



Nonlocal cross-diffusion systems for multi-species populations and networks[☆]



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ABSTRACT

Nonlocal cross-diffusion systems on the torus, arising in population dynamics and neuroscience, are analyzed. The global existence of weak solutions, the weak–strong uniqueness, and the localization limit are proved. The kernels are assumed to be in detailed balance. The proofs are based on entropy estimates coming from Shannon-type and Rao-type entropies, while the weak–strong uniqueness result follows from the relative entropy method. The existence and uniqueness theorems hold for nondifferentiable, only integrable kernels. The associated local cross-diffusion system, derived in the localization limit, is also discussed.

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1. Introduction

In this paper, we consider the following nonlocal cross-diffusion system:

$$\partial_t u_i - \sigma \Delta u_i = \operatorname{div}(u_i \nabla p_i[u]), \quad t > 0, \quad u_i(0) = u_i^0 \quad \text{in } \mathbb{T}^d, \quad i = 1, \dots, n, \quad (1)$$

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where $\sigma > 0$ is the diffusion coefficient, \mathbb{T}^d is the d -dimensional torus ($d \geq 1$) and $p_i[u]$ is a nonlocal operator given by

$$p_i[u](x) = \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}(x-y)u_j(y)dy, \quad i = 1, \dots, n, \tag{2}$$

$K_{ij} : \mathbb{T}^d \rightarrow \mathbb{R}$ are the kernel functions (extended periodically to \mathbb{R}^d), and $u = (u_1, \dots, u_n)$ is the solution vector.

Following [33,43], in the case $K_{ij} = a_{ij}K$ with $a_{ij} \in \mathbb{R}$, this model describes the dynamics of a population with n species. Here, each species can detect other species over a spatial neighborhood by nonlocal sensing, described by the kernel function K . The coefficient a_{ij} is a measure of the strength of attraction (if $a_{ij} < 0$) or repulsion (if $a_{ij} > 0$) of the i th species to/from the j th species. A typical choice of K is the characteristic function 1_B of a ball B centered at the origin. The authors of [33] proved the local existence of a unique (strong) solution to (1)–(2) for $d \geq 1$ under the condition that K is twice differentiable, and this solution can be extended globally in one space dimension. However, the condition on K excludes the case $K = 1_B$. One objective of this paper is to extend the results of [33] by proving the global existence of weak solutions to (1)–(2) and a weak–strong uniqueness result in any space dimension and for nondifferentiable kernels K_{ij} . As in [33], we consider Eqs. (1) on the torus; see Remark 9 for the whole space case or bounded domains.

Another motivation of the present research comes from the work [16], where the model (1)–(2) was rigorously derived from interacting many-particle systems in a mean-field-type limit. As a by-product of the limit, the local existence of smooth solutions to (1)–(2) could be shown under the assumption that K_{ij} is smooth. Under the same condition, the so-called localization limit was proved, i.e., if K_{ij} converges to the delta distribution times some factor a_{ij} , the solution to the nonlocal system (1)–(2) converges to a solution to the model (1) with

$$p_i[u] = \sum_{j=1}^n a_{ij}u_j, \quad i = 1, \dots, n. \tag{3}$$

We note that the local system was first introduced in [31] in the case of two species. In this paper, we generalize the results of [16] by imposing “minimal” conditions on the initial datum u^0 and the kernels K_{ij} .

A third motivation comes from neuroscience. Indeed, following [4,32], we are interested in the study of deterministic nonlocal models of the form (1)–(2) obtained as the mean-field limit of stochastic systems describing the evolution of the states of neurons belonging to different populations. When the number of neurons becomes very large, the solutions of the generalized Hodgkin–Huxley model of [4] can be described in the mean-field limit by a probability distribution u_i for the i th species, which solves the McKean–Vlasov–Fokker–Planck equation of the type

$$\partial_t u_i = \sigma \Delta u_i + \operatorname{div} \sum_{j=1}^n \int_{\mathbb{T}^d} M_{ij}(x,y)u_i(x)u_j(y)dx dy, \quad i = 1, \dots, n, \tag{4}$$

where we simplified the diffusion part involving σ . In neural network theory, the kernel function $M_{ij}(x,y)$ describes the weight of a connection between the node x associated to species i and node y associated to species j . In the present work, we simplify the problem further by assuming that the interaction kernels M_{ij} have the special form $M_{ij}(x,y) = \nabla K_{ij}(x-y)$. Our main objective is to advance the theory of nonlocal cross-diffusion systems. This theoretical study can allow us in the future to design and analyze efficient numerical schemes for the discretization of (1)–(2).

Most nonlocal models studied in the literature describe a single species. A simple example is $\partial_t u = \operatorname{div}(uv)$ with $v = \nabla(K * u)$. This corresponds to the mass continuity equation for the density u with a nonlocal velocity v . An L^p theory for this equation was provided in [7], while the Wasserstein gradient-flow structure was explored in [13]. In the machine learning context, the equation can be seen as the mean-field limit of infinitely many hidden network units [40,46]. Besides, in addition to the cited papers, there exist some works

dealing with the existence of solutions to multispecies nonlocal systems of the form (1)–(2). Indeed, for two species and symmetrizable cross-interaction potentials (i.e. $K_{12} = \alpha K_{21}$ for some $\alpha > 0$) without diffusion $\sigma = 0$, a complete existence and uniqueness theory for measure solutions to (1)–(2) in the whole space with smooth convolution kernels was established in [24] using the Wasserstein gradient-flow theory. A nonlocal system with size exclusion was analyzed in [6], using entropy methods. In [31], a nonlocal version of the Shigesada–Kawasaki–Teramoto (SKT) cross-diffusion system, where the diffusion operator is replaced by an integral diffusion operator, was analyzed. Finally, closer to our study, in [26], the authors show the existence of weak solutions to a nonlocal version of the SKT system, where the nonlocalities are similar to the ones considered in this paper. Assuming some regularity on the convolution kernels, their proof is based on the so-called duality method [23,39]. They also proved a localization limit result.

All these works, except [26], are concerned with two-species models. Compared to previous results, we allow for an arbitrary number of species and nondifferentiable kernel functions. In the following section, we explain our approach in detail.

1.1. Description of our approach

The mathematical difficulties are the cross-diffusion terms and the nonlocality, which exclude the application of standard techniques like maximum principles and regularity theory. For instance, it is well known that nonlocal diffusion operators generally do not possess regularizing effects on the solution [3]. The key of our analysis is the observation that the nonlocal system possesses, like the associated local one, two entropies, namely the Shannon-type entropy H_1 [45] and the Rao-type entropy H_2 [44],

$$H_1(u) = \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i (\log u_i - 1) dx,$$

$$H_2(u) = \frac{1}{2} \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) u_i(x) u_j(y) dx dy,$$

if the kernels are in detailed balance and positive definite in the sense specified below and for suitable numbers $\pi_i > 0$. A formal computation that is made rigorous below shows that the following entropy inequalities hold:

$$\frac{dH_1}{dt} + 4\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i |\nabla \sqrt{u_i}|^2 dx = - \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) \nabla u_i(x) \cdot \nabla u_j(y) dx dy, \tag{5}$$

$$\frac{dH_2}{dt} + \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i |\nabla p_i[u]|^2 dx = -\sigma \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) \nabla u_i(x) \cdot \nabla u_j(y) dx dy. \tag{6}$$

These computations are valid if K_{ij} is in detailed balance, which means that there exist $\pi_1, \dots, \pi_n > 0$ such that

$$\pi_i K_{ij}(x - y) = \pi_j K_{ji}(y - x) \quad \text{for all } i, j = 1, \dots, n, \ x, y \in \mathbb{T}^d. \tag{7}$$

We recognize these identities as a generalized detailed-balance condition for the Markov chain associated to $(K_{ij}(x - y))$ (for fixed $x - y$), and in this case, (π_1, \dots, π_n) is the corresponding reversible measure. The functionals H_1 and H_2 are Lyapunov functionals if $(\pi_i K_{ij})$ is positive definite in the sense

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) v_i(x) v_j(y) dx dy \geq 0 \quad \text{for all } v_i, v_j \in L^2(\mathbb{T}^d). \tag{8}$$

This condition generalizes the usual definition of positive definite kernels to the multi-species case [11]. Examples of kernels that satisfy (7) and (8) are given in Remark 1. The positive definiteness condition

for kernels is essential in reproducing kernel Hilbert theory [42]. From a PDE viewpoint, this condition can be replaced by the smoothness assumption $K_{ij} \in C^1$; see, e.g., [24,33]. Because of the nonlocality, we cannot conclude $L^2(\mathbb{T}^d)$ estimates for u_i and ∇u_i like in the local case; see [38] and Appendix B. We deduce from (5) only bounds for $u_i \log u_i$ in $L^1(\mathbb{T}^d)$ and $\sqrt{u_i}$ in $H^1(\mathbb{T}^d)$.

These bounds are not sufficient to pass to the limit in the approximate problem. In particular, we cannot identify the limit of the product $u_i \nabla p_i[u]$, since u_i and $\nabla p_i[u]$ are elements in larger spaces than $L^2(\mathbb{T}^d)$. We solve this issue by exploiting the uniform $L^2(\mathbb{T}^d)$ bound for $\sqrt{u_i} \nabla p_i[u]$ from (6) and prove a ‘‘compensated compactness’’ lemma (see Lemma 13 in Appendix A): If $u_\varepsilon \rightarrow u$ strongly in $L^p(\mathbb{T}^d)$, $v_\varepsilon \rightharpoonup v$ weakly in $L^p(\mathbb{T}^d)$, and $u_\varepsilon v_\varepsilon \rightharpoonup w$ weakly in $L^p(\mathbb{T}^d)$ for some $1 < p < 2$ then $uv = w$. The estimates from (5)–(6) are the key for the proof of the global existence of weak solutions to (1)–(2).

As a second result, we prove the weak–strong uniqueness of solutions, i.e., if u is a weak solution to (1)–(2) satisfying $u_i \in L^2(0, T; H^1(\mathbb{T}^d))$ and if v is a ‘‘strong’’ solution to this problem with the same initial data, then $u(t) = v(t)$ for a.e. $t \geq 0$. The proof uses the relative entropy

$$H(u|v) = \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i (u_i(\log u_i - 1) - u_i \log v_i + v_i) dx,$$

a variant of which was used in [30] for reaction–diffusion systems in the context of renormalized solutions and later extended to Shigesada–Kawasaki–Teramoto systems [19]. The recent work [34] generalizes this approach to more general Shigesada–Kawasaki–Teramoto as well as energy–reaction–diffusion systems. Originally, the relative entropy method was devised for conservation laws to estimate the L^2 distance between two solutions [22,27]. Up to our knowledge, we apply these techniques for the first time to nonlocal cross-diffusion systems. The idea is to differentiate $H(u|v)$ and to derive the inequality

$$H(u(t)|v(t)) \leq C \sum_{i=1}^n \int_0^t \|u_i - v_i\|_{L^1(\mathbb{T}^d)}^2 ds \quad \text{for } t > 0.$$

The Csiszár–Kullback–Pinsker inequality [37, Theorem A.2] allows us to estimate the relative entropy from below by $\|u_i(t) - v_i(t)\|_{L^1(\mathbb{T}^d)}^2$, up to some factor. Then Gronwall’s lemma implies that $u_i(t) = v_i(t)$ for a.e. $t > 0$. The application of this inequality is different from the proof in [19,30,34], where the relative entropy is estimated from below by $|u_i - v_i|^2$ on the set $\{u_i \leq K\}$. The difference originates from the nonlocal terms. Indeed, if K_{ij} is bounded,

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (u_i - v_i)(x) (u_j - v_j)(y) dx dy \leq C \sum_{i=1}^n \left(\int_{\mathbb{T}^d} |u_i - v_i| dx \right)^2,$$

leading to an estimate in $L^1(\mathbb{T}^d)$. In the local case, the associated estimate yields an $L^2(\mathbb{T}^d)$ estimate:

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d} \pi_i a_{ij} (u_i - v_i)(x) (u_j - v_j)(x) dx \leq C \sum_{i=1}^n \int_{\mathbb{T}^d} |u_i - v_i|^2 dx.$$

As the densities u_i may be only nonnegative, we cannot use $\pi_i \log u_i$ as a test function in (1) to compute $dH(u|v)/dt$. This issue is overcome by regularizing the entropy by using $\log(u_i + \varepsilon)$ for some $\varepsilon > 0$ as a test function and then to pass to the limit $\varepsilon \rightarrow 0$.

We observe that the uniqueness of weak solutions to cross-diffusion systems is a delicate task, and there are only few results in the literature. Most of the results are based on the fact that the total density $\sum_{i=1}^n u_i$ satisfies a simpler equation for which uniqueness can be shown; see [6,18]. A duality method for a nonlocal version of the Shigesada–Kawasaki–Teramoto system was used in [31]. In [5], a weak–strong uniqueness result on a cross-diffusion system, based on L^2 estimates, was shown.

The bounds obtained in the proof of our existence result are independent of the kernels, such that we can perform the localization limit, our third main result. For this, we assume that $K_{ij} = B_{ij}^\varepsilon \rightarrow a_{ij}\delta_0$ as $\varepsilon \rightarrow 0$ in the sense of distributions, where δ_0 is the Dirac delta distribution. Then, if u^ε is a weak solution to (1)–(2), we prove that $u_i^\varepsilon \rightarrow u_i$ strongly in $L^1(\mathbb{T}^d \times (0, T))$, and the limit u solves the local system (1) and (3). As a by-product, we obtain the global existence of weak solutions to this problem; see Appendix B for the precise statement.

We summarize our main results:

- global existence of weak solutions to the nonlocal system (1)–(2) for nondifferentiable positive definite kernels in detailed balance;
- weak–strong uniqueness of solutions to the nonlocal system;
- localization limit to the local system (1) and (3).

The papers [14,24,25] analyze nonlocal cross-diffusion systems for two species similar to our model. In [14], the steady states are characterized and numerical simulations are presented, while the works [24,25] prove the existence of weak measure solutions under a global Lipschitz condition on K_{ij} and ∇K_{ij} . The originality of the present paper is that we impose only integrability conditions on K_{ij} and that we have (slightly) weakened positive definiteness to detailed balance. The weak–strong uniqueness result and the localization limit are also new in this context.

The paper is organized as follows. Our hypotheses and main results are made precise in Section 2. The global existence of weak solutions to the nonlocal system and some regularity results are proved in Section 3. The weak–strong uniqueness result is shown in Section 4. In Section 5, the localization limit, based on the a priori estimates of Section 3, is performed. Finally, we collect some auxiliary lemmas in Appendix A and state a global existence result for the local system (1) and (3) in Appendix B.

2. Main results

We collect the main theorems which are proved in the subsequent sections. We impose the following hypotheses:

- (H1) Data: Let $d \geq 1$, $T > 0$, $\sigma > 0$, and $u^0 \in L^2(\mathbb{T}^d)$ satisfies $u_i^0 \geq 0$ in \mathbb{T}^d , $i = 1, \dots, n$.
- (H2) Regularity: $K_{ij} \in L^s(\mathbb{T}^d)$ for $i, j = 1, \dots, n$, where $s = d/2$ if $d > 2$, $s > 1$ if $d = 2$, and $s = 1$ if $d = 1$.
- (H3) Detailed balance: There exist $\pi_1, \dots, \pi_n > 0$ such that $\pi_i K_{ij}(x - y) = \pi_j K_{ji}(y - x)$ for all $i, j = 1, \dots, n$ and for a.e. $x, y \in \mathbb{T}^d$.
- (H4) Positive definiteness: For all $v_1, \dots, v_n \in L^2(\mathbb{T}^d)$, it holds that

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) v_i(x) v_j(y) dx dy \geq 0.$$

We need the same diffusivity σ for all species, since otherwise we cannot prove that the Rao-type functional H_2 is a Lyapunov functional. The reason is the mixing of the species in the definition of H_2 . The regularity $K_{ij} \in L^s(\mathbb{T}^d)$ of the interaction kernels is required to prove the weak convergence of $(u_i^\delta \nabla p_i[u^\delta])$ in $L^1(\mathbb{T}^d)$, where (u^δ) is an approximate sequence. More precisely, the regularity for K_{ij} implies that $\nabla p_i[u^\delta] \rightharpoonup \nabla p_i[u]$ weakly in $L^d(\mathbb{T}^d)$ (if $d \geq 3$) and, as a consequence of the entropy estimate, we have $u_i^\delta \rightarrow u_i$ strongly in $L^r(\mathbb{T}^d)$ for $r < d/(d - 2)$. Thus, $u_i^\delta \nabla p_i[u^\delta] \rightharpoonup u_i \nabla p_i[u]$ weakly in $L^\rho(\mathbb{T}^d)$ for $1 \leq \rho < d/(d - 1)$.

Remark 1 (*Kernels Satisfying Hypotheses (H2)–(H4)*). Kernels satisfying Hypothesis (H4) with $n = 1$ can be characterized by Mercer’s theorem [11,41]. A simple example satisfying Hypotheses (H2)–(H4) is $K_{ij} = a_{ij}1_B$ for $i, j = 1, \dots, n$, where 1_B is the characteristic function of the ball B around the origin and there exist $\pi_1, \dots, \pi_n > 0$ such that $(\pi_i a_{ij}) \in \mathbb{R}^{n \times n}$ is symmetric and positive definite.

Another example that satisfies Hypotheses (H2)–(H4) is given by the Gaussian kernel $B(|x - y|) = (2\pi)^{-d/2} \exp(-|x - y|^2/2)$. We define for $i, j = 1, \dots, n$ and $x, y \in \mathbb{R}^d$,

$$K_{ij}(x - y) = B_{ij}^\varepsilon(x - y) := \frac{a_{ij}}{(2\pi\varepsilon^2)^{d/2}} \exp\left(-\frac{|x - y|^2}{2\varepsilon^2}\right),$$

where $\varepsilon > 0$ and $a_{ij} \geq 0$ are such that the matrix $(\pi_i a_{ij})$ is symmetric and positive definite for some $\pi_i > 0$. Thus, Hypothesis (H3) holds. Hypothesis (H4) can be verified as follows. The identity

$$\frac{e^{-|x-y|^2/(2\varepsilon^2)}}{(2\pi\varepsilon^2)^{d/2}} = \int_{\mathbb{R}^d} \frac{e^{-|x-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}} \frac{e^{-|y-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}} dz,$$

shows that

$$\begin{aligned} & \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) v_i(x) v_j(y) dx dy = \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i a_{ij} \frac{e^{-|x-y|^2/(2\varepsilon^2)}}{(2\pi\varepsilon^2)^{d/2}} v_i(x) v_j(y) dx dy \\ & = \sum_{i,j=1}^n \pi_i a_{ij} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \left(\frac{e^{-|x-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}} v_i(x) \right) dx \int_{\mathbb{T}^d} \left(\frac{e^{-|y-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}} v_j(y) \right) dy dz \\ & \geq \frac{\alpha}{(\pi\varepsilon^2)^d} \sum_{i=1}^n \int_{\mathbb{R}^d} \left(\int_{\mathbb{T}^d} e^{-|x-z|^2/\varepsilon^2} v_i(x) dx \right)^2 dz \geq 0, \end{aligned}$$

where $\alpha > 0$ is the smallest eigenvalue of $(\pi_i a_{ij})$. This proves the positive definiteness of K_{ij} . Note that $B_{ij}^\varepsilon \rightarrow a_{ij} \delta_0$ as $\varepsilon \rightarrow 0$ in the sense of distributions.

We can construct further examples from the Gaussian kernel. For instance,

$$K_{ij}(x - y) = \frac{a_{ij}}{1 + |x - y|^2}, \quad i, j = 1, \dots, n, \quad x, y \in \mathbb{R}^d,$$

satisfies Hypothesis (H4), since

$$\frac{1}{1 + |x - y|^2} = \int_0^\infty e^{-s(1 + |x - y|^2)} ds,$$

and $B(x - y) = \exp(-s(1 + |x - y|^2))$ is positive definite. \square

We call $u = (u_1, \dots, u_n)$ a *weak solution* to (1)–(2) if for all test functions $\phi_i \in L^{d+2}(0, T; W^{1,d+2}(\mathbb{T}^d))$, $i = 1, \dots, n$, it holds that

$$\int_0^T \langle \partial_t u_i, \phi_i \rangle dt + \sigma \int_0^T \int_{\mathbb{T}^d} \nabla u_i \cdot \nabla \phi_i dx dt = - \int_0^T \int_{\mathbb{T}^d} u_i \nabla p_i[u] \cdot \nabla \phi_i dx dt, \tag{9}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $W^{1,d+2}(\mathbb{T}^d)'$ and $W^{1,d+2}(\mathbb{T}^d)$, and the initial datum $u_i(0) = u_i^0$ is satisfied in the sense of $W^{1,d+2}(\mathbb{T}^d)'$.

First, we show the global existence of weak solutions. Let $Q_T = \mathbb{T}^d \times (0, T)$.

Theorem 2 (Global Existence). *Let Hypotheses (H1)–(H4) hold. Then there exists a global weak solution $u = (u_1, \dots, u_n)$ to (1)–(2) satisfying $u_i \geq 0$ in Q_T and*

$$\begin{aligned} u_i^{1/2} & \in L^2(0, T; H^1(\mathbb{T}^d)), \quad u_i \in L^{1+2/d}(Q_T) \cap L^q(0, T; W^{1,q}(\mathbb{T}^d)), \\ \partial_t u_i & \in L^q(0, T; W^{-1,q}(\mathbb{T}^d)), \quad u_i \nabla p_i[u] \in L^q(Q_T), \end{aligned} \tag{10}$$

where $q = (d + 2)/(d + 1)$ and $i = 1, \dots, n$. The initial datum in (1) is satisfied in the sense of $W^{-1,q}(\mathbb{T}^d) := W^{1,d+2}(\mathbb{T}^d)'$. Moreover, the following entropy inequalities hold:

$$H_1(u(t)) + 4\sigma \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i |\nabla u_i^{1/2}|^2 dx ds \leq H_1(u^0), \tag{11}$$

$$H_2(u(t)) + \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i u_i |\nabla p_i[u]|^2 dx ds \leq H_2(u^0). \tag{12}$$

Unfortunately, we cannot treat vanishing diffusion $\sigma = 0$, since this would not allow us to derive suitable gradient bounds. However, we can allow for arbitrarily small $\sigma > 0$, which means that cross-diffusion may dominate diffusion, or in other words, the cross-diffusion term is generally *not* just a perturbation.

Imposing more regularity on the kernel functions, we can derive $H^1(\mathbb{T}^d)$ regularity for u_i , which is needed for the weak–strong uniqueness result.

Proposition 3 (Regularity). *Let Hypotheses (H1)–(H4) hold and let $\nabla K_{ij} \in L^{d+2}(\mathbb{T}^d)$ for $i, j = 1, \dots, n$. Then there exists a weak solution $u = (u_1, \dots, u_n)$ to (1)–(2) satisfying $u_i \geq 0$ in \mathbb{T}^d and*

$$u_i \in L^2(0, T; H^1(\mathbb{T}^d)), \quad \partial_t u_i \in L^2(0, T; H^{-1}(\mathbb{T}^d)), \quad \nabla p_i[u] \in L^\infty(0, T; L^\infty(\mathbb{T}^d)).$$

Moreover, if additionally $\nabla K_{ij}, \Delta K_{ij} \in L^\infty(\mathbb{T}^d)$ and $m_0 \leq u_i^0 \leq M_0$ in \mathbb{T}^d then $0 < m_0 e^{-\lambda t} \leq u_i(t) \leq M_0 e^{\lambda t}$ in \mathbb{T}^d for $t > 0$, where $\lambda > 0$ depends on ΔK_{ij} and u^0 .

The proof of the $H^1(\mathbb{T}^d)$ regularity is based on standard L^2 estimates if $\nabla K_{ij} \in L^\infty(\mathbb{T}^d)$. The difficulty is the reduced regularity $\nabla K_{ij} \in L^{d+2}(\mathbb{T}^d)$, which requires some care. Indeed, the test function u_i in the weak formulation of (1) leads to a cubic term, which is reduced to a subquadratic term for ∇u_i by combining the Gagliardo–Nirenberg inequality and the uniform $L^1(\mathbb{T}^d)$ bound for u_i .

Similar lower and upper bounds as in Proposition 3 were obtained in [26] with a different proof. Since the L^∞ bounds depend on the derivatives of K_{ij} , they do not carry over in the localization limit to the local system. In fact, it is an open problem whether the local system (1) and (3) possesses *bounded* weak solutions. The proposition also holds for kernel functions $K_{ij}(x, y)$ that are used in neural network theory; see Remark 10.

Theorem 4 (Weak–strong Uniqueness). *Let $K_{ij} \in L^\infty(\mathbb{R}^d)$ and $c \leq u_i^0 \leq C$ in \mathbb{T}^d for $i, j = 1, \dots, n$ with constants $0 < c \leq C < \infty$. Let u be a nonnegative weak solution to (1)–(2) satisfying (10) as well as $u_i \in L^2(0, T; H^1(\mathbb{T}^d)) \cap H^1(0, T; H^{-1}(\mathbb{T}^d))$, and let $v = (v_1, \dots, v_n)$ be a “strong” solution to (1)–(2), i.e. a weak solution to (1)–(2) satisfying*

$$c \leq v_i \leq C \quad \text{in } Q_T, \quad \partial_t v_i \in L^2(0, T; H^{-1}(\mathbb{T}^d)), \quad v_i \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}^d)),$$

and having the same initial data as u . Then $u(x, t) = v(x, t)$ for a.e. $(x, t) \in \mathbb{T}^d \times (0, T)$.

The existence of a strong solution v_i to (1)–(2) was proved in [16, Prop. 1], but only locally in time and in the whole space setting. While the proof can be adapted to the case of a torus, it is less clear how to extend it globally in time. Theorem 4 cannot be extended in a straightforward way to the whole-space case since $v_i \geq c > 0$ would be nonintegrable. In the case of the Maxwell–Stefan cross-diffusion system on a bounded domain $\Omega \subset \mathbb{R}^d$, it is possible to relax the lower bound to $v_i > 0$ a.e. and $\log v_i \in L^2(0, T; H^1(\Omega))$ [35]. The proof could be possibly extended to the whole space, but the computations in [35] are made rigorous by exploiting the specific structure of the Maxwell–Stefan diffusion coefficients.

For the localization limit, we choose the radial kernel

$$K_{ij}^\eta(x - y) = \frac{a_{ij}}{\eta^d} B\left(\frac{|x - y|}{\eta}\right), \quad i, j = 1, \dots, n, \quad x, y \in \mathbb{T}^d, \tag{13}$$

where $\eta > 0$, $B \in C^0(\mathbb{R})$, $\text{supp}(B) \subset (-1, 1)$, $\int_{-1}^1 B(|z|) dz = 1$, and $a_{ij} \geq 0$ is such that $(\pi_i a_{ij})$ is symmetric and positive definite for some $\pi_i > 0$, $i = 1, \dots, n$.

Theorem 5 (Localization Limit). *Let K_{ij}^η be given by (13) and satisfying Hypothesis (H4). Let u^η be the weak solution to (1)–(2), constructed in Theorem 2. Then there exists a subsequence of (u^η) that is not relabeled such that, as $\eta \rightarrow 0$,*

$$u^\eta \rightarrow u \quad \text{strongly in } L^2(0, T; L^{d/(d-1)}(\mathbb{T}^d)),$$

if $d \geq 2$ and strongly in $L^2(0, T; L^r(\mathbb{T}^d))$ for any $r < \infty$ if $d = 1$. Moreover, u is a nonnegative weak solution to (1) and (3).

The existence of global weak solutions to (1) and (3) can be proved for any bounded domain Ω with Lipschitz boundary $\partial\Omega$ imposing no-flux boundary conditions; see Appendix B.

3. Global existence for the nonlocal system

We prove the global existence of a nonnegative weak solution u to (1)–(2) and show the regularity properties of Proposition 3. Since the proof is based on the entropy method similar to [37, Chapter 4], we sketch the standard arguments and focus on the derivation of uniform estimates. We assume throughout this section that Hypotheses (H1)–(H4) hold.

Step 1: Solution of an approximated system. Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $\delta > 0$, and $m \in \mathbb{N}$ with $m > d/2 + 1$. We proceed by induction over $k \in \mathbb{N}$. Let $u^{k-1} \in L^2(\mathbb{T}^d; \mathbb{R}^n)$ be given. (The superindex k refers to the time step $t_k = k\tau$.) Set $u_i(w) = \exp(w_i/\pi_i) > 0$. We wish to find $w^k \in H^m(\mathbb{T}^d; \mathbb{R}^n)$ to the approximated system

$$\begin{aligned} & \frac{1}{\tau} \int_{\mathbb{T}^d} (u(w^k) - u^{k-1}) \cdot \phi \, dx + \sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \nabla u_i(w^k) \cdot \nabla \phi_i \, dx + \delta b(w^k, \phi) \\ & = - \sum_{i=1}^n \int_{\mathbb{T}^d} u_i(w^k) \nabla p_i[u(w^k)] \cdot \nabla \phi_i \, dx, \end{aligned} \tag{14}$$

for $\phi = (\phi_1, \dots, \phi_n) \in H^m(\mathbb{T}^d; \mathbb{R}^n)$. The bilinear form

$$b(w^k, \phi) = \int_{\mathbb{T}^d} \left(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha \phi + w \cdot \phi \right) dx,$$

is coercive on $H^m(\mathbb{T}^d; \mathbb{R}^n)$, i.e. $b(w^k, w^k) \geq C \|w^k\|_{H^m(\mathbb{T}^d)}^2$ for some $C > 0$, as a consequence of the generalized Poincaré–Wirtinger inequality (see Lemma 15 in Appendix A). By a fixed-point argument, using a mapping of the type $L^\infty(\mathbb{T}^d) \rightarrow H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, in the entropy variable w^k , it is sufficient to derive a uniform bound for all fixed points w_i^k in $H^m(\mathbb{T}^d)$ (see [37, Section 4.4] for details). To this end, let $w_i^k \in H^m(\mathbb{T}^d)$ be such a fixed point. We use the admissible test function $\phi_i = w_i^k = \pi_i \log u_i^k$ (with $u_i^k := u_i(w^k)$) in (14):

$$\begin{aligned} & \sum_{i=1}^n \frac{\pi_i}{\tau} \int_{\mathbb{T}^d} (u_i^k - u_i^{k-1}) \cdot \log u_i^k \, dx + 4\sigma \sum_{i=1}^n \pi_i \int_{\mathbb{T}^d} |\nabla (u_i^k)^{1/2}|^2 \, dx + \delta b(w^k, w^k) \\ & = - \sum_{i=1}^n \int_{\mathbb{T}^d} u_i^k \nabla p_i[u^k] \cdot \nabla w_i^k \, dx = - \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i \nabla p_i[u^k] \cdot \nabla u_i^k \, dx, \end{aligned}$$

where we used the identity $u_i^k \nabla w_i^k = \pi_i \nabla u_i^k$. An integration by parts gives

$$\int_{\mathbb{T}^d} \nabla K_{ij}(x - y) u_j^k(y) \, dy = \int_{\mathbb{T}^d} \nabla K_{ij}(z) u_j^k(x - z) \, dz = \int_{\mathbb{T}^d} K_{ij}(x - y) \nabla u_j^k(y) \, dy. \tag{15}$$

Thus, in view of definition (2) of $p_i[u^k]$ and Hypothesis (H4),

$$\begin{aligned} & \sum_{i=1}^n \frac{\pi_i}{\tau} \int_{\mathbb{T}^d} (u_i^k - u_i^{k-1}) \cdot \log u_i^k dx + 4\sigma \sum_{i=1}^n \pi_i \int_{\mathbb{T}^d} |\nabla(u_i^k)^{1/2}|^2 dx + \delta b(w^k, w^k) \\ &= - \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) \nabla u_j^k(y) \cdot \nabla u_i^k(x) dx dy \leq 0. \end{aligned}$$

The convexity of $f(z) = z(\log z - 1)$ for $z \geq 0$ implies that $f(z) - f(y) \leq f'(z)(z - y)$ for $y, z > 0$. Using this inequality to estimate the first integral on the left-hand side of the displayed inequality and the coercivity of $b(w^k, w^k)$ to estimate the third term, we find that

$$\begin{aligned} & \frac{1}{\tau} \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i (u_i^k (\log u_i^k - 1) - u_i^{k-1} (\log u_i^{k-1} - 1)) dx \\ &+ 4\sigma \sum_{i=1}^n \pi_i \|\nabla(u_i^k)^{1/2}\|_{L^2(\mathbb{T}^d)}^2 + \delta C \sum_{i=1}^n \|w_i^k\|_{H^m(\mathbb{T}^d)}^2 \leq 0, \end{aligned} \tag{16}$$

where C is the coercivity constant of the bilinear form $b(\cdot, \cdot)$. This provides an estimate uniform in the fixed points w^k in $H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, necessary to conclude the fixed-point argument and giving the existence of a solution $w^k \in H^m(\mathbb{T}^d; \mathbb{R}^n)$ to (14). This defines $u^k := u(w^k)$, finishing the induction step.

To derive further uniform estimates, we wish to use $\psi_i = \pi_i p_i[u^k]$ as a test function in (14). However, we cannot estimate the term $\delta b(w^k, \psi)$ appropriately. Therefore, we perform the limits $\delta \rightarrow 0$ and $\tau \rightarrow 0$ separately.

Step 2: Limit $\delta \rightarrow 0$. Let us first list some uniform bounds and convergence results.

Lemma 6. *Let $u^\delta = (u_1^\delta, \dots, u_n^\delta)$ with $u_i^\delta = u_i(w^k)$ be a solution to (14) and let $w_i^\delta = \pi_i \log u_i^\delta$ for $i = 1, \dots, n$ (slightly abusing the notation). Then there exists a constant $C > 0$, independent of δ , such that*

$$\|(u_i^\delta)^{1/2}\|_{L^{r_1}(\mathbb{T}^d)} + \|\nabla u_i^\delta\|_{L^{r_2}(\mathbb{T}^d)} \leq C,$$

where $r_1 = 2d/(d - 2)$ and $r_2 = d/(d - 1)$ if $d > 2$, $r_1 < \infty$ and $r_2 < 2$ if $d = 2$, $r_1 = \infty$ and $r_2 = 2$ if $d = 1$. Moreover, it holds, up to a subsequence, as $\delta \rightarrow 0$ that

$$\begin{aligned} u_i^\delta &\rightarrow u_i \quad \text{strongly in } L^r(\mathbb{T}^d), \quad r < r_1/2, \\ \nabla u_i^\delta &\rightharpoonup \nabla u_i \quad \text{weakly in } L^{r_2}(\mathbb{T}^d), \\ \delta w_i^\delta &\rightarrow 0 \quad \text{strongly in } H^m(\mathbb{T}^d). \end{aligned}$$

Proof. Estimate (16) and the Poincaré–Wirtinger inequality show that $(u_i^\delta)^{1/2}$ is uniformly bounded in $H^1(\mathbb{T}^d)$ and, by Sobolev’s embedding, in $L^{r_1}(\mathbb{T}^d)$. Therefore, $\nabla u_i^\delta = 2(u_i^\delta)^{1/2} \nabla(u_i^\delta)^{1/2}$ is uniformly bounded in $L^{r_2}(\mathbb{T}^d)$. By Sobolev’s embedding, (u_i^δ) is relatively compact in $L^r(\mathbb{T}^d)$ for $r < r_1/2$, and there exists a subsequence that is not relabeled such that, as $\delta \rightarrow 0$, the claimed convergences of u_i^δ and ∇u_i^δ hold. We deduce from (16) that $\sqrt{\delta} w_i^\delta$ is uniformly bounded in $H^m(\mathbb{T}^d)$, hence $\delta w_i^\delta \rightarrow 0$ in $H^m(\mathbb{T}^d)$ as $\delta \rightarrow 0$. This ends the proof. \square

Thanks to Lemma 6, we have, up to a subsequence, $u_i^\delta \rightarrow u_i$ a.e. and (u_i^δ) is dominated by some function in $L^r(\mathbb{T}^d)$. By dominated convergence, $p_i[u^\delta] \rightarrow p_i[u]$ a.e. Young’s convolution inequality (see Lemma 11 in Appendix A) shows that, for $d > 2$,

$$\begin{aligned} \|p_i[u^\delta]\|_{L^\infty(\mathbb{T}^d)} &\leq \sum_{j=1}^n \left\| \int_{\mathbb{T}^d} K_{ij}(\cdot - y) u_j^\delta(y) dy \right\|_{L^\infty(\mathbb{T}^d)} \\ &\leq \sum_{j=1}^n \|K_{ij}\|_{L^{d/2}(\mathbb{T}^d)} \|u_j^\delta\|_{L^{d/(d-2)}(\mathbb{T}^d)} \leq C. \end{aligned}$$

Here and in the following, $C > 0$ denotes a constant independent of δ with values changing from line to line. In a similar way, we can prove that $(p_i[u^\delta])$ is bounded in $L^r(\mathbb{T}^d)$ for any $r < \infty$ if $d = 2$ and in $L^\infty(\mathbb{T}^d)$ if $d = 1$, assuming that $K_{ij} \in L^1(\mathbb{T}^d)$; see Hypothesis (H2). Lemma 12 in Appendix A implies that $p_i[u^\delta] \rightarrow p_i[u]$ strongly in $L^r(\mathbb{T}^d)$ for any $r < \infty$. Furthermore, if $d > 2$, we use again Young’s convolution inequality and Lemma 6 to show that

$$\begin{aligned} \|\nabla p_i[u^\delta]\|_{L^{r_3}(\mathbb{T}^d)} &\leq \sum_{j=1}^n \|K_{ij}\|_{L^{d/2}(\mathbb{T}^d)} \|\nabla u_j^\delta\|_{L^{d/(d-1)}(\mathbb{T}^d)} \\ &\leq \max_{1 \leq j \leq n} \|K_{ij}\|_{L^{d/2}(\mathbb{T}^d)} \sum_{j=1}^n \|\nabla u_j^\delta\|_{L^{d/(d-1)}(\mathbb{T}^d)} \leq C, \end{aligned}$$

where $r_3 = d$. Similar computations show that for $d = 2$, we have $\nabla p_i[u^\delta]$ is bounded in $L^{r_3}(\mathbb{T}^d)$ for some $r_3 > 2$ and for $r_3 = 2$ if $d = 1$. Hence, for a subsequence,

$$\nabla p_i[u^\delta] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^{r_3}(\mathbb{T}^d).$$

Combining this with the strong convergence $u_i^\delta \rightarrow u_i$ in $L^r(\mathbb{T}^d)$ for $r < r_1/2$, we conclude that the product converges:

$$u_i^\delta \nabla p_i[u^\delta] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(\mathbb{T}^d).$$

We deduce from the uniform bounds $\|u_i^\delta\|_{L^{r_1/2}(\mathbb{T}^d)} \leq C$ and $\|\nabla p_i[u^\delta]\|_{L^{r_3}(\mathbb{T}^d)} \leq C$ that $(u_i^\delta \nabla p_i[u^\delta])$ is bounded in $L^{\min\{2, d/(d-1)\}}(\mathbb{T}^d)$ and

$$u_i^\delta \nabla p_i[u^\delta] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^{\min\{2, d/(d-1)\}}(\mathbb{T}^d).$$

Thus, we can pass to the limit $\delta \rightarrow 0$ in (14) to conclude that $u_i^k := u_i \geq 0$ for $i = 1, \dots, n$ solves

$$\frac{1}{\tau} \int_{\mathbb{T}^d} (u^k - u^{k-1}) \cdot \phi dx + \sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \nabla u_i^k \cdot \nabla \phi_i dx = - \sum_{i=1}^n \int_{\mathbb{T}^d} u_i^k \nabla p_i[u^k] \cdot \nabla \phi_i dx, \tag{17}$$

for all test functions $\phi_i \in W^{1, r_3}(\mathbb{T}^d)$. Observe that $p_i[u^k]$ is an element of the space $W^{1, r_3}(\mathbb{T}^d)$ and is an admissible test function; this will be used in the next step.

Step 3: Uniform estimates. We introduce the piecewise constant in time functions $u^{(\tau)}(x, t) = u^k(x)$ for $x \in \mathbb{T}^d$ and $t \in ((k-1)\tau, k\tau]$, $k = 1, \dots, N$. At time $t = 0$, we set $u_i^{(\tau)}(\cdot, 0) = u_i^0$. Furthermore, we use the time-shift operator $(S_\tau u^{(\tau)})(x, t) = u^{k-1}(x)$ for $x \in \mathbb{T}^d$, $t \in ((k-1)\tau, k\tau]$. Then, summing (17) over k , we obtain

$$\begin{aligned} &\frac{1}{\tau} \int_0^T \int_{\mathbb{T}^d} (u^{(\tau)} - S_\tau u^{(\tau)}) \cdot \phi dx dt + \sigma \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} \nabla u_i^{(\tau)} \cdot \nabla \phi_i dx dt \\ &= - \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} u_i^{(\tau)} \nabla p_i[u^{(\tau)}] \cdot \nabla \phi_i dx dt, \end{aligned}$$

for piecewise constant functions $\phi : (0, T) \rightarrow W^{1, r_3}(\mathbb{T}^d; \mathbb{R}^n)$ and, by density, for all functions $\phi \in L^2(0, T; W^{1, r_3}(\mathbb{T}^d; \mathbb{R}^n))$. Summing the entropy inequality (16) from $k = 1, \dots, N$, it follows that

$$H_1(u^{(\tau)}(T)) + 4\sigma \sum_{i=1}^n \int_0^T \pi_i \|\nabla (u_i^{(\tau)})^{1/2}\|_{L^2(\mathbb{T}^d)}^2 dt \leq H_1(u^0). \tag{18}$$

These bounds allow us to derive further estimates. Since the $L^1 \log L^1$ bound dominates the $L^1(\mathbb{T}^d)$ norm, we deduce from the Poincaré–Wirtinger inequality that

$$\|u_i^{(\tau)} \log u_i^{(\tau)}\|_{L^\infty(0, T; L^1(\mathbb{T}^d))} + \|(u_i^{(\tau)})^{1/2}\|_{L^2(0, T; H^1(\mathbb{T}^d))} \leq C(u^0), \quad i = 1, \dots, n.$$

This implies, by the Gagliardo–Nirenberg inequality with $\theta = d/(d + 2)$, that

$$\begin{aligned} \|u_i^{(\tau)}\|_{L^{1+2/d}(Q_T)}^{1+2/d} &= \int_0^T \|(u_i^{(\tau)})^{1/2}\|_{L^{2+4/d}(\mathbb{T}^d)}^{2+4/d} dt \\ &\leq C \int_0^T \|(u_i^{(\tau)})^{1/2}\|_{H^1(\mathbb{T}^d)}^{\theta(2d+4)/d} \|(u_i^{(\tau)})^{1/2}\|_{L^2(\mathbb{T}^d)}^{(1-\theta)(2d+4)/d} dt \\ &\leq C \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\mathbb{T}^d))}^{2/d} \int_0^T \|(u_i^{(\tau)})^{1/2}\|_{H^1(\mathbb{T}^d)}^2 dt \leq C(u^0, d), \end{aligned} \tag{19}$$

and by Hölder’s inequality with $q = (d + 2)/(d + 1)$,

$$\begin{aligned} \|\nabla u_i^{(\tau)}\|_{L^q(Q_T)} &= 2\|(u_i^{(\tau)})^{1/2}\nabla(u_i^{(\tau)})^{1/2}\|_{L^q(Q_T)} \\ &\leq 2\|(u_i^{(\tau)})^{1/2}\|_{L^{2+4/d}(Q_T)} \|\nabla(u_i^{(\tau)})^{1/2}\|_{L^2(Q_T)} \leq C. \end{aligned} \tag{20}$$

These bounds are not sufficient to pass to the limit $\tau \rightarrow 0$, since we also need uniform bounds on $u_i^\tau p_i[u^\tau]$ and on the discrete time derivative. To derive further estimates, we use the test function $\phi_i = \pi_i p_i[u^k] \in W^{1,r_3}(\mathbb{T}^d)$ in (17):

$$\begin{aligned} &\frac{1}{\tau} \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i (u_i^k(x) - u_i^{k-1}(x)) K_{ij}(x - y) u_j^k(y) dx dy \\ &+ \sigma \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) \nabla u_i^k(x) \cdot \nabla u_j^k(y) dx dy = - \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i^k |\nabla p_i[u^k]|^2 dx. \end{aligned} \tag{21}$$

Because of the positive definiteness of $\pi_i K_{ij}$, the second term on the left-hand side is nonnegative. Exploiting the symmetry and positive definiteness of $\pi_i K_{ij}$ (Hypotheses (H3)–(H4)), the first integral can be estimated from below as

$$\begin{aligned} &\frac{1}{2\tau} \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (u_i^k(x) u_j^k(y) - u_i^{k-1}(x) u_j^{k-1}(y) \\ &+ (u_i^k(x) - u_i^{k-1}(x))(u_j^k(y) - u_j^{k-1}(y))) dx dy \\ &\geq \frac{1}{2\tau} \sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (u_i^k(x) u_j^k(y) - u_i^{k-1}(x) u_j^{k-1}(y)) dx dy \\ &= \frac{1}{\tau} (H_2(u^k) - H_2(u^{k-1})). \end{aligned}$$

Therefore, we infer from (21) that

$$H_2(u^k) + \tau \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i^k |\nabla p_i[u^k]|^2 dx \leq H_2(u^{k-1}),$$

and summing this inequality from $k = 1, \dots, N$, we have

$$H_2(u^{(\tau)}(T)) + \sum_{i=1}^n \pi_i \int_0^T \|(u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}]\|_{L^2(\mathbb{T}^d)}^2 dt \leq H_2(u^0). \tag{22}$$

The previous bound allows us to derive an estimate for the discrete time derivative.

Lemma 7. *Let $u^{(\tau)}$ be a previously obtained solution to the weak formulation*

$$\begin{aligned} &\frac{1}{\tau} \int_0^T \int_{\mathbb{T}^d} (u^{(\tau)} - S_\tau u^{(\tau)}) \cdot \phi dx dt + \sigma \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} \nabla u_i^{(\tau)} \cdot \nabla \phi_i dx dt \\ &= - \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} u_i^{(\tau)} \nabla p_i[u^{(\tau)}] \cdot \nabla \phi_i dx dt \end{aligned} \tag{23}$$

for all functions $\phi \in L^2(0, T; W^{1,r_3}(\mathbb{T}^d; \mathbb{R}^n))$. Then there exists a constant $C > 0$ independent of τ such that

$$\tau^{-1} \|u^{(\tau)} - S_\tau u^{(\tau)}\|_{L^q(0, T; W^{2, (d+2)/2}(\mathbb{T}^d)')} \leq C, \tag{24}$$

where $q = (d + 2)/(d + 1)$.

Proof. Estimates (19) and (22) imply that

$$u_i^{(\tau)} \nabla p_i[u^{(\tau)}] = (u_i^{(\tau)})^{1/2} \cdot (u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}],$$

is uniformly bounded in $L^q(Q_T)$, where $q = (d + 2)/(d + 1)$. Let $\phi \in L^{q'}(0, T; W^{2, (d+2)/2}(\mathbb{T}^d))$, where $q' = d + 2$. Then $1/q + 1/q' = 1$ and

$$\begin{aligned} & \left| \frac{1}{\tau} \int_0^T \int_{\mathbb{T}^d} (u^{(\tau)} - S_\tau u^{(\tau)}) \cdot \phi dx dt \right| \\ & \leq \sigma \sum_{i=1}^n \|u_i^{(\tau)}\|_{L^{1+2/d}(Q_T)} \|\Delta \phi_i\|_{L^{(d+2)/2}(Q_T)} + \sum_{i=1}^n \|u_i^{(\tau)} \nabla p_i[u^{(\tau)}]\|_{L^q(Q_T)} \|\nabla \phi_i\|_{L^{q'}(Q_T)} \\ & \leq C \|\phi\|_{L^{q'}(0, T; W^{2, (d+2)/2}(\mathbb{T}^d))}. \end{aligned}$$

We conclude that

$$\tau^{-1} \|u^{(\tau)} - S_\tau u^{(\tau)}\|_{L^q(0, T; W^{2, (d+2)/2}(\mathbb{T}^d)')} \leq C,$$

which finishes the proof of the lemma. \square

Step 4: Limit $\tau \rightarrow 0$. Estimates (18) and (24) allow us to apply the Aubin–Lions compactness lemma in the version of [8] to conclude the existence of a subsequence that is not relabeled such that, as $\tau \rightarrow 0$,

$$u_i^{(\tau)} \rightarrow u_i \quad \text{strongly in } L^2(0, T; L^{d/(d-1)}(\mathbb{T}^d)), \quad i = 1, \dots, n,$$

if $d \geq 2$ and strongly in $L^2(0, T; L^r(\mathbb{T}^d))$ for any $r < \infty$ if $d = 1$. Strictly speaking, the version of [8] holds for the continuous time derivative, but the technique of [28] shows that the conclusion of [8] also holds for the discrete time derivative. Then, maybe for another subsequence, $u_i^{(\tau)} \rightarrow u_i$ a.e. in Q_T , and we deduce from (19) that $u_i^{(\tau)} \rightarrow u_i$ strongly in $L^r(Q_T)$ for all $r < 1 + 2/d$ (see Lemma 12 in Appendix A). Furthermore, we obtain from (18), (20), (22), and (24) the convergences

$$\begin{aligned} \nabla u_i^{(\tau)} &\rightharpoonup \nabla u_i \quad \text{weakly in } L^q(Q_T), \quad i = 1, \dots, n, \\ \tau^{-1} (u^{(\tau)} - S_\tau u^{(\tau)}) &\rightharpoonup \partial_t u_i \quad \text{weakly in } L^q(0, T; W^{2, (d+2)/2}(\mathbb{T}^d)'), \\ (u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}] &\rightharpoonup z_i \quad \text{weakly in } L^2(Q_T), \end{aligned} \tag{25}$$

where $z_i \in L^2(Q_T)$ and $q = (d + 2)/(d + 1)$. Since $u_i^{(\tau)} \geq 0$, we infer that $u_i \geq 0$ in Q_T . It remains to identify the limit z_i , which is stated in the following result.

Lemma 8. *Let $u^{(\tau)}$ be the previously obtained solution to (23). Then the weak limit z_i in (25) can be identified as $z_i = u_i^{1/2} \nabla p_i[u]$, i.e., as $\tau \rightarrow 0$,*

$$(u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}] \rightharpoonup u_i^{1/2} \nabla p_i[u] \quad \text{weakly in } L^2(Q_T).$$

Proof. We show first that

$$\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^q(Q_T).$$

Indeed, it follows from the strong convergence of $(u_i^{(\tau)})$ that (up to a subsequence) $K_{ij}(x - y) \times u_j^{(\tau)}(y, t) \rightarrow K_{ij}(x - y)u_j(y, t)$ for a.e. $(y, t) \in Q_T$ and for a.e. $x \in \mathbb{T}^d$. Hence, because of the uniform bounds, $p_i[u^{(\tau)}] \rightarrow p_i[u]$ a.e. in Q_T . We deduce from Young’s convolution inequality and the uniform bound for $\nabla u_i^{(\tau)}$ in $L^q(Q_T)$ that $\nabla p_i[u^{(\tau)}]$ is uniformly bounded in $L^q(Q_T)$. Therefore,

$$\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^q(Q_T).$$

When $d = 2$, we have the convergences $\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u]$ weakly in $L^{4/3}(Q_T)$ and $(u_i^{(\tau)})^{1/2} \rightarrow u_i^{1/2}$ strongly in $L^4(Q_T)$, which is sufficient to pass to the limit in the product and to identify it with z_i . However, this argument fails when $d > 2$, and we need a more sophisticated proof. The div–curl lemma does not apply, since the exponents of the Lebesgue spaces, in which the convergences of $(u_i^{(\tau)})^{1/2}$ and $\nabla p_i[u^{(\tau)}]$ take place, are not conjugate for $d > 2$. Also the generalization [10, Theorem 2.1] to nonconjugate exponents cannot be used for general d .

Our idea is to exploit the fact that the product converges in a space better than L^1 . Then Lemma 13 in Appendix A immediately implies that

$$(u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}] \rightharpoonup u_i^{1/2} \nabla p_i[u] \quad \text{weakly in } L^q(Q_T).$$

In fact, estimate (22) implies that this convergence holds in $L^2(Q_T)$, which finishes the proof of Lemma 8. \square

Combining Lemma 8 and the strong convergence of $(u_i^{(\tau)})^{1/2}$ in $L^2(Q_T)$ gives

$$u_i^{(\tau)} \nabla p_i[u^{(\tau)}] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(Q_T).$$

In view of the uniform bounds for $(u_i^{(\tau)})^{1/2}$ in $L^{2+4/d}(Q_T)$ and of $(u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}]$ in $L^2(Q_T)$, the product $u_i^{(\tau)} \nabla p_i[u^{(\tau)}]$ is uniformly bounded in $L^q(Q_T)$. Thus, the previous weak convergence also holds in $L^q(Q_T)$.

Step 5: End of the proof. The convergences of the previous step allow us to pass to the limit $\tau \rightarrow 0$ in (17), yielding

$$\int_0^T \langle \partial_t u_i, \phi_i \rangle dt + \sigma \int_0^T \int_{\mathbb{T}^d} \nabla u_i \cdot \nabla \phi_i dx dt = - \int_0^T \int_{\mathbb{T}^d} u_i \nabla p_i[u] \cdot \nabla \phi_i dx dt,$$

for all smooth test functions. Because of $\nabla u_i, u_i \nabla p_i[u] \in L^q(Q_T)$, a density argument shows that the weak formulation holds for all $\phi \in L^{q'}(0, T; W^{1,q'}(\mathbb{T}^d))$, recalling that $q' = d + 2$. Then $\partial_t u_i$ lies in the space $L^q(0, T; W^{-1,q}(\mathbb{T}^d))$, where $W^{-1,q}(\mathbb{T}^d) := W^{1,q'}(\mathbb{T}^d)'$. The proof that $u(\cdot, 0)$ satisfies the initial datum can be done exactly as in [36, p. 1980]. Finally, using the convexity of H_1 and the lower semi-continuity of convex functions, the entropy inequalities (18) and (22) become (11) and (12), respectively, in the limit $\tau \rightarrow 0$. This ends the proof of Theorem 2.

Remark 9 (Whole Space and Bounded Domains). We believe that the whole space $\Omega = \mathbb{R}^d$ can be treated by using the techniques of [17], under the assumption that a moment of u^0 is bounded, i.e. $\int_{\mathbb{R}^d} u_i^0(x)(1 + |x|^2)^{\alpha/2} dx < \infty$ for a suitable $\alpha > 0$. Indeed, standard estimates show that $u_i^\varepsilon(1 + |x|^2)^{\alpha/2}$ is bounded in $L^\infty(0, T; L^1(\mathbb{R}^d))$, where (u_i^ε) is a sequence of approximating solutions. By the previous proof, $(\sqrt{u_i^\varepsilon})$ is bounded in $L^2(0, T; H^1(\mathbb{R}^d))$, and estimate (20) shows that (u_i^ε) is bounded in $L^2(0, T; W^{1,q}(\mathbb{R}^d))$ with $q = (d+2)/(d+1)$. Since $W^{1,q}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d; (1 + |x|^2)^{\alpha/2})$ is compactly embedded in $L^r(\mathbb{R}^d)$ for $r < 2q/(2 - q)$ and $1 \leq q < 2$ (by adapting the proof of [12, Lemma 1]), we can apply the Aubin–Lions lemma, concluding that, up to a subsequence, $u_i^\varepsilon \rightarrow u$ strongly in $L^2(0, T; L^2(\mathbb{R}^d))$.

The case of bounded domains $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ seems to be more delicate. We assume no-flux boundary conditions to recover the weak formulation (9). The problem comes from the treatment of

the boundary integrals when integrating by parts. For instance, we need to integrate by parts in $\nabla p_i[u]$ (see (15)) and to control the integral

$$q_i[u](x) := \sum_{j=1}^n \int_{\partial\Omega} K_{ij}(x-y)u_j(y)\nu(y)dy,$$

where ν is the exterior unit normal vector of $\partial\Omega$. If $K_{ij} \in L^\infty(\mathbb{R}^d)$, we may estimate this integral by $\|u_i\|_{L^1(\partial\Omega)} \leq C\|u_i\|_{W^{1,1}(\Omega)}$. Consequently, $\|q_i[u]\|_{L^\infty(\Omega)} \leq C\sum_{j=1}^n \|u_j\|_{W^{1,1}(\Omega)}$. The integral $q_i[u]$ appears in the weak formulation, for instance, as

$$\left| \sigma \int_{\Omega} \nabla u_i \cdot q_i[u] dx \right| \leq C \sum_{j=1}^n \|\nabla u_j\|_{W^{1,1}(\Omega)}^2 \leq 2C \sum_{j=1}^n \|u_j\|_{L^1(\Omega)} \|\nabla u_j^{1/2}\|_{L^2(\Omega)}^2,$$

and this integral cannot generally be absorbed by the corresponding term in (16) except if $\|u_j^0\|_{L^1(\Omega)}$ is sufficiently small. \square

Proof of Proposition 3. The proof of the $H^1(\mathbb{T}^d)$ regularity requires an approximate scheme different from that one used in the proof of Theorem 2. Given $u^{k-1} \in L^2(\mathbb{T}^d, \mathbb{R}^n)$ with $u_i^{k-1} \geq 0$, we wish to find $u^k \in H^1(\mathbb{T}^d; \mathbb{R}^n)$ such that

$$\frac{1}{\tau} \int_{\mathbb{T}^d} (u_i^k - u_i^{k-1})\phi_i dx + \sigma \int_{\mathbb{T}^d} \nabla u_i^k \cdot \nabla \phi_i dx + \int_{\mathbb{T}^d} \frac{(u_i^k)^+}{1 + \delta(u_i^k)^+} \nabla p_i[u^k] \cdot \nabla \phi_i dx = 0, \tag{26}$$

for $\phi_i \in H^1(\mathbb{T}^d)$, where $\delta > 0$ and $z^+ = \max\{0, z\}$. Since $\nabla K_{ij} \in L^{d+2}(\mathbb{T}^d)$, $\nabla p_i[u^k]$ can be bounded in $L^{d+2}(\mathbb{T}^d)$ in terms of the $L^1(\mathbb{T}^d)$ norm of u^k . Thus, the last term on the left-hand side is well defined. The existence of a solution to this discrete scheme is proved by a fixed-point argument, and the main step is the derivation of uniform estimates. First, we observe that the test function $(u_i^k)^- = \min\{0, u_i^k\}$ yields

$$\begin{aligned} & \frac{1}{\tau} \int_{\mathbb{T}^d} (u_i^k - u_i^{k-1})(u_i^k)^- dx + \sigma \int_{\mathbb{T}^d} |\nabla (u_i^k)^-|^2 dx \\ &= - \int_{\mathbb{T}^d} \frac{(u_i^k)^+}{1 + \delta(u_i^k)^+} \nabla p_i[u^k] \cdot \nabla (u_i^k)^- dx = 0, \end{aligned}$$

and consequently, $(u_i^k)^- = 0$ in \mathbb{T}^d . Thus, $u_i^k \geq 0$ and we can remove the plus sign in (26). Second, we use the test function u_i^k in (26) and sum over $i = 1, \dots, n$:

$$\frac{1}{\tau} \sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^k - u_i^{k-1})u_i^k dx + \sigma \sum_{i=1}^n \int_{\mathbb{T}^d} |\nabla u_i^k|^2 dx = - \sum_{i=1}^n \int_{\mathbb{T}^d} \frac{u_i^k}{1 + \delta u_i^k} \nabla p_i[u^k] \cdot \nabla u_i^k dx. \tag{27}$$

The first integral becomes

$$\sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^k - u_i^{k-1})u_i^k dx \geq \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{T}^d} ((u_i^k)^2 - (u_i^{k-1})^2) dx.$$

The right-hand side in (27) is estimated by Hölder’s inequality and Young’s convolution inequality:

$$\begin{aligned} - \sum_{i=1}^n \int_{\mathbb{T}^d} \frac{u_i^k}{1 + \delta u_i^k} \nabla p_i[u^k] \cdot \nabla u_i^k dx &\leq \sum_{i=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)} \|\nabla p_i[u^k]\|_{L^{d+2}(\mathbb{T}^d)} \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C_K \sum_{i,j=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)} \|u_j^k\|_{L^1(\mathbb{T}^d)} \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}, \end{aligned}$$

where $C_K > 0$ depends on the $L^{d+2}(\mathbb{T}^d)$ norm of ∇K_{ij} but not on δ . Taking the test function $\phi_i = 1$ in (26) shows that $\|u_i^k\|_{L^1(\mathbb{T}^d)} = \|u_i^0\|_{L^1(\mathbb{T}^d)}$ is uniformly bounded. This allows us to reduce the cubic expression on the right-hand side of the previous inequality to a quadratic one. This is the key idea of the proof. Combining the previous arguments, (26) becomes

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^k)^2 dx - \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^{k-1})^2 dx + \tau\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} |\nabla u_i^k|^2 dx \\ & \leq \tau C \sum_{i=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)} \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)} \\ & \leq \frac{1}{2} \tau\sigma \sum_{i=1}^n \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^2 + \tau C \sum_{i=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}^2. \end{aligned}$$

The Gagliardo–Nirenberg and Poincaré–Wirtinger inequalities (see Lemmas 14 and 15 in Appendix A) show that

$$\begin{aligned} \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}^2 & \leq C \|u_i^k\|_{H^1(\mathbb{T}^d)}^{2\theta} \|u_i^k\|_{L^1(\mathbb{T}^d)}^{2(1-\theta)} \\ & \leq C (\|\nabla u_i^k\|_{L^2(\mathbb{T}^d)} + \|u_i^k\|_{L^1(\mathbb{T}^d)})^{2\theta} \|u_i^k\|_{L^1(\mathbb{T}^d)}^{2(1-\theta)} \\ & \leq C(u^0) \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^{2\theta} + C(u^0), \end{aligned}$$

where $\theta = d(d+4)/(d+2)^2 < 1$. We deduce from Young’s inequality that for any $\varepsilon > 0$,

$$C(u^0) \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^{2\theta} \leq \theta \varepsilon^{1/\theta} \|\nabla u_i^k\|^2 + (1-\theta) \varepsilon^{-1/(1-\theta)} C(u^0)^{1/(1-\theta)}.$$

After setting $C(\varepsilon) = C(u^0) + \varepsilon^{-1/(1-\theta)} C(u^0)^{1/(1-\theta)}$, we find that

$$\|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}^2 \leq \varepsilon^{1/\theta} \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^2 + C(\varepsilon).$$

Therefore, choosing $\varepsilon > 0$ sufficiently small, we infer that

$$\frac{1}{2} \sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^k)^2 dx - \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^{k-1})^2 dx + \frac{1}{4} \tau\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} |\nabla u_i^k|^2 dx \leq \tau C(\varepsilon), \tag{28}$$

This provides a uniform $H^1(\mathbb{T}^d)$ estimate for u^k . Defining the fixed-point operator as a mapping from $L^2(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$, the compact embedding $H^1(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d)$ implies the compactness of this operator (see [37, Chapter 4] for details). This shows that (26) possesses a weak solution $u^k \in H^1(\mathbb{T}^d)$.

In order to pass to the limit $(\delta, \tau) \rightarrow 0$, we need uniform estimates for the piecewise constant in time functions $u_i^{(\tau)}$, using the same notation as in the proof of Theorem 2. Estimate (28) provides uniform bounds for $u_i^{(\tau)}$ in $L^\infty(0, T; L^2(\mathbb{T}^d))$ and $L^2(0, T; H^1(\mathbb{T}^d))$. By the Gagliardo–Nirenberg inequality, $(u_i^{(\tau)})$ is bounded in $L^{2+4/d}(Q_T)$. By Young’s convolution inequality,

$$\sup_{t \in (0, T)} \|\nabla p_i[u^{(\tau)}(t)]\|_{L^\infty(\mathbb{T}^d)} \leq \sum_{j=1}^n \|\nabla K_{ij}\|_{L^{d+2}(\mathbb{T}^d)} \sup_{t \in (0, T)} \|u_j^{(\tau)}\|_{L^q(\mathbb{T}^d)} \leq C,$$

where $q = (d+2)/(d+1)$. Thus, $(\nabla p_i[u^{(\tau)}])$ is bounded in $L^\infty(0, T; L^\infty(\mathbb{T}^d))$. From these estimates, we can derive a uniform bound for the discrete time derivative. Using the uniform estimates $\|u_i^{(\tau)}\|_{L^2(0, T; H^1(\mathbb{T}^d))} \leq C$ and $\|\nabla p_i[u^{(\tau)}]\|_{L^\infty(0, T; L^\infty(\mathbb{T}^d))} \leq C$, we can prove in a similar way as in the proof of Theorem 2 that

$$\tau^{-1} \|u^{(\tau)} - S_\tau u^{(\tau)}\|_{L^2(0, T; H^{-1}(\mathbb{T}^d))} \leq C.$$

Therefore, by the Aubin–Lions lemma [28], up to a subsequence, as $(\delta, \tau) \rightarrow 0$,

$$u_i^{(\tau)} \rightarrow u_i \quad \text{strongly in } L^2(Q_T),$$

and this convergence even holds in $L^r(Q_T)$ for any $r < 2 + 4/d$. We can show as in the proof of Theorem 2 that $p_i[u^{(\tau)}] \rightarrow p_i[u]$ a.e. and consequently, for a subsequence, $\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u]$ weakly in $L^2(Q_T)$. We infer that

$$u_i^{(\tau)} \nabla p_i[u^{(\tau)}] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(Q_T).$$

Omitting the details, it follows that $u = (u_1, \dots, u_n)$ is a weak solution to (1)–(2) satisfying $u_i \in L^2(0, T; H^1(\mathbb{T}^d))$ for $i = 1, \dots, n$.

Next, we show the lower and upper bounds for u_i . Define $M(t) = M_0 e^{\lambda t}$, where $\lambda > 0$ will be specified later. Recall that we assume $\nabla K_{ij} \in L^\infty(\mathbb{T}^d)$, and $0 < m_0 \leq u_i^0 \leq M_0$ in \mathbb{T}^d (see Proposition 3). Hence, we can apply Young’s convolution inequality with $p = 1$ and $q, r = \infty$ and estimate

$$\|\nabla p_i[u]\|_{L^\infty(\mathbb{T}^d)} \leq \sum_{j=1}^n \|\nabla K_{ij}\|_{L^\infty(\mathbb{T}^d)} \|u_j\|_{L^1(\mathbb{T}^d)} \leq C.$$

Then, with the test function $e^{-\lambda t}(u_i - M)^+(t) = e^{-\lambda t} \max\{0, (u_i - M)(t)\}$ in the weak formulation of (26), we deduce from

$$\partial_t u_i e^{-\lambda t} (u_i - M)^+ = \frac{1}{2} \partial_t \{e^{-\lambda t} [(u_i - M)^+]^2\} + \frac{\lambda}{2} e^{-\lambda t} [(u_i - M)^+]^2 + \lambda e^{-\lambda t} M (u_i - M)^+$$

that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^d} e^{-\lambda t} (u_i - M)^+(t)^2 dx + \sigma \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} |\nabla (u_i - M)^+|^2 dx ds \\ &= - \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} (u_i - M) \nabla p_i[u] \cdot \nabla (u_i - M)^+ dx ds - \frac{\lambda}{2} \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} [(u_i - M)^+]^2 dx ds \\ & \quad - \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} M \nabla p_i[u] \cdot \nabla (u_i - M)^+ dx ds - \lambda \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} M (u_i - M)^+ dx ds. \end{aligned}$$

We write $(u_i - M) \nabla (u_i - M)^+ = \frac{1}{2} \nabla [(u_i - M)^+]^2$ and integrate by parts in the first and third terms of the right-hand side:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^d} e^{-\lambda s} (u_i - M)^+(t)^2 dx + \sigma \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} |\nabla (u_i - M)^+|^2 dx ds \\ & \leq \frac{1}{2} (\|\Delta p_i[u]\|_{L^\infty(0, T; L^\infty(\mathbb{T}^d))} - \lambda) \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} [(u_i - M)^+]^2 dx ds \\ & \quad + (\|\Delta p_i[u]\|_{L^\infty(0, T; L^\infty(\mathbb{T}^d))} - \lambda) \int_0^t \int_{\mathbb{T}^d} e^{-\lambda s} M (u_i - M)^+ dx ds. \end{aligned}$$

By Young’s convolution inequality and the regularity assumptions on K_{ij} ,

$$\|\Delta p_i[u]\|_{L^\infty(0, T; L^\infty(\mathbb{T}^d))} \leq C \sum_{j=1}^n \|u_j\|_{L^\infty(0, T; L^1(\mathbb{T}^d))} \leq C_0.$$

Therefore, choosing $\lambda \geq C_0$, it follows that

$$\int_{\mathbb{T}^d} e^{-\lambda t} (u_i - M)^+(t)^2 dx \leq 0,$$

and we infer that $e^{-\lambda t}(u_i - M)^+(t) = 0$ and $u_i(t) \leq M(t) = M_0 e^{\lambda t}$ for $t > 0$. The inequality $u_i(t) \geq m(t) := m_0 e^{-\lambda t}$ is proved in the same way, using the test function $e^{-\lambda t}(u_i - m)^- = e^{-\lambda t} \min\{0, u_i - m\}$. \square

Remark 10. Proposition 3 holds true for more general kernel functions of the type $K_{ij}(x, y)$. In that case, we need the regularity $\nabla_x K_{ij} \in L_y^\infty L_x^{d+2} \cap L_x^\infty L_y^{d+2}$ to apply the Young inequality for kernels; see [47, Theorem 0.3.1]. For the lower and upper bounds of the solution, we additionally need the regularity $\nabla_x K_{ij}, \Delta_x K_{ij} \in L_x^\infty L_y^\infty$. \square

4. Weak-strong uniqueness for the nonlocal system

In this section, we prove Theorem 4. Let u be a nonnegative weak solution and v be a positive “strong” solution to (1)–(2), i.e., $v = (v_1, \dots, v_n)$ is a weak solution to (1)–(2) satisfying

$$0 < c \leq v_i \leq C \quad \text{in } Q_T, \quad \partial_t v_i \in L^2(0, T; H^{-1}(\mathbb{T}^d)), \quad v_i \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}^d)).$$

In particular, we have $\log v_i, \nabla \log v_i \in L^\infty(0, T; L^\infty(\mathbb{T}^d))$. Then, for $0 < \varepsilon < 1$, we define the regularized relative entropy density

$$h_\varepsilon(u|v) = \sum_{i=1}^n \pi_i ((u_i + \varepsilon)(\log(u_i + \varepsilon) - 1) - (u_i + \varepsilon) \log v_i + v_i),$$

and the associated relative entropy

$$H_\varepsilon(u|v) = \int_{\mathbb{T}^d} h_\varepsilon(u|v) dx.$$

Step 1: Preparations. We compute

$$\frac{\partial h_\varepsilon}{\partial u_i}(u|v) = \pi_i \log(u_i + \varepsilon) - \pi_i \log v_i, \quad \frac{\partial h_\varepsilon}{\partial v_i}(u|v) = -\pi_i \frac{u_i + \varepsilon}{v_i} + \pi_i.$$

Then we have

$$\nabla \frac{\partial h_\varepsilon}{\partial u_i} = \frac{\pi_i}{u_i + \varepsilon} \nabla u_i - \frac{\pi_i}{v_i} \nabla v_i, \quad \nabla \frac{\partial h_\varepsilon}{\partial v_i} = -\frac{\pi_i}{v_i} \nabla u_i + \frac{\pi_i(u_i + \varepsilon)}{v_i^2} \nabla v_i.$$

The second function is an admissible test function for the weak formulation of (1), satisfied by v_i , since $\nabla u_i \in L^2(Q_T)$ and $\nabla v_i \in L^\infty(Q_T)$. Strictly speaking, the first function is not an admissible test function for the weak formulation of (1), satisfied by u_i , since it needs test functions in $W^{1,d+2}(\mathbb{T}^d)$. However, the nonlocal term becomes with this test function

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} K_{ij}(x - y) \nabla u_j(y) \cdot \frac{\nabla u_i(x)}{u_i(x) + \varepsilon} dx dy,$$

which is finite since K_{ij} is essentially bounded and $\nabla u_i \cdot \nabla u_j \in L^1(Q_T)$. Thus, by a suitable approximation argument, the following computation can be made rigorous. We find that

$$\begin{aligned} \frac{d}{dt} H_\varepsilon(u|v) &= \sum_{i=1}^n \left(\left\langle \partial_t u_i, \frac{\partial h_\varepsilon}{\partial u_i}(u|v) \right\rangle + \left\langle \partial_t v_i, \frac{\partial h_\varepsilon}{\partial v_i}(u|v) \right\rangle \right) \\ &= -\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \left(\nabla u_i \cdot \nabla \frac{\partial h_\varepsilon}{\partial u_i}(u|v) + \nabla v_i \cdot \nabla \frac{\partial h_\varepsilon}{\partial v_i}(u|v) \right) dx \\ &\quad - \sum_{i=1}^n \int_{\mathbb{T}^d} \left(u_i \nabla p_i[u] \cdot \nabla \frac{\partial h_\varepsilon}{\partial u_i}(u|v) + v_i \nabla p_i[v] \cdot \nabla \frac{\partial h_\varepsilon}{\partial v_i}(u|v) \right) dx \\ &= -\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i \left| \frac{\nabla u_i}{\sqrt{u_i + \varepsilon}} - \sqrt{u_i + \varepsilon} \frac{\nabla v_i}{v_i} \right|^2 dx \\ &\quad - \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i \left(\frac{u_i}{u_i + \varepsilon} \nabla p_i[u] \cdot \nabla u_i - \frac{u_i}{v_i} \nabla p_i[u] \cdot \nabla v_i - \nabla p_i[v] \cdot \nabla u_i \right. \\ &\quad \left. + \frac{u_i + \varepsilon}{v_i} \nabla p_i[v] \cdot \nabla v_i \right) dx. \end{aligned}$$

The first integral is nonpositive. Thus, an integration over $(0, t)$ gives

$$\begin{aligned}
 & H_\varepsilon(u(t)|v(t)) - H_\varepsilon(u(0)|v(0)) \\
 & \leq - \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i(u_i + \varepsilon) \nabla(p_i[u] - p_i[v]) \cdot \nabla \log \frac{u_i + \varepsilon}{v_i} dx ds \\
 & \quad + \varepsilon \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i \nabla p_i[u] \cdot \nabla \log \frac{u_i + \varepsilon}{v_i} dx ds =: I_1 + I_2.
 \end{aligned} \tag{29}$$

Step 2: Estimation of I_1 and I_2 . Inserting the definition of p_i ,

$$\begin{aligned}
 \nabla(p_i[u] - p_i[v])(x) &= \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}(x - y) \nabla(u_j - v_j)(y) dy \\
 &= \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}(x - y) \left((u_j + \varepsilon)(y) \nabla \log \frac{u_j + \varepsilon}{v_j}(y) \right. \\
 & \quad \left. + (u_j - v_j)(y) \nabla \log v_j(y) + \varepsilon \nabla \log v_j(y) \right) dy,
 \end{aligned}$$

leads to

$$\begin{aligned}
 I_1 &= - \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) \left((u_i + \varepsilon)(x)(u_j + \varepsilon)(y) \nabla \log \frac{u_j + \varepsilon}{v_j}(y) \right. \\
 & \quad \left. \times \nabla \log \frac{u_i + \varepsilon}{v_i}(x) + (u_i + \varepsilon)(x)(u_j - v_j)(y) \nabla \log v_j(y) \cdot \nabla \log \frac{u_i + \varepsilon}{v_i}(x) \right) dx dy ds \\
 & \quad - \varepsilon \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (u_i + \varepsilon)(x) \nabla \log v_j(y) \cdot \nabla \log \frac{u_i + \varepsilon}{v_i}(x) dx dy ds \\
 & =: I_{11} + I_{12}.
 \end{aligned}$$

Setting

$$U_i = (u_i + \varepsilon) \nabla \log \frac{u_i + \varepsilon}{v_i}, \quad V_i = \frac{1}{2} (u_i - v_i) \nabla \log v_i,$$

we can formulate the first integral as

$$\begin{aligned}
 I_{11} &= - \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (U_i(x) \cdot U_j(y) + 2U_i(x) \cdot V_j(y)) dx dy ds \\
 &= - \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (U_i + V_i)(x) \cdot (U_j + V_j)(y) dx dy ds \\
 & \quad + \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) V_i(x) \cdot V_j(y) dx dy ds \\
 &\leq \frac{1}{4} \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x - y) (u_i - v_i)(x)(u_j - v_j)(y) \nabla \log v_i(x) \cdot \nabla \log v_j(y) dx dy ds \\
 &\leq \frac{1}{4} \max_{i,j=1,\dots,n} \|\pi_i K_{ij}\|_{L^\infty(\mathbb{T}^d)} \max_{k=1,\dots,n} \|\nabla \log v_k\|_{L^\infty(Q_T)}^2 \\
 & \quad \times \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} |(u_i - v_i)(x)| dx \int_{\mathbb{T}^d} |(u_j - v_j)(y)| dy ds \\
 &\leq C \sum_{i=1}^n \int_0^t \left(\int_{\mathbb{T}^d} |u_i - v_i| dx \right)^2 ds,
 \end{aligned}$$

using the symmetry and positive definiteness of $\pi_i K_{ij}$ as well as the regularity assumptions on K_{ij} and $\nabla \log v_i$. This estimate is crucial, as it will allow us later to apply Gronwall’s lemma to conclude the equality of $u_i(t)$ and $v_i(t)$. The second integral I_{12} is estimated as

$$\begin{aligned} I_{12} &= -\varepsilon \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) \nabla \log v_j(y) \cdot (\nabla u_i - (u_i + \varepsilon) \nabla \log v_i)(x) dx dy ds \\ &\leq \varepsilon C \sum_{i,j=1}^n \|\nabla \log v_j\|_{L^\infty(Q_T)} \int_0^t \int_{\mathbb{T}^d} (|\nabla u_i| + (u_i + 1) |\nabla \log v_i|) dx ds \\ &\leq \varepsilon C \sum_{i=1}^n (\|\nabla u_i\|_{L^1(Q_T)} + \|u_i\|_{L^1(Q_T)} + 1) \leq \varepsilon C. \end{aligned}$$

We conclude that

$$I_1 \leq C \sum_{i=1}^n \int_0^t \left(\int_{\mathbb{T}^d} |u_i - v_i| dx \right)^2 ds + \varepsilon C.$$

It remains to estimate I_2 . Here we need the improved regularity $\nabla u_i \in L^2(Q_T)$. Inserting the definition of $p_i[u]$, we have

$$I_2 = \varepsilon \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) \nabla u_j(y) \cdot \nabla \log \frac{u_i + \varepsilon}{v_i}(x) dx dy ds.$$

Since

$$\varepsilon |\nabla u_j(y) \cdot \nabla \log(u_i + \varepsilon)(x)| = 2\varepsilon \left| \nabla u_j(y) \cdot \frac{\nabla \sqrt{u_i + \varepsilon}}{\sqrt{u_i + \varepsilon}}(x) \right| \leq 2\sqrt{\varepsilon} |\nabla u_j(y)| |\nabla \sqrt{u_i(x)}|,$$

we find that

$$I_2 \leq C \sum_{i,j=1}^n (\varepsilon \|\nabla u_j\|_{L^1(Q_T)} + \sqrt{\varepsilon} \|\nabla u_j\|_{L^2(Q_T)} \|\nabla \sqrt{u_i}\|_{L^2(Q_T)}) \leq \sqrt{\varepsilon} C.$$

We summarize the estimates for I_1 and I_2 and conclude from (29) that

$$H_\varepsilon(u(t)|v(t)) - H_\varepsilon(u(0)|v(0)) \leq C \sum_{i=1}^n \int_0^t \left(\int_{\mathbb{T}^d} |u_i - v_i| dx \right)^2 ds + \sqrt{\varepsilon} C. \tag{30}$$

Step 3: Limit $\varepsilon \rightarrow 0$. We perform first the limit in $H_\varepsilon(u(t)|v(t))$. We claim that for $0 < \varepsilon < 1$,

$$|(u_i + \varepsilon)(\log(u_i + \varepsilon) - 1)| \leq u_i(\log u_i + 1) + 1 + e^{-2}.$$

Indeed, set $f(s) := (u_i + \varepsilon)(\log(u_i + \varepsilon) - 1)$ and $g(s) := u_i(\log u_i + 1) + 1 + e^{-2}$ for $s \geq 0$. A computation shows that $|f(s)| \leq 1$ and $g(s) \geq 1$ on the interval $[0, e - \varepsilon]$. Thus, $|f(s)| \leq g(s)$ on this interval. Next, it holds that $0 = f(e - \varepsilon) < g(e - \varepsilon)$ and $f'(s) = \log(s + \varepsilon) = \log(1 + \varepsilon/s) + \log s < 2 + \log s = g'(s)$ for $s > e - \varepsilon$. This implies that $f(s) < g(s)$ for $s > e - \varepsilon$ and proves the claim. Since $u_i \in L^2(0, T; H^1(\mathbb{T}^d)) \cap H^1(0, T; H^{-1}(\mathbb{T}^d)) \hookrightarrow C^0([0, T]; L^2(\mathbb{T}^d))$, we infer that

$$|(u_i + \varepsilon)(\log(u_i + \varepsilon) - 1)| \leq u_i(\log u_i + 1) + C \in L^\infty(0, T; L^1(\mathbb{T}^d)).$$

Therefore, by dominated convergence, as $\varepsilon \rightarrow 0$,

$$\sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i (u_i(t) + \varepsilon)(\log(u_i(t) + \varepsilon) - 1) dx \rightarrow \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i(t)(\log u_i(t) - 1) dx,$$

and this convergence holds for a.e. $t \in (0, T)$. Furthermore, the assumption $c \leq v_i \leq C$ in Q_T shows that $\log v_i$ is bounded in Q_T and hence,

$$\sum_{i=1}^n \pi_i (-(u_i + \varepsilon) \log v_i + v_i) \leq C(v) \left(\sum_{i=1}^n u_i + 1 \right) \in L^\infty(0, T; L^1(\mathbb{T}^d)).$$

By dominated convergence again,

$$\sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i(-u_i(t) + \varepsilon) \log v_i(t) + v_i(t) dx \rightarrow \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i(-u_i(t) \log v_i(t) + v_i(t)) dx.$$

This shows that for a.e. $t \in (0, T)$,

$$\begin{aligned} H_\varepsilon(u(t)|v(t)) &\rightarrow H(u(t)|v(t)) \quad \text{as } \varepsilon \rightarrow 0, \text{ where} \\ H(u|v) &= \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i(u_i(\log u_i - 1) - u_i \log v_i + v_i) dx, \end{aligned}$$

and $H_\varepsilon(u(0)|v(0)) = H_\varepsilon(u^0|u^0) \rightarrow 0$. Then we deduce from (30) in the limit $\varepsilon \rightarrow 0$ that

$$H(u(t)|v(t)) \leq C \sum_{i=1}^n \int_0^t \|u_i - v_i\|_{L^1(\mathbb{T}^d)}^2 ds. \tag{31}$$

The constant C depends on the L^∞ -norm of $\nabla \log v_i$, which is bounded thanks to our assumptions.

Taking the test function $\phi_i = 1$ in the weak formulation of (1), we find that $\int_{\mathbb{T}^d} u_i^0 dx = \int_{\mathbb{T}^d} u_i(t) dx$ for all $t > 0$. Since u and v have the same initial data, it follows that $\int_{\mathbb{T}^d} u_i(t) dx = \int_{\mathbb{T}^d} v_i(t) dx$ for all $t > 0$. Thus, by the classical Csiszár–Kullback–Pinsker inequality [37, Theorem A.2], we have

$$\begin{aligned} H(u|v) &= \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i \log \frac{u_i}{v_i} dx + \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i (v_i - u_i) dx = \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i \log \frac{u_i}{v_i} dx \\ &\geq 2 \left(\int_{\mathbb{T}^d} u_i^0 dx \right)^{-1} \sum_{i=1}^n \|u_i - v_i\|_{L^1(\mathbb{T}^d)}^2 = C(u^0) \sum_{i=1}^n \|u_i - v_i\|_{L^1(\mathbb{T}^d)}^2, \end{aligned}$$

We infer from (31) that

$$\sum_{i=1}^n \|(u_i - v_i)(t)\|_{L^1(\mathbb{T}^d)}^2 \leq C \int_0^t \sum_{i=1}^n \|u_i - v_i\|_{L^1(\mathbb{T}^d)}^2 ds.$$

Gronwall’s inequality implies that $\|(u_i - v_i)(t)\|_{L^1(\mathbb{T}^d)} = 0$ and hence $u_i(t) = v_i(t)$ in \mathbb{T}^d for a.e. $t > 0$ and $i = 1, \dots, n$.

5. Localization limit

We prove Theorem 5. Let u^η be the nonnegative weak solution to (1)–(2) with kernel (13), constructed in Theorem 2. The entropy inequalities (11) and (12) as well as the proof of Theorem 2 show that all estimates are independent of η . More precisely, the right-hand side of (12) depends on K_{ij}^η , but in view of [9, Theorem 4.22], it holds that $K_{ij}^\eta * u_j \rightarrow a_{ij} u_j$ as $\eta \rightarrow 0$ and therefore, we can bound the right-hand side of (12) uniformly in η . Therefore, for $i = 1, \dots, n$ (see (18)–(20), (22)–(24)),

$$\begin{aligned} \|u_i^\eta \log u_i^\eta\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \|u_i^\eta\|_{L^{1+2/d}(Q_T)} + \|u_i^\eta\|_{L^q(0,T;W^{1,q}(\mathbb{T}^d))} \leq C, \\ \|(u_i^\eta)^{1/2}\|_{L^2(0,T;H^1(\mathbb{T}^d))} + \|\partial_t u_i^\eta\|_{L^q(0,T;W^{1,d+2}(\mathbb{T}^d)')} + \|(u_i^\eta)^{1/2} \nabla p_i^\eta[u^\eta]\|_{L^2(Q_T)} \leq C, \end{aligned}$$

where $C > 0$ is independent of η , $q = (d + 2)/(d + 1)$, and $p_i^\eta[u^\eta] = \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}^\eta(x - y) u_j^\eta(y) dy$. We infer from the Aubin–Lions lemma in the version of [8,28] that there exists a subsequence (not relabeled) such that, as $\eta \rightarrow 0$,

$$u_i^\eta \rightarrow u_i \quad \text{strongly in } L^2(0, T; L^{d/(d-1)}(\mathbb{T}^d)), \quad i = 1, \dots, n, \tag{32}$$

if $d \geq 2$ and strongly in $L^2(0, T; L^r(\mathbb{T}^d))$ for any $r < \infty$ if $d = 1$. Moreover,

$$\nabla u_i^\eta \rightharpoonup \nabla u_i \quad \text{weakly in } L^q(Q_T), \quad i = 1, \dots, n, \tag{33}$$

$$\partial_t u_i^\eta \rightharpoonup \partial_t u_i \quad \text{weakly in } L^q(0, T; W^{1, d+2}(\mathbb{T}^d)'), \tag{34}$$

$$(u_i^\eta)^{1/2} \nabla p_i^\eta[u^\eta] \rightharpoonup z_i \quad \text{weakly in } L^2(Q_T), \tag{35}$$

where $z_i \in L^2(Q_T)$ for $i = 1, \dots, n$.

As in Section 3, the main difficulty is the identification of z_i with $u_i^{1/2} \nabla p_i[u]$, where $p_i[u] := \sum_{j=1}^n a_{ij} \nabla u_j$. Since the kernel functions also depend on η , the proof is different from that one in Section 3. We claim that

$$\nabla p_i^\eta[u^\eta] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^q(Q_T). \tag{36}$$

Indeed, let $\phi \in L^{q'}(Q_T; \mathbb{R}^n)$, where $q' = d + 2$ satisfies $1/q + 1/q' = 1$. We compute

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{T}^d} (\nabla p_i^\eta[u^\eta] - \nabla p_i[u]) \cdot \phi \, dx dt \right| \\ &= \left| \sum_{j=1}^n \int_0^T \int_{\mathbb{T}^d} \left(\int_{\mathbb{T}^d} K_{ij}^\eta(x-y) \nabla u_j^\eta(y, t) \, dy \right) \cdot \phi(x, t) \, dx dt \right. \\ & \quad \left. - \sum_{j=1}^n \int_0^T \int_{\mathbb{T}^d} a_{ij} \nabla u_j(y, t) \cdot \phi(y, t) \, dy dt \right| \\ &\leq \sum_{j=1}^n \left| \int_0^T \int_{\mathbb{T}^d} \left(\int_{\mathbb{T}^d} K_{ij}^\eta(x-y) \phi(x, t) \, dx - a_{ij} \phi(y, t) \right) \cdot \nabla u_j^\eta(y, t) \, dy dt \right| \\ & \quad + \sum_{j=1}^n a_{ij} \left| \int_0^T \int_{\mathbb{T}^d} \nabla(u_j^\eta - u_j)(y, t) \cdot \phi(y, t) \, dy dt \right| \\ &\leq \sum_{j=1}^n \left\| \int_{\mathbb{T}^d} K_{ij}^\eta(\cdot - y) \phi(y) \, dy - a_{ij} \phi \right\|_{L^{q'}(Q_T)} \|\nabla u_j^\eta\|_{L^q(Q_T)} \\ & \quad + \sum_{j=1}^n a_{ij} \left| \int_0^T \int_{\mathbb{T}^d} \nabla(u_j^\eta - u_j)(y, t) \cdot \phi(y, t) \, dy dt \right|. \end{aligned}$$

Since B has compact support in \mathbb{R} , we can apply Lemma 16 to infer that the first term on the right-hand side, formulated as the convolution $K_{ij}^\eta * \phi - a_{ij} \phi$ (slightly abusing the notation), converges to zero strongly in $L^{q'}(\mathbb{R}^d)$ as $\eta \rightarrow 0$. Thus, taking into account the weak convergence (33), convergence (36) follows.

Because of the convergences (32), (35), and (36), we can apply Lemma 13 in Appendix A to infer that $z_i = u_i^{1/2} \nabla p_i[u]$. Therefore,

$$u_i^\eta \nabla p_i^\eta[u^\eta] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(Q_T).$$

Estimate (35) shows that the convergence holds in $L^q(Q_T)$. This convergence as well as (33) and (34) allow us to perform the limit $\eta \rightarrow 0$ in the weak formulation of (1), proving that u solves (1) and (3).

Appendix A. Auxiliary results

We recall the Young convolution inequality (the proof in [9, Theorem 4.33] also applies to the torus).

Lemma 11 (Young’s Convolution Inequality). *Let $1 \leq p \leq q \leq \infty$ be such that $1 + 1/q = 1/p + 1/r$ and let $K \in L^r(\mathbb{T}^d)$ (extended periodically to \mathbb{R}^d). Then for any $v \in L^p(\mathbb{T}^d)$,*

$$\left\| \int_{\mathbb{T}^d} K(\cdot - y) v(y) \, dy \right\|_{L^q(\mathbb{T}^d)} \leq \|K\|_{L^r(\mathbb{T}^d)} \|v\|_{L^p(\mathbb{T}^d)}.$$

The next result is a consequence of Vitali’s lemma and is well known. We recall it for the convenience of the reader.

Lemma 12. *Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded domain, $1 < p < \infty$, and $u_\varepsilon, u \in L^1(\Omega)$ be such that (u_ε) is bounded in $L^p(\Omega)$ and $u_\varepsilon \rightarrow u$ a.e. in Ω . Then $u_\varepsilon \rightarrow u$ strongly in $L^r(\Omega)$ for all $1 \leq r < p$.*

Proof. We have for any $M > 0$,

$$\int_{\{u_\varepsilon \geq M\}} |u_\varepsilon|^r dx = \int_{\{u_\varepsilon \geq M\}} |u_\varepsilon|^p |u_\varepsilon|^{-(p-r)} dx \leq M^{-(p-r)} \int_{\Omega} |u_\varepsilon|^p dx \leq CM^{-(p-r)} \rightarrow 0,$$

as $M \rightarrow \infty$. Thus, (u_ε) is uniformly integrable. Since convergence a.e. implies convergence in measure, we conclude with Vitali’s convergence theorem. \square

The following lemma specifies conditions under which the limit of the product of two converging sequences can be identified.

Lemma 13. *Let $p > 1$ and let $u_\varepsilon \geq 0, u_\varepsilon \rightarrow u$ strongly in $L^p(\mathbb{T}^d), v_\varepsilon \rightharpoonup v$ weakly in $L^p(\mathbb{T}^d)$, and $u_\varepsilon v_\varepsilon \rightharpoonup w$ weakly in $L^p(\mathbb{T}^d)$ as $\varepsilon \rightarrow 0$. Then $w = uv$.*

The lemma is trivial if $p \geq 2$. We apply it in Section 3 with $1 < p < 2$. Note that the strong convergence of (u_ε) cannot be replaced by weak convergence. A simple counter-example is given by $u_\varepsilon(x) = \exp(2\pi i x/\varepsilon) \rightarrow 0, v_\varepsilon(x) = \exp(-2\pi i x/\varepsilon) \rightarrow 0$ weakly in $L^2(-1, 1)$, but $u_\varepsilon v_\varepsilon \equiv 1 \neq 0 \cdot 0$.

Proof. We define the truncation function $T_1 \in C^2([0, \infty))$ satisfying $T_1(s) = s$ for $0 \leq s \leq 1, T_1(s) = 2$ for $s > 3$, and T_1 is nondecreasing and concave in the interval $[1, 3]$. Furthermore, we set $T_k(s) = kT_1(s/k)$ for $s \geq 0$ and $k \in \mathbb{N}$. The strong convergence of (u_ε) implies the existence of a subsequence that is not relabeled such that $u_\varepsilon \rightarrow u$ a.e. Hence, $T_k(u_\varepsilon) \rightarrow T_k(u)$ a.e. Since T_k is bounded for fixed $k \in \mathbb{N}$, we conclude by dominated convergence that $T_k(u_\varepsilon) \rightarrow T_k(u)$ strongly in $L^r(\mathbb{T}^d)$ for any $r < \infty$. Because of the uniqueness of the limit, the convergence holds for the whole sequence. Thus, $T_k(u_\varepsilon)v_\varepsilon \rightharpoonup T_k(u)v$ weakly in $L^1(\mathbb{T}^d)$. Writing $\overline{z_\varepsilon}$ for the weak limit of a sequence (z_ε) (if it exists), this result means that $\overline{T_k(u_\varepsilon)v_\varepsilon} = T_k(u)v$ and the assumption translates to $\overline{u_\varepsilon v_\varepsilon} = w$. Consequently, $w - T_k(u)v = \overline{(u_\varepsilon - T_k(u_\varepsilon))v_\varepsilon}$. Then we can estimate

$$\begin{aligned} \|w - T_k(u)v\|_{L^1(\mathbb{T}^d)} &\leq \sup_{0 < \varepsilon < 1} \int_{\mathbb{T}^d} |u_\varepsilon - T_k(u_\varepsilon)| |v_\varepsilon| dx \leq \sup_{0 < \varepsilon < 1} \int_{\{|u_\varepsilon| > k\}} |u_\varepsilon| |v_\varepsilon| dx \\ &\leq \frac{1}{k^{p-1}} \sup_{0 < \varepsilon < 1} \int_{\{|u_\varepsilon| > k\}} |u_\varepsilon|^p |v_\varepsilon| dx \leq \frac{1}{k^{p-1}} \sup_{0 < \varepsilon < 1} \int_{\mathbb{T}^d} |u_\varepsilon|^p (1 + |v_\varepsilon|^p) dx \leq \frac{C}{k^{p-1}}. \end{aligned}$$

where we used the properties $u_\varepsilon = T_k(u_\varepsilon)$ for $u_\varepsilon \leq k$ and $|u_\varepsilon - T_k(u_\varepsilon)| \leq |u_\varepsilon|$ due to $T_k(u_\varepsilon) \leq u_\varepsilon$ for $u_\varepsilon > k$. The constant C depends on the bounds for u_ε in $L^p(\mathbb{T}^d)$ and $u_\varepsilon v_\varepsilon$ in $L^p(\mathbb{T}^d)$. We infer that shows that $T_k(u)v \rightarrow w$ strongly in $L^1(\mathbb{T}^d)$ and (for a subsequence) a.e. as $k \rightarrow \infty$. Since $T_k(u)v = uv$ in $\{|u| \leq k\}$

for any $k \in \mathbb{N}$ and $\text{meas}\{|u| > k\} \leq \|u\|_{L^1(\mathbb{T}^d)}/k \rightarrow 0$, we infer in the limit $k \rightarrow \infty$ that $w = uv$ a.e. in \mathbb{T}^d . \square

For the convenience of the reader, we recall the Gagliardo–Nirenberg inequality [48, Appendix A, (54a)] and the Poincaré–Wirtinger inequality [29, Sec. 5.8.1, Theorem 1].

Lemma 14 (Gagliardo–Nirenberg Inequality). *Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded domain with Lipschitz boundary. Let $m \in \mathbb{N}$, $\beta \in \mathbb{N}_0^d$ be such that $0 \leq |\beta| \leq m - 1$, let $1 \leq p, q, r \leq \infty$, and let $\theta \in [0, 1]$ be such that*

$$\frac{1}{p} = \frac{|\beta|}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\theta + (1 - \theta)\frac{1}{q}, \quad \frac{|\beta|}{m} \leq \theta \leq 1.$$

Then there exists a constant $C > 0$ such that for all $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, there holds

$$\|D^\beta u\|_{L^p(\Omega)} \leq C \|u\|_{W^{m,r}(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}.$$

If $1 < r < \infty$ and $m - |\beta| - d/r$ is a nonnegative integer, then the inequality holds only for $|\beta|/m \leq \theta < 1$.

Lemma 15 (Poincaré–Wirtinger Inequality). *Let $\Omega \subset \mathbb{R}^d$ ($d \geq q$) be a bounded domain with Lipschitz boundary and let $1 \leq p \leq \infty$. Then there exists a constant $C > 0$ such that for all $u \in W^{1,p}(\Omega)$,*

$$\left\| u - \frac{1}{|\Omega|} \int_{\Omega} u dx \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

In particular, we have

$$\|u\|_{L^p(\Omega)} \leq C(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}).$$

The following lemma states the convergence of a convolution with a sequence of mollifiers [9, Theorem 4.22].

Lemma 16. *Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d)$, and let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of mollifiers, i.e.*

$$\rho_n \in C_0^\infty(\mathbb{R}^d), \quad \text{supp } \rho_n \subset \overline{B(0, 1/n)}, \quad \int \rho_n dx = 1, \quad \rho_n \geq 0 \text{ on } \mathbb{R}^d.$$

*Then $\rho_n * f \rightarrow f$ in $L^p(\mathbb{R}^d)$ as $n \rightarrow \infty$.*

Appendix B. Local cross-diffusion system

The existence of global weak solutions to the local system (1) and (3) in any bounded polygonal domain was shown in [38] by analyzing a finite-volume scheme. For existence results on related systems, we refer to, e.g., [1,2,15,21]. For completeness, we state the assumptions and the theorem and indicate how the result can be proved using the techniques of Section 3. We assume that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with Lipschitz boundary $\partial\Omega$, $T > 0$, and $u^0 \in L^2(\Omega)$ satisfies $u_i^0 \geq 0$ in Ω for $i = 1, \dots, n$. We set $Q_T = \Omega \times (0, T)$.

Theorem 17 (Existence for the Local System). *Let $\sigma > 0$, $a_{ij} \geq 0$, and let the matrix $(u_i a_{ij})$ be positively stable for all $u_i > 0$, $i = 1, \dots, n$. Assume that there exist $\pi_1, \dots, \pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i, j = 1, \dots, n$. Then there exists a global weak solution to (1) and (3) with no-flux boundary conditions, satisfying $u_i \geq 0$ in Q_T and*

$$u_i \in L^2(0, T; H^1(\Omega)) \cap L^{2+4/d}(Q_T), \quad \partial_t u_i \in L^q(0, T; W^{-1,q}(\Omega)),$$

for $i = 1, \dots, n$, where $q = (d + 2)/(d + 1)$. The initial datum in (1) is satisfied in the sense of $W^{-1,q}(\Omega)$. Moreover, the following entropy inequalities are satisfied:

$$\begin{aligned} \frac{dH_1}{dt} + 4\sigma \sum_{i=1}^n \int_{\Omega} \pi_i |\nabla \sqrt{u_i}|^2 dx + \alpha \sum_{i=1}^n \int_{\Omega} |\nabla u_i|^2 dx &\leq 0, \\ \frac{dH_2^0}{dt} + \sum_{i=1}^n \int_{\Omega} \pi_i u_i |\nabla p_i[u]|^2 dx + \alpha\sigma \sum_{i=1}^n \int_{\Omega} |\nabla u_i|^2 dx &\leq 0, \end{aligned} \tag{37}$$

where $\alpha > 0$ is the smallest eigenvalue of $(\pi_i a_{ij})$ and $H_2^0(u) := \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} \pi_i a_{ij} u_i u_j dx \geq 0$.

We call a matrix *positively stable* if all eigenvalues have a positive real part. This condition means that (1) is parabolic in the sense of Petrovskii, which is a minimal condition to ensure the local solvability [2]. Inequalities (5)–(6) and (37) reveal a link between the entropy structures of the nonlocal and local systems. This link was explored recently in detail for related systems in [26].

Proof. If Ω is the torus, the theorem is a consequence of the localization limit (Theorem 5). If Ω is a bounded domain, the result can be proved by using the techniques of the proof of Theorem 2. In fact, the proof is simpler since the problem is local. The entropy identities are (formally)

$$\begin{aligned} \frac{dH_1}{dt} + 4\sigma \sum_{i=1}^n \int_{\Omega} \pi_i |\nabla \sqrt{u_i}|^2 dx &= - \sum_{i,j=1}^n \int_{\Omega} \pi_i a_{ij} \nabla u_i \cdot \nabla u_j dx, \\ \frac{dH_2^0}{dt} + \sum_{i=1}^n \int_{\Omega} \pi_i u_i |\nabla p_i[u]|^2 dx &= -\sigma \sum_{i,j=1}^n \int_{\Omega} \pi_i a_{ij} \nabla u_i \cdot \nabla u_j dx. \end{aligned} \tag{38}$$

We claim that the matrix $(\pi_i a_{ij})$ is positive definite. Let $A_1 := \text{diag}(u_i/\pi_i)$ and $A_2 := (\pi_i a_{ij})$. Then A_1 is symmetric and positive definite; by our assumptions, A_2 is symmetric and $A_1 A_2 = (u_i a_{ij})$ is positively stable. Therefore, by [20, Prop. 3], A_2 is positive definite. We infer that the right-hand sides in (38) are nonpositive, and we derive estimates for an approximate family of u_i in $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H^1(\Omega))$. By the Gagliardo–Nirenberg inequality, this yields bounds for u_i in $L^{2+4/d}(Q_T)$. Consequently, $u_i \nabla p_i[u]$ is bounded in $L^q(Q_T)$, where $q = (d + 2)/(d + 1)$ (we can even choose $q = 4(d + 2)/(3d + 4)$), and $\partial_t u_i$ is bounded in $L^q(0, T; W^{-1,q}(\Omega))$. These estimates are sufficient to deduce from the Aubin–Lions lemma the relative compactness for the approximate family of u_i in $L^2(Q_T)$. The limit in the approximate problem, similar to (14), shows that the limit satisfies (1) and (3). Finally, using the lower semicontinuity of convex functions and the norm, the weak limit in the entropy inequalities leads to (37). \square

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