Coupling Flexible Structure and Viscous, Compressible Fluid: A Case for Model Order Reduction?

YES, but which approach will work?

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Considering viscous effects in acoustics is important in cases where the size of the viscous boundary layer is not negligible. This is often the case on small scales, e.g. in MEMS devices, or small features.
We employ the finite element method to solve the governing equations: The linearised mass and momentum conservation equations for a viscous, compressible fluid, coupled to the balance of momentum for a linear elastic solid. Since the computational effort is high, we attempt a model order reduction procedure using the standard reduced basis method.

We consider a 2D example of an elastic, fluid-filled container with two chambers connected by a narrow neck. The container is placed on top of a flexible pillar, which is excited at its base. Dimensions and material parameters are selected to ensure strong fluid-structure interaction.

Fluid

viscous
compressible
small perturbations
no background flow

$$\rho_\mathrm{f}\kappa_s \frac{\partial p}{\partial t} + \rho_\mathrm{f}\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$$

pressure, velocity, isentropic compressibility

$$\rho_\mathrm{f}\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_\mathrm{f} = \boldsymbol{0}$$

background density

$$\boldsymbol{\sigma}_\mathrm{f} = -p\boldsymbol{I} + \boldsymbol{\tau}$$

stress tensor, viscous stress

Newtonian fluid

$$\boldsymbol{\tau} = \lambda\dot{\boldsymbol{\epsilon}}_\mathrm{v} + 2\mu\dot{\boldsymbol{\epsilon}}_\mathrm{d}$$

volumetric strain rate $\frac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I}$

deviatoric strain rate $\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_\mathrm{v}$

bulk viscosity, dynamic viscosity

Solid

linear elastic
small displacements

$$\rho_\mathrm{s}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_\mathrm{s} = \boldsymbol{0}$$

displacement

Hooke's law

$$\boldsymbol{\sigma}_\mathrm{s} = \boldsymbol{C} : \boldsymbol{\epsilon}$$

stress tensor, stiffness tensor, strain tensor

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right)$$

small strains

traction

$$\boldsymbol{\sigma}_\mathrm{s}\cdot\boldsymbol{n} = \boldsymbol{t}$$

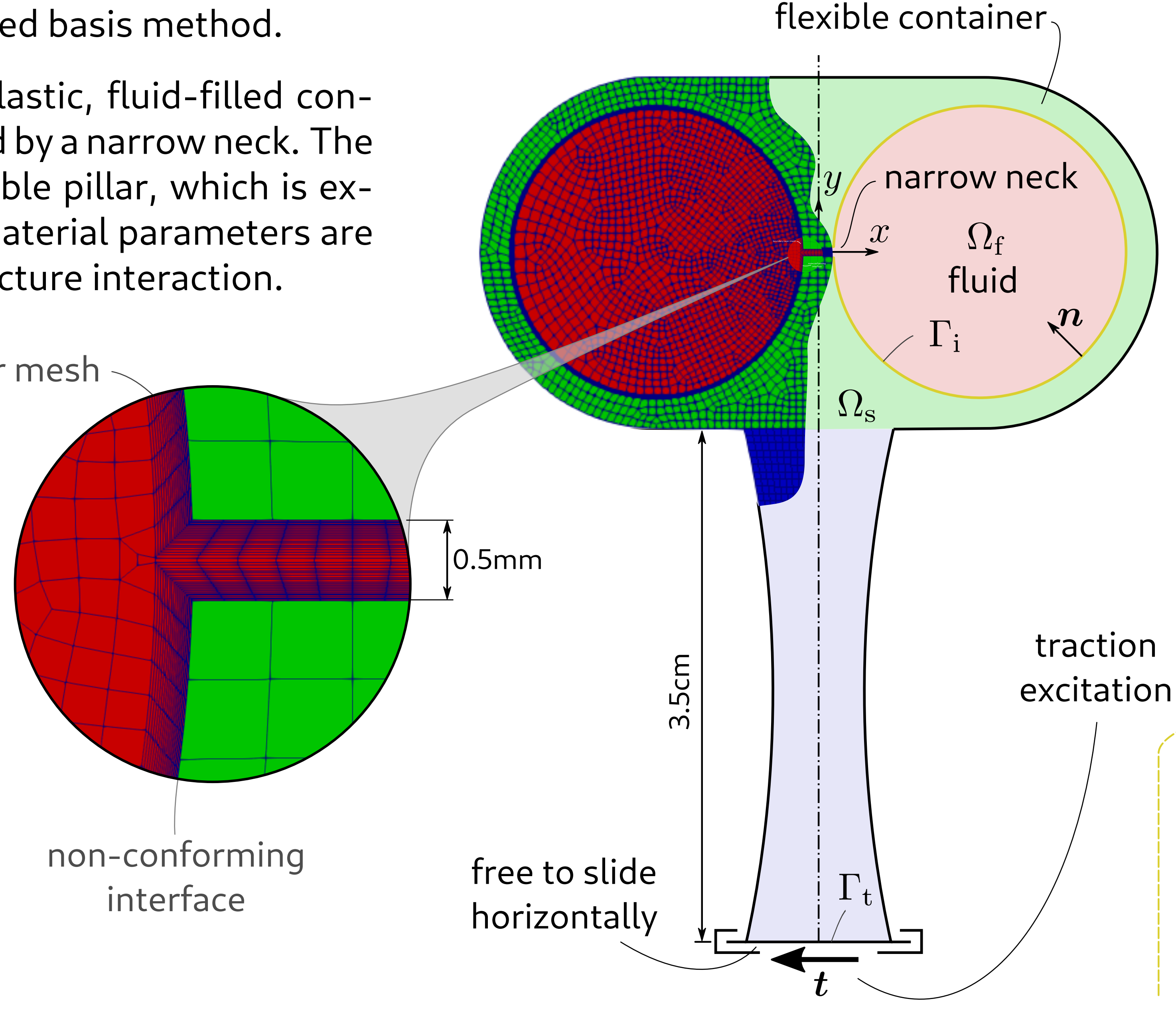
Coupling

on fluid-solid interface

$$\boldsymbol{\sigma}_\mathrm{s}\cdot\boldsymbol{n} = \boldsymbol{\sigma}_\mathrm{f}\cdot\boldsymbol{n}$$ dynamic equilibrium

normal vector

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t}$$ kinematic condition



openCFS

The formulation is implemented in the open source FEM software openCFS.

Find p, $\boldsymbol{v}$ and $\boldsymbol{u}$ such that

$$\int_{\Omega_\mathrm{f}} p'\nabla\cdot\boldsymbol{v}\,\mathrm{d}\Omega + \int_{\Omega_\mathrm{f}} p'\rho_\mathrm{f}\kappa_\mathrm{f}\frac{\partial p}{\partial t}\,\mathrm{d}\Omega \int_{\Omega_\mathrm{f}} \rho_\mathrm{f}\boldsymbol{v}'\cdot\frac{\partial \boldsymbol{v}}{\partial t}\,\mathrm{d}\Omega + \int_{\Omega_\mathrm{f}} \nabla\boldsymbol{v}' : \boldsymbol{\sigma}_\mathrm{f}\,\mathrm{d}\Omega + \int_{\Omega_\mathrm{s}} \boldsymbol{u}'\cdot\rho_\mathrm{s}\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,\mathrm{d}\Omega + \int_{\Omega_\mathrm{s}} \nabla\boldsymbol{u}' : \boldsymbol{\sigma}_\mathrm{f}\,\mathrm{d}\Omega - \int_{\Gamma_\mathrm{i}} (\boldsymbol{u}'-\boldsymbol{v}')\cdot\boldsymbol{\sigma}_\mathrm{s}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma + \beta\int_{\Gamma_\mathrm{i}} (\boldsymbol{u}'-\boldsymbol{v}')\cdot(\frac{\partial\boldsymbol{u}}{\partial t} - \boldsymbol{v})\,\mathrm{d}\Gamma = \int_{\Gamma_\mathrm{t}} \boldsymbol{u}'\cdot\boldsymbol{t}\,\mathrm{d}\Gamma$$

for all test functions p', $\boldsymbol{v}'$, $\boldsymbol{u}'$.

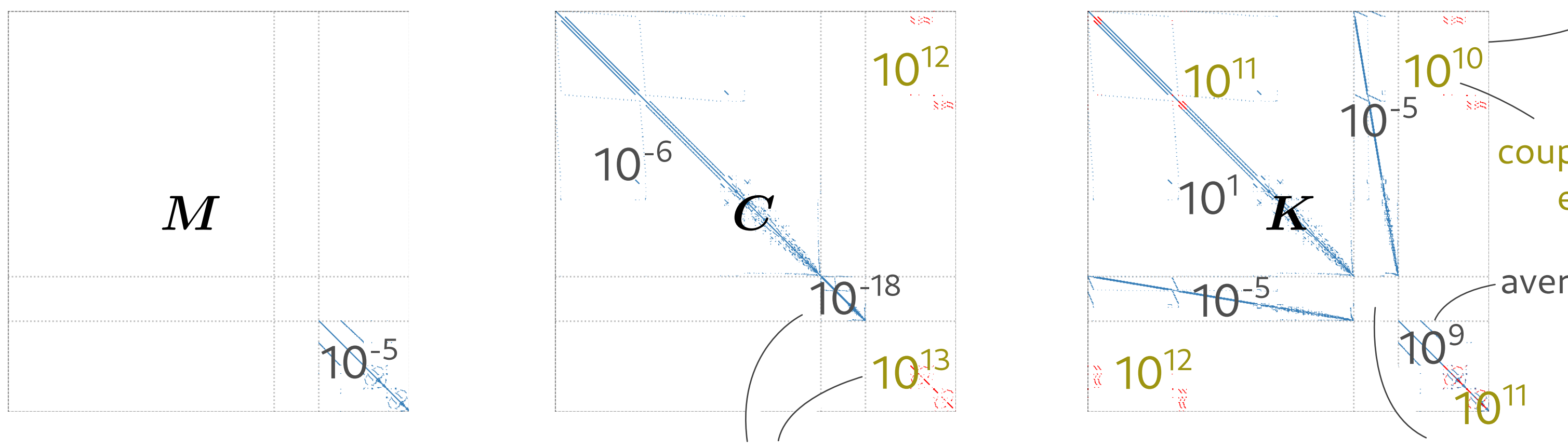
Discrestisation with finite elements:
Order q for pressure and q+1 for velocity, variable order for displacement.

frequency domain formulation

$$\boldsymbol{x}(t) = \hat{\boldsymbol{x}}e^{st}$$

$$\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{M}_{uu} \end{bmatrix} s^2 + \begin{bmatrix} \boldsymbol{C}_{vv} & 0 & \boldsymbol{C}_{vu} \\ 0 & \boldsymbol{C}_{pp} & 0 \\ 0 & 0 & \boldsymbol{C}_{uu} \end{bmatrix} s + \begin{bmatrix} \boldsymbol{K}_{vv} & \boldsymbol{K}_{vp} & \boldsymbol{K}_{vu} \\ \boldsymbol{K}_{pv} & 0 & 0 \\ \boldsymbol{K}_{uv} & 0 & \boldsymbol{K}_{uu} \end{bmatrix}\right) \begin{bmatrix} \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{p}} \\ \hat{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hat{\boldsymbol{f}}_u \end{bmatrix}$$

~76k degrees of freedom



average coupling/penalty entry size

average entry size

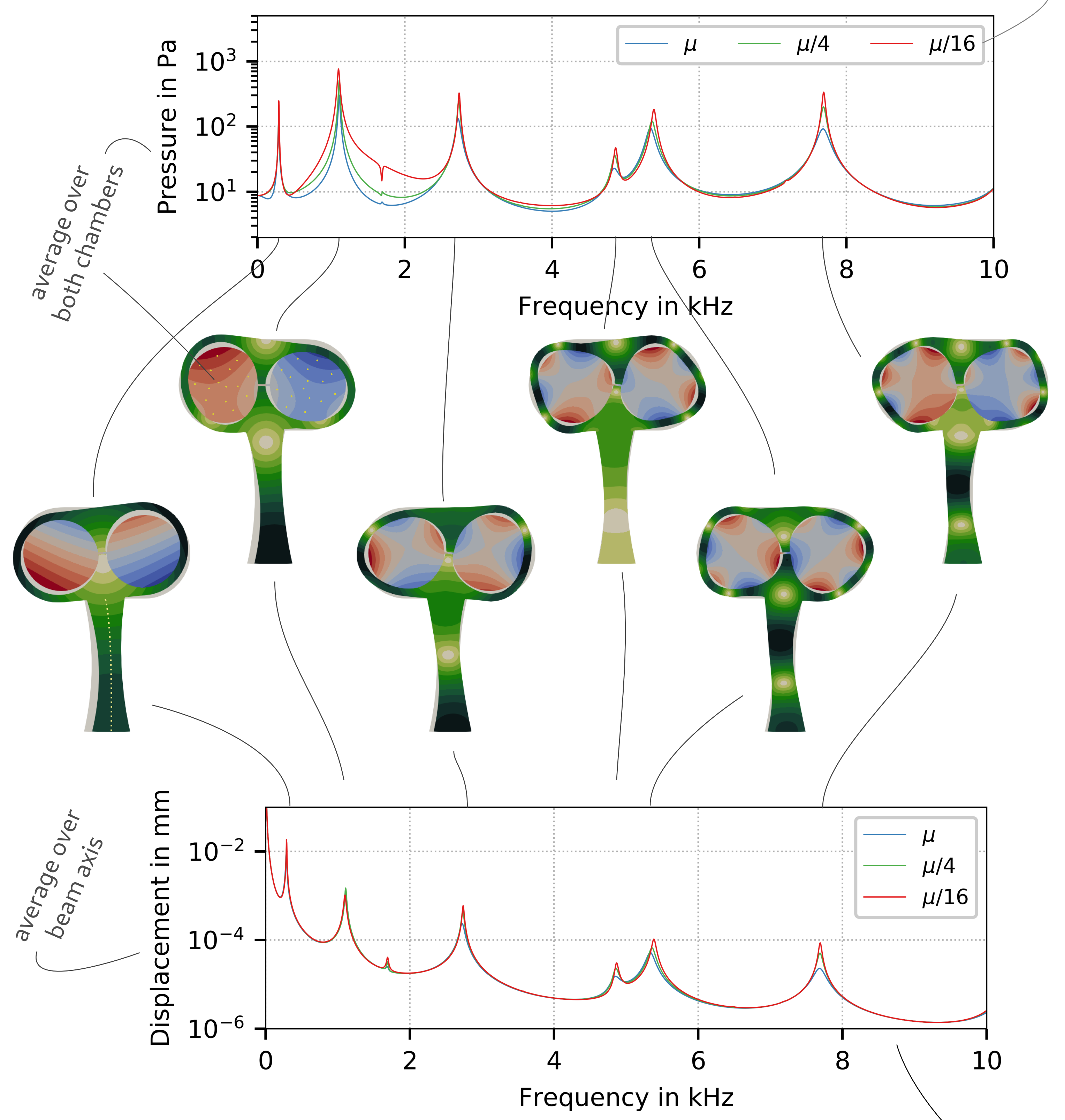
singular

ill-conditioned

unsymmetric & singular

System Behaviour

Direct solution of the full system ... for different dynamic viscosities



The solution of the full system is computationally expensive. Distinct peaks are clearly visible in the transfer functions, and the viscosity determines the amount of damping. Despite the badly conditioned system matrices the results seem accurate and physically plausible.

Model Order Reduction

using the standard reduced basis method

0 transform to first order system $\hat{\boldsymbol{y}} = s\hat{\boldsymbol{x}}$

$$\left(\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{K} & -\boldsymbol{C} \end{bmatrix} - s\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix}\right)\begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \hat{\boldsymbol{f}} \end{bmatrix}$$

L1: first companion form

$$\begin{bmatrix} -\boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} - s\begin{bmatrix} \boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

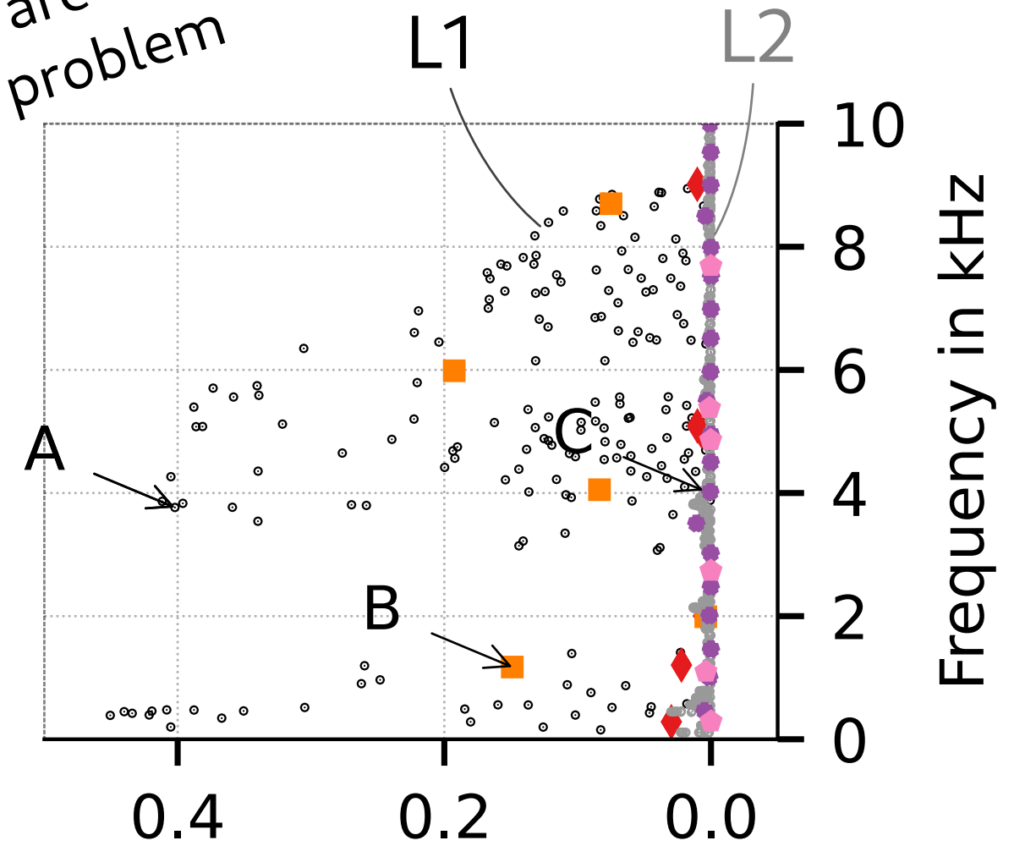
L2: second companion

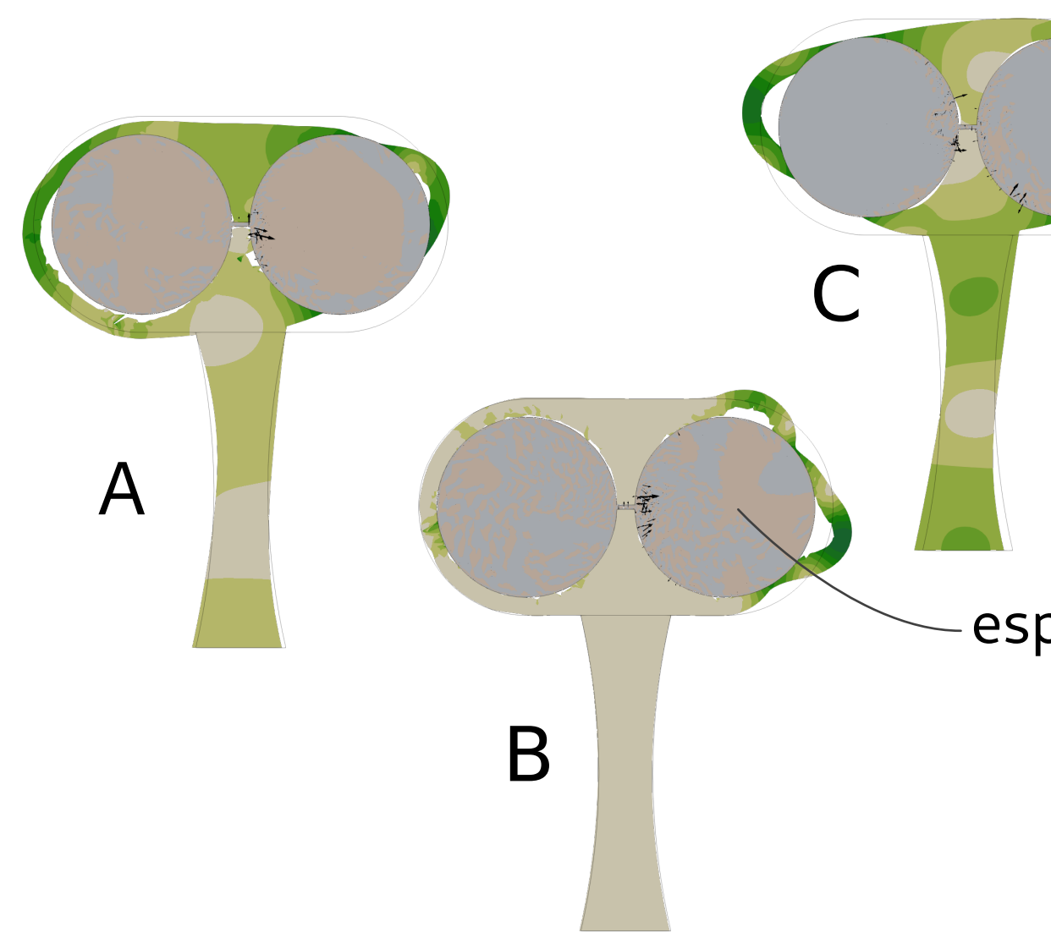
1 solve generalised eigenvalue problem

$$(\boldsymbol{A}_i - s\boldsymbol{B}_i)\,\hat{\boldsymbol{z}} = \boldsymbol{0}$$

both matrices are singular ill-posed problem

deviating results on repeated ARPACK runs





computed modes seem inaccurate

especially in fluid domain

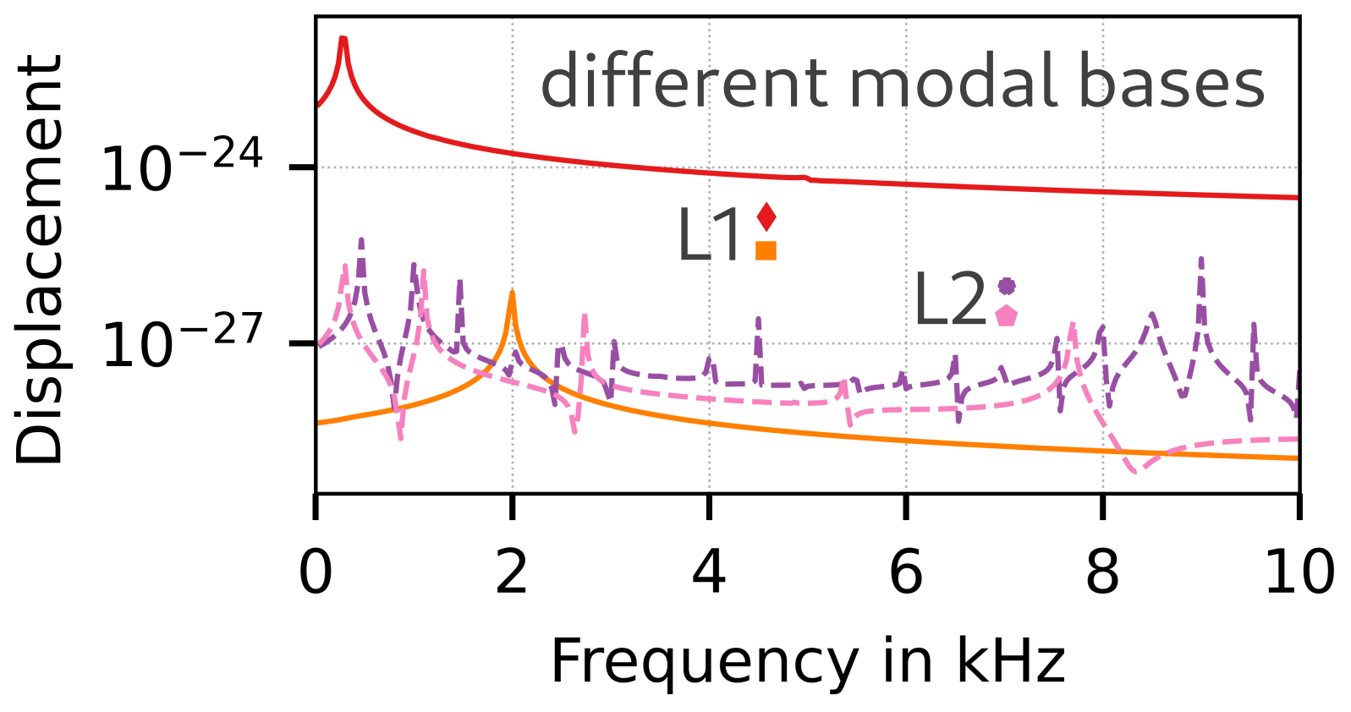
two sets from L1 or L2

2 select some eigenvectors to build a modal basis $\hat{\boldsymbol{z}} \approx \boldsymbol{Z}\hat{\boldsymbol{\eta}}$

3 project and solve reduced system

$$\boldsymbol{Z}^{-1}(\boldsymbol{A}_i - j\omega\boldsymbol{B}_i)\,\boldsymbol{Z}\hat{\boldsymbol{\eta}} = \boldsymbol{Z}^{-1}\hat{\boldsymbol{\zeta}}$$

no agreement with reference solution from full system



Model order reduction using the standard modal basis approch does not work for the coupled system. Possibly a suitable reduction base can be found from the modes of the un-coupled fluid and solid systems, respectively.
If you have suggestions, do not hesitate to get in touch ...

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F.Tisseur, K.Meerbergen: The quadratic eigenvalue problem.

F.Toth, M.Kaltenbacher: Coupling of incompressible free-surface flow, acoustic fluid and flexible structure via a modal basis.

F.Toth et.al. www.opencfs.org

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