsymbol{\xi}} \in F_{\lambda,h}, \quad |\widehat{\boldsymbol{M}}_h(\boldsymbol{\xi}) - \widehat{\boldsymbol{M}}_h(\widehat{\boldsymbol{\xi}})| \leq C\lambda\, |\boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}|. \tag{3.50}$$

Moreover, the measure of the complement of the set $F_{\lambda,h}$ is controlled as follows

$$\mathscr{L}^3(\mathbb{R}^3 \setminus F_{\lambda,h}) \leq \frac{C}{\lambda^2} \int_{|\nabla \widehat{\boldsymbol{M}}_h| \geq \lambda/2} |\nabla \widehat{\boldsymbol{M}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \frac{C(\varepsilon, \vartheta)}{\lambda^2 h}, \tag{3.51}$$

where we used (3.49).

Going back to the first integral on the right-hand side of (3.47), using the change-of-variable formula, we compute

$$\begin{aligned} \int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} &= \frac{1}{h} \int_{\Omega^\varepsilon_{\vartheta h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{h} \int_{\Omega^\varepsilon_{\vartheta h} \cap F_{\lambda,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &\quad + \frac{1}{h} \int_{\Omega^\varepsilon_{\vartheta h} \setminus F_{\lambda,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi}. \end{aligned} \tag{3.52}$$

Thanks to (3.51) and to the uniform boundedness of the integrand, the second integral on the right-hand side of (3.52) is simply estimated by

$$\frac{1}{h} \int_{\Omega^\varepsilon_{\vartheta h} \setminus F_{\lambda,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \frac{C}{h} \mathscr{L}^3(\mathbb{R}^3 \setminus F_{\lambda,h}) \leq \frac{C(\varepsilon, \vartheta)}{\lambda^2 h^2}. \tag{3.53}$$

For the first integral on the right-hand side of (3.52), using (3.50), the change-of-variable formula, the Poincaré inequality and (3.32) with $a = 2$, we compute

$$\begin{aligned} \frac{1}{h} \int_{\Omega^\varepsilon_{\vartheta h} \cap F_{\lambda,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} &= \frac{1}{h} \int_{\Omega^\varepsilon_{\vartheta h} \cap F_{\lambda,h}} |\widehat{\boldsymbol{M}}_h \circ \widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \widehat{\boldsymbol{M}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &\leq \frac{C\lambda^2}{h} \int_{\Omega^\varepsilon_{\vartheta h}} |\widehat{\boldsymbol{y}}_h \circ \boldsymbol{\pi}_h^{-1} - \boldsymbol{id}|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &= C\lambda^2 \int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h|^2 \,\mathrm{d}\boldsymbol{x} \\ &\leq C\lambda^2 h^{\beta-2}. \end{aligned} \tag{3.54}$$

Therefore, combining (3.52)–(3.54), we obtain

$$\int_{\Omega^\varepsilon_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} \leq C\lambda^2 h^{\beta-2} + C(\varepsilon, \vartheta)\lambda^{-2} h^{-2}.$$

As, taking $\lambda = h^{-\alpha}$ with $1 < \alpha < \beta/2 - 1$, the right-hand side goes to zero, as $h \to 0^+$, this concludes the proof of the claim. Note that such a α always exists thanks to the assumption $\beta > 6 > 4$. □

We now move to the the proof of the lower bound. For convenience, we highlight the results regarding the convergence of the magnetostatic energy. This has already been proved in [8, Proposition 4.7] by adapting the results in [11, 22]. For convenience of the reader, we briefly sketch the proof and we refer to the first paper for details. Recall the notation in (2.20).

Proposition 3.6 (Convergence of the magnetostatic energy). *Let $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ be with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h)$. Suppose that there exists $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S; \mathbb{S}^2)$ such that the following convergence holds, as $h \to 0^+$:*

$$\widehat{\boldsymbol{\eta}}_h := (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \to \widehat{\boldsymbol{\eta}} \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3), \text{ where } \widehat{\boldsymbol{\eta}} := \chi_\Omega \widehat{\boldsymbol{\zeta}}. \tag{3.55}$$

Then, the following equality holds:

$$E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}) = \lim_{h \to 0^+} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h). \tag{3.56}$$

Proof. Denote by $\widehat{\psi}_h \in V^{1,2}(\mathbb{R}^3)$ a stray field potential corresponding to $\widehat{\boldsymbol{q}}_h$. Thus, we have the following:

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla \widehat{\psi}_h \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi}. \tag{3.57}$$

By (2.21), there holds

$$||\nabla \widehat{\psi}_h||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq ||\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.$$

Then, taking the square at both sides and applying and the change-of-variable formula, we compute

$$\int_{\mathbb{R}^3} |\nabla \widehat{\psi}_h|^2 \, \mathrm{d}\boldsymbol{\xi} \leq \int_{\mathbb{R}^3} |\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h|^2 \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\eta}}_h|^2 \circ \boldsymbol{\pi}_h^{-1} \, \mathrm{d}\boldsymbol{\xi} = h \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\eta}}_h|^2 \, \mathrm{d}\boldsymbol{\xi}. \tag{3.58}$$

Define $\widehat{\mu}_h \coloneqq \widehat{\psi}_h \circ \boldsymbol{\pi}_h \in V^{1,2}(\mathbb{R}^3)$. From (3.58), using again the change-of-variable formula, we deduce

$$\int_{\mathbb{R}^3} |\nabla_h \widehat{\mu}_h|^2 \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} |\nabla \widehat{\psi}_h|^2 \circ \boldsymbol{\pi}_h \, \mathrm{d}\boldsymbol{x} = \frac{1}{h} \int_{\mathbb{R}^3} |\nabla \widehat{\psi}_h|^2 \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\eta}}_h|^2 \, \mathrm{d}\boldsymbol{\xi}.$$

As the right-hand side is uniformly bounded by (3.55), we deduce that $(\nabla_h \widehat{\mu}_h)$ is bounded in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. From this, we deduce two facts. First, there exists $\widehat{\nu} \in L^2(\mathbb{R}^3)$ such that, up to subsequences, $\partial_3 \widehat{\mu}_h / h \rightharpoonup \widehat{\nu}$ in $L^2(\mathbb{R}^3)$ and, in turn, $\partial_3 \widehat{\mu}_h \to 0$ in $L^2(\mathbb{R}^3)$, as $h \to 0^+$. Second, exploiting the Hilbert space structure of the quotient $V^{1,2}(\mathbb{R}^3)/\mathbb{R}$, we prove the existence of $\widehat{\mu} \in V^{1,2}(\mathbb{R}^3)$ such that, up to subsequences, there holds $\nabla \widehat{\mu}_h \rightharpoonup \nabla \widehat{\mu}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$, as $h \to 0^+$. These two facts together imply that $\partial_3 \widehat{\mu} = 0$ almost everywhere which, as $\nabla \widehat{\mu} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, yields $\nabla' \widehat{\mu} = \boldsymbol{0}'$ almost everywhere.

Now, testing again (3.57) with $\varphi = \widehat{\psi}_h$ and applying the change-of-variable formula, we write

$$\begin{aligned} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h) &= \frac{1}{2h} \int_{\mathbb{R}^3} \chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h \cdot \nabla \widehat{\psi}_h \, \mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \widehat{\boldsymbol{\eta}}_h \cdot \nabla_h \widehat{\mu}_h \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \widehat{\boldsymbol{\eta}}'_h \cdot \nabla' \widehat{\mu}_h \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathbb{R}^3} \widehat{\boldsymbol{\eta}}^3_h \, \frac{\partial_3 \widehat{\mu}_h}{h} \, \mathrm{d}\boldsymbol{x} \end{aligned}$$

From this, passing to the limit, we obtain

$$\lim_{h \to 0^+} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h) = \frac{1}{2} \int_\Omega \widehat{\zeta}^{\,3} \, \widehat{\nu} \, \mathrm{d}\boldsymbol{x}. \tag{3.59}$$

Thus, if we show that $\widehat{\nu} = \chi_\Omega \widehat{\zeta}^{\,3}$ almost everywhere, then (3.56) follows from (3.59). To check this, we go back to (3.57). Applying the change-of-variable formula, we deduce the following

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} \nabla_h \widehat{\mu}_h \cdot \nabla_h \varphi \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \widehat{\boldsymbol{\eta}}_h \cdot \nabla_h \varphi \, \mathrm{d}\boldsymbol{x}.$$

From this, multiplying by h and then passing to the limit, as $h \to 0^+$, we obtain

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} (\widehat{\nu} - \chi_\Omega \widehat{\zeta}^{\,3}) \, \partial_3 \varphi \, \mathrm{d}\boldsymbol{x} = 0.$$

Given the arbitrariness of φ, this entails that the function $\widehat{\nu} - \chi_\Omega \widehat{\zeta}^{\,3}$ does not depend on the third variable. However, as this function belongs to $L^2(\mathbb{R}^3)$, we necessarily have $\widehat{\nu} - \chi_\Omega \widehat{\zeta}^{\,3} = 0$ almost everywhere. ☐

The next result asserts the existence of a lower bound and, for future reference, it is presented in a more self-contained form.

Proposition 3.7 (Lower bound). *Let $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ be with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h)$. For every $h > 0$, set $\widehat{\boldsymbol{F}}_h \coloneqq \nabla_h \widehat{\boldsymbol{y}}_h$. Suppose that there exist a sequence $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S; SO(3))$ and maps $\widehat{\boldsymbol{G}} \in L^2(\Omega;\mathbb{R}^{3\times 3})$, $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{v} \in W^{2,2}(S)$ such that, as $h \to 0^+$, we have*

$$\widehat{\boldsymbol{G}}_h \coloneqq h^{-\beta/2} (\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h - \boldsymbol{I}) \rightharpoonup \widehat{\boldsymbol{G}} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{3.60}$$

and, for almost every $\boldsymbol{x} \in \Omega$, there holds

$$\widehat{\boldsymbol{G}}''(\boldsymbol{x}) = \mathrm{sym} \nabla \widehat{\boldsymbol{u}}(\boldsymbol{x}') + ((\nabla')^2 \widehat{v}(\boldsymbol{x}')) \, x_3. \tag{3.61}$$

Suppose also that there exist $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ *and* $\widehat{\boldsymbol{\nu}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ *such that the following convergences hold, as* $h \to 0^+$*:*

$$\widehat{\boldsymbol{\eta}}_h := (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \to \widehat{\boldsymbol{\eta}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3), \text{ where } \widehat{\boldsymbol{\eta}} := \chi_\Omega \widehat{\boldsymbol{\zeta}}; \tag{3.62}$$

$$\widehat{\boldsymbol{H}}_h := (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \nabla \widehat{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \rightharpoonup \widehat{\boldsymbol{H}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}), \text{ where } \widehat{\boldsymbol{H}} := \chi_\Omega (\nabla' \widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\nu}}); \tag{3.63}$$

$$\widehat{\boldsymbol{z}}_h := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \text{ in } L^1(\Omega;\mathbb{R}^3). \tag{3.64}$$

Then, the following inequality holds:

$$E_0(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \le \liminf_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{3.65}$$

Proof. We only have prove the following:

$$E_0^{\mathrm{el}}(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \le \liminf_{h\to 0^+} E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h); \tag{3.66}$$

$$E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}) \le \liminf_{h\to 0^+} E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h). \tag{3.67}$$

Indeed, thanks to (3.62), the limit in (3.56) holds by Proposition 3.6. Thus, combining (3.66)–(3.67) with (3.56), we establish (3.65).

We first focus on (3.66). This is proved similarly to [21, Corollary 2]. Recall (2.2)–(2.4) and, for simplicity, set $\widehat{\boldsymbol{\lambda}}_h := \widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h$ and $\widehat{\boldsymbol{K}}_h := \mathcal{K}_h(\widehat{\boldsymbol{F}}_h, \widehat{\boldsymbol{\lambda}}_h)$. Assumption (3.64) yields

$$\widehat{\boldsymbol{L}}_h := h^{-\beta/2}(\boldsymbol{I} - \widehat{\boldsymbol{K}}_h^{-1}) \to \widehat{\boldsymbol{L}} \text{ in } L^q(\Omega;\mathbb{R}^{3\times 3}) \text{ for every } 1 \le q < \infty, \text{ where } \widehat{\boldsymbol{L}} := \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}}. \tag{3.68}$$

Define $A_h := \{|\widehat{\boldsymbol{G}}_h| \le h^{-\beta/6}\}$, so that, by (3.60), $\chi_{A_h} \to 1$ in $L^1(\Omega)$. Note that, on A_h, there holds

$$\sqrt{\widehat{\boldsymbol{F}}_h^\top \widehat{\boldsymbol{F}}_h} = \sqrt{(\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h)^\top (\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h)} = \boldsymbol{I} + h^{\beta/2} \mathrm{sym}\, \widehat{\boldsymbol{G}}_h + O(h^{2\beta/3}). \tag{3.69}$$

Moreover, recalling (2.11), for $h \gg 1$ we have $h^{\beta/2}|\widehat{\boldsymbol{G}}_h| < \delta_\Phi$ on A_h. Using (3.69), we write

$$\begin{aligned}
\int_\Omega W_h(\widehat{\boldsymbol{F}}_h, \widehat{\boldsymbol{\lambda}}_h)\,\mathrm{d}\boldsymbol{x} &\ge \int_\Omega \chi_{A_h} W_h(\widehat{\boldsymbol{F}}_h, \widehat{\boldsymbol{\lambda}}_h)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \chi_{A_h} \Phi\left(\sqrt{\widehat{\boldsymbol{F}}_h^\top \widehat{\boldsymbol{F}}_h}\, \widehat{\boldsymbol{K}}_h^{-1}\right)\mathrm{d}\boldsymbol{x} \\
&\ge \int_\Omega \chi_{A_h} \Phi\left((\boldsymbol{I} + h^{\beta/2}\mathrm{sym}\,\widehat{\boldsymbol{G}}_h + O(h^{2\beta/3}))(\boldsymbol{I} - h^{\beta/2}\widehat{\boldsymbol{L}}_h)\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \chi_{A_h} \Phi\left(\boldsymbol{I} + h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3})\right)\mathrm{d}\boldsymbol{x} \\
&= \frac{h^\beta}{2}\int_\Omega Q_\Phi\left(\chi_{A_h}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{\beta/6})\right)\mathrm{d}\boldsymbol{x} \\
&+ \int_\Omega \chi_{A_h}\omega_\Phi\left(h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3})\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Thus

$$\begin{aligned}
E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h) &\ge \frac{1}{2}\int_\Omega Q_\Phi\left(\chi_{A_h}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{\beta/6})\right)\mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{h^\beta}\int_\Omega \chi_{A_h}\omega_\Phi\left(h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3})\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.70}$$

For the first integral on the right-hand side of (3.70), we exploit the convexity of the quadratic form Q_Φ. By (3.60) and (3.68), we have $\chi_{A_h}\mathrm{sym}\,\widehat{\boldsymbol{G}}_h \rightharpoonup \widehat{\boldsymbol{G}}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\chi_{A_h}\widehat{\boldsymbol{L}}_h \to \widehat{\boldsymbol{L}}$ in $L^1(\Omega;\mathbb{R}^{3\times 3})$, as $h \to 0^+$. Thus, by lower semicontinuity, we get

$$\begin{aligned}
\liminf_{h\to 0^+}\int_\Omega Q_\Phi\left(\chi_{A_h}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{\beta/6})\right)\mathrm{d}\boldsymbol{x} &\ge \int_\Omega Q_\Phi(\widehat{\boldsymbol{G}} - \widehat{\boldsymbol{L}})\,\mathrm{d}\boldsymbol{x} \\
&\ge \int_\Omega Q_\Phi^{\mathrm{red}}(\widehat{\boldsymbol{G}}'' - \widehat{\boldsymbol{L}}'')\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Thanks to equality in (3.115), this proves (3.66) once we show that the second integral on the right-hand side of (3.70) goes to zero, as $h \to 0^+$. To prove this, observe that

$$\begin{aligned}
&\left| \frac{1}{h^\beta} \int_\Omega \chi_{A_h} \omega_\Phi \left(h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3}) \right) \mathrm{d}\boldsymbol{x} \right| \\
&\qquad \leq \int_\Omega \frac{\left| \omega_\Phi \left(h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3}) \right) \right|}{|h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3})|^2} \, \frac{\chi_{A_h} |h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3})|^2}{h^\beta} \, \mathrm{d}\boldsymbol{x} \\
&\qquad \leq C \int_\Omega \frac{\left| \omega_\Phi \left(h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3}) \right) \right|}{|h^{\beta/2}(\mathrm{sym}\,\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{L}}_h) + O(h^{2\beta/3})|^2} \, \mathrm{d}\boldsymbol{x},
\end{aligned}$$

where the right-hand side goes to zero, as $h \to 0^+$, by the Dominated Convergence Theorem.

Then, we prove (3.67). Recalling (3.45), this follows immediately from (3.63). Indeed, by lower semicontinuity, we have

$$\begin{aligned}
\liminf_{h \to 0^+} E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h) &= \liminf_{h \to 0^+} \int_{\mathbb{R}^3} |\widehat{\boldsymbol{H}}_h|^2 \, \mathrm{d}\boldsymbol{x} \geq \int_{\mathbb{R}^3} |\widehat{\boldsymbol{H}}|^2 \, \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega |\nabla' \widehat{\boldsymbol{\zeta}}|^2 \, \mathrm{d}\boldsymbol{x} + \int_\Omega |\widehat{\boldsymbol{\nu}}|^2 \, \mathrm{d}\boldsymbol{x} \geq \int_S |\nabla' \widehat{\boldsymbol{\zeta}}|^2 \, \mathrm{d}\boldsymbol{x}'.
\end{aligned}$$

□

The existence of recovery sequences in ensured by the following result.

Proposition 3.8 (Recovery sequence). *Let $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$, $\widehat{v} \in W^{2,2}(S)$ and $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$. Then, there exists $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h)$ such that the following convergences hold, as $h \to 0^+$:*

$$\widehat{\boldsymbol{y}}_h \to \boldsymbol{\pi}_0 \ \text{in } W^{1,p}(\Omega;\mathbb{R}^3); \tag{3.71}$$

$$\widehat{\boldsymbol{F}}_h := \nabla_h \widehat{\boldsymbol{y}}_h \to \boldsymbol{I} \ \text{in } L^p(\Omega;\mathbb{R}^{3\times 3}); \tag{3.72}$$

$$\widehat{\boldsymbol{u}}_h := \mathcal{U}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{u}} \ \text{in } W^{1,2}(S;\mathbb{R}^3); \tag{3.73}$$

$$\widehat{v}_h := \mathcal{V}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{v} \ \text{in } W^{2,2}(S); \tag{3.74}$$

$$\widehat{\boldsymbol{w}}_h := \mathcal{W}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{w}} \ \text{in } W^{2,2}(S), \text{ where } \widehat{\boldsymbol{w}} := -\frac{1}{12} \begin{pmatrix} \nabla' \widehat{v} \\ 0 \end{pmatrix}; \tag{3.75}$$

$$\widehat{\boldsymbol{\eta}}_h := (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \to \widehat{\boldsymbol{\eta}} \ \text{in } L^q(\mathbb{R}^3;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty, \text{ where } \widehat{\boldsymbol{\eta}} := \chi_\Omega \widehat{\boldsymbol{\zeta}}; \tag{3.76}$$

$$\widehat{\boldsymbol{H}}_h := (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \nabla \widehat{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \to \widehat{\boldsymbol{H}} \ \text{in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}), \text{ where } \widehat{\boldsymbol{H}} := \chi_\Omega (\nabla' \widehat{\boldsymbol{\zeta}}, \boldsymbol{0}); \tag{3.77}$$

$$\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \to \widehat{\boldsymbol{\zeta}} \ \text{in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty; \tag{3.78}$$

$$\widehat{\boldsymbol{z}}_h := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \ \text{in } L^q(\Omega;\mathbb{R}^{3\times 3}) \text{ for every } 1 \leq q < \infty. \tag{3.79}$$

Moreover, the following equality holds:

$$E_0(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) = \lim_{h \to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{3.80}$$

Proof. For convenience of the reader, the proof is subdivided into three steps.

Step 1 (Approximation of the limiting state). By definition of Q_Φ^{red}, there exist $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \colon S \to \mathbb{R}^3$ such that

$$Q_\Phi^{\mathrm{red}}(\mathrm{sym}\,\nabla' \widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\zeta}}' \otimes \widehat{\boldsymbol{\zeta}}') = Q_\Phi(\widehat{\boldsymbol{\Lambda}} - \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}}), \qquad Q_\Phi^{\mathrm{red}}((\nabla')^2 \widehat{v}) = Q_\Phi(\widehat{\boldsymbol{\Theta}}), \tag{3.81}$$

where we set

$$\widehat{\boldsymbol{\Lambda}} := \left(\begin{array}{c|c} \mathrm{sym} \nabla' \widehat{\boldsymbol{u}} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + \widehat{\boldsymbol{a}} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{a}}, \qquad \widehat{\boldsymbol{\Theta}} := -\left(\begin{array}{c|c} (\nabla')^2 \widehat{v} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + \widehat{\boldsymbol{b}} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{b}}.$$

In particular, thanks to (2.12), we have $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \in L^2(S;\mathbb{R}^3)$. Let $(\widehat{\boldsymbol{u}}_j) \subset C^1(\overline{S};\mathbb{R}^2)$, $(\widehat{v}_j) \subset C^2(\overline{S})$ and $(\widehat{\boldsymbol{a}}_j), (\widehat{\boldsymbol{b}}_j) \subset C^1(\overline{S};\mathbb{R}^3)$ be such that the following convergences hold, as $j \to \infty$:

$$\widehat{\boldsymbol{u}}_j \to \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2), \qquad \widehat{v}_j \to \widehat{v} \text{ in } W^{2,2}(S), \tag{3.82}$$

$$\widehat{\boldsymbol{a}}_j \to \widehat{\boldsymbol{a}} \text{ in } L^2(S;\mathbb{R}^3), \qquad \widehat{\boldsymbol{b}}_j \to \widehat{\boldsymbol{b}} \text{ in } L^2(S;\mathbb{R}^3). \tag{3.83}$$

If we set

$$\widehat{\boldsymbol{\Lambda}}_j \coloneqq \left(\begin{array}{c|c} \mathrm{sym}\nabla'\widehat{\boldsymbol{u}}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{a}}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{a}}_j, \qquad \widehat{\boldsymbol{\Theta}}_j \coloneqq -\left(\begin{array}{c|c} (\nabla')^2\widehat{v}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{b}}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{b}}_j,$$

then, by (3.82)–(3.83), we immediately have

$$\widehat{\boldsymbol{\Lambda}}_j \to \widehat{\boldsymbol{\Lambda}} \text{ in } L^2(S;\mathbb{R}^{3\times 3}), \quad \widehat{\boldsymbol{\Theta}}_j \to \widehat{\boldsymbol{\Theta}} \text{ in } L^2(S;\mathbb{R}^{3\times 3}), \tag{3.84}$$

as $j \to \infty$. Moreover, by [25, Theorem 2.1], there exists $(\widehat{\boldsymbol{\zeta}}_j) \subset C^1(\overline{S};\mathbb{S}^2)$ such that

$$\widehat{\boldsymbol{\zeta}}_j \to \widehat{\boldsymbol{\zeta}} \text{ in } W^{1,2}(S;\mathbb{R}^3), \tag{3.85}$$

as $j \to \infty$. Thus, setting $\widehat{\boldsymbol{L}}_j \coloneqq \widehat{\boldsymbol{\zeta}}_j \otimes \widehat{\boldsymbol{\zeta}}_j$ and $\widehat{\boldsymbol{L}} \coloneqq \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}}$, there holds

$$\widehat{\boldsymbol{L}}_j \to \widehat{\boldsymbol{L}} \text{ in } L^2(S;\mathbb{R}^{3\times 3}), \tag{3.86}$$

as $j \to \infty$.

Step 2 (Construction of recovery sequences). Fix $j \in \mathbb{N}$. Deformations of the recovery sequence are constructed according con the classical ansatz of the linearized von Kármán regime [21]. For every $h > 0$, we define

$$\widehat{\boldsymbol{y}}_h^{(j)} \coloneqq \boldsymbol{\pi}_h + h^{\beta/2} \begin{pmatrix} \widehat{\boldsymbol{u}}_j \\ 0 \end{pmatrix} + h^{\beta/2-1} \begin{pmatrix} \boldsymbol{0}' \\ \widehat{v}_j \end{pmatrix} - h^{\beta/2} x_3 \begin{pmatrix} \nabla'\widehat{v}_j \\ 0 \end{pmatrix} + 2h^{\beta/2+1} x_3 \widehat{\boldsymbol{a}}_j + h^{\beta/2+1} x_3^2 \widehat{\boldsymbol{b}}_j. \tag{3.87}$$

Applying [12, Theorem 5.5-1] as in the proof of [8, Proposition 5.1], we show that, for $h \ll 1$ depending on j, the map $\widehat{\boldsymbol{y}}_h^{(j)}$ is everywhere injective.

Set $\widehat{\boldsymbol{F}}_h^{(j)} \coloneqq \nabla_h \widehat{\boldsymbol{y}}_h^{(j)}$. We compute

$$\begin{aligned} \widehat{\boldsymbol{F}}_h^{(j)} &= \boldsymbol{I} + h^{\beta/2} \left(\begin{array}{c|c} \nabla'\widehat{\boldsymbol{u}}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + h^{\beta/2-1} \left(\begin{array}{c|c} \boldsymbol{O}'' & -\nabla'\widehat{v}_j \\ \hline \nabla' v_j^\top & 0 \end{array}\right) - h^{\beta/2} x_3 \left(\begin{array}{c|c} (\nabla')^2\widehat{v}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) \\ &\quad + 2h^{\beta/2} \widehat{\boldsymbol{a}}_j \otimes \boldsymbol{e}_3 + 2h^{\beta/2} x_3 \widehat{\boldsymbol{b}}_j \otimes \boldsymbol{e}_3 + O(h^{\beta/2+1}). \end{aligned} \tag{3.88}$$

Recall the identity $(\boldsymbol{I}+\boldsymbol{F})^\top(\boldsymbol{I}+\boldsymbol{F}) = \boldsymbol{I} + 2\mathrm{sym}\,\boldsymbol{F} + \boldsymbol{F}^\top\boldsymbol{F}$ for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$. Thanks to the assumption $\beta > 6$, we obtain

$$\sqrt{\left(\widehat{\boldsymbol{F}}_h^{(j)}\right)^\top \widehat{\boldsymbol{F}}_h^{(j)}} = \boldsymbol{I} + h^{\beta/2}(\widehat{\boldsymbol{\Lambda}}_j + x_3\,\widehat{\boldsymbol{\Theta}}_j) + O(h^{\beta/2+1}), \tag{3.89}$$

where

$$\widehat{\boldsymbol{\Lambda}}_j \coloneqq \left(\begin{array}{c|c} \mathrm{sym}\nabla'\widehat{\boldsymbol{u}}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{a}}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{a}}_j, \qquad \widehat{\boldsymbol{\Theta}}_j \coloneqq -\left(\begin{array}{c|c} (\nabla')^2\widehat{v}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{b}}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{b}}_j.$$

From (3.89), using the expansion of the determinant close to the identity, we deduce

$$\det \widehat{\boldsymbol{F}}_h^{(j)} = 1 + O(h^{\beta/2}).$$

This entails $\det \widehat{\boldsymbol{F}}_h^{(j)} > 0$ for $h \ll 1$ and, in particular, $\det \nabla \widehat{\boldsymbol{y}}_h^{(j)} > 0$ almost everywhere. Thus, $\widehat{\boldsymbol{y}}_h^{(j)} \in \mathcal{Y}$. By (3.87), we have the estimate

$$||\widehat{\boldsymbol{y}}_h^{(j)} - \boldsymbol{\pi}_h||_{C^0(\overline{\Omega};\mathbb{R}^3)} \le C(j) h^{\beta/2-1},$$

where the constant $C(j) > 0$ depends on the maxima of the maps in (3.82)–(3.83). As $\beta/2 - 1 > 1$ by the assumption $\beta > 6$, arguing exactly as in Step 1 of the proof of Proposition 3.5, we establish the following:

$$\forall\, \varepsilon > 0,\ \forall\, 0 < \vartheta < 1,\ \exists\, \bar{h}(\varepsilon,\vartheta,j) > 0 : \forall\, 0 < h \leq \bar{h}(\varepsilon,\vartheta,j),\ \Omega^{\varepsilon}_{\vartheta h} \subset \Omega^{\widehat{\boldsymbol{y}}_h^{(j)}}, \tag{3.90}$$

$$\forall\, \varepsilon > 0,\ \forall\, \gamma > 1,\ \exists\, \underline{h}(\varepsilon,\gamma,j) > 0 : \forall\, 0 < h \leq \underline{h}(\varepsilon,\gamma,j),\ \Omega^{\widehat{\boldsymbol{y}}_h^{(j)}} \subset \Omega^{-\varepsilon}_{\gamma h}. \tag{3.91}$$

Let $\widehat{\boldsymbol{Z}}_j \in C^1(\mathbb{R}^2;\mathbb{R}^3)$ be an extension of $\widehat{\boldsymbol{\zeta}}_j$. As $|\widehat{\boldsymbol{\zeta}}_j| = 1$ on S, by continuity, there exists an open set $V_j \subset\subset \mathbb{R}^2$ with $S \subset\subset V_j$ such that $|\widehat{\boldsymbol{Z}}_j| > 1/2$ on V_j. By (3.91), there holds $\Omega^{\widehat{\boldsymbol{y}}_h^{(j)}} \subset V_j \times \mathbb{R}$ for $h \ll 1$ depending on j. Thus, we define

$$\widehat{\boldsymbol{m}}_h^{(j)} \coloneqq \left.\frac{\widehat{\boldsymbol{Z}}_j}{|\widehat{\boldsymbol{Z}}_j|}\right|_{\Omega^{\widehat{\boldsymbol{y}}_h^{(j)}}},$$

so that $\widehat{\boldsymbol{m}}_h \in C^1\left(\Omega^{\widehat{\boldsymbol{y}}_h};\mathbb{S}^2\right)$. In particular, this gives $\widehat{\boldsymbol{q}}_h^{(j)} \coloneqq (\widehat{\boldsymbol{y}}_h^{(j)}, \widehat{\boldsymbol{m}}_h^{(j)}) \in \mathcal{Q}$.
Now, the following convergences hold, as $h \to 0^+$:

$$\widehat{\boldsymbol{y}}_h^{(j)} \to \boldsymbol{\pi}_0 \text{ in } W^{1,p}(\Omega;\mathbb{R}^3); \tag{3.92}$$

$$\widehat{\boldsymbol{F}}_h^{(j)} \coloneqq \nabla_h \widehat{\boldsymbol{y}}_h^{(j)} \to \boldsymbol{I} \text{ in } L^p(\Omega;\mathbb{R}^{3\times 3}); \tag{3.93}$$

$$\widehat{\boldsymbol{u}}_h^{(j)} \coloneqq \mathcal{U}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \widehat{\boldsymbol{u}}_j \text{ in } W^{1,2}(S;\mathbb{R}^3); \tag{3.94}$$

$$\widehat{v}_h^{(j)} \coloneqq \mathcal{V}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \widehat{v}_j \text{ in } W^{2,2}(S); \tag{3.95}$$

$$\widehat{\boldsymbol{\eta}}_h^{(j)} \coloneqq (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h^{(j)}}} \widehat{\boldsymbol{m}}_h^{(j)}) \circ \boldsymbol{\pi}_h \to \widehat{\boldsymbol{\eta}}_j \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty, \text{ where } \widehat{\boldsymbol{\eta}}_j \coloneqq \chi_\Omega \widehat{\boldsymbol{\zeta}}_j; \tag{3.96}$$

$$\widehat{\boldsymbol{H}}_h^{(j)} \coloneqq (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h^{(j)}}} \nabla \widehat{\boldsymbol{m}}_h^{(j)}) \circ \boldsymbol{\pi}_h \to \widehat{\boldsymbol{H}}_j \text{ in } L^2(\Omega;\mathbb{R}^3), \text{ where } \widehat{\boldsymbol{H}}_j \coloneqq \chi_\Omega(\nabla' \widehat{\boldsymbol{\zeta}}_j, \boldsymbol{0}); \tag{3.97}$$

$$\widehat{\boldsymbol{m}}_h^{(j)} \circ \widehat{\boldsymbol{y}}_h^{(j)} \to \widehat{\boldsymbol{\zeta}}_j \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty; \tag{3.98}$$

$$\widehat{\boldsymbol{z}}_h^{(j)} \coloneqq \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \widehat{\boldsymbol{\zeta}}_j \text{ in } L^q(\Omega;\mathbb{R}^{3\times 3}) \text{ for every } 1 \leq q < \infty. \tag{3.99}$$

Claims (3.92)–(3.95) are checked by direct computation. Claim (3.98) follows from (3.92) and the continuity of $\widehat{\boldsymbol{Z}}_j$ by applying the Dominated Convergence Theorem. Then, (3.99) is obtained from (3.93) and (3.98) applying once more the Dominated Convergence Theorem. Observe that, by (3.90)–(3.91), there holds

$$\chi_{\boldsymbol{\pi}_h^{-1}\left(\Omega^{\boldsymbol{y}_h^{(j)}}\right)} \to \chi_\Omega \text{ almost everywhere,}$$

as $h \to 0^+$. From this, taking into account the definition of $\widehat{\boldsymbol{m}}_h^{(j)}$ and applying again the Dominated Convergence Theorem, we prove (3.96)–(3.97).

We now move to the convergence of the energy. Recall (2.2)–(2.4). Set $\widehat{\boldsymbol{K}}_h^{(j)} \coloneqq \mathcal{K}_h\left(\widehat{\boldsymbol{F}}_h^{(j)}, \widehat{\boldsymbol{\lambda}}_h^{(j)}\right)$, where $\widehat{\boldsymbol{\lambda}}_h^{(j)} \coloneqq \widehat{\boldsymbol{m}}_h^{(j)} \circ \widehat{\boldsymbol{y}}_h^{(j)}$. By (3.99) and the Dominated Convergence Theorem, we have

$$\widehat{\boldsymbol{L}}_h^{(j)} \coloneqq h^{-\beta/2}\left(\boldsymbol{I} - \left(\widehat{\boldsymbol{K}}_h^{(j)}\right)^{-1}\right) \to \widehat{\boldsymbol{L}}_j \text{ in } L^q(\Omega;\mathbb{R}^{3\times 3}) \text{ for every } 1 \leq q < \infty, \tag{3.100}$$

as $h \to 0^+$. Recall (3.89). Thus

$$\sqrt{\left(\widehat{\boldsymbol{F}}_h^{j}\right)^\top \widehat{\boldsymbol{F}}_h^{(j)}}\left(\widehat{\boldsymbol{K}}_h^{(j)}\right)^{-1} = \boldsymbol{I} + h^{\beta/2}\left(\widehat{\boldsymbol{\Lambda}}_j - \widehat{\boldsymbol{L}}_h^{(j)} + x_3\widehat{\boldsymbol{\Theta}}_j\right) + O(h^{\beta/2+1}).$$

Using (2.11), we compute

$$\begin{aligned} E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h^{(j)}) &= \frac{1}{h^\beta}\int_\Omega \Phi\left(\sqrt{\left(\widehat{\boldsymbol{F}}_h^{j}\right)^\top \widehat{\boldsymbol{F}}_h^{(j)}}\left(\widehat{\boldsymbol{K}}_h^{(j)}\right)^{-1}\right)\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\int_\Omega Q_\Phi\left(\widehat{\boldsymbol{\Lambda}}_j - \widehat{\boldsymbol{L}}_h^{(j)} + x_3\widehat{\boldsymbol{\Theta}}_j + O(h)\right)\mathrm{d}\boldsymbol{x} \\ &+ \frac{1}{h^\beta}\int_\Omega \omega_\Phi\left(h^{\beta/2}\left(\widehat{\boldsymbol{\Lambda}}_j - \widehat{\boldsymbol{L}}_h^{(j)} + x_3\widehat{\boldsymbol{\Theta}}_j\right) + O(h^{\beta/2+1})\right)\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Thanks to (3.100), applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned} \lim_{h\to 0^+} E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h^{(j)}) &= \frac{1}{2}\int_\Omega Q_\Phi(\widehat{\boldsymbol{\Lambda}}_j - \widehat{\boldsymbol{L}}_j + x_3\widehat{\boldsymbol{\Theta}}_j)\,\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\int_S Q_\Phi(\widehat{\boldsymbol{\Lambda}}_j - \widehat{\boldsymbol{L}}_j)\,\mathrm{d}\boldsymbol{x} + \frac{1}{24}\int_S Q_\Phi(\widehat{\boldsymbol{\Theta}}_j)\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.101}$$

From (3.97), using the change-of-variable formula, we deduce

$$\lim_{h\to 0^+} E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h^{(j)}) = \lim_{h\to 0^+}\left\|\widehat{\boldsymbol{H}}_h^{(j)}\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} = \left\|\widehat{\boldsymbol{H}}_j\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} = E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}_j). \tag{3.102}$$

Exploiting (3.96) and applying Proposition 3.6, we obtain

$$\lim_{h\to 0^+} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h^{(j)}) = E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}_j). \tag{3.103}$$

Step 3 (Diagonal argument). To conclude the proof, we employ a standard diagonal argument. First, from (3.84) and (3.86), we see that

$$\lim_{j\to\infty}\int_S Q_\Phi(\widehat{\boldsymbol{\Lambda}}_j - \widehat{\boldsymbol{L}}_j)\,\mathrm{d}\boldsymbol{x}' = \int_S Q_\Phi(\widehat{\boldsymbol{\Lambda}} - \widehat{\boldsymbol{L}})\,\mathrm{d}\boldsymbol{x}', \quad \lim_{j\to\infty}\int_S Q_\Phi(\widehat{\boldsymbol{\Theta}}_j)\,\mathrm{d}\boldsymbol{x}' = \int_S Q_\Phi(\widehat{\boldsymbol{\Theta}})\,\mathrm{d}\boldsymbol{x}', \tag{3.104}$$

while (3.85) immediately gives

$$\lim_{j\to\infty} E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}_j) = E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}), \qquad \lim_{j\to\infty} E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}_j) = E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}). \tag{3.105}$$

In view of (3.82)–(3.83), (3.85), (3.92)–(3.97) and (3.101)–(3.105), we select a subsequence (h_j) such that, setting $\widehat{\boldsymbol{q}}_{h_j} := (\widehat{\boldsymbol{y}}_{h_j}, \widehat{\boldsymbol{m}}_{h_j})$ with $\widehat{\boldsymbol{y}}_{h_j} := \widehat{\boldsymbol{y}}_{h_j}^{(j)}$ and $\widehat{\boldsymbol{m}}_{h_j} := \widehat{\boldsymbol{m}}_{h_j}^{(j)}$, the convergences in (3.71)–(3.77) hold for $h = h_j$, as $j\to\infty$, as well as

$$\lim_{j\to\infty} E_{h_j}^{\mathrm{el}}(\widehat{\boldsymbol{q}}_{h_j}) = \frac{1}{2}\int_S Q_\Phi(\widehat{\boldsymbol{\Lambda}} - \widehat{\boldsymbol{\zeta}}\otimes\widehat{\boldsymbol{\zeta}})\,\mathrm{d}\boldsymbol{x}' + \frac{1}{24}\int_S Q_\Phi(\widehat{\boldsymbol{\Theta}})\,\mathrm{d}\boldsymbol{x}' \tag{3.106}$$

and

$$\lim_{j\to\infty} E_{h_j}^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_{h_j}) = E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}), \qquad \lim_{j\to\infty} E_{h_j}^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_{h_j}) = E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}). \tag{3.107}$$

As the right-hand side of (3.106) equals $E_0^{\mathrm{el}}(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}})$ by (3.81), combining (3.106)–(3.107) we deduce (3.80). □

Remark 3.9 (Recovery sequence under clamped boundary conditions). Following the notation of Proposition 3.8, suppose that $\widehat{\boldsymbol{u}} \in W_0^{1,2}(S;\mathbb{R}^3)$ and $\widehat{v} \in W_0^{2,2}(S)$. In this case, the deformations of recovery sequence can be constructed to satisfy the clamped boundary condition $\widehat{\boldsymbol{y}}_h = \boldsymbol{\pi}_h$ on $\partial S \times I$ for every $h > 0$. To prove this, we argue exactly as in the proof of the Proposition but we take $(\widehat{\boldsymbol{u}}_j) \subset C_c^1(S;\mathbb{R}^2)$, $(\widehat{v}_j) \subset C_c^2(S)$ and $(\widehat{\boldsymbol{a}}_j), (\widehat{\boldsymbol{b}}_j) \subset C_c^1(S;\mathbb{R}^3)$. This observation is going to be exploited in the proof of Theorem 3.10.

We are finally ready to prove our first main result.

Proof of Theorem 3.1. We have to prove just the first part, since the second part has already been proved in Proposition 3.8. Recall (2.2)–(2.4). For simplicity, set

$$\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}_h, \quad \boldsymbol{\lambda}_h \coloneqq \boldsymbol{m}_h \circ \boldsymbol{y}_h, \quad \boldsymbol{K}_h \coloneqq \mathcal{K}_h(\boldsymbol{F}_h, \boldsymbol{\lambda}_h), \quad \boldsymbol{Y}_h \coloneqq \sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h} \boldsymbol{K}_h^{-1}.$$

Let $a \in \{2, p\}$. By (2.10) and (3.8), we deduce that $(\boldsymbol{Y}_h)$ is bounded in $L^a(\Omega;\mathbb{R}^{3\times 3})$. Then, by the uniform boundedness of $(\boldsymbol{K}_h)$, we obtain

$$||\boldsymbol{F}_h||_{L^a(\Omega;\mathbb{R}^{3\times3})} = \left\|\sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h}\right\|_{L^a(\Omega;\mathbb{R}^{3\times3})} = ||\boldsymbol{Y}_h \boldsymbol{K}_h||_{L^a(\Omega;\mathbb{R}^{3\times3})} \leq C\,||\boldsymbol{Y}_h||_{L^a(\Omega;\mathbb{R}^{3\times3})} \leq C. \tag{3.108}$$

As $|\boldsymbol{I} - \boldsymbol{K}_h^{-1}| \leq Ch^{\beta/2}$ by (2.4), there holds

$$\begin{aligned} \operatorname{dist}(\boldsymbol{F}_h; SO(3)) &= \operatorname{dist}\left(\sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h}; SO(3)\right) \\ &\leq \left|\sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h} - \boldsymbol{Y}_h\right| + \operatorname{dist}(\boldsymbol{Y}_h; SO(3)) \\ &\leq \left|\sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h}\right| \, |\boldsymbol{I} - \boldsymbol{K}_h^{-1}| + \operatorname{dist}(\boldsymbol{Y}_h; SO(3)) \\ &\leq Ch^{\beta/2} \left|\sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h}\right| + \operatorname{dist}(\boldsymbol{Y}_h; SO(3)). \end{aligned} \tag{3.109}$$

Given (3.108) and (3.109), assumption (2.8) yields

$$\begin{aligned} \int_\Omega \operatorname{dist}^a(\boldsymbol{F}_h; SO(3))\,\mathrm{d}\boldsymbol{x} &\leq Ch^{a\beta/2} \left\|\sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h}\right\|^a_{L^a(\Omega;\mathbb{R}^{3\times3})} + \int_\Omega \operatorname{dist}^a(\boldsymbol{Y}_h; SO(3))\,\mathrm{d}\boldsymbol{x} \\ &\leq Ch^\beta + Ch^\beta E^{\mathrm{el}}_h(\boldsymbol{q}_h) \\ &\leq Ch^\beta, \end{aligned} \tag{3.110}$$

where, in the last line, we employed (3.8). Recall the notation in (3.17). We apply Lemma 3.2 to each $\boldsymbol{y}_h$. This gives two sequences $(\boldsymbol{R}_h) \subset W^{1,p}(S; SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ satisfying

$$||\boldsymbol{F}_h - \boldsymbol{R}_h||_{L^a(\Omega;\mathbb{R}^{3\times3})} \leq CR_h(\boldsymbol{y}_h)^{1/a}, \qquad ||\nabla' \boldsymbol{R}_h||_{L^a(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}, \tag{3.111}$$

$$||\boldsymbol{R}_h - \boldsymbol{Q}_h||_{L^a(s;\mathbb{R}^{3\times3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}, \qquad ||\boldsymbol{F}_h - \boldsymbol{Q}_h||_{L^a(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}. \tag{3.112}$$

Now, denote by $\boldsymbol{c}_h \in \mathbb{R}^3$ the average of $\boldsymbol{Q}_h^\top \boldsymbol{y}_h - \boldsymbol{\pi}_h$ over Ω and consider the rigid motion $\boldsymbol{T}_h \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{T}_h(\boldsymbol{\xi}) \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{\xi} - \boldsymbol{c}_h$ for every $\boldsymbol{\xi} \in \mathbb{R}^3$. Set $\widetilde{\boldsymbol{y}}_h \coloneqq \boldsymbol{T}_h \circ \boldsymbol{y}_h$ and note that $\widetilde{\boldsymbol{y}}_h - \boldsymbol{\pi}_h$ has null average over Ω by the choice of $\boldsymbol{c}_h$. From (3.111)–(3.112), setting $\widetilde{\boldsymbol{R}}_h \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{R}_h$, we immediately deduce

$$||\widetilde{\boldsymbol{F}}_h - \widetilde{\boldsymbol{R}}_h||_{L^a(\Omega;\mathbb{R}^{3\times3})} \leq CR_h(\boldsymbol{y}_h)^{1/a}, \qquad ||\nabla' \widetilde{\boldsymbol{R}}_h||_{L^a(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}, \tag{3.113}$$

$$||\widetilde{\boldsymbol{R}}_h - \boldsymbol{I}||_{L^a(s;\mathbb{R}^{3\times3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}, \qquad ||\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}||_{L^a(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}. \tag{3.114}$$

In view of (3.110), there holds $R_h(\boldsymbol{y}_h) \leq Ch^\beta$. Thus, thanks to the assumption $\beta > 6 > 2$, we are in a position to apply Proposition 3.3 to $\widehat{\boldsymbol{y}}_h = \widetilde{\boldsymbol{y}}_h$ with $r_h = R_h(\boldsymbol{y}_h)$ and $e_h = h^\beta$. Therefore, there exist $\widetilde{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widetilde{v} \in W^{2,2}(S)$ such that, up to subsequences, (3.9)–(3.10) hold. From these convergences and from the first estimate in (3.113), applying Lemma 3.4 to $\widehat{\boldsymbol{y}}_h = \widetilde{\boldsymbol{y}}_h$, we see that there exists $\widetilde{\boldsymbol{G}} \in L^2(\Omega;\mathbb{R}^{3\times3})$ such that, up to subsequences, we have

$$\widetilde{\boldsymbol{G}}_h \coloneqq h^{-\beta/2}(\widetilde{\boldsymbol{R}}_h^\top \widetilde{\boldsymbol{F}}_h - \boldsymbol{I}) \rightharpoonup \widetilde{\boldsymbol{G}} \text{ in } L^2(\Omega;\mathbb{R}^{3\times3}), \tag{3.115}$$

as $h \to 0^+$, and, for almost every $\boldsymbol{x} \in \Omega$, there holds

$$\widetilde{\boldsymbol{G}}''(\boldsymbol{x}', x_3) = \operatorname{sym}\nabla\widetilde{\boldsymbol{u}}(\boldsymbol{x}') + ((\nabla')^2\widetilde{v}(\boldsymbol{x}'))x_3. \tag{3.116}$$

Define $\widetilde{\boldsymbol{m}}_h \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{m}_h \circ \boldsymbol{T}_h^{-1}$ and set $\widetilde{\boldsymbol{q}}_h = (\widetilde{\boldsymbol{y}}_h, \widetilde{\boldsymbol{m}}_h) \in \mathcal{Q}$. Exploiting the second estimate in (3.114), we apply Proposition 3.5 to $\widehat{\boldsymbol{q}}_h = \widetilde{\boldsymbol{q}}_h$. Thus, there exist $\widetilde{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ and $\widetilde{\boldsymbol{\nu}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\widetilde{\boldsymbol{\eta}}_h := (\chi_{\Omega^{\tilde{\boldsymbol{y}}_h}} \widetilde{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \to \widetilde{\boldsymbol{\eta}} \text{ in } L^q(\mathbb{R}^3;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty, \text{ where } \widetilde{\boldsymbol{\eta}} := \chi_\Omega \widetilde{\boldsymbol{\zeta}}; \tag{3.117}$$

$$\widetilde{\boldsymbol{H}}_h := (\chi_{\Omega^{\tilde{\boldsymbol{y}}_h}} \nabla \widetilde{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \rightharpoonup \widetilde{\boldsymbol{H}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}), \text{ where } \widetilde{\boldsymbol{H}} := \chi_\Omega (\nabla' \widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\nu}}); \tag{3.118}$$

Moreover, up to subsequences, (3.11) holds. From this and(3.115)–(3.118), by applying Proposition 3.7 to $\widehat{\boldsymbol{q}}_h = \widetilde{\boldsymbol{q}}_h$, we obtain

$$E_0(\widetilde{\boldsymbol{u}}.\widetilde{v}, \widetilde{\boldsymbol{\zeta}}) \leq \liminf_{h\to 0^+} E_h(\widetilde{\boldsymbol{q}}_h),$$

which, in view of Remark 2.1, gives (3.12). □

3.2. **Convergence of almost minimizers.** Henceforth, we consider applied loads determined by body forces and by an external magnetic field. For simplicity, applied surface forces are excluded. However, these can be easily included in the analysis. According to the assumption of *dead loads*, the work of mechanical forces is described by a Lagrangian term. Conversely, the energy contribution corresponding to the external magnetic field, usually called *Zeeman energy*, is of Eulerian type.

Given $h > 0$, let $\boldsymbol{f}_h \in L^2(S;\mathbb{R}^2)$, $\boldsymbol{g}_h \in L^2(S)$ and $\boldsymbol{h}_h \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ represent an horizontal force, a vertical force and an external magnetic field, respectively. The work of applied loads is determined by the functional $L_h \colon \mathcal{Q} \to \mathbb{R}$ defined by

$$L_h(\boldsymbol{y},\boldsymbol{m}) := \frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{h^\beta}\int_\Omega g_h\, y^3\,\mathrm{d}\boldsymbol{x} + \frac{1}{h}\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}_h \cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}, \tag{3.119}$$

so that the total energy $F_h \colon \mathcal{Q} \to \mathbb{R}$ reads

$$F_h(\boldsymbol{y},\boldsymbol{m}) := E_h(\boldsymbol{y},\boldsymbol{m}) - L_h(\boldsymbol{y},\boldsymbol{m}). \tag{3.120}$$

Regarding the asymptotic behaviour of the applied loads, we assume that there exist $\boldsymbol{f} \in L^2(S;\mathbb{R}^2)$, $g \in L^2(S)$ and $\boldsymbol{h} \in L^2(\mathbb{R}^3;\mathbb{R}^2)$ such that the following convergences hold, as $h \to 0^+$:

$$h^{-\beta/2}\boldsymbol{f}_h \rightharpoonup \boldsymbol{f} \text{ in } L^2(S;\mathbb{R}^2), \tag{3.121}$$

$$h^{-\beta/2-1} g_h \rightharpoonup g \text{ in } L^2(S), \tag{3.122}$$

$$\boldsymbol{h}_h \circ \boldsymbol{\pi}_h \rightharpoonup \chi_I \boldsymbol{h} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3). \tag{3.123}$$

We stress that the limiting magnetic field $\boldsymbol{h}$ is a priori assumed not to depend on the variable x_3.

The limiting total energy $F_0 \colon W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2) \to \mathbb{R}$ is defined as

$$F_0(\boldsymbol{u}, v, \boldsymbol{\zeta}) := E_0(\boldsymbol{u}, v, \boldsymbol{\zeta}) - L_0(\boldsymbol{u}, v, \boldsymbol{\zeta}), \tag{3.124}$$

where the functional $L_0 \colon W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2) \to \mathbb{R}$ is given by

$$L_0(\boldsymbol{u}, v, \boldsymbol{\zeta}) := \int_S \boldsymbol{f} \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x}' + \int_S g\, v\,\mathrm{d}\boldsymbol{x}' + \int_S \boldsymbol{h} \cdot \boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}'. \tag{3.125}$$

Note that the limiting total energy is purely Lagrangian.

Additionally, we henceforth impose some Dirichlet boundary conditions. For simplicity, we consider clamped boundary conditions by restricting ourselves to the class of admissible states

$$\mathcal{Q}_h := \left\{ (\boldsymbol{y},\boldsymbol{m}) : \ \boldsymbol{y} \in \mathcal{Y}_h,\ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2) \right\},$$

where, recalling (2.16), we set

$$\mathcal{Y}_h := \{ \boldsymbol{y} \in \mathcal{Y} : \ \boldsymbol{y} = \boldsymbol{\pi}_h \text{ on } \partial S \times I \}. \tag{3.126}$$

Accordingly, limiting admissible states belong to the class

$$\mathcal{Q}_0 := W^{1,2}_0(S;\mathbb{R}^2) \times W^{2,2}_0(S) \times W^{1,2}(S;\mathbb{S}^2). \tag{3.127}$$

However, as explained in Remark 3.11, more general Dirichlet boundary conditions can be considered.

Our second main result claims that, under the boundary conditions in (3.126), almost minimizers of the sequence (F_h) in (3.120) converge, as $h \to 0^+$, to minimizers of the energy F_0 in (3.124) in the class $\mathcal{Q}_0$.

Theorem 3.10 (Convergence of almost minimizers). *Assume $p > 3$ and $\beta > 6 \vee p$. Suppose that the elastic energy density W_h has the form in (2.2), where the function Φ satisfies (2.7)–(2.9), and that the applied loads satisfy (3.121)-(3.123). Let $(\boldsymbol{q}_h) \subset \mathcal{Q}$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \mathcal{Q}_h$ for every $h > 0$ be such that*

$$\lim_{h \to 0^+} \left\{ F_h(\boldsymbol{q}_h) - \inf_{\mathcal{Q}_h} F_h \right\} = 0. \tag{3.128}$$

Then, there exist $\boldsymbol{u} \in W^{1,2}_0(S;\mathbb{R}^2)$, $v \in W^{2,2}_0(S)$ and $\boldsymbol{\zeta} \in W^{1,2}(S;\mathbb{S}^2)$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{u}_h \coloneqq \mathcal{U}_h(\boldsymbol{q}_h) \rightharpoonup \boldsymbol{u} \text{ in } W^{1,2}(S;\mathbb{R}^3); \tag{3.129}$$

$$v_h \coloneqq \mathcal{V}_h(\boldsymbol{q}_h) \to v \text{ in } W^{1,2}(S); \tag{3.130}$$

$$\boldsymbol{z}_h \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h) \to \boldsymbol{\zeta} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{3.131}$$

Moreover, $(\boldsymbol{u}, v, \boldsymbol{\zeta}) \in \mathcal{Q}_0$ is a minimizer of F_0 in $\mathcal{Q}_0$.

We mention that the weak convergence in (3.129) can be improved to strong convergence by arguing as in [21, Subsection 7.2].

Remark 3.11 (More general boundary conditions). In Theorem 3.10, more general Dirichlet boundary conditions, like the ones in [32], can be considered. Precisely, let $\overline{\boldsymbol{u}} \in W^{1,\infty}(S;\mathbb{R}^2)$ and $\overline{v} \in W^{2,\infty}(S)$. For $h > 0$, let the deformation $\overline{\boldsymbol{y}}_h \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ be defined as

$$\overline{\boldsymbol{y}}_h \coloneqq \boldsymbol{\pi}_h + h^{\beta/2} \begin{pmatrix} \overline{\boldsymbol{u}} \\ 0 \end{pmatrix} + h^{\beta/2-1} \begin{pmatrix} \boldsymbol{0}' \\ \overline{v} \end{pmatrix} - h^{\beta/2} x_3 \begin{pmatrix} \nabla' \overline{v} \\ 0 \end{pmatrix}.$$

If $\Gamma \subset \partial S$ is given by a finite union of closed maximal connected subsets of ∂S with non-empty interior with respect to the topology of ∂S [32, Formula (16)], then Theorem 3.10 still holds by replacing the class $\mathcal{Y}_h$ in (3.126) with the set

$$\{ \boldsymbol{y} \in \mathcal{Y} : \ \boldsymbol{y} = \overline{\boldsymbol{y}}_h \text{ on } \Gamma \times I \}.$$

Accordingly, the limiting class $\mathcal{Q}_0$ in (3.127) needs to be replaced by the set

$$\left\{ (\boldsymbol{u}, v, \boldsymbol{\zeta}) \in W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2) : \ \boldsymbol{u} = \overline{\boldsymbol{u}} \text{ on } \Gamma, \ v = \overline{v} \text{ on } \Gamma, \ \nabla' v = \nabla' \overline{v} \text{ on } \Gamma \right\}.$$

The convergence of almost minimizers is proved more or less in the same way. The main changes concern the construction of recovery sequences, as we need to approximate the limiting averaged displacements $\boldsymbol{u}$ and v with regular maps satisfying the boundary conditions above. This is achieved by appealing to [20, Proposition A.2], which requires the above mentioned regularity of Γ. In our case of clamped boundary condition on the whole ∂S, this issue is easily solved as explained in Remark 3.9.

Remark 3.12 (Existence of minimizers for the reduced model). The existence of minimizer of F_0 in $\mathcal{Q}_0$ is a consequence of Theorem 3.10. However, under our assumptions, this can be established directly. First, note that the functional F_0 is lower semicontinuous with respect to the product weak topology in view of the convexity of Q^{red}_Φ. Thus, in order to apply the Direct Method, one only has to show that the functional F_0 is coercive on $\mathcal{Q}_0$. This is done by exploiting the positive definiteness of Q^{red}_Φ on symmetric matrices in (3.6) and applying Korn and Poincaré inequalities in view of the boundary conditions in (3.126).

Observe that in (3.129)–(3.131) compactness is obtained without composing with rigid motions. This improved compactness is due to the boundary conditions imposed in (3.126). To see this, we need a preliminary result which is inspired by [32, Lemma 13].

Lemma 3.13 (Clamped boundary conditions). *Let $(\boldsymbol{y}_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ with $\boldsymbol{y}_h = \boldsymbol{\pi}_h$ on $\partial S \times I$ for every $h > 0$ and let $(r_h) \subset \mathbb{R}$ with $r_h > 0$ for every $h > 0$ be such that $r_h \leq Ch$ for every $h > 0$ and $r_h/h^2 \to 0$, as $h \to 0^+$. For every $h > 0$, set $\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}_h$ and suppose that there exist $(\boldsymbol{R}_h) \subset W^{1,2}(S;SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ satisfying*

$$||\boldsymbol{F}_h - \boldsymbol{R}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, \qquad ||\nabla' \boldsymbol{R}_h||_{L^2(S;\mathbb{R}^{3\times 3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \tag{3.132}$$

$$||\boldsymbol{R}_h - \boldsymbol{Q}_h||_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \qquad ||\boldsymbol{F}_h - \boldsymbol{Q}_h||_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}. \tag{3.133}$$

Then, for every $h>0$, there holds

$$|\boldsymbol{Q}_h-\boldsymbol{I}|\le Ch^{-1}\sqrt{r_h} \tag{3.134}$$

and, in turn, we have

$$||\boldsymbol{R}_h-\boldsymbol{I}||_{L^2(S;\mathbb{R}^{3\times 3})}\le Ch^{-1}\sqrt{r_h}, \qquad ||\boldsymbol{F}_h-\boldsymbol{I}||_{L^2(S;\mathbb{R}^{3\times 3})}\le Ch^{-1}\sqrt{r_h}. \tag{3.135}$$

Moreover, denoting by $\boldsymbol{c}_h\in\mathbb{R}^3$ the average of $\boldsymbol{Q}_h^\top\boldsymbol{y}_h-\boldsymbol{\pi}_h$ over Ω, for every $h>0$, there holds

$$|\boldsymbol{c}_h|\le Ch^{-1}\sqrt{r_h}. \tag{3.136}$$

Proof. We first prove (3.134). Then, (3.133) and (3.134) immediately give (3.135). Similarly to the proof of Theorem 3.1, define $\widetilde{\boldsymbol{y}}_h:=\boldsymbol{Q}_h^\top\boldsymbol{y}_h-\boldsymbol{c}_h$ with $\boldsymbol{c}_h\in\mathbb{R}^3$ chosen so that $\widetilde{\boldsymbol{y}}_h-\boldsymbol{\pi}_h$ has null average over Ω. Set $\widetilde{\boldsymbol{F}}_h:=\nabla_h\widetilde{\boldsymbol{y}}_h$ and $\widetilde{\boldsymbol{R}}_h:=\boldsymbol{Q}_h^\top\boldsymbol{R}_h$. Assumptions (3.132)–(3.133) immediately yield

$$||\widetilde{\boldsymbol{F}}_h-\widetilde{\boldsymbol{R}}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})}\le C\sqrt{r_h}, \qquad ||\nabla'\widetilde{\boldsymbol{R}}_h||_{L^2(S;\mathbb{R}^{3\times 3\times 3})}\le Ch^{-1}\sqrt{r_h},$$
$$||\widetilde{\boldsymbol{R}}_h-\boldsymbol{I}||_{L^2(S;\mathbb{R}^{3\times 3})}\le Ch^{-1}\sqrt{r_h}, \qquad ||\widetilde{\boldsymbol{F}}_h-\boldsymbol{I}||_{L^2(S;\mathbb{R}^{3\times 3})}\le Ch^{-1}\sqrt{r_h}.$$

Therefore, we are in a position to apply Proposition 3.3 to $\widehat{\boldsymbol{y}}_h=\widetilde{\boldsymbol{y}}_h$ with $e_h=r_h$. Define $\widetilde{\boldsymbol{u}}_h\colon S\to\mathbb{R}^2$, $\widetilde{v}_h:S\to\mathbb{R}$ and $\widetilde{\boldsymbol{w}}_h\colon S\to\mathbb{R}^3$ by setting

$$\begin{aligned}
\widetilde{\boldsymbol{u}}_h(\boldsymbol{x}')&:=\frac{h^2}{r_h}\wedge\frac{1}{\sqrt{r_h}}\int_I(\widetilde{\boldsymbol{y}}_h'(\boldsymbol{x}',x_3)-\boldsymbol{x}')\,\mathrm{d}x_3,\\
\widetilde{v}_h(\boldsymbol{x}')&:=\frac{h}{\sqrt{r_h}}\int_I\widetilde{y}_h^3(\boldsymbol{x}',x_3)\,\mathrm{d}x_3,\\
\widetilde{\boldsymbol{w}}_h(\boldsymbol{x}')&:=\frac{1}{\sqrt{r_h}}\int_I x_3(\widetilde{\boldsymbol{y}}_h-\boldsymbol{\pi}_h)\,\mathrm{d}x_3,
\end{aligned}$$

for every $\boldsymbol{x}'\in S$. Thus, there exist $\widetilde{\boldsymbol{u}}\in W^{1,2}(S;\mathbb{R}^2)$ and $\widetilde{v}\in W^{2,2}(S)$ such that, up to subsequences, the following convergences hold, as $h\to 0^+$:

$$\widetilde{\boldsymbol{u}}_h\rightharpoonup\widetilde{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{3.137}$$

$$\widetilde{v}_h\to\widetilde{v} \text{ in } W^{1,2}(S); \tag{3.138}$$

$$\widetilde{\boldsymbol{w}}_h\rightharpoonup\widetilde{\boldsymbol{w}} \text{ in } W^{1,2}(S;\mathbb{R}^3), \text{ where } \widetilde{\boldsymbol{w}}:=-\frac{1}{12}\begin{pmatrix}\nabla'\widetilde{v}\\ 0\end{pmatrix}. \tag{3.139}$$

In particular, by the compactness of the trace operator, the traces of $\widetilde{\boldsymbol{u}}_h$, $\widetilde{v}_h$ and $\widetilde{\boldsymbol{w}}_h$ result uniformly bounded in the spaces $L^2(\partial S;\mathbb{R}^2)$, $L^2(\partial S)$ and $L^2(\partial S;\mathbb{R}^3)$, respectively.

Analogously, we define $\boldsymbol{u}_h\colon S\to\mathbb{R}^2$, $v_h\colon S\to\mathbb{R}$ and $\boldsymbol{w}_h\colon S\to\mathbb{R}^3$ by setting

$$\begin{aligned}
\boldsymbol{u}_h(\boldsymbol{x}')&:=\frac{h^2}{r_h}\wedge\frac{1}{\sqrt{r_h}}\int_I(\boldsymbol{y}_h'(\boldsymbol{x}',x_3)-\boldsymbol{x}')\,\mathrm{d}x_3,\\
v_h(\boldsymbol{x}')&:=\frac{h}{\sqrt{r_h}}\int_I y_h^3(\boldsymbol{x}',x_3)\,\mathrm{d}x_3,\\
\boldsymbol{w}_h(\boldsymbol{x}')&:=\frac{1}{\sqrt{r_h}}\int_I x_3(\boldsymbol{y}_h-\boldsymbol{\pi}_h)\,\mathrm{d}x_3,
\end{aligned}$$

for every $\boldsymbol{x}'\in S$. Note that all the maps $\boldsymbol{u}_h$, v_h and $\boldsymbol{w}_h$ satisfy homogeneous Dirichlet boundary conditions.

For simplicity, set $\boldsymbol{d}_h:=\boldsymbol{Q}_h\boldsymbol{c}_h$. We compute

$$\begin{pmatrix}h^{-2}r_h\vee\sqrt{r_h}\,\boldsymbol{u}_h\\ h^{-1}\sqrt{r_h}\,v_h\end{pmatrix}=(\boldsymbol{Q}_h-\boldsymbol{I})\,\boldsymbol{\pi}_0+\boldsymbol{Q}_h\begin{pmatrix}h^{-2}r_h\vee\sqrt{r_h}\,\widetilde{\boldsymbol{u}}_h\\ h^{-1}\sqrt{r_h}\,\widetilde{v}_h\end{pmatrix}+\boldsymbol{d}_h, \tag{3.140}$$

$$\sqrt{r_h}\boldsymbol{w}_h=\frac{h}{12}(\boldsymbol{Q}_h-\boldsymbol{I})\boldsymbol{e}_3+\sqrt{r_h}\boldsymbol{Q}_h\,\widetilde{\boldsymbol{w}}_h. \tag{3.141}$$

From (3.141), taking the norm in $L^2(\partial S;\mathbb{R}^3)$, we see that

$$|(\boldsymbol{Q}_h-\boldsymbol{I})\boldsymbol{e}_3|\le Ch^{-1}\sqrt{r_h} \tag{3.142}$$

and, in turn, we also have

$$|(\boldsymbol{Q}_h^\top - \boldsymbol{I})\boldsymbol{e}_3| \leq Ch^{-1}\sqrt{r_h}. \tag{3.143}$$

From (3.140), looking at the first two components and taking the norm in $L^2(\partial S;\mathbb{R}^2)$, we obtain

$$||(\boldsymbol{Q}''_h - \boldsymbol{I}'')\boldsymbol{x}' + \boldsymbol{d}'_h||_{L^2(\partial S;\mathbb{R}^2)} \leq Ch^{-1}\sqrt{r_h}, \tag{3.144}$$

where we used the assumption $r_h \leq Ch$ for every $h > 0$. Up to translations, we can assume that

$$\int_{\partial S} \boldsymbol{x}' \,\mathrm{d}\boldsymbol{l} = \boldsymbol{0}'.$$

In this case, (3.144) yields

$$|\boldsymbol{Q}''_h - \boldsymbol{I}''| \leq Ch^{-1}\sqrt{r_h}, \qquad |\boldsymbol{d}'_h| \leq Ch^{-1}\sqrt{r_h}. \tag{3.145}$$

Combining (3.142)–(3.143) and the first estimate in (3.145), we obtain (3.134). Looking at the third component of (3.140) and taking the norm in $L^2(\partial S)$, we deduce

$$|d_h^3| \leq Ch^{-1}\sqrt{r_h}. \tag{3.146}$$

Thus, (3.136) follows from the second estimate in (3.145) and (3.146). □

We now prove the compactness of sequences of admissible states with equi-bounded energy under clamped boundary conditions. This result is going to be instrumental also in the quasistatic setting.

Proposition 3.14 (Compactness). *Let $(\boldsymbol{q}_h) \subset \mathcal{Q}$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \mathcal{Q}_h$ be such that*

$$\sup_{h>0} \left\{ E_h^{\mathrm{el}}(\boldsymbol{q}_h) + E_h^{\mathrm{mag}}(\boldsymbol{q}_h) \right\} \leq C. \tag{3.147}$$

Then, there exist maps $\boldsymbol{u} \in W_0^{1,2}(S;\mathbb{R}^2)$, $v \in W_0^{2,2}(S)$, $\boldsymbol{\zeta} \in W^{1,2}(S;\mathbb{S}^2)$ and $\boldsymbol{\nu} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{y}_h \to \boldsymbol{\pi}_0 \text{ in } W^{1,p}(\Omega;\mathbb{R}^3); \tag{3.148}$$

$$\boldsymbol{u}_h \coloneqq \mathcal{U}_h(\boldsymbol{q}_h) \rightharpoonup \boldsymbol{u} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{3.149}$$

$$v_h \coloneqq \mathcal{V}_h(\boldsymbol{q}_h) \to v \text{ in } W^{1,2}(S); \tag{3.150}$$

$$\boldsymbol{w}_h \coloneqq \mathcal{W}_h(\boldsymbol{q}_h) \to \boldsymbol{w} \text{ in } W^{1,2}(S;\mathbb{R}^3), \text{ where } \boldsymbol{w} \coloneqq -1/12\,((\nabla' v)^\top, 0)^\top; \tag{3.151}$$

$$\boldsymbol{\eta}_h \coloneqq \chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\boldsymbol{y}_h})} \boldsymbol{m}_h \circ \boldsymbol{\pi}_h \to \boldsymbol{\eta} \text{ in } L^q(\mathbb{R}^3;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty, \text{ where } \boldsymbol{\eta} \coloneqq \chi_\Omega \boldsymbol{\zeta}; \tag{3.152}$$

$$\boldsymbol{H}_h \coloneqq \chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\boldsymbol{y}_h})} \nabla \boldsymbol{m}_h \circ \boldsymbol{\pi}_h \rightharpoonup \boldsymbol{H} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}), \text{ where } \boldsymbol{H} \coloneqq \chi_\Omega(\nabla'\boldsymbol{\zeta}, \boldsymbol{\nu}); \tag{3.153}$$

$$\boldsymbol{m}_h \circ \boldsymbol{y}_h \to \boldsymbol{\zeta} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty; \tag{3.154}$$

$$\boldsymbol{z}_h \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h) \to \boldsymbol{\zeta} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{3.155}$$

Moreover, there exists $(\boldsymbol{R}_h) \subset W^{1,p}(S;SO(3))$ such that, setting $\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}_h$ for every $h > 0$, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{F}_h \to \boldsymbol{I} \text{ in } L^p(\Omega;\mathbb{R}^{3\times 3}); \tag{3.156}$$

$$\boldsymbol{A}_h \coloneqq h^{-\beta/2+1}(\boldsymbol{R}_h - \boldsymbol{I}) \rightharpoonup \boldsymbol{A} \text{ in } W^{1,2}(S;\mathbb{R}^{3\times 3}), \text{ where } \boldsymbol{A} \coloneqq \left(\begin{array}{c|c} \boldsymbol{O}'' & -\nabla' v \\ \hline (\nabla' v)^\top & 0 \end{array}\right). \tag{3.157}$$

In particular, for every $h > 0$, the following estimates hold:

$$||\boldsymbol{F}_h - \boldsymbol{R}_h||_{L^a(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{\beta/a}, \qquad ||\nabla'\boldsymbol{R}_h||_{L^a(S;\mathbb{R}^{3\times 3\times 3})} \leq Ch^{\beta/a-1}, \tag{3.158}$$

$$||\boldsymbol{R}_h - \boldsymbol{I}||_{L^a(S;\mathbb{R}^{3\times 3})} \leq Ch^{\beta/a-1}, \qquad ||\boldsymbol{F}_h - \boldsymbol{I}||_{L^a(S;\mathbb{R}^{3\times 3})} \leq Ch^{\beta/a-1}, \tag{3.159}$$

Additionally, for every $h > 0$, there hold:

$$||\boldsymbol{u}_h||_{W^{1,2}(S;\mathbb{R}^2)} \leq C\left(\sqrt{E_h^{\mathrm{el}}(\boldsymbol{q}_h)} + 1\right), \qquad ||v_h||_{W^{1,2}(S)} \leq C\left(\sqrt{E_h^{\mathrm{el}}(\boldsymbol{q}_h)} + 1\right). \tag{3.160}$$

Proof. From (3.147), arguing as in (3.108)–(3.110), we establish the existence of $(\boldsymbol{R}_h) \subset W^{1,p}(S;SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ such that (3.111)–(3.112) hold for $a \in \{2,p\}$. Moreover, we see that $R_h(\boldsymbol{y}_h) \leq Ch^\beta$. Then, applying Lemma 3.13 with $r_h = R_h(\boldsymbol{y}_h)$, we obtain

$$||\boldsymbol{R}_h - \boldsymbol{I}||_{L^a(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}, \qquad ||\boldsymbol{F}_h - \boldsymbol{I}||_{L^a(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}R_h(\boldsymbol{y}_h)^{1/a}. \tag{3.161}$$

Given the bound on $R_h(\boldsymbol{y}_h)$, (3.111) and (3.161) immediately yield (3.158)–(3.159). Now, (3.156) follows immediately by the second estimate in (3.159) with $a = p$ and the assumption $\beta > p$. Moreover, (3.156) yields (3.148) by the Poincaré inequality. Claims (3.149)–(3.151) and (3.157) are proved by applying Proposition 3.3 to $\widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h$ with $r_h = R_h(\boldsymbol{y}_h)$ and $e_h = h^\beta$ in view of (3.111) and (3.161). Note that the boundary conditions on $\boldsymbol{u}$ and v follow from (3.149)–(3.151) thanks to the weak continuity of the trace operator. Also, (3.160) are deduced from (3.23) since $R_h(\boldsymbol{y}_h)/h^\beta \leq C(1 + E_h^{\mathrm{el}}(\boldsymbol{q}_h))$ by (3.110). Finally, thanks to (3.147) and the second estimate in (3.159), claims (3.152)–(3.155) follow by applying Proposition 3.5 to $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h$. □

The major difficulty in proving Theorem 3.10 is to deduce the equi-boundedness of the elastic energy starting from the equi-boundedness of the total energy. This is accomplished by arguing by contradiction, similarly to [32, Theorem 4].

Lemma 3.15 (Energy scaling). *Let $(\boldsymbol{q}_h) \subset \mathcal{Q}$ with $\boldsymbol{q}_h \in \mathcal{Q}_h$ be such that*

$$\sup_{h>0} F_h(\boldsymbol{q}_h) \leq C. \tag{3.162}$$

Then, there holds

$$\sup_{h>0} E_h(\boldsymbol{q}_h) \leq C. \tag{3.163}$$

Proof. For convenience, we introduce some further notation. Given $h > 0$ and $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, we set

$$I_h(\boldsymbol{q}) \coloneqq \int_\Omega W_h(\nabla_h \boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y})\,\mathrm{d}\boldsymbol{x},$$

$$J_h(\boldsymbol{q}) = I_h(\boldsymbol{q}) - \int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \int_\Omega g_h\, y^3\,\mathrm{d}\boldsymbol{x}.$$

First, we prove that $I_h(\boldsymbol{q}_h) \leq Ch^{\beta/2}$. Let $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h)$. Using (2.8) and (3.121) and applying Hölder and Poincaré inequalities, we compute

$$\begin{aligned} \frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}_h' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} &\leq \frac{1}{h^\beta}||\boldsymbol{f}_h||_{L^2(S;\mathbb{R}^2)}\,||\boldsymbol{y}_h - \boldsymbol{\pi}_h||_{L^2(\Omega;\mathbb{R}^3)} \\ &\leq \frac{C}{h^\beta}\left(||\nabla_h \boldsymbol{y}_h||_{L^2(\Omega;\mathbb{R}^{3\times 3})} + 1\right) \\ &\leq \frac{C}{h^\beta}\left(\sqrt{I_h(\boldsymbol{q}_h)} + 1\right). \end{aligned} \tag{3.164}$$

Analogously, exploiting (2.8) and (3.122), we obtain

$$\frac{1}{h^\beta}\int_\Omega g_h \cdot y_h^3\,\mathrm{d}\boldsymbol{x} \leq \frac{C}{h^\beta}\left(\sqrt{I_h(\boldsymbol{q}_h)} + 1\right). \tag{3.165}$$

By the change-of-variable formula, we have

$$\mathscr{L}^3(\boldsymbol{\pi}_h^{-1}(\Omega^{\boldsymbol{y}_h})) = \frac{1}{h}\mathscr{L}^3(\Omega^{\boldsymbol{y}_h}) = \frac{1}{h}\int_\Omega \det \nabla \boldsymbol{y}_h\,\mathrm{d}\boldsymbol{x} = \int_\Omega \det \nabla_h \boldsymbol{y}_h\,\mathrm{d}\boldsymbol{x}. \tag{3.166}$$

Recall (2.2)–(2.4) and set

$$\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}_h, \qquad \boldsymbol{\lambda}_h \coloneqq \boldsymbol{m}_h \circ \boldsymbol{y}_h, \qquad \boldsymbol{K}_h \coloneqq \mathcal{K}_h(\boldsymbol{F}_h, \boldsymbol{\lambda}_h), \qquad \boldsymbol{Y}_h \coloneqq \sqrt{\boldsymbol{F}_h^\top \boldsymbol{F}_h}\boldsymbol{K}_h^{-1}.$$

From (3.166), by (2.10), and the uniform boundedness of $(\boldsymbol{K}_h)$, applying the Young inequality we obtain

$$\begin{aligned}\mathscr{L}^3(\boldsymbol{\pi}_h^{-1}(\Omega^{\boldsymbol{y}_h})) &= \int_\Omega \det \boldsymbol{F}_h \,\mathrm{d}\boldsymbol{x} \leq C \int_\Omega |\boldsymbol{F}_h|^3 \,\mathrm{d}\boldsymbol{x} \leq C_1 \int_\Omega |\boldsymbol{F}_h|^p \,\mathrm{d}\boldsymbol{x} + C_2 \\ &\leq C_1 \int_\Omega |\boldsymbol{Y}_h|^p \,\mathrm{d}\boldsymbol{x} + C_2 \leq C_1 h^\beta E_h^{\mathrm{el}}(\boldsymbol{q}_h) + C_2 \leq C(I_h(\boldsymbol{q}_h)+1)\end{aligned} \tag{3.167}$$

Then, taking into account (3.123) and applying the change-of-variable formula and the Hölder inequality, we deduce

$$\begin{aligned}\frac{1}{h}\int_{\Omega^{\boldsymbol{y}_h}} \boldsymbol{h}_h \cdot \boldsymbol{m}_h \,\mathrm{d}\boldsymbol{\xi} &= \int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\boldsymbol{y}_h})} \boldsymbol{h}_h \circ \boldsymbol{\pi}_h \cdot \boldsymbol{m}_h \circ \boldsymbol{\pi}_h \,\mathrm{d}\boldsymbol{x} \\ &\leq ||\boldsymbol{h}_h \circ \boldsymbol{\pi}_h||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \sqrt{\mathscr{L}^3(\boldsymbol{\pi}_h^{-1}(\Omega^{\boldsymbol{y}_h}))} \\ &\leq C\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right).\end{aligned} \tag{3.168}$$

Combining (3.162) with (3.164)–(3.168) and using the Young inequality, we obtain

$$C \geq F_h(\boldsymbol{q}_h) \geq \frac{1}{h^\beta} I_h(\boldsymbol{q}_h) - \frac{C}{h^\beta}\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right) \geq \frac{1}{h^\beta}\left(C_1 I_h(\boldsymbol{q}_h) - C_2\right),$$

which gives $I_h(\boldsymbol{q}_h) \leq C$. This easily yields that $J_h(\boldsymbol{q}_h) \leq Ch^\beta$, since

$$h^{-\beta} J_h(\boldsymbol{q}_h) \leq F_h(\boldsymbol{q}_h) + \frac{1}{h}\int_{\Omega^{\boldsymbol{y}_h}} \boldsymbol{h}_h \cdot \boldsymbol{m}_h \,\mathrm{d}\boldsymbol{\xi} \leq C\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right) \leq C,$$

where we used (3.162) and (3.168). Moreover, combining (3.121)–(3.122) and (3.164)–(3.165), we get

$$\begin{aligned}I_h(\boldsymbol{q}_h) &= J_h(\boldsymbol{q}_h) + \int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}_h' - \boldsymbol{x}') \,\mathrm{d}\boldsymbol{x} + \int_\Omega g_h \cdot y_h^3 \,\mathrm{d}\boldsymbol{x} \\ &\leq Ch^\beta + Ch^{\beta/2} ||\boldsymbol{y}_h - \boldsymbol{\pi}_h||_{L^2(\Omega;\mathbb{R}^3)} \\ &\leq Ch^\beta + Ch^{\beta/2} \leq Ch^{\beta/2}.\end{aligned} \tag{3.169}$$

Now we claim that $I_h(\boldsymbol{q}_h) \leq Ch^\beta$. To prove this, we argue by contradiction. Let $e_h = I_h(\boldsymbol{q}_h)$ and suppose by contradiction that

$$\limsup_{h\to 0^+} \frac{e_h}{h^\beta} = +\infty. \tag{3.170}$$

Note that, by (3.169), there holds $e_h/h^2 \to 0$, as $h \to 0^+$, since $\beta > 6 > 4$. Let $r_h = R_h(\boldsymbol{y}_h)$, where we use the notation in (3.17). Recalling (2.8), we have $r_h \leq Ce_h$ so that we can apply Lemma 3.2 to each $\boldsymbol{y}_h$. Setting $\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}_h$, this gives $(\boldsymbol{R}_h) \subset W^{1,p}(S;SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ such that (3.132)–(3.133) hold. Thanks to Lemma 3.13, these imply (3.135). Thus, we are in a position to apply Proposition 3.3 to $\widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h$. Given $\boldsymbol{U}_h \colon S \to \mathbb{R}^2$ and $V_h \colon S \to \mathbb{R}$ defined by

$$\begin{aligned}\boldsymbol{U}_h(\boldsymbol{x}') &\coloneqq \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}} \int_I (\boldsymbol{y}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}')\,\mathrm{d}x_3, \\ V_h(\boldsymbol{x}') &\coloneqq \frac{h}{\sqrt{e_h}} \int_I y_h^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3,\end{aligned}$$

for every $\boldsymbol{x}' \in S$, there exist $\boldsymbol{U} \in W^{1,2}(S;\mathbb{R}^2)$ and $V \in W^{2,2}(S)$ such that

$$\begin{aligned}\boldsymbol{U}_h &\rightharpoonup \boldsymbol{U} \text{ in } W^{1,2}(S;\mathbb{R}^2), \\ V_h &\to V \text{ in } W^{1,2}(S).\end{aligned}$$

In particular, $(\boldsymbol{U}_h)$ is bounded in $W^{1,2}(S;\mathbb{R}^2)$ and (V_h) is bounded in $W^{1,2}(S)$. Exploiting (3.121)–(3.122), (3.170) and the assumption $\beta > 6 > 4$, we check that

$$\begin{aligned}&\lim_{h\to 0^+}\left\{\frac{1}{e_h}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}_h' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{e_h}\int_\Omega g_h y_h^3 \,\mathrm{d}\boldsymbol{x}\right\} \\ &\quad = \lim_{h\to 0^+}\left\{\frac{h^{\beta/2}}{e_h}\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\int_S h^{-\beta/2}\boldsymbol{f}_h \cdot \boldsymbol{U}_h \,\mathrm{d}\boldsymbol{x}' + \frac{h^{\beta/2}}{\sqrt{e_h}}\int_S h^{-\beta/2-1} g_h\, V_h \,\mathrm{d}\boldsymbol{x}'\right\} = 0.\end{aligned}$$

This yields

$$
\begin{aligned}
1 &= \lim_{h\to 0^+} \frac{1}{e_h} I_h(\boldsymbol{q}_h)\\
&= \lim_{h\to 0^+} \left\{ \frac{1}{e_h} J_h(\boldsymbol{q}_h) + \frac{1}{e_h}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}'_h - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{e_h}\int_\Omega g_h y^3_h \,\mathrm{d}\boldsymbol{x} \right\}\\
&= \lim_{h\to 0^+} \frac{1}{e_h} J_h(\boldsymbol{q}_h) \leq \lim_{h\to 0^+} \frac{Ch^\beta}{e_h} = 0,
\end{aligned}
$$

where we used (3.170). This provides the desired contradiction. Therefore, $I_h(\boldsymbol{q}_h) \leq Ch^\beta$ for $h \ll 1$.

At this point, we can apply once more Lemma 3.2 in combination with Lemma 3.13 and Proposition 3.3, this time with $e_h = h^\beta$. Thus, the maps $\boldsymbol{u}_h \coloneqq \mathcal{U}_h(\boldsymbol{q}_h)$ and $v_h \coloneqq \mathcal{V}_h(\boldsymbol{q}_h)$ are uniformly bounded in $W^{1,2}(S;\mathbb{R}^2)$ and $W^{1,2}(S)$, respectively. In particular, recalling (3.121)–(3.122), this gives

$$
\begin{aligned}
&\frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{y}'_h - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{h^\beta}\int_\Omega g_h\, y^3_h\,\mathrm{d}\boldsymbol{x}\\
&\qquad = \int_S h^{-\beta/2}\boldsymbol{f}_h\cdot\boldsymbol{u}_h\,\mathrm{d}\boldsymbol{x}' + \int_S h^{-\beta/2-1} g_h\, v_h\,\mathrm{d}\boldsymbol{x}'\\
&\qquad \leq ||h^{-\beta/2}\boldsymbol{f}_h||_{L^2(S;\mathbb{R}^2)}\,||\boldsymbol{u}_h||_{L^2(S;\mathbb{R}^2)} + ||h^{-\beta/2-1}g_h||_{L^2(S)}\,||v_h||_{L^2(S)} \leq C,
\end{aligned}
$$

which, together with (3.168), ensures that $L_h(\boldsymbol{q}_h) \leq C$ for $h \ll 1$. Finally, for every $h \ll 1$, we have

$$
E_h(\boldsymbol{q}_h) = F_h(\boldsymbol{q}_h) + L_h(\boldsymbol{q}_h) \leq C,
$$

which proves (3.163). □

We are now ready to prove our second main result.

Proof of Theorem 3.10. Given $h > 0$, let $\boldsymbol{n}_h \colon \Omega_h \to \mathbb{S}^2$ be constantly equal to some fixed $\boldsymbol{e} \in \mathbb{S}^2$, and set $\overline{\boldsymbol{q}}_h \coloneqq (\boldsymbol{\pi}_h, \boldsymbol{n}_h) \in \mathcal{Q}_h$. We claim that $F_h(\overline{\boldsymbol{q}}_h) \leq C$ and, in turn, $\inf_{\mathcal{Q}_h} F_h \leq C$. To see this, using (2.11), we compute

$$
\begin{aligned}
E^{\mathrm{el}}_h(\overline{\boldsymbol{q}}_h) &= \frac{1}{h^\beta}\int_\Omega \Phi\left(\boldsymbol{I} - \frac{h^{\beta/2}}{1+h^{\beta/2}}\boldsymbol{e}\otimes\boldsymbol{e}\right)\mathrm{d}\boldsymbol{x}\\
&= \frac{1}{2}\int_\Omega Q_\Phi\left(-\frac{1}{1+h^{\beta/2}}\boldsymbol{e}\otimes\boldsymbol{e}\right)\mathrm{d}\boldsymbol{x} + \frac{1}{h^\beta}\int_\Omega \omega_\Phi\left(-\frac{h^{\beta/2}}{1+h^{\beta/2}}\boldsymbol{e}\otimes\boldsymbol{e}\right)\mathrm{d}\boldsymbol{x} \leq C.
\end{aligned}
$$

Denote by $\overline{\psi}_h$ the stray field potential corresponding to $\overline{\boldsymbol{q}}_h$. By (2.21) and the change-of-variable formula, we have

$$
E^{\mathrm{mag}}_h(\overline{\boldsymbol{q}}_h) = \frac{1}{h}\int_{\Omega_h} |\nabla\overline{\psi}_h|^2\,\boldsymbol{\xi} \leq h^{-1}\mathscr{L}^3(\Omega_h) = \mathscr{L}^2(S).
$$

Thus, $E_h(\overline{\boldsymbol{q}}_h) \leq C$. Moreover, by (3.123) and the change-of-variable formula, there holds

$$
|L_h(\overline{\boldsymbol{q}}_h)| = \left|\frac{1}{h}\int_{\Omega_h}\boldsymbol{h}_h\cdot\boldsymbol{e}\,\mathrm{d}\boldsymbol{\xi}\right| = \int_\Omega |\boldsymbol{h}_h\circ\boldsymbol{\pi}_h|\,\mathrm{d}\boldsymbol{x} \leq C\,||\boldsymbol{h}_h\circ\boldsymbol{\pi}_h||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq C.
$$

Therefore, the claim is proved.

Thanks to (3.128), this yields $F_h(\boldsymbol{q}_h) \leq C$ for every $h > 0$. Then, by Lemma 3.15, for every $h > 0$, there holds $E_h(\boldsymbol{q}_h) \leq C$. By Proposition 3.14, there exist $\boldsymbol{u} \in W^{1,2}_0(S;\mathbb{R}^2)$ and $v \in W^{2,2}_0(S)$ such that (3.129)–(3.130) hold true. Moreover, there exists $\boldsymbol{\zeta} \in W^{1,2}(S;\mathbb{S}^2)$ and $\boldsymbol{\nu} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that (3.152)–(3.153) and (3.131) hold true. Set $\boldsymbol{F}_h \coloneqq \nabla_h \boldsymbol{y}_h$ for every $h > 0$. From (3.129)–(3.130) and from the first estimate in (3.158), applying Lemma 3.4 to $\widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h$ with $e_h = h^\beta$, we show that there exists a sequence $(\boldsymbol{R}_h) \subset W^{1,2}(S;\mathbb{R}^{3\times 3})$ and a map $\boldsymbol{G} \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that

$$
\boldsymbol{G}_h \coloneqq h^{-\beta/2}(\boldsymbol{R}_h^\top \boldsymbol{F}_h - \boldsymbol{I}) \rightharpoonup \boldsymbol{G} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{3.171}
$$

and, for almost every $\boldsymbol{x} \in \Omega$, there holds

$$
\boldsymbol{G}''(\boldsymbol{x}', x_3) = \mathrm{sym}\nabla'\boldsymbol{u}(\boldsymbol{x}') + ((\nabla')^2 v(\boldsymbol{x}'))x_3. \tag{3.172}
$$

In view of (3.131), (3.152)–(3.153) and (3.171)–(3.172), we apply Proposition 3.7 to $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h$ and we conclude that

$$E_0(\boldsymbol{u}, v, \boldsymbol{\zeta}) \leq \liminf_{h \to 0^+} E_h(\boldsymbol{q}_h). \tag{3.173}$$

Now, let $(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \mathcal{Q}_0$. By Proposition 3.8 and Remark 3.9, there exists $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ with $\widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h$ such that (3.80) holds. Thus, combining (3.80), (3.128) and (3.173), we get

$$E_0(\boldsymbol{u}, v, \boldsymbol{\zeta}) \leq \liminf_{h \to 0^+} E_h(\boldsymbol{q}_h) \leq \liminf_{h \to 0^+} E_h(\widehat{\boldsymbol{q}}_h) = E_0(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}).$$

Since $(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \mathcal{Q}_0$ is arbitrary, this proves that $(\boldsymbol{u}, v, \boldsymbol{\zeta})$ is a minimizer of F_0 on $\mathcal{Q}_0$. □

4. Quasistatic setting

In this last section, we study the quasistatic evolution of the system under dissipative effects. We consider evolutions driven by time-dependent applied loads. The framework is the theory of *rate-independent systems* [37].

We start describing the setting. Let $T > 0$. Given $h > 0$, let $\boldsymbol{f}_h \in W^{1,1}(0,T; L^2(S;\mathbb{R}^2))$, $g_h \in W^{1,1}(0,T; L^2(S))$ and $\boldsymbol{h}_h \in W^{1,1}(0,T; L^2(\mathbb{R}^3;\mathbb{R}^3))$ represent a time-dependent horizontal force, vertical force and external magnetic field, respectively. Without loss of generality, we assume that all these functions are absolutely continuous in time. The corresponding functional $\mathcal{L}_h \colon [0,T] \times \mathcal{Q} \to \mathbb{R}$ is defined by

$$\mathcal{L}_h(t, \boldsymbol{y}, \boldsymbol{m}) \coloneqq \frac{1}{h^\beta} \int_\Omega \boldsymbol{f}_h(t) \cdot (\boldsymbol{y}' - \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x} + \frac{1}{h^\beta} \int_\Omega g_h(t) \, y^3 \, \mathrm{d}\boldsymbol{x} + \frac{1}{h} \int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}_h(t) \cdot \boldsymbol{m} \, \mathrm{d}\boldsymbol{\xi} \tag{4.1}$$

and the total energy $\mathcal{F}_h \colon [0,T] \times \mathcal{Q} \to \mathbb{R}$ reads

$$\mathcal{F}_h(t, \boldsymbol{q}) = E_h(\boldsymbol{q}) - \mathcal{L}_h(t, \boldsymbol{q}). \tag{4.2}$$

The dissipation distance is defined using Lagrangian magnetizations. Recalling the notation introduced in (3.4), we define the dissipation distance $\mathcal{D}_h \colon \mathcal{Q} \times \mathcal{Q} \to [0, +\infty)$ by setting

$$\mathcal{D}_h(\boldsymbol{q}, \widehat{\boldsymbol{q}}) \coloneqq \int_\Omega |\mathcal{Z}_h(\boldsymbol{q}) - \mathcal{Z}_h(\widehat{\boldsymbol{q}})| \, \mathrm{d}\boldsymbol{x}. \tag{4.3}$$

Thus, the energy dissipated by an evolution $\boldsymbol{q} \colon [0,T] \to \mathcal{Q}$ in the time interval $[r,s] \subset [0,T]$ is given by

$$\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}; [r,s]) \coloneqq \sup \left\{ \sum_{i=1}^N \mathcal{D}_h(\boldsymbol{q}(t^i), \boldsymbol{q}(t^{i-1})) : \, \Pi = (t^0, \dots, t^N) \text{ partition of } [r,s] \right\}.$$

Here, by partition of the time interval $[r,s] \subset [0,T]$ we mean any finite ordered set $\Pi = (t^0, t^1, \dots, t^N) \subset [r,s]^N$ with $r = t^0 < t^1 < \cdots < t^N = s$. Also, we define the size of the partition as

$$\varnothing(\Pi) \coloneqq \max\{t^i - t^{i-1} : \, i = 1, \dots, N\}.$$

For the reduced model, we also have an evolution driven by time-dependent applied loads. Precisely, we assume that there exist $\boldsymbol{f} \in W^{1,1}(0,T; L^2(S;\mathbb{R}^2))$, $g \in W^{1,1}(0,T; L^2(S))$ and $\boldsymbol{h} \in W^{1,1}(0,T; L^2(\mathbb{R}^2;\mathbb{R}^3))$ such that, as $h \to 0^+$, the following convergences hold:

$$h^{-\beta/2} \boldsymbol{f}_h \to \boldsymbol{f} \text{ in } W^{1,1}(0,T; L^2(S;\mathbb{R}^2)); \tag{4.4}$$

$$h^{-\beta/2-1} g_h \to g \text{ in } W^{1,1}(0,T; L^2(S)); \tag{4.5}$$

$$\boldsymbol{h}_h \circ \boldsymbol{\pi}_h \to \chi_I \boldsymbol{h} \text{ in } W^{1,1}(0,T; L^2(\mathbb{R}^3;\mathbb{R}^3)). \tag{4.6}$$

Also here, we assume that the functions $\boldsymbol{f}, g$ and $\boldsymbol{h}$ are all absolutely continuous in time. In (4.6), we trivially set $\boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t) \coloneqq \boldsymbol{h}_h(t) \circ \boldsymbol{\pi}_h$ for every $t \in [0,T]$. In particular, $\boldsymbol{h}_h \circ \boldsymbol{\pi}_h \in W^{1,1}(0,T; L^2(\mathbb{R}^3;\mathbb{R}^3))$ and its time derivative is given by $\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h \in L^1(0,T; L^2(\mathbb{R}^3;\mathbb{R}^3))$, where we set $\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t) \coloneqq \dot{\boldsymbol{h}}_h(t) \circ \boldsymbol{\pi}_h$ for every $t \in [0,T]$. Note that the limiting magnetic field $\boldsymbol{h}$ is a priori assumed to be independent on the variable x_3.

We define the functional $\mathcal{L}_0\colon [0,T]\times W^{1,2}(S;\mathbb{S}^2)\to\mathbb{R}$ by setting

$$\mathcal{L}_0(t,\boldsymbol{u},v,\boldsymbol{\zeta}) \coloneqq \int_S \boldsymbol{f}(t)\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}' + \int_S g(t)\,v\,\mathrm{d}\boldsymbol{x}' + \int_S \boldsymbol{h}(t)\cdot\boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}, \tag{4.7}$$

so that the limiting total energy $\mathcal{F}_0\colon [0,T]\times W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)\times W^{1,2}(S;\mathbb{S}^2)\to\mathbb{R}$ reads

$$\mathcal{F}_0(t,\boldsymbol{u},v,\boldsymbol{\zeta}) \coloneqq E_0(\boldsymbol{u},v,\boldsymbol{\zeta}) - \mathcal{L}_0(t,\boldsymbol{u},v,\boldsymbol{\zeta}). \tag{4.8}$$

The dissipation distance $\mathcal{D}_0\colon W^{1,2}(S;\mathbb{S}^2)\times W^{1,2}(S;\mathbb{S}^2)\to[0,+\infty)$ for the reduced model is defined as

$$\mathcal{D}_0(\boldsymbol{\zeta},\widehat{\boldsymbol{\zeta}}) = \int_S |\boldsymbol{\zeta}-\widehat{\boldsymbol{\zeta}}|\,\mathrm{d}\boldsymbol{x}', \tag{4.9}$$

so that the energy dissipated by $\boldsymbol{\zeta}\colon [0,T]\to W^{1,2}(S;\mathbb{S}^2)$ in $[r,s]\subset[0,T]$ is given by

$$\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{\zeta};[r,s]) \coloneqq \sup\left\{\sum_{i=1}^N \mathcal{D}_0(\boldsymbol{\zeta}(t^i),\boldsymbol{\zeta}(t^{i-1})) :\ \Pi=(t^0,\dots,t^N) \text{ partition of } [r,s]\right\}.$$

With a slight abuse of notation, for $\boldsymbol{q}_0=(\boldsymbol{u},v,\boldsymbol{\zeta}),\widehat{\boldsymbol{q}}_0=(\widehat{\boldsymbol{u}},\widehat{v},\widehat{\boldsymbol{\zeta}})\in W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)\times W^{1,2}(S;\mathbb{S}^2)$, we will equivalently write $\mathcal{D}_0(\boldsymbol{q}_0,\widehat{\boldsymbol{q}}_0)$ or $\mathcal{D}_0(\boldsymbol{\zeta},\widehat{\boldsymbol{\zeta}})$.

In the theory of rate-independent systems, one defines *energetic solutions* as time evolutions satisfying two requirements: a global stability condition and an energy balance [37, Definition 2.1.2]. These solutions are usually constructed by time discretizations considering piecewise constant functions defined by means of incremental minimization problems [37, Subsection 2.1.2]. As mentioned in Section 3, for $h>0$ and $t\in[0,T]$ fixed, the functional $\mathcal{F}_h(t,\cdot)$ does not necessarily admit a minimum in $\mathcal{Q}_h$. Therefore, we consider instead a relaxed version of the incremental minimization problem for which solutions always exist. This has already been considered in [38] and [39].

Definition 4.1 (Approximate incremental minimization problem). Given $h>0$, let $\Pi_h=(t_h^0,\dots,t_h^{N_h})$ be a partition of $[0,T]$, let $\sigma_h>0$ and let $\boldsymbol{q}_h^0\in\mathcal{Q}_h$. The *approximate incremental minimization problem* (AIMP) determined by Π_h with tolerance σ_h and initial datum $\boldsymbol{q}_h^0$ reads as follows: for every $i\in\{1,\dots,N_h\}$, find $\boldsymbol{q}_h^i\in\mathcal{Q}_h$ such that

$$\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i)+\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i) \le (t_h^i-t_h^{i-1})\sigma_h + \inf_{\mathcal{Q}_h}\left\{\mathcal{F}_h(t_h^i,\cdot)+\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\cdot)\right\}. \tag{4.10}$$

We underline that, by definition, solutions of the AIMP satisfy the clamped boundary condition and that the infimum in (4.10) is determined on $\mathcal{Q}_h$.

In order to state our third main result, we give the definition of energetic solution for the reduced model. Again, the boundary conditions are incorporated in the definition.

Definition 4.2 (Energetic solution). An *energetic solution* for the reduced model is a function $\boldsymbol{q}_0\colon[0,T]\to\mathcal{Q}_0$ such that the function $t\mapsto\partial_t\mathcal{F}_0(t,\boldsymbol{q}_0(t))$ is integrable and, for every $t\in[0,T]$, the following *global stability* and *energy balance* hold:

$$\forall\,\widehat{\boldsymbol{q}}_0\in\mathcal{Q}_0,\quad \mathcal{F}_0(t,\boldsymbol{q}_0(t))\le\mathcal{F}_0(t,\widehat{\boldsymbol{q}}_0)+\mathcal{D}_0(\boldsymbol{q}_0(t),\widehat{\boldsymbol{q}}_0), \tag{4.11}$$

$$\mathcal{F}_0(t,\boldsymbol{q}_0(t))+\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]) = \mathcal{F}_0(0,\boldsymbol{q}_0(0))+\int_0^t \partial_t\mathcal{F}_0(\tau,\boldsymbol{q}_0(\tau))\,\mathrm{d}\tau. \tag{4.12}$$

Our third main result claims that, for a sequence of partitions whose sizes vanish jointly with the thickness of the plate together with a sequence of tolerances, solutions of the approximate incremental minimization problem, or better their piecewise constant interpolants, converge to energetic solutions for the reduced model. As a byproduct, we deduce the existence of energetic solutions for the reduced model. Recall the definition of the total energy in (4.1)–(4.2) and (4.8)–(4.7) and of the dissipation in (4.3) and (4.9).

Theorem 4.3 (Convergence of solutions of the AIMP). *Assume $p>3$ and $\beta>6\vee p$. Suppose that the elastic energy density W_h has the form in (2.2), where the function Φ satisfies (2.7)–(2.9) and that the applied loads satisfy (4.4)–(4.6). Let (Π_h) be a sequence of partitions of $[0,T]$ such that $\varnothing(\Pi_h)\to 0$,* *as $h \to 0^+$, and let $(\sigma_h) \subset \mathbb{R}$ with $\sigma_h > 0$ be such that $\sigma_h \to 0$, as $h \to 0^+$. Let $(\boldsymbol{q}_h^0) \subset \mathcal{Q}$ with $\boldsymbol{q}_h^0 = (\boldsymbol{y}_h^0, \boldsymbol{m}_h^0) \in \mathcal{Q}_h$ for every $h > 0$ be such that the following holds:*

$$\forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(0, \boldsymbol{q}_h^0) \leq \mathcal{F}_h(0, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^0, \widehat{\boldsymbol{q}}_h). \tag{4.13}$$

Moreover, assume that there exist $\boldsymbol{u}^0 \in W_0^{1,2}(S;\mathbb{R}^2)$, $v \in W_0^{2,2}(S)$ and $\boldsymbol{\zeta}^0 \in W^{1,2}(S;\mathbb{S}^2)$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{u}_h^0 \coloneqq \mathcal{U}_h(\boldsymbol{q}_h^0) \rightharpoonup \boldsymbol{u}^0 \ \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.14}$$

$$v_h^0 \coloneqq \mathcal{V}_h(\boldsymbol{q}_h^0) \to v^0 \ \text{ in } W^{1,2}(S); \tag{4.15}$$

$$\boldsymbol{z}_h^0 \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h^0) \to \boldsymbol{\zeta}^0 \ \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{4.16}$$

$$\mathcal{F}_h(0, \boldsymbol{q}_h^0) \to \mathcal{F}_0(0, \boldsymbol{q}_0^0), \text{ where } \boldsymbol{q}_0^0 \coloneqq (\boldsymbol{u}^0, v^0, \boldsymbol{\zeta}^0) \in \mathcal{Q}_0. \tag{4.17}$$

For every $h > 0$, consider a solution of the AIMP determined by Π_h with tolerance σ_h and initial datum $\boldsymbol{q}_h^0$ according to Definition 4.1 and denote by $\boldsymbol{q}_h \colon [0,T] \to \mathcal{Q}_h$ its right-continuous piecewise constant interpolant. Then, there exists a measurable function $\boldsymbol{q}_0 \colon [0,T] \to \mathcal{Q}_0$ with $\boldsymbol{q}_0(t) = (\boldsymbol{u}(t), v(t), \boldsymbol{\zeta}(t))$ for every $t \in [0,T]$ such that $\boldsymbol{q}_0(0) = \boldsymbol{q}_0^0$ and, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\forall\, t \in [0,T], \quad \boldsymbol{z}_h(t) \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h(t)) \to \boldsymbol{\zeta}(t) \ \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{4.18}$$

$$\forall\, t \in [0,T], \quad \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t]) \to \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]); \tag{4.19}$$

$$\forall\, t \in [0,T], \quad \mathcal{F}_h(t, \boldsymbol{q}_h(t)) \to \mathcal{F}_0(t, \boldsymbol{q}_0(t)); \tag{4.20}$$

$$\partial_t \mathcal{F}_h(\cdot, \boldsymbol{q}_h) \to \partial_t \mathcal{F}_0(\cdot, \boldsymbol{q}_0) \ \text{ in } L^1(0,T). \tag{4.21}$$

Additionally, $\mathcal{F}_0(\cdot, \boldsymbol{q}_0) \in BV([0,T])$, $\boldsymbol{\zeta} \in BV([0,T]; L^1(\Omega;\mathbb{R}^{3\times 3}))$ and $\boldsymbol{q}_0$ is an energetic solution for the reduced model according to Definition 4.2.

We mention that Theorem 4.3 still holds true under the more general Dirichlet boundary conditions described in Remark 3.11.

In the statement of Theorem 4.3, the measurability of the map $\boldsymbol{q}_0 \colon [0,T] \to \mathcal{Q}_0$ is meant with respect to the Borel σ-algebra of $\mathcal{Q}_0$, where the latter space is equipped with the product weak topology. By weak topology of $W^{1,2}(S;\mathbb{S}^2)$ we mean the topology induced on this class by the weak topology of $W^{1,2}(S;\mathbb{R}^3)$.

Remark 4.4 (Time-dependent boundary conditions). So far, we are not able to treat time-dependent Dirichlet boundary conditions in the proof of Theorem 4.3. Indeed, given the Eulerian character of some of the energy terms, the approach developed in [19, Section 4] seems not to be applicable in our setting. However, time-dependent boundary conditions can be included in the analysis in a relaxed form as we will briefly discuss.

For every $h > 0$, replace $\mathcal{Q}_h$ by $\mathcal{Q}$ and consider $W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2)$ in place of $\mathcal{Q}_0$. Let $\overline{\boldsymbol{u}} \in W^{1,1}(0,T;W^{1,\infty}(S;\mathbb{R}^2))$ and $\overline{v} \in W^{1,1}(0,T;W^{2,\infty}(S))$. Analogously to Remark 3.11, for every $h > 0$, we define the time-dependent deformation $\overline{\boldsymbol{y}}_h \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))$ by setting

$$\overline{\boldsymbol{y}}_h(t) \coloneqq \boldsymbol{\pi}_h + h^{\beta/2} \begin{pmatrix} \overline{\boldsymbol{u}}(t) \\ 0 \end{pmatrix} + h^{\beta/2-1} \begin{pmatrix} \boldsymbol{0}' \\ \overline{v}(t) \end{pmatrix} - h^{\beta/2} x_3 \begin{pmatrix} \nabla' \overline{v}(t) \\ 0 \end{pmatrix}$$

for every $t \in [0,T]$. Let $\Gamma \subset \partial S$ be measurable with respect to the one-dimensional Hausdorff measure and such that $\mathscr{H}^1(\Gamma) > 0$. For every $h > 0$, we impose the boundary condition

$$\forall\, t \in [0,T], \quad \boldsymbol{y} = \overline{\boldsymbol{y}}_h(t) \text{ on } \Gamma \times I$$

in a relaxed form by augmenting the energy $\mathcal{F}_h$ by the term

$$(t, \boldsymbol{y}, \boldsymbol{m}) \mapsto \frac{1}{h^{\beta/2}} \int_{\Gamma \times I} |\boldsymbol{y}' - \overline{\boldsymbol{y}}_h'(t)| \,\mathrm{d}\boldsymbol{a} + \frac{1}{h^{\beta/2-1}} \int_{\Gamma \times I} |\boldsymbol{y}^3 - \overline{\boldsymbol{y}}_h^3(t)| \,\mathrm{d}\boldsymbol{a}.$$

The scalings are chosen in such a way that the corresponding term in the reduced model, which has to be added to $\mathcal{F}_0$, is given by

$$(t, \boldsymbol{u}, v, \boldsymbol{\zeta}) \mapsto \int_{\Gamma \times I} |\boldsymbol{u} - \overline{\boldsymbol{u}}(t) + x_3(\nabla' v - \nabla' \overline{v}(t))| \,\mathrm{d}\boldsymbol{a} + \int_{\Gamma} |v - \overline{v}(t)| \,\mathrm{d}\boldsymbol{l}.$$

This latter term imposes in a relaxed form of the following limiting boundary conditions:

$$\forall\, t \in [0,T], \quad \boldsymbol{u} = \overline{\boldsymbol{u}}(t) \text{ on } \Gamma, \quad v = \overline{v}(t) \text{ on } \Gamma, \quad \nabla' v = \nabla' \,\overline{v}(t) \text{ on } \Gamma.$$

Clearly, Lemma 3.13 must be suitably modified. Note that, contrary to Remark 3.11, no regularity assumption on Γ is required in this case.

Remark 4.5 (Existence of energetic solutions for the reduced model). As a byproduct of Theorem 4.3, we obtain the existence of energetic solutions for the reduced model. However, under our assumption, this can be established directly. Indeed, the limiting total energy $\mathcal{F}_0$ satisfies suitable compactness properties in view of the coercivity of Q_Φ^{red} noted in (3.6) and the dissipation distance $\mathcal{D}_0$ is continuous on the sublevels of $\mathcal{F}_0$, so that the existence of energetic solutions for the reduced model can be proved following the usual scheme [37, Theorem 2.1.6].

Before moving to the proof of Theorem 4.3, we briefly mention an alternative approach to study rate-independent evolutions in connection to our dimension reduction problem in the spirit of evolutionary Γ-convergence [38]. Assume that, for every $h > 0$, there exists an energetic solution $\boldsymbol{q}_h$ for the bulk model with $\boldsymbol{q}_h(0) = \boldsymbol{q}_h^0$ for some initial datum $\boldsymbol{q}_h^0 \in \mathcal{Q}_h$. Analogously to (4.11)–(4.12), this is defined as a function $\boldsymbol{q}_h \colon [0,T] \to \mathcal{Q}_h$ such that the function $t \mapsto \partial_t \mathcal{F}_h(t, \boldsymbol{q}_h(t))$ is integrable and, for every $t \in [0,T]$, there hold:

$$\forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(t, \boldsymbol{q}_h(t)) \leq \mathcal{F}_h(t, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h(t), \widehat{\boldsymbol{q}}_h),$$

$$\mathcal{F}_h(t, \boldsymbol{q}_h(t)) + \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,t]) = \mathcal{F}_h(0, \boldsymbol{q}_h^0) + \int_0^t \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h(\tau))\,\mathrm{d}\tau.$$

If we assume (4.14)–(4.17), then there exists a measurable function $\boldsymbol{q}_0 \colon [0,T] \to \mathcal{Q}_0$ satisfying $\boldsymbol{q}_0(0) = (\boldsymbol{u}^0, v^0, \boldsymbol{\zeta}^0)$ such that (4.18)–(4.21) hold and $\boldsymbol{q}_0$ is an energetic solution for the reduced model as in Definition 4.2. The proof of this fact is very similar to the proof of Theorem 4.3.

In the present work, we did not pursue this approach since, in our setting, we are not able to prove the existence of energetic solutions for the bulk model, even if we assume the polyconvexity of the elastic energy density in its first argument. In that case, the functional $\mathcal{F}_h(t, \cdot)$ admits global minimizers in $\mathcal{Q}_h$ for every fixed $t \in [0,T]$, so that the incremental minimization problem [37, Section 2.1.2] is actually solvable. However, we cannot prove the existence of energetic solutions for the bulk model following the usual scheme [37, Theorem 2.1.6]. Indeed, given a sequence of piecewise constant interpolants determined by solutions of incremental minimization problems corresponding to a sequence of partitions of the time interval with vanishing size, we cannot show that the limiting evolution, identified by compactness arguments, satisfies the required global stability condition. This is essentially due to a lack of compactness which does not ensure the continuity of the dissipation distance $\mathcal{D}_h$ on the sublevels of the energy $\mathcal{F}_h$. This kind of situation is typical of large-strain theories.

A practicable way to overcome this issue has been recently proposed in [9] and relies the notion of gradient polyconvexity [6]. Recallin (2.16), the idea is to restrict the class of admissible deformation to the set

$$\left\{ \boldsymbol{y} \in \mathcal{Y} : \ \mathrm{cof} \nabla \boldsymbol{y} \in BV(\Omega; \mathbb{R}^{3\times 3}) \right\},$$

and to regularize the energy by adding the higher-order term

$$\boldsymbol{y} \mapsto |D(\mathrm{cof} \nabla \boldsymbol{y})|(\Omega),$$

where $|D(\mathrm{cof} \nabla \boldsymbol{y})|(\Omega)$ denotes the total variation of the tensor-valued measure $D(\mathrm{cof} \nabla \boldsymbol{y})$. In this way, the dissipation distance $\mathcal{D}_h$ results continuous on the sublevels of the regularized energy, so that the existence of energetic solutions can be proved following the aforementioned scheme [37, Theorem 2.1.6]. Some natural growth conditions of the elastic energy density Φ with respect to the Jacobian determinant are required. Note that, in this regularized setting, we do not need any polyconvexity assumption on the elastic energy density. We refer to [6] or [29] for details.

The reminder of the section is devoted to the proof of Theorem 4.3. We start with some preliminary results. The first one shows that, when restricted to sublevel sets, the total energy satisfies suitable controls with respect to time.

Lemma 4.6 (Time-control of the total energy). *Let $(\widehat{t}_h) \subset [0,T]$ and let $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ with $\widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h$ for every $h>0$ be such that*

$$\sup_{h>0} \mathcal{F}_h(\widehat{t}_h, \widehat{\boldsymbol{q}}_h) \leq C. \tag{4.22}$$

Then, we have

$$\sup_{h>0} E_h(\widehat{\boldsymbol{q}}_h) \leq C. \tag{4.23}$$

In particular, there exists two constants $C>0$ and $c>0$ such that, for every $h>0$, the following estimate holds:

$$\forall\, t \in [0,T], \quad |\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| \leq \kappa_h(t)(\mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)+c), \tag{4.24}$$

where we set

$$\kappa_h(t) := C\left(||h^{-\beta/2}\dot{\boldsymbol{f}}_h(t)||_{L^2(S;\mathbb{R}^2)} + ||h^{-\beta/2-1}\dot{g}_h(t)||_{L^2(S)} + ||\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \right). \tag{4.25}$$

Moreover, for every $h>0$, there hold:

$$\forall\, s,t \in [0,T], \qquad \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)+c \leq (\mathcal{F}_h(s,\widehat{\boldsymbol{q}}_h)+c)\mathrm{e}^{|K_h(t)-K_h(s)|}, \tag{4.26}$$

$$\forall\, s,t \in [0,T], \qquad |\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| \leq \kappa_h(t)(\mathcal{F}_h(s,\widehat{\boldsymbol{q}}_h)+c)\mathrm{e}^{|K_h(t)-K_h(s)|}, \tag{4.27}$$

where we set

$$K_h(t) := \int_0^t \kappa_h(\tau)\,\mathrm{d}\tau. \tag{4.28}$$

Note that, by (4.4)–(4.6), the sequence (κ_h) converges in $L^1(0,T)$ and hence it is equi-integrable. As a consequence, the sequence (K_h) is equi-continuous.

Proof. From (4.22), arguing exactly as in Lemma 3.15, we deduce (4.23). Note that the sequence $(\widehat{t}_h)$ plays no role, as the sequences $(h^{-\beta/2}\boldsymbol{f}_h)$, $(h^{-\beta/2-1}g_h)$ and $(\boldsymbol{h}_h \circ \boldsymbol{\pi}_h)$ are uniformly bounded in $C^0([0,T];L^2(S;\mathbb{R}^2))$, $C^0([0,T];L^2(S))$ and $C^0([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))$ by (4.4)–(4.6) and the Morrey embedding. Henceforth, we will use this fact without further mention.

Set $\widehat{\boldsymbol{u}}_h := \mathcal{U}_h(\widehat{\boldsymbol{q}}_h)$ and $\widehat{v}_h := \mathcal{V}_h(\widehat{\boldsymbol{q}}_h)$. By Proposition 3.14, for every $h>0$, there hold:

$$||\widehat{\boldsymbol{u}}_h||_{W^{1,2}(S;\mathbb{R}^2)} \leq C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right), \qquad ||\widehat{v}_h||_{W^{1,2}(S)} \leq C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right). \tag{4.29}$$

In this case, for every $t\in[0,T]$, using the Hölder inequality, we compute

$$\begin{aligned}\left|\frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h(t)\cdot(\widehat{\boldsymbol{y}}_h' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}\right| &= \int_S h^{-\beta/2}|\boldsymbol{f}_h(t)|\,|\widehat{\boldsymbol{u}}_h|\,\mathrm{d}\boldsymbol{x}' \\ &\leq ||h^{-\beta/2}\boldsymbol{f}_h(t)||_{L^2(S;\mathbb{R}^2)}\,||\widehat{\boldsymbol{u}}_h||_{L^2(S;\mathbb{R}^2)} \\ &\leq C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right),\end{aligned} \tag{4.30}$$

and analogously

$$\begin{aligned}\left|\frac{1}{h^\beta}\int_\Omega g_h(t)\cdot\widehat{y}^3_h\,\mathrm{d}\boldsymbol{x}\right| &= \int_S h^{-\beta/2-1}|g_h(t)|\,|\widehat{v}_h|\,\mathrm{d}\boldsymbol{x}' \\ &\leq ||h^{-\beta/2-1}g_h(t)||_{L^2(S)}\,||\widehat{v}_h||_{L^2(S)} \\ &\leq C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right).\end{aligned} \tag{4.31}$$

Analogously to (3.167)–(3.168), using the change-of-variable formula and the Hölder inequality, for every $t\in[0,T]$, we obtain

$$\begin{aligned}\left|\frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} \boldsymbol{h}_h(t)\cdot\widehat{\boldsymbol{m}}_h\,\mathrm{d}\boldsymbol{\xi}\right| &= \int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} |\boldsymbol{h}_h\circ\boldsymbol{\pi}_h(t)|\,\mathrm{d}\boldsymbol{x} \\ &\leq ||\boldsymbol{h}_h\circ\boldsymbol{\pi}_h(t)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\sqrt{\mathscr{L}^3(\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h}))} \\ &\leq C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right).\end{aligned} \tag{4.32}$$

Combining (4.30)–(4.32), we deduce

$$|\mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h)| \leq C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right), \tag{4.33}$$

which gives

$$\begin{aligned}\mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h) &\geq E_h(\widehat{\boldsymbol{q}}_h) - |\mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h)| \\ &\geq E_h(\widehat{\boldsymbol{q}}_h) - C\left(\sqrt{E_h(\widehat{\boldsymbol{q}}_h)}+1\right) \\ &\geq C_1\, E_h(\widehat{\boldsymbol{q}}_h) - C_2,\end{aligned} \tag{4.34}$$

where, in the last line, we used the Young inequality. For simplicity, set

$$l_h(t) \coloneqq ||h^{-\beta/2}\dot{\boldsymbol{f}}_h(t)||_{L^2(S;\mathbb{R}^2)} + ||h^{-\beta/2-1}\dot{g}_h(t)||_{L^2(S)} + ||\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.$$

With computations analogous to (4.30)–(4.32), having the time-derivatives of the applied loads in place of the applied loads themselves, we obtain

$$|\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| = |\partial_t \mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h)| \leq C\, l_h(t)\,\sqrt{E_h(\widehat{\boldsymbol{q}}_h)+1}. \tag{4.35}$$

Thus, from (4.34) and (4.35), using the Young inequality, we have

$$|\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| \leq C\, l_h(t)\,\left(E_h(\widehat{\boldsymbol{q}}_h)+1\right) \leq C\, l_h(t)\,\left(\mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)+c\right),$$

where $c>0$. Setting $\kappa_h(t) \coloneqq C\, l_h(t)$, this gives (4.24). Finally, (4.26) follows from (4.24) by applying the Gronwall inequality while (4.27) is obtained by combining (4.24) and (4.26). □

In the next result, we collect the main properties of solutions of the AIMP. We employ the notation introduced in (4.25) and (4.28).

Lemma 4.7 (Solutions of the AIMP). *Let (Π_h) be a sequence of partitions of $[0,T]$ and let $(\sigma_h) \subset \mathbb{R}$ with $\sigma_h > 0$ for every $h>0$ be bounded. Also, let $(\boldsymbol{q}_h^0) \subset \mathcal{Q}$ with $\boldsymbol{q}_h^0 \in \mathcal{Q}_h$ for every $h>0$ be such that*

$$\sup_{h>0} \mathcal{F}_h(0,\boldsymbol{q}_h^0) \leq C. \tag{4.36}$$

For every $h>0$, let $\Pi_h = (t_h^0,\dots,t_h^{N_h})$ and let $(\boldsymbol{q}_h^1,\dots,\boldsymbol{q}_h^{N_h}) \in \mathcal{Q}_h^{N_h}$ be a solution of the AIMP determined by Π_h with tolerance σ_h and initial datum $\boldsymbol{q}_h^0$ according to Definition 4.1. Then, we have

$$\sup_{h>0}\sup_{i\in\{0,\dots,N_h\}} \mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) \leq C \tag{4.37}$$

and

$$\sup_{h>0}\sup_{i\in\{0,\dots,N_h\}} E_h(\boldsymbol{q}_h^i) \leq C. \tag{4.38}$$

In particular, for every $h>0$ and $i \in \{1,\dots,N_h\}$, the following estimates hold:

$$\forall\, t\in[0,T], \qquad |\partial_t \mathcal{F}_h(t,\boldsymbol{q}_h^i)| \leq \kappa_h(t)\left(\mathcal{F}_h(t,\boldsymbol{q}_h^i)+c\right), \tag{4.39}$$

$$\forall\, s,t\in[0,T], \qquad \mathcal{F}_h(t,\boldsymbol{q}_h^i)+c \leq \left(\mathcal{F}_h(s,\boldsymbol{q}_h^i)+c\right)\mathrm{e}^{|K_h(t)-K_h(s)|}, \tag{4.40}$$

$$\forall\, s,t\in[0,T], \qquad |\partial_t \mathcal{F}_h(t,\boldsymbol{q}_h^i)| \leq \kappa_h(t)\left(\mathcal{F}_h(s,\boldsymbol{q}_h^i)+c\right)\mathrm{e}^{|K_h(t)-K_h(s)|}. \tag{4.41}$$

Moreover, for every $h>0$ and $i\in\{1,\dots.N_h\}$, there hold:

$$\forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) \leq (t_h^i - t_h^{i-1})\sigma_h + \mathcal{F}_h(t_h^i,\widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^i,\widehat{\boldsymbol{q}}_h), \tag{4.42}$$

$$\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) + \mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i) \leq (t_h^i - t_h^{i-1})\sigma_h + \mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1}) + \int_{t_h^{i-1}}^{t_h^i} \partial_t \mathcal{F}_h(\tau,\boldsymbol{q}_h^{i-1})\,\mathrm{d}\tau, \tag{4.43}$$

$$\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) + c + \sum_{j=1}^{i} \mathcal{D}_h(\boldsymbol{q}_h^{j-1},\boldsymbol{q}_h^j) \leq \left(\mathcal{F}_h(0,\boldsymbol{q}_h^0)+c+t_i\sigma_h\right)\mathrm{e}^{K_h(t_h^i)}. \tag{4.44}$$

Additionally, if the initial datum $\boldsymbol{q}_h^0$ satisfies (4.13), then, for every $h>0$ and $i\in\{1,\dots,N_h\}$, there holds

$$\begin{aligned}|\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i)-\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1})+\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i)|&\leq\left(t_h^i-t_h^{i-2}\right)\sigma_h\\&+\left(\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1})+c\right)\left(\mathrm{e}^{K_h(t_h^i)-K_h(t_h^{i-1})}-1\right),\end{aligned}\tag{4.45}$$

where, for convenience of notation, we set $t_h^{-1}\coloneqq 0$.

Proof. Let $h>0$ and $i\in\{1,\dots,N_h\}$. For simplicity, set $\sigma_h^i\coloneqq(t_h^i-t_h^{i-1})\sigma_h$. By (4.10), given $\widehat{\boldsymbol{q}}_h\in\mathcal{Q}_h$, we have

$$\begin{aligned}\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i)&\leq\sigma_h^i+\mathcal{F}_h(t_h^i,\widehat{\boldsymbol{q}}_h)+\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\widehat{\boldsymbol{q}}_h)-\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i)\\&\leq\sigma_h^i+\mathcal{F}_h(t_h^i,\widehat{\boldsymbol{q}}_h)+\mathcal{D}_h(\boldsymbol{q}_h^i,\widehat{\boldsymbol{q}}_h),\end{aligned}\tag{4.46}$$

where, in the last line, we used the triangle inequality. This proves (4.42).

We check (4.43). For simplicity, set $f_h^i\coloneqq\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i)$ and $d_h^i\coloneqq\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i)$. By (4.10), applying the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}f_h^i-f_h^{i-1}+d_h^i&\leq\sigma_h^i-f_h^{i-1}+\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^{i-1})\\&=\sigma_h^i+\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^{i-1})-\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1})\\&=\sigma_h^i+\int_{t_h^{i-1}}^{t_h^i}\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h^{i-1})\,\mathrm{d}\tau,\end{aligned}\tag{4.47}$$

which gives (4.43).

Let $\overline{\boldsymbol{q}}_h\in\mathcal{Q}_h$ be defined as in the proof of Theorem 3.10. By (4.6) and the Morrey embedding, which gives the boundedness of $(\boldsymbol{h}_h\circ\boldsymbol{\pi}_h)$ in $C^0([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))$, we have

$$\sup_{h>0}\sup_{t\in[0,T]}\mathcal{F}_h(t,\overline{\boldsymbol{q}}_h)\leq C.$$

Testing (4.42) with $\overline{\boldsymbol{q}}_h$ and taking into account (4.36) and the boundedness of the dissipation, we have

$$\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i)\leq\sigma_h^i+\mathcal{F}_h(t_h^i,\overline{\boldsymbol{q}}_h)+\mathcal{D}_h(\boldsymbol{q}_h^i,\overline{\boldsymbol{q}}_h)\leq C.$$

This proves (4.37).

Now, for the sake of clarity, we specify the sequence (h_ℓ) such that $h_\ell\to0^+$, as $\ell\to\infty$, in place of $h>0$. For every $\ell\in\mathbb{N}$, let $(\boldsymbol{q}_{h_\ell}^1,\dots,\boldsymbol{q}_{h_\ell}^{N_{h_\ell}})\in\mathcal{Q}_{h_\ell}^{N_{h_\ell}}$ be a solution of the AIMP determined by $\Pi=(t_{h_\ell}^0,\dots,t_{h_\ell}^{N_{h_\ell}})$ with tolerance σ_{h_ℓ} and initial datum $\boldsymbol{q}_{h_\ell}^0\in\mathcal{Q}_{h_\ell}$ according to Definition 4.1. In order to be able to use the estimates in Lemma 4.6, we proceed as follows. Define the sequences $(k_m)\subset\mathbb{R}$ and $(\widehat{\boldsymbol{q}}_{k_m})\subset\mathcal{Q}$ by setting

$$k_m\coloneqq\begin{cases}h_1&\text{if } m\leq N_{h_1}+1,\\h_\ell&\text{if } \sum_{n=1}^{\ell-1}N_{h_n}+\ell-1<m\leq\sum_{n=1}^{\ell}N_{h_n}+\ell\text{ for some }\ell\in\mathbb{N},\end{cases}$$

and

$$\widehat{\boldsymbol{q}}_{k_m}\coloneqq\begin{cases}q_{h_1}^{m-1}&\text{if } m\leq N_{h_1}+1,\\\boldsymbol{q}_{h_\ell}^{m+\ell-2-\sum_{n=1}^{\ell-1}N_{h_n}}&\text{if } \sum_{n=1}^{\ell-1}N_{h_n}+\ell-1<m\leq\sum_{n=1}^{\ell}N_{h_n}+\ell\text{ for some }\ell\in\mathbb{N}.\end{cases}$$

The first terms of the two sequences are simply given by

$$\underbrace{h_1,h_1,\dots,h_1}_{N_{h_1}+1\text{ times}},\underbrace{h_2,h_2,\dots,h_2}_{N_{h_2}+1\text{ times}},\underbrace{h_3,h_3,\dots,h_3}_{N_{h_3}+1\text{ times}},\dots$$

and

$$\boldsymbol{q}_{h_1}^0,\boldsymbol{q}_{h_1}^1,\dots,\boldsymbol{q}_{h_1}^{N_{h_1}},\boldsymbol{q}_{h_2}^0,\boldsymbol{q}_{h_2}^1,\dots,\boldsymbol{q}_{h_2}^{N_{h_2}},\boldsymbol{q}_{h_3}^0,\boldsymbol{q}_{h_3}^1,\dots,\boldsymbol{q}_{h_3}^{N_{h_3}},\dots$$

respectively. Note that $k_m \to 0^+$, as $m \to \infty$, and $\widehat{\boldsymbol{q}}_{k_m} \in \mathcal{Q}_{k_m}$ for every $m \in \mathbb{N}$. Also, let $(\widehat{t}_{k_m})$ be any sequence in $[0,T]$. By (4.36)–(4.37), we have

$$\sup_{m\in\mathbb{N}} \mathcal{F}_{k_m}(\widehat{t}_{k_m}, \widehat{\boldsymbol{q}}_{k_m}) \leq C.$$

Applying Lemma 4.6, we deduce that

$$\sup_{m\in\mathbb{N}} E_{k_m}(\widehat{\boldsymbol{q}}_{k_m}) \leq C,$$

and, for every $m \in \mathbb{N}$, the following estimates hold:

$$\begin{aligned}
\forall\, t \in [0,T], \qquad & |\partial_t \mathcal{F}_{k_m}(t, \widehat{\boldsymbol{q}}_{k_m})| \leq \kappa_{k_m}(t) \left(\mathcal{F}_{k_m}(t, \widehat{\boldsymbol{q}}_{k_m}) + c\right), \\
\forall\, s,t \in [0,T], \qquad & \mathcal{F}_{k_m}(t, \widehat{\boldsymbol{q}}_{k_m}) + c \leq \left(\mathcal{F}_{k_m}(s, \widehat{\boldsymbol{q}}_{k_m}) + c\right) \mathrm{e}^{|K_{k_m}(t) - K_{k_m}(s)|}, \\
\forall\, s,t \in [0,T], \qquad & |\partial_t \mathcal{F}_{k_m}(t, \widehat{\boldsymbol{q}}_{k_m})| \leq \kappa_{k_m}(t) \left(\mathcal{F}_{k_m}(s, \widehat{\boldsymbol{q}}_{k_m}) + c\right) \mathrm{e}^{|K_{k_m}(t) - K_{k_m}(s)|}.
\end{aligned}$$

Therefore, recalling the definition of $(\widehat{\boldsymbol{q}}_{k_m})$, we see that, for every $\ell \in \mathbb{N}$ and for every $i \in \{1, \dots, N_{h_\ell}\}$, the estimates in (4.38)–(4.41) hold with $h = h_\ell$.

Henceforth, for simplicity, we will go back writing h as subscript without specifying the sequence of thicknesses. Let $h > 0$ and let $i \in \{1, \dots, N_h\}$. Set $K_h^i := K_h(t_h^i)$. We show (4.44). By (4.40), we have

$$f_h^i + c \leq \sigma_h^i + (f_h^{i-1} + c)\mathrm{e}^{K_h^i - K_h^{i-1}}$$

from which, by induction, we prove

$$f_h^i + c \leq \left(f_h^0 + c + \sum_{j=1}^{i} \mathrm{e}^{-K_h^j} \sigma_h^j \right) \mathrm{e}^{K_h^i}, \tag{4.48}$$

where we set $f_h^0 := \mathcal{F}_h(0, \boldsymbol{q}_h^0)$. From (4.43), using (4.41), we obtain

$$\begin{aligned}
f_h^i - f_h^{i-1} + d_h^i &\leq \sigma_h^i + \int_{t_h^{i-1}}^{t_h^i} \partial_h \mathcal{F}_h(\tau, \boldsymbol{q}_h^{i-1})\, \mathrm{d}\tau \\
&\leq \sigma_h^i + (f_h^{i-1} + c) \int_{t_h^{i-1}}^{t_h^i} \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h(t_h^{i-1})}\, \mathrm{d}\tau \\
&\leq \sigma_h^i + (f_h^{i-1} + c) \left(\mathrm{e}^{K_h(t_h^i) - K_h(t_h^{i-1})} - 1 \right).
\end{aligned} \tag{4.49}$$

Summing (4.49), with j in place of i, for $j \in \{1, \dots, i\}$ and using (4.48), with $j-1$ in place of i, we obtain

$$\begin{aligned}
f_h^i + \sum_{j=1}^{i} d_h^j + c &\leq f_h^0 + c + \sum_{j=1}^{i} \sigma_h^j + \sum_{j=1}^{i} (f_h^{j-1} + c) \left(\mathrm{e}^{K_h^j - K_h^{j-1}} - 1 \right) \\
&\leq f_h^0 + c + \sum_{j=1}^{i} \sigma_h^j + \sum_{j=1}^{i} \left(f_h^0 + c + \sum_{k=1}^{j-1} \mathrm{e}^{-K_h^k} \sigma_h^k \right) \left(\mathrm{e}^{K_h^j} - \mathrm{e}^{K_h^{j-1}} \right) \\
&= \sum_{j=1}^{i} \sigma_h^j + (f_h^0 + c)\mathrm{e}^{K_h^i} + \sum_{j=1}^{i} \left(\mathrm{e}^{K_h^j} - \mathrm{e}^{K_h^{j-1}} \right) \sum_{k=1}^{j-1} \mathrm{e}^{-K_h^k} \sigma_h^k \\
&\leq \sum_{j=1}^{i} \sigma_h^j + (f_h^0 + c)\mathrm{e}^{K_h^i} + \sum_{j=1}^{i} \left(\mathrm{e}^{K_h^j} - \mathrm{e}^{K_h^{j-1}} \right) \sum_{k=1}^{i} \sigma_h^k \\
&= \left(f_h^0 + c + \sum_{j=1}^{i} \sigma_h^i \right) \mathrm{e}^{K_h^i},
\end{aligned}$$

which is (4.44).

Finally, we prove (4.45). Testing (4.42) for $i-1$ if $i>1$ or (4.13) if $i=1$ both with $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^i$, we have

$$f_h^{i-1} \leq \sigma_h^{i-1} + \mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^i) + d_h^i.$$

Here, in the second case, consider $\sigma_h^0 \coloneqq 0$. From this, employing the Fundamental Theorem of Calculus and (4.41), we compute

$$\begin{aligned} f_h^{i-1} - f_h^i - d_h^i &\leq \sigma_h^{i-1} - \left(\mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i) - \mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^i)\right) \\ &= \sigma_h^{i-1} - \int_{t_h^{i-1}}^{t_h^i} \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h^i)\,\mathrm{d}\tau \\ &\leq \sigma_h^{i-1} + (f_h^{i-1}+c) \int_{t_h^{i-1}}^{t_h^i} \kappa_h(\tau) \mathrm{e}^{K_h(\tau)-K_h^{i-1}}\,\mathrm{d}\tau \\ &\leq \sigma_h^{i-1} + (f_h^{i-1}+c)\left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1\right). \end{aligned} \tag{4.50}$$

Combining (4.49)–(4.50), as $\sigma_h^{i-1} + \sigma_h^i = (t_h^i - t_h^{i-2})\sigma_h$, we obtain (4.45).

□

To ease the exposition, we present a simple result about the convergence of the work of applied loads.

Lemma 4.8 (Convergence of the work of applied loads). *Let $(t_h) \subset [0,T]$ and let $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{Q}$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}_h)$. Suppose that $t_h \to t$ for some $t \in [0,T]$. Also, suppose that there exist $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$, $\widehat{v} \in W^{2,2}(S)$ and $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ such that*

$$\widehat{\boldsymbol{u}}_h \coloneqq \mathcal{U}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.51}$$

$$\widehat{v}_h \coloneqq \mathcal{V}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{v} \text{ in } W^{1,2}(S); \tag{4.52}$$

$$\widehat{\boldsymbol{\eta}}_h \coloneqq (\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h) \circ \boldsymbol{\pi}_h \to\to \chi_\Omega \widehat{\boldsymbol{\zeta}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3). \tag{4.53}$$

Then, the following equality holds:

$$\mathcal{L}_0(t, \widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) = \lim_{h\to 0^+} \mathcal{L}_h(t_h, \widehat{\boldsymbol{q}}_h). \tag{4.54}$$

Proof. Applying the change-of-variable formula, we write

$$\begin{aligned} \mathcal{L}_h(t_h, \widehat{\boldsymbol{q}}_h) - \mathcal{L}_h(t, \widehat{\boldsymbol{q}}_h) &= \int_S h^{-\beta/2}(\boldsymbol{f}_h(t_h) - \boldsymbol{f}_h(t)) \cdot \widehat{\boldsymbol{u}}_h \,\mathrm{d}\boldsymbol{x}' \\ &\quad + \int_S h^{-\beta/2-1}(g_h(t_h) - g_h(t))\,\widehat{v}_h \,\mathrm{d}\boldsymbol{x}' \\ &\quad + \int_{\mathbb{R}^3} (\boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t_h) - \boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t)) \cdot \widehat{\boldsymbol{\eta}}_h \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.55}$$

By (4.4)–(4.6) and by the Morrey embedding, there hold

$$\begin{aligned} h^{-\beta/2}\boldsymbol{f}_h(t) &\to \boldsymbol{f}(t) \text{ in } L^2(S;\mathbb{R}^2), \\ h^{-\beta/2-1} g_h(t) &\to g(t) \text{ in } L^2(S), \\ \boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t) &\to \chi_I \boldsymbol{h}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3). \end{aligned}$$

These, together with (4.51)–(4.53), yield

$$\lim_{h\to 0^+} \mathcal{L}_h(t, \widehat{\boldsymbol{q}}_h) = \mathcal{L}_0(t, \widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}).$$

Therefore, in order to check (4.54), we only have to prove that the right-hand side of (4.55) converges to zero, as $h \to 0^+$. By the Hölder inequality, we compute

$$
\begin{aligned}
|\mathcal{L}_h(t_h, \widehat{\boldsymbol{q}}_h) - \mathcal{L}_h(t, \widehat{\boldsymbol{q}}_h)| &\leq ||h^{-\beta/2}(\boldsymbol{f}_h(t_h) - \boldsymbol{f}_h(t))||_{L^2(S;\mathbb{R}^2)}\, ||\widehat{\boldsymbol{u}}_h||_{L^2(S;\mathbb{R}^2)} \\
&\quad + ||h^{-\beta/2-1}(g_h(t_h) - g_h(t))||_{L^2(S)}\, ||\widehat{v}_h||_{L^2(S)} \\
&\quad + ||\boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t_h) - \boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\, ||\widehat{\boldsymbol{\eta}}_h||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
&\leq C\, ||h^{-\beta/2}(\boldsymbol{f}_h(t_h) - \boldsymbol{f}_h(t))||_{L^2(S;\mathbb{R}^2)} \\
&\quad + C\, ||h^{-\beta/2-1}(g_h(t_h) - g_h(t))||_{L^2(S)} \\
&\quad + C\, ||\boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t_h) - \boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}
\end{aligned}
$$

where, in the last line, we used the boundedness of $(\widehat{\boldsymbol{u}}_h)$, $(\widehat{v}_h)$ and $(\widehat{\boldsymbol{\eta}}_h)$ in $L^2(S;\mathbb{R}^2)$, $L^2(S)$ and $L^2(\mathbb{R}^3;\mathbb{R}^3)$, respectively, which follows from (4.51)–(4.53). Applying the Fundamental Theorem of Calculus for the Bochner integral, we compute

$$
\begin{aligned}
||h^{-\beta/2}(\boldsymbol{f}_h(t_h) - \boldsymbol{f}_h(t))||_{L^2(S;\mathbb{R}^2)} &= \left\| \int_t^{t_h} h^{-\beta/2} \dot{\boldsymbol{f}}_h(\tau)\,\mathrm{d}\tau \right\|_{L^2(S;\mathbb{R}^2)} \\
&\leq \int_t^{t_h} ||h^{-\beta/2} \dot{\boldsymbol{f}}_h(\tau)||_{L^2(S;\mathbb{R}^2)}\,\mathrm{d}\tau \\
&\leq C \int_t^{t_h} \kappa_h(\tau)\,\mathrm{d}\tau, \\
||h^{-\beta/2-1}(g_h(t_h) - g_h(t))||_{L^2(S;\mathbb{R}^2)} &= \left\| \int_t^{t_h} h^{-\beta/2-1} \dot{g}_h(\tau)\,\mathrm{d}\tau \right\|_{L^2(S;\mathbb{R}^2)} \\
&\leq \int_t^{t_h} ||h^{-\beta/2-1} \dot{g}_h(\tau)||_{L^2(S;\mathbb{R}^2)}\,\mathrm{d}\tau \\
&\leq C \int_t^{t_h} \kappa_h(\tau)\,\mathrm{d}\tau, \\
||\boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t_h) - \boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} &= \left\| \int_t^{t_h} \dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(\tau)\,\mathrm{d}\tau \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
&\leq \int_t^{t_h} ||\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(\tau)||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\leq C \int_t^{t_h} \kappa_h(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Thus, we have

$$
|\mathcal{L}_h(t_h, \widehat{\boldsymbol{q}}_h) - \mathcal{L}_h(t_h, \widehat{\boldsymbol{q}}_h)| \leq C \int_t^{t_h} \kappa_h(\tau)\,\mathrm{d}\tau.
$$

As the right-hand side goes to zero, as $h \to 0^+$, by the equi-integrability of (κ_h), the proof is concluded. □

We are finally ready to prove our third main result.

Proof of Theorem 4.3. The proof follows the well established scheme in [37, Theorem 2.1.6] and it is subdivided into six steps.

Step 1 (A priori estimates). For every $h > 0$, let $(\boldsymbol{q}_h^1, \dots, \boldsymbol{q}_h^{N_h}) \in \mathcal{Q}_h^{N_h}$ be the solution of the AIMP determined by $\Pi_h = (t_h^0, \dots, t_h^{N_h})$ with tolerance $\sigma_h > 0$ and initial datum $\boldsymbol{q}_h^0 \in \mathcal{Q}_h$. We define the piecewise constant interpolant $\boldsymbol{q}_h \colon [0,T] \to \mathcal{Q}_h$ by setting

$$
\boldsymbol{q}_h(t) := \begin{cases} \boldsymbol{q}_h^{i-1} & \text{if } t_h^{i-1} \leq t < t_h^i \text{ for some } i \in \{1, \dots, N_h\}, \\ \boldsymbol{q}_h^{N_h} & \text{if } t = T. \end{cases}
$$

Let $t \in [0,T]$. Given $h>0$, let $i \in \{1,\dots,N_h\}$ be such that $t_h^{i-1} \leq t < t_h^i$. By definition, $\boldsymbol{q}_h(t)=\boldsymbol{q}_h^{i-1}$ and $\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t])=\sum_{j=1}^{i-1}\mathcal{D}_h(\boldsymbol{q}_h^{j-1},\boldsymbol{q}_h^j)$. Thus, using (4.40) and (4.44), we obtain

$$
\begin{aligned}
\mathcal{F}_h(t,\boldsymbol{q}_h(t))+c+\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t]) &\leq (\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1})+c)\mathrm{e}^{K_h(t)-K_h(t_h^{i-1})}+\sum_{j=1}^{i-1}\mathcal{D}_h(\boldsymbol{q}_h^{j-1},\boldsymbol{q}_h^j)\\
&\leq \left(\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1})+c+\sum_{j=1}^{i-1}\mathcal{D}_h(\boldsymbol{q}_h^{j-1},\boldsymbol{q}_h^j)\right)\mathrm{e}^{K_h(t)-K_h(t_h^{i-1})}\\
&\leq (\mathcal{F}_h(0,\boldsymbol{q}_h^0)+c+t_h^{i-1}\sigma_h)\left(\mathrm{e}^{K_h(t)}-1\right)\\
&\leq (\mathcal{F}_h(0,\boldsymbol{q}_h^0)+c+T\sigma_h)\left(\mathrm{e}^{K_h(T)}-1\right).
\end{aligned}
$$

As, by (4.6) and (4.17), the sequences $(\mathcal{F}_h(0,\boldsymbol{q}_h^0))$, (σ_h) and $(K_h(T))$ are all bounded, we deduce

$$
\sup_{h>0}\sup_{t\in[0,T]}\mathcal{F}_h(t,\boldsymbol{q}_h(t))+c+\sup_{h>0}\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,T])\leq C \tag{4.56}
$$

For every $h>0$ and $t\in[0,T]$, define $f_h(t):=\mathcal{F}_h(t,\boldsymbol{q}_h(t))$ and $\boldsymbol{z}_h(t):=\mathcal{Z}_h(\boldsymbol{q}_h(t))$. From (4.56), we immediately get

$$
\sup_{h>0}\sup_{t\in[0,T]}f_h(t)+\sup_{h>0}\mathrm{Var}_{L^1(\Omega;\mathbb{R}^3)}(\boldsymbol{z}_h;[0,T])\leq C. \tag{4.57}
$$

We now establish a uniform bound for the total variation of f_h. For simplicity, for every $h>0$ and $i\in\{1,\dots,N_h\}$, set

$$
f_h^i:=\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i),\quad d_h^i:=\mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i),\quad K_h^i:=K_h(t_h^i),\quad \sigma_h^i:=(t_h^i-t_h^{i-1})\sigma_h.
$$

Again, for notational convenience, we set $t_h^{-1}:=0$. Recall that, in view of (4.56), for every $h>0$ and $i\in\{1,\dots,N_h\}$ there holds $f_h^i\leq C$. Denote by $[f_h]^i$ the jump of f_h at time t_h^i. Exploiting the continuity in time of $\mathcal{F}_h$ and using (4.41) and (4.45), which holds true in view of (4.13), we compute

$$
\begin{aligned}
|[f_h]^i| &= \left|\lim_{s\to0^+}\left\{f_h(t_h^i)-f_h(t_h^i-s)\right\}\right|\\
&= \left|\lim_{s\to0^+}\left\{\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i)-\mathcal{F}_h(t_h^i-s,\boldsymbol{q}_h^{i-1})\right\}\right|\\
&= |f_h^i-f_h^{i-1}-(\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^{i-1})-\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1}))|\\
&\leq |f_h^i-f_h^{i-1}|+\int_{t_h^{i-1}}^{t_h^i}|\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h^{i-1})|\,\mathrm{d}\tau\\
&\leq |f_h^i-f_h^{i-1}|+(f_h^{i-1}+c)\int_{t_h^{i-1}}^{t_h^i}\kappa_h(\tau)\mathrm{e}^{K_h(\tau)-K_h^{i-1}}\,\mathrm{d}\tau\\
&\leq |f_h^i-f_h^{i-1}|+(M+c)\left(\mathrm{e}^{K_h^i-K_h^{i-1}}-1\right)\\
&\leq d_h^i+\sigma_h^{i-1}+\sigma_h^i+2(M+c)\left(\mathrm{e}^{K_h^i-K_h^{i-1}}-1\right).
\end{aligned} \tag{4.58}
$$

For $t_h^{i-1}\leq t<t_h^i$, we have $f_h(t)=\mathcal{F}_h(t,\boldsymbol{q}_h^{i-1})$ and, in turn, $\dot{f}_h(t)=\partial_t\mathcal{F}_h(t,\boldsymbol{q}_h^{i-1})$. Using (4.41) once more, we compute

$$
\begin{aligned}
\int_{t_h^i}^{t_h^{i-1}}|\dot{f}_h(\tau)|\,\mathrm{d}\tau &= \int_{t_h^i}^{t_h^{i-1}}|\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h^{i-1})|\,\mathrm{d}\tau\\
&\leq (f_h^{i-1}+c)\int_{t_h^i}^{t_h^{i-1}}\kappa_h(\tau)\mathrm{e}^{K_h(\tau)-K_h^{i-1}}\,\mathrm{d}\tau\\
&\leq (M+c)\left(\mathrm{e}^{K_h^i-K_h^{i-1}}-1\right).
\end{aligned} \tag{4.59}
$$

Thus, combining (4.58)–(4.59), we obtain

$$\begin{aligned}\operatorname{Var}(f_h;[0,T]) &= \sum_{i=1}^{N_h}\left\{|[f_h]^i| + \int_{t_h^i}^{t_h^{i-1}} |\dot f_h(\tau)|\,\mathrm{d}\tau\right\}\\ &\leq \sum_{i=1}^{N_h}\left\{d_h^i + \sigma_h^i + \sigma_h^{i-1} + 3(M+c)\left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1\right)\right\}\\ &\leq \operatorname{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,T]) + 2T\sigma_h + 3(M+c)\left(\mathrm{e}^{K_h(T)} - 1\right).\end{aligned} \tag{4.60}$$

Here, in the last line, we used that

$$\begin{aligned}\sum_{i=1}^{N_h}\left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1\right) &= \sum_{i=1}^{N_h}\mathrm{e}^{-K_h^{i-1}}\left(\mathrm{e}^{K_h^i} - \mathrm{e}^{K_h^{i-1}}\right)\\ &\leq \sum_{i=1}^{N_h}\left(\mathrm{e}^{K_h^i} - \mathrm{e}^{K_h^{i-1}}\right)\\ &= \mathrm{e}^{K_h(T)} - 1.\end{aligned}$$

Thus, in view of (4.56) and the boundedness of (σ_h) and $(K_h(T))$, from (4.60) we deduce

$$\sup_{h>0}\operatorname{Var}(f_h;[0,T]) \leq C. \tag{4.61}$$

Step 2 (Compactness). By (4.57)–(4.61), the sequence (f_h) is bounded in $BV([0,T])$. Thus, by the Helly Compactness Theorem, it admits a pointwise convergent subsequence.

By Lemma 4.7, we have

$$\sup_{h>0} E_h(\boldsymbol{q}_h(t)) = \sup_{h>0}\sup_{i\in\{0,\dots,N_h\}} E_h(\boldsymbol{q}_h^i) \leq C. \tag{4.62}$$

In particular, by Proposition 3.14, for every fixed $t\in[0,T]$ the sequence $(\boldsymbol{z}_h(t))$ admits a limit point in $L^1(\Omega;\mathbb{R}^3)$. This, together with (4.57), allows us to argue as in [36, Theorem 5.1].

Set $\delta_h(t) \coloneqq \operatorname{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t])$ for every $h>0$ and $t\in[0,T]$. By the previous considerations, there exist a subsequence (h_ℓ), a bounded increasing function $\delta\colon[0,T]\to[0,+\infty)$ and two functions $f\in BV([0,T])$ and $\boldsymbol{z}\in BV([0,T];L^1(\Omega;\mathbb{R}^3))$ such that, as $\ell\to\infty$, the following convergences hold:

$$\forall\, t\in[0,T],\quad f_{h_\ell}(t)\to f(t), \tag{4.63}$$

$$\forall\, t\in[0,T],\quad \delta_{h_\ell}(t)\to\delta(t), \tag{4.64}$$

$$\forall\, t\in[0,t],\quad \boldsymbol{z}_{h_\ell}(t)\to\boldsymbol{z}(t)\ \text{ in } L^1(\Omega;\mathbb{R}^3). \tag{4.65}$$

To construct the limiting function $\boldsymbol{q}_0\colon[0,T]\to\mathcal{Q}_0$, we proceed as follows. For every $h>0$ and $t\in[0,T]$, let $\boldsymbol{q}_h(t) = (\boldsymbol{y}_h(t),\boldsymbol{m}_h(t))$. Then, set

$$\begin{aligned}\boldsymbol{u}_h(t) &\coloneqq \mathcal{U}_h(\boldsymbol{q}_h(t)), & v_h(t) &\coloneqq \mathcal{V}_h(\boldsymbol{q}_h(t)),\\ \boldsymbol{\eta}_h(t) &\coloneqq (\chi_{\Omega^{\boldsymbol{y}_h(t)}}\boldsymbol{m}_h(t))\circ\boldsymbol{\pi}_h, & \boldsymbol{H}_h(t) &\coloneqq (\chi_{\Omega^{\boldsymbol{y}_h(t)}}\nabla\boldsymbol{m}_h(t))\circ\boldsymbol{\pi}_h.\end{aligned}$$

Recalling (4.62), by Proposition 3.14, for every fixed $t\in[0,T]$ there exists a map $\boldsymbol{R}_h(t)\in W^{1,p}(S;SO(3))$ such that the following estimates hold

$$||\boldsymbol{F}_h(t)-\boldsymbol{R}_h(t)||_{L^a(\Omega;\mathbb{R}^{3\times3})} \leq Ch^{\beta/a}, \qquad ||\nabla'\boldsymbol{R}_h(t)||_{L^a(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{\beta/a-1}, \tag{4.66}$$

$$||\boldsymbol{R}_h(t)-\boldsymbol{I}||_{L^a(S;\mathbb{R}^{3\times3})} \leq Ch^{\beta/a-1}, \quad ||\boldsymbol{F}_h(t)-\boldsymbol{I}||_{L^a(\Omega;\mathbb{R}^{3\times3})} \leq Ch^{\beta/a-1}, \tag{4.67}$$

where $\boldsymbol{F}_h(t)\coloneqq\nabla_h\boldsymbol{y}_h(t)$ and $a\in\{2,p\}$. We set

$$\boldsymbol{A}_h(t)\coloneqq h^{-\beta/2+1}(\boldsymbol{R}_h(t)-\boldsymbol{I}), \qquad \boldsymbol{G}_h(t)\coloneqq h^{-\beta/2}(\boldsymbol{R}_h(t)^\top\boldsymbol{F}_h(t)-\boldsymbol{I}).$$

For a suitable constant $C>0$, we define $\mathcal{X}$ as the set of all quintuples

$$(\widehat{\boldsymbol{u}},\widehat{v},\widehat{\boldsymbol{\eta}},\widehat{\boldsymbol{H}},\widehat{\boldsymbol{A}})\in W_0^{1,2}(S;\mathbb{R}^2)\times W_0^{1,2}(S)\times L^2(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})\times W^{1,2}(S;\mathbb{R}^{3\times3})$$

satisfying

$$||\widehat{\boldsymbol{u}}||_{W^{1,2}(S;\mathbb{R}^2)} + ||\widehat{v}||_{W^{1,2}(S)} + ||\widehat{\boldsymbol{\eta}}||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + ||\widehat{\boldsymbol{H}}||_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} + ||\widehat{\boldsymbol{A}}||_{W^{1,2}(S;\mathbb{R}^{3\times 3})} \leq C. \tag{4.68}$$

The space $\mathcal{X}$ is endowed with the product topology, where the spaces $W^{1,2}(S)$ and $L^2(\mathbb{R}^3;\mathbb{R}^3)$ are equipped with the strong topology and all the other spaces are equipped with the weak topology. This makes $\mathcal{X}$ a complete and separable metric space. The constant $C>0$ in (4.68) is chosen according to (4.62) in order to define the set-valued map $P_h\colon [0,T]\to\mathcal{P}(\mathcal{X})$ as $P_h(t) \coloneqq \{\boldsymbol{p}_h(t)\}$, where

$$\boldsymbol{p}_h(t) \coloneqq (\boldsymbol{u}_h(t), v_h(t), \boldsymbol{\eta}_h(t), \boldsymbol{H}_h(t), \boldsymbol{A}_h(t)).$$

Consider the set-valued map $P\colon [0,T]\to\mathcal{P}(\mathcal{X})$ defined by

$$P(t) \coloneqq \operatorname*{Lim\,sup}_{\ell\to\infty} P_{h_\ell}(t) = \left\{ \widehat{\boldsymbol{p}}\in\mathcal{X} :\ \exists\,(h_{\ell^t_m}) \text{ subsequence} :\ \boldsymbol{p}_{h_{\ell^t_m}}(t)\to\widehat{\boldsymbol{p}} \text{ in } \mathcal{X}\right\}, \tag{4.69}$$

where the superior limit is meant in the sense of Kuratowski [3, Definition 1.1.1]. By [3, Theorem 8.2.5], this set-valued map is measurable and, by definition, $P(t)\subset\mathcal{X}$ is closed for every $t\in[0,T]$. We claim that $P(t)\neq\emptyset$ for every $t\in[0,T]$. Indeed, for $t\in[0,T]$ fixed, this follows from (4.62) by applying Proposition 3.14 to the sequence $(\boldsymbol{q}_h(t))$. Therefore, by [3, Theorem 8.1.3], there exists a measurable selection $\boldsymbol{p}\colon [0;T]\to\mathcal{X}$ of P, so that $\boldsymbol{p}(t)\in P(t)$ for every $t\in[0,T]$. Let $\boldsymbol{p}(t) = (\boldsymbol{u}(t), v(t), \boldsymbol{\eta}(t), \boldsymbol{H}(t), \boldsymbol{A}(t))$. By (4.69), for every $t\in[0,T]$, there exists a subsequence $(h_{\ell^t_m})$ such that the following convergences hold, as $m\to\infty$:

$$\boldsymbol{u}_{h_{\ell^t_m}}(t) \rightharpoonup \boldsymbol{u}(t) \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.70}$$

$$v_{h_{\ell^t_m}}(t) \to v(t) \text{ in } W^{1,2}(S); \tag{4.71}$$

$$\boldsymbol{\eta}_{h_{\ell^t_m}}(t) \to \boldsymbol{\eta}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3); \tag{4.72}$$

$$\boldsymbol{H}_{h_{\ell^t_m}}(t) \rightharpoonup \boldsymbol{H}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.73}$$

$$\boldsymbol{A}_{h_{\ell^t_m}}(t) \rightharpoonup \boldsymbol{A}(t) \text{ in } W^{1,2}(S;\mathbb{R}^{3\times 3}). \tag{4.74}$$

Thanks to (4.62), applying Proposition 3.14 and appealing to the Urysohn property, we deduce several facts. First, $v(t)\in W^{2,2}_0(S)$ and the limit in (4.74) has the form

$$\boldsymbol{A}(t) = \left(\begin{array}{c|c} \boldsymbol{O}'' & -\nabla' v(t) \\ \hline (\nabla' v(t))^\top & 0 \end{array}\right). \tag{4.75}$$

Second, we have the following:

$$\begin{aligned} &\boldsymbol{G}_{h_{\ell^t_m}}(t) \rightharpoonup \boldsymbol{G}(t) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \text{ as } m\to\infty, \text{ for some } \boldsymbol{G}(t)\in L^2(\Omega;\mathbb{R}^{3\times 3}) \text{ satisfying} \\ &\boldsymbol{G}''(t,\boldsymbol{x}',x_3) = \operatorname{sym}\nabla'\boldsymbol{u}(t,\boldsymbol{x}') + ((\nabla')^2 v(t,\boldsymbol{x}'))x_3 \text{ for almost every } \boldsymbol{x}\in\Omega. \end{aligned} \tag{4.76}$$

Third, there exist $\boldsymbol{\zeta}(t)\in W^{1,2}(S;\mathbb{S}^2)$ and $\boldsymbol{\nu}(t)\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that the limits in (4.72)–(4.73) have the form

$$\boldsymbol{\eta}(t) = \chi_\Omega\boldsymbol{\zeta}(t), \qquad \boldsymbol{H}(t) = \chi_\Omega(\nabla'\boldsymbol{\zeta}, \boldsymbol{\nu}(t)). \tag{4.77}$$

Moreover, we obtain

$$\boldsymbol{z}_{h_{\ell^t_m}}(t) \to \boldsymbol{\zeta}(t) \text{ in } L^{p/2},$$

as $m\to\infty$, which, given (4.65), entails that $\boldsymbol{z}(t) = \boldsymbol{\zeta}(t)$ for every $t\in[0,T]$. From the measurability of v and $\boldsymbol{A}$ with respect to the Borel σ-algebras given by the strong topology of $W^{1,2}(S)$ and the weak topology of $W^{1,2}(S;\mathbb{R}^{3\times 3})$, respectively, and (4.75) we deduce that the function v is actually measurable with respect to the Borel σ-algebra given by the weak topology of $W^{2,2}(S)$. Similarly, from the measurability of $\boldsymbol{\eta}$ and $\boldsymbol{H}$ with respect to the Borel σ-algebras given by the strong topology of $L^2(\mathbb{R}^3;\mathbb{R}^3)$ and the weak topology $L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$, respectively, we conclude that the function $\boldsymbol{\zeta}\colon [0,T]\to W^{1,2}(S;\mathbb{S}^2)$ in (4.77) is measurable with respect to the Borel σ-algebra given by the weak topology of $W^{1,2}(S;\mathbb{S}^2)$. Therefore, the function $\boldsymbol{q}_0\colon [0,T]\to\mathcal{Q}_0$ defined by $\boldsymbol{q}_0(t) \coloneqq (\boldsymbol{u}(t), v(t), \boldsymbol{\zeta}(t))$ is measurable with respect to the Borel σ-algebra given by the product weak topology of $\mathcal{Q}_0$.

Concerning initial conditions, by definition, $\boldsymbol{u}_h(0) = \boldsymbol{u}_h^0$, $v_h(0) = v_h^0$ and $\boldsymbol{z}_h(0) = \boldsymbol{z}_h^0$. Thus, combining (4.14)–(4.16) with (4.65) and (4.70)–(4.71), we see that $\boldsymbol{q}_0(0) = \boldsymbol{q}_0^0$, where $\boldsymbol{q}_0^0 := (\boldsymbol{u}^0, v^0, \boldsymbol{\zeta}^0)$.

For every $h > 0$ and $t \in [0,T]$, define $\vartheta_h(t) := \partial_t \mathcal{F}_h(t, \boldsymbol{q}_h(t))$. Note that, since $\boldsymbol{q}_h$ is piecewise constant, the measurability of ϑ_h follows from the measurability of $\partial_t \mathcal{F}_h(\cdot, \boldsymbol{q})$ for every $\boldsymbol{q} \in \mathcal{Q}$. By (4.39) and (4.57), we have

$$|\vartheta_h(t)| \leq \kappa_h(t)\,(f_h(t) + c) \leq (M + c)\kappa_h(t). \tag{4.78}$$

As the sequence (κ_h) is equi-integrable, so is (ϑ_h). Thus, by virtue of the Dunford-Pettis Theorem [18, Theorem 2.54], we assume that $\vartheta_{h_\ell} \rightharpoonup \vartheta$ in $L^1(0,T)$ for some $\vartheta \in L^1(0,T)$. For every $t \in [0,T]$, define

$$\bar{\vartheta}(t) := \limsup_{\ell \to \infty} \vartheta_{h_\ell}(t).$$

By (4.78), since (κ_h) converges in $L^1(0,T)$, we see that $\bar{\vartheta} \in L^1(0,T)$. Then, applying the Reverse Fatou Lemma [41, Corollary 5.35], we check that $\vartheta(t) \leq \bar{\vartheta}(t)$ for almost every $t \in [0,T]$.

To identify the function $\bar{\vartheta}$, we fix $t \in [0,T]$ and we recall (4.72). Without loss of generality, we assume that $\vartheta_{h_{\ell^t_m}}(t) \to \bar{\vartheta}(t)$, as $m \to \infty$. Also, in view of (4.4)–(4.6), we suppose that

$$\begin{aligned} h^{-\beta/2} \dot{\boldsymbol{f}}_{h_{\ell^t_m}}(t) &\to \dot{\boldsymbol{f}}(t) \text{ in } L^2(S;\mathbb{R}^2), \\ h^{-\beta/2-1} \dot{g}_{h_{\ell^t_m}}(t) &\to \dot{g}(t) \text{ in } L^2(S), \\ \dot{\boldsymbol{h}}_{h_{\ell^t_m}} \circ \boldsymbol{\pi}_{h_{\ell^t_m}}(t) &\to \chi_I \dot{\boldsymbol{h}}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3), \end{aligned}$$

as $m \to \infty$. Using the change-of-variable formula and passing to the limit exploiting (4.70)–(4.72), we obtain

$$\begin{aligned} \vartheta_{h_{\ell^t_m}}(t) &= -\int_S h^{-\beta/2} \dot{\boldsymbol{f}}_{h_{\ell^t_m}}(t) \cdot \boldsymbol{u}_{h_{\ell^t_m}}(t)\,\mathrm{d}\boldsymbol{x}' - \int_S h^{-\beta/2-1} \dot{g}_{h_{\ell^t_m}}(t)\, v_{h_{\ell^t_m}}(t)\,\mathrm{d}\boldsymbol{x}' \\ &\quad - \int_{\mathbb{R}^3} \dot{\boldsymbol{h}}_{h_{\ell^t_m}} \circ \boldsymbol{\pi}_{h_{\ell^t_m}}(t) \cdot \boldsymbol{\eta}_{h_{\ell^t_m}}(t)\,\mathrm{d}\boldsymbol{x} \\ &\quad \to -\int_S \dot{\boldsymbol{f}}(t) \cdot \boldsymbol{u}(t)\,\mathrm{d}\boldsymbol{x}' - \int_S \dot{g}(t)\, v(t)\,\mathrm{d}\boldsymbol{x}' - \int_S \dot{\boldsymbol{h}}(t) \cdot \boldsymbol{\zeta}(t)\,\mathrm{d}\boldsymbol{x}' = \partial_t \mathcal{F}_0(t, \boldsymbol{q}_0(t)), \end{aligned}$$

as $m \to \infty$. Thus, we conclude that $\bar{\vartheta}(t) = \partial_t \mathcal{F}_0(t, \boldsymbol{q}_0(t))$ for almost every $t \in [0,T]$. In particular, the function $\partial_t \mathcal{F}_0(\cdot, \boldsymbol{q}_0)$ is measurable.

Step 3 (Stability). We claim that the function $\boldsymbol{q}_0$ satisfies (4.11). To see this, recall (4.42). Using the triangle inequality, we deduce the following:

$$\forall\, t_h \in \Pi_h,\ \forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(t_h, \boldsymbol{q}_h(t_h)) \leq \varnothing_h \sigma_h + \mathcal{F}_h(t_h, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h(t_h), \widehat{\boldsymbol{q}}_h), \tag{4.79}$$

where we set $\varnothing_h := \varnothing(\Pi_h)$. Fix $t \in [0,T]$ and let $(h_{\ell^t_m})$ be a subsequence such that (4.70)–(4.74) and (4.76) hold. Henceforth, for the sake of brevity, we will simply write m in place of $h_{\ell^t_m}$ as subscript. Define $\tau_m(t) := \{s \in \Pi_m : \ s \leq t\}$ and note that $\tau_m(t) \to t$, as $m \to \infty$, since $\varnothing_m \to 0$, as $m \to \infty$. By definition, we have $\boldsymbol{q}_m(t) = \boldsymbol{q}_m(\tau_m(t))$. Let $\widehat{\boldsymbol{q}}_0 \in \mathcal{Q}_0$ and let $(\widehat{\boldsymbol{q}}_h)$ a corresponding recovery sequence with $\widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h$ given by Proposition 3.8 and Remark 3.9. By (4.79), there holds

$$\mathcal{F}_m(\tau_m(t), \boldsymbol{q}_m(t)) \leq \varnothing_m \sigma_m + \mathcal{F}_m(t, \widehat{\boldsymbol{q}}_m) + \mathcal{D}_m(\boldsymbol{q}_m(t), \widehat{\boldsymbol{q}}_m). \tag{4.80}$$

Recall (4.65), (4.72)–(4.73) and (4.76). Applying Proposition 3.7 and Lemma 4.8 to the sequence $(\boldsymbol{q}_m(t))$, we obtain

$$\mathcal{F}_0(t, \boldsymbol{q}_0(t)) \leq \liminf_{m \to \infty} \mathcal{F}_m(\tau_m(t), \boldsymbol{q}_m(t)). \tag{4.81}$$

Similarly, by (3.79)–(3.76), (3.80), (4.65) and (4.72), employing Lemma 4.8, we have

$$\mathcal{F}_0(t, \widehat{\boldsymbol{q}}_0) + \mathcal{D}_0(\boldsymbol{q}_0(t), \widehat{\boldsymbol{q}}_0) = \lim_{m \to \infty} \left\{ \mathcal{F}_m(t, \widehat{\boldsymbol{q}}_m) + \mathcal{D}_m(\boldsymbol{q}_m(t), \widehat{\boldsymbol{q}}_m) \right\}. \tag{4.82}$$

Thus, combining (4.80)–(4.82), we get

$$\begin{aligned}\mathcal{F}_0(t,\boldsymbol{q}_0(t)) &\leq \liminf_{m\to\infty} \mathcal{F}_m(\tau_m(t),\boldsymbol{q}_m(t))\\ &\leq \liminf_{m\to\infty}\{\varnothing_m\sigma_m+\mathcal{F}_m(\tau_m(t),\widehat{\boldsymbol{q}}_m)+\mathcal{D}_m(\boldsymbol{q}_m(t),\widehat{\boldsymbol{q}}_m)\}\\ &= \mathcal{F}_0(t,\widehat{\boldsymbol{q}}_0)+\mathcal{D}_0(\boldsymbol{q}_0(t),\widehat{\boldsymbol{q}}_0),\end{aligned}$$

which proves (4.11) for t fixed.

Step 4 (Upper energy estimate). We prove the following:

$$\forall\, t\in[0,T],\quad \mathcal{F}_0(t,\boldsymbol{q}_0(t))+\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t])\leq \mathcal{F}_0(0,\boldsymbol{q}_0^0)+\int_0^t \partial_t\mathcal{F}_0(\tau,\boldsymbol{q}_0(\tau))\,\mathrm{d}\tau. \tag{4.83}$$

First, fix $t\in[0,T]$ and let $h>0$. Note that $\tau_h(t)\leq t$. Using the Fundamental Theorem of Calculus, (4.41) and (4.56), we estimate

$$\begin{aligned}|\mathcal{F}_h(t,\boldsymbol{q}_h(t))-\mathcal{F}_h(\tau_h(t),\boldsymbol{q}_h(\tau_h(t)))| &\leq \int_{\tau_h(t)}^t |\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau_h(t)))|\,\mathrm{d}\tau\\ &\leq (\mathcal{F}_h(\tau_h(t),\boldsymbol{q}_h(\tau_h(t)))+c)\int_{\tau_h(t)}^t \kappa_h(\tau)\mathrm{e}^{K_h(\tau)-K_h(\tau_h(t))}\,\mathrm{d}\tau\\ &= (C+c)\left(\mathrm{e}^{K_h(t)-K_h(\tau_h(t))}-1\right).\end{aligned} \tag{4.84}$$

Then, recall (4.43). Using the triangle inequality, we obtain the following:

$$\begin{aligned}\forall\, t_h\in\Pi_h,\quad \mathcal{F}_h(t_h,\boldsymbol{q}_h(t_h))+\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t_h]) &\leq t_h\sigma_h\\ &\quad+\mathcal{F}_h(0,\boldsymbol{q}_h^0)+\int_0^{t_h}\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{4.85}$$

Note that $\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t])=\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,\tau_h(t)])$. Employing (4.84)–(4.85), we compute

$$\begin{aligned}\mathcal{F}_h(t,\boldsymbol{q}_h(t))+\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t]) &\leq \mathcal{F}_h(\tau_h(t),\boldsymbol{q}_h(\tau_h(t)))+\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,\tau_h(t)])\\ &\quad+(C+c)\left(\mathrm{e}^{K_h(t)-K_h(\tau_h(t))}-1\right)\\ &\leq \tau_h(t)\sigma_h+\mathcal{F}_h(0,\boldsymbol{q}_h^0)+\int_0^{\tau_h(t)}\vartheta_h(\tau)\,\mathrm{d}\tau\\ &\quad+(C+c)\left(\mathrm{e}^{K_h(t)-K_h(\tau_h(t))}-1\right).\end{aligned} \tag{4.86}$$

Now, recalling (4.65), (4.72)–(4.73) and (4.76), applying Proposition 3.7 and Lemma 4.8 to $\widehat{\boldsymbol{q}}_h=\boldsymbol{q}_{h_{\ell_m^t}}(t)$, we obtain

$$\mathcal{F}_0(t,\boldsymbol{q}_0)\leq \liminf_{m\to\infty}\mathcal{F}_m(t,\boldsymbol{q}_m(t)), \tag{4.87}$$

while, given (4.65), by the lower semicontinuity of the total variation, we have

$$\begin{aligned}\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]) &= \mathrm{Var}_{L^1(\Omega;\mathbb{R}^3)}(\boldsymbol{z};[0,t])\\ &\leq \liminf_{\ell\to\infty}\mathrm{Var}_{L^1(\Omega;\mathbb{R}^3)}(\boldsymbol{z}_{h_\ell};[0,t])\\ &= \liminf_{\ell\to\infty}\mathrm{Var}_{\mathcal{D}_{h_\ell}}(\boldsymbol{q}_{h_\ell};[0,t]).\end{aligned} \tag{4.88}$$

Thus, for $h = h_{\ell^t_m}$, taking the inferior limit in (4.86), as $m \to \infty$, and recalling (4.63)–(4.64), we obtain

$$\begin{aligned}
\mathcal{F}_0(t, \boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) &\leq \liminf_{m\to\infty} \{\mathcal{F}_m(t, \boldsymbol{q}_m(t)) + \mathrm{Var}_{\mathcal{D}_m}(\boldsymbol{q}_m; [0,t])\} \\
&= \liminf_{m\to\infty} \{f_m(t) + \delta_m(t)\} \\
&= f(t) + \delta(t) \\
&\leq \mathcal{F}_0(0, \boldsymbol{q}_0^0) + \int_0^t \vartheta(\tau)\,\mathrm{d}\tau \\
&\leq \mathcal{F}_0(0, \boldsymbol{q}_0^0) + \int_0^t \bar{\vartheta}(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{4.89}$$

This proves (4.83).

Step 5 (Lower energy estimate). We prove the following:

$$\forall\, t \in [0,T], \quad \mathcal{F}_0(t, \boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) \geq \mathcal{F}_0(\boldsymbol{0}, \boldsymbol{q}_0^0) + \int_0^t \partial_t \mathcal{F}_0(\tau, \boldsymbol{q}_0(\tau))\,\mathrm{d}\tau. \tag{4.90}$$

This is deduced from (4.11) by arguing as in [37, Proposition 2.1.23]. Then, combining (4.83)–(4.90), we establish (4.12).

Step 6 (Improved convergence). As (4.18) has been already proven, we are left to show (4.19)–(4.21). By (4.89)–(4.90), for every $t \in [0,T]$ there holds

$$\mathcal{F}_0(t, \boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0, [0,t]) = f(t) + \delta(t). \tag{4.91}$$

Fix $t \in [0,T]$. Recalling (4.63) and (4.87), we have

$$\mathcal{F}_0(t, \boldsymbol{q}_0(t)) \leq \liminf_{m\to\infty} \mathcal{F}_m(t, \boldsymbol{q}_m(t)) = \liminf_{m\to\infty} f_m(t) = f(t),$$

while, from (4.64) and (4.88), we obtain

$$\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) \leq \liminf_{\ell\to\infty} \mathrm{Var}_{\mathcal{D}_{h_\ell}}(\boldsymbol{q}_{h_\ell}; [0,t]) = \liminf_{\ell\to\infty} \delta_{h_\ell}(t) = \delta(t).$$

Therefore, (4.91) entails $f(t) = \mathcal{F}_0(t, \boldsymbol{q}_0(t))$ and $\delta(t) = \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t])$, so that (4.19)–(4.20) are proved. From (4.89)–(4.90), given the arbitrariness of $t \in [0,T]$, we also deduce that $\vartheta(t) = \bar{\vartheta}(t)$ for almost every $t \in [0,T]$. Finally, arguing as in the proof of [19, Lemma 3.5], we establish (4.21). □

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