Discrete Mathematics 344 (2021) 112158

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A best possible result for the square of a 2-block to be hamiltonian

Jan Ekstein [a], Herbert Fleischner [b]

[a] Department of Mathematics and European Centre of Excellence NTIS - New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Pilsen, Technická 8, 306 14 Plzeň, Czech Republic
[b] Institute of Logic and Computation, Algorithms and Complexity Group, Technical University of Vienna, Favoritenstrasse 9-11, 1040 Wien, EU, Austria

ARTICLE INFO

Article history:
Received 4 June 2019
Received in revised form 22 June 2020
Accepted 7 September 2020
Available online 18 September 2020

Keywords:
Square of graphs
Hamiltonian cycles

ABSTRACT

It is shown that for any choice of four different vertices $x_1, \ldots, x_4$ in a 2-block G of order $p > 3$, there is a hamiltonian cycle in G^2 containing four different edges x_iy_i of $E(G)$ for certain vertices y_i, $i = 1, 2, 3, 4$. This result is best possible.

1. Introduction

As for standard terminology, we refer to the book by Bondy and Murty, [2], and to the papers quoted in the references.

The *square* of a graph G, denoted G^2, is the graph obtained from G by joining any two nonadjacent vertices which have a common neighbor, by an edge. Fairly recent development in hamiltonian graph theory has shown a resurgence of interest in hamiltonian cycles and paths in the square of 2-connected graphs (which we call 2-blocks for short). In particular, short proofs have been found for two results of the second author of the present paper, [10,11]. And more recently, in [1] the authors develop algorithms which are linear in $|E(G)|$ and produce a hamiltonian cycle, a hamiltonian path joining arbitrary vertices u and v respectively, in G^2. Moreover, they develop an algorithm running in $O(|V(G)|^2)$ time and producing cycles of arbitrary length from 3 to $|V(G)|$.

Also very recently it was shown in [3] and [8] that a 2-block has the $\mathcal{F}_4$ property; that is, given vertices x_1, x_2, x_3, x_4 in the 2-block G, there is a hamiltonian path in G^2 joining x_1 and x_2 and traversing distinct edges x_3y_3 and x_4y_4 of G (see Theorem 7). The proof of this result is very long and is based on techniques developed by Fleischner in [5–7] and by Fleischner and Hobbs in [9]. It remains to be shown whether one can find a much shorter proof of this result. However, this result will be of importance in the proof of the main result of the current paper.

We start with a definition.

Definition 1. A graph G is said to have the $\mathcal{H}_k$ property if for any given vertices $x_1, \ldots, x_k$ there is a hamiltonian cycle in G^2 containing distinct edges $x_1y_1, \ldots, x_ky_k$ of G.

We note in passing that G having the $\mathcal{F}_4$ property implies that G has the $\mathcal{H}_3$ property; clearly, choose x_1, x_2, x_3 arbitrarily and a different x_4 adjacent to some x_1 for $i \in \{1, 2, 3\}$ in G, say $i = 1$. A hamiltonian path in G^2 joining x_1 and x_4 and containing edges x_2y_2 and x_3y_3 of G yields a hamiltonian cycle containing these two edges of G and x_1x_4 which lies also in G.

E-mail addresses: ekstein@kma.zcu.cz (J. Ekstein), fleischner@ac.tuwien.ac.at (H. Fleischner).

https://doi.org/10.1016/j.disc.2020.112158

The main result of this paper is the following.

Theorem 2. *Given a 2-block G on at least four vertices, then G has the $\mathcal{H}_4$ property, and there are 2-blocks of arbitrary order greater than 4 without the $\mathcal{H}_5$ property.*

This theorem and the $\mathcal{F}_4$ property of 2-blocks are key to describe the most general block-cut vertex structure a graph G may have in order to guarantee that G^2 is hamiltonian, hamiltonian connected, respectively. This will be done in follow-up papers.

Moreover Theorem 2 gives the positive answer to Conjecture 5.4 stated in [4] as an immediate corollary.

Corollary 3. *Let G be a connected graph such that its block-cutvertex graph $bc(G)$ is homeomorphic to a star in which the center c corresponds to a block B_c of G. If B_c contains at most 4 cutvertices, then G^2 is hamiltonian.*

2. Preliminaries

However, before proving Theorem 2 we mention several concepts and results which we need to make use of, and we prove a lemma.

A graph G is an edge-critical block, if $\kappa(G) = 2$ and $\kappa(G - e) = 1$ for any edge e of G. Let $D(G)$ be the set of edges uv where both $d_G(u) \geq 3$ and $d_G(v) \geq 3$. If $D(G) = \emptyset$, then every edge of G is incident to a vertex of degree 2; we call such a graph a *DT-graph*.

Theorem 4 (*[6]*)**.** *Let G be an edge-critical block. Then exactly one of the following two statements is true:*

(1) G is a DT-block.
(2) There is an edge f in $D(G)$ such that at least one of the endblocks of $G - f$ is a DT-block.

The basic result about hamiltonicity of the square of a 2-block is given by the following theorem.

Theorem 5 (*[7]*)**.** *Suppose v and w are two arbitrarily chosen vertices of a 2-block G. Then G^2 contains a hamiltonian cycle C such that the edges of C incident to v are in G and at least one of the edges of C incident to w is in G. Furthermore, if v and w are adjacent in G, then these are three different edges.*

Let $\mathrm{bc}(G)$ denote the block-cutvertex graph of G. Blocks corresponding to leaves of $\mathrm{bc}(G)$ are called *endblocks*. Note that a block in a graph G is either a 2-block or a bridge of G. The graph G is called *blockchain* if $\mathrm{bc}(G)$ is a path. Let G be a blockchain. We denote its blocks $B_1, B_2, \ldots, B_k$ and cutvertices $c_1, c_2, \ldots, c_{k-1}$ such that $c_i \in V(B_i) \cap V(B_{i+1})$, for $i = 1, 2, \ldots, k - 1$. A blockchain G is called *trivial*, if $E(\mathrm{bc}(G)) = \emptyset$, otherwise it is called *non-trivial*. Note that only B_1 and B_k are endblocks of a non-trivial blockchain G. *An inner block* is a block of G containing exactly 2 cutvertices. *An inner vertex* is a vertex in G which is not a cutvertex of G.

The first author proved in [4] the following theorem dealing hamiltonicity of the square of a blockchain graph.

Theorem 6 (*[4]*)**.** *Let G be a blockchain and let u_1, u_2 be arbitrary inner vertices which are contained in different endblocks of G.*

Then G^2 contains a hamiltonian cycle C such that, for $i = 1, 2$,

- *if u_i is contained in a 2-block, then both edges of C incident with u_i are in G, and*
- *if u_i is not contained in a 2-block, then exactly one edge of C incident with u_i is in G.*

Let G be a connected graph. By a *uv-path* we mean a path from u to v in G. If a uv-path is hamiltonian, we call it a *uv-hamiltonian path*. Let $A = \{x_1, x_2, \ldots, x_k\}$ be a set of k (≥ 3) distinct vertices in G. An x_1x_2-hamiltonian path in G^2 which contains $k - 2$ distinct edges $x_iy_i \in E(G)$, $i = 3, \ldots, k$, is said to be $\mathcal{F}_k$. A graph G is said to have the $\mathcal{F}_k$ property if, for any set $A = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, there is an $\mathcal{F}_k$ x_1x_2-hamiltonian path in G^2.

Theorem 7 (*[8]*)**.** *Let G be a 2-block. Then G has the $\mathcal{F}_4$ property.*

A graph G is said to have the strong $\mathcal{F}_3$ property if, for any set of 3 vertices $\{x_1, x_2, x_3\}$ in G, there is an x_1x_2-hamiltonian path in G^2 containing distinct edges $x_3z_3, x_iz_i \in E(G)$ for a given $i \in \{1, 2\}$. Such an x_1x_2-hamiltonian path in G^2 is called a strong $\mathcal{F}_3$ x_1x_2-hamiltonian path.

Theorem 8 (*[8]*)**.** *Every 2-block has the strong $\mathcal{F}_3$ property.*

The following lemma is frequently used in the proofs below.

Lemma 9. *Let G be a non-trivial blockchain. We choose*

- *$c_0 \in V(B_1)$, $c_k \in V(B_k)$ which are not cutvertices;*
- *$u_i \in V(B_i)$ (if any) which is not a cutvertex and $v_i \in V(B_i)$ such that $u_i \neq v_i$, $u_1 \neq c_0$ and $u_k \neq c_k$, for $i = 1, 2, \ldots, k$.*

Then G^2 contains a c_0c_k-hamiltonian path P such that there exist distinct edges u_iu_i' v_iv_i' $\in E(B_i) \cap E(P)$ (if u_i exists), $i = 1, 2, \dots, k$.

Proof. If B_i is 2-connected, then let P_i be an $\mathcal{F}_4$ $c_{i-1}c_i$-hamiltonian path in B_i^2 containing 2 distinct edges u_iu_i', $v_iv_i' \in E(B_i)$ for $v_i \notin \{c_{i-1}, c_i\}$ by Theorem 7; and let P_i be a strong $\mathcal{F}_3$ $c_{i-1}c_i$-hamiltonian path in B_i^2 containing 2 distinct edges u_iu_i', $v_iv_i' \in E(B_i)$ for $v_i \in \{c_{i-1}, c_i\}$ by Theorem 8, respectively.

If $B_i = c_{i-1}c_i$, then we set $P_i = B_i$. Note that in this case u_i does not exist and $v_i \in \{c_{i-1}, c_i\}$.

Then $P = \cup_{i=1}^{k} P_i$ is a c_0c_k-hamiltonian path in G^2 as required. □

The concept of EPS-graphs plays a central role in proofs of hamiltonicity in the square of a DT-graph (see [5]). We use this concept also in one part of the proof of Theorem 2. Let G be a graph. An *EPS-graph* is a spanning connected subgraph S of G which is the edge-disjoint union of an Eulerian graph E (which may be disconnected) and a linear forest P. For $S = E \cup P$, let $d_E(v)$, $d_P(v)$ denote the degree of v in E, P, respectively.

Fleischner and Hobbs introduced in [9] the concept of W-soundness of a cycle. Let W be a set of vertices of G. A cycle K is called *W-maximal* if $|V(K') \cap W| \leq |V(K) \cap W|$ for any cycle K' of G. Let K be a cycle of G and let W be a set of vertices of G. A blockchain P of $G - K$ is *a W-separated K-to-K blockchain based on vertex x* if a vertex of W is a cut vertex of P, both endblocks B and B' of P include vertices of K, $V(B) \cap V(K) = \{x\}$, no vertex of K is a cutvertex of P, and $(V(P) \cap V(K)) - \{x\} \subseteq V(B')$. For a given path $p = v_1, v_2, \dots, v_{n-1}, v_n$ we let $F(p) = v_1$, $L(p) = v_n$.

Definition 10. A cycle K in G is *W-sound* if it is W-maximal, $|W| = 5$ and the following hold:

(1) $|V(K) \cap W| \geq 4$; or
(2) $|V(K) \cap W| = 3$ and the following situation does not prevail; there are two W-separated K-to-K blockchains P and Q of $G - K$ based on a vertex w of W such that $V(P) \cap V(Q) = \{w\}$ and if p is a shortest path in P from w to a vertex of K different from w and q is the same for Q, then there is a subsequence $w, w', L(p), L(q), w'', w$ of K where w' and w'' are in $W - \{w\}$; or
(3) $|V(K) \cap W| = 2$ and the following situation does not prevail; there are three W-separated K-to-K blockchains P_1, P_2 and P_3 of $G - K$ based on a single vertex a of $V(K) - W$, such that $V(P_i) \cap V(P_j) = \{a\}$ whenever i and j are distinct elements of $\{1, 2, 3\}$, and if p_i is a shortest path in P_i from a to a vertex of K different from a for each $i \in \{1, 2, 3\}$, then there is a subsequence $a, w', L(p_1), L(p_2), L(p_3), w'', a$ of K where $\{w', w''\} = V(K) \cap W$.

We observe that Definition 10 is basically the content of Lemma 1 in [9]. That is, said lemma guarantees that for every choice $W \subseteq V(G)$ with $|W| = 5$ in a 2-block G of order at least 5, there is a W-sound cycle in G.

Theorem 11 (*[9]*)**.** *Let G be a 2-block and W a set of five distinct vertices in G, and let K be a W-sound cycle in G. Then there is an EPS-graph $S = E \cup P$ of G such that $K \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$.*

3. Proof of Theorem 2

Proof. First we prove that G has the $\mathcal{H}_4$ property. We proceed by contradiction supposing that $|V(G)| + |E(G)|$ is minimal. It follows that G is an edge-critical block and in particular $|V(G)| \geq 5$. We distinguish cases by the number of edges in $D(G)$. The reader is advised to draw figures where he/she deems it necessary to follow our case distinctions.

Case 1. $|D(G)| > 0$. By Theorem 4, let $f = x'x \in D(G)$ be an edge where both $d_G(x') \geq 3$ and $d_G(x) \geq 3$. Then $G - f$ is a blockchain and both endblocks B', B of $G - f$ are 2-blocks. Set $X = \{x_1, x_2, x_3, x_4\}$. Without loss of generality assume that $|X \cap (V(B) - y)| \leq 2$ (otherwise we consider B' instead of B); i.e., at most $x_1, x_2 \in V(B) - y$, say, where $x, y \in V(B)$ and y is a cutvertex of $G - f$. We distinguish the following 3 subcases.

Subcase 1.1: $|X \cap (V(B) - y)| = 2$; i.e., $x_1, x_2 \in V(B) - y$.

Then B^2 has an xy-hamiltonian path P_1 containing different edges x_1y_1, x_2y_2 of $E(G)$ for certain y_1, y_2 by Theorem 7 or by Theorem 8 if $x_1 = x$ or $x_2 = x$; and $(G - B)^2$ has an xy-hamiltonian path P_2 containing different edges x_3y_3, x_4y_4 of $E(G)$ for certain y_3, y_4 by Lemma 9. Now $P_1 \cup P_2$ is a required hamiltonian cycle in G^2, a contradiction. Note that $x_3, x_4 \in V(B') - y'$ where $y' \in V(B')$ is a cutvertex of $G - f$, otherwise we can use B' instead of B and x_3 or x_4 instead of x_1 or x_2 (see *Subcase 1.2* or *Subcase 1.3* below).

Subcase 1.2: $|X \cap (V(B) - y)| = 1$; i.e., $x_1 \in V(B) - y$ and $x_2 \notin V(B) - y$.

(1.2.1) Assume that x_2, x_3, x_4 are not inner vertices of G in the same block of $G - B$. We proceed very similar as in *Subcase 1.1*; we use only the strong $\mathcal{F}_3$ property in B, and $G - B$ is a non-trivial blockchain. Hence we can apply Lemma 9 except if $x = x_1$, some $x_i = y$ for $i \in \{2, 3, 4\}$, say $i = 2$, and x_3, x_4 are inner vertices in the same endblock of $G - B$ which also contains x_2.

If $x = x_1$, $x_2 = y$, and x_3, x_4 are inner vertices in the same endblock of $G - B$ which also contains x_2, then B^2 has an x_2x_1-hamiltonian path P_1 containing different edges x_2y_2, uv of $E(G)$ for certain y_2, u, v by Theorem 8, and $(G - B)^2$ has an x_2x_1-hamiltonian path P_2 containing different edges x_1x', x_3y_3, x_4y_4 of $E(G)$ for certain y_3, y_4 by Lemma 9. Again, $P_1 \cup P_2$ is a required hamiltonian cycle in G^2, a contradiction.

(1.2.2) Assume that x_2, x_3, x_4 are inner vertices of G in the same block B^* of $G - B$.

Clearly, B^2 contains a hamiltonian cycle H_B containing 3 different edges $y'y, x_1'x_1, x''x$ of $E(B)$ for certain vertices y', x_1', x'' by Theorem 7 (starting with a corresponding $\mathcal{F}_4$ $x''x$-hamiltonian path in B^2) if $x \neq x_1$, and $y'y, x_1'x, x''x$ of $E(B)$ for certain vertices y', x_1', x'' by Theorem 5 if $x = x_1$.

Let G_1 be the component of $G - B^* - xx'$ containing B and $y^* = V(B^*) \cap V(G_1)$. Note that G_1 is a trivial or non-trivial blockchain.

(a) If $y^* = y$, then $G_1 = B$ and we set $H_{G_1} = H_B$ (see above).

(b) If $y^* \neq y$, then either $G_1 - B = y^*y$ or $(G_1 - B)^2$ contains a hamiltonian cycle C containing edges $y_1^*y^*, y''y$ of $E(G_1 - B)$ for certain y_1^*, y'' by applying Theorem 5 or Theorem 6.

Now we set

$$H_{G_1} = (H_B - y'y) \cup y'y^*$$

and $y_1^* = y$ if $G_1 - B = y^*y$; and

$$H_{G_1} = (H_B \cup C - \{y'y, y''y\}) \cup y'y''$$

if $G_1 - B \neq y^*y$.

Note that the edge $y_1^*y^* \in E(G_1)$ is contained in H_{G_1} in both cases.

Clearly, $|V(B^*)| + |E(B^*)| < |V(G)| + |E(G)|$. Hence $(B^*)^2$ contains a hamiltonian cycle H_{B^*} containing four different edges $y_2^*y^*, x_2x_2', x_3x_3', x_4x_4'$ of $E(B^*)$ for certain vertices y_2^*, x_i', $i = 2, 3, 4$.

Let $z \in V(B^*)$ be the cutvertex of $G - x'x$ different from y^*.

(A) $x' = z$. Then

$$(H_{G_1} \cup H_{B^*} - \{y_2^*y^*, y_1^*y^*\}) \cup \{y_1^*y_2^*\}$$

is a required hamiltonian cycle in G^2 containing four different edges x_ix_i', of $E(G)$, $i = 1, 2, 3, 4$, a contradiction.

(B) $x' \neq z$

If $d_{G-B^*}(z) = 1$, then we set $G_2 = G - G_1 - B^* - z_1z$ where z_1 is the unique neighbor of z in $G - B^*$; otherwise we set $G_2 = G - G_1 - B^*$. Note that G_2 is a trivial or non-trivial blockchain and $G_2 = x'x$ is not possible because of $d_G(x') > 2$.

We apply Theorem 6 such that either $(G_2)^2$ contains a hamiltonian cycle H_{G_2} with $x'x \in E(H_{G_2})$ if $z \notin V(G_2)$, or $(G_2)^2$ contains a hamiltonian cycle H containing the edge $x'x$ and different edges z_1z, z_2z of G_1 for certain z_1, z_2 if $z \in V(G_2)$. In the latter case we set $H_{G_2} = (H - \{z_1z, z_2z\}) \cup z_1z_2$. Then

$$(H_{G_1} \cup H_{G_2} \cup H_{B^*} - \{y_2^*y^*, y_1^*y^*, x'x, x''x\}) \cup \{y_1^*y_2^*, x''x'\}$$

is again a hamiltonian cycle in G^2 containing four different edges x_ix_i' of $E(G)$, $i = 1, 2, 3, 4$, a contradiction.

Subcase 1.3: $|X \cap (V(B) - y)| = 0$; i.e., $x_1, x_2 \notin V(B) - y$.

Let G_1 be a graph which arises from G by replacing B with a path p of length 3, say $p = x, a, b, y$. Then $|V(G_1)| + |E(G_1)| < |V(G)| + |E(G)|$ since B is not a triangle because G is edge-critical. Hence $(G_1)^2$ contains a hamiltonian cycle H_1 containing four different edges x_iy_i of $E(G_1)$ for certain vertices y_i, $i = 1, 2, 3, 4$, and as many edges as possible of G_1.

In the following we shall proceed in a manner very similar to the proof in [6] that the square of a 2-block is hamiltonian. However, in order to avoid total dependence of the reader on the knowledge or study of [6], we shall describe and partially repeat the procedure employed in that paper. In particular, we shall quote the cases with the numbering of [6].

This yields the consideration of 13 cases on how the hamiltonian cycle H_1 traverses vertices of the path p. As in [6], Cases 3, Case 4, Case 12, and Case 13 are contradictory to the maximality of the number of edges of G_1 belonging to H_1; and Case 6 can be reduced to Case 10, Case 8 to Case 7, Case 10 to Case 9 and Case 11 to Case 5. Note that by the reductions we preserve the existence of the edges x_iy_i even if $x_i \in \{x', y\}$ for $i \in \{1, 2, 3, 4\}$.

The remaining 5 cases are (using the labeling of vertices x', x, a, b, y instead of x, w, a, b, v in [6]):

Case 1. $H_1 = \dots, x, a, b, y, \dots$

Case 2. $H_1 = \dots, x, a, b, y', \dots$

Case 5. $H_1 = \dots, x', a, b, x, \dots$

Case 7. $H_1 = \dots, x', a, y, \dots, y', b, x$

Case 9. $H_1 = \dots, x', a, y, b, x \dots$;

and $y'y$ is an edge of G.

In order to extend H_1 to H in G^2 in these five cases with H having the required property, one can proceed in the same way as it has been done in [6]. However, we deem it necessary to show explicitly that no problems arise under the stronger condition of this theorem (similarly as in [7]).

Case 1. By Theorem 8, B^2 has an xy-hamiltonian path P starting with an edge yy^* of $E(B)$ and containing an edge uv of B for certain vertices u, v. Replace in H_1 the path p with a hamiltonian path P and we get a hamiltonian cycle H as required.

Case 2. Take P as in Case 1 and replace in H_1 the path x, a, b, y' with $(P - yy^*) \cup y'y^*$ and again we get a hamiltonian cycle H as required. Note that H contains all edges of G belonging to H_1.

Case 5. By Theorem 5, B^2 contains a hamiltonian cycle H_B such that both edges of H_B incident to y (say yy^*, yy^{**}) are in B and at least one of the edges of H_B incident to x (say xx^*) is in B. We set

$$H^* = (H_B - \{yy^*, yy^{**}\}) \cup y^*y^{**}$$

which does not contain y, and replace in H_1 the path x', a, b, x with $(H^* - xx^*) \cup x'x^*$, thus obtaining a hamiltonian cycle H in G^2 which has the same behavior in all vertices of $G_1 - \{a, b\} \subset G$ as H_1.

Case 7. Take H_B as in Case 5 and replace in H_1 the path x', a, y with the path $P_1 \cup x^*x'$ where $P_1 \subset H_B$ is the path from y to x^* and does not contain x; and replace in H_1 the path y', b, x with the path $P_2 \cup y't$ where $t \in \{y^*, y^{**}\}$ and $P_2 \subset H_B$ is the path from x to t and does not contain any of y, x^*. Again we get a hamiltonian cycle H as required.

Case 9. Take H_B as in Case 5 and replace in H_1 the path x', a, y, b, x with $(H_B - xx^*) \cup x'x^*$, thus obtaining a hamiltonian cycle H in G^2 which has the same behavior in all vertices of $G_1 - \{a, b, y\} \subset G$ as H_1 and both edges of H incident to y are in G.

In all cases we obtained a hamiltonian cycle H in G^2 containing four different edges x_ix_i', of $E(G)$ (in most cases we have $x_i' = y_i$; see the first paragraph of this subcase 1.3), $i = 1, 2, 3, 4$, a contradiction.

Case 2. $|D(G)| = 0$. That is, G is a DT-graph.

(a) Suppose $N(x_i) \subseteq V_2(G)$ for every $i = 1, 2, 3, 4$.

Set $W' = \{x_1, x_2, x_3, x_4\}$ and let K be a W'-maximal cycle in G. Observe that $|V(K)| \geq 4$ since an edge-critical block on at least 4 vertices cannot contain a triangle.

If $|W' \cap V(K)| = 4$, then we choose x_5 arbitrary in $V(G) - W'$. If $|W' \cap V(K)| = 3$, then we choose x_5 arbitrary in $V(K) - W'$. If $|W' \cap V(K)| = 2$, then we choose an arbitrary 2-valent vertex x_5 in $V(K) - W'$ which exists because all neighbors of x_i are 2-valent.

We set $W = W' \cup \{x_5\}$. Then K is W-sound in G unless $|W \cap V(K)| = 3$ and forbidden situation (2) in Definition 10 arises. That is, without loss of generality $x_1, x_2 \in V(K)$ and there exist W-separated K-to-K blockchains P, Q based on x_i, $i \in \{1, 2\}$, $P \cap Q = x_i$, and paths p, q in P, Q, respectively, such that there is a subsequence $x_i, w', L(p), L(q), w'', x_i$, where $\{w', w''\} = \{x_{3-i}, x_5\}$ and $x_3, x_4 \in V(p) \cup V(q)$. Then there is a cycle K' containing x_i, x_3, x_4, a contradiction to the W'-maximality of K.

By Theorem 11, G contains an EPS-graph $S = E \cup P$ such that $K \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$. If there is no adjacent pair x_i, x_j for $i, j \in \{1, 2, 3, 4\}$, we use S and an algorithm in [5] to obtain a hamiltonian cycle in G^2 with the required properties, a contradiction. However, if there is an adjacent pair, say x_1, x_2, then $d_G(x_1) = d_G(x_2) = 2$ and $d_P(x_1) = d_P(x_2) = 0$ and we can proceed with the cycle K containing x_1, x_2, x_3 to obtain a required hamiltonian cycle in G^2 as before, a contradiction.

(b) Without loss of generality suppose that $N(x_4) \nsubseteq V_2(G)$.

Hence $\deg_G(x_4) = 2$. Let $P_4 = y_4x_4z_1...z_k$ be a unique path in G such that $d_G(y_4) > 2$, $d_G(z_k) > 2$ and $d_G(z_i) = 2$, for $i = 1, 2, \ldots, k-1$. We set $G^- = G - \{x_4, z_1, \ldots, z_{k-1}\}$, where $z_0 = x_4$ if $k = 1$.

(b1) Assume that G^- is 2-connected.

If $x_i \in V(G^-) - \{y_4, z_k\}$ for $i = 1, 2, 3$, then $|V(G)| + |E(G)| > |V(G^-)| + |E(G^-)|$ and hence $(G^-)^2$ has a hamiltonian cycle H^- containing different edges $x_iy_i, z_kw_4 \in E(G)$, $i = 1, 2, 3$. It is easy to see that we can extend H^- to a hamiltonian cycle H in G^2 such that H contains edges x_iy_i, x_4z_1, for $i = 1, 2, 3$, a contradiction.

Suppose $x_3 \notin V(G^-) - \{y_4, z_k\}$. If $\{\{x_1, x_2, x_3\} \cap \{y_4, z_k\} \neq \emptyset\}$, then without loss of generality $x_3 \in \{y_4, z_k\}$. By Theorem 7 or Theorem 8, $(G^-)^2$ contains a y_4z_k-hamiltonian path P^- and P^- contains distinct edges x_iy_i of G if $x_i \in V(G^-)$ for $i = 1, 2$. Then $P^- \cup P_4$ is a hamiltonian cycle in G^2 with the required properties, a contradiction.

(b2) Assume that G^- is not 2-connected.

Then G^- is a non-trivial blockchain with y_4, z_k in distinct endblocks and y_4, z_k are not cutvertices.

Assume not all x_1, x_2, x_3 are inner vertices in the same block. Then we apply Lemma 9 to get a y_4z_k-hamiltonian path P^- in $(G^-)^2$ with distinct edges $x_iy_i \in E(G^-)$, $i = 1, 2, 3$. Note than x_i could be y_4 or z_k. Then again $P^- \cup P_4$ is a hamiltonian cycle in G^2 with the required properties, a contradiction.

Now assume that x_1, x_2, x_3 are inner vertices in the same block B. Then there exists an end block B^* of G^- such that $x_i \notin V(B^*)$, $i = 1, 2, 3$. A graph G' arises from G by the replacement of B^* by a path p of length 3. Hence $|V(G)| + |E(G)| > |V(G')| + |E(G')|$ and we denote by H' a hamiltonian cycle in $(G')^2$ containing edges x_iw_i, $i = 1, 2, 3, 4$, and as many edges of G' as possible.

We proceed in the same manner as in Subcase 1.3 (note that in this case none of x_i, $i = 1, 2, 3, 4$, is on p) to get a hamiltonian cycle in G^2 with required properties, a contradiction.

Finally we want to show that Theorem 2 is best possible, i.e., we construct an infinite family of graphs which do not satisfy the $\mathcal{H}_5$ property. For this purpose start with an arbitrary 2-block G and fix different vertices $x_1, x_2 \in V(G)$.

Define

$$H = G \cup \{y_1, y_2, \ldots, y_t; t \geq 3\} \cup \{x_iy_j : 1 \leq i \leq 2, 1 \leq j \leq t\},$$

where $\{y_1, \ldots, y_t\} \cap V(G) = \emptyset$. Then H is a 2-block. However, H does not have the $\mathcal{H}_5$ property: indeed, there is no hamiltonian cycle C in H^2 containing edges of H incident to x_1, x_2, y_1, y_2, y_3 because of the neighbors of y_1, y_2, y_3 in H which are x_1 and x_2 only; that is x_1 or x_2 would be incident to three edges of $C \cap H$, which is impossible. □

4. Conclusion

We introduced the concept of the $\mathcal{H}_k$ property and proved that every 2-block has the $\mathcal{H}_4$ property but not the $\mathcal{H}_5$ property in general. Similarly in [8] it is proved that every 2-block has the $\mathcal{F}_4$ property but not the $\mathcal{F}_5$ property in general. Moreover, a 2-block G having the $\mathcal{F}_k$ property implies that G has the $\mathcal{H}_{k-1}$ property for $k = 3, 4, \ldots$. Hence we conclude that Theorems 2 and 7 are best possible with respect to hamiltonicity and hamiltonian connectedness in the square of a 2-block.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This publication was supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports, and by FWF, Austria project P27615-N25.

References

[1] S. Alstrup, A. Georgakopoulos, E. Rotenberg, C. Thomassen, Carsten, A Hamiltonian cycle in the square of a 2-connected graph in linear time, in: Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, Philadelphia, PA, 2018, pp. 1645–1649, (English summary).
[2] J.A. Bondy, U.S.R. Murty, Graph Theory, in: Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008.
[3] G.L. Chia, J. Ekstein, H. Fleischner, Revisiting the Hamiltonian theme in the square of a block: The case of DT-graphs, J. Combin. 9 (1) (2018) 119–161.
[4] J. Ekstein, Hamiltonian cycles in the square of a graph, Electron. J. Combin. 18 (2011) #P203.
[5] H. Fleischner, On spanning subgraphs of a connected bridgeless graph and their application to DT-graphs, J. Combin. Theory 16 (1974) 17–28.
[6] H. Fleischner, The square of every two-connected graph is hamiltonian, J. Combin. Theory 16 (1974) 29–34.
[7] H. Fleischner, In the square of graphs, Hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concept, Monatsh. Math. 82 (1976) 125–149.
[8] H. Fleischner, G.L. Chia, Revisiting the hamiltonian theme in the square of a block: The general case, J. Combin. 10 (1) (2019) 163–201.
[9] H. Fleischner, A.M. Hobbs, Hamiltonian total graphs, Math. Nachr. 68 (1975) 59–82.
[10] A. Georgakopoulos, A short proof of fleischner's theorem, Discrete Math. 309 (2009) 6632–6634.
[11] J. Müttel, D. Rautenbach, A short proof of the versatile version of fleischner's theorem, Discrete Math. 313 (2013) 1929–1933.