*TPLP*: Page 1–51.  
doi:10.1017/S1471068421000132

# *Reasoning on DL-Lite$_{\mathcal{R}}$ with Defeasibility in ASP*

LORIS BOZZATO
*Fondazione Bruno Kessler, Via Sommarive 18, 38123 Trento, Italy*
(*e-mail:* bozzato@fbk.eu)

THOMAS EITER
*Technische Universität Wien, Favoritenstraße 9-11, A-1040 Vienna, Austria*
(*e-mail:* eiter@kr.tuwien.ac.at)

LUCIANO SERAFINI
*Fondazione Bruno Kessler, Via Sommarive 18, 38123 Trento, Italy*
(*e-mail:* serafini@fbk.eu)

*submitted 12 May 2020; revised 30 June 2021; accepted 4 July 2021*

**Abstract**

Reasoning on defeasible knowledge is a topic of interest in the area of description logics, as it is related to the need of representing exceptional instances in knowledge bases. In this direction, in our previous works we presented a framework for representing (contextualized) OWL RL knowledge bases with a notion of justified exceptions on defeasible axioms: reasoning in such framework is realized by a translation into ASP programs. The resulting reasoning process for OWL RL, however, introduces a complex encoding in order to capture reasoning on the negative information needed for reasoning on exceptions. In this paper, we apply the justified exception approach to knowledge bases in *DL-Lite*$_{\mathcal{R}}$, that is, the language underlying OWL QL. We provide a definition for *DL-Lite*$_{\mathcal{R}}$ knowledge bases with defeasible axioms and study their semantic and computational properties. In particular, we study the effects of exceptions over unnamed individuals. The limited form of *DL-Lite*$_{\mathcal{R}}$ axioms allows us to formulate a simpler ASP encoding, where reasoning on negative information is managed by direct rules. The resulting materialization method gives rise to a complete reasoning procedure for instance checking in *DL-Lite*$_{\mathcal{R}}$ with defeasible axioms.[1]

*KEYWORDS*: defeasible knowledge, description logics, answer set programming, justifiable exceptions

## 1 Introduction

Representing defeasible information is a topic of interest in the area of description logics (DLs), as it is related to the need of accommodating the presence of exceptional instances in knowledge bases. This interest led to different proposals for non-monotonic features in DLs based on different notions of defeasibility, for example, Bonatti *et al.* (2015), Bonatti *et al.* (2006), Britz and Varzinczak (2016), Casini and Straccia (2010), Giordano

[1] This paper is an extended and revised version of a conference paper appearing in the proceedings of the *3rd International Joint Conference on Rules and Reasoning (RuleML+RR 2019)* Bozzato *et al.* (2019a).
* We thank the reviewers for their constructive comments and suggestions to improve this paper.

et al. (2013), Pensel and Turhan (2018). In this direction, we presented in Bozzato et al. (2018) an approach to represent defeasible information in contextualized DL knowledge bases by introducing a notion of *justifiable exceptions* that has been inspired by ideas in Buccafurri et al. (1999): general *defeasible axioms* can be overridden by more specific exceptional instances if their application would provably lead to inconsistency. For example[2], we can express that *"in general, concerts are expensive"* as a defeasible concept inclusion $\mathit{Concert} \sqsubseteq \mathit{Expensive}$. However, for a specific instance *free_concert* of *Concert* representing a free concert we may want to "override" the defeasible axiom (i.e. disregard its application): in our approach this is possible, provided that one can prove a set of assertions $\{\mathit{Concert}(\mathit{free\_concert}), \neg \mathit{Expensive}(\mathit{free\_concert})\}$ that justify the exception for this individual.

In our seminal paper Bozzato et al. (2018), we concentrated on reasoning over $\mathcal{SROIQ}$-RL based knowledge bases: $\mathcal{SROIQ}$-RL corresponds to the language of the OWL RL profile of the *Web Ontology Language (OWL)* Motik et al. (2009) and allows for tractable reasoning. In particular, this language emerges from the $\mathcal{SROIQ}$ language Horrocks et al. (2006) and dl-programs Grosof et al. (2003). Remarkably, $\mathcal{SROIQ}$-RL can be seen as an intersection of DLs and Horn logic programs.

In Bozzato et al. (2018) reasoning in $\mathcal{SROIQ}$-RL knowledge bases is realized by a translation to datalog (under answer sets semantics), which provides a complete *materialization calculus* in the style of Krötzsch (2010) for instance checking and conjunctive query (CQ) answering. While the translation covers the full $\mathcal{SROIQ}$-RL language, it needs a complex encoding to represent reasoning on exceptions. In particular, it relies on proofs by contradiction to ensure completeness in presence of negative disjunctive information. In fact, negative disjunctive information is not easily expressible in datalog: for example, from $A \sqcap B \sqsubseteq C$ and $\neg C(a)$ we can derive $(\neg A \sqcup \neg B)(a)$, which is not directly representable by datalog rules. Also, a naive use of disjunction in rule heads does not overcome this problem. For this reason, in Bozzato et al. (2018) inference on negative literals is obtained as an encoding of a "test" for contradiction of such literals in the deduction rules of the datalog translation.

In this paper, we consider the case of knowledge bases with defeasible axioms in the description logic $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ Calvanese et al. (2007), which corresponds to the language underlying the OWL QL Profile Motik et al. (2009). As in the case of $\mathcal{SROIQ}$-RL, also $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ is a Horn logic and thus can be related to logic programs. In fact, $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ is a class of existential rules and falls then into the linear fragment Calì et al. (2012).

It is indeed interesting to show the applicability of our defeasible reasoning approach to the well-known *DL-Lite* family: by adopting $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ as the base logic, we need to take unnamed individuals introduced by existential quantifiers into account, especially for the justifications of exceptions. Defeasible axioms like $\mathrm{D}(\mathit{Concert} \sqsubseteq \exists \mathit{hasOrganizer})$, which says that concerts have some organizer, allow for a smooth handling. On the other hand, if we have axioms like $\exists \mathit{hasOrganizer}^- \sqsubseteq \mathit{Organizer}$, which informally assigns a type, and $\mathrm{D}(\mathit{Organizer} \sqsubseteq \mathit{Company})$, then overridings can happen over unnamed individuals relative to this axiom. The problem for reasoning with such unnamed elements is that they can have different interpretations in different models of the knowledge base, while, for determining the applicability of a defeasible axiom or its overriding, we need to identify the exceptional domain elements.

[2] see (Bozzato et al. 2018, Example 2).

Moreover, with respect to the translation to datalog, we show that due to the restricted form of its axioms, the *DL-Lite*$_{\mathcal{R}}$ language allows us to give a less involved datalog encoding in which reasoning on negative information is directly encoded in datalog rules; for more background, we refer to the discussion on "justification safeness" in Bozzato *et al.* (2018).

The choice of studying the application of our methods to *DL-Lite*$_{\mathcal{R}}$ knowledge bases is indeed motivated by the interest in covering the OWL QL fragment of OWL 2, which is relevant from an application perspective. More importantly, from a formal perspective, *DL-Lite*$_{\mathcal{R}}$ allows for the use of unnamed individuals in inverse roles which are not available in $\mathcal{EL}_{\bot}$ Bozzato *et al.* (2019b) and $\mathcal{SROIQ}$-RL Bozzato *et al.* (2018): thus, the techniques for managing unnamed individuals, especially in exceptions, need to be adapted to the expressivity of *DL-Lite*$_{\mathcal{R}}$. Another reason of our interest in *DL-Lite*$_{\mathcal{R}}$ stands in the fact that it is a standard DL which falls in the fragment where no reasoning on disjunctive negative information is needed for deductions on exceptions: this shows a notable example of "justification safe" language which, as noted above, allows us to formulate a simpler version of the datalog encoding for instance checking.

The contributions of this paper can be summarized as follows:

- In Section 3 we provide a definition of defeasible DL knowledge base (DKB) with justified models that draws from the definition of *Contextualized Knowledge Repositories (CKR)* Bozzato *et al.* (2012); Bozzato and Serafini (2013); Serafini and Homola (2012) with defeasible axioms provided in Bozzato *et al.* (2018). This allows us to concentrate on the defeasible reasoning aspects without considering the aspects related to context representation. In the case of *DL-Lite*$_{\mathcal{R}}$, we consider the effects of reasoning with unnamed individuals and of their admission in the exceptions of defeasible axioms. In particular, we consider models in which exceptions can only occur on individuals named in the DKB (called *exception safety*).
- In Section 4, we study the semantic properties of DKB models. In particular, in the case of exception safe DKBs, we show that their models preserve conditions from Bozzato *et al.* (2018) that allow us to concentrate on minimal models that are restricted to the individual names occurring in the knowledge base. These properties are important to verify the feasibility of the reasoning method based on the datalog translation that we provide in the later sections.
- For exception safe DKBs based on *DL-Lite*$_{\mathcal{R}}$, we provide in Section 5 a translation to datalog (under answer set semantics Gelfond and Lifschitz (1991)) that alters the translation in Bozzato *et al.* (2014), Bozzato *et al.* (2018) and prove its correctness for instance checking. Notably, the fact that reasoning on negative disjunctive information is not needed allows us to provide a simpler translation without the involving "test" environments mechanism of Bozzato *et al.* (2018). The datalog translation for *DL-Lite*$_{\mathcal{R}}$ DKBs is included in the latest version of the *CKR*ew *(CKR datalog rewriter)* prototype Bozzato *et al.* (2018), which is available online.[3]
- In Section 6 we provide complexity results for reasoning problems on exception safe DKBs based on *DL-Lite*$_{\mathcal{R}}$. Deciding satisfiability of such a DKB with respect to justified models is tractable, while inference of an axiom under cautious (i.e. certainty)

[3] http://ckrew.fbk.eu/.

semantics is co-NP-complete in general. Moreover, CQ answering is shown to be $\Pi_2^p$-complete.

– In Section 7 we discuss how reasoning on unnamed exceptional instances affects the complexity, and in particular how the notion of exception safety can be generalized in a way such that the techniques developed and the results obtained can be lifted to this setting. We present for this the class of $n$-derivation exception (de) safe programs, which for a few ($n$ bounded by a constant) unnamed individuals in exceptions stays within the same complexity and for polynomially many faces an increase by at most one level in the polynomial hierarchy. Furthermore, we discuss how it is possible to extend the current datalog translation to manage unnamed elements under de-safety.

With respect to the initial conference paper presented at *RuleML+RR 2019* Bozzato *et al.* (2019a), this version of the paper extends the work by including a study of properties of DKB models and justifications over unnamed individuals in Sections 3 and 7, where the notion of $n$-de safe DKBs is introduced and the extension of results to this setting is discussed. The current paper also includes a more detailed study for semantic properties of DKB models (Section 4) and complexity of reasoning problems (Section 6). With respect to the ASP translation, rules have been slightly simplified; moreover, the current translation has been implemented in the CKR*ew* prototype. Finally, with respect to the conference paper, we provide further details and comparisons on related work in Section 8. To increase readability and the comprehension of the contributions, additional details and proofs of the results are reported in the Appendix.

## 2 Preliminaries

**Description logics and *DL-Lite*$_\mathcal{R}$ language.** We assume the common definitions of description logics Baader *et al.* (2003) and the definition of the logic *DL-Lite*$_\mathcal{R}$ Calvanese *et al.* (2007): we summarize in the following the basic definitions used in this work. For ease of reference, we present in Table A1 in the Appendix the details of syntax and semantics of *DL-Lite*$_\mathcal{R}$.

A *DL vocabulary* $\Sigma$ consists of the mutually disjoint countably infinite sets NC of *atomic concepts*, NR of *atomic roles*, and NI of *individual constants*. Intuitively, concepts represent classes of objects (e.g. *PhDStudent*), roles represent binary relations across objects (e.g. *hasCourse*), and individual names identify specific elements of the domain (e.g. *bob*). Complex *concepts* are then recursively defined as the smallest sets containing all concepts that can be inductively constructed using the constructors of the considered DL language (see, e.g. Table A1 for *DL-Lite*$_\mathcal{R}$).

A *DL-Lite*$_\mathcal{R}$ *knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ consists of: a TBox $\mathcal{T}$ containing *general concept inclusion (GCI)* axioms $C \sqsubseteq D$ where $C, D$ are concepts, of the form:

$$C := A \mid \exists R \qquad\qquad D := A \mid \neg C \mid \exists R,$$

where $A \in$ NC and $R \in$ NR;[4] an RBox $\mathcal{R}$ containing *role inclusion (RIA)* axioms $S \sqsubseteq R$, reflexivity, irreflexivity, inverse and role disjointness axioms, where $S, R$ are roles; and an ABox $\mathcal{A}$ composed of assertions of the forms $D(a)$, $R(a, b)$, with $R \in$ NR and $a, b \in$ NI.

[4] In the following, we will use $C$ to denote a left side concept and $D$ as a right side concept.

*Example 1*
The TBox $\mathcal{T}$ may include concept inclusion expressions such as $PhDStudent \sqsubseteq \neg\exists hasCourse$; the RBox $\mathcal{R}$ may contain a role inclusion $hasAdvisor \sqsubseteq isStudentOf$; and finally, the ABox $\mathcal{A}$ may contain assertions $\neg Professor(bob), hasAdvisor(bob, alice)$.[5] ◇

A *DL interpretation* is a pair $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ where $\Delta^\mathcal{I}$ is a non-empty set called *domain* and $\cdot^\mathcal{I}$ is the *interpretation function* which assigns denotations for language elements: $a^\mathcal{I} \in \Delta^\mathcal{I}$, for $a \in \mathrm{NI}$; $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, for $A \in \mathrm{NC}$; $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, for $R \in \mathrm{NR}$. The interpretation of nonatomic concepts and roles is defined by the evaluation of their description logic operators (see Table A1 and Calvanese *et al.* (2007) for DL-Lite$_\mathcal{R}$). An interpretation $\mathcal{I}$ *satisfies* an axiom $\phi$, denoted $\mathcal{I} \models_{\mathrm{DL}} \phi$, if it verifies the respective semantic condition, in particular: for $\phi = D(a)$, $a^\mathcal{I} \in D^\mathcal{I}$; for $\phi = R(a,b)$, $\langle a^\mathcal{I}, b^\mathcal{I} \rangle \in R^\mathcal{I}$; for $\phi = C \sqsubseteq D$, $C^\mathcal{I} \subseteq D^\mathcal{I}$ (resp. for RIAs). $\mathcal{I}$ is a *model* of $\mathcal{K}$, denoted $\mathcal{I} \models_{\mathrm{DL}} \mathcal{K}$, if it satisfies all axioms of $\mathcal{K}$.

*Example 2 (cont'd)*
An interpretation $\mathcal{I}$ satisfies $\neg Professor(bob)$ if $bob^\mathcal{I} \notin Professor^\mathcal{I}$, and $\mathcal{I}$ satisfies $PhDStudent \sqsubseteq \neg\exists hasCourse$ if, for every element $d$ of $PhDStudent^\mathcal{I}$, there does not exist some domain element $e$ such that $\langle d, e \rangle \in hasCourse^\mathcal{I}$. ◇

Without loss of generality, we adopt the *standard name assumption (SNA)* in the DL context (see de Bruijn *et al.* (2008), Eiter *et al.* (2008) for more details). That is, we assume an infinite subset $\mathrm{NI}_S \subseteq \mathrm{NI}$ of individual constants, called *standard names* s.t. in every interpretation $\mathcal{I}$ we have (i) $\Delta^\mathcal{I} = \mathrm{NI}_S^\mathcal{I} = \{c^\mathcal{I} \mid c \in \mathrm{NI}_S\}$; (ii) $c^\mathcal{I} \neq d^\mathcal{I}$, for every distinct $c, d \in \mathrm{NI}_S$. Thus, we may assume that $\Delta^\mathcal{I} = \mathrm{NI}_S$ and $c^\mathcal{I} = c$ for each $c \in \mathrm{NI}_S$. The *unique name assumption (UNA)* corresponds to assuming $c \neq d$ for all constants in $\mathrm{NI} \setminus \mathrm{NI}_S$ resp. occurring in the knowledge base.[6] We confine here to knowledge bases without reflexivity axioms. The reason is that reflexivity allows one to derive positive properties for any (named and unnamed) individual, thus complicating the treatment of defeasible axioms.

**Datalog programs and Answer Sets.** We express our rules in *datalog with negation* under answer sets semantics. In fact, we use here two kinds of negation[7]: strong ("classical") negation $\neg$ and weak *(default) negation* `not` under the interpretation of answer sets semantics Gelfond and Lifschitz (1991); the latter is in particular needed for representing defeasibility.

A *signature* is a tuple $\langle \mathbf{C}, \mathbf{P} \rangle$ of a finite set $\mathbf{C}$ of *constants* and a finite set $\mathbf{P}$ of *predicates*. We assume a set $\mathbf{V}$ of *variables*; the elements of $\mathbf{C} \cup \mathbf{V}$ are *terms*. An *atom* is of the form $p(t_1, \ldots, t_n)$ where $p \in \mathbf{P}$ and $t_1, \ldots, t_n$, are terms. A *literal* $l$ is either a *positive literal* $p$ or a *negative literal* $\neg p$, where $p$ is an atom and $\neg$ is strong negation. Literals of the form $p$, $\neg p$ are *complementary*. We denote with $\neg.l$ the opposite of literal

[5] For simplicity, in the following examples, we may represent knowledge bases as set of axioms with implicit separation of TBox, RBox and ABox.
[6] Under the SNA, equality between elements can be embedded using a binary predicate $\approx$ that satisfies the usual congruence axioms Fitting (1996).
[7] Strong negation can be easily emulated using fresh atoms and weak negation resp. constraints. While it does not yield higher expressiveness, it is more convenient for presentation.

$l$, that is, $\neg.p = \neg p$ and $\neg.\neg p = p$ for an atom $p$. A (datalog) rule $r$ is an expression:

$$a \leftarrow b_1, \ldots, b_k, \texttt{not}\, b_{k+1}, \ldots, \texttt{not}\, b_m, \tag{1}$$

where $a, b_1, \ldots, b_m$ are literals. We denote with $\mathit{Head}(r)$ the head $a$ of rule $r$ and with $\mathit{Body}(r) = \{b_1, \ldots, b_k, \texttt{not}\, b_{k+1}, \ldots, \texttt{not}\, b_m\}$ the body of $r$, respectively. A (datalog) *program* $P$ is a finite set of rules. An atom (rule etc.) is *ground* if no variables occur in it. A *ground substitution* $\sigma$ for $\langle \mathbf{C}, \mathbf{P} \rangle$ is any function $\sigma : \mathbf{V} \to \mathbf{C}$; the *ground instance* of an atom (rule, etc.) $\chi$ from $\sigma$, denoted $\chi\sigma$, is obtained by replacing in $\chi$ each occurrence of variable $v \in \mathbf{V}$ with $\sigma(v)$. A *fact* $H$ is a ground rule $r$ with empty body. The *grounding* of a rule $r$, $\mathit{grnd}(r)$, is the set of all ground instances of $r$, and the *grounding* of a program $P$ is $\mathit{grnd}(P) = \bigcup_{r \in P} \mathit{grnd}(r)$.

Given a program $P$, the *(Herbrand) universe* $U_P$ of $P$ is the set of all constants occurring in $P$ and the *(Herbrand) base* $B_P$ of $P$ is the set of all the ground literals constructable from the predicates in $P$ and the constants in $U_P$. An *interpretation* $I \subseteq B_P$ is any satisfiable subset of $B_P$ (i.e. not containing complementary literals); a literal $l$ is *true* in $I$, denoted $I \models l$, if $l \in I$, and $l$ is *false* in $I$ if $\neg.l$ is true. Given a rule $r \in \mathit{grnd}(P)$, we say that $\mathit{Body}(r)$ is true in $I$, denoted $I \models \mathit{Body}(r)$, if (i) $I \models b$ for each literal $b \in \mathit{Body}(r)$ and (ii) $I \not\models b$ for each literal $\texttt{not}\, b \in \mathit{Body}(r)$. A rule r is *satisfied* in $I$, denoted $I \models r$, if either $I \models \mathit{Head}(r)$ or $I \not\models \mathit{Body}(r)$. An interpretation $I$ is a *model* of $P$, denoted $I \models P$, if $I \models r$ for each $r \in \mathit{grnd}(P)$; moreover, $I$ is *minimal*, if $I' \not\models P$ for each subset $I' \subset I$.

Given an interpretation $I$ for $P$, the *reduct* of $P$ w.r.t. $I$ Gelfond and Lifschitz (1991), denoted by $G_I(P)$, is the set of rules obtained from $\mathit{grnd}(P)$ by (i) removing every rule $r$ such that $I \models l$ for some $\texttt{not}\, l \in \mathit{Body}(r)$; and (ii) removing the NAF part from the bodies of the remaining rules. Then, $I$ is an *answer set* of $P$, if $I$ is a minimal model of $G_I(P)$; the minimal model is unique and exists iff $G_I(P)$ has some model. Moreover, if $M$ is an answer set for $P$, then $M$ is a minimal model of $P$. We say that a literal $a \in B_P$ is a *consequence* of $P$ and write $P \models a$ if every answer set $M$ of $P$ fulfills $M \models a$.

## 3 DL knowledge base with justifiable exceptions

In this paper we concentrate on reasoning over a DL knowledge base enriched with *defeasible axioms*, whose syntax and interpretation are analogous to Bozzato *et al.* (2018). With respect to the contextual framework presented in Bozzato *et al.* (2018), this corresponds to reasoning inside a single local context: while this simplifies the presentation of defeasibility aspects and the resulting reasoning method for the case of *DL-Lite*$_\mathcal{R}$, it can be generalized to the original case of multiple local contexts.

**Syntax.** Given a DL language $\mathcal{L}_\Sigma$ based on a DL vocabulary $\Sigma = \mathrm{NC} \cup \mathrm{NR} \cup \mathrm{NI}$, a *defeasible axiom* is any expression of the form $\mathrm{D}(\alpha)$, where $\alpha \in \mathcal{L}_\Sigma$.

We denote with $\mathcal{L}^{\mathrm{D}}_\Sigma$ the DL language extending $\mathcal{L}_\Sigma$ with the set of defeasible axioms in $\mathcal{L}_\Sigma$. On the base of such language, we provide our definition of knowledge base with defeasible axioms.

*Definition 1* (*defeasible knowledge base, DKB*)
A *defeasible knowledge base (DKB)* $\mathcal{K}$ on a vocabulary $\Sigma$ is a DL knowledge base over $\mathcal{L}^{\mathrm{D}}_\Sigma$.

In the following, we tacitly consider DKBs based on *DL-Lite*$_{\mathcal{R}}$.

*Example 3*
We introduce a simple example showing the definition and interpretation of a defeasible existential axiom. In the organization of a university research department, we want to specify that "in general" department members need also to teach at least a course. On the other hand, PhD students, while recognized as department members, are not allowed to hold a course. We can represent this scenario as a DKB $\mathcal{K}_{dept}$ where:

$$\mathcal{K}_{dept}: \left\{ \begin{array}{l} \mathrm{D}(\mathit{DeptMember} \sqsubseteq \exists \mathit{hasCourse}), \mathit{Professor} \sqsubseteq \mathit{DeptMember}, \\ \mathit{PhDStudent} \sqsubseteq \mathit{DeptMember}, \mathit{PhDStudent} \sqsubseteq \neg\exists \mathit{hasCourse}, \\ \mathit{Professor}(\mathit{alice}),\ \mathit{PhDStudent}(\mathit{bob}) \end{array} \right\}.$$

Intuitively, we want to override the fact that there exists some course assigned to the PhD student *bob*. On the other hand, for the individual *alice* no overriding should happen and the defeasible axiom can be applied. ◇

**Semantics.** We can now define a model based interpretation of DKBs, in particular by providing a semantic characterization to defeasible axioms.

Similarly to the case of $\mathcal{SROIQ}$-RL in Bozzato *et al.* (2018), we can express *DL-Lite*$_{\mathcal{R}}$ knowledge bases in first-order (FO) logic, where every axiom $\alpha \in \mathcal{L}_\Sigma$ is translated into an equivalent FO sentence $\forall \vec{x}.\phi_\alpha(\vec{x})$ where $\vec{x}$ contains all free variables of $\phi_\alpha$ depending on the type of the axiom. The translation, depending on the axiom types, is natural and can be defined analogously to the FO translation presented in Bozzato *et al.* (2018).[8] In the case of existential axioms of the kind $\alpha = A \sqsubseteq \exists R$, the FO translation $\phi_\alpha(\vec{x})$ is defined as:

$$A(x_1) \rightarrow R(x_1, f_R(x_1))\,;$$

that is, we introduce a Skolem function $f_R(x_1)$ which represents new "existential" individuals. Formally, for every atomic role $R \in \mathrm{NR}$ we define a Skolem function $f_R$. In particular, for a set of individual names $N \subseteq \mathrm{NI}$, we will write $sk(N)$ to denote the extension of $N$ with the set of Skolem constants for elements in $N$, that is, for each name $a \in N$, $sk(N)$ also contains $f_R(a)$ for each $f_R$ as above.

After this transformation, the resulting formulas $\phi_\alpha(\vec{x})$ amount semantically to Horn formulas, since left side concepts $C$ can be expressed by an existential positive FO formula, and right side concepts $D$ by a conjunction of Horn clauses. The following property from (Bozzato *et al.* 2018, Section 3.2) is then preserved for *DL-Lite*$_{\mathcal{R}}$ knowledge bases.

*Lemma 1*
For a DL knowledge base $\mathcal{K}$ on $\mathcal{L}_\Sigma$, its FO translation $\phi_{\mathcal{K}} := \bigwedge_{\alpha\in\mathcal{K}} \forall\vec{x}.\phi_\alpha(\vec{x})$ is semantically equivalent to a conjunction of universal Horn clauses.

We remark that the introduction of Skolem functions does not allow us to work on proper Herbrand models of the original language as in Bozzato *et al.* (2018), since they introduce new Skolem terms in the language. As we will see in the following, exceptions on these elements need further conditions to be defined.

With these considerations on the definition of FO translation, we can now provide our definition of axiom instantiation:

[8] A FO translation for *DL-Lite*$_{\mathcal{R}}$ axioms is provided in Appendix B.

Table 1. *(Minimal) clashing sets for DL-Lite$_\mathcal{R}$ clashing assumptions.*

| | | | |
|---|---|---|---|
| $\langle A(a), a\rangle$ : | $\{\neg A(a)\}$ | $\langle R \sqsubseteq T, (e_1, e_2)\rangle$ : | $\{R(e_1, e_2), \neg T(e_1.e_2)\}$ |
| $\langle \neg A(a), a\rangle$ : | $\{A(a)\}$ | $\langle \mathrm{Dis}(R, S), (e_1, e_2)\rangle$ : | $\{R(e_1, e_2), S(e_1, e_2)\}$ |
| $\langle R(a, b), (a, b)\rangle$ : | $\{\neg R(a, b)\}$ | $\langle \mathrm{Inv}(R, S), (e_1, e_2)\rangle$ : | $\{R(e_1, e_2), \neg S(e_2, e_1)\}$, $\{\neg R(e_1, e_2), S(e_2, e_1)\}$ |
| $\langle A \sqsubseteq B, e\rangle$ : | $\{A(e), \neg B(e)\}$ | $\langle \mathrm{Irr}(R), e\rangle$ : | $\{R(e, e)\}$ |
| $\langle A \sqsubseteq \neg B, e\rangle$ : | $\{A(e), B(e)\}$ | | |
| $\langle \exists R \sqsubseteq B, e\rangle$ : | $\{\exists R(e), \neg B(e)\}$ | | |
| $\langle A \sqsubseteq \exists R, e\rangle$ : | $\{A(e), \neg \exists R(e)\}$ | | |

*Definition 2* (*axiom instantiation*)
Given an axiom $\alpha \in \mathcal{L}_\Sigma$ with FO translation $\forall \vec{x}.\phi_\alpha(\vec{x})$, the instantiation of $\alpha$ with a tuple $\mathbf{e}$ of individuals in NI, written $\alpha(\mathbf{e})$, is the specialization of $\alpha$ to $\mathbf{e}$, that is, $\phi_\alpha(\mathbf{e})$, depending on the type of $\alpha$.

Note that, since we are assuming standard names, this basically means that we can express instantiations (and exceptions) to any element of the domain (identified by a standard name in $\mathrm{NI}_S$). We next introduce clashing assumptions and clashing sets.

*Definition 3* (*clashing assumptions and sets*)
A *clashing assumption* is a pair $\langle \alpha, \mathbf{e}\rangle$ s.t. $\alpha(\mathbf{e})$ is an instantiation for an axiom $\alpha \in \mathcal{L}_\Sigma$. A *clashing set* for a clashing assumption $\langle \alpha, \mathbf{e}\rangle$ is a satisfiable set $S$ that consists of ABox assertions over $\mathcal{L}_\Sigma$ and negated ABox assertions of the forms $\neg C(a)$ and $\neg R(a, b)$ such that $S \cup \{\alpha(\mathbf{e})\}$ is unsatisfiable.

A clashing assumption $\langle \alpha, \mathbf{e}\rangle$ represents that $\alpha(\mathbf{e})$ is not satisfiable, while a clashing set $S$ provides an assertional "justification" for the assumption of local overriding of $\alpha$ on $\mathbf{e}$. In Table 1 we show the form of clashing sets for axioms of *DL-Lite*$_\mathcal{R}$. For example, in the case of an atomic concept inclusion defeasible axiom $\mathrm{D}(A \sqsubseteq B)$ in a context $\mathsf{c}$, a clashing assumption $\langle A \sqsubseteq B, e\rangle$ states the assumption that $A \sqsubseteq B$ is not satisfiable for $e$ in $\mathsf{c}$; a clashing set $S = \{A(e), \neg B(e)\}$ provides a justification for the assumption on the overriding of $A \sqsubseteq B$ on $e$ in $\mathsf{c}$. We can then extend the notion of DL interpretation with a set of clashing assumptions.

*Definition 4* (*CAS interpretation*)
A *CAS interpretation* is a structure $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi\rangle$ where $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I}\rangle$ is a DL interpretation for $\Sigma$ and $\chi$ is a set of clashing assumptions.

By extending the notion of satisfaction with respect to CAS interpretations, we can disregard the application of defeasible axioms to the exceptional elements in the sets of clashing assumptions. For convenience, we call two DL interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ NI-*congruent*, if $c^{\mathcal{I}_1} = c^{\mathcal{I}_2}$ holds for every $c \in \mathrm{NI}$.

*Definition 5* (*CAS model*)
Given a DKB $\mathcal{K}$, a CAS interpretation $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi\rangle$ is a CAS model for $\mathcal{K}$ (denoted $\mathcal{I}_{CAS} \models \mathcal{K}$), if the following holds:

(i) for every $\alpha \in \mathcal{L}_\Sigma$ in $\mathcal{K}$, $\mathcal{I} \models \alpha$;
(ii) for every $\mathrm{D}(\alpha) \in \mathcal{K}$ (where $\alpha \in \mathcal{L}_\Sigma$), with $|\vec{x}|$-tuple $\vec{d}$ of elements in $\mathrm{NI}_\Sigma$ such that $\vec{d} \notin \{\mathbf{e} \mid \langle \alpha, \mathbf{e}\rangle \in \chi\}$, we have $\mathcal{I} \models \phi_\alpha(\vec{d})$.

We say that a clashing assumption $\langle \alpha, \mathbf{e} \rangle \in \chi$ is *justified* for a *CAS*-model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$, if some clashing set $S = S_{\langle \alpha, \mathbf{e} \rangle}$ exists such that, for every CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ of $\mathcal{K}$ that is NI-congruent with $\mathcal{I}_{CAS}$, it holds that $\mathcal{I}' \models S_{\langle \alpha, \mathbf{e} \rangle}$. We then consider as DKB models only the CAS models where all clashing assumptions are justified.

*Definition 6* (*justified CAS model and DKB model*)
A *CAS*-model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of a DKB $\mathcal{K}$ is *justified*, if every $\langle \alpha, \mathbf{e} \rangle \in \chi$ is justified. An interpretation $\mathcal{I}$ is a *DKB model* of $\mathcal{K}$ (in symbols, $\mathcal{I} \models \mathcal{K}$), if $\mathcal{K}$ has some justified *CAS*-model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$.

*Example 4*
Reconsidering $\mathcal{K}_{dept}$ in Example 3, a CAS model providing the intended interpretation of defeasible axioms is $\mathcal{I}_{CAS_{dept}} = \langle \mathcal{I}, \chi_{dept} \rangle$ where $bob^\mathcal{I} \neq alice^\mathcal{I}$ and $\chi_{dept} = \{\langle \alpha, bob \rangle\}$ with $\alpha = DeptMember \sqsubseteq \exists hasCourse$. The fact that this model is justified is verifiable considering that for the clashing set $S = \{DeptMember(bob), \neg\exists hasCourse(bob)\}$ we have $\mathcal{I} \models S$. On the other hand, note that a similar clashing assumption for *alice* is not justifiable: it is not possible from the contents of $\mathcal{K}_{dept}$ to derive a clashing set $S'$ such that $S' \cup \{\alpha(alice)\}$ is unsatisfiable. By Definition 5, this allows us to apply $\alpha$ to this individual as expected and thus $\mathcal{I} \models \exists hasCourse(alice)$. ◇

*Example 5* (*Nixon Diamond*)
Note that different combinations of clashing assumptions can lead to different and alternative justified CAS models and thus alternative DKB models. We can show this by considering the classic example of the *Nixon Diamond* as presented in (Bonatti *et al.* 2015, Example 9) (see also the example in (Bozzato *et al.* 2018, Section 7.4)). Let $\mathrm{K}_{nd}$ be a DKB defined as follows:

$$\mathcal{K}_{nd} : \left\{ \begin{array}{l} \mathrm{D}(Quacker \sqsubseteq Pacifist), \mathrm{D}(Republican \sqsubseteq \neg Pacifist), \\ Quacker(nixon),\ Republican(nixon) \end{array} \right\}.$$

This DKB has two possible overridings of the two defeasible axioms (having the same priority), which lead to two possible DKB models $\mathcal{I}_1$ and $\mathcal{I}_2$. In particular, in model $\mathcal{I}_1$ we have a clashing assumption $\chi_1 = \{\langle Republican \sqsubseteq \neg Pacifist, nixon \rangle\}$ that is justified by the clashing set $\{Republican(nixon), Pacifist(nixon)\}$: in this model we have that $\mathcal{I}_1 \models Pacifist(nixon)$. Similarly, in model $\mathcal{I}_2$ we have the clashing assumption $\chi_2 = \{\langle Quaker \sqsubseteq Pacifist, nixon \rangle\}$ with clashing set $\{Quaker(nixon), \neg Pacifist(nixon)\}$: then, in this model $\mathcal{I}_2 \models \neg Pacifist(nixon)$.

Thus, we obtain that $\mathcal{K}_{nd} \not\models Pacifist(nixon)$ and $\mathcal{K}_{nd} \not\models \neg Pacifist(nixon)$. Similarly, the approach presented in Bonatti *et al.* (2015) can not derive $Pacifist(nixon)$ or $\neg Pacifist(nixon)$: however, as we have shown in Bozzato *et al.* (2018), differently from this approach, in our semantics we can use the alternative models to enable "reasoning by cases". For example, consider the DKB $\mathcal{K}'_{nd}$ obtained from $\mathcal{K}_{nd}$ by substituting $\mathrm{D}(Republican \sqsubseteq \neg Pacifist)$ with the axioms $\mathrm{D}(Republican \sqsubseteq Hawk)$, $Hawk \sqsubseteq \neg Pacifist$, $Hawk \sqsubseteq Activist$, $Pacifist \sqsubseteq Activist$. Then, differently from Bonatti *et al.* (2015), even with alternative models given by the possible instantiation of clashing assumptions, we obtain that $\mathcal{K}'_{nd} \models Activist(nixon)$. ◇

We are interested in DKB models $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ in which clashing assumptions $\langle \alpha, \mathbf{e} \rangle \in \chi$ are admitted over individuals that are both named and unnamed in the knowledge base. That is, we want to admit also exceptions over individuals introduced by existential axioms. However, we should have the means to limit the existence of such individuals in exceptions in order to control the reasoning from such models.

Let us denote by $N_\mathcal{K}$ the individuals occurring in $\mathcal{K}$. A condition that allows us to control the number of unnamed individuals in models is the following.

*Definition 7* ($n$*-boundedness*)
A CAS interpretation $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ for a DKB $\mathcal{K}$ is $n$*-bounded* for $n \geq 0$, if in $\chi$ at most $n$ elements occur that are not named by $\mathcal{K}$, that is, it holds that $|uni_\mathcal{K}(\mathcal{I}_{CAS})| \leq n$ where $uni_\mathcal{K}(\mathcal{I}_{CAS}) = \{e_1^\mathcal{I}, \ldots, e_k^\mathcal{I} \in \mathrm{NI}_S \mid \langle \alpha, (e_1, \ldots, e_k) \rangle \in \chi\} \setminus \{c^\mathcal{I} \mid c \in N_\mathcal{K}\}$.

If this condition holds, we can show that unnamed individuals appearing in clashing assumptions can be always linked to named individuals from the DKB. Given a DKB $\mathcal{K}$, let us denote by $\mathcal{K}_s$ the knowledge base where all defeasible axioms are turned into strict axioms.

*Lemma 2*
Suppose that $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ is a justified CAS model of a DKB $\mathcal{K}$ and that an element $e$ occurring in $\langle \alpha, \mathbf{e} \rangle \in \chi$ is not named by $\mathcal{K}$. Then, there exists a role chain $R_1^\mathcal{I}(e_0, e_1), \ldots, R_m^\mathcal{I}(e_{m-1}, e_m)$ where $e_0 = a^\mathcal{I}$ for some $a \in N_\mathcal{K}$, $e_m = e$, and $e_{i+1} = f_{R_{i+1}}(e_i)$.

That is, elements in clashing assumptions that are not named by the DKB must be linked to it by a "Skolem chain".

As our main definition for limiting models to $n$-bounded CAS interpretations, we consider the following syntactic condition restricting unnamed individuals appearing in clashing sets.

*Definition 8* ($n$*-derivation exception (de) safety*)
A DKB $\mathcal{K}$ is $n$*-derivation exception (de) safe*, if $m \leq n$ Skolem terms $t_1, \ldots, t_m$ exist such that for every positive assertion $D(e_1)$ resp. $R(e_1, e_2)$ from a possible clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ for any $D(\alpha) \in \mathcal{K}$ and atom $D(t_1')$ resp. $R(t_1', t_2')$ that is derivable from $\mathcal{K}_s$ (in FO under Skolemization), it holds that we have $t_1' \in N_K \cup \{t_1, \ldots, t_m\}$ resp. $t_1', t_2' \in N_K \cup \{t_1, \ldots, t_m\}$.

In particular, for $n = 0$ we obtain that no exception on an unnamed individual can be derived: in this case, we say that $\mathcal{K}$ is *exception safe*. If $\mathcal{K}$ is acyclic,[9] then it is $n$-de safe for some $n$ that is exponential in the size of $\mathcal{K}$ in general, which drops to polynomial if derivations are feasible in constantly many steps.

*Example 6* (*Ex. 4 cont'd*)
Reconsider the CAS model $\mathcal{I}_{CAS_{dept}} = \langle \mathcal{I}, \chi_{dept} \rangle$ where $\chi_{dept} = \{\langle \alpha, bob \rangle\}$ with $bob^\mathcal{I} \neq alice^\mathcal{I}$, and $\alpha = \mathit{DeptMember} \sqsubseteq \exists \mathit{hasCourse}$. If we make $\alpha$ strict, we cannot derive a clashing set $S = \{\mathit{DeptMember}(e), \neg\exists \mathit{hasCourse}(e)\}$ where $e$ is an unnamed individual; to derive $\mathit{DeptMember}(e)$, it would require some axiom $\exists R^- \sqsubseteq \mathit{DeptMember}$ where some unnamed individual is introduced by some axiom $A \sqsubseteq \exists R$; however, no such former axioms can be derived, and thus $\mathcal{K}_{dept}$ is exception safe. ◇

[9] $\mathcal{K}$ is acyclic, if there is no sequence of axioms $E_0 \sqsubseteq E_1$, $E_1 \sqsubseteq E_2, \ldots, E_{k-1} \sqsubseteq E_k$ such that $E_k = E_0$.

However, if $\mathcal{K}$ is cyclic, it may be not $n$-de safe for every $n \geq 0$.

*Example 7*
Let us consider the DKB $\mathcal{K} = \{\mathit{Employee} \sqsubseteq \exists \mathit{hasSupervisor}, \exists \mathit{hasSupervisor}^- \sqsubseteq \mathit{Employee}, \mathrm{D}(\exists \mathit{hasSupervisor}^- \sqsubseteq \bot), \mathit{Employee}(\mathit{alice})\}$.[10] Informally, every employee and so *alice* has a supervisor that is an employee, and unless provable to the contrary, an individual is not a supervisor. This KB has an infinite feed of Skolem terms $f^n(\mathit{alice})$, $n \geq 1$ into the defeasible axiom by the chain $\mathit{hasSupervisor}(\mathit{alice}, f(\mathit{alice}))$, $\mathit{hasSupervisor}(f(\mathit{alice}), f(f(\mathit{alice}))), \ldots$. Thus, $\mathcal{K}$ is not $n$-de safe for any $n \geq 0$. It has two non-isomorphic DKB models: one, $\mathcal{I}^1_{CAS}$, where we have an exception to $\mathrm{D}(\exists \mathit{hasSupervisor}^- \sqsubseteq \bot)$ for *alice* (thus the model is 0-bounded) and $f(\mathit{alice}) = \mathit{alice}$, and another one, $\mathcal{I}^2_{CAS}$, where we have an exception for $f(\mathit{alice})$ and $\mathit{alice} \neq f(\mathit{alice})$, $f(\mathit{alice}) = f(f(\mathit{alice}))$ (the model is 1-bounded). No longer Skolem chain of three different elements is possible: then two exceptions would be needed, which then are however not provable. If we add the assertion $\neg\exists \mathit{hasSupervisor}^-(\mathit{alice})$ to $\mathcal{K}$ stating that *alice* is not a supervisor, then only the DKB model $\mathcal{I}^2_{CAS}$ remains; adding the assertion $\mathit{Employee}(\mathit{bob})$ instead, we obtain under UNA a further DKB model $\mathcal{I}^3_{CAS}$ with an exception of $\mathrm{D}(\exists \mathit{hasSupervisor}^- \sqsubseteq \bot)$ for *bob*; an exception for both *alice* and *bob* is infeasible as this would not be justifiable. $\diamond$

A syntactic property of DKBs that is useful to be verified is related to the reachability of unnamed elements in derivations.

*Definition 9* (*n-chain safety*)
A DKB $\mathcal{K}$ is *n-chain safe*, if from $\mathcal{K}_s$ only role chains $R_1(a, t_1), \ldots, R_m(t_{m-1}, t_m)$ where $a \in N_{\mathcal{K}}$ and the $t_1, \ldots, t_m$ are distinct Skolem terms can be derived such that $m \leq n$.

This condition, in particular, is verified in the case that $\mathcal{K}$ is acyclic: in this case the maximum length of chains is determined by the chains of existential axioms in $\mathcal{K}$.

If a DKB $\mathcal{K}$ is $n$-chain safe then it is also $m$-de safe for some $m$ that is exponentially bounded by $n$. On the other hand, $\mathcal{K}$ may be $n$-de safe but not $m$-chain safe for any $m \geq 0$: in the latter case, recursion through axioms that do not feed into defeasible axioms occur. For instance, if we drop in Example 7 the defeasible axiom $\mathrm{D}(\exists \mathit{hasSupervisor}^- \sqsubseteq \bot)$, then the resulting $\mathcal{K}$ is trivially exception safe but not $n$-chain bounded. For our purposes, we shall call a DKB $\mathcal{K}$ *recursive*, if $\mathcal{K}$ is not $n$-de bounded for any $n \geq 0$.

In case of exception safe (i.e. 0-de safe) DKBs we obtain the following result.

*Proposition 1*
Let $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ be a CAS model of DKB $\mathcal{K}$ and let $\mathcal{K}'$ result from $\mathcal{K}$ by pushing equality w.r.t. $\mathcal{I}$, that is, replace all $a, b \in N_{\mathcal{K}}$ s.t. $a^{\mathcal{I}} = b^{\mathcal{I}}$ by one representative. If $\mathcal{K}'$ is exception safe, then $\mathcal{I}_{CAS}$ can be justified only if every $\langle \alpha, \mathbf{e} \rangle \in \chi$ is over $N_{\mathcal{K}}$.

We remark that the condition of exception safety can be tested in polynomial time, by non-deterministically unfolding the axioms (resolution-style, or forward in a chase). In fact, we obtain the following result.

*Proposition 2*
Deciding whether a given DKB $\mathcal{K}$ is exception safe is feasible in NLogSpace, and whether it is $n$-de safe in PTime, if $n$ is bounded by a polynomial in the size of $\mathcal{K}$.

[10] Here, $\bot$ is the empty concept ("falsity") emulated by $\bot \sqsubseteq A$, $\bot \sqsubseteq \neg A$ for a fresh concept name $A$.

Syntactic classes ensuring exception safety can be singled out: simple examples of these can be the class of DKBs containing no existential axioms or the class where no inverse roles and no defeasible role axioms appear in the DKB.

Checking chain-safety is tractable, similarly to testing exception safety.

*Proposition 3*
Deciding whether a given DKB $\mathcal{K}$ is $n$-chain safe, where $n \geq 0$, is feasible in NLogSpace.

We remark that both checking exceptions and $n$-chain safety are in fact NLogSpace-complete, as the hardness is inherited from the NLogSpace-completeness of *DL-Lite*$_\mathcal{R}$ (which holds already in the absence of existential axioms).

## 4 Semantic properties

DKB models have interesting semantic properties similar to those exhibited by CKR models in Bozzato *et al.* (2018). In this section we provide a review of such properties: in particular, these results are important to show the feasibility of the reasoning approach presented in Section 5.

For example, we can prove that justified CAS models have a non-monotonic behavior with respect to the contents of DKBs, cf. (Bozzato *et al.* 2018, Prop. 4, non-monotonicity).

*Proposition 4 (non-monotonicity)*
Suppose $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ is a justified CAS model of a DKB $\mathcal{K}'$. Then, $\mathcal{I}_{CAS}$ is not necessarily a justified CAS model of every $\mathcal{K} \subset \mathcal{K}'$.

*Proof*
This property can be easily verified by considering the interpretation of defeasible axioms and their justification. Let us suppose that $\mathrm{D}(A \sqsubseteq B) \in \mathcal{K}$ (cases for other defeasible axioms can be shown similarly) and $\{A(c), \neg B(c)\} \subseteq \mathcal{K}$. If we consider a justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ for $\mathcal{K}$, then the defeasible axiom is not applied to the exceptional instance $c$ in the interpretation: that is, $\langle A \sqsubseteq B, c \rangle \in \chi$ and $S = \{A(c), \neg B(c)\}$ is a clashing set for such exception. However, if we consider $\mathcal{K}' = \mathcal{K} \setminus \{\neg B(c)\}$, then in $S$ is no longer verified by models of $\mathcal{K}'$: thus the clashing assumption $\langle A \sqsubseteq B, c \rangle$ can no longer be justified and $\mathcal{I}_{CAS}$ is not a justified model for $\mathcal{K}'$. □

Another property of justified CAS models that we can show is non-redundancy of justifications, cf. (Bozzato *et al.* 2018, Prop. 6, minimality of justification). Basically, this means that in justified models clashing assumptions are minimal, in the sense that no assumption can be omitted.

*Proposition 5 (non-redundancy)*
Suppose $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ and $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi' \rangle$ are NI-congruent justified CAS models of a DKB $\mathcal{K}$, then $\chi' \not\subset \chi$ holds.

*Proof*
Let us consider $\langle \alpha, \mathbf{e} \rangle \in \chi \setminus \chi'$. Then, given that $\mathcal{I}'_{CAS}$ is a model for $\mathcal{K}$, it holds that $\mathcal{I}' \models \alpha(\mathbf{e})$ (i.e. $\mathbf{e}$ is not an exceptional instance of the defeasible axiom $\mathrm{D}(\alpha)$). Given that

all clashing assumptions in $\chi$ are justified, then there exists a clashing set $S = S_{\langle\alpha,\mathbf{e}\rangle}$ for the clashing assumption $\langle\alpha,\mathbf{e}\rangle$ such that $\mathcal{I} \models S$. Moreover, by the definition of justification, for every other $\mathcal{I}''_{CAS} = \langle\mathcal{I}'',\chi\rangle$ for $\mathcal{K}$ that is NI-congruent with $\mathcal{I}_{CAS}$, it holds that $\mathcal{I}'' \models S$.

Let us consider $\mathcal{I}'''_{CAS} = \langle\mathcal{I}',\chi\rangle$. Given that $\mathcal{I}'_{CAS} \models \mathcal{K}$ and $\chi' \subseteq \chi$, we have that also $\mathcal{I}'''_{CAS} \models \mathcal{K}$. Moreover, since $\mathcal{I}'''_{CAS}$ is NI-congruent with $\mathcal{I}_{CAS}$, we have $\mathcal{I}' \models S$. Then, $\mathcal{I}' \models S \cup \{\alpha(\mathbf{e})\}$: however this contradicts the fact that $S$ is a clashing set for $\alpha(\mathbf{e})$. This proves the fact that $\chi' \not\subset \chi$ must hold. □

We remark that, as a consequence of this property, exceptions in DKB models are minimal (i.e. *minimally justified*): thus, in our approach this minimality property is derived from the definition of the interpretation of defeasible axioms and it is not explicitly required in its definition.

As shown in Lemma 1, *DL-Lite*$_\mathcal{R}$ knowledge bases can be represented as Horn theories: however, differently from $\mathcal{SROIQ}$-RL (Bozzato *et al.* 2018, Prop. 7, intersection property), the use of Skolem functions does not allow us to properly preserve the intersection property of Horn theories and a revised notion of model intersection is needed, cf. (de Bruijn *et al.* 2011, Proof for Prop. 6.7). Formally, for two NI-congruent DL interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$, we denote by $\mathcal{I}_1 \widetilde{\cap}_\mathcal{N} \mathcal{I}_2$ the NI-congruent "intersection" interpretation over a set of ground terms $\mathcal{N}$ (i.e. a set of individual names and the possible instantiations of Skolem functions over them) defined as follows:

- $\Delta^\mathcal{J} = \{[t^{\mathcal{I}_1}, t^{\mathcal{I}_2}] \mid t \in \mathcal{N}\}$;
- $t^\mathcal{J} = [t^{\mathcal{I}_1}, t^{\mathcal{I}_2}]$, for $t \in \mathcal{N}$;
- $[d_1, d_2] \in C^\mathcal{J}$ iff $d_1 \in C^{\mathcal{I}_1}$ and $d_2 \in C^{\mathcal{I}_2}$, for $C \in \mathrm{NC}$;
- $([d_1, d_2], [e_1, e_2]) \in R^\mathcal{J}$ iff $(d_1, e_1) \in R^{\mathcal{I}_1}$ and $(d_2, e_2) \in R^{\mathcal{I}_2}$, for $R \in \mathrm{NR}$;

where we abbreviate $\mathcal{J} = \mathcal{I}_1 \widetilde{\cap}_\mathcal{N} \mathcal{I}_2$. Note that, while for the NI-congruence we have that $a^{\mathcal{I}_1} = a^{\mathcal{I}_2}$ for individual names $a \in \mathcal{N} \cap \mathrm{NI}$, it is not necessarily true that $t^{\mathcal{I}_1} = t^{\mathcal{I}_2}$ for some Skolem term $t \in \mathcal{N}$: in this case, by the definition above, in the "intersection" interpretation we consider $t^\mathcal{J}$ (i.e. $t^{\mathcal{I}_1 \widetilde{\cap}_\mathcal{N} \mathcal{I}_2}$) as the "conjunction" of the interpretations of $t$ in the two models. Extending this construction to CAS interpretations, we need to ensure that the interpretation of the joined interpretations is coherent on exceptions. Namely, we require the following property:

$$\text{If } t^{\mathcal{I}_i} = e \text{ and some clashing assumption } \langle\alpha,\mathbf{e}\rangle \text{ with } e \in \mathbf{e} \text{ exists, then } t^{\mathcal{I}_1} = t^{\mathcal{I}_2}. \quad (*)$$

Then, the following result can be shown:

*Proposition 6*
Let $\mathcal{I}^i_{CAS} = \langle\mathcal{I}_i,\chi\rangle$, $i \in \{1,2\}$, be NI-congruent CAS models of a DKB $\mathcal{K}$ fulfilling $(*)$. Then, $\mathcal{I}_{CAS} = \langle\mathcal{I},\chi\rangle$, where $\mathcal{I} = \mathcal{I}_1 \widetilde{\cap}_\mathcal{N} \mathcal{I}_2$ and $\mathcal{N}$ includes all individual names occurring in $\mathcal{K}$ and for each element $e$ occurring in $\chi$ some $t \in \mathcal{N}$ such that $t^{\mathcal{I}_1} = e (= t^{\mathcal{I}_2})$, is also a CAS model of $\mathcal{K}$.[11] Furthermore, if some $\mathcal{I}^i_{CAS}$, $i \in \{1,2\}$, is justified and $\mathcal{K}$ is exception safe, then $\mathcal{I}_{CAS}$ is justified.

A consequence of this result is that a least justified CAS model exists for exception safe DKBs relative to a *name assignment*, which we define as any interpretation $\nu : \mathrm{NI} \to \Delta$

[11] Technically, we view here $[e, e] \in \Delta^\mathcal{I}$ as $e$; the assumption ensures we have infinitely many standard names left.

of the individual constants on the domain $\Delta$ (respecting SNA). The name assignment of a CAS interpretation $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ is the one induced by $\mathrm{NI}^{\mathcal{I}}$. We call a clashing assumption $\chi$ for a DKB $\mathcal{K}$ *satisfiable* (resp., *justified*) for a name assignment $\nu$, if $\mathcal{K}$ has some CAS model (resp., justified CAS model) $\mathcal{I}_{CAS}$ with name assignment $\nu$. Then, by using the construction of "intersection" interpretations over $CAS$-models of $\mathcal{K}$ for a given satisfiable $\chi$, we obtain the following result. Denote for a CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ by $At_{\mathcal{N}}(\mathcal{I}_{CAS}) = \{A(t), R(t,t') \mid t,t' \in \mathcal{N}, A \in \mathrm{NC},\ R \in \mathrm{NR}, \mathcal{I} \models A(t), \mathcal{I} \models R(t,t')\}$ the set of all atomic concepts and roles over $\mathcal{N}$ satisfied by $\mathcal{I}_{CAS}$, and define $\mathcal{I}_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}'_{CAS}$ for CAS models $\mathcal{I}_{CAS}$ and $\mathcal{I}'_{CAS}$ by $At_{\mathcal{N}}(\mathcal{I}_{CAS}) \subseteq At_{\mathcal{N}}(\mathcal{I}'_{CAS})$.

*Corollary 1* (*least model property*)
If a clashing assumption $\chi$ for an exception safe DKB $\mathcal{K}$ is satisfiable for name assignment $\nu$, then $\mathcal{K}$ has an $\subseteq_{\mathcal{N}}$-least (unique minimal)[12] CAS model $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu) = \langle \hat{\mathcal{I}}, \chi \rangle$ on $\mathcal{N}$ that contains all Skolem terms of individual constants, that is, for every CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ relative to $\nu$, it holds that $\mathcal{I}_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}'_{CAS}$. Furthermore, $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ is justified if $\chi$ is justified.

We note that, moreover, a Skolem term $t$ over an individual constant $c$ can occur in the least model $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ if and only if it has an alias in $\mathcal{K}$, that is, some $c' \in N_{\mathcal{K}}$ such that $\nu(c') = \nu(c)$ exists; thus, $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ is fully characterized by its restriction to $N_{\mathcal{K}}$. We also note that we can reason independently from (in)equalities that emerge from the name assignment $\nu$ regarding exceptions, since modulo $\nu$, no new Skolem terms can occur in derived positive atoms. This also means that (in)equalities do not affect (relative to $\nu$) the conditions for $n$-de safety.

As in the case of $\mathcal{SROIQ}$-RL knowledge bases in Bozzato *et al*. (2018), in order to formulate a reasoning method it is important to show that also in *DL-Lite*$_{\mathcal{R}}$ knowledge bases (and DKBs) we can concentrate on reasoning over the *named* part of an interpretation. Notably, in the case of *DL-Lite*$_{\mathcal{R}}$ we need to extend this notion to consider the interpretation of Skolem individuals.

We say $\mathcal{I}$ is *named* relative to $\mathcal{N} \subseteq sk(\mathrm{NI}) \setminus \mathrm{NI}_S$, if (i) $C^{\mathcal{I}} \subseteq \mathcal{N}^{\mathcal{I}}$ and $R^{\mathcal{I}} \subseteq \mathcal{N}^{\mathcal{I}} \times \mathcal{N}^{\mathcal{I}}$ for each $C \in \mathrm{NC}$ and $R \in \mathrm{NR}$; (ii) for every Skolem function $f$ and $a \in \mathcal{N} \cap \mathrm{NI}$, $f(a)^{\mathcal{I}} \in \mathcal{N}^{\mathcal{I}}$. Moreover, for a *DL-Lite*$_{\mathcal{R}}$ knowledge base $\mathcal{K}$, if $c^{\mathcal{I}} \neq d^{\mathcal{I}}$ for any distinct $c, d \in \mathcal{N}$ and $\mathcal{N}$ includes $sk(N_{\mathcal{K}})$ (i.e. all constants that occur in $\mathcal{K}$ and their Skolem constants), we call $\mathcal{I}$ a *pseudo Herbrand interpretation* for $\mathcal{K}$ relative to $\mathcal{N}$.

Let for any $N \subseteq \mathrm{NI} \setminus \mathrm{NI}_S$ the *$N$-restriction of $\mathcal{I}$*, denoted by $\mathcal{I}^N$, be the interpretation that results from $\mathcal{I}$ by: (i) restricting $C^{\mathcal{I}}$ to $sk(N)^{\mathcal{I}}$ for all $C \in \mathrm{NC}$ and $R^{\mathcal{I}}$ to $sk(N)^{\mathcal{I}} \times sk(N)^{\mathcal{I}}$ for every $R \in \mathrm{NR}$; (ii) redirecting any role of the type $(e_1, e_2) \in R^{\mathcal{I}}$ with $e_1 \in sk(N)$ and $e_1 \notin sk(N)$ to $(e_1, g) \in R^{\mathcal{I}}$ with $g \neq e_1$ and $g \in sk(N)^{\mathcal{I}} \setminus N^{\mathcal{I}}$ (i.e. $g$ is the interpretation of some Skolem term on $N$). Then, we can obtain the following lemma over such interpretation restrictions.

*Lemma 3*
Suppose $\mathcal{I}$ is a model of a *DL-Lite*$_{\mathcal{R}}$ knowledge base $\mathcal{K}$ and $N \subseteq \mathrm{NI} \setminus \mathrm{NI}_S$ includes all individuals occurring in $\mathcal{K}$. Then, the $N$-restriction $\mathcal{I}^N$ is named w.r.t. $sk(N)$ and a model of $\mathcal{K}$.

[12] We consider uniqueness modulo equivalence, that is, that $\mathcal{I}_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}'_{CAS}$ and $\mathcal{I}'_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}_{CAS}$

In the case of exception safe DKBs, this property can be extended to CAS interpretations $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of DKB $\mathcal{K}$. Considering $N$ that includes each individual constant that occurs in $\mathcal{K}$, a CAS interpretation $\mathcal{I}^N_{CAS} = \langle \mathcal{I}^N, \chi^N \rangle$ can be obtained from $\mathcal{I}_{CAS}$ by (i) replacing $\mathcal{I}$ with its $N$-restriction $\mathcal{I}^N$, (ii) removing each clashing assumption $\langle \alpha, \mathbf{d} \rangle$ from $\chi$ where $\mathbf{d}$ is not over $N$, and (iii) interpreting each constant symbol $c \in sk(\mathrm{NI}) \setminus (sk(N) \cup \mathrm{NI}_S)$ by some arbitrary element not in $sk(N)^{\mathcal{I}}$. In particular, we consider the case of $N = N_{\mathcal{K}}$ consisting of the individual constants that occur in $\mathcal{K}$. Then, we obtain:

*Theorem 1* (*named model focus*)
Let $\mathcal{I}_{CAS}$ be a CAS model of an exception safe DKB $\mathcal{K}$ and suppose $N_{\mathcal{K}} \subseteq N \subseteq \mathrm{NI} \setminus \mathrm{NI}_S$. Then, also $\mathcal{I}^N_{CAS}$, and in particular $\mathcal{I}^{N_{\mathcal{K}}}_{CAS}$, is a CAS model for $\mathcal{K}$. Furthermore, $\mathcal{I}^N_{CAS}$ is justified if $\mathcal{I}_{CAS}$ is justified, and every clashing assumption $\langle \alpha, \mathbf{e} \rangle$ in $\mathcal{I}^N_{CAS}$ is justified by some clashing set $S$ formulated with constants from $sk(N)$.

*Proof*
Suppose that $\mathcal{I}_{CAS} \models \mathcal{K}$, with $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$. Then, by the Definition 5 of CAS model, if we consider the restriction $\mathcal{I}^N_{CAS} = \langle \mathcal{I}^N, \chi^N \rangle$, from Lemma 3 we directly obtain condition (i) on strict axioms in $\mathcal{K}$, that is, for every $\alpha \in \mathcal{L}_\Sigma$ in $\mathcal{K}$, $\mathcal{I}^N \models \alpha$. We can prove also the satisfaction of condition (ii) on the interpretation of defeasible axioms. Let $\mathbf{d} \notin \{\mathbf{e} \mid \langle \alpha, \mathbf{e} \rangle \in \chi^N\}$ for $\mathrm{D}(\alpha) \in \mathcal{K}$. If $\mathbf{d}$ is over $N$, then by Lemma 3 we obtain that $\mathcal{I} \models \alpha(\mathbf{d})$ and thus $\mathcal{I}^N \models \alpha(\mathbf{d})$. Otherwise, if $\mathbf{d}$ is not over $N$, then as noted in the proof of previous lemma, in the translation to Horn clauses $\phi_\alpha(\mathbf{d})$ there must be a clause $\gamma_i(\mathbf{d}, \vec{x}_i)$ where some constant outside $N$ occurs in the antecedent. This causes $\gamma_i(\mathbf{d}, \vec{x}_i)$ to evaluate to false for every assignment on $\vec{x}_i$.

We can show that if $\mathcal{I}_{CAS}$ is justified, then $\mathcal{I}^N_{CAS}$ is also justified. Let us assume that $\mathcal{I}^N_{CAS}$ is not justified: then there exists a $\langle \alpha, \mathbf{e} \rangle \in \chi^N$ (and thus also $\langle \alpha, \mathbf{e} \rangle \in \chi$) that is not justified. By the definition of justification, this means that for every clashing set $S = S_{\langle \alpha, \mathbf{e} \rangle}$ for $\langle \alpha, \mathbf{e} \rangle$, there exists some CAS model $\mathcal{I}^{N'}_{CAS} = \langle \mathcal{I}^{N'}, \chi^N \rangle$ of $\mathcal{K}$ that is NI-congruent to $\mathcal{I}^N_{CAS}$ and $\mathcal{I}^{N'} \not\models S$. In particular, this must hold for the clashing set $S$ providing the justification of $\langle \alpha, \mathbf{e} \rangle$ in $\mathcal{I}_{CAS}$. Consider then the interpretation $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$, corresponding to changing the interpretations of symbols in $\mathcal{I}$ to the interpretation of $\mathcal{I}^{N'}$: then, $\mathcal{I}'_{CAS}$ is NI-congruent with $\mathcal{I}_{CAS}$, but $\mathcal{I}'_{CAS} \not\models S$. This contradicts the fact that $\mathcal{I}_{CAS}$ is justified: thus, $\mathcal{I}^N_{CAS}$ is justified as well.

Finally, the fact that every $\langle \alpha, \mathbf{e} \rangle$ in $\mathcal{I}^N_{CAS}$ is justified by some clashing set $S$ over $sk(N)$ can be verified by considering that $S$ can be expressed in (a grounding of) Horn clauses and $\mathcal{I}^N_{CAS} \models S$. Thus an equivalent renaming of constants in $S$ over $sk(N)$ can be provided. □

The following property is useful in order to prove the correctness of justifications: the result provides a characterization of justification based on the least model $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ for a clashing assumption set $\chi$ and a name assignment $\nu$.

*Theorem 2* (*justified CAS characterization*)
Let $\chi$ be a satisfiable clashing assumptions set for an exception safe DKB $\mathcal{K}$ and name assignment $\nu$. Then, $\chi$ is justified iff $\langle \alpha, \mathbf{e} \rangle \in \chi$ implies some clashing set $S = S_{\langle \alpha, \mathbf{e} \rangle}$ exists such that

Table 2. *Normal form for $\mathcal{K}$ axioms from $\mathcal{L}_\Sigma$*

**Strict axioms:** for $A, B \in \mathrm{NC}$, $R, S \in \mathrm{NR}$, $a, b \in \mathrm{NI}$:

$A(a)$ $R(a,b)$ $A \sqsubseteq B$ $A \sqsubseteq \neg B$ $\exists R \sqsubseteq A_{\exists R}$ $A_{\exists R} \sqsubseteq \exists R$

$R \sqsubseteq S$ $\mathrm{Dis}(R,S)$ $\mathrm{Inv}(R,S)$ $\mathrm{Irr}(R)$

**Defeasible axioms:** for $A, B \in \mathrm{NC}$, $R, S \in \mathrm{NR}$:

$\mathrm{D}(A \sqsubseteq B)$ $\mathrm{D}(R \sqsubseteq S)$ $\mathrm{Inv}(R,S)$ $\mathrm{Irr}(R)$

(i) $\hat{\mathcal{I}} \models \beta$, for each positive $\beta \in S$, where $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu) = (\hat{\mathcal{I}}, \chi)$, and
(ii) no CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ with name assignment $\nu$ exists s.t. $\mathcal{I} \models \beta$ for some $\neg\beta \in S$.

In the following sections, we concentrate on reasoning in exception safe DKBs under UNA (on elements of $\mathcal{K}$); we will discuss the possible extensions for more general DKBs.

## 5 Datalog translation for *DL-Lite*$_\mathcal{R}$ DKB

We present a datalog translation for reasoning on *DL-Lite*$_\mathcal{R}$ DKBs which refines the translation provided in Bozzato *et al.* (2018). The translation provides a reasoning method for positive instance queries w.r.t. entailment on DKB models for exception safe DKBs. An important aspect of this translation is that, due to the form of *DL-Lite*$_\mathcal{R}$ axioms, no inference on disjunctive negative information is needed for the reasoning on derivations of clashing sets. Thus, reasoning by contradiction using "test environments" is not needed and we can directly encode negative reasoning as rules on negative literals: with respect to the discussion in Bozzato *et al.* (2018), we can say that *DL-Lite*$_\mathcal{R}$ thus represents an inherently "justification safe" fragment which then allows us to formulate such a direct datalog encoding. With respect to the interpretation of right-hand side existential axioms, we follow the approach of Krötzsch (2010): for every axiom of the kind $\alpha = A \sqsubseteq \exists R$, an auxiliary abstract individual $aux^\alpha$ is added in the translation to represent the class of all $R$-successors introduced by $\alpha$.

We introduce a *normal form* for axioms of *DL-Lite*$_\mathcal{R}$ (in Table 2) which allows us to simplify the formulation of reasoning rules. We can provide rules to transform any *DL-Lite*$_\mathcal{R}$ DKB into normal form and show that the rewritten DKB is equivalent to the original (see Lemma 4 and the discussion following in Appendix C.3.1). In the normalization, we introduce new concept names $A_{\exists R}$ to simplify the management of existential formulas $\exists R$ in rules for defeasible axioms: we assume that, for every role $R$, axioms $A_{\exists R} \sqsubseteq \exists R$, $\exists R \sqsubseteq A_{\exists R}$ are added to the DKB. Note that, with respect to the previous formulation of normal form provided in Bozzato *et al.* (2018), we further simplified the case for defeasible assertions and negative inclusions, as they can be represented using (defeasible) class and role inclusions with auxiliary symbols.

*Lemma 4*
Every DKB $\mathcal{K}$ can be transformed in linear time into an equivalent DKB $\mathcal{K}'$ which has modulo auxiliary symbols the same DKB models, and such that $n$-de safety and $n$-chain safety are preserved.

**Translation rules overview.** We can now present the components of our datalog translation for *DL-Lite*$_\mathcal{R}$ based DKBs. As in the original formulation in Bozzato *et al.* (2014), Bozzato *et al.* (2018), which extended the encoding without defeasibility proposed in Bozzato and Serafini (2013) (inspired by the materialization calculus in Krötzsch (2010)), the translation includes sets of *input rules* (which encode DL axioms and signature in datalog), *deduction rules* (datalog rules providing instance level inference) and *output rules* (that encode, in terms of a datalog fact, the ABox assertion to be proved). The translation is composed by the following sets of rules:

*DL-Lite*$_\mathcal{R}$ *input and output rules:* rules in $I_{dlr}$ encode as datalog facts the *DL-Lite*$_\mathcal{R}$ axioms and signature of the input DKB. For example, in the case of existential axioms,[13] these are translated by rule (idlr-supex) as $A \sqsubseteq \exists R \mapsto \{\mathtt{supEx}(A, R, aux^\alpha)\}$. Note that this rule, in the spirit of Krötzsch (2010), introduces an auxiliary element $aux^\alpha$, which intuitively represents the class of all new $R$-successors generated by the axiom $\alpha$. Similarly, output rules in $O$ encode in datalog the ABox assertions to be proved. These rules are provided in Table 3.

*DL-Lite*$_\mathcal{R}$ *deduction rules:* rules in $P_{dlr}$ (in Table 3) add deduction rules for ABox reasoning. In the case of existential axioms, the rule (pdlr-supex) introduces a new relation to the auxiliary individual as follows:

$$\mathtt{tripled}(x, r, x') \leftarrow \mathtt{supEx}(y, r, x'), \mathtt{instd}(x, y).$$

In this translation the reasoning on negative information is directly encoded by "contrapositive" versions of the rules. For example, with respect to previous rule, we have the negative rule (pdlr-nsupex):

$$\neg\mathtt{instd}(x, y) \leftarrow \mathtt{supEx}(y, r, w), \mathtt{const}(x), \mathtt{all\_nrel}(x, r),$$

where $\mathtt{all\_nrel}(x, r)$ verifies that $\neg\mathtt{triple}(x, r, y)$ holds for all $\mathtt{const}(y)$ by an iteration over all constants.

*Defeasible axioms input translations*: the set of input rules $I_{\mathrm{D}}$ (shown in Table 4) provides the translation of defeasible axioms $\mathrm{D}(\alpha)$ in the DKB: in other words, they are used to specify that the axiom $\alpha$ needs to be considered as defeasible. For example, $\mathrm{D}(A \sqsubseteq B)$ is translated to $\mathtt{def\_subclass}(A, B)$. Note that, by the definition of the normal form, the existential axioms are "compiled out" from defeasible axioms (i.e. defeasible existential axioms can be expressed by using the newly added $A_{\exists R}$ concepts).

*Overriding rules:* rules for defeasible axioms provide the different conditions for the correct interpretation of defeasibility: the overriding rules define conditions (corresponding to clashing sets) for recognizing an exceptional instance. For example, for axioms of the form $\mathrm{D}(A \sqsubseteq B)$, the translation introduces the rule (ovr-subc):

$$\mathtt{ovr}(\mathtt{subClass}, x, y, z) \leftarrow \mathtt{def\_subclass}(y, z), \mathtt{instd}(x, y), \neg\mathtt{instd}(x, z).$$

Note that in this version of the calculus, the reasoning on negative information (of the clashing sets) is directly encoded in the deduction rules. Overriding rules in $P_{\mathrm{D}}$ are shown in Table 4.

*Defeasible application rules:* another set of rules in $P_{\mathrm{D}}$ defines the defeasible application of such axioms: intuitively, defeasible axioms are applied only to instances that have not

[13] Note that, by the normal form above, this kind of axioms is in the form $A_{\exists R} \sqsubseteq \exists R$.

Table 3. *DL-Lite$_\mathcal{R}$ input, deduction and output rules*

***DL-Lite$_\mathcal{R}$* input translation** $I_{dlr}(S)$

| | | | |
|---|---|---|---|
| (idlr-nom) | $a \in \mathrm{NI} \mapsto \{\mathtt{nom}(a)\}$ | (idlr-subex) | $\exists R \sqsubseteq B \mapsto \{\mathtt{subEx}(R, B)\}$ |
| (idlr-cls) | $A \in \mathrm{NC} \mapsto \{\mathtt{cls}(A)\}$ | (idlr-supex) | $A \sqsubseteq \exists R \mapsto \{\mathtt{supEx}(A, R, aux^\alpha)\}$ |
| (idlr-rol) | $R \in \mathrm{NR} \mapsto \{\mathtt{rol}(R)\}$ | (idlr-subr) | $R \sqsubseteq S \mapsto \{\mathtt{subRole}(R, S)\}$ |
| (idlr-inst) | $A(a) \mapsto \{\mathtt{insta}(a, A)\}$ | (idlr-dis) | $\mathrm{Dis}(R, S) \mapsto \{\mathtt{dis}(R, S)\}$ |
| (idlr-triple) | $R(a, b) \mapsto \{\mathtt{triplea}(a, R, b)\}$ | (idlr-inv) | $\mathrm{Inv}(R, S) \mapsto \{\mathtt{inv}(R, S)\}$ |
| (idlr-subc) | $A \sqsubseteq B \mapsto \{\mathtt{subClass}(A, B)\}$ | (idlr-irr) | $\mathrm{Irr}(R) \mapsto \{\mathtt{irr}(R)\}$ |

***DL-Lite$_\mathcal{R}$* deduction rules** $P_{dlr}$

| | |
|---|---|
| (pdlr-instd) | $\mathtt{instd}(x, z) \leftarrow \mathtt{insta}(x, z).$ |
| (pdlr-tripled) | $\mathtt{tripled}(x, r, y) \leftarrow \mathtt{triplea}(x, r, y).$ |
| (pdlr-subc) | $\mathtt{instd}(x, z) \leftarrow \mathtt{subClass}(y, z), \mathtt{instd}(x, y).$ |
| (pdlr-supnot) | $\neg\mathtt{instd}(x, z) \leftarrow \mathtt{supNot}(y, z), \mathtt{instd}(x, y).$ |
| (pdlr-subex) | $\mathtt{instd}(x, z) \leftarrow \mathtt{subEx}(v, z), \mathtt{tripled}(x, v, x').$ |
| (pdlr-supex) | $\mathtt{tripled}(x, r, x') \leftarrow \mathtt{supEx}(y, r, x'), \mathtt{instd}(x, y).$ |
| (pdlr-subr) | $\mathtt{tripled}(x, w, x') \leftarrow \mathtt{subRole}(v, w), \mathtt{tripled}(x, v, x').$ |
| (pdlr-dis1) | $\neg\mathtt{tripled}(x, u, y) \leftarrow \mathtt{dis}(u, v), \mathtt{tripled}(x, v, y).$ |
| (pdlr-dis2) | $\neg\mathtt{tripled}(x, v, y) \leftarrow \mathtt{dis}(u, v), \mathtt{tripled}(x, u, y).$ |
| (pdlr-inv1) | $\mathtt{tripled}(y, v, x) \leftarrow \mathtt{inv}(u, v), \mathtt{tripled}(x, u, y).$ |
| (pdlr-inv2) | $\mathtt{tripled}(y, u, x) \leftarrow \mathtt{inv}(u, v), \mathtt{tripled}(x, v, y).$ |
| (pdlr-irr) | $\neg\mathtt{tripled}(x, u, x) \leftarrow \mathtt{irr}(u), \mathtt{const}(x).$ |
| (pdlr-nsubc) | $\neg\mathtt{instd}(x, y) \leftarrow \mathtt{subClass}(y, z), \neg\mathtt{instd}(x, z).$ |
| (pdlr-nsupnot) | $\mathtt{instd}(x, y) \leftarrow \mathtt{supNot}(y, z), \neg\mathtt{instd}(x, z).$ |
| (pdlr-nsubex) | $\neg\mathtt{tripled}(x, v, x') \leftarrow \mathtt{subEx}(v, z), \mathtt{const}(x'), \neg\mathtt{instd}(x, z).$ |
| (pdlr-nsupex) | $\neg\mathtt{instd}(x, y) \leftarrow \mathtt{supEx}(y, r, w), \mathtt{const}(x), \mathtt{all\_nrel}(x, r).$ |
| (pdlr-nsubr) | $\neg\mathtt{tripled}(x, v, x') \leftarrow \mathtt{subRole}(v, w), \neg\mathtt{tripled}(x, w, x').$ |
| (pdlr-ninv1) | $\neg\mathtt{tripled}(y, v, x) \leftarrow \mathtt{inv}(u, v), \neg\mathtt{tripled}(x, u, y).$ |
| (pdlr-ninv2) | $\neg\mathtt{tripled}(y, u, x) \leftarrow \mathtt{inv}(u, v), \neg\mathtt{tripled}(x, v, y).$ |
| (pdlr-allnrel1) | $\mathtt{all\_nrel\_step}(x, r, y) \leftarrow \mathtt{first}(y), \neg\mathtt{tripled}(x, r, y).$ |
| (pdlr-allnrel2) | $\mathtt{all\_nrel\_step}(x, r, y) \leftarrow \mathtt{all\_nrel\_step}(x, r, y'), \mathtt{next}(y', y), \neg\mathtt{tripled}(x, r, y).$ |
| (pdlr-allnrel3) | $\mathtt{all\_nrel}(x, r) \leftarrow \mathtt{last}(y), \mathtt{all\_nrel\_step}(x, r, y).$ |

**Output translation** $O(\alpha)$

| | |
|---|---|
| (o-concept) | $A(a) \mapsto \{A(a)\}$ |
| (o-role) | $R(a, b) \mapsto \{R(a, b)\}$ |

been recognized as exceptional. For example, the rule (app-subc) applies a defeasible concept inclusion $\mathrm{D}(A \sqsubseteq B)$:

$$\mathtt{instd}(x, z) \leftarrow \mathtt{def\_subclass}(y, z), \mathtt{instd}(x, y), \mathtt{not}\ \mathtt{ovr}(\mathtt{subClass}, x, y, z).$$

Defeasible application rules are provided in Table 4.

**Translation process.** Given a DKB $\mathcal{K}$ in *DL-Lite$_\mathcal{R}$* normal form, a program $PK(\mathcal{K})$ that encodes query answering for $\mathcal{K}$ is obtained as:

$$PK(\mathcal{K}) = P_{dlr} \cup P_{\mathrm{D}} \cup I_{dlr}(\mathcal{K}) \cup I_{\mathrm{D}}(\mathcal{K}).$$

Moreover, $PK(\mathcal{K})$ is completed with a set of supporting facts about constants: for every literal $\mathtt{nom}(c)$ or $\mathtt{supEx}(a, r, c)$ in $PK(\mathcal{K})$, $\mathtt{const}(c)$ is added to $PK(\mathcal{K})$. Then, given an arbitrary enumeration $c_0, \ldots, c_n$ s.t. each $\mathtt{const}(c_i) \in PK(\mathcal{K})$, the facts

Table 4. *Input and deduction rules for defeasible axioms*

**Input rules for defeasible axioms $I_{\mathrm{D}}(S)$**

(id-subc) $\mathrm{D}(A \sqsubseteq B) \mapsto \{\, \mathtt{def\_subclass}(A, B). \,\}$
(id-subr) $\mathrm{D}(R \sqsubseteq S) \mapsto \{\, \mathtt{def\_subr}(R, S). \,\}$
(id-inv) $\mathrm{D}(\mathrm{Inv}(R, S)) \mapsto \{\, \mathtt{def\_inv}(R, S). \,\}$
(id-irr) $\mathrm{D}(\mathrm{Irr}(R)) \mapsto \{\, \mathtt{def\_irr}(R). \,\}$

**Deduction rules for defeasible axioms $P_{\mathrm{D}}$: overriding rules**

(ovr-subc) $\mathtt{ovr}(\mathtt{subClass}, x, y, z) \leftarrow \mathtt{def\_subclass}(y, z), \mathtt{instd}(x, y), \neg\mathtt{instd}(x, z).$
(ovr-subr) $\mathtt{ovr}(\mathtt{subRole}, x, y, r, s) \leftarrow \mathtt{def\_subr}(r, s), \mathtt{tripled}(x, r, y), \neg\mathtt{tripled}(x, s, y).$
(ovr-inv1) $\mathtt{ovr}(\mathtt{inv}, x, y, r, s) \leftarrow \mathtt{def\_inv}(r, s), \mathtt{tripled}(x, r, y), \neg\mathtt{tripled}(y, s, x).$
(ovr-inv2) $\mathtt{ovr}(\mathtt{inv}, x, y, r, s) \leftarrow \mathtt{def\_inv}(r, s), \mathtt{tripled}(y, s, x), \neg\mathtt{tripled}(x, r, y).$
(ovr-irr) $\mathtt{ovr}(\mathtt{irr}, x, r) \leftarrow \mathtt{def\_irr}(r), \mathtt{tripled}(x, r, x).$

**Deduction rules for defeasible axioms $P_{\mathrm{D}}$: application rules**

(app-subc) $\mathtt{instd}(x, z) \leftarrow \mathtt{def\_subclass}(y, z), \mathtt{instd}(x, y), \mathtt{not}\, \mathtt{ovr}(\mathtt{subClass}, x, y, z).$
(app-subr) $\mathtt{tripled}(x, w, y) \leftarrow \mathtt{def\_subr}(v, w), \mathtt{tripled}(x, v, y), \mathtt{not}\, \mathtt{ovr}(\mathtt{subRole}, x, y, v, w).$
(app-inv1) $\mathtt{tripled}(y, v, x) \leftarrow \mathtt{def\_inv}(u, v), \mathtt{tripled}(x, u, y), \mathtt{not}\, \mathtt{ovr}(\mathtt{inv}, x, y, u, v).$
(app-inv2) $\mathtt{tripled}(x, u, y) \leftarrow \mathtt{def\_inv}(u, v), \mathtt{tripled}(y, v, x), \mathtt{not}\, \mathtt{ovr}(\mathtt{inv}, x, y, u, v).$
(app-irr) $\neg\mathtt{tripled}(x, u, x) \leftarrow \mathtt{def\_irr}(u), \mathtt{const}(x), \mathtt{not}\, \mathtt{ovr}(\mathtt{irr}, x, u).$
(app-nsubc) $\neg\mathtt{instd}(x, y) \leftarrow \mathtt{def\_subclass}(y, z), \neg\mathtt{instd}(x, z), \mathtt{not}\, \mathtt{ovr}(\mathtt{subClass}, x, y, z).$
(app-nsubr) $\neg\mathtt{tripled}(x, v, y) \leftarrow \mathtt{def\_subr}(v, w), \neg\mathtt{tripled}(x, w, y), \mathtt{not}\, \mathtt{ovr}(\mathtt{subRole}, x, y, v, w).$
(app-ninv1) $\neg\mathtt{tripled}(y, v, x) \leftarrow \mathtt{def\_inv}(u, v), \neg\mathtt{tripled}(x, u, y), \mathtt{not}\, \mathtt{ovr}(\mathtt{inv}, x, y, u, v).$
(app-ninv2) $\neg\mathtt{tripled}(x, u, y) \leftarrow \mathtt{def\_inv}(u, v), \neg\mathtt{tripled}(y, v, x), \mathtt{not}\, \mathtt{ovr}(\mathtt{inv}, x, y, u, v).$

$\mathtt{first}(c_0), \mathtt{last}(c_n)$ and $\mathtt{next}(c_i, c_{i+1})$ with $0 \leq i < n$ are added to $PK(\mathcal{K})$. Query answering $\mathcal{K} \models \alpha$ is then obtained by testing whether the (instance) query, translated to datalog by $O(\alpha)$, is a consequence of $PK(\mathcal{K})$, that is, whether $PK(\mathcal{K}) \models O(\alpha)$ holds.

Note that we use a linear ordering of constants in an encoding by means of the predicates $\mathtt{first}, \mathtt{last}$ and $\mathtt{next}$, which allows us to verify universal sentences over all constants (in our case, negation on roles), by walking through them starting at the first constant over the next one until the last constant is reached. We note that verifying universal sentences can also accomplished by means of aggregates in ASP Alviano and Faber (2018): however, we chose to use this simpler method in order to keep the standard interpretation of ASP programs.

**Correctness.** The presented translation procedure provides a sound and complete materialization calculus for instance checking on *DL-Lite*$_{\mathcal{R}}$ DKBs in normal form.

As in Bozzato *et al.* (2018), the proof for this result can be verified by establishing a correspondence between minimal justified models of $\mathcal{K}$ and answer sets of $PK(\mathcal{K})$. Besides the simpler structure of the final program, the proof is simplified by the direct formulation of rules for negative reasoning. Another new aspect of the proof in the case of *DL-Lite*$_{\mathcal{R}}$ resides in the management of existential axioms, since there is the need to define a correspondence between the auxiliary individuals in the translation and the interpretation of existential axioms in the semantics: we follow the approach of Krötzsch (2010), where in building the correspondence with justified models, auxiliary constants $aux^{\alpha}$ are mapped to the class of Skolem individuals for existential axioms $\alpha$. We remark that this collective encoding of unnamed individuals is possible since, in the case of exception safe DKBs, no exceptions can appear on such individuals: thus, differently from named individuals (which need to single out exceptional elements of the domain), there is no need to identify single unnamed elements of the domain.

As in Bozzato *et al.* (2018), in our translation we consider UNA on elements of $\mathcal{K}$ and *named models*, that is, interpretations restricted to $sk(N_{\mathcal{K}})$. Thus, we can show the correctness result on the least model for $\mathcal{K}$ with respect to a set of clashing assumptions $\chi$, that will be denoted by $\hat{\mathcal{I}}(\chi)$.

Let $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ be a justified named CAS model. We define the set of overriding assumptions $OVR(\mathcal{I}_{CAS}) = \{\, \mathtt{ovr}(p(\mathbf{e})) \mid \langle \alpha, \mathbf{e} \rangle \in \chi,\ I_{dlr}(\alpha) = p \,\}$. Given a CAS interpretation $\mathcal{I}_{CAS}$, we define a corresponding interpretation $I(\mathcal{I}_{CAS})$ for $PK(\mathcal{K})$ by including the following atoms in it:

(1) all facts of $PK(\mathcal{K})$;
(2) $\mathtt{instd}(a, A)$, if $\mathcal{I} \models A(a)$ and $\neg\mathtt{instd}(a, A)$, if $\mathcal{I} \models \neg A(a)$;
(3) $\mathtt{tripled}(a, R, b)$, if $\mathcal{I} \models R(a, b)$ and $\neg\mathtt{tripled}(a, R, b)$, if $\mathcal{I} \models \neg R(a, b)$;
(4) $\mathtt{tripled}(a, R, aux^{\alpha})$, if $\mathcal{I} \models \exists R(a)$ for $\alpha = A \sqsubseteq \exists R$;
(5) $\mathtt{all\_nrel}(a, R)$ if $\mathcal{I} \models \neg\exists R(a)$;
(6) each ovr-literal from $OVR(\mathcal{I}_{CAS})$;

The next proposition shows that the least models of $\mathcal{K}$ can be represented by the answer sets of the program $PK(\mathcal{K})$.

*Proposition 7*
Let $\mathcal{K}$ be an exception safe DKB in *DL-Lite*$_{\mathcal{R}}$ normal form. Then:

(i) for every (named) justified clashing assumption $\chi$, the interpretation $S = I(\hat{\mathcal{I}}(\chi))$ is an answer set of $PK(\mathcal{K})$;
(ii) every answer set $S$ of $PK(\mathcal{K})$ is of the form $S = I(\hat{\mathcal{I}}(\chi))$ where $\chi$ is a (named) justified clashing assumption for $\mathcal{K}$.

The correctness of the translation with respect to instance checking is obtained as a direct consequence of Proposition 7.

*Theorem 3*
Let $\mathcal{K}$ be an exception safe DKB in *DL-Lite*$_{\mathcal{R}}$ normal form, and let $\alpha \in \mathcal{L}_{\Sigma}$ such that the output translation $O(\alpha)$ is defined. Then, $\mathcal{K} \models \alpha$ iff $PK(\mathcal{K}) \models O(\alpha)$.

**Prototype implementation.** A proof-of-concept implementation of the presented datalog translation for *DL-Lite*$_{\mathcal{R}}$ DKBs has been included in the latest version of the CKR*ew* *(CKR datalog rewriter)* prototype Bozzato *et al.* (2018). CKR*ew* is a Java-based command line application that accepts as input RDF files representing (contextualized) knowledge bases with defeasible axioms and produces as output a single `.dlv` text file with the datalog rewriting for the input KB. The current version of the prototype includes an option to accept as input a single RDF file containing a *DL-Lite*$_{\mathcal{R}}$ DKB (represented as OWL axioms in the normal form of Table 2) and apply the datalog translation presented above.

The latest version of CKR*ew*, together with sample RDF files implementing the knowledge base of Example 3, is available on-line at: `http://ckrew.fbk.eu/`.

## 6 Complexity of reasoning problems

In this section, we turn to the computational complexity of reasoning from a DKB. As in the previous section, we shall pay special attention to adopting the UNA on knowledge bases, in particular when we consider lower complexity bounds. As UNA on DKBs is easy to express in *DL-Lite*$_{\mathcal{R}}$, the results will carry over to the case without assumptions.

### *6.1 Satisfiability*

We first consider the satisfiability problem, that is, deciding whether a given *DL-Lite*$_{\mathcal{R}}$ DKB has some DKB model. As it turns out, defeasible axioms do not increase the complexity with respect to satisfiability of *DL-Lite*$_{\mathcal{R}}$, due to the following property.

*Proposition 8*
Let $\mathcal{K}$ be a normalized *DL-Lite*$_{\mathcal{R}}$ DKB, and let $\chi' = \{\langle \alpha, \mathbf{e} \rangle \mid D(\alpha) \in \mathcal{K}$, $\mathbf{e}$ is over standard names$\}$ be the clashing assumption with all exceptions possible. Then, $\mathcal{K}$ has some justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ such that $\chi \subseteq \chi'$ iff $\mathcal{K}$ has some CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi' \rangle$.

That is, a DKB $\mathcal{K}$ has a DKB model iff the *DL-Lite*$_{\mathcal{R}}$ KB consisting of the non-defeasible axioms in $\mathcal{K}$ has a model. We note that in the argument for this proposition, no particular NI-congruence is considered. Conditions such as UNA or other equivalence relations over the individuals in $\mathcal{K}$ can be accommodated (using Horn axioms).

Thus, DKB satisfiability testing with arbitrary exceptions boils down to testing whether $\mathcal{K}$ is satisfiable if all defeasible axioms are dropped, which is tractable.

*Theorem 4*
Deciding whether a given arbitrary *DL-Lite*$_{\mathcal{R}}$ DKB $\mathcal{K}$ has some DKB model is NLogSpace-complete in combined complexity and FO rewritable in data complexity.

*Proof*
We can normalize $\mathcal{K}$ efficiently in linear time (and in fact logspace) while preserving exception safety, so we may assume $\mathcal{K}$ is of this form. We then can test whether $\mathcal{K}$ with defeasible axioms dropped, which is an ordinary *DL-Lite*$_{\mathcal{R}}$ KB, is satisfiable; it is well-known that this is feasible in NLogSpace Calvanese *et al.* (2007). The NLogSpace-hardness is inherited from the combined complexity of KB satisfiability in *DL-Lite*$_{\mathcal{R}}$, which is NLogSpace-complete.

As regards data complexity, it is well-known that instance checking and satisfiability testing for *DL-Lite*$_{\mathcal{R}}$ are FO rewritable Calvanese *et al.* (2007); this has been shown by a reformulation algorithm, which informally unfolds the axioms $\alpha(\vec{x})$ (i.e. performs resolution viewing axioms as clauses), such that deriving an instance $A(a)$ reduces to presence of certain assertions in the ABox. We can use the same rewriting and apply it to $\mathcal{K}$ with all defeasible axioms dropped. □

We note that while satisfiability is tractable for arbitrary DKB models in general, this does not necessarily hold under restrictions on exceptions, as the construction in the proof of Proposition 8 depends on the enumeration; in particular, deciding the existence of some DKB model with no exceptions involving unnamed individuals (i.e. of a 0-bounded justified DKB model) is intractable; this can be shown, for example, by an adaption of an NP-hardness proof for 0-bounded justified model existence for defeasible $\mathcal{EL}_\bot$ context knowledge repositories in Bozzato *et al.* (2019b). On the other hand, under a condition that ensures that some DKB model is 0-bounded if any DKB model exists, we retain tractability. In particular:

*Corollary 2*
Deciding whether a given exception-safe DKB $\mathcal{K}$ has some DKB model is NLogSpace-complete in combined complexity and FO rewritable in data complexity.

In passing, we remark that for exception safe DKBs $\mathcal{K}$, checking whether an interpretation $\mathcal{I}$ is a DKB model of $\mathcal{K}$ is tractable (as follows from the ASP encoding), as is constructing some arbitrary DKB model; however, we focus in the sequel here on inference.

### 6.2 Entailment checking

As regards inference, entailment checking of axioms from the DKB models of an exception safe DKB is intractable: there can be exponentially many justified clashing assumptions for such models, even under UNA; finding a DKB model that violates an axiom turns out to be difficult.

*Theorem 5*
Given an exception safe DKB $\mathcal{K}$ and an axiom $\alpha$, deciding whether $\mathcal{K} \models \alpha$ is co-NP-complete; this holds also for data complexity and instance checking, that is, $\alpha$ is an assertion of the form $A(a)$.

*Proof (Sketch)*
To refute $\mathcal{K} \models \alpha$, we need to show that some justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$ exists such that $\mathcal{I} \not\models \alpha$. Without loss of generality, we assume that $\alpha$ is normalized.

Given $\chi$ and a name assignment $\nu$, we can prove the refutation depending on the type of $\alpha$. For example, if $\alpha$ is an inclusion axiom $A \sqsubseteq B$, then we need to show that for some element $e$ it holds that $\mathcal{I} \models A(e)$ and $\mathcal{I} \models \neg B(e)$. To deal with this, we first incorporate $\nu$ into $\mathcal{K}$, by pushing (in)equalities w.r.t. $\nu$ (replace all equal constants by one representative, add axioms that enforce inequalities $a \neq b$, for example, stating $A_a(a)$, $\neg A_b(b)$ where $A_a$ and $A_b$ are fresh concept names). We then add to $\mathcal{K}$ the axioms $Aux \sqsubseteq A$, $Aux \sqsubseteq \neg B$. We may then assume without loss of generality that $\mathcal{I} \models Aux(e)$, that is, $\mathcal{I} \not\models \neg Aux(e)$. We next add to $\mathcal{K}$ an assertion $A_e(a_e)$, where $A_e$ and $a_e$ are a fresh concept and individual name, respectively; this serves to give $e$ a name if it is outside the elements named in $\mathcal{I}$ by Skolem terms. We then check whether $\neg Aux(a_e)$ is not derivable from the resulting DKB $\mathcal{K}'$ under $\chi$; this holds iff some $\mathcal{I}$ with $e$ not named by some Skolem term of $\mathcal{K}$ exists. Otherwise, $e$ must be named by some Skolem term $t$ of $\mathcal{K}$. We thus check that for none such $t$, $\neg Aux(t)$ is derivable from $\mathcal{K}'$ under $\chi$; the depth of $t$ can be polynomially bounded. The checks can be done in non-deterministic logspace, and thus deciding $\mathcal{K} \not\models \alpha$ under $\chi$ is feasible in polynomial time. The cases for other forms of $\alpha$ can be shown similarly and are described in the full proof in Appendix C.4.

Thus, to decide $\mathcal{K} \not\models \alpha$, we can guess a justified clashing assumption $\chi$ over $N_{\mathcal{K}}$ together with a clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ for each $\langle \alpha, \mathbf{e} \rangle \in \chi$ for a name assignment $\nu$. We then check relative to $\nu$ (i) that $\chi$ is satisfiable, (ii) that all $S_{\langle \alpha, \mathbf{e} \rangle}$ are derivable from $\mathcal{K}$ under $\chi$

and $\nu$, and (iii) that $\mathcal{K} \not\models \alpha$. Each of the steps (i)–(iii) is feasible in polynomial time. Consequently, the entailment problem $\mathcal{K} \models \alpha$ is in co-NP.

The co-NP-hardness can be shown by a reduction from inconsistency-tolerant reasoning from *DL-Lite*$_\mathcal{R}$ KBs under AR-semantics Lembo *et al.* (2010). The result for *DL-Lite*$_\mathcal{R}$ KBs can be easily extended to exception safe DKBs and to data complexity (further details are provided in the full proof in Appendix C.4). □

We observe that the co-NP-hardness proof in Lembo *et al.* (2010) used many role restrictions and inverse roles; for combined complexity, co-NP-hardness of entailment in absence of any role names can be derived from results about propositional circumscription.

*Proposition 9*
Given a DKB $\mathcal{K}$, deciding whether $\mathcal{K} \models \alpha$ is co-NP-hard even if no roles occur in $\mathcal{K}$ and $\alpha$ is an assertion $A(a)$.

While the proof of Proposition 9 establishes co-NP-hardness of entailment for combined complexity under UNA when roles are absent (and for the case without UNA as well), this setting has tractable data complexity: we can consider the axioms for individuals $a$ separately, and if the GCI axioms are fixed only few axioms per individual exist. This also holds if role axioms but no existential restrictions are permitted, as we can concentrate on the pairs $a, b$ and $b, a$ of individuals. The question remains how much of the latter is possible while staying tractable.

### *6.3 Conjunctive query answering*

A *conjunctive query (CQ)* is a formula $Q(\vec{x}) = \exists \vec{y}.\gamma(\vec{x}, \vec{y})$ where $\vec{x}$, $\vec{y}$ are disjoint lists of different variables and $\gamma(\vec{x}, \vec{y}) = \gamma_1 \wedge \cdots \wedge \gamma_m$ is a conjunction of atoms $\gamma_i = \alpha_i(\vec{t_i})$, $1 \leq i \leq m$ where $\alpha_i$ is either a concept name or a role name and $\vec{t_i}$ is a tuple of variables from $\vec{x} \cup \vec{y}$ and individual constants that matches the arity of $\alpha_i$. The CQ is *Boolean* (a *BCQ*), if $\vec{x}$ is empty.

A CAS interpretation $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ satisfies a BCQ $Q$, denoted $\mathcal{I}_{CAS} \models Q$, if a query matches, that is, some substitution $\vartheta : \vec{y} \to \mathrm{NI}_s$ exists such that $\mathcal{I} \models \alpha_i(\vec{t_i}\vartheta)$ for all $i = 1, \ldots, m$. A DKB $\mathcal{K}$ entails $Q$, denoted $\mathcal{K} \models Q$, if every DKB model of $\mathcal{K}$ entails $Q$. The *(certain) answers* for a general CQ $Q(\vec{x})$ are then as usual the tuples $\vec{c}$ of individual names such that $\mathcal{K} \models Q(\vec{c})$.

*Example 8* (*Example 4 cont'd*)
Consider the CQ $Q(x) = \exists y.\mathit{DeptMember}(x) \wedge \mathit{hasCourse}(x, y)$ on the DKB $\mathcal{K}_{dept}$ in Example 3. In Example 4, we discussed that $\mathcal{K}_{dept}$ has a justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ with an exception for *bob* on the axiom $\alpha = \mathit{DeptMember} \sqsubseteq \exists \mathit{hasCourse}$, while *alice* has no exception. Thus, the query $Q$ has a match in $\mathcal{I}_{CAS}$ by $\vartheta = \{x \mapsto \mathit{alice}^\mathcal{I}, y \mapsto f^\mathcal{I}_{\mathit{hasCourse}}(x)\}$, where $f^\mathcal{I}_{\mathit{hasCourse}}$ is the Skolem function in $\mathcal{I}$; in fact, in every such justified CAS model $\mathcal{I}_{CAS}$ the query has this match. If *alice* and *bob* are regarded different, that is, under the unique name assumption, no other justified CAS model exists; thus *alice* is the (only) certain answer of the query. ◇

Deciding whether a BCQ $Q$ has a query match in a *DL-Lite*$_\mathcal{R}$ KB is known to be NP-complete, cf. Calvanese *et al.* (2007). As multiple (even exponentially many) clashing assumptions may lead to different DKB models, and as for each such assumption the query must have a match, BCQ answering from exception safe DKBs is at the second level of the polynomial hierarchy.

*Theorem 6*
Given an exception safe DKB $\mathcal{K}$ and a Boolean CQ $Q$, deciding whether $\mathcal{K} \models Q$ is (i) $\Pi^p_2$-complete in combined complexity and (ii) co-NP-complete in data complexity.

*Proof (Sketch)*
To start with (i), as for membership in $\Pi^p_2$, to refute $Q$ we can guess for a justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ such that $\mathcal{I}_{CAS} \not\models Q$ the clashing assumption $\chi$ on $N_\mathcal{K}$ and a name assignment $\nu$. Since $\mathcal{K}$ is exception safe, we can decide in NLogSpace whether $\chi$ is satisfiable relative to $\nu$ using the entailment method for positive and negative assertions in the proof of Theorem 5; note that $\nu$ can be pushed to $\mathcal{K}$ and indeed can give rise to a desired justified CAS model $\mathcal{I}_{CAS}$ of $\mathcal{K}$. We then can use an NP oracle to check whether for some polynomial number of Skolem terms $ST$, where the number depends on $Q$ and $\mathcal{K}$, the query has a match on $N_\mathcal{K} \cup ST$ in a least CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$; to this end, each atom $A(t)$ resp. $R(t, t')$ in the match must be derived by applying the axioms (that is, by unraveling $\mathcal{I}^{N_\mathcal{K}}_{CAS}$); this will ensure that a match exists in each CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$. If the oracle answer is no, then some $\mathcal{I}_{CAS}$ such that $\mathcal{I}_{CAS} \not\models Q$ exists. Consequently, refuting $\mathcal{K} \models Q$ is in $\Sigma^p_2$, which proves the membership part.

The $\Pi^p_2$-hardness of (i) is shown by a reduction from a generalization of deciding whether a graph is 3-colorable: given an (undirected) graph $G = (V, E)$, can every color assignment to the nodes of degree 1 in $G$ (i.e. source nodes) be extended to a 3-coloring of $G$? This problem is $\Pi^p_2$-complete (see Lemma 5 in Appendix C.4). The construction for such reduction is provided in the full proof in Appendix C.4.

(ii) As for data complexity, we note that the check where $Q$ has no match in any $\mathcal{I}''_{CAS}$ is feasible in polynomial time, as the number of variables in the query is fixed and thus only constantly many Skolem terms $ST$ have to be added to $N_\mathcal{K}$ for a query match in a least CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$, for which only polynomially many possibilities exist; furthermore, the inference of atoms $A(t)$ resp. $R(t, t')$ is feasible in polynomial time. Hence, the problem is in co-NP. The co-NP-hardness follows from Theorem 5. □

## 7 Reasoning on unnamed individuals

In the sections above, we have concentrated on exception safe DKBs, where no exceptions on unnamed individuals are possible. However, this is not a real limitation in principle, as unnamed individuals may be named. Specifically, we note the following property of $n$-bounded CAS models (recall from Definition 7 that $uni_\mathcal{K}(\mathcal{I}_{CAS})$ are the domain elements in clashing assumptions not named by individuals in $\mathcal{K}$):

*Proposition 10*
Let $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ be a CAS model of a DKB $\mathcal{K}$ such that $uni_\mathcal{K}(\mathcal{I}_{CAS}) = \{e_1, \ldots, e_m\}$. Let $c_1, \ldots, c_m$ be fresh individual names, and $A$ be a fresh concept. Then, $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ where $c_i^{\mathcal{I}'} = e_i$, $i = 1, \ldots, m$ and $A^{\mathcal{I}'} = \{e_1, \ldots, e_m\}$ is a CAS model of $\mathcal{K}' = \mathcal{K} \cup \{A(c_1), \ldots, A(c_m)\}$.

That is, we can name unnamed individuals in clashing assumptions, and in this way turn an $n$-bounded CAS model into a 0-bounded CAS model. In particular, if $n$ is polynomial in the size of $\mathcal{K}$ we can do this with polynomial overhead.

However, when we reason from a DKB $\mathcal{K}$ under a (named) clashing assumption $\chi$, the issue rises whether for a Skolem term $f(t)$ and a clashing assumption $\langle \alpha, e \rangle$, where $e$ is named by an individual $a$, say, the exception is applicable (if $f(t) = a$) or not (if $f(t) \neq a$) in a model. This in fact complicates reasoning, and to decide whether a given partial CAS interpretation $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ where all exceptions are individuals in $\mathcal{K}$ can be extended to some justified CAS model of $\mathcal{K}$ is no longer easy to accomplish. If $\mathcal{K}$ is $n$-de safe, we have to consider such terms $f_1(t_1), \ldots, f_m(t_m)$ for $m \leq n$ and possibly collapse them with some individuals in $\mathcal{K}$. This leads to an exponential explosion, even if $n = O(|\mathcal{K}|^k)$ is polynomial in the size of $\mathcal{K}$. As it turns out, already deciding whether $\mathcal{K}$ has for $\chi$ some CAS model is intractable in this setting, which can be decided by a proper guess of the (in)equality for all $f_i(t_i)$. More precisely, the following property can be shown.

*Proposition 11*
Given an $n$-de safe DKB $\mathcal{K}$, where $n$ is polynomial in the size of $\mathcal{K}$, and a clashing assumption $\chi$ defined on $N_{\mathcal{K}}$, deciding whether $\mathcal{K}$ has (i) some arbitrary CAS model resp. (ii) some justified CAS model of form $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ is NP-complete resp. $D^p$-complete[14] in general but feasible in polynomial time if $n$ is bounded by a constant.

As regards properties of justified CAS models, Proposition 6 readily generalizes from exception safe to $n$-de safe DKBs if the (in)equalities of the Skolem terms $t_1, \ldots, t_m$ with individuals are fixed; hence, a least justified CAS model exists relative to such fixed (in)equalities and a name assignment $\nu$.

The intractability in Proposition 11 holds even under data complexity and when $\mathcal{K}$ is $k$-chain bounded for a small constant $k$. Furthermore, we obtain as a side result from the proof that DKB model checking, that is, deciding whether an interpretation $\mathcal{I}$ is a DKB model of a given $\mathcal{K}$, is co-NP-hard and for such DKBs co-NP-complete.

As a consequence of the previous result, reasoning when a few (constantly many chains) to exceptions exist is not more expensive than if no such chains exist; for polynomially many chains, axiom inference gets more expensive.

*Theorem 7*
Given an $n$-de safe DKB $\mathcal{K}$, where $n$ is bounded by a polynomial in $|\mathcal{K}|$, (i) deciding $\mathcal{K} \models \alpha$ for an axiom $\alpha$ and (ii) BCQ answering $\mathcal{K} \models Q$ are both $\Pi^p_2$-complete. In case $n$ is bounded by a constant, (i) is co-NP-complete while (ii) remains $\Pi^p_2$-hard.

The results in the theorem hold in fact under data complexity. Intuitively, the complexity of BCQ-Answering does not increase in the $n$-de bounded case, as checking whether a guess for a clashing assumption $\chi$ that allows to refute the query $Q$ does not add further complexity in general, since checking whether a query $Q$ has no match in an ABox is already co-NP-hard. It does so, however, if the query $Q$ is just an assertion $A(a)$.

**Towards an ASP encoding.** A possible approach for extending the ASP encoding in Section 5 in this regard, then, would be as follows. We may consider the different "equality

[14] $D^p$ consists loosely speaking of the "conjunction" of independent instances $I_1$ and $I_2$ of two problems in NP and co-NP, respectively (e.g. SAT-UNSAT).

environments", where each environment $e$ is given by the condition $f_i(t_i) = a_{i_j}$ for some individual $a_{i_j}$, using test environments similar as in Bozzato *et al.* (2018): for each $f_i(t_i)$, we introduce a predicate $sk_i$ and an argument $x_i$ in the predicates for the derivation, where $sk_i$ and $x_i$ can take any individual name $a_i$ or $f_i(t_i)$; we add propagation rules that push equalities, of form

$$\texttt{instd}(x_i, z, x_1, \ldots, x_m) \leftarrow \texttt{instd}(x, z, x_1, \ldots, x_m), sk_i(x). \quad (2)$$

$$\neg\texttt{instd}(x_i, z, x_1, \ldots, x_m) \leftarrow \neg\texttt{instd}(x, z, x_1, \ldots, x_m), sk_i(x). \quad (3)$$

and similar for `tripled`; furthermore, with a technique similar as in the rules (pdlr-allnrel1) – (pdlr-allnrel3) for `all_nrel` we can check whether a derivation succeeds for all environments. As an alternative, we may consider using (recursive) aggregates to perform this check. Furthermore, the set of auxiliary constants is extended with further constants that allow to build the Skolem paths $f_1(t_1)$, ..., $f_m(t_m)$. For $m$ bounded by a constant, this would lead to a fixed program, where the Skolem chains are provided as data; the latter might be determined inside an ASP encoding as well, which however is more involving.

## 8 Related work

The relation of the justified exception approach to nonmonotonic description logics was discussed in Bozzato *et al.* (2018), where in particular an in-depth comparison w.r.t. typicality in DLs Giordano *et al.* (2013), normality Bonatti *et al.* (2011) and overriding Bonatti *et al.* (2015) was given. A distinctive feature of our approach, linked to the interpretation of exception candidates as different clashing assumptions, is the possibility to "reason by cases" inside the alternative justified models (as we have demonstrated over the Nixon Diamond problem in Example 5). Note that we do not consider a preference ordering across defeasible axioms, but all alternative interpretations that justify their clashing assumptions (cf. also (Bozzato *et al.* 2018) where a preference is defined by the KB contextual structure).

In particular, compared to Bozzato *et al.* (2018), in this paper we work on a different language: particularly, $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ allows for reasoning with unnamed individuals and their use in inverse roles, as detailed in the sections above. Moreover, we are not considering contextual aspects of the previous works, for example, knowledge propagation by *eval* operator and local interpretation of knowledge. Note that, with respect to the notion of defeasibility expressed on the CKR contexts structure, we have a slightly different interpretation for DKBs: while in CKR we defined defeasibility over the inheritance from more general to more specific contexts, in DKBs we consider exceptions on the "local" application of defeasible axioms to the elements of the knowledge base. We remark that the results shown for DKBs could be then extended to CKRs in order to study the interaction of $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ features with contextual structures and overriding preferences.

The introduction of non-monotonic features in the *DL-Lite* family and, more in general, to low complexity DLs has been the subject of many works, mostly with the goal of preserving the low complexity properties of the base logic in the extension. For example, in Bonatti *et al.* (2011) an in-depth study of the complexity of reasoning with circumscription in $\mathit{DL\text{-}Lite}_{\mathcal{R}}$ and $\mathcal{EL}$ was presented: the idea is to verify whether syntactic restrictions of these languages can be useful to limit the complexity of the non-monotonic version of these languages. The work considers defeasibility on inclusion axioms of the kind $C \sqsubseteq_n D$, which intuitively can be read as "an instance of $C$ is *normally* an instance of $D$". Conflicts across defeasible inclusions are solved by providing a priority on such axioms: an option to define priority is to use the specificity of defeasible inclusions, that is $C_1 \sqsubseteq_n D_1$ is preferred to $C_2 \sqsubseteq_n D_2$ if $C_1$ is subsumed by $C_2$. Different fragments of *DL-Lite*$_\mathcal{R}$ and $\mathcal{EL}$ are considered to limit the complexity of the reasoning problems. In the case of *DL-Lite*$_\mathcal{R}$, it is shown that such syntactic restrictions allow to limit the complexity of instance checking to $\Pi_2^p$. For $\mathcal{EL}$, its extension to circumscription is ExpTime-hard and more restrictions are needed to limit its complexity to the second level of the polynomial hierarchy.

Similarly, in Giordano *et al.* (2011) the authors studied the complexity of the application of their typicality approach to low complexity description logics. They note that the introduction of the typicality operator to $\mathcal{ALC}$ leads to an increase in complexity of reasoning (query entailment becomes Co-Nexp$^{NP}$): thus, their goal is to find (fragments of) low level description logics where the extension to typicality has a limited impact on the complexity of entailment. In Giordano *et al.* (2011), an extension to typicality of the DLs *DL-Lite*$_c$ and $\mathcal{EL}^\perp$ is proposed and their complexity properties are studied. It is shown that, in the case of $\mathcal{EL}^\perp$, the extension with typical concept inclusions (called $\mathcal{EL}^\perp T_{min}$) is ExpTime-hard. However, by limiting to *left local* KBs in $\mathcal{EL}^\perp$ (i.e. using a fragment of $\mathcal{EL}^\perp$ that restricts the form of left side concepts in concept inclusions), one can show that complexity can be limited to $\Pi_2^p$. Similarly, the extension of *DL-Lite*$_c$ can be shown to have the same $\Pi_2^p$ complexity upper bound. Notably, the complexity bounds for $\mathcal{EL}^\perp$ and *DL-Lite*$_c$ match the ones proved in Bonatti *et al.* (2011).

A recent work in this direction is Pensel and Turhan (2017), Pensel and Turhan (2018), where a defeasible version of $\mathcal{EL}^\perp$ was obtained: as an interesting parallel with the work presented in our paper, the goal of such work is to overcome issues with the approach by Casini and Straccia (2010) on the interpretation of defeasible properties on quantified concepts, especially in nested expressions. This approach is based on an extension of classical canonical models of $\mathcal{EL}^\perp$, called *typicality models*, where multiple representatives for each concept are used to identify different versions of the same concept under different levels of typicality. Using typicality models the authors show that they can obtain stronger versions of rational and relevant entailment that do not neglect defeasible information in nested quantifications. The authors also present a reasoning algorithm for instance checking under the proposed semantics by a variant of the materialization-based approach that only uses the expressivity of $\mathcal{EL}_\perp$. Finally, the computational complexity of the defeasible subsumption and instance checking under the different semantics is investigated: in particular, the definition of the materialization method extending the reasoning to defeasibility while keeping the expressivity of the base logic $\mathcal{EL}_\perp$ provides the evidence that complexity of reasoning need not to increase in this logic. This approach is different from ours, which works on all models and uses factual justifications that need to be derived: on the other hand, canonical models are useful for characterization and implementation, thus we could investigate some of the results in these papers in the extension of our work.

*Example 9* (*Inheritance blocking*)
In studying the properties of rational and minimal relevant closure, Pensel and Turhan (2018) highlight that rational closure suffers from the problem of *inheritance blocking*,

intuitively the effect where all properties of superclasses of an exceptional class $C$ are not inherited, even if they are not related to the “exceptionality” of $C$. In the case of our semantics, we have shown in Bozzato *et al.* (2018) by means of the *Situs inversus* example from Bonatti *et al.* (2015) that we can deal with property inheritance at the instance level.

We show that inheritance can be also preserved when reasoning involves unnamed individuals, by rephrasing Example 3.2 from Pensel and Turhan (2018). Consider the DKB $\mathcal{K}_{org}$ defined as:

$$\mathcal{K}_{org}: \left\{ \begin{array}{l} \mathit{Boss} \sqsubseteq \mathit{Worker},\ \mathit{Boss} \sqsubseteq \neg\exists \mathit{hasSuperior},\ \exists \mathit{hasSuperior}^{-} \sqsubseteq \mathit{Boss} \\ \mathrm{D}(\mathit{Worker} \sqsubseteq \exists \mathit{hasSuperior}),\ \mathrm{D}(\mathit{Worker} \sqsubseteq \mathit{Productive}), \\ \mathrm{D}(\mathit{Boss} \sqsubseteq \mathit{Responsible}),\ \mathit{Worker}(\mathit{bob}),\ \neg \mathit{Boss}(\mathit{bob}) \end{array} \right\}.$$

Similarly to Example 7, this DKB admits a model $\mathcal{I}_{CAS}$ where we have an exception on $\alpha = \mathit{Worker} \sqsubseteq \exists \mathit{hasSuperior}$ for the (unnamed) boss $f(\mathit{bob})$ of $\mathit{bob}$, with $f(\mathit{bob}) \neq \mathit{bob}$. However, we have no reason to override the other properties of $\mathit{Boss}$ on $f(\mathit{bob})$, thus we have $\mathcal{I}_{CAS} \models \mathit{Responsible}(f(\mathit{bob}))$ and $\mathcal{I}_{CAS} \models \mathit{Productive}(f(\mathit{bob}))$. Also, if we add $\neg \mathit{Productive}(\mathit{bob})$ to $\mathcal{K}_{org}$, the overriding of the respective axiom does not influence the applicability of $\alpha$ to $\mathit{bob}$.

As shown by Pensel and Turhan (2018), rational closure by the materialization-based approach of Casini and Straccia (2010) fails to derive information on such existential individuals; thus, for example, it can not derive $\mathit{Productive}(f(\mathit{bob}))$. Similar considerations on inheritance blocking can be drawn for DLs with typicality as those presented in Giordano *et al.* (2011). ◇

Another recent work about reasoning on non-monotonic versions of the $\mathcal{EL}$ family is Casini *et al.* (2019). The paper considered the logic $\mathcal{ELO}_{\bot}$ (i.e. the extension of $\mathcal{EL}_{\bot}$ with nominals) and studies the problem of (non-monotonic) concept subsumption, where the non-monotonic aspects are represented via rational closure. The authors provided a polynomial time subsumption algorithm for $\mathcal{ELO}_{\bot}$ under rational closure that, notably, reduces the problem to a series of classical monotonic subsumption tests in the same language. This allows to use the customary (monotonic) $\mathcal{EL}$ based reasoners to implement the reasoning method.

A recent approach related to our work is Eiter *et al.* (2016), in which inconsistency-tolerant query answering over a set of existential rules was studied. The authors considered removing errors from ontological axioms that lead to inconsistency, with the possibility to specify a set of axioms that should not be touched. They introduced two semantics for BCQ Answering on existential rules, in which a maximal set of designated rules *(GR semantics)* or rule instances *(LGR semantics)* is kept while maintaining consistency. This dually corresponds to the inherent minimality of clashing assumptions in justified DKB models, with the difference that in Eiter *et al.* (2016) no proof of a clashing set or a similar certificate is required for removing a rule instance. Exceptions may be harder to obtain under LGR semantics; for example, in Example 7, no exception for *alice* to the defeasible axiom is possible, and thus only models with an unnamed supervisor for *alice* exist. Notably, for LGR semantics minimal removal checking is polynomial for such rule sets, while testing whether a corresponding clashing assumption is justified is intractable (cf. Proposition 11). Closer relationship with LGR semantics remains to be clarified in future work.

## 9 Conclusion

In this paper, we considered the justified exception approach in Bozzato *et al.* (2018) for reasoning on *DL-Lite*$_\mathcal{R}$ KBs with defeasible axioms. With respect to our previous works, we had to consider the problem of reasoning with unnamed individuals introduced by existential axioms, especially when they are involved in the reasoning over exceptions: we provided different characterizations of DKB models with respect to the use of unnamed individuals and their presence in exceptions. Considering DKBs where exceptions can appear only on named individuals, we studied the semantic properties of DKB models and we analyzed the complexity of the main reasoning problems. We have shown that the limited language of *DL-Lite*$_\mathcal{R}$ allows us to formulate a direct datalog translation to reason on derivations for negative information in instance checking. Finally, we provided some insights in the case of reasoning with exceptions on existential individuals and a direction for extending the datalog translation in this regard.

While the focus of this work lies mostly in the area of Description Logics, where we extend current languages to deal with non-monotonicity, we note that this work also shows the strength of LP and ASP technologies: in particular, we have that ASP provides a practical way to encode and solve tasks such as conjunctive query answering in other logical systems. Moreover, by comparing our previously defined ASP encodings, we note that by the declarative nature of ASP, the management of the new aspects of *DL-Lite*$_\mathcal{R}$ can be encoded by an adaptation of the program rules. This shows some flexibility of the rule-based approach, which is in particular valuable for developing prototypical implementations.

As discussed in the previous sections, reasoning with unnamed individuals in exceptions can be further studied to obtain more insights, possibly with a refinement or further elaboration of the ASP encoding sketched in Section 7. In particular, it would be interesting to explore possible variants of the exception semantics that we considered here, from comparison with and inspired by related work such as the one discussed in Section 8. Using Skolem terms in exceptions, which underlies the approach in Eiter *et al.* (2016), may be an option, but the consequences will have to be carefully considered, as issues with Skolemization in non-monotonic reasoning are folklore.

The complexity results obtained for $n$-de safe DKBs imply that some encoding in ASP is possible that uses predicates of arity bounded by a constant, in contrast to the rules (2)–(3) sketched in Section 7, with rule bodies that have a variable (possible large) number of literals. In other contexts, such bounded-arity encodings proved to be useful using solvers based on decomposition techniques Bichler *et al.* (2016). It thus would be interesting to see whether this approach could be fruitfully used for encoding DKB reasoning with unnamed individuals as well.

Finally, we plan to apply the current results on *DL-Lite*$_\mathcal{R}$ in the framework of CKR with hierarchies as in Bozzato *et al.* (2018), for which the current results have to be extended to the respective setting. Imposing preference on exceptions in the hierarchy may however increase the complexity and thus require language constructs that offer increased expressivity, such as optimization (e.g. by weak constraints) or disjunction in rules heads.

**Competing interests declaration.** The authors declare none.

## References


Alviano, M. and Faber, W. 2018. Aggregates in answer set programming. *Künstliche Intelligence 32,* 2–3, 119–124.

Baader, F., Calvanese, D., McGuinness, D., Nardi, D. and Patel-Schneider, P., Eds. 2003. *The Description Logic Handbook*. Cambridge University Press.

Bichler, M., Morak, M. and Woltran, S. 2016. The power of non-ground rules in answer set programming. *Theory and Practice of Logic Programming 16,* 5-6, 552–569.

Bonatti, P. A., Faella, M., Petrova, I. and Sauro, L. 2015. A new semantics for overriding in description logics. *Artificial Intelligence 222*, 1–48.

Bonatti, P. A., Faella, M. and Sauro, L. 2011. Defeasible inclusions in low-complexity DLs. *Journal of Artificial Intelligence Research 42*, 719–764.

Bonatti, P. A., Lutz, C. and Wolter, F. 2006. Description logics with circumscription. See Doherty *et al*. (2006), 400–410.

Bozzato, L., Eiter, T. and Serafini, L. 2014. Contextualized knowledge repositories with justifiable exceptions. In *27th International Workshop on Description Logics (DL2014)*, M. Bienvenu, M. Ortiz, R. Rosati, and M. Simkus, Eds. CEUR Workshop Proceedings, vol. 1193. CEUR-WS.org, 112–123.

Bozzato, L., Eiter, T. and Serafini, L. 2018. Enhancing context knowledge repositories with justifiable exceptions. *Artificial Intelligence 257*, 72–126.

Bozzato, L., Eiter, T. and Serafini, L. 2019a. Reasoning on *DL-Lite*$_{\mathcal{R}}$ with defeasibility in ASP. In *3rd International Joint Conference on Rules and Reasoning (RuleML+RR 2019)*, P. Fodor, M. Montali, D. Calvanese, and D. Roman, Eds. Lecture Notes in Computer Science, vol. 11784. Springer, 19–35.

Bozzato, L., Eiter, T. and Serafini, L. 2019b. Reasoning with justifiable exceptions in $\mathcal{EL}_{\bot}$ contextualized knowledge repositories. In *Description Logic, Theory Combination, and All That - Essays Dedicated to Franz Baader on the Occasion of His 60th Birthday*, C. Lutz, U. Sattler, C. Tinelli, A. Turhan, and F. Wolter, Eds. Lecture Notes in Computer Science, vol. 11560. Springer, 110–134.

Bozzato, L., Homola, M. and Serafini, L. 2012. Towards more effective tableaux reasoning for CKR. In *25th International Workshop on Description Logics (DL2012)*, Y. Kazakov, D. Lembo, and F. Wolter, Eds. CEUR Workshop Proceedings, vol. 846. CEUR-WS.org, 114–124.

Bozzato, L. and Serafini, L. 2013. Materialization calculus for contexts in the semantic web. In *26th International Workshop on Description Logics (DL2013)*, T. Eiter, B. Glimm, Y. Kazakov, and M. Krötzsch, Eds. CEUR Workshop Proceedings, vol. 1014. CEUR-WS.org, 552–572.

Bozzato, L., Serafini, L. and Eiter, T. 2018. Reasoning with justifiable exceptions in contextual hierarchies. In *16th International Conference on Principles of Knowledge Representation and Reasoning (KR 2018)*, M. Thielscher, F. Toni, and F. Wolter, Eds. AAAI Press, 329–338.

Britz, K. and Varzinczak, I. J. 2016. Introducing role defeasibility in description logics. In *15th European Conference on Logics in Artificial Intelligence (JELIA 2016)*, L. Michael and A. C. Kakas, Eds. LNCS, vol. 10021. 174–189.

Buccafurri, F., Faber, W. and Leone, N. 1999. Disjunctive logic programs with inheritance. In *16th International Conference on Logic Programming (ICLP 1999)*, D. D. Schreye, Ed. MIT Press, 79–93.

Cadoli, M. and Lenzerini, M. 1994. The complexity of propositional closed world reasoning and circumscription. *Journal of Computer and System Sciences 48,* 2, 255–310.

Calì, A., Gottlob, G. and Lukasiewicz, T. 2012. A general datalog-based framework for tractable query answering over ontologies. *J. Web Semant. 14*, 57–83.

Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M. and Rosati, R. 2007. Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. *J. Automated Reasoning 39,* 3, 385–429.

Casini, G. and Straccia, U. 2010. Rational closure for defeasible description logics. See Janhunen and Niemelä (2010), 77–90.

Casini, G., Straccia, U. and Meyer, T. 2019. A polynomial time subsumption algorithm for nominal safe $\mathcal{ELO}_\bot$ under rational closure. *Information Science 501*, 588–620.

de Bruijn, J., Eiter, T., Polleres, A. and Tompits, H. 2011. Embedding nonground logic programs into autoepistemic logic for knowledge-base combination. *ACM Transactions on Computational Logic 12,* 3, 20:1–20:39.

de Bruijn, J., Eiter, T. and Tompits, H. 2008. Embedding approaches to combining rules and ontologies into autoepistemic logic. In *11th International Conference on Principles of Knowledge Representation and Reasoning (KR 2008)*, G. Brewka and J. Lang, Eds. AAAI Press, 485–495.

Doherty, P., Mylopoulos, J. and Welty, C. A., Eds. 2006. *10th International Conference on Principles of Knowledge Representation and Reasoning (KR 2006)*. AAAI Press.

Eiter, T., Ianni, G., Lukasiewicz, T., Schindlauer, R. and Tompits, H. 2008. Combining answer set programming with description logics for the semantic web. *Artif. Intell. 172,* 12–13, 1495–1539.

Eiter, T., Lukasiewicz, T. and Predoiu, L. 2016. Generalized consistent query answering under existential rules. In *15th International Conference on Principles of Knowledge Representation and Reasoning (KR 2016)*, C. Baral, J. P. Delgrande, and F. Wolter, Eds. AAAI Press, 359–368.

Fitting, M. 1996. *First-Order Logic and Automated Theorem Proving*, 2nd ed. Graduate Texts in Computer Science. Springer.

Gelfond, M. and Lifschitz, V. 1991. Classical negation in logic programs and disjunctive databases. *New Generation Computing 9,* 3/4, 365–386.

Giordano, L., Gliozzi, V., Olivetti, N. and Pozzato, G. L. 2011. Reasoning about Typicality in Low Complexity DLs: The Logics $\mathcal{EL}^\bot T_{min}$ and DL-Lite$_c T_{min}$. In *22nd International Joint Conference on Artificial Intelligence (IJCAI 2011)*, T. Walsh, Ed. IJCAI/AAAI, 894–899.

Giordano, L., Gliozzi, V., Olivetti, N. and Pozzato, G. L. 2013. A non-monotonic description logic for reasoning about typicality. *Artificial Intelligence 195*, 165–202.

Grosof, B. N., Horrocks, I., Volz, R. and Decker, S. 2003. Description logic programs: Combining logic programs with description logic. In *12th International World Wide Web Conference (WWW 2003)*, G. Hencsey, B. White, Y. R. Chen, L. Kovács, and S. Lawrence, Eds. ACM, 48–57.

Horrocks, I., Kutz, O., and Sattler, U. 2006. The even more irresistible $\mathcal{SROIQ}$. See Doherty *et al.* (2006), 57–67.

Janhunen, T. and Niemelä, I., Eds. 2010. *12th European Conference on Logics in Artificial Intelligence (JELIA 2010)*. Lecture Notes in Computer Science, vol. 6341. Springer.

Krötzsch, M. 2010. Efficient inferencing for OWL EL. See Janhunen and Niemelä (2010), 234–246.

Lembo, D., Lenzerini, M., Rosati, R., Ruzzi, M. and Savo, D. F. 2010. Inconsistency-tolerant semantics for description logics. In *4th International Conference on Web Reasoning and Rule Systems (RR 2010)*, P. Hitzler and T. Lukasiewicz, Eds. Lecture Notes in Computer Science, vol. 6333. Springer, 103–117.

Lifschitz, V., Pearce, D. and Valverde, A. 2001. Strongly equivalent logic programs. *ACM Transactions on Computational Logic 2,* 4, 526–541.

Motik, B., Fokoue, A., Horrocks, I., Wu, Z., Lutz, C. and Grau, B. C. 2009. OWL 2 Web Ontology Language Profiles. W3C recommendation, W3C. October. http://www.w3.org/TR/2009/REC-owl2-profiles-20091027/

Pensel, M. and Turhan, A. 2017. Including quantification in defeasible reasoning for the description logic $\mathcal{EL}_\bot$. In *14th International Conference on Logic Programming and*

*Nonmonotonic Reasoning (LPNMR 2017)*, M. Balduccini and T. Janhunen, Eds. Lecture Notes in Computer Science, vol. 10377. Springer, 78–84.

Pensel, M. and Turhan, A. 2018. Reasoning in the defeasible description logic $\mathcal{EL}_\bot$ - computing standard inferences under rational and relevant semantics. *International Journal of Approximate Reasoning 103*, 28–70.

Serafini, L. and Homola, M. 2012. Contextualized knowledge repositories for the semantic web. *Journal of Web Semantics 12*, 64–87.

Woltran, S. 2008. A common view on strong, uniform, and other notions of equivalence in answer-set programming. *Theory and Practice of Logic Programming 8,* 2, 217–234.

## Appendix A Description logic *DL-Lite*$_\mathcal{R}$

In Table A1 we present the syntax and semantics of operators included in the description logic *DL-Lite*$_\mathcal{R}$. Further rules for the composition of axioms in *DL-Lite*$_\mathcal{R}$ are specified in Section 2.

## Appendix B FO translation for *DL-Lite*$_\mathcal{R}$

In Table B1 we provide a FO translation for axioms in *DL-Lite*$_\mathcal{R}$. Given a *DL-Lite*$_\mathcal{R}$ axiom $\alpha$ in $\mathcal{L}_\Sigma$, the formula $\forall \vec{x}.\phi_\alpha(\vec{x})$, where $\vec{x} = x_1, x_2, \ldots, x_n$ is a list of variables, expresses $\alpha$ as a first-order formula. The translation rules for $\phi_\alpha(\vec{x})$ are recursively defined by set of rules $\beta_E(\vec{x}, x_c)$ for left side and $\gamma_E(\vec{x}, x_c)$ for right side expressions, shown at the bottom of Table B1.

By the definition of this translation, Lemma 1 can then be proved analogously to the case of the FO translation for $\mathcal{SROIQ}$-RL provided in (Bozzato *et al*. 2018, Appendix A.2). Intuitively, it is possible to show that, using the provided translation, every *DL-Lite*$_\mathcal{R}$ axiom can be expressed as a universal Horn sentence $\forall \vec{x}.\phi_\alpha(\vec{x})$, where $\vec{x} = x_1, \ldots, x_n$ is a list of free variables. Hence, $\phi_\alpha(\vec{x})$ can be written as $\phi_\alpha(\vec{x}) = \bigwedge_{i=1}^{\ell} \forall \vec{x}_i.\gamma_i(\vec{x}, \vec{x}_i)$, where each $\gamma_i$ is a Horn clause of the form

$$\gamma_i(\vec{x}, \vec{x}_i) = p_1(\vec{x}, \vec{x}_{i,1}) \wedge \cdots \wedge p_k(\vec{x}, \vec{x}_{i,k}) \rightarrow p_0(\vec{x}, \vec{x}_{i,0}), \tag{B1}$$

where (i) each $p_i$ is a concept name or a role name with possibly $p_0 = \bot$ (falsum); and (ii) each variable in $\vec{x}, \vec{x}_{i,j}$ occurs in the antecedent (safety), and $\vec{x}_i = \vec{x}_{i,0}, \ldots, \vec{x}_{i,k}$.

## Appendix C Proofs of Main Results

### C.1 *DL knowledge base with justifiable exceptions*

*Proposition 1*
Let $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ be a CAS model of DKB $\mathcal{K}$ and let $\mathcal{K}'$ result from $\mathcal{K}$ by pushing equality w.r.t. $\mathcal{I}$, that is, replace all $a, b \in N_\mathcal{K}$ s.t. $a^\mathcal{I} = b^\mathcal{I}$ by one representative. If $\mathcal{K}'$ is exception safe, then $\mathcal{I}_{CAS}$ can be justified only if every $\langle \alpha, \mathbf{e} \rangle \in \chi$ is over $N_\mathcal{K}$.

*Proof*
Suppose $\mathcal{I}_{CAS}$ is justified and some $\langle \alpha, \mathbf{e} \rangle \in \chi$ is not over $N_\mathcal{K}$, that is, for some $e$ in $\mathbf{e}$ we have $e \notin N_\mathcal{K}^\mathcal{I}$. Then, by definition of justification, some clashing set $S$ for $\langle \alpha, \mathbf{e} \rangle$

Table A.1. *Syntax and Semantics of DL-Lite$_{\mathcal{R}}$ operators, where $A$ is any atomic concept, $C$ and $D$ are any concepts, $P$ and $R$ are any atomic roles, $S$ and $Q$ are any (possibly complex) roles, $a$ and $b$ are any individual constants.*

| **Concept constructors** | Syntax | Semantics |
|---|---|---|
| Atomic concept | $A$ | $A^{\mathcal{I}}$ |
| Complement | $\neg C$ | $\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ |
| Existential restriction | $\exists R$ | $\{x \in \Delta^{\mathcal{I}} \mid \exists y. \langle x, y\rangle \in R^{\mathcal{I}}\}$ |
| **Role constructors** | Syntax | Semantics |
| Atomic role | $R$ | $R^{\mathcal{I}}$ |
| Role complement | $\neg S$ | $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \setminus S^{\mathcal{I}}$ |
| Inverse role | $R^-$ | $\{\langle y, x\rangle \mid \langle x, y\rangle \in R^{\mathcal{I}}\}$ |
| **Axioms** | Syntax | Semantics |
| Concept inclusion | $C \sqsubseteq D$ | $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ |
| Role inclusion | $S \sqsubseteq Q$ | $S^{\mathcal{I}} \subseteq Q^{\mathcal{I}}$ |
| Role disjointness | $\mathrm{Dis}(P, R)$ | $P^{\mathcal{I}} \cap R^{\mathcal{I}} = \emptyset$ |
| Role inverse | $\mathrm{Inv}(P, R)$ | $P^{\mathcal{I}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R^{\mathcal{I}}\}$ |
| Reflexivity assertion | $\mathrm{Ref}(R)$ | $\{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} \subseteq R^{\mathcal{I}}$ |
| Irreflexivity assertion | $\mathrm{Irr}(R)$ | $R^{\mathcal{I}} \cap \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} = \emptyset$ |
| Concept assertion | $C(a)$ | $a^{\mathcal{I}} \in C^{\mathcal{I}}$ |
| Role assertion | $S(a, b)$ | $\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in S^{\mathcal{I}}$ |

with $\mathbf{e}$ not over $N_{\mathcal{K}}$ is satisfied in all CAS models $\mathcal{I}'_{CAS}$ of $\mathcal{K}$ that are NI-congruent with $\mathcal{I}_{CAS}$. This means that $S$ can be derived with axiom unfolding restricted by the clashing assumptions in $\chi$. But then $S$ can also be derived without restrictions, and thus from the knowledge base $\mathcal{K}'_s$. However, this means that $\mathcal{K}'$ is not exception safe, which is a contradiction. □

*Proposition 2*
Deciding whether a given DKB $\mathcal{K}$ is exception safe is feasible in NLogSpace, and whether it is $n$-de safe in PTime, if $n$ is bounded by a polynomial in the size of $\mathcal{K}$.

*Proof*
Apparently, $\mathcal{K}$ is not exception safe, if some Skolem term $t_1$ resp. Skolem terms $t_1, t_2$ exists such that an atom $D(t_1)$ resp. $R(t_1, t_2)$ can be derived from the first-order rewriting $\phi_{\mathcal{K}_s}$ of $\mathcal{K}_s$ such that an assertion $D(e_1)$ resp. $R(e_1, e_2)$ occurs in a clashing set for some possible exception $\langle \alpha, \vec{e}\rangle$ to a defeasible axiom $\mathrm{D}(\alpha)$ in $\mathcal{K}$. Such $D(e_1)$ resp. $R(e_1, e_2)$ can be guessed and the derivation of $D(t_1)$ resp. $R(t_1, t_2)$ be non-deterministically checked by applying iteratively the axioms and deriving atoms $\alpha_0(\vec{t^0})$, $\alpha_1(\vec{t^1})$, $\dots$, $\alpha_i(\vec{t^i}) = D(t_1)$ resp. $\alpha_i(\vec{t^i}) = R(t_1, t_2)$; at each step, it is sufficient to store merely the type of each argument (Skolem term, $N_K$ individual) of $\alpha_j$ and to require that $t_1$ resp. some of $t_1, t_2$ is a Skolem term. This is feasible in non-deterministic logspace.

Table B.1. *Translation $\phi_\alpha(\vec{x})$, $\vec{x} = x_1, \dots, x_n$, of DL-Lite$_\mathcal{R}$ axioms $\alpha$ in $\mathcal{L}_\Sigma$ to first-order logic.*

**Translation $\phi_\alpha(\vec{x})$ for axioms $\alpha$**

| | |
|---|---|
| $D(a) \mapsto \gamma_D(a)$ | $\mathrm{Dis}(R,S) \mapsto (R(x_1,x_2) \to \neg S(x_1,x_2)) \wedge (S(x_1,x_2) \to \neg R(x_1,x_2))$ |
| $R(a,b) \mapsto R(a,b)$ | |
| $\neg R(a,b) \mapsto \neg R(a,b)$ | $\mathrm{Inv}(R,S) \mapsto (R(x_1,x_2) \to S(x_2,x_1)) \wedge (S(x_1,x_2) \to R(x_2,x_1))$ |
| $C \sqsubseteq D \mapsto \beta_C(x_1) \to \gamma_D(x_1)$ | $\mathrm{Ref}(R) \mapsto R(x_1,x_1)$ |
| $R \sqsubseteq T \mapsto R(x_1,x_2) \to T(x_1,x_2)$ | $\mathrm{Irr}(R) \mapsto \neg R(x_1,x_1)$ |

**Translation $\beta_E(\vec{x})$ for (left side) expressions $E$**

$A \mapsto A(x_1)$
$\exists R \mapsto R(x_1,x_2)$

**Translation $\gamma_E(\vec{x})$ for (right side) expressions $E$**

$A \mapsto A(x_1)$
$\neg C_1 \mapsto \neg\beta_{C_1}(x_1)$
$\exists R \mapsto R(x_1, f_R(x_1))$

As for $n$-de safety where $n > 0$, one can first similarly check whether a derivation with a cycle is possible, such that atoms $D(t_1)$ and $D(t_1')$ resp. $R(t_1,t_2)$ and $R(t_1',t_2')$ can be derived where $t_1$ is a subterm of $t_1'$ resp. $t_1$ is a subterm of $t_1'$ or $t_2$ is a subterm of $t_2'$. This can be done with additional book-keeping in nondeterministic logspace.

If such a cycle exists, then $\mathcal{K}$ is not $n$-de safe for any $n \geq 0$. Otherwise, we can systematically enumerate the terms $t_1$ resp. $t_1, t_2$ in a lexicographic fashion. To this end, we determine for an individual $a \in N_\mathcal{K}$ all assertions $R_i(a, f_{R_i}(a))$ that hold for it; each gives rise to a child $f_{R_i}(a)$, and by repeated application (where again all assertions $R_i(f_{R_i}(a), f_{R_j}(a))$ are determined) we obtain a tree whose depth is linearly bounded. We can traverse this tree in a depth first manner where, before expanding, we ask at the current node whether some atom $D(t_1)$ resp. $R(t_1,t_2)$ as above is reachable (which then contains some new Skolem term not seen so far). This test is, like computing all assertions $R(t, f_R(t))$ for $t$ feasible in nondeterministic logspace. In this way, the number of nodes explored until $n+1$ different terms are found is polynomial in $n$, and the effort for each node is polynomial; as there are linearly many starting nodes, the overall effort is polynomial in $n$. Thus if $n$ is polynomially bounded in the size of $\mathcal{K}$, the overall effort is polynomial in the size of $\mathcal{K}$ as well. □

*Proposition 3*
Deciding whether a given DKB $\mathcal{K}$ is $n$-chain safe, where $n \geq 0$, is feasible in NLogSpace.

*Proof*
This test can be made by an algorithm similar to the one checking exception safety. It nondeterministically builds a chain $R_1(a,t_1), \dots, R_m(t_{m-1},t_m)$ starting from the assertions in $\mathcal{K}_s$, where it records at each point just the predicate name $R_i$ and the type of the arguments. It increases a counter whenever by applying some axiom a new Skolem term $f_{R_i}(t)$ is introduced; if the counter exceeds $n$, then $\mathcal{K}$ is not $n$-chain safe. Since logarithmic workspace is sufficient for the book-keeping, the result follows. □

### *C.2 Semantic properties*

*Proposition 6*

Let $\mathcal{I}^i_{CAS} = \langle \mathcal{I}_i, \chi \rangle$, $i \in \{1, 2\}$, be NI-congruent CAS models of a DKB $\mathcal{K}$ fulfilling ($*$). Then, $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$, where $\mathcal{I} = \mathcal{I}_1 \widetilde{\cap}_{\mathcal{N}} \mathcal{I}_2$ and $\mathcal{N}$ includes all individual names occurring in $\mathcal{K}$ and for each element $e$ occurring in $\chi$ some $t \in \mathcal{N}$ such that $t^{\mathcal{I}_1} = e (= t^{\mathcal{I}_2})$, is also a CAS model of $\mathcal{K}$. Furthermore, if some $\mathcal{I}^i_{CAS}$, $i \in \{1, 2\}$, is justified and $\mathcal{K}$ is exception safe, then $\mathcal{I}_{CAS}$ is justified.

*Proof*

By Lemma 1, we have that the FO translation $\phi_{\mathcal{K}}$ of a DKB $\mathcal{K}$ is equivalent to a conjunction of Horn clauses. Since $\mathcal{I}^i_{CAS} = \langle \mathcal{I}_i, \chi \rangle$, $i \in \{1, 2\}$ are CAS models of $\mathcal{K}$, we can consider their "intersection" model $\mathcal{I} = \mathcal{I}_1 \widetilde{\cap}_{\mathcal{N}} \mathcal{I}_2$ over the set $\mathcal{N}$ corresponding to $N_{\mathcal{K}}$ extended with its grounding on Skolem functions. We can prove that $\mathcal{I}_{CAS}$ is indeed a model for $\mathcal{K}$: let us suppose that in $\mathcal{I}$ we have that some Horn clause $\gamma(\vec{x}) = a_1(\vec{x}), \ldots, a_n(\vec{x}) \rightarrow b(\vec{x})$ from $\phi_{\mathcal{K}}$ is violated for some variable assignment $\sigma$ for $\vec{x} = x_1, \ldots, x_k$. Since all elements of $\mathcal{I}_1 \widetilde{\cap}_{\mathcal{N}} \mathcal{I}_2$ are named by some term in $\mathcal{N}$, the assignment is of the form $\sigma(x_i) = t_i$ with $t_i \in \mathcal{N}$ for $i \in \{1, \ldots, k\}$. Since each $t_i$ is interpreted in $\mathcal{I}$ by $[t^{\mathcal{I}_1}, t^{\mathcal{I}_2}]$, by the interpretation of concepts and roles in $\mathcal{I}$ it follows that each $a_i\sigma$ is true in both $\mathcal{I}_1$ and $\mathcal{I}_2$. Furthermore, as $\gamma(\vec{x})$ is violated for $\sigma$, the axiom $\alpha$ in $\mathcal{K}$ that led to $\gamma(\vec{x})$ does not have an exception for $\sigma$ in $\mathcal{I}_{CAS}$; hence, from the property ($*$) it follows that $\gamma(\vec{x})$ has neither for the assignment $\sigma_1(x_i) = t_i^{\mathcal{I}_1}$, $i \in \{1, \ldots, k\}$, an exception in $\mathcal{I}^1_{CAS}$ nor for the assignment $\sigma_2(x_i) = t_i^{\mathcal{I}_2}$, $i \in \{1, \ldots, k\}$, in $\mathcal{I}^2_{CAS}$; thus $b(\vec{x})$ is true for $\sigma_i$ in $\mathcal{I}_i$, $i = 1, 2$. By construction, this means $b(\vec{x})$ is true for $\sigma$ in $\mathcal{I}$, and thus $\gamma(\vec{x})$ is satisfied for $\sigma$ in $\mathcal{I}$, which is a contradiction. Hence $\mathcal{I}_{CAS} = \langle \mathcal{I}_1 \widetilde{\cap}_{\mathcal{N}} \mathcal{I}_2, \chi \rangle$ is also a CAS model for $\mathcal{K}$.

With respect to justification, let us assume without loss of generality that $\mathcal{I}^1_{CAS}$ is justified. Hence, if $\langle \alpha, \mathbf{e} \rangle \in \chi$, then we have that there exists a clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ for this clashing assumption such that for every NI-congruent $\mathcal{I}'_{CAS}$ it holds that $\mathcal{I}'_{CAS} \models S_{\langle \alpha, \mathbf{e} \rangle}$. Moreover, considering $\mathcal{K}$ to be exception safe, all exceptions in $\chi$ are named by individuals. Then, for the justified model $\mathcal{I}^1_{CAS}$, for every Skolem term $t \in \mathcal{N}$, we must have $t^{\mathcal{I}_1} \neq c^{\mathcal{I}_1}$ if $c \in$ NI appears in an exception. Thus, since $\mathcal{I}_{CAS}$ is NI-congruent with $\mathcal{I}^1_{CAS}$ and property ($*$) holds, it follows that also the intersection model $\mathcal{I}_{CAS}$ is justified. □

*Corollary 1 (least model property)*

If a clashing assumption $\chi$ for an exception safe DKB $\mathcal{K}$ is satisfiable for name assignment $\nu$, then $\mathcal{K}$ has an $\subseteq_{\mathcal{N}}$-least (unique minimal) CAS model $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu) = \langle \hat{\mathcal{I}}, \chi \rangle$ on $\mathcal{N}$ that contains all Skolem terms of individual constants, that is, for every CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ relative to $\nu$, it holds that $\mathcal{I}_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}'_{CAS}$. Furthermore, $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ is justified if $\chi$ is justified.

*Proof*

Given that $\chi$ is satisfiable, consider any CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ for $\mathcal{K}$ with name assignment $\nu$. Then, some model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ such that $\mathcal{I}'_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}_{CAS}$ exists that is founded, that is, has the following property:

If $R(e_1, e_2)$ is true in $\mathcal{I}'_{CAS}$, then a sequence $\alpha_1(\mathbf{e}_1) = R(e_1, e_2), \alpha_2(\mathbf{e}_2), \ldots, \alpha_k(\mathbf{e}_k)$ of atoms with domain elements such that:

(1) $\alpha_i(\mathbf{e}_i)$ is obtained from $\alpha_{i+1}(\mathbf{e}_{i+1})$ by applying an axiom (resp. the rule for it) where $\alpha_i(\mathbf{e}_i)$ is the head and $\alpha_{i+1}(\mathbf{e}_{i+1})$ is the body, and both are satisfied;
(2) some $A(t_1)$ or $S(t_1, t_2)$ in $\mathcal{K}$ exists such that $\alpha_k(\mathbf{e}_k) = A(t_1^{\mathcal{I}'})$ resp. $\alpha_k(\mathbf{e}_k) = S(t_1^{\mathcal{I}'}, t_2^{\mathcal{I}'})$.

If this were not the case for some $R(e_1, e_2)$ in $\mathcal{I}_{CAS}$, that is, its truth is not "founded" in the facts in $\mathcal{K}$, we could set $R(e_1, e_2)$ and, recursively along all possible such chains, all $\alpha_i(\mathbf{e}_i)$ to false and obtain a smaller model; eliminating all such $R(e_1, e_2)$ (at once or alternatively in an iterative process) yields a founded model $\mathcal{I}'_{CAS}$.

Now if $R(e_1, e_2)$ is true in $\mathcal{I}'_{CAS}$, starting from $A(t_1)$ resp. $S(t_1, t_2)$, we can apply the same axioms (rules) as in (1) w.r.t. $\mathcal{K}_s$, using Skolemization and obtain $R(t, t')$. Since $\mathcal{K}$ is exception safe, if the first (resp. second) argument of $R$ can appear in a clashing assumption, $t$ (resp. $t'$) must not be a Skolem term, but a constant. This is analogous for $A(e_1)$ where $A$ can occur in a clashing assumption.

Let now $\mathcal{I}^1_{CAS} = \langle \mathcal{I}_1, \chi \rangle$ and $\mathcal{I}^2_{CAS} = \langle \mathcal{I}_2, \chi \rangle$ be two founded models of $\mathcal{K}$ relative to $\nu$. Considering condition $(*)$, if $t^{\mathcal{I}_1} = e$ holds where $e$ occurs in a clashing assumption, then $t = c$ must hold for some constant $c$ (in fact, for some $c \in \mathcal{K}$). As $\mathcal{I}^2_{CAS}$ interprets constants in the same way, we thus have $t^{\mathcal{I}_1} = t^{\mathcal{I}_2}$ so the intersection property holds. Thus by Proposition 6, $\mathcal{I}_{CAS} = \langle \mathcal{I}_1 \widetilde{\cap}_{\mathcal{N}} \mathcal{I}_2, \chi \rangle$ is also a model of $K$. We claim that $\mathcal{I}^1_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}_{CAS}$ holds, that is, for each atom $\alpha \in At_{\mathcal{N}}(\mathcal{I}^1_{CAS})$, $\mathcal{I}^1 \models \alpha$ implies $\mathcal{I} \models \alpha$. Towards a contradiction, suppose $\mathcal{I}^1 \models \alpha$ but $\mathcal{I}' \not\models \alpha$, and consider $\alpha = R(t, t')$. As $\mathcal{I}^1_{CAS}$ is founded, $R(t, t')$ must result, modulo $\nu$, from a derivation chain of an atom $R(e_1, e_2)$ showing foundedness. The bottom of this chain, $\alpha_k(\vec{t}) = A(t_1)$ resp. $\alpha_k = S(t_1, t_2)$ is a fact in $\mathcal{K}$, and thus $\mathcal{I}' \models \alpha_k(\vec{t})$. By an inductive argument, we obtain that the axiom that has been applied to derive $\alpha_{i-1}(\mathbf{e}_{i-1})$ from $\alpha_i(\mathbf{e}_i)$ in the chain from $\alpha_k(\mathbf{e}_k)$ to $\alpha_1 = R(e_1, e_2)$ in $\mathcal{I}^1$ can be applied in $\mathcal{I}'$ to derive the same $\alpha_{i-1}(\vec{t}_{i-1})$ from $\alpha_i(\vec{t}_i)$ as in $\mathcal{I}'$, as the axiom is also applicable to $\alpha_i(\vec{t}_i)$ in $\mathcal{I}^2$. Thus, $\mathcal{I}' \models \alpha_1$, that is, $\mathcal{I}' \models R(t, t')$, which is a contradiction. The case of $\alpha = A(t_1)$ is analogous. Thus, $\mathcal{I}^1_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}_{CAS}$ holds, which means by transitivity that $\mathcal{I}^1_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}^2_{CAS}$. As $\mathcal{I}^2_{CAS}$ can be an arbitrary founded CAS model, it follows that $\mathcal{I}^1_{CAS}$ is a $\subseteq_{\mathcal{N}}$-least founded CAS model; and as for every CAS model $\mathcal{I}_{CAS}$ some founded CAS model $\mathcal{I}^2_{CAS}$ exists such that $\mathcal{I}^2_{CAS} \subseteq_{\mathcal{N}} \mathcal{I}_{CAS}$, $\mathcal{I}^1_{CAS}$ is a $\subseteq_{\mathcal{N}}$-least CAS model $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ of $\mathcal{K}$, as as claimed. By symmetry, also has $\mathcal{I}^2_{CAS}$ this property. Justification of $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ is immediate from Proposition 6. □

*Lemma 3*

Suppose $\mathcal{I}$ is a model of a *DL-Lite*$_{\mathcal{R}}$ knowledge base $\mathcal{K}$ and $N \subseteq \mathrm{NI} \setminus \mathrm{NI}_S$ includes all individuals occurring in $\mathcal{K}$. Then, the $N$-restriction $\mathcal{I}^N$ is named w.r.t. $sk(N)$ and a model of $\mathcal{K}$.

*Proof*

By Lemma 1, we have that $\phi_{\mathcal{K}}$ represents the contents of $\mathcal{K}$ as a first-order formula. As discussed in Appendix B, this formula can be written as a conjunction $\phi_\alpha(\vec{x}) = \bigwedge_{i=1}^{\ell} \forall \vec{x}_i . \gamma_i(\vec{x}, \vec{x}_i)$, where each $\gamma_i(\vec{x}, \vec{x}_i)$ is a Horn clause. By construction, $\mathcal{I}^N$ is named relative to $sk(N)$. Moreover, in any assignment $\theta : \vec{x}_i \mapsto \mathrm{NI}_s$, $\gamma_i\theta$ evaluates to false in $\mathcal{I}^N$

whenever there is some $\theta(x) \notin sk(N)$ with $x \in \vec{x}_i$. It follows that, if $\gamma_i$ is verified under $\mathcal{I}$ and $\theta$, then it is also verified by $\mathcal{I}^N$ and $\theta$. This implies that if $\mathcal{I} \models \phi_\mathcal{K}$ then $\mathcal{I}^N \models \phi_\mathcal{K}$, and thus $\mathcal{I}^N$ is a named model for $\mathcal{K}$. □

*Theorem 2 (justified CAS characterization)*
Let $\chi$ be a satisfiable clashing assumptions set for an exception safe DKB $\mathcal{K}$ and name assignment $\nu$. Then, $\chi$ is justified iff $\langle \alpha, \mathbf{e} \rangle \in \chi$ implies some clashing set $S = S_{\langle \alpha, \mathbf{e} \rangle}$ exists such that

(i) $\hat{\mathcal{I}} \models \beta$, for each positive $\beta \in S$, where $\hat{\mathcal{I}}_\mathcal{K}(\chi, \nu) = (\hat{\mathcal{I}}, \chi)$, and
(ii) no CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ with name assignment $\nu$ exists s.t. $\mathcal{I} \models \beta$ for some $\neg\beta \in S$.

*Proof*
Since $\chi$ is satisfiable, then for Corollary 1 there exists a least CAS model $\hat{\mathcal{I}}_\mathcal{K}(\chi, \nu) = (\hat{\mathcal{I}}, \chi)$.

Then, we can show that the justification is characterized by the conditions on the validity of clashing sets. Let us suppose that $\chi$ is justified. Then, $\hat{\mathcal{I}}_\mathcal{K}(\chi, \nu)$ is justified: this implies that, by definition, for every $\langle \alpha, \mathbf{e} \rangle \in \chi$ there exists a clashing set $S = S_{\langle \alpha, \mathbf{e} \rangle}$ such that, for every CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ that is NI-congruent with $\hat{\mathcal{I}}_\mathcal{K}(\chi, \nu)$, we have that $\mathcal{I}' \models S$. This directly verifies items (i) and (ii).

On the other hand, let us suppose that for every $\langle \alpha, \mathbf{e} \rangle \in \chi$ there exists some clashing set $S = S_{\langle \alpha, \mathbf{e} \rangle}$ such that items (i) and (ii) hold. Considering a CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi \rangle$ that is NI-congruent with $\hat{\mathcal{I}}_\mathcal{K}(\chi, \nu)$, it holds that from (i) $\mathcal{I}' \models \beta$ for every positive $\beta \in S$. Moreover, condition (ii) implies that $\mathcal{I}' \models \neg\beta$ for every negative assertion $\neg\beta \in S$. Thus, $\langle \alpha, \mathbf{e} \rangle \in \chi$ is justified in $\chi$: hence, $\hat{\mathcal{I}}_\mathcal{K}(\chi, \nu)$ is justified. □

## C.3 Datalog translation for DL-Lite$_\mathcal{R}$ DKB

### C.3.1 Normal form

*Lemma 4*
Every DKB $\mathcal{K}$ can be transformed in linear time into an equivalent DKB $\mathcal{K}'$ which has modulo auxiliary symbols the same DKB models, and such that $n$-de safety and $n$-chain safety are preserved.

*Proof*
Using the new concept names $A_{\exists R}$, all strict inclusion axioms $C \sqsubseteq D$ can be easily expressed in the given form. Negative concept assertions $\neg C(a)$ can be expressed by $A_a(a)$ and $A_a \sqsubseteq \neg C$, where $A_a$ is a fresh concept name. Similarly, negative role assertions $\neg R(a, b)$ can be expressed by $R'(a, b)$ and $\mathrm{Dis}(R, R')$ where $R'$ is a fresh role name.

As regards defeasible axioms, we can express $\mathrm{D}(\neg C(a))$ by $A_a(a)$ and $\mathrm{D}(A_a \sqsubseteq \neg C)$, where $A_a$ is as above, and furthermore $\mathrm{D}(A \sqsubseteq \neg B)$ by $\mathrm{D}(A \sqsubseteq A')$ and $B \sqsubseteq \neg A'$, where $A'$ is a fresh concept name. Finally, we can express $\mathrm{D}(\neg R(a, b))$ by $R'(a, b)$, $\mathrm{D}(R' \sqsubseteq S)$, and $\mathrm{Dis}(R, S)$, where $R'$ and $S$ are fresh role names. It can be seen that for each justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of the original DKB, some justified CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi' \rangle$ for the DKB obtained by a rewriting step exists, where $\chi'$ is obtained by modifying exceptions in $\chi$ according to the rewriting in the obvious way. In the same way, one obtains from each justified CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi' \rangle$ of the rewritten DKB a justified CAS model of the original DKB. The rewriting does not use existential axioms, and thus $n$-de safety and $n$-chain safety are preserved. □

With respect to the equivalence result defined in Lemma 4, one interesting aspect would be *strong equivalence* Lifschitz *et al.* (2001) between the original DKB $\mathcal{K}$ and the transformed DKB $\mathcal{K}'$. In the non-monotonic setting, two theories $\Gamma$ and $\Gamma'$ are strongly equivalent if, for any other theory $H$, $\Gamma \cup H$ and $\Gamma' \cup H$ have the same models. For our purposes, we would need to carefully define and analyze this notion, taking auxiliary symbols into account, cf. Woltran (2008). The very limited form of changes by the normal form transformation and the syntax of $DL\text{-}Lite_{\mathcal{R}}$ suggest that a relativized equivalence may prevail, possible under adaptations.

### *C.3.2 Translation correctness*

Let $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ be a justified named CAS model. We define the set of overriding assumptions $OVR(\mathcal{I}_{CAS}) = \{\, \mathtt{ovr}(p(\mathbf{e})) \mid \langle \alpha, \mathbf{e} \rangle \in \chi,\ I_{dlr}(\alpha) = p \,\}$. Given a CAS interpretation $\mathcal{I}_{CAS}$, we can define a corresponding interpretation $S = I(\mathcal{I}_{CAS})$ for $PK(\mathcal{K})$: the construction is similar to the one in Bozzato *et al.* (2018), by extending it to negative literals and providing an interpretation for existential individuals:

(1). $l \in S$, if $l \in PK(\mathcal{K})$;
(2). $\mathtt{instd}(a, A) \in S$, if $\mathcal{I} \models A(a)$ and $\neg\mathtt{instd}(a, A) \in S$, if $\mathcal{I} \models \neg A(a)$;
(3). $\mathtt{tripled}(a, R, b) \in S$, if $\mathcal{I} \models R(a, b)$ and $\neg\mathtt{tripled}(a, R, b) \in S$, if $\mathcal{I} \models \neg R(a, b)$;
(4). $\mathtt{tripled}(a, R, aux^{\alpha}) \in S$, if $\mathcal{I} \models \exists R(a)$ for $\alpha = A \sqsubseteq \exists R$;
(5). $\mathtt{all\_nrel}(a, R) \in S$ if $\mathcal{I} \models \neg\exists R(a)$;
(6). $\mathtt{ovr}(p(\mathbf{e})) \in S$, if $\mathtt{ovr}(p(\mathbf{e})) \in OVR(\mathcal{I}_{CAS})$;

*Proposition 7*
Let $\mathcal{K}$ be an exception safe DKB in $DL\text{-}Lite_{\mathcal{R}}$ normal form. Then:

(i). for every (named) justified clashing assumption $\chi$, the interpretation $S = I(\hat{\mathcal{I}}(\chi))$ is an answer set of $PK(\mathcal{K})$;
(ii). every answer set $S$ of $PK(\mathcal{K})$ is of the form $S = I(\hat{\mathcal{I}}(\chi))$ where $\chi$ is a (named) justified clashing assumption for $\mathcal{K}$.

*Proof*
We consider $S = I(\hat{\mathcal{I}}(\chi))$ built as above and reason over the reduct $G_S(PK(\mathcal{K}))$ of $PK(\mathcal{K})$ with respect to $S$. By definition, the reduct $G_S(PK(\mathcal{K}))$ is the set of rules resulting from the ground instances of rules of $PK(\mathcal{K})$ after the removal of (i) every rule $r$ such that $S \models l$ for some NAF literal $\mathtt{not}\, l \in Body(r)$; and (ii) the NAF part (i.e. `ovr` literals) from the bodies of the remaining rules. Basically, $G_S(PK(\mathcal{K}))$ contains all ground rules from $PK(\mathcal{K})$ that are not falsified by some NAF literal in $S$: in particular, this excludes application rules for the axiom instances that are recognized as overridden.

Item (i) can be proved by showing that given a justified $\chi$, $S$ is an answer set for $G_S(PK(\mathcal{K}))$ (and thus $PK(\mathcal{K})$). The proof follows the same reasoning of the one in

(Bozzato *et al.* 2018, Lemma 6), where the fact that $I(\hat{\mathcal{I}}(\chi))$ satisfies rules of the form (pdlr-supex) in $PK(\mathcal{K})$ is verified by the condition (4) on existential formulas in the construction of the model above.

We first show that $S \models G_S(PK(\mathcal{K}))$: for every rule instance $r \in G_S(PK(\mathcal{K}))$ we show that $S \models r$ holds by examining the possible rule forms that occur in the reduction. In the following we show some of the most relevant cases, while the other cases can be proven with similar reasoning. Assuming that $S \models Body(r)$ for a rule instance $r$ in $grnd(PK(\mathcal{K}))$, we show that $Head(r) \in S$.

- **(pdlr-instd):** then $\texttt{insta}(a, A) \in S$ and, by definition of the translation, $A(a) \in \mathcal{K}$. This implies that $\hat{\mathcal{I}} \models A(a)$ and thus $\texttt{instd}(a, A)$ is added to $S$.
- **(pdlr-subc):** then $\{\texttt{subClass}(A, B), \texttt{instd}(a, A)\} \subseteq S$. By definition of the translation we have $A \sqsubseteq B \in \mathcal{K}$. Then, for the construction of $S = I(\hat{\mathcal{I}}(\chi))$, $\hat{\mathcal{I}} \models A(a)$. This implies that $\hat{\mathcal{I}} \models B(a)$ and $\texttt{instd}(a, B)$ is added to $S$.
- **(pdlr-supex):** then $\{\texttt{supEx}(A, R, aux^\alpha), \texttt{instd}(a, A)\} \subseteq S$ (with $aux^\alpha$ a new constant relative to the considered existential axiom $\alpha$). By definition of the input translation, we have that $A \sqsubseteq \exists R \in \mathcal{K}$. Moreover, by the construction of $S$, we have $\hat{\mathcal{I}} \models A(a)$. This implies that $\hat{\mathcal{I}} \models \exists R(a)$: thus, by the conditions defining the construction of $S$, $\texttt{tripled}(a, R, aux^\alpha)$ is added to $S$.
- **(pdlr-nsupex):** then $\{\texttt{supEx}(A, R, aux^\alpha), \texttt{const}(a), \texttt{all\_nrel}(a, R)\} \subseteq S$. By definition of the translation, we have that $A \sqsubseteq \exists R \in \mathcal{K}$ and $a$ is a constant (i.e. either an individual name appearing in $\mathcal{K}$ or any other auxiliary existential constant $aux^\beta$). Moreover, by the construction of $S$, since $\texttt{all\_nrel}(a, R) \in S$ it holds that $\hat{\mathcal{I}} \models \neg\exists R(a)$. This implies that $\hat{\mathcal{I}} \models \neg A(a)$ and then $\neg\texttt{instd}(a, A)$ is added to $S$.
- **(ovr-subc):** then $\{\texttt{def\_subclass}(A, B), \texttt{instd}(a, A), \neg\texttt{instd}(a, B)\} \subseteq S$. By definition of the translation, we have that $\mathrm{D}(A \sqsubseteq B) \in \mathcal{K}$ and, by the construction of $S$, $\hat{\mathcal{I}} \models A(a)$ and $\hat{\mathcal{I}} \models \neg B(a)$. Thus, $\hat{\mathcal{I}}$ satisfies the clashing set $\{A(a), \neg B(a)\}$ for the clashing assumption $\langle A \sqsubseteq B, a\rangle$. This implies that $\langle A \sqsubseteq B, a\rangle \in \chi$ and by construction $\texttt{ovr}(\texttt{subClass}, a, A, B)$ is added to $S$.
- **(app-subc):** then $\{\texttt{def\_subclass}(A, B), \texttt{instd}(a, A)\} \subseteq S$. As $r \in G_S(PK(\mathcal{K}))$, we have that $\texttt{ovr}(\texttt{subClass}, a, A, B) \notin OVR(\hat{\mathcal{I}}(\chi))$ and hence $\langle A \sqsubseteq B, a\rangle \notin \chi$. By definition, $A \sqsubseteq B \in \mathcal{K}$ and, by the construction of $S$, $\hat{\mathcal{I}} \models A(a)$. Thus, for the definition of $CAS$-model and the semantics, $\texttt{instd}(a, B)$ is added to $S$.

To show that $S$ is indeed an answer set for $G_S(PK(\mathcal{K}))$, we have to prove its minimality with respect to the rules in the reduction. We can show that no model $S' \subseteq S$ of $G_S(PK(\mathcal{K}))$ such that $S' \neq S$ can exist: as $\hat{\mathcal{I}}(\chi)$ is the least model of $\mathcal{K}$ w.r.t. $\chi$, $S'$ can not be a proper subset of $S$ on any of the facts from the input translations, nor on the derivable instance level facts ($\texttt{instd}$, $\texttt{tripled}$ or their negation). Thus, $S'$ needs to contain all atoms on $\texttt{ovr}$ from $S$, as for every corresponding clashing assumption $\langle\alpha, \mathbf{e}\rangle \in \chi$ the body of some overriding rule in $PK(\mathcal{K})$ that encodes a clashing set for $\langle\alpha, \mathbf{e}\rangle$ will be satisfied. Thus, $S' = S$ must hold.

For item (ii), we can show that from any answer set $S$ we can build a justified model $\mathcal{I}_S$ for $\mathcal{K}$ such that $S = I(\hat{\mathcal{I}}(\chi))$ holds. The model can be defined similarly to the original proof in Bozzato *et al.* (2018), but we need to consider auxiliary individuals in the domain of $\mathcal{I}_S$. Considering an answer set $S$ for $PK(\mathcal{K})$, we can build a model $\mathfrak{I}_S = \langle \mathcal{I}_S, \chi_S \rangle$ as follows:

- $\Delta^{\mathcal{I}_S} = \{c \mid c \in \mathrm{NI}_\Sigma\} \cup \{aux^\alpha \mid \alpha = A \sqsubseteq \exists R \in \mathcal{K}\}$;
- $a^{\mathcal{I}_S} = a$, for every $a \in \mathrm{NI}_\Sigma$ and auxiliary constant of the kind $aux^\alpha$;
- $A^{\mathcal{I}_S} = \{d \in \Delta^{\mathcal{I}_S} \mid S \models \mathtt{instd}(d, A)\}$, for every $A \in \mathrm{NC}$;
- $R^{\mathcal{I}_S} = \{(d, d') \in \Delta^{\mathcal{I}_S} \times \Delta^{\mathcal{I}_S} \mid S \models \mathtt{tripled}(d, R, d')\}$ for $R \in \mathrm{NR}$;

Finally, $\chi_S = \{\langle \alpha, \mathbf{e} \rangle \mid I_{dlr}(\alpha) = p, \mathtt{ovr}(p(\mathbf{e})) \in S\}$. We need to show that $\mathfrak{I}_S$ is a least justified CAS model for $\mathcal{K}$, that is:

(a) for every $\alpha \in \mathcal{L}_\Sigma$ in $\mathcal{K}$, $\mathfrak{I}_S \models \alpha$;
(b) for every $\mathrm{D}(\alpha) \in \mathcal{K}$ (where $\alpha \in \mathcal{L}_\Sigma$), with $|\vec{x}|$-tuple $\vec{d}$ of elements in $\mathrm{NI}_\Sigma$ such that $\vec{d} \notin \{\mathbf{e} \mid \langle \alpha, \mathbf{e} \rangle \in \chi\}$, we have $\mathfrak{I}_S \models \phi_\alpha(\vec{d})$.

The claim can then be proven by considering the effect of deduction rules for existential axioms in $G_S(PK(\mathcal{K}))$: auxiliary individuals provide the domain elements in $\mathcal{I}_S$ needed to verify this kind of axioms.

In particular, condition (a) can be shown by cases considering the form of all of the (strict) axioms $\beta \in \mathcal{L}_\Sigma$ that can occur in $\mathcal{K}$. We show in the following some of the cases (the others are similar):

- Let $\beta = A(a) \in \mathcal{K}$, then, by rule (pdlr-instd), $S \models \mathtt{instd}(a, A)$. This directly implies that $a^{\mathcal{I}_S} \in A^{\mathcal{I}_S}$.
- Let $\beta = A \sqsubseteq B \in \mathcal{K}$, then $S \models \mathtt{subClass}(A, B)$. If $d \in A^{\mathcal{I}_S}$, then by definition $S \models \mathtt{instd}(d, A)$: by rule (pdlr-subc) we obtain that $S \models \mathtt{instd}(d, B)$ and thus $d \in B^{\mathcal{I}_S}$. On the other hand, let us assume that $e \in \neg B^{\mathcal{I}_S}$ with $S \models \neg\mathtt{instd}(e, B)$: then, by the negative rule (pdlr-nsubc) we have that $S \models \neg\mathtt{instd}(e, A)$. This implies that $e \in \neg A^{\mathcal{I}_S}$, since otherwise we would have $\mathtt{instd}(e, A) \in S$ and $S$ would be inconsistent. Note that if we assume $e \in \neg B^{\mathcal{I}_S}$ but $S \not\models \neg\mathtt{instd}(e, B)$, we also obtain that $e \in \neg A^{\mathcal{I}_S}$, otherwise by (pdlr-instd) and the definition of $\mathcal{I}_S$ we would derive that $e \in \neg B^{\mathcal{I}_S}$.
- Let $\beta = A \sqsubseteq \exists R \in \mathcal{K}$, then $S \models \mathtt{supEx}(A, R, aux^\beta)$. If $d \in A^{\mathcal{I}_S}$, then by definition $S \models \mathtt{instd}(d, A)$. By rule (pdlr-supex) we obtain that $S \models \mathtt{tripled}(d, R, aux^\beta)$. By the definition of $\mathcal{I}_S$, this means that there exists an $e \in \Delta^{\mathcal{I}_S}$ such that $(d, e) \in R^{\mathcal{I}_S}$: this implies that $\mathcal{I}_S \models \exists R(d)$. On the other hand, if we consider an $e \in \Delta^{\mathcal{I}_S}$ such that $e \in \neg\exists R^{\mathcal{I}_S}$ and $S \models \neg\mathtt{tripled}(e, R, c)$ for each $\mathtt{const}(c)$ in $S$ (i.e. for each $c \in \Delta^{\mathcal{I}_S}$). Then, by the definition of $\mathcal{I}_S$ and the rules (pdlr-allnrel1) – (pdlr-allnrel3) it holds that $S \models \mathtt{all\_nrel}(e, R)$. By the negative rule (pdlr-nsupex), we have that $S \models \neg\mathtt{instd}(e, A)$. As above, we have that $e \in \neg A^{\mathcal{I}_S}$, as otherwise we would have $\mathtt{instd}(e, A) \in S$ and $S$ would be inconsistent.

For condition (b), let us assume that $\mathrm{D}(\beta) \in \mathcal{K}$ with $\beta \in \mathcal{L}_\Sigma$. Let $\beta = A \sqsubseteq B$. Then, by definition of the translation, we have that $S \models \mathtt{def\_subclass}(A, B)$. Let us suppose that $b^{\mathcal{I}_S} \in A^{\mathcal{I}_S}$: then $S \models \mathtt{instd}(b, A)$. Supposing that $\langle A \sqsubseteq B, b \rangle \notin \chi_S$, then by definition $\mathtt{ovr}(\mathtt{subClass}, b, A, B) \notin OVR(\hat{\mathfrak{I}}(\chi))$. By the definition of the reduction, the corresponding instantiation of rule (app-subc) has not been removed from $G_S(PK(\mathcal{K}))$. This implies that $S \models \mathtt{instd}(b, B)$ and thus $b^{\mathcal{I}_S} \in B^{\mathcal{I}_S}$. The contrapositive case for $b^{\mathcal{I}_S} \in \neg B^{\mathcal{I}_S}$ and the cases for the other forms of defeasible axioms can be proved similarly.

Thus, $\mathcal{I}_S$ is a CAS model of $\mathcal{K}$: moreover, we can show that $\mathcal{I}_S = \hat{\mathcal{I}}(\chi_S)$ holds. Assuming that $\mathcal{I} \subset \mathcal{I}_S$ is a CAS model of $\mathcal{K}$ with clashing assumption $\chi_S$, we can construct an interpretation $S' \subset S$ such that $S' \models G_S(PK(\mathcal{K}))$, by removing (at least) one instance-level fact $\mathtt{instd}(d, A)$ or $\mathtt{tripled}(d, R, d')$ from $S$. However, this would contradict that $S$ is an answer set of $PK(\mathcal{K})$: thus, it holds that $\mathcal{I}_S = \hat{\mathcal{I}}(\chi_S)$.

The justification of $\chi_S$ follows by verifying that the new formulation of overriding rules correctly encodes the possible clashing sets for the input defeasible axioms. Formally, since any $\langle \alpha, \mathbf{e} \rangle \in \chi_S$ is due to $\mathtt{ovr}(p(\mathbf{e})) \in S$ and $\mathtt{ovr}(p(\mathbf{e}))$ is derived from the reduct $G_S(PK(\mathcal{K}))$, it follows that $S$ must satisfy some overriding rule $r$ for $p(\mathbf{e})$. This means that $\mathcal{I}_S$ must satisfy the clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ for $\langle \alpha, \mathbf{e} \rangle$ encoded by the rule $r$. By the property defined in Theorem 2 it follows that the clashing assumption $\langle \alpha, \mathbf{e} \rangle$ is justified: thus, $\chi_S$ is justified. □

## *C.4 Complexity of reasoning problems*

### *C.4.1 Satisfiability*

*Lemma 5*
Given a graph $G = (V, E)$, deciding whether every color assignment to the nodes of degree 1 in $G$ is can be extended to a 3-coloring of $G$ is $\Pi^p_2$-complete.

*Proof*
The problem is in $\Pi^p_2$, since a guess for a coloring $\kappa$ of the degree 1 nodes of $E$ that can not be extended to a 3-coloring of $G$ can be checked with a call to a co-NP-oracle.

The $\Pi^p_2$-hardness is shown by a reduction from evaluating QBFs of the form $\Phi = \forall X \exists Y\, E$, where $E = \bigwedge_i C_i$ is in CNF and each clause $C_i$ has size 3; to this end, a reduction of 3SAT to 3-colorability can be easily generalized, for example, the one in https://www.cs.princeton.edu/courses/archive/spring07/cos226/lectures/23Reductions.pdf. In this and similar reductions, a graph $G$ is constructed from $E$ that is 3-colorable iff $E$ is satisfiable. This graph has nodes $v_p$ and $v_{\neg p}$ that correspond to the literals $p$, $\neg p$ of the variables $p$ occurring in $E$, such that color assignments to $v_p$ resp. $v_{\neg p}$ correspond to truth assignments to the literals; furthermore, it has a distinguished node $B$, such that $v_p$, $v_{\neg p}$, and $B$ form a triangle for each $p$.

We may assume that $G$ has no node of degree 1 (else we would involve that node with two fresh nodes in a triangle). To encode $X$, we connect to each $v_x$ for $x \in X$ a fresh node $v'_x$, which becomes a source. Furthermore, we connect fresh nodes $z_1$ and $z_2$ to $B$ and to each other (forming a triangle), and connect further fresh nodes $z'_1$ and $z'_2$ to $z_1$ and $z_2$, respectively. The purpose of the latter gadget is to ensure that in each 3-coloring of the resulting graph $G'$, the node $B$ must have the color of either $z'_1$ or $z'_2$. This will then allow to easily reduce checking whether for an assignment $\sigma$ to $X$, $E(\sigma(X), Y)$ is satisfiable to an 3-coloring extension test.

The claim is that $\Phi$ evaluates to true iff every color assignment $\kappa$ to the nodes $v'_p$ is extendible to a 3-coloring of $G'$.

($\Leftarrow$) Suppose every coloring to the $v'_p$ is extendible to a 3-coloring of $G'$. Then, if we color $z'_1$ and $z'_2$ with $b$, also $B$ is colored with $b$. If we have a truth assignment $\sigma$ to $X$ and set $\kappa(v_p) = g$ if $\sigma(p) =$ true and $\kappa(v_p) = r$ if $\sigma(p) =$ false, the extending 3-coloring must encode a truth assignment $\mu$ such that $E(\sigma(X), \mu(Y))$ evaluates to true, by the properties of the graph $G$.

($\Rightarrow$) Conversely, suppose $\Phi$ evaluates to true, and consider and coloring $\kappa$ of the degree-1 nodes. We can extend $\kappa$ to a 3-coloring of $G'$ as follows. Suppose the node $z'_1$ has the color $b$; then we can color $z_1$, $z_2$ and $B$ such that the latter also has the color $b$. Now for each $x \in X$ we color $v_x$ with $g$ and $v_{\neg x}$ with $r$ if $n'_x$ has color $r$, and $v_x$ with $r$ and $v_{\neg x}$ with $g$ otherwise; thus, the coloring of the nodes $v_x$, $v_{\neg x}$ reflects a truth assignment to $X$. As $\Phi$ evaluates to true, we can color all other $v_p$ and $v_{\neg p}$ nodes, as well as the auxiliary nodes such that we obtain a 3-coloring of $G'$. □

*Proposition 8*

Let $\mathcal{K}$ be a normalized *DL-Lite*$_\mathcal{R}$ DKB, and let $\chi' = \{\langle \alpha, \mathbf{e} \rangle \mid D(\alpha) \in \mathcal{K}$, $\mathbf{e}$ is over standard names$\}$ be the clashing assumption with all exceptions possible. Then, $\mathcal{K}$ has some justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ such that $\chi \subseteq \chi'$ iff $\mathcal{K}$ has some CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi' \rangle$.

*Proof*

($\Rightarrow$) Every justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$ is a CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$, and since $\chi \subseteq \chi'$, also $\mathcal{I}'_{CAS} = \langle \mathcal{I}, \chi' \rangle$ is a CAS model of $\mathcal{K}$ as every exception $\langle \alpha, \mathbf{e} \rangle$ in $\mathcal{I}_{CAS}$ is also made in $I'_{CAS}$ (and possibly more exceptions are made).

($\Leftarrow$) Suppose that $\mathcal{K}$ has some CAS model of the form $\mathcal{I}'_{CAS} = \langle \mathcal{I}, \chi' \rangle$. We then can construct some justified CAS model of $\mathcal{K}$ by trying to remove, one by one, the clashing assumptions $\langle \alpha, \mathbf{e} \rangle$ in $\chi'$. To this end, let $\Delta_0$ consist of the FO translation $\forall \vec{x} \phi_\alpha(\vec{x})$ of all non-defeasible axioms in $\mathcal{K}$, plus all instances $\phi_\alpha(\mathbf{e})$ of defeasible axioms $D(\alpha)$ in $\mathcal{K}$ such that $\langle \alpha, \mathbf{e} \rangle \notin \chi_0$. Furthermore, let $ex_1, ex_2, \ldots, ex_i = \langle \alpha_i, \mathbf{e}_i \rangle, \ldots$ be a (possibly infinite) enumeration of $\chi_0$, and let $\chi_0 = \chi$. We then build the sequences theories $\Delta_i$ and clashing sets $\chi_i$, $i \geq 1$ inductively as follows:

$$(\Delta_{i+1}, \chi_{i+1}) = \begin{cases} (\Delta_i, \chi_i) & \text{if } \Delta_i \cup \{\phi_{\alpha_i}(\mathbf{e}_i)\} \text{ is unsatisfiable} \\ (\Delta_i \cup \{\phi_{\alpha_i}(\mathbf{e}_i)\}, \chi_i \setminus \{\langle \alpha_i, \mathbf{e}_i \rangle\} & \text{otherwise} \end{cases}.$$

We obtain then that $\Delta = \bigcup_{i \geq 0} \Delta_i$ is satisfiable iff $\Delta_0$ is satisfiable, and in the latter case it satisfies $\chi = \bigcap_{i \geq 0} \chi_i$; furthermore, if $\langle \alpha_{i+1}, \mathbf{e}_{i+1} \rangle$ was not removed, then $\Delta_i \models \neg \phi_{\alpha_i}(\mathbf{e}_i)$ and thus $\Delta \models \neg \phi_{\alpha_i}(\mathbf{e}_i)$. The formula $\neg \phi_{\alpha_i}(\mathbf{e}_i)$ amounts for axioms $\alpha_i$ of the defeasible (normal) form $A \sqsubseteq (\neg)B$, $R \sqsubseteq S$, $\mathrm{Dis}(R, S)$, and $\mathrm{Irr}(R)$, to the minimal clashing set in Table 1; for $\mathrm{Inv}(R, S)$, it amounts to $R(e_1, e_2) \wedge \neg S(e_2, e_1) \vee \neg R(e_1, e_2) \wedge S(e_2, e_1)$, which is logically equivalent to $(R(e_1, e_2) \vee S(e_2, e_1)) \wedge (\neg R(e_1, e_2) \vee \neg S(e_2, e_1))$. As $\Delta$ is Horn and $\Delta \models R(e_1, e_2) \vee S(e_2, e_1)$, it follows that either $\Delta \models R(e_1, e_2)$ or $\Delta \models S(e_2, e_1)$ holds; thus either $\Delta \models R(e_1, e_2) \wedge \neg S(e_2, e_1)$ or $\Delta \models \neg R(e_1, e_2) \wedge S(e_2, e_1)$ holds, that is, one of the two clashing sets is indeed derived.

Consequently, every model $\mathcal{I}$ of $\Delta$ gives rise to a justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$. □

### *C.4.2 Entailment checking*

*Theorem 5*

Given an exception safe DKB $\mathcal{K}$ and an axiom $\alpha$, deciding $\mathcal{K} \models \alpha$ is co-NP-complete; this holds also for data complexity and instance checking, that is, $\alpha$ is of the form $A(a)$ for some assertion $A(a)$.

*Proof*
To refute $\mathcal{K} \models \alpha$, we need to show that some justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$ exists such that $\mathcal{I} \not\models \alpha$. Without loss of generality, we assume that $\alpha$ is normalized.

Depending on the type of $\alpha$, given $\chi$ and a name assignment $\nu$, a we push the latter first to $\mathcal{K}$ and then proceed as follows.

(a) If $\alpha$ is an assertion $A(a)$, $R(a,b)$, $\neg A(a)$, or $\neg R(a,b)$ we can check the existence of $\mathcal{I}_{CAS}$ by the use of Theorem 1, and we can assume that $\mathcal{I}$ is named relative to $N$ with $N = sk(N_\mathcal{K})$. That is, we can use the materialization calculus relative to $\chi$, and verify given a justified $\chi$ that $\alpha$ can not be derived.

(b) If $\alpha$ is an inclusion axiom $A \sqsubseteq B$, then we need to show that for some element $e$ it holds that $\mathcal{I} \models A(e)$ and $\mathcal{I} \models \neg B(e)$. To deal with this, we first add to $\mathcal{K}$ the axioms $Aux \sqsubseteq A$, $Aux \sqsubseteq \neg B$; this does not compromise exception safety nor that $\chi$ is justified. We may then assume without loss of generality that $\mathcal{I} \models Aux(e)$, that is, $\mathcal{I} \not\models \neg Aux(e)$.
We next add to $\mathcal{K}$ an assertion $A_e(a_e)$, where $A_e$ and $a_e$ are a fresh concept and individual name, respectively; this serves to give $e$ a name if it is outside the elements named in $\mathcal{I}$ by Skolem terms. This addition again does neither compromise exception safety nor that $\chi$ is justified, and $\mathcal{I}$ can be adjusted to it.
We then check whether $\neg Aux(a_e)$ is not derivable from the resulting DKB $\mathcal{K}'$ under $\chi$; this holds iff some $\mathcal{I}$ with $e$ not named by some Skolem term of the $\mathcal{K}$ exists.
Otherwise, $e$ must be named by some Skolem term $t$ of $\mathcal{K}$. We thus check that for none such $t$, $\neg Aux(t)$ is derivable from $\mathcal{K}'$ under $\chi$; the depth of $t$ can be polynomially bounded.
The checks can be done in nondeterministic logspace, and thus deciding $\mathcal{K} \not\models \alpha$ under $\chi$ is feasible in polynomial time.

(c) if $\alpha$ is a role inclusion axiom $R \sqsubseteq S$, then we need to show that for some elements $e, e'$ it holds that $\mathcal{I} \models R(e,e')$ and $\mathcal{I} \models \neg S(e,e')$. Similarly as for $A \sqsubseteq B$ above, we can use an auxiliary role $Aux$ and axioms $Aux \sqsubseteq R$, $Aux \sqsubseteq \neg S$, proceed with assuming that $\mathcal{I} \not\models \neg Aux(e,e')$, introduce $a_e, a_{e'}$ for $e, e'$ in $\mathcal{K}$, and test then that from $\mathcal{K}'$ under $\chi$, $\neg Aux(t,t')$ is not derivable, where $t, t'$ range over the Skolem terms of depth bounded by a polynomial and $a_e$ resp. $a_{e'}$, where $e = e'$ must be respected. Again, this allows us to show that $\mathcal{K} \not\models \alpha$ under $\chi$ in polynomial time.

(d) in all other cases, we can proceed similarly as in a) and b), as to refute $\alpha$, we need to show that for some element $e$ (resp. elements $e, e'$), $\mathcal{I}$ satisfies some literals $\alpha_i(e)$, $1 = 1, \ldots, k$. (resp. $\alpha_i(e,e')$, $1 = 1, \ldots, k$). In particular, for $A \sqsubseteq \neg B$: $A(e)$, $B(e)$; for $\mathrm{Dis}(R,S)$: $R(e,e')$, $S(e,e')$; for $\mathrm{Irr}(R)$: $R(e,e')$; for $\mathrm{Inv}(R,S)$: either $R(e,e')$, $\neg S(e,e')$ or $\neg R(e,e')$, $S(e,e')$. Along the same lines as above we can introduce an auxiliary concept resp. role $Aux$, individual names $a_e$, $a_{e'}$ etc. and decide $\mathcal{K} \not\models \alpha$ under $\chi$ in polynomial time.

Thus, to decide $\mathcal{K} \not\models \alpha$, we can guess a justified clashing assumption $\chi$ over $N_\mathcal{K}$ together with a clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ for each $\langle \alpha, \mathbf{e} \rangle \in \chi$ and check (i) that $\chi$ is satisfiable, (ii) that all $S_{\langle \alpha, \mathbf{e} \rangle}$ are derivable from $\mathcal{K}$ under $\chi$, and (iii) that $\mathcal{K} \not\models \alpha$. Each of the steps (i)–(iii) is feasible in polynomial time. Consequently, the entailment problem $\mathcal{K} \models \alpha$ is in co-NP.

co-NP-*hardness.* The co-NP-hardness can be shown by a reduction from inconsistency-tolerant reasoning from *DL-Lite*$_\mathcal{R}$ KBs under AR-semantics Lembo *et al.* (2010). Given a *DL-Lite*$_{\mathcal{R}}$ KB $\mathcal{K} = \mathcal{A} \cup \mathcal{T}$ with ABox $\mathcal{A}$ and TBox $\mathcal{T}$, a repair is a maximal subset $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{K}' = \mathcal{A}' \cup \mathcal{T}$ is satisfiable; an assertion $\alpha$ is AR-entailed by $\mathcal{K}$, if $\mathcal{K}' \models \alpha$ for every repair $\mathcal{K}'$ of $\mathcal{K}$. As shown by Lembo *et al.*, deciding AR-entailment is co-NP-hard; this continues to hold under UNA and if all assertions involve only concept resp. role names.

Let $\hat{\mathcal{K}} = \mathcal{T} \cup \{D(\alpha) \mid \alpha \in \mathcal{A}\}$, that is, all assertions from $\mathcal{K}$ are defeasible. As easily seen, under this assumption the maximal repairs $\mathcal{A}'$ correspond to the justified clashing assumptions by $\chi = \{\langle \alpha, \mathbf{e} \rangle \mid \alpha(\mathbf{e}) \in \mathcal{A} \setminus \mathcal{A}'\}$. Thus, $\mathcal{K}$ AR-entails $\alpha$ iff $\hat{\mathcal{K}} \models \alpha$, proving co-NP-hardness. Furthermore, in order to establish the result for exception safety without looking further into the structure of $\mathcal{K}$, we may apply the normal form transformation of Lemma 4; as defeasible assertions $\mathrm{D}(A(a))$ are transformed to $A'(a)$ and $\mathrm{D}(A' \sqsubseteq A)$ and since $A'$ does not occur on the right hand side of any axiom, no Skolem terms can be derived that feeds into the positive literal of the clashing set $\{A'(a), \neg A(a)\}$; similarly, $\mathrm{D}(R(a,b))$ is translated to $R'(a,b)$ and $\mathrm{D}(R' \sqsubseteq R)$ and similarly exception safety is warranted.

As Lembo *et al.* proved the co-NP-hardness under data complexity, with the normal form transformation (which then requires merely the addition of the assertions $A'(a)$ resp. $R'(a,b)$, or a renaming of the symbols) the claimed result for data complexity follows.□

*Proposition 9*
Given a DKB $\mathcal{K}$, deciding where $\mathcal{K} \models \alpha$ is co-NP-hard even if no roles occur in $\mathcal{K}$ and $\alpha$ is an assertion $A(a)$.

*Proof*
Cadoli and Lenzerini (Theorem 16 1994) showed that given a positive propositional 2CNF $F$ over variables $V$, deciding whether an atom $z$ is a circumscriptive consequence of $F$ is co-NP-hard if all variables except $z$ are minimized. In circumscription, the latter is denoted as $\mathit{CIRC}(F;P;Z) \models z$ where $P = V \setminus Z$ and $Z = \{z\}$, where $\mathit{CIRC}(F;P;Z)$ is defined as a QBF with free variables $V$; semantically, $\mathit{CIRC}(F;P;Z)$ captures the $P;Z$-minimal models of $F$, which are the models $M$ of $F$ for which no model $M'$ of $F$ exists such that (a) $M \cap Q = M' \cap Q$, where $Q = V - (P \cup Z)$, and (b) $M' \cap P \subset M \cap P$.

We reduce the inference $\mathit{CIRC}(F;P,Q;Z) \models z$ to entailment $\mathcal{K} \models A(a)$, where the variables in $V$ are used as concept names, as follows.

- For each clause $c = x \vee y$ in $F$, we add to $\mathcal{K}$ an axiom $x \sqsubseteq \neg y$ if $z \neq x, y$ and an axiom $x \sqsubseteq z$ (resp. $y \sqsubseteq z$) if $z = y$ (resp. $x = z$). Informally, we flip in this representation the polarity of all variables except $z$, in order to obtain *DL-Lite*$_{\mathcal{R}}$ axioms.
- Furthermore, for each variable $x \neq z$, we add a defeasible assertion $D(x(a))$, where $a$ is a fixed individual.

This construction effects that the justified DKB models of $\mathcal{K}$ correspond to the models of $\mathit{CIRC}(F;P,\emptyset;\{z\})$, where the minimality of exceptions in justified DKB models emulates the minimality of circumscription models. Formally, $\mathcal{K} \models z(a)$ iff $\mathit{CIRC}(F;P,\emptyset;\{z\}) \models z$.

($\Leftarrow$) Suppose $\mathcal{K} \not\models z(a)$; then, some justified model $\mathfrak{I} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$ exists such that $\mathcal{I} \not\models z(a)$. Let $M = \{v \in V \mid v \neq z, \mathcal{I} \not\models v(a)\}$; we claim that $M$ is a $P;Z$-minimal model of $F$. Suppose this is not the case. Then, some model $M'$ of $F$ exists such that

(a) and (b) hold. Then, some variable $v \in (M \cap P) \setminus (M' \cap P)$ exists, which means that $F \cup \{\neg x \mid x \neq z, x \in V - M\} \not\models v$. As $\mathcal{I} \not\models v(a)$, an exception to $D(v(a))$ was made in $\chi$. However, by switching $v(a)$ in $\mathcal{I}$ to true, we obtain an NI-congruent CAS interpretation $\mathcal{I}'$ that satisfies $\mathcal{K}$ relative to $\chi$. This means that the exception to $D(v(a))$ is not justified, which is a contradiction.

($\Rightarrow$) Suppose that $CIRC(F; P, \emptyset; \{z\}) \not\models z$, that is, some $P; Z$-minimal model $M$ of $F$ exists such that $M \not\models z$. We define $\mathcal{I} = \langle \mathcal{I}, \chi \rangle$ where $\mathcal{I} \models v(a)$ iff $v \in V \setminus M$ and $\chi$ contains all exceptions for $D(v(a))$ where $v \in M$. Similarly as in the if-case, it is argued that $\mathcal{I}$ is a justified model of $\mathcal{K}$. As $\mathcal{I} \not\models z(a)$, it follows that $\mathcal{K} \not\models z(a)$. This concludes the proof of the claim.

Similarly as in the proof of Theorem 5, the defeasible assertions $D(x(a))$ can be moved to defeasible axioms $D(c \sqsubseteq x)$ with a single assertion $c(a)$. □

### *C.4.3 Conjunctive query answering*

*Theorem 6*
Given an exception safe DKB $\mathcal{K}$ and a Boolean CQ $Q$, deciding whether $\mathcal{K} \models Q$ is (i) $\Pi^p_2$-complete in combined complexity and (ii) co-NP-complete in data complexity.

*Proof*
To start with (i), as for membership in $\Pi^p_2$, to refute $Q$ we can guess for a justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ such that $\mathcal{I}_{CAS} \not\models Q$ the clashing assumption $\chi$ on $N_{\mathcal{K}}$ and a name assignment $\nu$, which we can push to the knowledge base. Since $\mathcal{K}$ is exception safe, we can decide in NLogSpace whether $\chi$ is satisfiable relative to $\nu$ (which can be pushed to $\mathcal{K}$) and can indeed give rise to a desired justified CAS model $\mathcal{I}_{CAS}$ of $\mathcal{K}$. We then can use an NP oracle to check whether for some polynomial number of Skolem terms $ST$, where the number depends on $Q$ and $\mathcal{K}$, the query has a match on $N_{\mathcal{K}} \cup ST$ in the least CAS model $\hat{\mathcal{I}}_{\mathcal{K}}(\chi, \nu)$ of $\mathcal{K}$; to this end, each atom $A(t)$ resp. $R(t, t')$ in the match must be derived by applying the axioms (that is, by unraveling $\mathcal{I}^{N_{\mathcal{K}}}_{CAS}$); this will ensure that a match exists in each CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$. If the oracle answer is no, then some $\mathcal{I}_{CAS}$ such that $\mathcal{I}_{CAS} \not\models Q$ exists. Consequently, refuting $\mathcal{K} \models Q$ is in $\Sigma^p_2$, which proves the membership part.

The $\Pi^p_2$-hardness of (i) is shown by a reduction from a generalization of deciding whether a graph is 3-colorable: given an (undirected) graph $G = (V, E)$, can every color assignment to the nodes of degree 1 in $G$ (i.e. source nodes) be extended to a 3-coloring of $G$? This problem is $\Pi^p_2$-complete (see Lemma 5).

We construct a DKB $\mathcal{K}$ as follows. We use roles $R$, $R_r$, $R_g$, $R_b$, and $E$, and as individual names $r, g, b$ and each $v \in V$, where we assume that names are unique (this can be easily enforced by adding further auxiliary axioms). Informally, $R$ and the $R_c$ serve to encode color assignments to nodes and $E$ to represent the edges of the graph. We add to $\mathcal{K}$ the following axioms:

- defeasible axioms $D(R_r(v, r))$, $D(R_g(v, g))$, $D(R_b(v, b))$, for each node $v$ of degree $\neq 1$;
- $R_r \sqsubseteq R$, $R_g \sqsubseteq R$, $R_b \sqsubseteq R$;
- $\exists R_r \sqsubseteq \neg \exists R_g$, $\exists R_r \sqsubseteq \neg \exists R_b$, $\exists R_g \sqsubseteq \neg \exists R_b$;

and the assertions

- $R(v,r)$ $R(v,g)$, $R(v,b)$ for each non-source $v$;
- $E(r,g)$, $E(r,b)$, $E(g,b)$, $E(g,r)$, $E(b,r)$, $E(b,g)$.

Intuitively, we must make for each source node $v$ an exception to two of the three axioms $D(R_r(v,r))$, $D(R_g(v,g))$, $D(R_b(v,b))$, and in this way assign a color to $v$. For example, for assigning red ($r$) the exceptions are $R_g(v,g))$ and $R_b(v,b)$ which have the minimal clashing sets $\{\neg R_g(v,g)\}$ and $\{\neg R_b(v,b)\}$, respectively; for the other assignments this is analogous. Every choice $\kappa$ of a coloring for the sources in $G$ thus gives rise to a natural justified clashing assumption $CAS_\kappa$.

The Boolean query that we construct is

$$Q = \exists \vec{y} \bigwedge_{v \in V} R(v, y_v) \wedge \bigwedge_{e=(v,v') \in E} E(y_v, y_{v'}).$$

Informally, the graph $G$ is encoded in $Q$, where the variables $y_v$ range over the colors of the nodes $v$; with $R(v,y_v)$ we pick a color for a match where for sources only the color chosen by $\kappa$ is available, while for the other nodes all three colors $r$, $g$, $b$ are available. The $E$-atoms enforce that adjacent nodes must have different color.

It is then not difficult to verify that $\mathcal{K} \models Q$ holds under UNA iff for every coloring $\kappa$ of the sources, we have $\mathcal{I}_{CAS_\kappa} \models Q$, that is, the coloring $\kappa$ of the sources can be extended to a 3-coloring of the whole graph $G$. As $\mathcal{K}$ is clearly constructable from $G$ in polynomial time, this proves $\Pi^p_2$-hardness.

(ii) As for data complexity, we note that the check where $Q$ has no match in any $\mathcal{I}''_{CAS}$ is feasible in polynomial time, as the number of variables in the query is fixed and thus only constantly many Skolem terms $ST$ have to be added to $N_\mathcal{K}$ for a query match in the least CAS model $\hat{\mathcal{I}}_\mathcal{K}(\chi,\nu)$ of $\mathcal{K}$, for which only polynomially many possibilities exist; furthermore, the inference of atoms $A(t)$ resp. $R(t,t')$ is feasible in polynomial time. Hence, the problem is in co-NP. The co-NP-hardness follows from Theorem 5. □

### *C.5 Complexity of reasoning problems with unnamed individuals*

*Proposition 11*
Given an $n$-de safe DKB $\mathcal{K}$, where $n$ is polynomial in the size of $\mathcal{K}$, and a clashing assumption $\chi$ defined on $N_\mathcal{K}$, deciding whether $\mathcal{K}$ has (i) some arbitrary CAS model resp. (ii) some justified CAS model of form $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ is NP-complete resp. $D^p$-complete in general but feasible in polynomial time if $n$ is bounded by a constant.

*Proof*
We can compute the (polynomially many) Skolem terms $t_i$, $i = 1,\ldots,m \leq n$ that feed into clashing assumptions for $\mathcal{K}$ using the algorithm in the proof of Proposition 2 for deciding $n$-de safety in polynomial time.

As for (i), in order to show that some CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ of $\mathcal{K}$ exists, we need to show that no inconsistency can be derived from $\mathcal{K}$ under $\chi$ relative to some name assignment $\nu$ (which can be pushed to $\mathcal{K}$).

We guess for each $i = 1,\ldots,m$ whether $t_i = a_j$ holds for one or none of the individuals $a_1,\ldots,a_n$ that name exceptions in $\chi$. Relative to this guess, we then decide in polynomial time whether $\mathcal{K}$ is satisfiable.

To this end, we slightly modify a common algorithm that decides unsatisfiability in absence of defeasible axioms by nondeterministically deriving opposite assertions $D(t), \neg D(t)$ resp. $R(t,t'), \neg(t,t')$ from at most two assertions in $\mathcal{K}$ in polynomially many steps using logarithmic workspace; these assertions are either over $N_\mathcal{K}$, the same Skolem term $t$ or $t' = f_R(t)$ for some role $R$, where outside $N_\mathcal{K}$ we need not store $t$. In the extension, we keep track of which terms $\bar{t}, \bar{t}'$ in the assertion $(\neg)D(\bar{t}_h)$ resp. $(\neg)R(\bar{t}_h, \bar{t}'_h)$ derived in step $h$ are among the subterms of some $t_1, \ldots, t_m$; and if $\bar{t} = t_i$ resp. $\bar{t}' = t_i$ and $t_i = a_j$ is in the guess, then $D(a_j)$, $R(a_j, \bar{t}')$ resp. $R(\bar{t}, a_j)$ can be derived.

As there are only polynomially many subterms of $t_1, \ldots, t_m$ (otherwise $\mathcal{K}$ would be recursive, thus not $n$-de bounded), the bookkeeping for respecting the subterms is feasible in logarithmic work space (each subterm $t$ may have an identifier $id(t)$, and a table computed before hand holds $(id(t), R, id(f_R(t))))$.

As the algorithm works in polynomial time, we can decide in this way also whether $\mathcal{K}$ is satisfiable under $\chi$ relative to a name assignment $\nu$.

As for (ii), we must in addition to (i) check that for each clashing assumption $\langle \alpha, \mathbf{e} \rangle$ in $\chi$, some clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ can be derived. We utilize here a similar guess and check algorithm as in (i) to decide whether a given assertion $\alpha$ is not derivable from $\mathcal{K}$ under $\chi$; that is, we guess $t_i = a_j$ for all terms $t_i$ and then check in nondeterministic logspace that $\alpha$ can not be derived. Hence, deciding that for some $\langle \alpha, \mathbf{e} \rangle$ no clashing set $S_{\langle \alpha, \mathbf{e} \rangle}$ is derivable is in NP, which implies that the additional check is in co-NP. Consequently, in case (ii) membership in $D^p$ follows.

If $n$ is bounded by a constant, then in the algorithm above the guess for the equalities $t_i = a_j$ can be eliminated, by cycling through all (polynomially) many possibilities, which results in PTime membership.

For the hardness proofs, for (i) we reduce deciding 3-colorability of a graph $G = (V, E)$ to deciding CAS model existence; we provide for this a construction that can be reused for (ii).

We construct $\mathcal{K}$ as follows. For each edge $e_i = (v_{i,1}, v_{i,2})$ in $E = \{e_1, \ldots, e_m\}$, we introduce two individuals $v_{i,1}$ and $v_{i,2}$, and for each $v_{i,j}$ we introduce three further individuals $col_{i,j}$, $c1_{i,j}$ and $c2_{i,j}$. Informally, the latter three individuals will serve to take the three colors red, green, and blue by roles $R_r$, $R_g$, and $R_b$ such that the color assigned to $col_{i,j}$ will be the color assigned to the occurrence of the node $v_{i,j}$ in the edge $e_i$.

The alignment of colors assigned to $v_{i,j}$ and $v_{i',j'}$ that represent the same node $v_k$ in $V = \{v_1, \ldots, v_n\}$ will be ensured with the help of an auxiliary node $check_{v_k}$. To this end, the nodes $col_{i,j}$ and $col_{i',j'}$ will send their assigned colors to this node using roles *RCheck*, *GCheck* and *BCheck*, which tests for their equality. That $v_{i,1}$ and $v_{i,2}$ are colored differently will be checked with the help of auxiliary roles *RNeighbor*, *GNeighbor*, *BNeighbor*.

Finally, we use an individual *esc* that allows us to model a state in which no $v_{i,j}$ has a color assigned. This state however, requires an exception to an axiom.

In the construction, we use a domain predicate, expressed by a concept *Dom* that will be asserted for all individuals in $\mathcal{K}$ and enforced to be false for all other elements; roles between individuals can only be in the domain, thus do not involve unnamed individuals. Furthermore, we shall restrict roles between individuals by negative role assertions.

The DKB $\mathcal{K}$ consists of the following assertions, axioms and defeasible axioms:

1. $V(v_{i,j})$ for all $v_{i,j}$ and $Dom(a)$ for each $a = v_{i,j}, col_{i,j}$, $c1_{i,j}$, $c2_{i,j}, e$, $check_{v_k}$, $esc$, $esc'$;
2. $V \sqsubseteq \exists R$, $V \sqsubseteq \exists G$, $V \sqsubseteq \exists B$;
3. $\exists R^- \sqsubseteq \neg\exists G^-$, $\exists G^- \sqsubseteq \neg\exists B^-$, $\exists R^- \sqsubseteq \neg\exists B^-$;
4. $\exists R^- \sqsubseteq \exists RNeighbor$, $\exists G^- \sqsubseteq \exists GNeighbor$, $\exists B^- \sqsubseteq \exists BNeighbor$;
5. $\exists R^- \sqsubseteq \neg\exists RNeighbor^-$, $\exists G^- \sqsubseteq \neg\exists GNeighbor^-$, $\exists B^- \sqsubseteq \neg\exists BNeighbor^-$;
6. $\exists R^- \sqsubseteq \exists RCheck$, $\exists G^- \sqsubseteq \exists GCheck$, $\exists B^- \sqsubseteq \exists BCheck$;
7. $\exists RCheck^- \sqsubseteq \neg\exists GCheck^-$, $\exists GCheck^- \sqsubseteq \neg\exists BCheck^-$, $\exists RCheck^- \sqsubseteq \neg\exists BCheck^-$
8. $\exists R^- \sqsubseteq \neg E$;
9. for each role $X$, we add the axiom $\exists X^- \sqsubseteq Dom$ and limit its range by adding $\neg X(a, b)$ for each pair $(a, b)$ of individuals from above that is not allowed as follows:
   - for $R, G, B$ we allow $(v_{i,j}, col_{i,j})$, $(v_{i,j}, c1_{i,j})$, $(v_{i,j}, c2_{i,j})$, and for $R$ in addition $(v_{i,j}, esc)$;
   - for $RCheck$, $GCheck$, $BCheck$, we allow $(col_{i,j}, check_k)$, $(c1_{i,j}, c1_{i,j})$, $(c2_{i,j}, c2_{i,j})$, were $v_{i,j} = v_k$, and $(esc, esc)$;
   - for $RNeighbor$, $GNeighbor$, $BNeighbor$, we allow $(col_{i,j}, col_{i',j'})$, $(col_{i',j'}, col_{i,j})$, where $v_{i,j} = v_k$, $v_{i',j'} = v'_k$, and $(v_k, v'_k) \in E$; furthermore $(c1_{i,j}, c2_{i,j})$ and $(c2_{i,j}, c1_{i,j})$, for all $v_{i,j}$, and $(esc, esc')$;
10. $\mathrm{D}(Dom \sqsubseteq \bot)$ and $\mathrm{D}(E(esc))$.

We define the clashing assumption $\chi$ to have an exception of $\mathrm{D}(Dom \sqsubseteq \bot)$ for all individuals $a$ where have asserted $Dom(a)$ above.

We note that $\mathcal{K}$ has for $\chi' = \chi \cup \{\langle E(esc), ()\rangle\}$ (i.e. when making also an exception to $\mathrm{D}(E(esc))$) a CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi'\rangle$: if in $\mathcal{I}'$ the atomic concept instances are those mentioned in (1), and the role instances atoms are: $R(v_{i,j}, esc)$, $RCheck(esc, esc')$, $RNeighbor(esc, esc')$; $G(v_{i,j}, c1_{i,j})$, $GCheck(c1_{i,j}, c2_{i,j})$; and $B(v_{i,j}, c2_{i,j})$, $VCheck(c2_{i,j}, c1_{i,j})$ for all $v_{i,j}$; then by defining the Skolem functions appropriately we can obtain a CAS model which is named by $N_{\mathcal{K}}$.

On the other hand, it turns out that $\mathcal{K}$ has some CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi\rangle$ (i.e. when making no exception to $\mathrm{D}(E(esc))$) under UNA iff $G$ is 3-colorable.

To see this, if $G$ is 3-colorable, then we can reassign the roles $R(v_{i,j}, esc)$, $G(v_{i,j}, c1_{i,j})$, and $B(v_{i,j}, c2_{i,j})$ in $\mathcal{I}'$ to $C(v_{i,j}, col_{i,j})$, $C_1(v_{i,j}, c1_{i,j})$, and $C_2(v_{i,j}, c2_{i,j})$ to where $C$ is the color of the node $v_k$ such that $v_{i,j} = v_k$ in the 3-coloring, and $C_1$ and $C_2$ are the other two colors. This then requires to set up, in a determined way, for $col_{i,j}$ the role instances $CCheck(col_{i,j}, check_{i,j})$ and $CNeighbor(col_{i,j}, col_{i',j'})$, where $v_{i,j} = v_k$ and $v_{i',j'} = v_{k'}$ and $(v_k, v'_k) \in E$, for $c1_{i,j}$ the role instances $C_1Check(c1_{i,j}, c1_{i,j})$, $C_1Neighbor(c1_{i,j}, c2_{i,j})$, and for for $c2_{i,j}$ analogously the role instances $C_2Check(c2_{i,j}, c2_{i,j})$, $C_2Neighbor(c2_{i,j}, c1_{i,j})$. Finally, we make $E(esc)$ true (further roles $RCheck(esc, esc')$, $RNeighbor(esc, esc')$ could be removed). The resulting CAS interpretation $\mathcal{I}_{CAS}$ is then a model of $\mathcal{K}$.

Conversely, if $\mathcal{K}$ has some CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi\rangle$, then $E(esc)$ is satisfied in it. Hence no role $R(v_{i,j}, esc)$ is satisfied in $\mathcal{I}_{CAS}$, which means that for each $v_{i,j}$, for some color $C$, $C \in \{R, G, B\}$ the role $C(v_{i,j}, col_{i,j})$ holds in $\mathcal{I}$. As then the role $CCheck(col_{i,j}, check_k)$ also holds, by the axioms in (7) all $v_{i',j'}$ that correspond to the node $v_k$ will have the same color $C$. Furthermore, the axiom $\exists C^- \sqsubseteq \neg\exists CNeighbor^-$ is satisfied, which implies that $v_{1,2}$ has a different color $C' \neq C$. Thus, we obtain from $\mathcal{I}_{CAS}$ a 3-coloring of the graph $G$. This proves the NP-hardness of case (i).

We show the $D^p$-hardness of case (ii) by a reduction from 3COL-3UNCOL, that is, given graphs $G_1$ and $G_2$, decide whether $G_1$ is 3-colorable and $G_2$ is not 3-colorable. We observe that the DKB $\mathcal{K}$ defined for the graph $G$ above has some justified CAS model of the form $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \cup \{\langle E(esc), ()\rangle\}\rangle$ iff $G$ is not 3-colorable.

We thus take for $G_1$ and $G_2$ two copies $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively of the construction as above (using disjoint vocabularies), and set $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ and $\chi = \chi_1 \setminus \{\langle E_1(esc_1), ()\rangle\} \cup \chi_2$. Then, $\mathcal{K}$ some justified CAS model of form $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi\rangle$ iff $G_1$ is 3-colorable and $G_2$ is not 3-colorable.

As easily seen, $\mathcal{K}$ is acyclic and its TBox is the same for each graph $G$. Hence, $\mathcal{K}$ is $k$-chain bounded for some constant $k$ and thus also $n$-de bounded for some $n$ polynomial in $|\mathcal{K}'|$. This proves the result under the stated restrictions, which moreover also holds under data complexity. □

We remark that from the proof of Proposition 11, we obtain that DKB model checking, that is, decide whether an interpretation $\mathcal{I}$ is a DKB model of an DKB $\mathcal{K}$ is co-NP-hard, as the CAS model $\mathcal{I}'_{CAS}$ for the DKB $\mathcal{K}$ constructed for the graph $G$ is justified iff $G$ is not 3-colorable. On the other hand, for $n$-de safe $\mathcal{K}$ where $n$ is bounded by a polynomial in the size of $\mathcal{K}$, the problem is in co-NP since it reduces to checking whether the clashing assumption $\chi$ that contains all instances of axioms of $\mathcal{K}$ over $N_{\mathcal{K}}$ that are violated by $\mathcal{I}$, is justified. That is, for such DKBs, the model checking problem is co-NP-complete.

*Theorem 7*
Given an $n$-de safe DKB $\mathcal{K}$, where $n$ is bounded by a polynomial in $|\mathcal{K}|$, (i) deciding $\mathcal{K} \models \alpha$ for an axiom $\alpha$ and (ii) BCQ answering $\mathcal{K} \models Q$ are both $\Pi^p_2$-complete. In case $n$ is bounded by a constant, (i) is co-NP-complete while (ii) remains $\Pi^p_2$-hard.

*Proof*
Regarding the $\Pi^p_2$-membership results, to show in (i) that $\mathcal{K} \not\models \alpha$, we can similarly proceed as in Theorem 5 and guess a clashing assumption $\chi$ for $\mathcal{K}$ on $N_{\mathcal{K}}$ and a name assignment $\nu$ then check by Proposition 11 with an NP-oracle that some justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi\rangle$ exists relative to $\nu$. If so, we check whether $\mathcal{K} \not\models \alpha$ relative to $\chi$ and $\nu$ using an NP oracle, where we proceed depending on the type of $\alpha$ as follows:

– If $\alpha$ is a positive or negative assertion, then we use the guess and check algorithm described in the proof of case (ii) of Proposition 11.
– In the other cases, we proceed similarly as in the proof of Theorem 5: we introduce an auxiliary concept resp. role $Aux$, fresh individual names $a_e$ resp. $a_{e'}$ and check that, relative to $\chi$ and a guess for the equalities $t_i = a_j$ of the Skolem terms $t_1, \ldots, t_m$ that feed into clashing assumptions for $\mathcal{K}$ (which can be computed in polynomial time, cf. proof of Proposition 2) to the individuals $a_1, \ldots, a_n$ that name exceptions in $\chi$, we can not derive $\neg Aux(t)$ resp. $\neg Aux(t, t')$ for some terms $t, t'$ that range over the Skolem terms of polynomially bounded depth and $a_e$ resp. $a_{e'}$. The algorithm in the proof of item (i) of Proposition 11 can be adjusted to this end, so that it runs in polynomial time.

Hence, deciding $\mathcal{K} \not\models \alpha$ is in $\Sigma_2^p$, and thus deciding $\mathcal{K} \models \alpha$ is in $\Pi_2^p$.

In (ii), to show $\mathcal{K} \not\models Q$ we likewise guess a clashing assumption $\chi$ on $N_\mathcal{K}$ and a name assignment $\nu$, and we check using an NP oracle that some justified CAS model of form $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ relative to $\nu$ exists.

We then compute the (polynomially many) Skolem terms $t_i$, $i = 1, \ldots, m$ that feed into clashing assumptions, which is feasible in polynomial time (cf. proof of Proposition 2). We then guess for each $t_i$ whether $t_i = a_j$ holds for a single $a_j$ or none, where $a_1, \ldots, a_n$ are all the individuals that name exceptions in $\chi$. We then can use an NP oracle to guess polynomially many Skolem terms $ST$ that are connected to $N_\mathcal{K}$, including the subterms of all $t_i$, where the number depends on $Q$ and $\mathcal{K}$, and a match of the query $Q$ on $N_\mathcal{K} \cup ST$, for which we test the derivability of each atom $A(t)$ resp. $R(t, t')$ in the match. By the least model property, this will ensure that this partial model given by the match can indeed be extended to a model

It follows that deciding $\mathcal{K} \not\models Q$ is in $\Sigma_2^p$, which means deciding $\mathcal{K} \models Q$ is in $\Pi_2^p$.

As for (i) in case $n$ is bounded by a constant $k$, we can by Proposition 11 eliminate the NP oracle and obtain membership in co-NP.

To show the $\Pi_2^p$-hardness for (i), we extend the encoding of graph non-3-colorability in the proof of Proposition 11 in order to encode the constrained 3-colorability problem of Lemma 5.

We first note that the DKB $\mathcal{K}$ constructed for the graph $G = (V, E)$ allows under UNA for a justified CAS model of form $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ such that $\langle E(esc), () \rangle \in \chi'$ iff $G$ is not 3-colorable. To see this, recall that the constructed $\chi$ is justified iff $G$ is not 3-colorable. By construction of $\mathcal{K}$, each justified CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi' \rangle$ of $\mathcal{K}$ such that $\langle E(esc), () \rangle \in \chi'$ must satisfy $\chi \subseteq \chi'$; thus by non-redundancy of clashing assumptions (Proposition 5), it follows that $\chi = \chi'$. Furthermore, we have that $\mathcal{K} \models \neg E(esc)$ if $G$ is not 3-colorable and $\mathcal{K} \models E(esc)$ otherwise.

Suppose now that $v_{d_1}, \ldots, v_{d_m}$ are the nodes in $G$ of degree 1. We use additional concepts $S$, $F_R$, $F_G$, $F_B$ and add the following assertions, axioms and defeasible axioms:

1. $S(check_{d_j})$, for all $j = 1, \ldots, m$
2. $\mathrm{D}(S \sqsubseteq F_R)$, $\mathrm{D}(S \sqsubseteq F_G)$, $\mathrm{D}(S \sqsubseteq F_B)$
3. $F_R \sqsubseteq \neg F_G$, $F_G \sqsubseteq \neg F_B$, $F_R \sqsubseteq \neg F_B$
4. $F_C \sqsubseteq \neg \exists C_1Check^-$, $F_C \sqsubseteq \neg \exists C_2Check^-$ where $C \in \{R, G, B\}$ and $C_1$, $C_2$ are the remaining colors $\{R, G, B\} \setminus \{C\}$.

Intuitively, (1)-(3) allow us to select one of the colors for each node $v_{d_j}$ of degree 1. This selection must be in alignment with possible color checks via $CCheck$ roles issued by nodes $check_{i',j'}$ that correspond to $v_{d_j}$, that is, if color $C$ is selected, then only incoming $CCheck$ arcs are possible for $check_{d_j}$. Furthermore, by non-redundancy of clashing assumptions some color for $v_{d_j}$ must be selected, as $\neg F_C(check_{i',j'})$ can not be proven for all three colors $C$ simultaneously from the axioms (3) and (4) because $CCheck^-$ can be true at $check_{i,j}$ for at most one color $C$. Thus, each clashing assumption of a justified CAS model of the the resulting DKB $\mathcal{K}'$ must encode a color assignment $\rho$ to the nodes $v_{d_1}, \ldots, v_{d_m}$.

For an arbitrary such $\rho$, we obtain a CAS model $\mathcal{I}^\rho_{CAS} = \langle \mathcal{I}^\rho, \chi^\rho \rangle$ of $\mathcal{K}'$ from the candidate justified CAS model $\mathcal{I}_{CAS} = \langle \mathcal{I}, \chi \rangle$ described in the proof of Proposition 11 by making, for each $v_{d_j}$, $check_{d_j}$ an instance of $D$ and of $F_C$ where $C = \rho(v_{d_j})$ is the color of $v_{d_j}$ and adding $\langle D \sqsubseteq F_{C_1}, check_{d_j} \rangle$, $\langle D \sqsubseteq F_{C_2}, check_{d_j} \rangle$ to $\chi$ for the remaining colors $C_1$ and $C_2$. It then holds that $\mathcal{I}^{\rho}_{CAS}$ is a justified CAS model of $\mathcal{K}'$ iff the coloring $\rho$ is not extendable to a 3-coloring of the full graph $G$.

By construction of $\mathcal{K}'$ and the non-redundancy of clashing assumptions, it follows that every justified CAS model $\mathcal{I}'_{CAS} = \langle \mathcal{I}', \chi' \rangle$ of $\mathcal{K}'$ such that $\langle E(esc), () \rangle \in \chi'$ must under UNA be of the form $\chi' = \chi^{\rho}$ for some coloring $\rho$. It follows that $\mathcal{K}' \not\models E(esc)$ iff some coloring $\rho$ is not extendable to a 3-coloring of the full graph $G$; hence deciding $\mathcal{K} \models E(esc)$ is $\Pi^p_2$-hard.

Like $\mathcal{K}$ in the proof of Proposition 11, also DKB $\mathcal{K}'$ is acyclic and its TBox is the same for each graph $G$, and thus along the same lines the result holds under the stated restrictions and under data complexity.

The hardness results for the other cases follow from the results on exception safe DKBs. □